Linear algebra for data science

Day 1 27/5/23

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Outcome

Module learning outcomes:

1. Participants will be able to identify relationships between variables in large datasets
2. Participants will be able to identify information sufficiency in terms of both equations and variables
3. Participants will be able to understand basic linear algebra concepts that underlie the complicated data analytics algorithms

Linear Algebra

System of Equations

What about $Q_{in} = Q_{out}$?

Q_{in}

Q_1

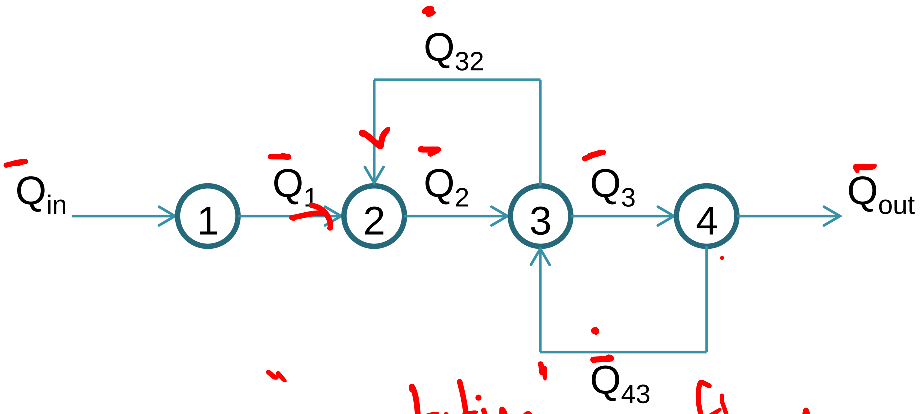
Q_2

Q_{32}

Q_3

Q_{43}

Q_{out}



How many "quantities" or flows are here?

How many equations could be formed from this?

Equation: Conservation In = out at every 'junction'

1, 2, 3, 4

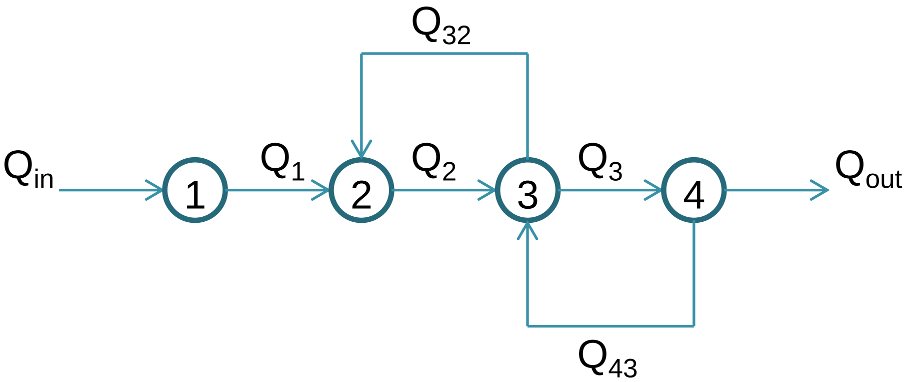
Jn 1: $Q_{in} = Q_1$

2: $Q_1 + Q_{32} = Q_2$

3: $Q_2 + Q_{43} = Q_{32} + Q_3$

4: $Q_3 = Q_{43} + Q_{out}$

System of Equations



4 equation

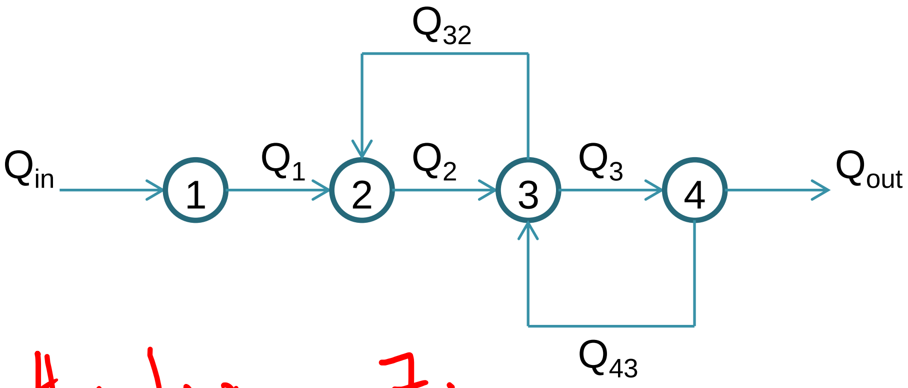
Adding up

flows (unknown): 7

- $Q_{in} = \cancel{Q_1}$
- $\cancel{Q_1}+\cancel{Q_{32}} = \cancel{Q_2}$
- $\cancel{Q_2}+\cancel{Q_{43}} = \cancel{Q_{32}}+\cancel{Q_3}$
- $\cancel{Q_3} = Q_{out}+\cancel{Q_{43}}$

$Q_{in} = Q_{out}$ is therefore redundant

System of Equations



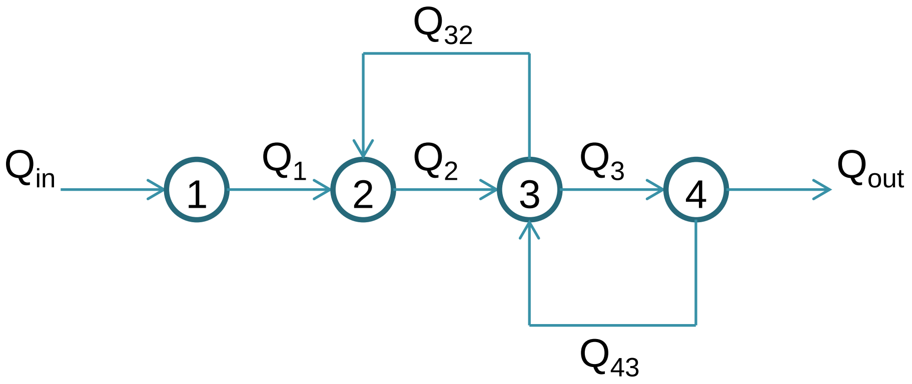
unknowns = 7.

- $Q_{in} = Q_1$
- $Q_1+Q_{32} = Q_2$
- $Q_2+Q_{43} = Q_{32}+Q_3$
- $Q_3 = Q_{out}+Q_{43}$

un = 4

- Can we solve the equations when –
    - $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$
    - $Q_{in} = 10$, $Q_{out} = 10$ and $Q_3 = 7$
    - $Q_{in} = 10$, $Q_{out} = 11$ and $Q_3 = 7$

System of Equations



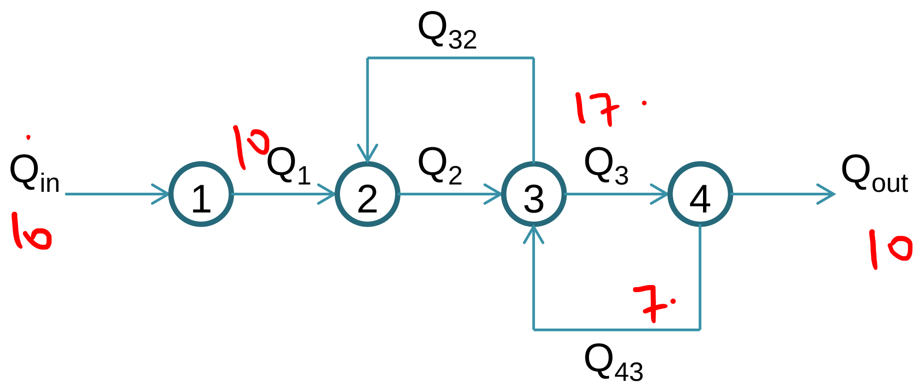
- $Q_{in} = Q_1$ = 10
- 10 $Q_1 + \overset{5}{Q_{32}} = Q_2$ = 15
- $\underset{15}{Q_2} + \underset{3}{Q_{43}} = \overset{5}{Q_{32}} + Q_3$ 13
- $Q_3 = Q_{out} + Q_{43}$

13 = 10 3

- Can we solve the equations when $Q_{in} = 10$, $Q_{32} = 5$ and $Q_{43} = 3$?

Unique solution

System of Equations



- $Q_{in} = Q_1$
- $Q_1+Q_{32} = Q_2$
- $Q_2+Q_{43} = Q_{32}+Q_3$
- $Q_3 = Q_{out}+Q_{43}$

7

- Can we solve the equations when Q_{in} = 10, Q_{out} = 10 and Q_3 = 17?

Q_{32}. Q_2 In 2: $10 + Q_{32} = Q_2$.

Infinite number of solutions

In 3: $Q_2 + 7 = 17 + Q_{32}$

- Loop involving Q_2 and Q_{32} : both are unmeasured
- Given measurements consistent with equations

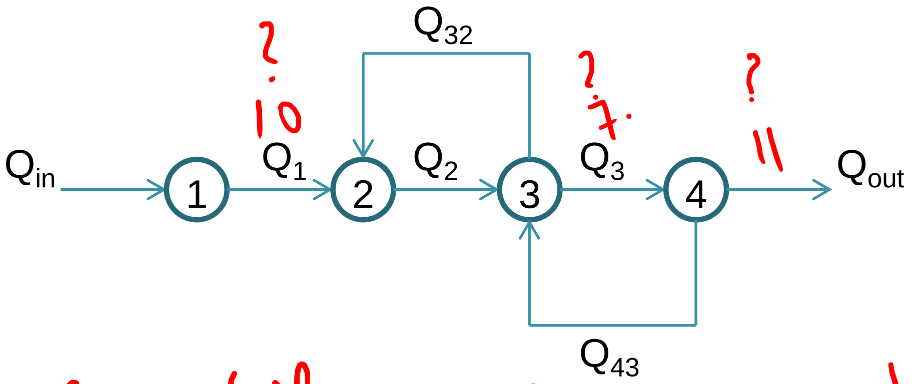
1 eqn. 2 unknowns → $\boxed{Q_{32} - Q_2 = 10}$

Infinite

$Q_{32}=10$ $Q_2=0$

$Q_{32}=11$ $Q_2=1$

System of Equations



- $Q_{in} = Q_1$
- $Q_1+Q_{32} = Q_2$
- $Q_2+Q_{43} = Q_{32}+Q_3$
- $Q_3 = Q_{out}+Q_{43}$

Eqn: tell me $Q_{in} = Q_{out}$!

- Can we solve the equations when Q_{in} = 10, Q_{out} = 11 and Q_3 = 7?

$w \propto \frac{1}{\sigma^2}$ $\sigma^2 \uparrow$ $w \downarrow$

$\sigma^2 \downarrow$ $w \uparrow$

No solution !

$$\frac{w_1 Q_{in} + w_2 Q_{out}}{w_1 + w_2}$$

Equations are inconsistent !

- Given information not consistent with equations

What is a reasonable "fix"? 10.5 (?) $\frac{10+11}{2}$

System of Equations: Key Concept

Understanding when do we have:

7 unknowns 4 eqn.

- Unique solution ✓
- No solution ✓
- Infinite number of solutions ✓

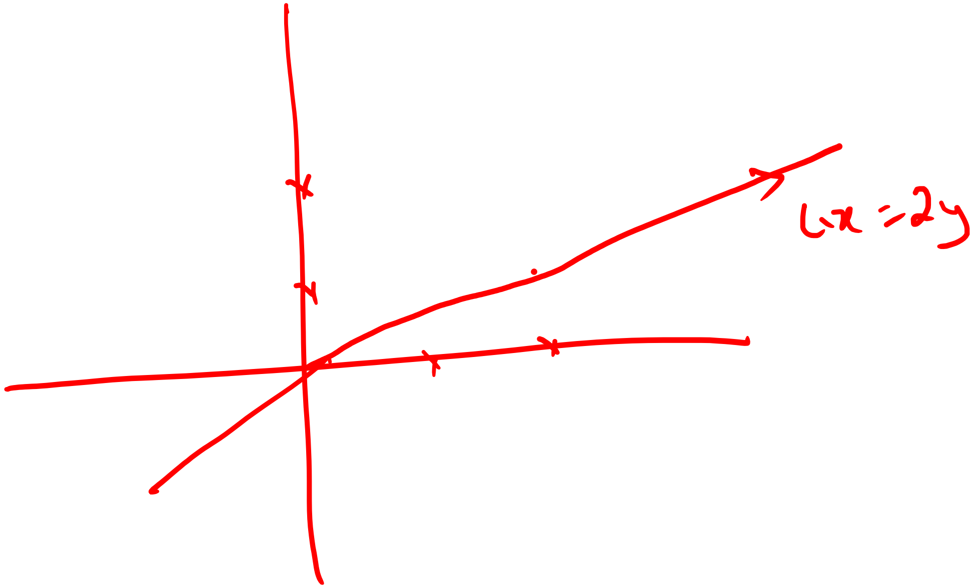
Solving Simultaneous Linear Equations

Solve the two linear equations:

$4x - 2y = 0$; $-2x + 4y = 6$

$4x = 2y$

$x = \frac{1}{2} y$



Solving Simultaneous Linear Equations

Elimination (High–school method)

$$\begin{array}{r} 4\cancel{x} - 2y = 0 \\ 2(-\cancel{2}x + 4y = 6) \\ \hline 6y = 12 \end{array}$$

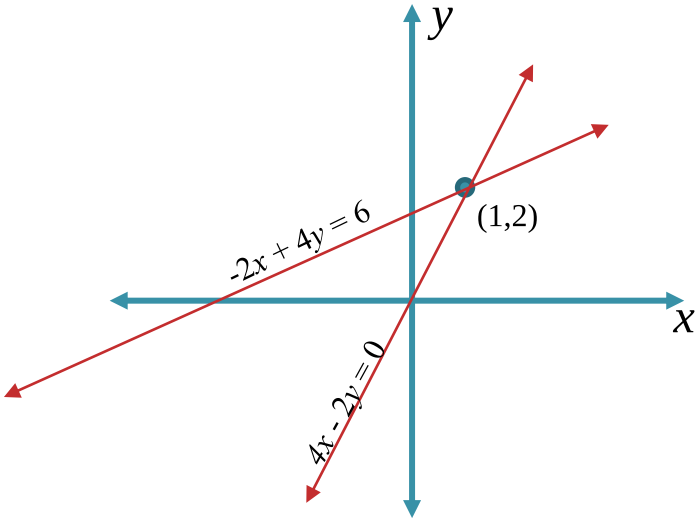
$y = 2$

$\Rightarrow y = 2$ $4x = 2y$

$\Rightarrow x = 1$ $x = \frac{2y}{4} = \frac{2\times 2}{4}$

(1,2) is the solution

Graphical method



(1,2) is the solution

Geometry of Linear Equations

- We may view a system of linear equations in three different ways –
  - Matrix form – $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ where **A** forms a matrix with the coefficients of the unknowns and ***x*** forms a matrix with the unknowns and **b**, a matrix with the values in the R.H.S
  - Row picture – viewing one equation at a time
  - Column picture – two separate equations as one vector equation

Geometry of Linear Equations

Geometry for a system of 2 equations:

Matrix form

- Consider two linear equations:

$$4x - 2y = 0$$

$$-2x + 4y = 6$$

$$\underset{2\times2}{A:}\ \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \underbrace{\overset{c1 \qquad c2}{\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}}}_{2\times2} \underset{\substack{\text{vector} \\ 2\times1}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times1}{\begin{bmatrix} 0 \\ 6 \end{bmatrix}}$$

- A matrix is a rectangular arrangement of numbers in rows and columns
- Rows run horizontally and columns run vertically

- Order of a matrix: $m \times n$ where m is the # of rows and n is the # of columns

$$4x - 2y = 0$$

$$-2x + 4y = 6$$

Geometry of Linear Equations

<u>Matrix form</u>

$$\overbrace{\begin{bmatrix}4 & -2\\ -2 & 4\end{bmatrix}}^{A}\overset{x}{\begin{bmatrix}x\\ y\end{bmatrix}}=\begin{bmatrix}0\\ 6\end{bmatrix}$$

- This is of the form $\underline{A}\underset{?}{\boxed{\mathbf{x}}} = \underline{\mathbf{b}}$

  where –

  - **A** – matrix with coefficients of the unknowns
  - ***x*** – unknowns
  - **b** – R.H.S of the equations

- *n* equations and *n* unknowns ⇒ $n \times n$ matrix (square matrix) (2×2).

- *m* equations and *n* unknowns ⇒ $m \times n$ matrix (rectangular matrix)

Geometry of Linear Equations

Row picture

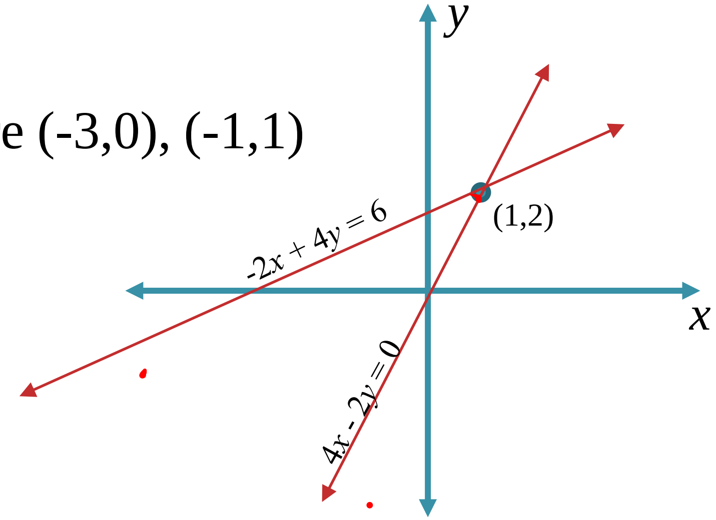
$$4x - 2y = 0 \;;\; -2x + 4y = 6$$

- Taking one row at a time and plotting it in the x–y plane

Few points that satisfy $4x - 2y = 0$ are (0,0), (1,2) and (1/2, 1)

Few points that satisfy $-2x + 4y = 6$ are (-3,0), (-1,1) and (1,2)

So the solution of the system is (1,2)



Geometry of Linear Equations

Geometry for a system of 3 equations

- Consider a system of 3 equations:

$$\begin{aligned} x + 2y + z &= 6 \\ 6x - 2y &= 4 \\ -3x - y + 4z &= 8 \end{aligned}$$

Matrix form

$$\underbrace{\begin{bmatrix}1 & 2 & 1\\ 6 & -2 & 0\\ -3 & -1 & 4\end{bmatrix}}_{\substack{A \\ 3\times3}} \underset{\substack{X \\ 3\times1}}{\begin{bmatrix}x\\ y\\ z\end{bmatrix}} = \underset{\substack{b \\ 3\times1}}{\begin{bmatrix}6\\ 4\\ 8\end{bmatrix}}$$

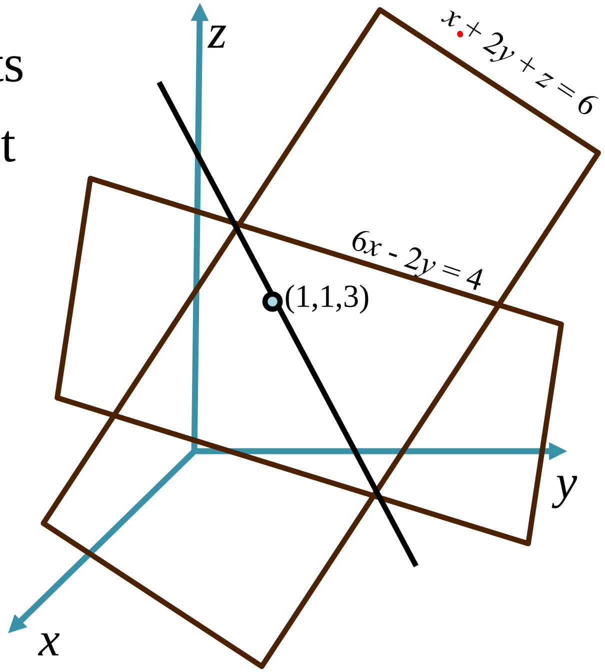
Geometry of Linear Equations

Row picture

$$\underline{x + 2y + z = 6}\ ;\ 6x - 2y = 4\ ;\ -3x - y + 4z = 8$$

$6x - 2y + 0z = 4$

- Each equation describes a plane in 3 dimensions. The intersection of the first plane with the second plane is a line
- The 3rd plane (not shown in the figure) intersects the line of intersection of the other two planes at a point (1,1,3)
- Solution for the system of equation is (1,1,3)



Types of Solutions

A system of linear equations can have –

3 d.

2(d)

planes intersect at a point

- Unique solution

X^n. lines

or

2 planes are ||

or

1 plane is || to intersection of other 2 planes

- No solution

|| lines

or

Planes are coincident

or

line of intersection

- Infinite number of solutions

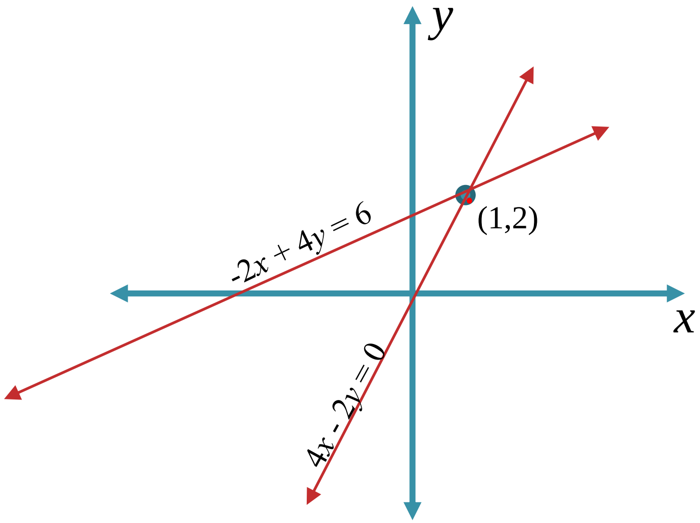
coincident lines

Types of Solutions

2 dimensional case

Unique solution

- This is the nice case where the system will have a point of intersection and hence a unique solution. $4x - 2y = 0$ and $-2x + 4y = 6$ has a unique solution (1,2)



One solution $(x, y) = (1,2)$

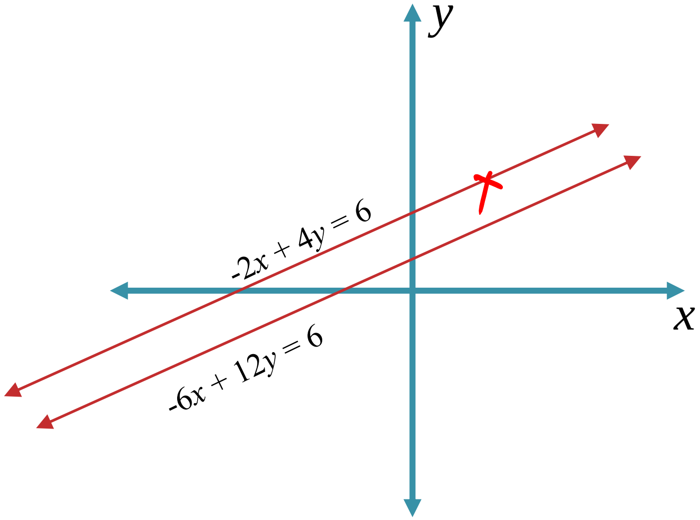
Types of Solutions

No solution

- A system has no solution if the equations are inconsistent. For example, $-2x + 4y = 6$ and $-6x + 12y = 6$ has no solution

$$-6x + 12y = 6 \Rightarrow -2x + 4y = 2$$

which contradicts with the first equation and hence the system has no solution



Parallel: No solution

Types of Solutions

Infinite number of solutions

- The other case is when one equation is just some multiple of the other. Then we will get infinite number of solutions

$-2x + 4y = 6$; $(-4x + 8y = 12)$

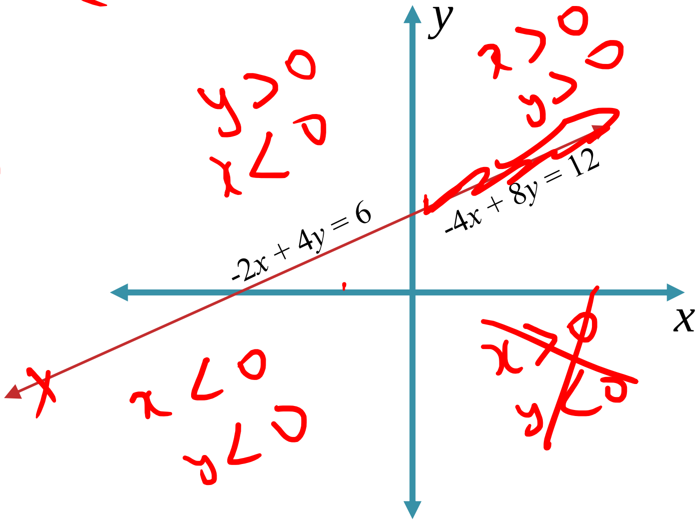
2nd eqn: 2(1st eqn)

$x = 0.1$

$y = \frac{6 + 2(0.1)}{4}$

$= \frac{6+2}{4}$

$x = -1, y = 1$



Whole line of solutions

$-2x + 4y = 6.$

$x = 0 \quad y = 3/2.$

$x = 1 \quad y = \frac{6 + 2}{4}$

$= 2.$

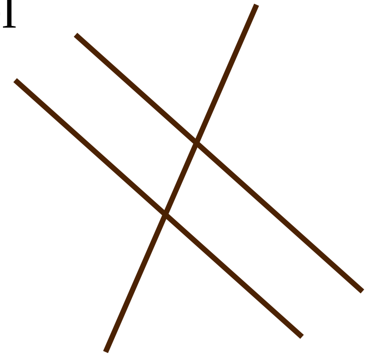
$y = 0 \quad x = -3.$

$4y = \frac{6 + 2x}{4} = \frac{6 - 2}{4} = 1.$

Types of Solutions

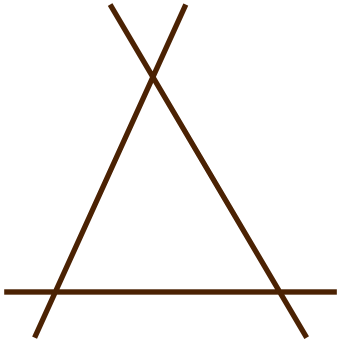
Singular case for three dimensions

- Two planes may be parallel



Two parallel planes: No solution

- One plane is parallel to the line of intersection of the other two planes

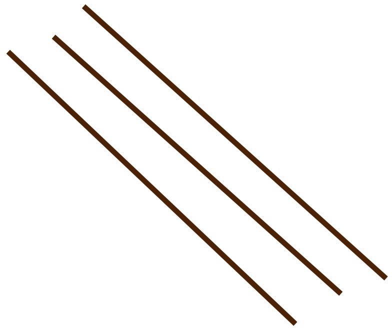


No intersection: No solution

Types of Solutions

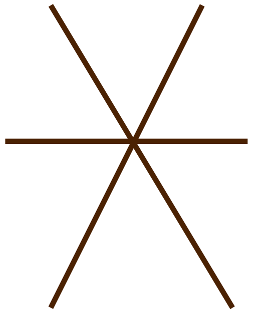
<u>Singular case for three dimensions</u>

- Three parallel planes



All planes parallel: No solution or a whole plane of solutions

- One equation is just the sum of the other two equations, the three planes have a whole line in common



Line of intersection: Infinite # of solutions

Geometry of Linear Equations

Vector

- A vector is defined as an ordered collection of numbers

$\begin{bmatrix} \vdots \end{bmatrix}$ $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

- Elements of a vector arranged as a column ⟶ column vector

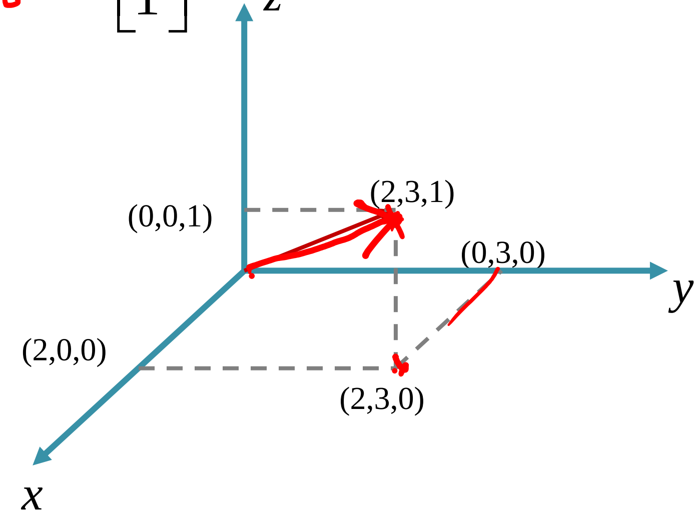
$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

- Elements of a vector arranged as a row ⟶ row vector
- If a vector v contains three real numbers say, $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, then v belongs to the vector space R^3

$\mathbb{R}$ (real numbers)

- The vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ a \end{bmatrix}$ are not the same

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} a \\ c \\ b \end{bmatrix}$

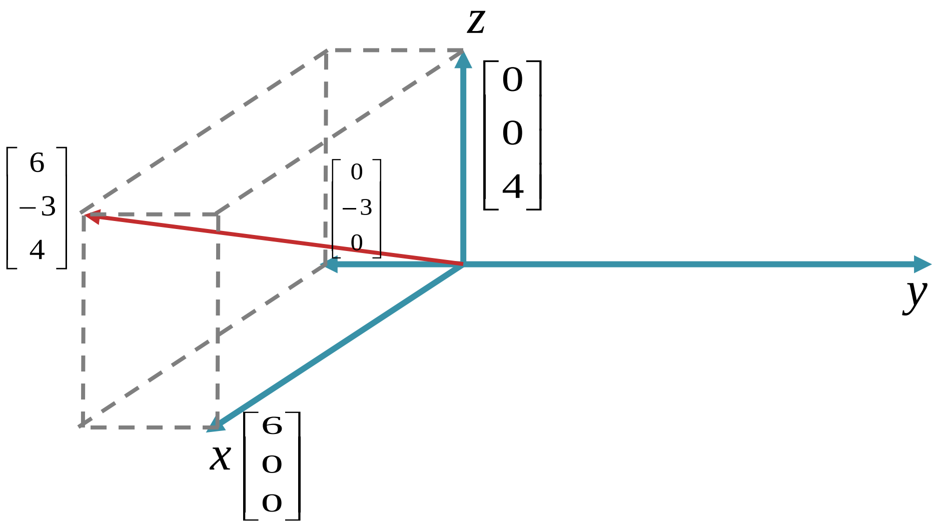


Vector Addition

Addition of a vector $\begin{bmatrix}6\\-3\\4\end{bmatrix}$ is done component by component and can be written as –

$$\begin{bmatrix}6\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-3\\0\end{bmatrix}+\begin{bmatrix}0\\0\\4\end{bmatrix}=\begin{bmatrix}6\\-3\\4\end{bmatrix}$$

$$\begin{bmatrix}6\\-3\\4\end{bmatrix}+\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}7\\-1\\7\end{bmatrix}$$



Geometry of Linear Equations

Column picture

$$4x - 2y = 0 \;\; ; \;\; -2x + 4y = 6$$

$4x + -2y = 0$

$-2x + 4y = 6$.

RHS.

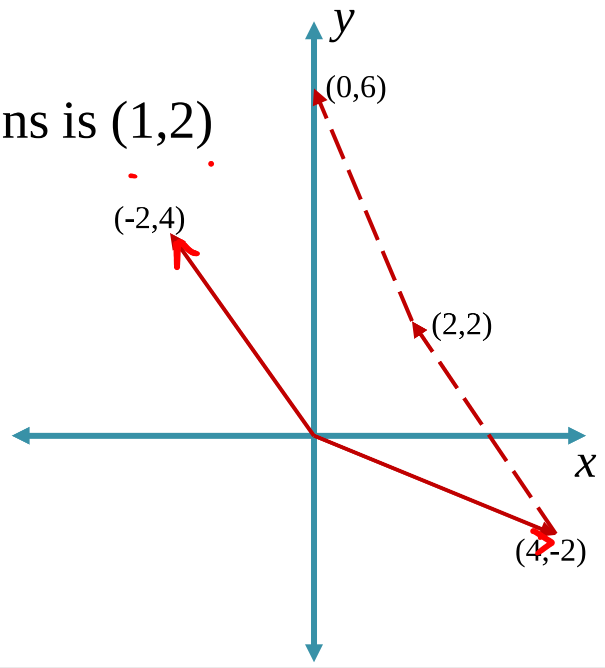
- Column picture: Linear combination of columns

$$x\begin{bmatrix}4\\-2\end{bmatrix} + y\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$$

We know that the solution for the two equations is (1,2)

Substitute them

$$1\begin{bmatrix}4\\-2\end{bmatrix} + 2\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$$



Vector: ordered set of real numbers

Vector addition v.

Scalar: (Real number). c: if c = 1

$$\underset{=}{c}\begin{bmatrix}1\\2\\3\end{bmatrix} \qquad (1\cdot)\begin{bmatrix}1\\2\\3\end{bmatrix} \qquad 2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix}$$

c = 1

c = 2

c = -1 $\begin{bmatrix}-1\\-2\\-3\end{bmatrix}$

c = -2 $\begin{bmatrix}-2\\-4\\-6\end{bmatrix}$

c = · $\begin{bmatrix}-3\\-6\\-9\end{bmatrix}$

Geometry of Linear Equations

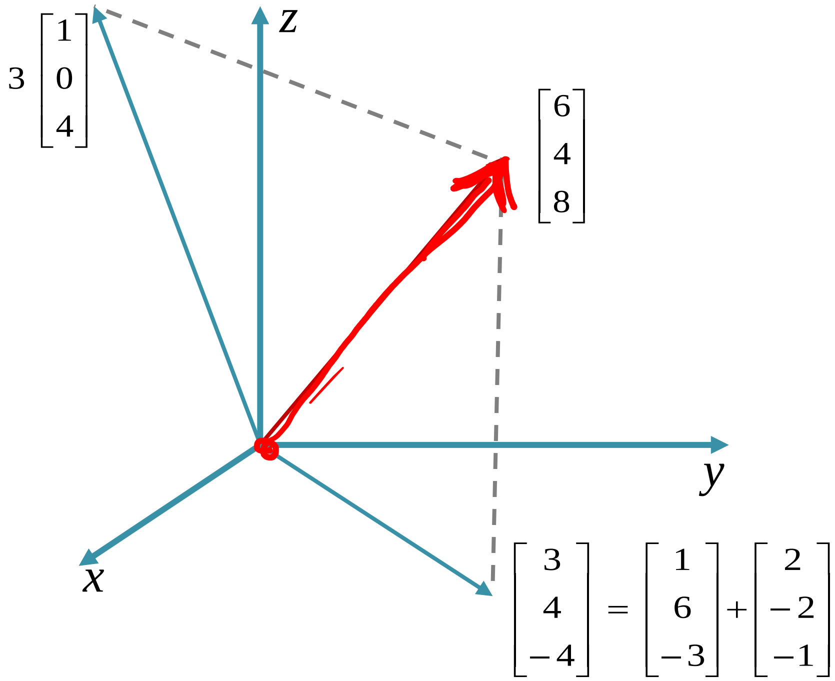
Column picture

$x + 2y + z = 6$; $6x - 2y = 4$; $-3x - y + 4z = 8$

0z

$$x\begin{bmatrix}1\\6\\-3\end{bmatrix}+y\begin{bmatrix}2\\-2\\-1\end{bmatrix}+z\begin{bmatrix}1\\0\\4\end{bmatrix}=\begin{bmatrix}6\\4\\8\end{bmatrix}$$

Solution for the system of equation is (1,1,3)

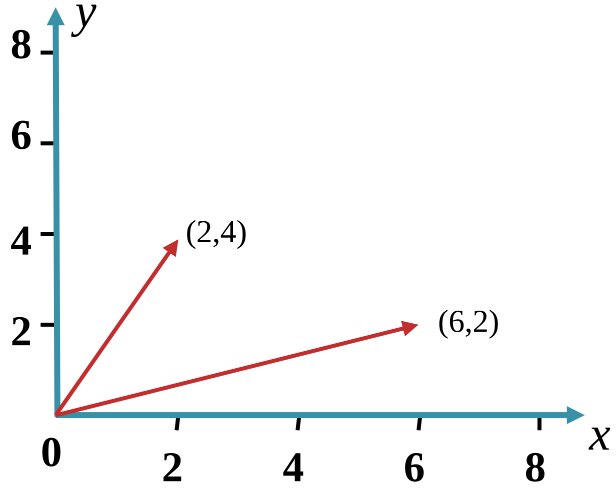


Vector Space

- Let V be a set of all vectors that lie in the first quadrant of R^2 and F be R

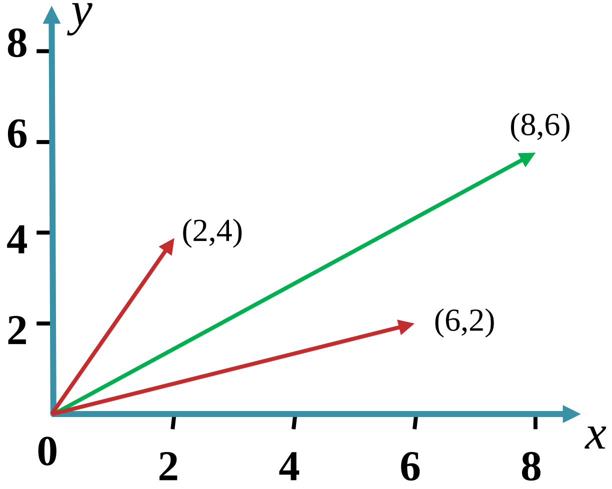
- Consider two vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 2 \end{bmatrix} \in V$

- It can easily be noted that both these vectors lie in the first quadrant of R^2



Vector Space

- Addition: $\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \in V$
- The resulting vector also lies in the first quadrant of R^2



Vector Space

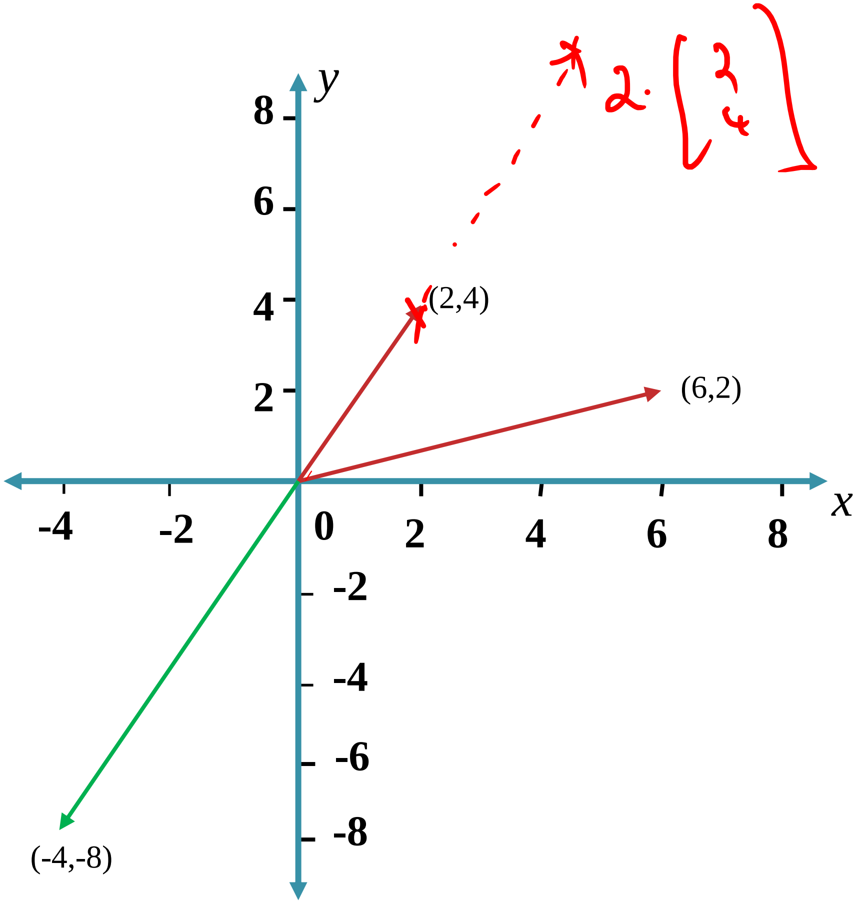
- Scalar multiplication: $a \in F$ where F is a field R.

  Consider a = -2

  Then, $-2 \times \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

- It is clear that the resulting vector is outside V and hence it is not a vector space

- So, the first quadrant of R^2 is not a vector space whereas, R^2 is a vector space



Vector Space

- Let *V* be a set of vectors and *F* be a field. Then *V* is called a vector space over a field *F* if the following requirements are met

$$\forall x, y \in V, (x + y) \in V$$

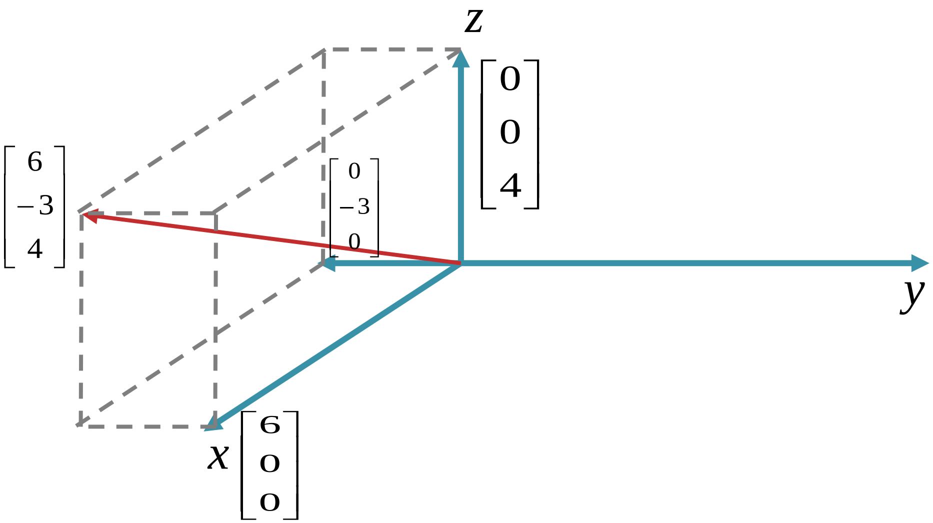
$$\forall x \in V \text{ and } \forall a \in F, a \times x \in V$$

- In other words, a set of vectors is called a vector space if the set is closed under vector addition and scalar multiplication of a vector

Linear Combination

- The vector $\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$ can be expressed as a linear combination of vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as below

$$\begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Linear Combination

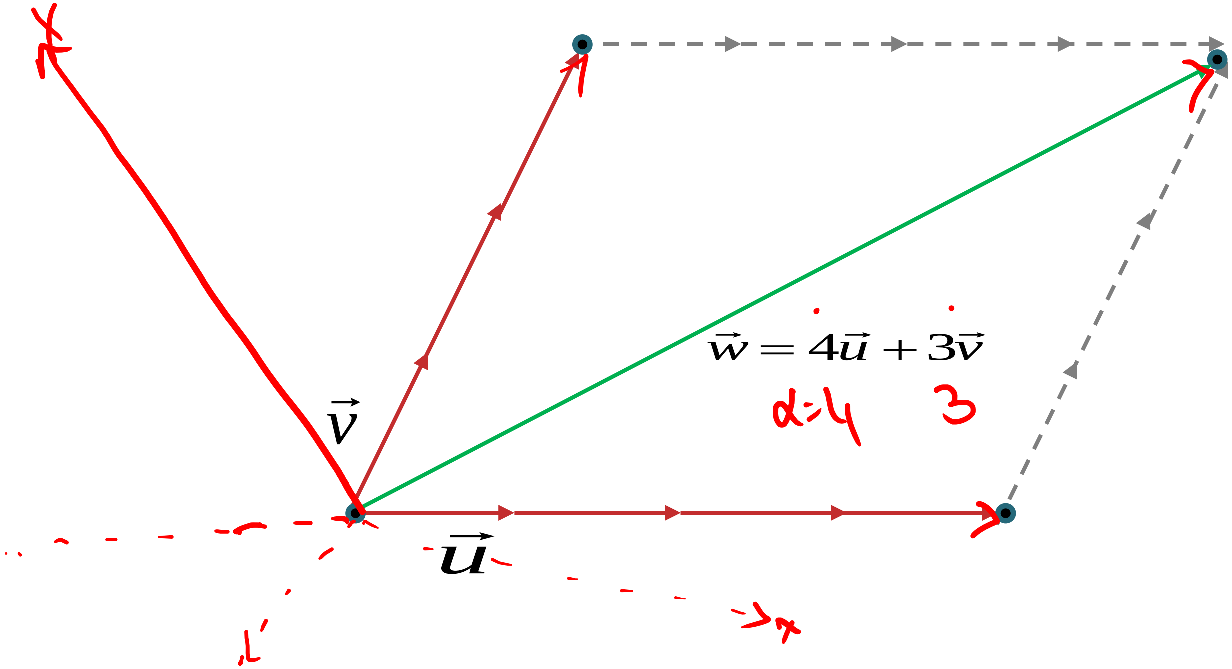
- A vector v can be written as a **linear combination** of vectors $u_1, u_2, u_3, \ldots, u_n$ such that

$$v = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$$

where $c_1, c_2, \ldots, c_n$ are all scalars

Linear Combination

- Geometrically, we may see the linear combination as –



$\alpha u + \beta v$

α, β are scalars (real numbers)

- We have the vector $\vec{w}$ which is expressed as a linear combination of the other two vectors $\vec{u}$ and $\vec{v}$

All possible linear combinations gives me the plane

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $u = 2v.$

$\alpha u + \beta v.$ α, β are scalars.

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha + 2\beta \\ 2\alpha + 4\beta \end{bmatrix} = \cancel{2} \left[\cancel{\alpha + \beta} \right.$$

All possible combinations give me a line ONLY.

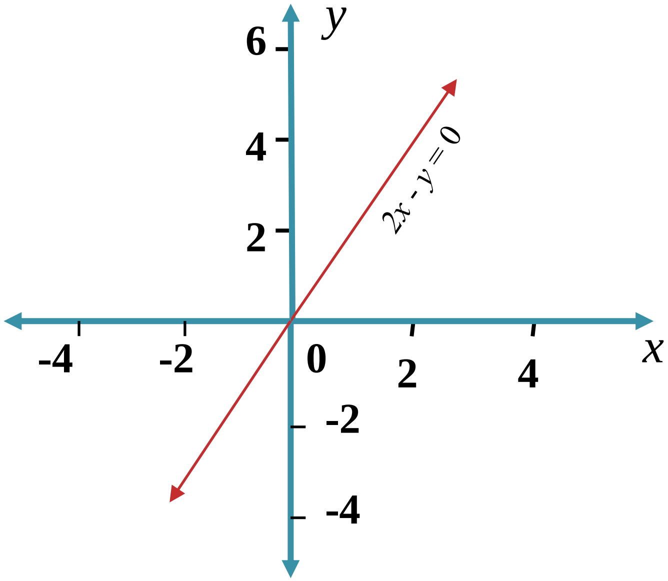
$$\underline{(\alpha + 2\beta)} \begin{bmatrix} 1 \\ \underline{2} \end{bmatrix}$$

Subspace

- R^2 satisfies the conditions for a vector space – vector addition and scalar multiplication
- What about the line $2x - y = 0$? Does it satisfy the conditions?

  Answer: Yes

  Any line which passes through the origin in R^2 is a subspace of R^2



Subspace

- A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space –
  - Vector addition: For any vectors x, y in the subspace, $x + y$ is in the subspace
  - Scalar multiplication: For any scalar c, cx is in the subspace

- Column space: Contains all linear combinations of the columns of **A**

- Row space: Contains all linear combinations of the rows of **A**

Subspace

Is column space a subspace?

- Column space contains all linear combinations of the columns of **A**
- A = $\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$; Column space of A contains all linear combinations of

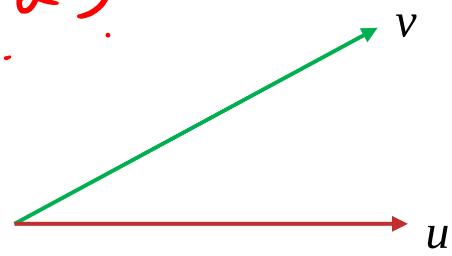
$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

- If we take all the linear combinations of the columns in A, it will fill a plane in $R^3 \Rightarrow$ it is a subspace of R^3

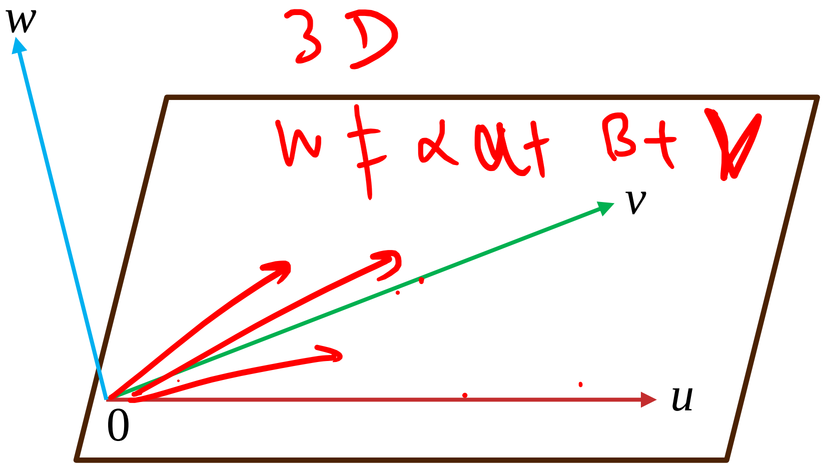
Linear Dependence & Independence

- Linearly independent:

2D



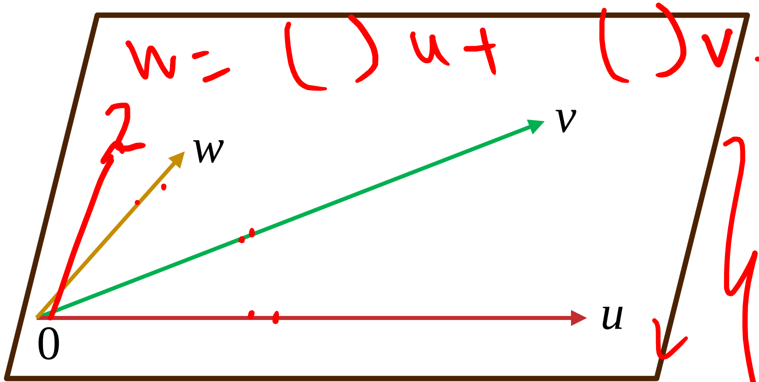
3D



$w \neq \alpha u + \beta v$

- Linearly dependent:

u, w are l. d.



w = ( ) u + ( ) v +2

2

u = ( ) v + ( ) w + 2

v = ( ) u + ( ) w + 2

If $u = 2v$, u, v are linearly dependent.

u, v, w.

$u = \alpha v + \beta (w)$, u, v, w are linearly dependent.

Linear Independence

- A set of vectors $\{v_1, v_2, v_3, \ldots, v_n\}$ is said to be **linearly independent** if $\underbrace{c_1 = c_2 = c_3 = \ldots = c_n}_{\text{scalars}} = 0$ is the only solution to the following equation

$$\overset{(0).}{c_1} v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = 0$$

one sol^n. is $c_1 = c_2 = c_3 = \quad c_n = 0$

- Any one vector in the set cannot be expressed as a linear combinations of the rest of the vectors in the set

- If any other combination gives zero, they are **linearly dependent** and at least one of them is a linear combination of the others

u, v, w are linearly dependent. eg.

$u = v + w.$ (given)

$u - v - w = 0$

$c_1 u + c_2 v + c_3 w = 0$

$(1)u + (-1)v + (-1)w = 0$

one sol^n $c_1 = c_2 = c_3 = 0$

$c_1 = 1$

$c_2 = -1$

$c_3 = -1.$

$$\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \quad \begin{bmatrix} \alpha \\ -\alpha \\ -\alpha \end{bmatrix}$$

α is a scalar

$\cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

you get non zero sol^n!

linearly dependent !

$u, \quad v, \quad w.$

$2u + v + w = 0$

somehow I have found

$c_1 = 2$

$c_2 = 1$

$c_3 = 1$

$v = -2u - w.$

or

$u = \frac{-v - w}{2}$

or

$w = -2u - v$

u, v, w are

d. D!

u, v, w

$c_1 u + c_2 v + c_3 w = 0$

only sol^n is $c_1 = c_2 = c_3 = 0$

$c_1 u = - c_2 v - c_3 w.$

$(0) u = -(0) v + ( ) w.$ Cannot divide by 0

I cannot write
u as a combination of v, w

nor v " " u, w

nor w " " u, v

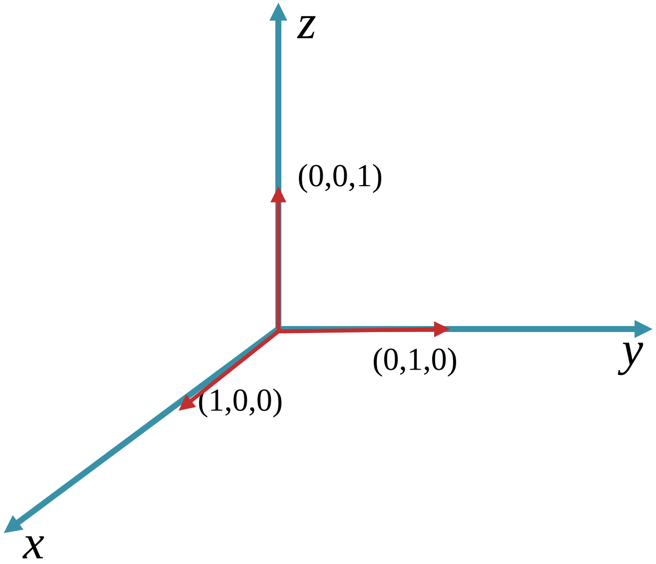
Linear Independence

Example:

- Consider the set $V = \{ [1,0,0]^T, [0,1,0]^T, [0,0,1]^T \}$. Now we represent the zero vector as

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The only values c_1, c_2 and c_3 can take in the above equation is zero. So, the set V is linearly independent



Linear Dependence & Independence

2 D!

Example:

- Consider the set **U** = {[1,0,0]^T , [0,1,0]^T , [1,1,0]^T}

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

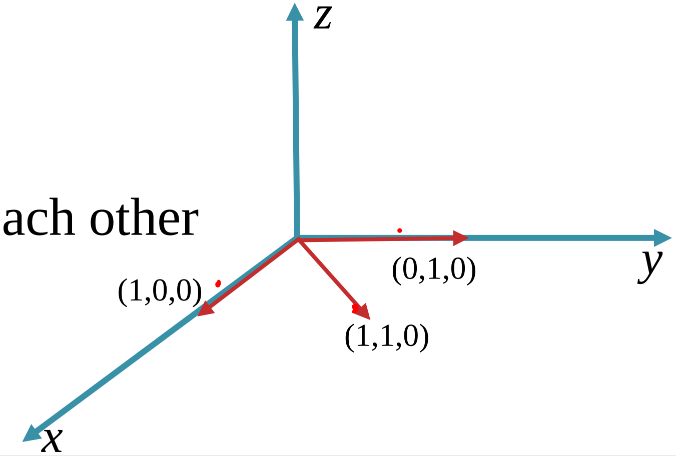
$c_1 = c_2 = -2$

$c_3 = 2$

Here, in addition to the solution $c_1 = c_2 = c_3 = 0$, there exists other solutions such as $c_1 = c_2 = -1$ and $c_3 = 1$. So they are linearly dependent on each other. Hence one can be expressed in terms of the rest. For instance, here,

$$\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Hence the vectors are linearly dependent on each other



No

Are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ l.i ?

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$c_1 = -2c_2$.

$\left.\begin{array}{l} c_2 = 1 \\ c_1 = -2. \end{array}\right\}$ Non zero soln exists.

So l D.

$\boxed{u = 2v.}$

YES.

Are $\begin{bmatrix}1\\2\end{bmatrix}$ $\begin{bmatrix}2\\2\end{bmatrix}$ l.i

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$c_1 + 2c_2 = 0$

$2c_1 + 2c_2 = 0$

only soln. $\left\{\begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}\right\}$

l.I

Linear Dependence & Independence

- So the vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are linearly dependent ✓
- The vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ are linearly dependent or independent?
- Answer: Linearly independent
- Zero is the only value that c_1 and c_2 could take in

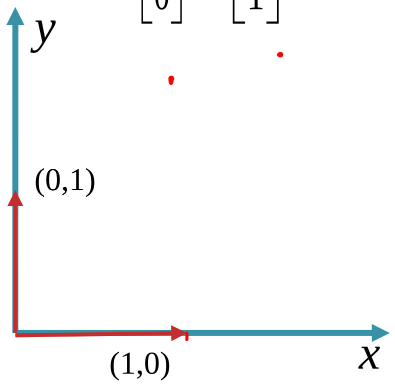
$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Basis & Dimension of a Vector Space

- **Basis**: A basis for a vector space is a set of vectors having two properties at once: Sufficient to generate every other vector !
  - It is linearly independent
  - It spans the space, i.e. a linear combination of the basis vectors can generate any other vector in the vector space
- In more general terms, a basis is a linearly independent spanning set
- Standard basis:

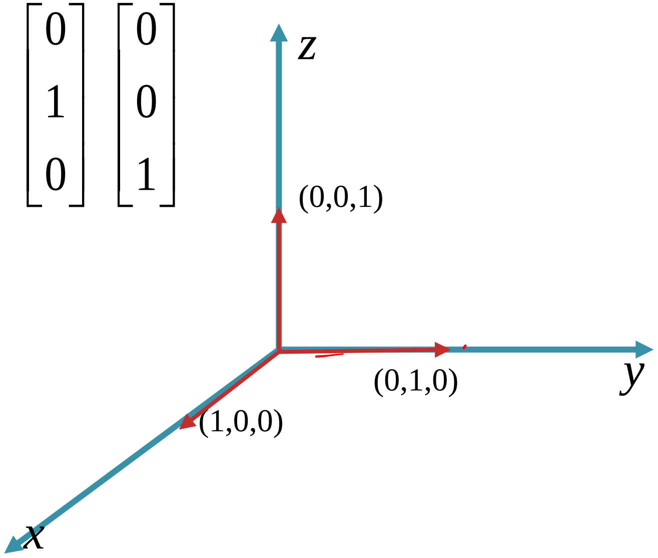
2D:

$$\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\end{bmatrix}$$



3D:

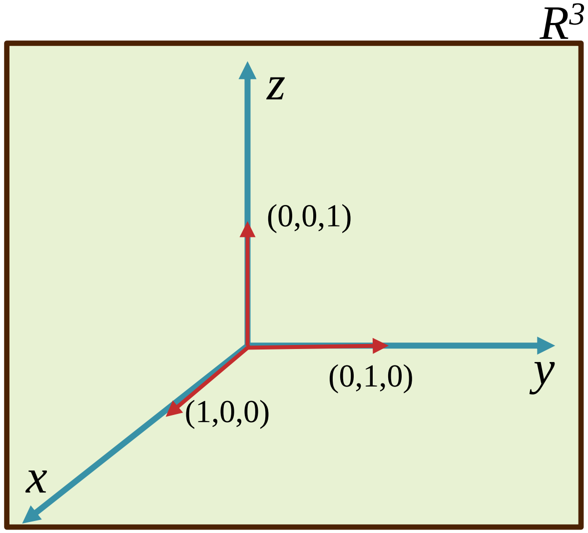
$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$



Basis & Dimension of a Vector Space

- **Span:** Span of a vector set *V* refers to the vector space generated by all possible linear combinations of vectors present in *V*

  Example: Consider the vector set $V = \{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\}$. The span of *V* is R^3 . This means that the possible linear combinations of the vectors of *V* fill the complete R^3 space



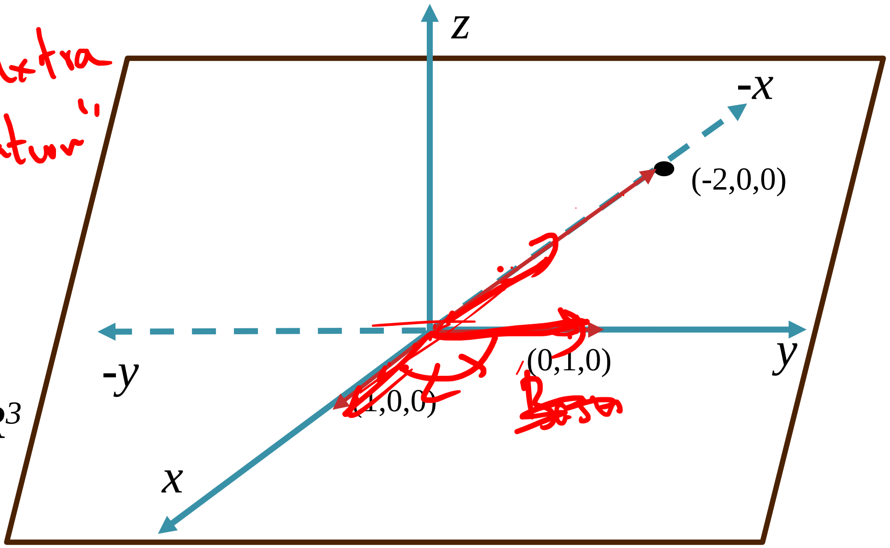
- **Dimension:** The maximum number of linearly independent vectors that can be obtained from a vector space is called the **dimension** of that vector space

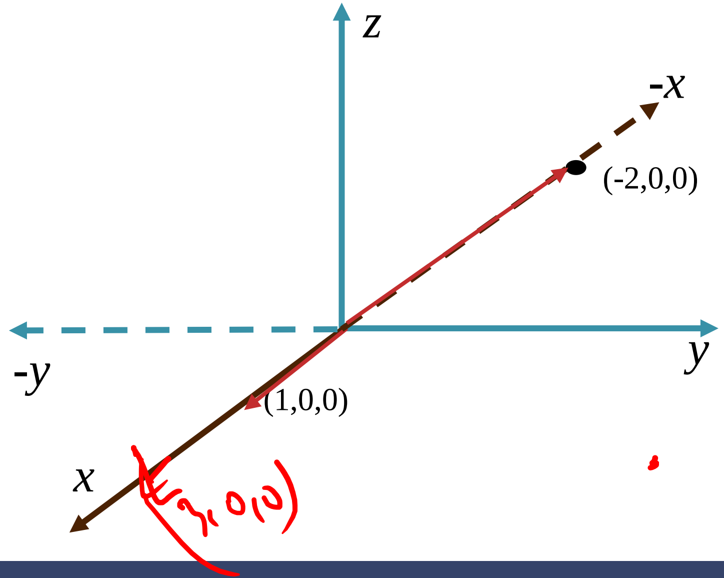
Basis & Dimension of a Vector Space

- V = {(1,0,0) , (0,1,0) , (-2,0,0)} "not giving any extra information"
- Spans a plane (the x–y plane) in R^3
- The vector (-2,0,0) and (1,0,0) are linearly dependent
- Hence, the vector set V doesn't form a basis in R^3

✓

- ***Dimension* = 2**



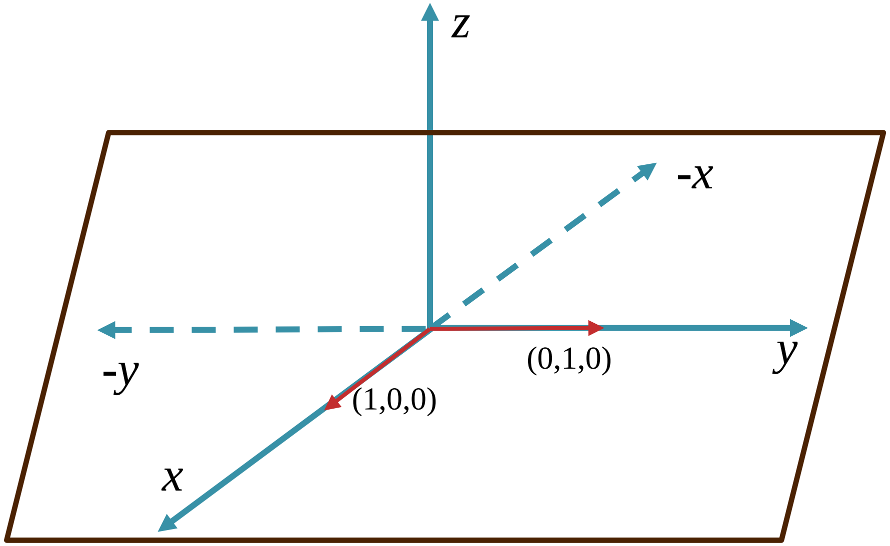


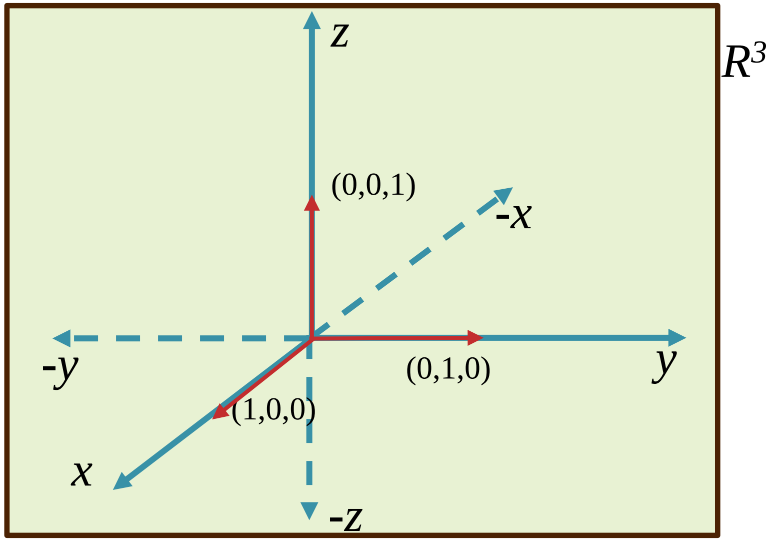
- V= {(1,0,0) , (-2,0,0)} (3,0,0)
- Spans only a line (1 dimensional)
- Linearly dependent
- Hence V doesn't form a basis in R^3
- ***Dimension* = 1**

only 1 vector is sufficient.

Basis & Dimension of a Vector Space

- V= {(1,0,0) , (0,1,0)}
- Spans a plane (the x–y plane) in R^3
- Linearly independent
- Hence, V is a basis for the x–y plane
- ***Dimension = 2***





- V = {(1,0,0) , (0,1,0) , (0,0,1)}
- Spans the whole space R^3
- Linearly independent
- So, V is a basis for the space R^3
- ***Dimension = 3***

Rank of a Matrix

- Rank of a matrix refers to the number of linearly independent rows or columns of the matrix: Dimension of column space= Dimension of row space

- It can also be viewed as the number of pivots in Gaussian elimination process

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

Number of linearly independent rows = 2

⇒ Rank of the matrix = 2

A: m×n m rows, n columns.

Row rank: # independent rows

Column rank: # independent columns

Row rank = column rank! = rank of matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 4 & 8 \end{bmatrix}$$

row rank = 1

column rank also = 1

A: m×n

m = 2

n = 4

2nd row = 2(1st row)

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 5 \end{bmatrix}$$

each is a 2D vector

Max possible vectors w 2D = 2.

row rank = 2 (as rows are independent)

column rank : (4 columns) max possible column rank = 2 :

Max possible row rank = 2

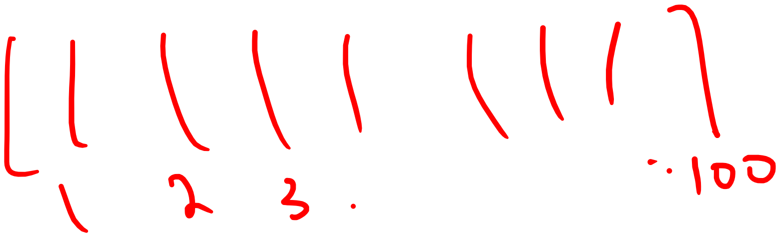
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

rv

A: ~~too~~ 2×100.

Max rank of matrix (2). Why? max rowrank = 2



max col. rank = 2

$\text{rank}(A) \leq 2$.

A: 50×100

$\text{rank}(A) = 50$

A: $m \times n$ $\text{rank}(A) \leq \min(m,n)$

A: $75 \times \boxed{25}$.

$\text{rank}(A) \leq 25$

row rank ≤ 75

col rank ≤ 25

$$A: \begin{array}{c|cccc} & D1 & D2 & D3 & D4 \\ \hline P1 \to & 1 & 2 & 3 & 4 \\ P2 \to & 2 & 4 & 6 & 8 \end{array}$$

Data is collected.

$\text{rank}(A) = 1 < \min(2,4)$

Even though there are 4 depots & 2 products, sales of product are "dependent" ∴.

Relationship between P1 & P2.

rank =1 tells you the sales are "magically" related to each other

Existence and Uniqueness of a solution

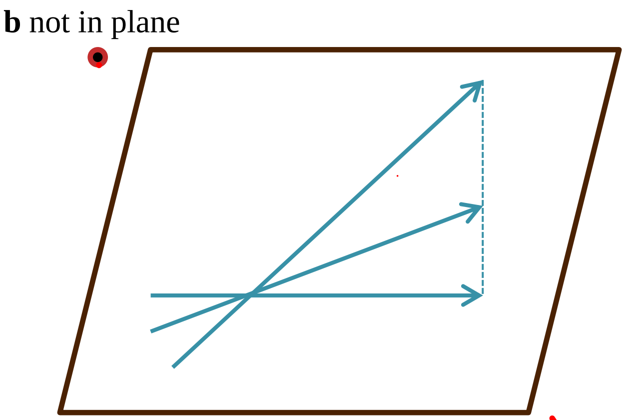
- **Ax=b**

$m\times n \quad n\times 1 \quad m\times 1$

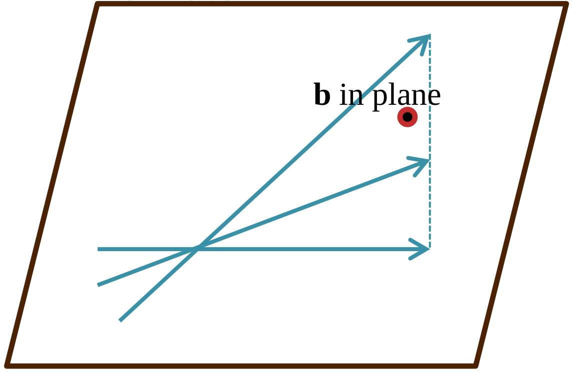
- When does solution exist?

b cannot be written as combination of columns of A

No solution



At least one solution exists



columns of A are (n)

Augmented $\left[\quad \vdots \quad b \right] > \text{rank}(A)$

$A_u = (m, n+1)$

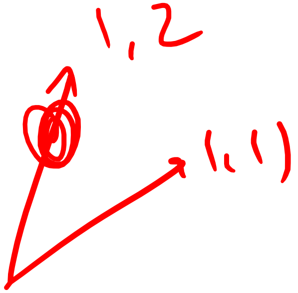
$\text{rank}(A) \leq \min(m, n+1)$

$$A : \underbrace{\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}}_{\text{rank}(A) = 1} \begin{bmatrix}x\\ y\end{bmatrix} = b = \begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & 1\\ 2 & 4 & 1\end{array}\right]$$

rank of Aug = 2 ∴

∴ No sol^n exists!



$\begin{bmatrix}1\\ 1\end{bmatrix}$ cannot be expressed as combination of columns of A

∴ So no solution

$m=2,\ n=2.$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$\text{rank}(A) = 1 < n.$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}}$$

$\text{rank}(A) = 1 < (2)$

$n=2$

sol: exists!

How many sol^n. ?

∞

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \Big[$$

$\text{rank}(A) = 2 = n.$

$$\left[\begin{array}{cc|c} 1 & 1 & b \\ 1 & 2 & \end{array}\right].$$

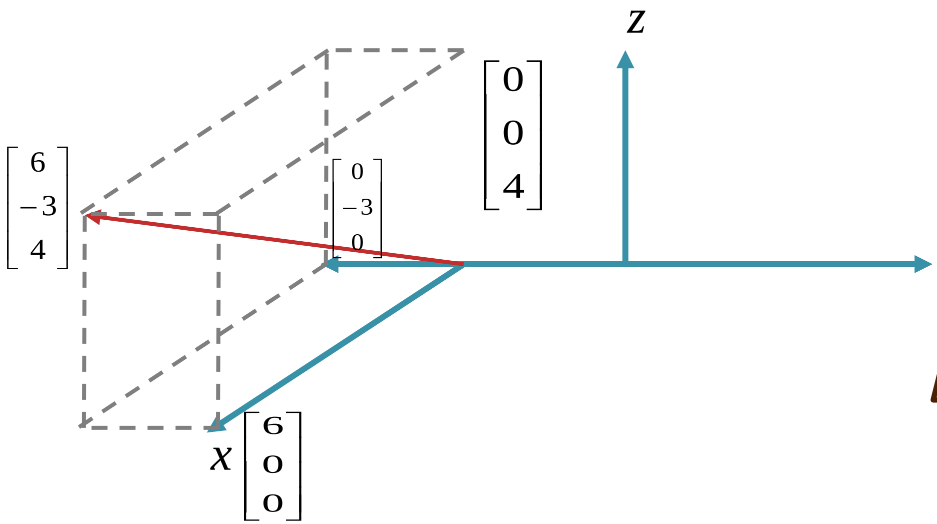
$\text{rank}(A_{aug}) = 2 = n.$

Sol^n. $\underbrace{\text{always exists}}_{\text{unique sol}^n.}$!

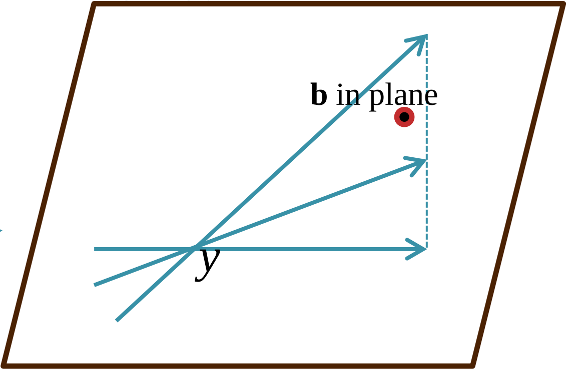
Existence and Uniqueness of a solution

- Uniqueness?

Unique solution



Infinite # solution exist



Null Space

- The null space of a matrix **A** consists of all vectors ***x*** such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

- The set of solutions (***x***'s) in $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is itself a vector space which is called as a null space of **A**

$$N(\mathbf{A}) = \{\, x \in R^n \,/\, \mathbf{A}\boldsymbol{x} = \mathbf{0} \,\}$$

- If the column vectors are linearly independent, the null space contains only the zero vector
- The null space of an invertible matrix contains only zero vector

Null Space

To find null space for the matrix A = $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

- Null space: $N(\mathbf{A}) = \{x \in R^n \,/\, \mathbf{A}\boldsymbol{x} = \mathbf{0}\}$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \end{bmatrix} \Rightarrow$$

$$\begin{tikzcd}
\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 3 & 0 \end{array}\right] \arrow[r, "R_3 = R_3 - R_1"'] & \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \arrow[d, "R_3 = R_3 - R_2"'] \\
\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \arrow[l, "R_1 = R_1 - R_2"']
\end{tikzcd}$$

Null Space

- We have the equations:

$$x_1 + 2x_3 + x_4 = 0 \Rightarrow x_1 = -2x_3 - x_4$$

$$x_2 + x_4 = 0 \Rightarrow x_2 = -x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- N(A) = scalar multiples of the vectors $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Null Space

- **If the vectors are linearly independent, the null space contains only the zero vector**
- The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent
- What is the null space of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$?
- $\mathbf{A}\boldsymbol{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .What are the values of x_1 and x_2?

Null Space

- We have –

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $x_1 = 0$ and $x_2 = 0$ is the only solution that the system of equations could take

⇒ If the vectors are linearly independent, the null space contains only the zero vector

Null Space

- **If the vectors are linearly independent, the null space contains only the zero vector. What about the dependent vectors?**
- Consider two dependent vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- What is the null space of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$?
- $\mathbf{A}\boldsymbol{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- x_1= ? and x_2= ?

Null Space

- We have –

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

- We have the equations –

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- We have $N(A)$ = scalar multiples of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- **If the vectors are linearly dependent, the null space also contains non-zero vector(s)**

Null Space: Cause of Non-Uniqueness

- Solutions of **A***x* = **b** with **A** having n columns.

- Solution does not exist if **b** does not belong to the column space of **A**

- At least one solution exists if **b** belongs to the column space of **A**
    - Solution unique if null space of **A** has only the zero vector
    - Infinite solutions if null space of **A** has non-zero vectors

Existence and Uniqueness of a solution

A : $m \times n$.

- **A***x* = **b** has no solution or is inconsistent iff ✓
rank(**A**) < rank(augmented matrix [**A b**])
[**b** does not belong to column space of **A**]

- **A***x* = **b** has a unique solution iff
rank(**A**) = rank(augmented matrix [**A b**]) = *n*
[**b** belongs to column space of **A** and null space of **A** has only the zero vector]

- **A***x* = **b** has infinitely many solutions iff
rank(**A**) = rank(augmented matrix [**A b**]) < *n*
[**b** belongs to column space of **A** and null space of **A** has non-zero vectors as well]

$m = 3$

$n = 2$ $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 2 \end{bmatrix}.$

$b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$

$\text{rank}(A) = 2 = n$

$\text{rank}(A_{aug}) = 2 = n.$

$x = 1$

$y = 1$

$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\text{rank}(A_{aug} \overset{?}{=})$

$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$\text{ran}(A_{aug}) = 2$ Sol^n exits!

or $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\text{rank}(A_{aug}) = 3$

$A: m \times n:$ $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Augmented matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ rank $=$ rank (A)

RHS is "contained" in A!

$A: \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

rank $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix} = 2 \neq$ rank (A)

No solution

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

rank (A) = 2 rank (Aug) = 2

unique solution exists !

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 3 \end{bmatrix}$$

- We will include the right hand side as an extra column to **A**. That matrix is called as an ‘Augmented matrix’ –

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 1 & 2 & 12 \\ 1 & 0 & 1 & 3 \end{array}\right]$$

- The system is inconsistent as the rank(**A**) < rank(augmented matrix)

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 6 \end{bmatrix}$$

- The system has a unique solution as the rank(**A**) = rank(augmented matrix) = *n*

Existence and Uniqueness of a solution

- Consider the system,

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix}$$

- The system has infinitely many solutions as the rank(**A**) = rank(augmented matrix) $< n$