Hypothesis Tests

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Example

- Is the average height of students in our class 5’ 7”?

students in IIT Madras ~ 10k studs + 20k online.

Sample: $\boxed{5'6''}$ $\Rightarrow$ $\mu = 5'7''$? Yes / No.

$5'6.9''$ $\Rightarrow$ $\mu = 5'7''$?

Sampling dist. $\Rightarrow$ {CLT / t-dist ...} $\Rightarrow$ interpretations

100% certain.

Confidence $(1-\alpha)$

Hypothesis tests.

How to use these sampling distributions to draw conclusion?

$H_0: \mu = 5'7''$

$H_a: \mu \neq 5'7''$, $\mu < 5'7''$, $\mu > 5'7''$ } 3 options

- Hypothesis testing
  - Concerned with two distinct choices:
    - Null Hypothesis (H_0)
    - Alternate hypothesis (H_1)
  - Test whether to accept or reject H_0 using various test statistics.

    not reject
  - Two types of errors:

α - significance level.

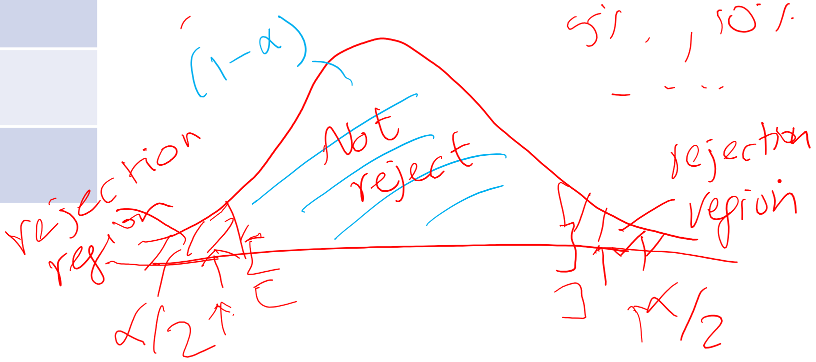
$P(\text{type I error}) = \alpha$

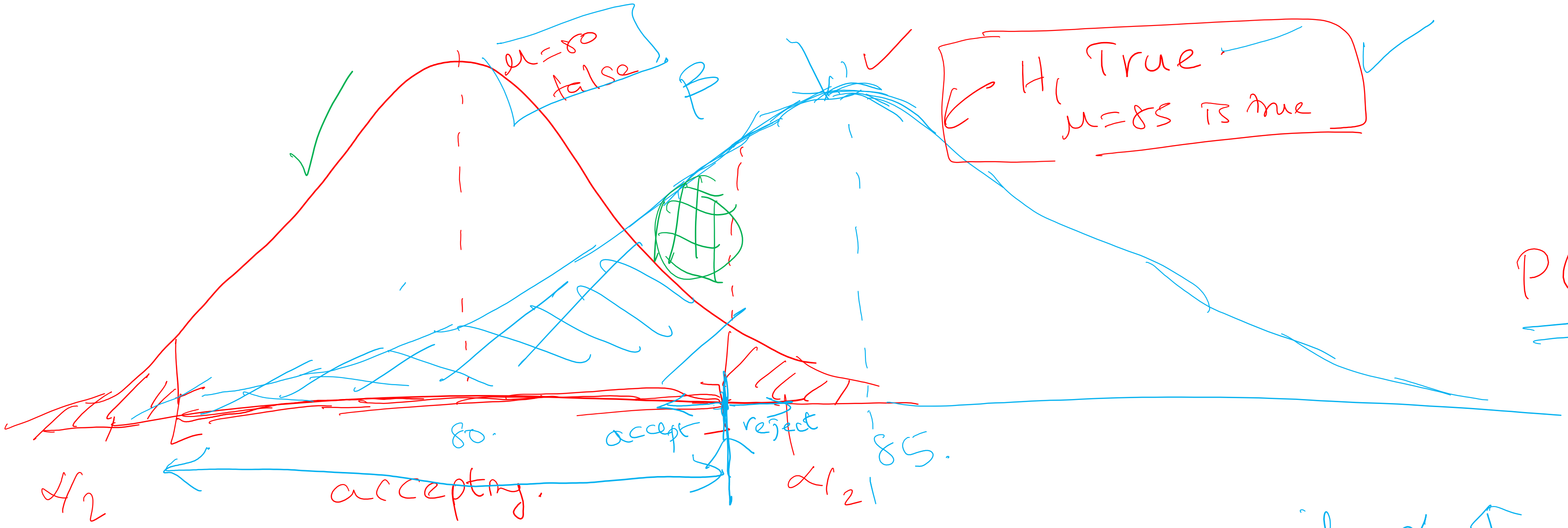
$Z_\alpha = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Two possibilities	**Decision**	
	Accept H_0	Reject H_0
H_0 True	Correct !	Type I error
H_1 True	Type II error	Correct !

$P(\text{type II error}) = \beta$

5%, 10%





$H_0: \mu = 80$.

$H_a: \mu \neq 80$

P(type II error) = H_1 is true but did not reject H_0.

$\cancel{(1-\alpha)}$

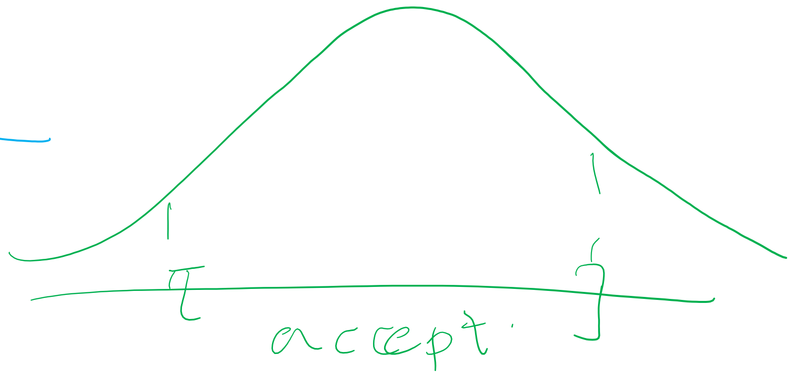
if $\alpha \uparrow \Rightarrow \beta \downarrow$

β: Power of a test.

$\boxed{\text{Sample size}}$

to have more precise intervals

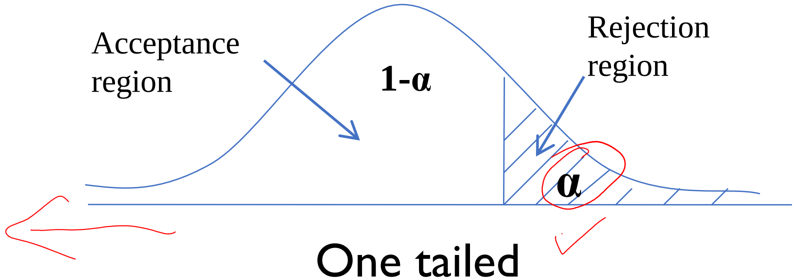
$\Rightarrow \alpha \downarrow$ & $\beta \downarrow$



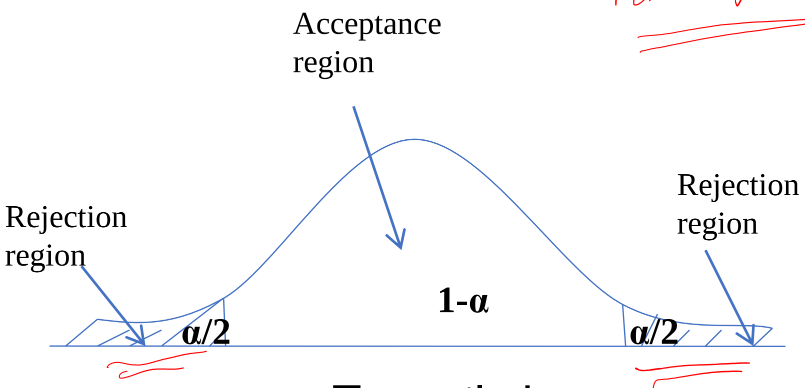
Testing the hypothesis

- One tail or two tail? $H_a: \mu < 80$

$H_a: \mu \neq 80$



One tailed



Two tailed

- Confidence level: 1-α : probability that the computed estimate will lie in the acceptance region
- Level of significance: α :probability that the computed estimate will lie in the rejection region

Distribution statistics in hypothesis testing

- *Que.No.1: The spot speed at a particular location in an expressway are known to be normally distributed with a mean of 80km/hr. and std. dev. of 15km/hr. A new radar speed meter was bought by traffic dept. and a set of 100 observations were taken. The mean speed observed was 77.3km/hr. Is there any evidence to prove that :*

  (i) the new speed meter might have been faulty

  (ii) the new speed meter is showing lesser speed than actual. Assume 5% level of significance.

Solution to Ques.No.1(i)

Here we have to test:

H_0: The speedometer is not faulty (μ=80km/hr.)

against

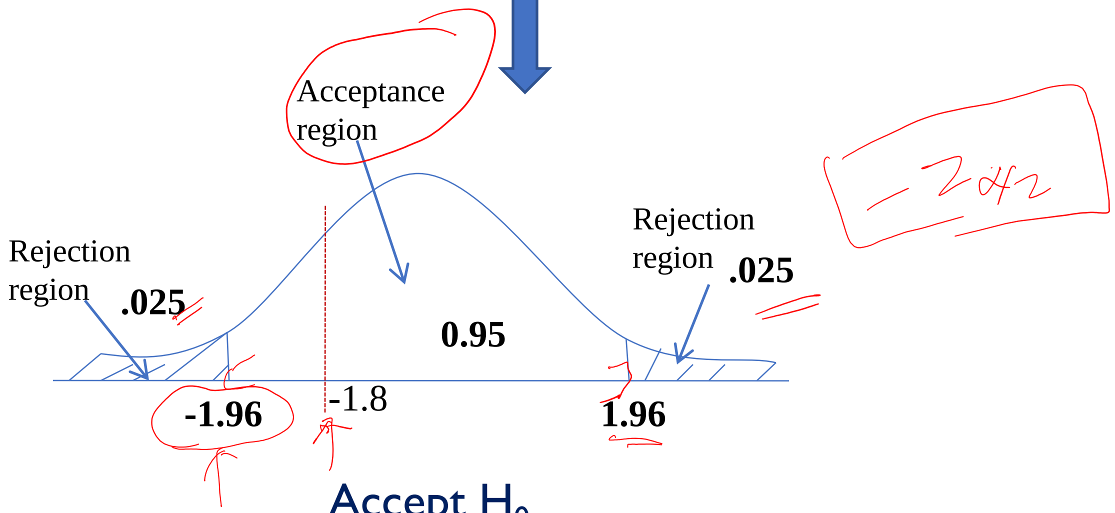
H_1: The speedometer is faulty ($\mu \neq$80km/hr. i.e either >80 or <80)

Two -tailed

Given α = 5%

n=100, large sample

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = \frac{77.3 - 80}{15 / \sqrt{100}} = -1.8$$



Accept H_0

Inference: The speedometer is not faulty

Solution to Ques.No.1(ii)

Here we have to test:

H_0: $\mu = 80$km/hr.

against

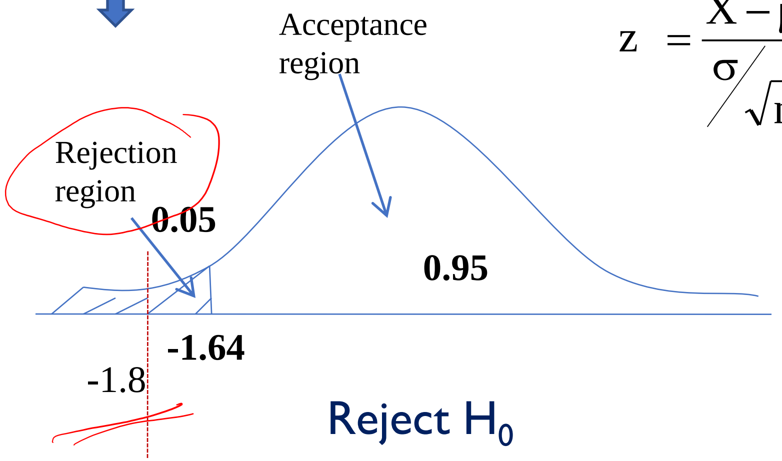
H_1: $\mu < 80$

One tailed

Given α = 5%

n=100, large sample

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{77.3 - 80}{15/\sqrt{100}} = -1.8$$



Reject H_0

Inference: The new speedometer is showing lesser speed than actual

Distribution statistics in hypothesis testing

- *Que. No. 2: The mean spot speed of 15 vehicles observed on a Sunday at a particular roadway was 81.2km/hr. The mean speeds of all vehicles at this location as per previous records was 75.5 km/hr. and std. dev. 10.2km/hr. Is there sufficient evidence to show that the speeds of vehicles on that Sunday was higher than the average speed? Take level of significance as 5%*

Solution to Ques.No.2

t-stat instead of z-stat
t-dist (n-1) dof

Here we have to test:

H_0: $\mu = 75.5$ km/hr.

against

H_1: $\mu > 75.5$ km/hr.

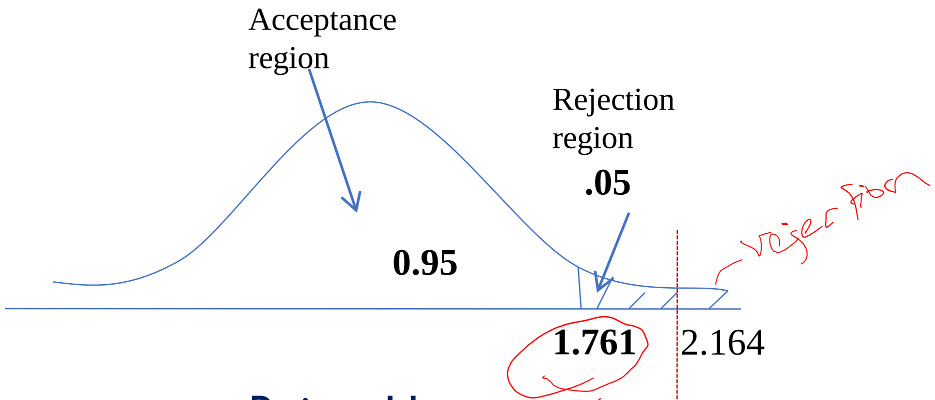
One tailed

Given $\alpha = 5\%$

n=15, small sample

Also sample std. dev. is given, hence use t-statistics

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{75.5 - 81.2}{10.2/\sqrt{15}} = \boxed{2.164}$$



Reject H_0

Inference: The speeds of vehicles on that Sunday is higher than the average speed

Distribution statistics in hypothesis testing

- *Ques. No.3: Two samples of speed data collected are as follows:*

  For sample 1, mean speed is 74.3km/hr. and std. dev. is 7km/hr. (n_1=120)

  For sample 2, mean speed is 72.5km/hr. and std. dev. is 8km/hr. (n_2=120)

  Is there any evidence to prove that the mean speed reduced by more than 0.5km/hr. when using these samples? Assume level of significance as 10%.

Solution to Ques.No.3

Two samples and hence concerned with two means μ_1 and μ_2

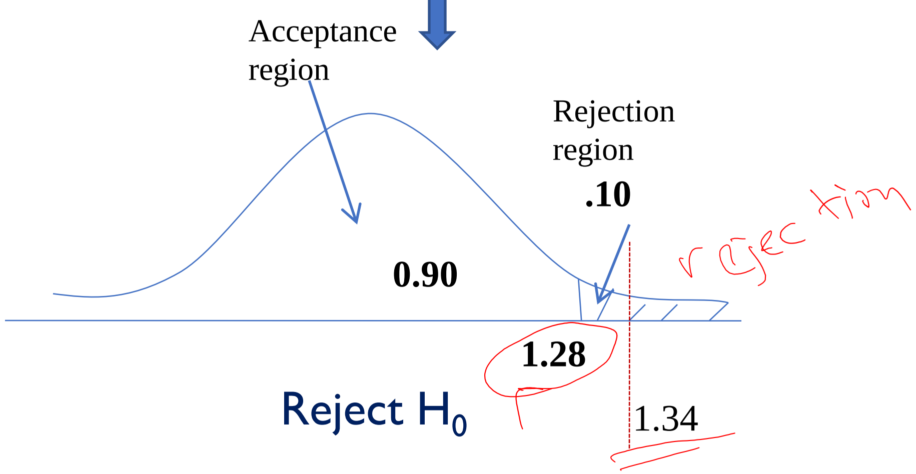
Have to test:

H_0: $\mu_1 - \mu_2 = 0.5$ km/hr.

against

H_1: $\mu_1 - \mu_2 > 0.5$ km/hr.

One tailed



Reject H_0

Given α = 10%

$n_1 = n_2 = 120$, large sample

For test concerning two means, z-statistics is given by,

$$z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$z = \frac{(74.3 - 72.5) - (0.5)}{\sqrt{\frac{7^2}{120} + \frac{8^2}{120}}} = 1.34$$

Inference: the mean speed reduced by more than 0.5km/hr.

Distribution statistics in hypothesis testing

- *Que.No.4:For a given vehicle speed data sample of size 20, the standard deviation observed was 12.5km/hr and speeds were assumed to follow normal distribution. The data can be used only if the standard deviation is less than or equal to 10km/hr. Check whether the data can be accepted at 5% level of significance.*

Solution to Ques.No.4

Problem is related to the sampling distribution of variance

Have to test:

H_0: $\sigma = 10$km/hr.

against

H_1: $\sigma > 10$km/hr.

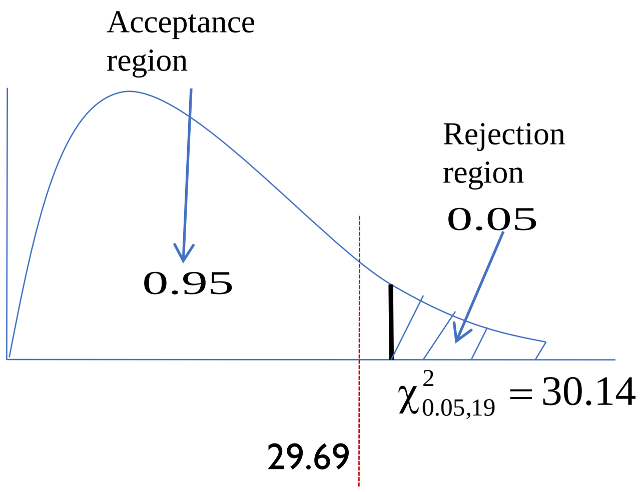
Given $\alpha = 5\%$

Degrees of freedom = sample size-1 =19

χ^2 statistics for variance is:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

$$= \frac{(20-1)12.5^2}{10^2} = 29.69$$



Accept H_0

Inference: The given speed data can be accepted

Distribution statistics – Hypothesis testing

- *Que.No.6: Every minute vehicle count data was collected for a period of 65 minutes. Determine at 95% confidence level , whether the data follows a Poisson distribution.*

No. of arrival	Observed frequency
0	2
1	6
2	7
3	12
4	13
5	9
6	9
7	4
8	2
9	1

To test the fit of data to a particular distribution,

'GOODNESS OF FIT' test

χ^2 test statistic

Solution to Que.No.6

'If Poisson is correct $E_i \approx O_i$
not $E_i \not\approx O_i$

H_0: Data follows poisson distribution
H_1: Data not follows poisson distribution

O_i: Observed frequency
E_i: Expected frequency

Poisson probability:

pdf

$$p(x) = \frac{e^{-\nu}\nu^{x}}{x!}$$

$\nu = \lambda t$

ν = mean number of arrival = 260/65 =4

pdf

$$\boxed{p(x) = \frac{e^{-4}4^{x}}{x!}}$$

Arrival (x_i)	Obsv. freq (min)	Total no. of veh.	Prob. $p(x_i)$	E_i (prob.*65)
0	2	0	0.018	1.17
1	6	6	.0733	4.76
2	7	14	0.1465	9.52
3	12	36	0.1954	12.7
4	13	52	0.1954	12.7
5	9	45	0.1563	10.16
6	9	54	0.1042	6.77
7	4	28	0.0595	3.87
8	2	16	0.0298	1.94
9	1	9	0.0132	0.858
	∑ =65	∑ =260		

Goodness of fit – solution to Que.No.6

At least 5 groups and at least 5 nos. in each group

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

$= 2.31$

Degrees of freedom

= N-1-g = 5

7-1-1 = 5 d.o.f

g - no. of statistics used to calculate E_i ; here only ν

$\chi^2_{0.05,5} = 11.07 > 2.31$

5%

Accept H_0

5 d.o.f

No. of arrival	Observed frequency (mintute), O_i			Expected frequency (E_i)		$(O_i-E_i)^2/E_i$
0	2	8	1	1.17	5.93	0.7189
1	6			4.76		
2	7		2	9.52		0.6671
3	12		3	12.7		0.0386
4	13		4	12.7		0.007
5	9		5	10.16		0.132
6	9		6	6.77		0.7345
7	4	7	7	3.87	6.67	0.0165
8	2			1.94		
9	1			0.858		

N=7

$\sum = 2.31$

Inference: The given data follows poisson distribution

$$\frac{(O_i - E_i)^2}{E}$$

TEST STATISTICS **Hint: μ_0 = population mean** **σ_0 = population std. dev.**	**H_1**	**Reject H_0 if**
Large sample – concerning mean $H_0 : \mu = \mu_0$		
$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$	$\mu < \mu_0$	$z < -z_\alpha$
	$\mu > \mu_0$	$z > z_\alpha$
	$\mu \neq \mu_0$	$z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
Small sample – concerning mean $H_0 : \mu = \mu_0$		
$$t = \frac{\overline{X} - \mu}{s / \sqrt{n}}$$	$\mu < \mu_0$	$t < -t_\alpha$
	$\mu > \mu_0$	$t > t_\alpha$
	$\mu \neq \mu_0$	$t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

TEST STATISTICS **Hint: μ_0 = population mean** **σ_0 = population std. dev.**	**H_1**	**Reject H_0 if**
Comparison of sample mean $H_0: \mu_1 - \mu_2 = \delta$		
$$z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$	$\mu_1 - \mu_2 < \delta$	$z < -z_\alpha$
	$\mu_1 - \mu_2 > \delta$	$z > z_\alpha$
	$\mu_1 - \mu_2 \neq \delta$	$z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
One variance $H_0: \sigma^2 = \sigma_0^2$		
$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$	$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi_\alpha^2$
	$\sigma^2 < \sigma_0^2$	$\chi^2 > \chi_{1-\alpha}^2$
	$\sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi_{1-\alpha/2}^2$ or $\chi^2 > \chi_{\alpha/2}^2$

TEST STATISTICS Hint: μ_0 = population mean б$_0$ = population std. dev.		H_1	Reject H_0 if
Two variance $H_0 : \sigma_1^2 = \sigma_2^2$			
F	s_1^2 / s_2^2	$\sigma_1^2 > \sigma_2^2$	$F > F_{\alpha, n_1 - 1, n_2 - 1}$
	s_2^2 / s_1^2	$\sigma_1^2 < \sigma_2^2$	$F > F_{\alpha, n_2 - 1, n_1 - 1}$
	$s_{large}^2 / s_{small}^2$	$\sigma_1^2 \neq \sigma_2^2$	$F > F_{\alpha, n_{large} - 1, n_{small} - 1}$
Underlying distribution H_0 : Data follows given distribution			
$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$		Data not follows given distribution	$\chi^2 > \chi_\alpha^2$