LINEAR PROGRAMMING (optimization)

Simplex Method

objective function : linear

$x_1, x_2 \ldots x_n$: $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\min \underbrace{(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}\ + c_0)$

$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\underset{c^T}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$

$\min \ c^T x$ s.t.

such that or subject to

LP – STANDARD FORMULATION

- Objective function and constraints are linear:

$$\min_{\mathbf{x}} \underbrace{\sum_{i=1}^{n} c_i x_i = \mathbf{c}^T\mathbf{x}}_{\text{linear}} = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$st$$

$$\underbrace{\sum_{j=1}^{n} a_{ij} x_j = b_i}_{\text{linear}} \quad i = 1 \dots m \equiv \underset{m\times n}{\mathbf{A}}\,\underset{n\times 1}{\mathbf{x}} = \underset{m\times 1}{\mathbf{b}}$$

$$x_j \geq 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \\ a_{m1} & & \dots & a_{mn} \end{bmatrix} \qquad A: m\times n$$

$$\underbrace{\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{matrix}}_{Ax}$$

LP PROBLEMS

- Blending
- Transportation
- Resource allocation
- Production planning

SIMPLE EXAMPLE

- Assume that to stay healthy you have to take 4 units of vitamin D per day
- There are two natural sources for vitamin D. In source 1 you get 1 unit of vitamin D per gram and in source 2 you get 2 units of vitamin D
- The cost of source 1 per gram is 2 cost units and source 2 per gram is 3 cost units
- The question is how many grams (x) of food source 1 and how many grams (y) of food source 2 should one consume every day to stay healthy ?

$$\min_{x,y} 2x + 3y$$

$$s.t. x + 2y \geq 4; x \geq 0; y \geq 0$$

- The optimal solution is x* =0, y*=2

$x + 2y \geq 4$

$x \geq 0 \quad y \geq 0$

	S1	S2
	x gm	y gm
vitamin/gm	1	2
Actual vitamin	x	2y

	S1	S2
vitamin / gm	1	2
cost / gm	2	3.
Actual bought	x	y.
vitamin	x	$2y$
Actual cost	$2x$	$3y$.

Total vitamin $x + 2y$

Total cost $2x + 3y$.

min $2x + 3y$

s.t.,

$x + 2y \geq 4$

$x \geq 0$

$y \geq 0$

	S1	S2	vitamin	cost.
×	1	1	$1+2(1) = 3$	$2+3 = 5$
✓	2	1	$2+2(1) = 4$.	$2(2)+3 = 7$.
✓	1	2.	$1+2.(2) = 5$.	$2+6 = 8$
optimal ✓	0	2	$0+2(2) = 4$	$2.3 = 6$

Is there a diet that gives at least 4 units vitamin but has lower cost?

$x^* = 0$ is optimal solution

$y^* = 2$

A: Corner

B: Corner.

O: Origin X

C Interior X

D. Boundary.

E Interior

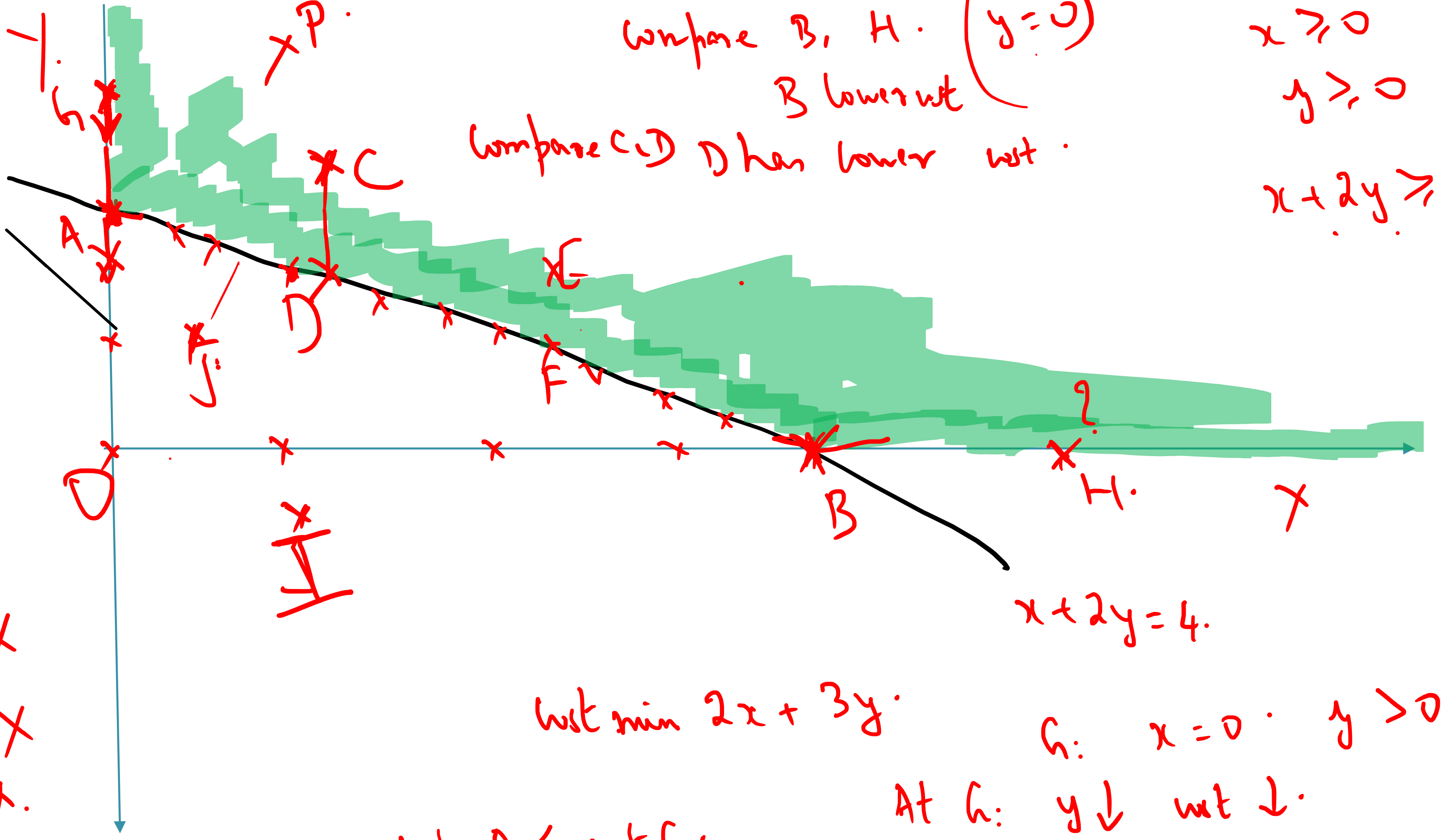
F Boundary

G Boundary X

H Boundary. X

I. Infeasible X.

J Infeasible X



At H $y=0$.

Compare B, H. ($y=0$)

B lower cost

Compare C,D D has lower cost.

$x \geq 0$

$y \geq 0$

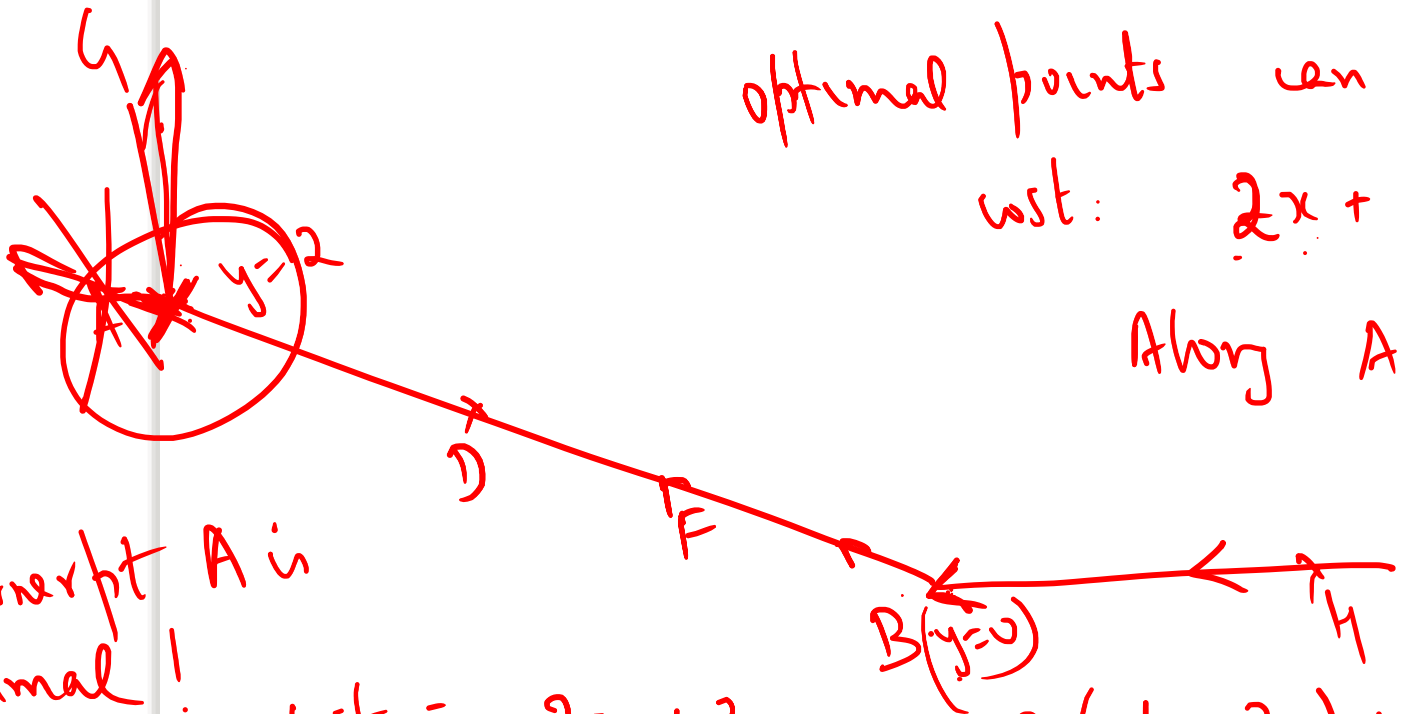
$x+2y \geq 4$.

cost min $2x+3y$.

G: $x=0$. $y>0$

At G: $y\downarrow$ cost $\downarrow$.

cost A < cost G.



optimal points can lie only on AB (not interior)

cost: $2x + 3y$.

Along AB. $x + 2y = 4$

Replace x by $4-2y$

Corner pt A is optimal!

cost $= 2x + 3y = 2(4-2y) + 3y = 8 - y$. along AB.

If y increase y, cost ↓.

Stop at A:

B → F → D → A. cost ↓ $(8-y)$ ↓.

Stop at A: At: $y = 2$, $x = 0$. Optimal solution

Along AG. $x = 0$.

cost $= 3y$.

y↑ cost↑.

ANOTHER EXAMPLE

- Two products A and B are produced. A sells at Rs. 0.50/unit and B sells at Rs. 0.6/unit.
- Time in mins for processing per unit product in 3 stages - blending, cooking and packing.

	Product	Blending	Cooking	Packing
x_1	A	1	5	3
x_2	B	2	4	1

- Equipment availability : Blending : 14 hrs, Cooking : 40 hrs, Packaging : 15 hrs.
- Objective : Maximize ~~profit~~ income : max $0.5x_1 + 0.6x_2$ min $-0.5x_1 - 0.6x_2$.

Blending time : $(x_1 + 2x_2)_{min} \leq (14 \times 60)$ min

$3x_1 + x_2 \leq 900$

Cooking time $5x_1 + 4x_2 \leq 40 \times 60$

LP FORMULATION

~~cost~~ income

- Objective Min $-0.5x_1 - 0.6x_2$ (maximize ~~profit~~)
- Constraints on availability of equipment:

$$x_1 + 2x_2 \leq 840 \quad \text{(blending)}$$

$$5x_1 + 4x_2 \leq 2400 \quad \text{(cooking)}$$

$$3x_1 + x_2 \leq 900 \quad \text{(packing)}$$

- Non-negativity constraints

$$x_1, x_2 \geq 0$$

Slack/Surplus Variables

- Convert constraints to equalities by adding slack/surplus variables

$x_1 + 2x_2 + s_1 = 840$ (blending)

$5x_1 + 4x_2 + s_2 = 2400$ (cooking)

$3x_1 + x_2 + s_3 = 900$ (packing)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 5 & 4 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- Non-negativity constraints

$$x_1, x_2, \; s_1, s_2, s_3 \geq 0$$

GRAPHICAL REPRESENTATION

O: (0, 0)

A: $x_1 = 0$ $x_2 = \frac{840}{2} = 420$ (0, 420)

min $-0.5x_1 - 0.6x_2$

C: $x_2 = 0$ $x_1 = \frac{900}{3} = 300$ (300, 0)

$x_1 + 2x_2 \le 840$

B: ?

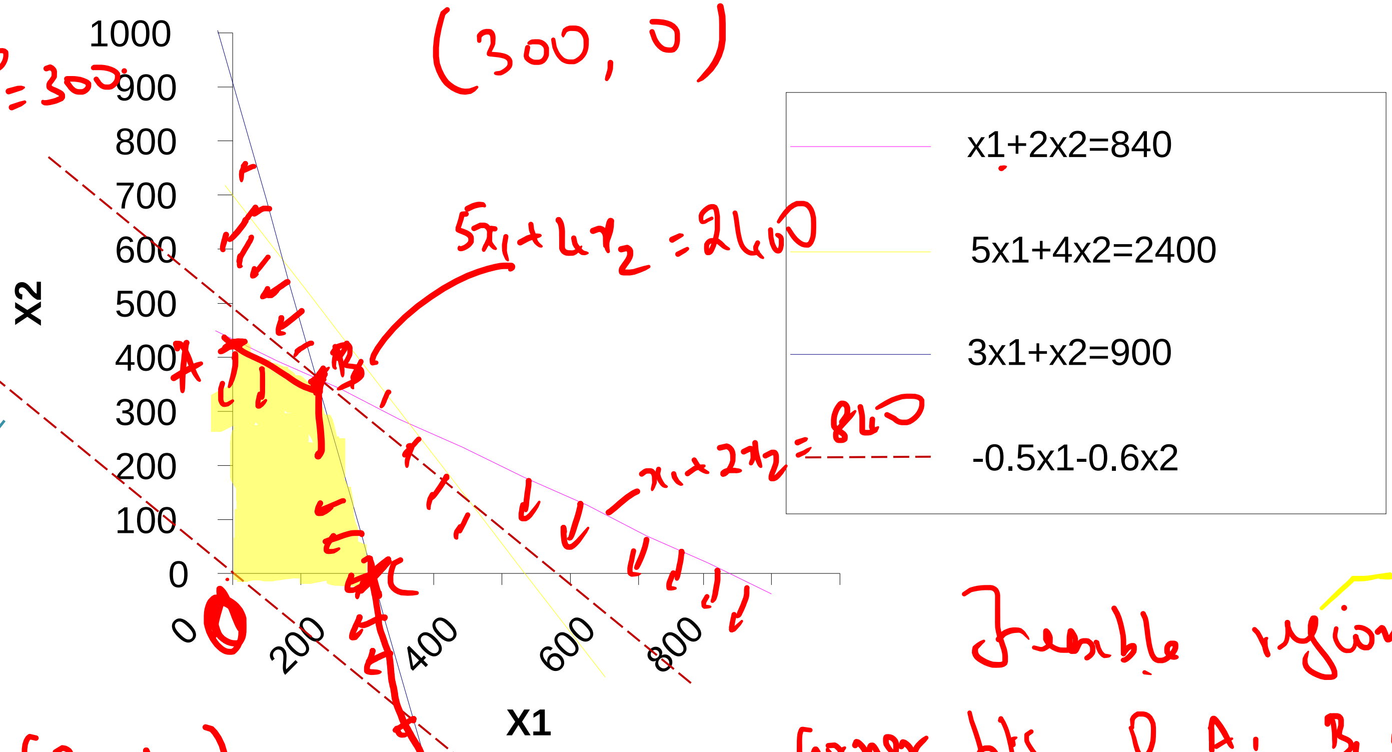
$x_1 + 2x_2 = 840$

$3x_1 + x_2 = 900$

Increasing f

$x_1 = ?$ 192

$x_2 = ?$ 324



$x_1 \ge 0$ $x_2 \ge 0$.

Feasible region

Corner pts O, A, B, C

$x_1 + 2x_2 = 192 + 2(324)$
$= 840$.

$(324) + 3(192) = 900$

Corner pts:

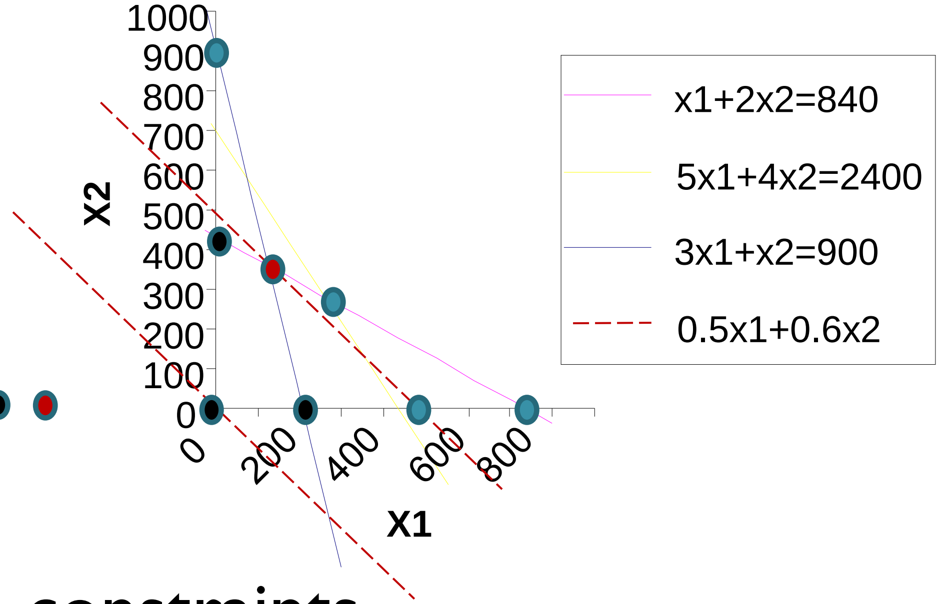
O, (0,0)

A (0,1420)

B (

LP - OVERVIEW

- n = 5 variables, m = 3 constraints (in the equality form after adding slack/surplus)
- Optimum solution is at some vertex
- Vertex : Intersection of constraints.
  - Number of zero variables at a vertex
    = *n-m* = 2 called as non-basic variables
  - The other variables are solved using the constraints
  - This solution is called a basic solution and if it is feasible it is called the basic feasible solution
  - Optimum is at a feasible vertex



LP - OVERVIEW

- Geometrically
  - Basic feasible solutions are the vertices
- Fundamental concept in Linear Programming
  - If there exists a feasible solution then there exists a basic feasible solution (corner point)
  - If there exists a optimal solution then there exists a basic optimal solution (corner point)
- As a result the optimum solution can be obtained by just searching the vertices
- Simplex algorithm jumps from one vertex to another through an algebraic procedure

SIMPLEX - OVERVIEW

- Simplex starts with a basic feasible solution
- We will start the explanation of simplex assuming that we have an initial basic feasible solution
- This is called the phase 2 of simplex
- An initial basic feasible solution itself will be generated as a simplex problem
- This is called phase 1 of simplex

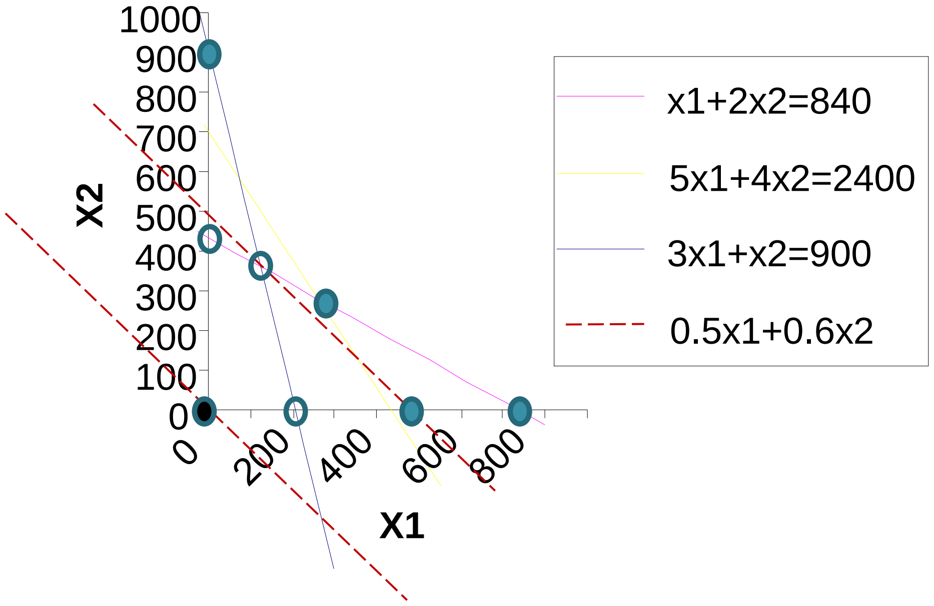
SIMPLEX - OVERVIEW

- Remember, basic feasible solutions are the vertices
- Simplex moves from one basic feasible solution to another as long as improvement to the objective function can be made
- The procedure terminates if
  - No further improvement can be made
  - Cost function can be improved without any bound

LP – PHASE 2

- Start with a feasible vertex • : *n-m* variables as non-basic ($\mathbf{x}_N = x1, x2$) and others basic ($\mathbf{x}_B = s1, s2, s3$)
- $x1 = x2 = 0$

$$\mathbf{A} = \begin{array}{cc|ccc} \mathrm{N} & & & \mathrm{B} & \\ \end{array} \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 0 \\ 5 & 4 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{array}\right]$$



LP – BASIC FEASIBLE SOLUTION

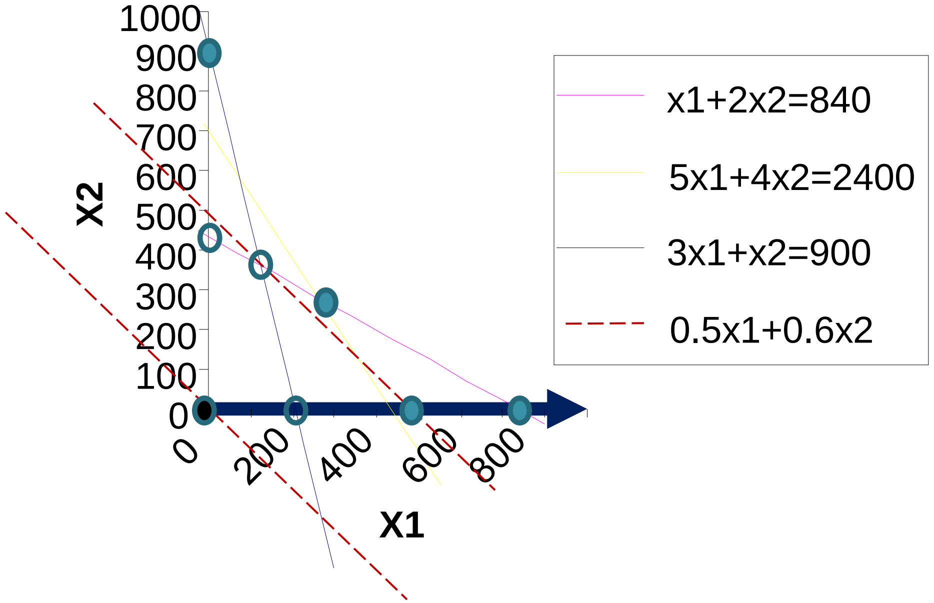
- Columns of **A** corresponding to $\mathbf{x}_B$ denoted as *m x m* matrix **B** (basis matrix) and the columns corresponding to $\mathbf{x}_N$ denoted as *m x (n-m)* matrix N
- Solution for $\mathbf{x}_B$: $\mathbf{B}\mathbf{x}_B = \mathbf{b}$; $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ **=>**

$$\begin{array}{cc} & \begin{array}{ccccc} & \mathbf{N} & & \mathbf{B} & \end{array} \\ \mathbf{A} = & \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 5 & 4 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_N \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} 800 \\ 2400 \\ 900 \end{bmatrix} \Rightarrow s_1 = 800, s_2 = 2400, s_3 = 900 \end{array}$$

- Objective function $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B{}^T\mathbf{x}_B + \mathbf{c}_N{}^T\mathbf{x}_N = 0$

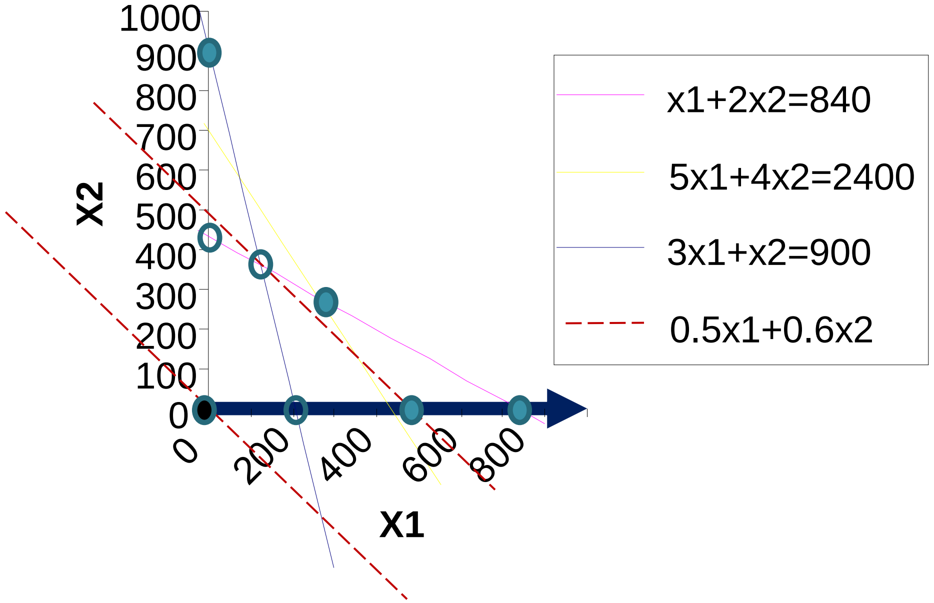
LP – FEASIBLE SEARCH DIRECTION

- Choose a non-basic variable and increase it to θ. => Travel along edge of the polyhedron (simplex)



LP – EFFECT OF ENETERING VARIABLE ON BASIC SOLUTION

- x1 > 0 travel along x-axis
- Examine how entering variable x_1 affects the basic solution



Column corresponding to x_1

$$Bx_B^{new} + \theta a_1 = b$$

$$x_B^{new} = x_B^{old} - \theta B^{-1} a_1$$

$$\begin{bmatrix} s_1^{new} \\ s_2^{new} \\ s_3^{new} \end{bmatrix} = \begin{bmatrix} 800 \\ 2400 \\ 900 \end{bmatrix} - \theta \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

LP – CHOOSING ENTERING VARIABLE

- Examine how entering variable affects the objective function

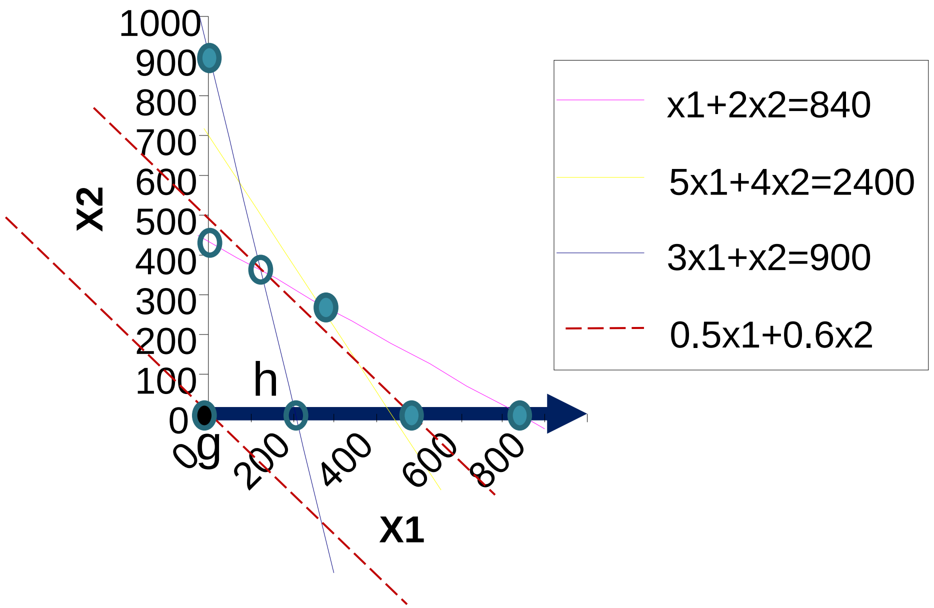
$$c_B^T x_B^{new} + \theta c_1 = c_B^T x_B^{old} + \theta\underbrace{(c_1 - c_B^T B^{-1} a_1)}_{d_1 \text{ (reduced cost of } x_1)}$$

$$f^{new} = f^{old} + \theta d_1$$

- If d_1 is negative then objective function decreases
- Compute d_j for all non-basic variables and chose a variable which has most negative d_j as entering variable (denoted as x_p)
- If all d_j's are positive $\Rightarrow$ OPTIMUM SOLUTION FOUND

LP – CHOOSING LEAVING VARIABLE

- Examine how entering variable x_p affects the basic solution



Column corresponding to x_p

$$Bx_B^{new} + \theta a_p = b$$

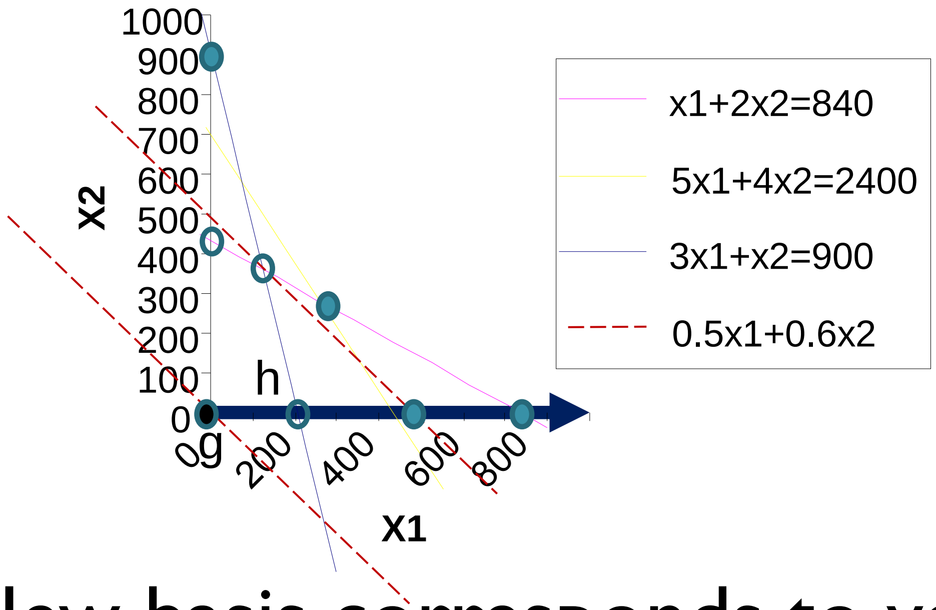
$$x_B^{new} = x_B^{old} - \theta B^{-1} a_p$$

$$\begin{bmatrix} s_1^{new} \\ s_2^{new} \\ s_3^{new} \end{bmatrix} = \begin{bmatrix} 800 \\ 2400 \\ 900 \end{bmatrix} - \theta \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

- As θ increases some basic variables may decrease
- Maximum value of θ is when one of the basic variable becomes zero first (minimum ratio test). In above case s_3 corresponding to vertex h.
- If all basic variables increase then SOLUTION UNBOUNDED

LP – CHANGING BASIS

- Leaving variable denoted as x_q
- Replace column of x_q in B with column a_p



Column of x_q replaced corresponding to x_p

$$B^{new} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

- New basis corresponds to vertex h

ITERATION

- Drop x_q from basic set and replace it with x_p
- Update Basis matrix and basic feasible solution
- Iterate. Algorithm will converge (either OPTMIMUM or UNBOUNDED)

LP – PHASE I

- Find a feasible vertex solution
- Many approaches are possible
- One approach is to modify problem with more variables so that an obvious feasible solution is identified
- Solve a LP with a different objective function (artificial objective function)
- The solution to this problem will provide a basic feasible solution for the original problem or determine if the problem is infeasible

Sensitivity Analysis

- Is the optimal solution sensitive to changes in objective function and the right hand side of the constraints?
- Good to identify the robustness of the solutions to fluctuations in data

Sensitivity to cost coefficients

- Bounds on cost coefficients
  - The optimal solution will remain unchanged as long as
    - None of the reduced costs for the non-basic variables in the optimal solution become negative for a change in the original cost coefficient
  - The value of the objective function will change if the coefficient multiplies a variable whose value is nonzero.

Sensitivity Analysis for b

- Any change to the value of b for a constraint with zero slack or surplus (s_i) – active constraint - will change the optimal solution.

- Any change to the value of b for an inactive constraint that is less than its slack or surplus (s_i), will cause no change in the optimal solution.