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CHAPTER TWO
RANDOM VARIABLE AND PROBABILITY DISTRIBUTIONS

- **Definition** : Given a random experiment with a sample space S, a function X that assigns to each element s in S one and only one real number $X(s) = x$ is called a **random variable(r.v.)**.
  - A random variable is a numerical description of the outcomes of an experiment or a numerical valued function defined on sample space, usually denoted by capital letters.
- **Example** : Let X be the number of defective Laptop products when three Laptops are tested.

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Let $D = Defective$ and $N = None\ defective\ item$

The sample space S will be;

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

Let X be a function defined on S such that:

$$X(DDD) = 3,$$

$$X(DDN) = X(DND) = X(NDD) = 2,$$

$$X(DNN) = X(NDN) = X(NND) = 1,$$

$$X(NNN) = 0$$

- The function X assigns a real number to each element s in S. Thus X is a random variable. $R_x = (0,1,2,3)$ is called the range space of X.

Classification of Random variables

- Random variables may be divided into two types:
  - *Discrete random variables* and
  - *Continuous random variables*.

❑ Discrete Random Variable

❖ Discrete random variable assume only a countable number of values.

Examples:

- Number of attempts to solve a coding challenge.
- Number of users on a website.
- Number of lines of code written.

➢ For a random variable X of discrete type, we associate a number $P(x_i), 0 \leq P(x_i) \leq 1$ for $i = 1,2,3, \dots .$ Which is probability of x_i

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- The function $P(X) = P(X = x)$ is called the probability mass function (pmf) if $P(x), x\epsilon R$ satisfy the following properties.

   I. $P(x_i) \geq 0 \; for \; all \; x\epsilon R$;

   II. $\sum_{x\epsilon R} P(x) = 1$;

   III. $P(X = x) = F(x)$

- The collection of pairs $\{x_i, P(x_i)\}$ for $i = 1,2,3, \ldots$ is called a probability distribution.

- Since X is a discrete random variable here we call it a discrete probability distribution.

Example-1

- Consider an experiment of testing PC products in a company. Let X be the number of defective products . Construct the probability distribution of X when three PCs are tested.

Solution:

- First identify the possible value that X can assume.
- Calculate the probability of each possible distinct value of X and express X in the form of frequency distribution.

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- From our previous Example
- ✓ $X = (0,1,2,3)$
- ✓ Sample space $=2^n = 2^3$=8 OR

$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$=8

- ✓ and the probability distribution of X (number of defective products) is given by:

$X = x$	0	1	2	3
$p(X = x)$	1/8	3/8	3/8	1/8

Exercise

❖ Let X be a random variable denoting the number of bits equal to 1 in an 8 bit random binary number. Find the sample space and construct the probability distribution of X. What is the probability that at most 3 bits ON?

Continuous random variables

- A sample space contains an infinite number of outcomes (non countably infinite) equal to the number of points on a line segment, it is called continuous sample space.
- Let X denote a random variable with space R, an interval or union of intervals. Such a set R is called continuous sample space.
- The random variable X is called a continuous $r.v.$,and X is said to have a continuous distribution denoted as $f(x)$.

Examples:

- CPU processing time.
- Mark of a student.

Cont'd

- The function $f(x)$ is called probability density function (pdf) of a $r.v.X$, if it satisfy the following properties:
  I. $f(x) \geq 0, \ x \epsilon R;$
  II. $\int f(x)dx = 1$
  III. $P(a \leq X \leq b) = \int_a^b f(x)dx = 1$
- Probability means area for continuous random variable.
- A continuous random variable has a probability of zero of assuming any one of its values. i.e. if X is continuous $P(x = a) = 0$

Example-1

- Consider the function

$$f(x)\begin{cases} c(2x - x^2) & 0 \le x \le 2, c > 0 \\ 0 & elsewhere \end{cases}$$

I. Find the value of c so that $f(x)$ defines a probability density function.

II. Find $P(0 \le x \le 1)$ and $P(\frac{1}{2} \le x \le \frac{2}{3})$

Solution

I) For a probability density function

$\int_{-\infty}^{\infty} f(x)dx = 1$, $(note\ f(x) \ge 0\ for\ all)$. So

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$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{2} c(2x - x^2)dx = c\left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= c\lfloor 4 - {}^{8}/_{3}\rfloor = {}^{4c}/_{3} = 1 =$$

$$c = {}^{3}/_{4}$$

II) $P(0 \leq x \leq 1) = \int_0^1 f(x)dx = \int_0^1 {}^{3}/_{4}\,(2x - x^2)dx$

$$= {}^{3}/_{4}\left[x^2 - \frac{x^3}{3}\right]_0^1$$

$${}^{3}/_{4}\lfloor 1 - {}^{1}/_{3}\rfloor = {}^{1}/_{2}$$

III) $P(\frac{1}{2} \leq x \leq \frac{2}{3})$ =??

Exercise

Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{6}x + k, & 0 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$$

I. Evaluate k

II. Find $P(1 \leq x \leq 2)$

Properties of Probability Distribution

1) $p(x) \geq 0, \qquad if\ X\ is\ discrete.$

$f(x) \geq 0, \qquad if\ X\ is\ continuous.$

2) $\sum_x p(X = x) = 1, \qquad if\ X\ is\ discrete.$

$$\int f(x)dx = 1, \qquad if\ X\ is\ continuous$$

Two Dimensional Random Variables

- We may define two or more random variables on the same sample space .
- Let S be the sample space associated with a random experiment E. Let $X \; and \; Y$ be two real random variables defined on the same space.
- Let $X = X(s) and \; Y = Y(s)$ be two functions assigning a real number to each outcomes $s \in S$. Then (x, y) is called two dimensional random variable.

Example: Suppose in a communication system, X is a transmitted signal and Y is the corresponding noisy received signal. Then (x, y) is a joint random variable.

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Note

I. If the possible values of (X, Y) are finite or countable infinite, (X, Y) is called a two dimensional discrete random variable.

II. If (X, Y) can assume all values in a specific region R in the xy-plane, (X, Y) is called a two dimensional continuous random variable.

Joint Probability Mass Function (Discrete case)

If (X, Y) is a two dimensional discrete random variable such that $P(X = x_i, Y = y_i) = p(x, y)$, then $p(x, y)$ is called the probability mass function of (X, Y) provided that

I. $p(x, y) \geq 0 \;\; for\ all\ x, y$

II. $\sum_R p(x, y) = 1$

III. $P(X = x, Y = y) = F(x, y)$

Example: Suppose that two machines are used for particular task in the morning and for a different task in the afternoon. Let X and Y represent the number of times that a machine breaks down in the morning and in the afternoon respectively. The table below gives the joint probability distribution of (x,y)

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		Y			
		0	1	2	Total
X	0	0.25	0.15	0.10	0.50
	1	0.10	0.08	0.07	0.25
	2	0.05	0.07	0.13	0.25
	Total	0.40	0.30	0.30	1

a) What is the probability that the machine breaks down equal number of times in the morning and in the afternoon?

b) What is the probability that the machine breaks down greater number of times in the morning and in the afternoon?

Solution

a) $p(x = y) = p(x = 0, y = 0) + p(x = 1, y = 1) + p(x = 2, y = 2) = 0.25 + 0.08 + 0.13 = 0.46$

b) $p(x > y) = p(x = 1, y = 0) + p(x = 2, y = 0) + p(x = 2, y = 1) = 0.15 + 0.10 + 0.07 = 0.32$

Joint Probability Density function (Continuous Case)

If (X, Y) is a two-dimensional continuous RV then $f(x.y)$ is called the joint pdf (X, Y), provided that $f(x.y)$ satisfies the following conditions.

I. $f(x, y) \geq 0 \quad for\ all\ x, y;$

II. $\iint f(x, y)dx\, dy = 1\ ;$

III. $P\{a < X \leq b, c < Y \leq d\} = \int_a^b \int_c^d f(x, y)dydx$

Example

- Let (X, Y) be a continuous RV with a joint pdf $f(x, y) =$

$$f(x,y) = \begin{cases} c \ \ if\ 0 < x < 2\,, 0 < y < 4 \\ 0 \qquad\qquad otherwise \end{cases}$$

a) Determine c

b) Find $P(x < 1, y < 3)$

Solution

a. $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)dxdy = \int\limits_{0}^{4}\int\limits_{0}^{2} cdxdy = 1 \Rightarrow \int\limits_{0}^{4}\left[\int\limits_{0}^{2} cdx\right]dy = c\int\limits_{0}^{4}(x|_{0}^{2})dy = 2c\int\limits_{0}^{4} dy = 2c(4-0) = 8c = 1 \Leftrightarrow c = \frac{1}{8}$

$$f(x,y) = \begin{cases} \frac{1}{8} \ if\ 0 < x < 2,\ 0 < y < 4 \\ 0 \qquad otherwise \end{cases}$$

b. $p(x < 1, y < 3) = \int\limits_{-\infty}^{3}\int\limits_{-\infty}^{1} f(x,y)dxdy = \int\limits_{0}^{3}\int\limits_{0}^{1}\frac{1}{8}dxdy = \int\limits_{0}^{3}\left[\int\limits_{0}^{1}\frac{1}{8}dx\right]dy = \frac{1}{8}\int\limits_{0}^{3}(x|_{0}^{1})dy = \frac{1}{8}\int\limits_{0}^{3} dy = \frac{1}{8}(3-0) = \frac{3}{8}$

Marginal, Conditional Probability distributions and Independent of RVs

- Marginal probability Distribution

With two dimensional random variable (x, y) we associate one dimensional random variable namely X or Y. that is we may be interested in the probability distribution of X or Y only. We call such probabilities distribution as marginal probability distribution.

a. **In the discrete case**

Let X assumes the values $x_1, x_2, x_3, \ldots, x_m$ and Y assumes the values $y_1, y_2, y_3, \ldots, y_n$ then the probability of an event $X = x_i$ and $Y = y_j$ for $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

$P(X = x) = P\{(X = x_i \text{ and } Y = y_1) \text{ or } (X = x_i \text{ and } Y = y_2) \text{ or}$ etc.

$$= p(x = x_i) = \sum_j p\{x = x_i, y = y_j\}$$ is called the marginal probability function of X.

The collection of pairs $\{X_i, P_{i.}\}$, i = 1, 2, 3,..., m is called the marginal probability distribution of X.

Similarly; $P(Y = y) = P\{(Y = y_j \text{ and } X = x_1) \text{ or } (Y = y_j \text{ and } X = x_2) \text{ or}$ etc.}

$$= p(y = y_j) = \sum_i p\{x = x_i, y = y_j\}$$ is called the marginal probability function of Y.

The collection of pairs $\{y_j, P_{.j}\}$, j = 1, 2, 3,..., n is called the marginal probability distribution of Y.

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b. In the continuous case

Let (X,Y) be a two dimensional continuous random vector with joint pdf f(x,y). Then the individual or marginal distribution of X and Y are defined by the pdf's

$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$ and $h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$, respectively.

Example 1: Recall the example on machine operation. Find the marginal distribution of x and y.

Solution: Marginal of x, $p(x = x_i) = \sum_j p\{x = x_i, y = y_j\}$

If x = 0, $p(x = 0) = p(x = 0, y = 0) + p(x = 0, y = 1) + p(x = 0, y = 2) = 0.25 + 0.10 + 0.05 = 0.40$.

Similarly find $p(x = x_i)$ for i = 1,2.

$$p(x = x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Marginal of y, $p(y = y_i) = \sum_i p\{= x_i, y = y_j\}$

If y = 0, $p(y = 0) = p(x = 0, y = 0) + p(x = 1, y = 0) + p(x = 2, y = 0) = 0.25 + 0.15 + 0.10 = 0.50$.

Similarly find $p(y = y_j)$ for j = 1,2.

$$p(y = y_i) = \begin{cases} 0.5 & \text{if } x = 0 \\ 0.25 & \text{if } x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

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Example 2: Let (x, y) be a two dimensional continuous random variable with joint pdf

$$f(x, y) = \begin{cases} \frac{1}{8} & \text{if } 0 < x < 2, 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal of x and y.

Solution:

Marginal of x

$$g(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^4 \frac{1}{8} dy = \frac{1}{8} y \Big|_0^4 = \frac{1}{2}$$

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Marginal of y

$$h(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^2 \frac{1}{8} dx = \frac{1}{8} x \Big|_0^2 = \frac{1}{4}$$

$$h(y) = \begin{cases} \frac{1}{4} & \text{if } 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Conditional Distribution Function

When two random variables are defined in a random experiment, knowledge one can change the probabilities of the other.

a. **In the discrete case**

Definition: Given two discrete random variables X and Y with joint pmf P(x, y) the conditional pmf of X given Y is defined as: $P(X = x_i \mid Y = y_j) = \dfrac{p(X = x_i, Y = y_j)}{p(Y = y_j)} = \dfrac{P(x_i, y_j)}{P(y_j)}$ similarly the conditional pmf of Y given X is defined as: $P(Y = y_j \mid X = x_i) = \dfrac{p(X = x_i, Y = y_j)}{p(X = x_i)} = \dfrac{P(x_i, y_j)}{P(x_i)}$.

b. **In the continuous case**

Definition: Let X and Y denote two random variables with joint probability density function $f(x, y)$ and marginal densities $g(x)$, $h(y)$ then the conditional pdf of X given Y = y, is defined as $g(x \mid y) = \dfrac{f(x, y)}{h(y)}$, $h(y) > 0$, similarly the conditional pdf of Y given X = x, is defined as $h(y \mid x) = \dfrac{f(x, y)}{g(x)}$, $g(x) > 0$.

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Example 1: In a binary communication channel, let x denote the bit sent by the transmitter and let y denote the bit received at the other end of the channel. X is a discrete random variable with two possible outcomes (0, 1) and Y is a discrete random variable with two possible outcomes (0, 1). Due to noise in the channel we don’t always have y = x. A joint probability distribution is given as:

P(x, y) =

		X	
		0	1
Y	0	0.45	0.03
	1	0.05	0.47

i. Find marginal of x and y
ii. Evaluate the conditional of x and y

Independent Random Variables

Two random variables X and Y are said to be independent if:

a. **Discrete:** Let X and Y denote two discrete random variables

$P(x,y) = P(X)*P(Y)$ or equivalently

$P(X = x_i \mid Y = y_j) = P(X = x_i)$ and $P(Y = y_j \mid X = x_i) = P(Y = y_j)$

b. **Continuous:** Let X and Y denote two continuous random variables

$f(x, y) = g(x)*h(y)$ or equivalently $g(x \mid y) = g(x)$ and $h(y \mid x) = h(y)$

Example 1: The joint probability mass function (pmf) of X and Y is

P (x, y) =

	Y	0	1	2
X	0	0.1	0.04	0.02
	1	0.08	0.2	0.06
	2	0.06	0.14	0.3

Compute the marginal pmf X and of Y, P $[X \leq 1, Y \leq 1]$ and check if X and Y are independent.

Exercise

The joint probability mass function of (X, Y) is given by $P(x, y) = k(2x + 3), x = 0,1,2, y = 1,2,3$ then find the marginal probability distribution of x and y