Point Estimation: A single numerical value $\hat{\theta}$ of a statistic θ

Unbiased: $E[\hat{\theta}] = \theta$ / Bias of $\hat{\theta} = E[\hat{\theta}] - \theta$

E.g. (7-4) Estimator for variance — $\boxed{S^2 = \dfrac{\Sigma (x_i - \bar{x})^2}{(n-1)}}$ ← estimator

$\theta - \sigma^2$

$\hat{\theta} - s^2$

$$E[s^2] = E\left[\frac{\Sigma (x_i - \bar{x})^2}{n-1}\right] = \frac{1}{n-1} E[\Sigma (x_i - \bar{x})^2]$$

$$= \frac{1}{n-1}\left[\Sigma E[x_i^2] - nE[\bar{x}^2]\right]$$

$$n\sigma^2 \quad - \quad \sigma^2$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} (\cancel{\mu^2} + \sigma^2) - \cancel{n}\left(\cancel{\mu^2} + \frac{\sigma^2}{\cancel{n}}\right)\right]$$

$$= \frac{\cancel{n-1}}{\cancel{n-1}}\sigma^2 = \sigma^2 \quad \text{Unbiased!}$$

$\sigma^2 = E[x_i^2] - \mu^2$

$\mu^2 + \sigma^2 = E[x_i^2]$

$\dfrac{\sigma^2}{n} = E[\bar{x}^2] - \mu^2$

Maximum Likelihood Estimation (MLE)

Suppose X is a RV with a prob. dist. $f(x, \theta)$ where θ is an unknown parameter. $x_1, x_2, \ldots x_n$ – observed random values

Likelihood function $L(\theta) = f(x_1; \theta) \cdot f(x_2; \theta) \cdots f(x_n; \theta)$

MLE of θ gives the value of θ that maximizes $L(\theta)$

E.g. (7-10) Let X be a Bernoulli RV $f(x; \underline{p}) = \begin{cases} p^x (1-p)^{1-x} & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Given a random sample of size n:

$$L(p) = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} \cdots p^{x_n}(1-p)^{1-x_n} = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

Easier to work log of $L(p)$ [log-likelihood fn]

$$\ln L(p) = \sum x_i \ln p + (n - \sum x_i) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum x_i}{p} - \frac{(n - \sum x_i)}{(1-p)} = 0 \qquad \frac{d^2 \ln L(p)}{dp^2} < 0 \text{ - maxi}$$

$$\Rightarrow \boxed{\hat{p} = \frac{\sum x_i}{n}}$$

E.g. 7-13

$X \sim N(\mu, \sigma^2)$ μ, σ^2 – unknown parameters

Random sample of size n $X_1, X_2 \dots X_n$

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

$$\ln L(\mu, \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$$

Differentiate wrt μ, σ^2

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2} \Sigma (x_i - \mu) = 0.$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \Sigma (x_i - \mu)^2 = 0$$

$$\frac{\Sigma x_i}{\sigma^2} - \frac{n\mu}{\sigma^2} = 0 \Rightarrow \hat{\mu} = \frac{\Sigma x_i}{n} \quad (= \bar{x}\,!)$$

$$\hat{\sigma}^2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2 \Rightarrow \text{biased estimator!}$$

$$E[\hat{\sigma}^2] = \frac{n-1}{n} \sigma^2$$

$$\text{Bias} = E[\hat{\sigma}^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{\sigma^2}{n}$$

However, for large n, bias $\to 0$. $\quad n \to \infty, \frac{\sigma^2}{n} \to 0$

Asymptotically unbiased estimator