NLP (Non-linear Programming)

$$\min \quad f(x) \text{ - } \mid \text{linear} \mid \text{non-linear}$$

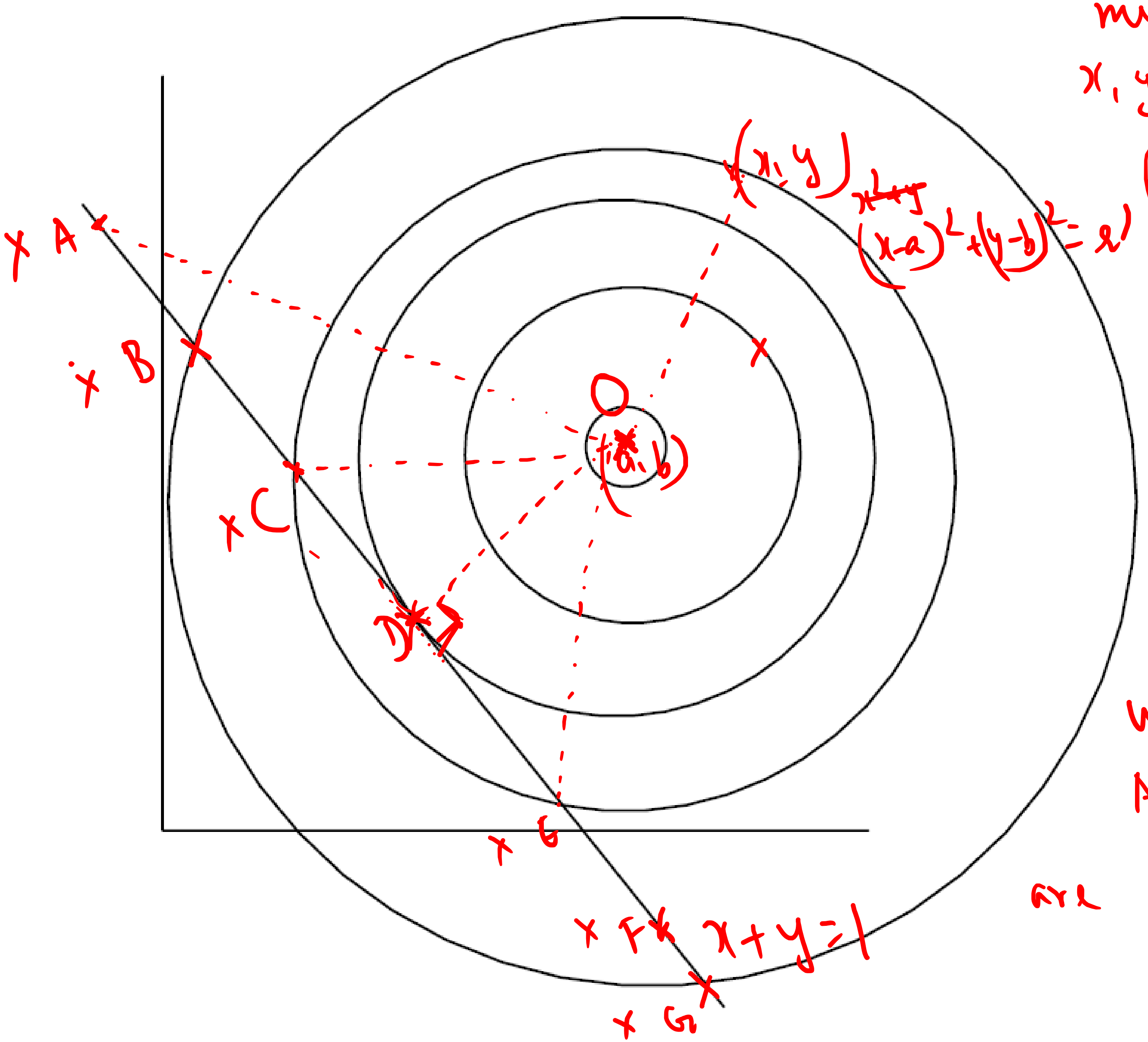
$$\text{st to} \quad \underbrace{g(x)}_{\text{linear} \mid \text{nonlinear fun}} = 0$$

EQUALITY CONSTRAINTS

$$\min \quad f(x).$$

$$\underbrace{h(x) \le 0}$$

(INEQUALITY CONSTRAINT)



$$\min_{x,y} \quad (x-a)^2 + (y-b)^2.$$

(Distance)^2 of (x, y) from (a, b)

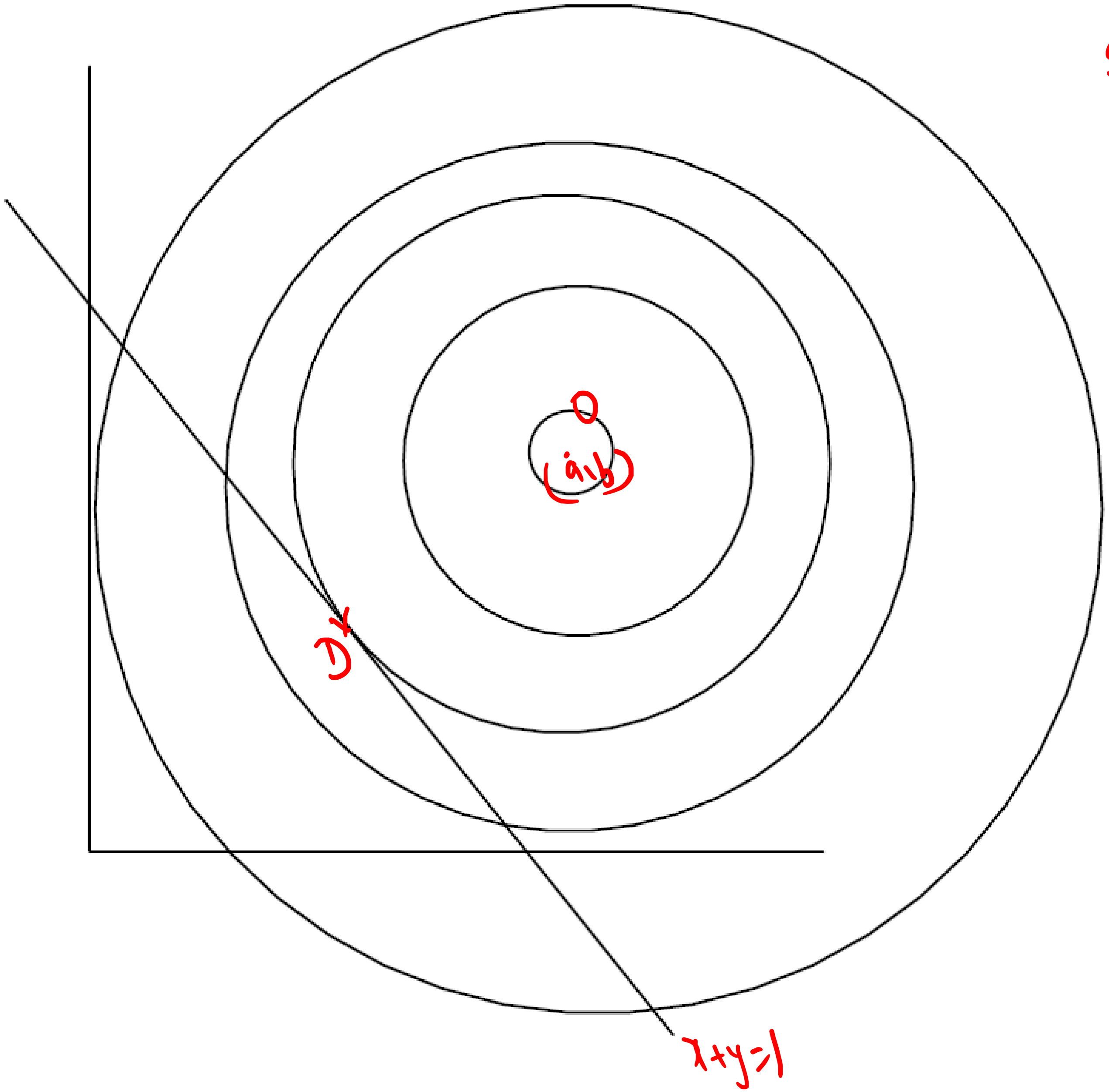
unconstrained minimum is (a, b).

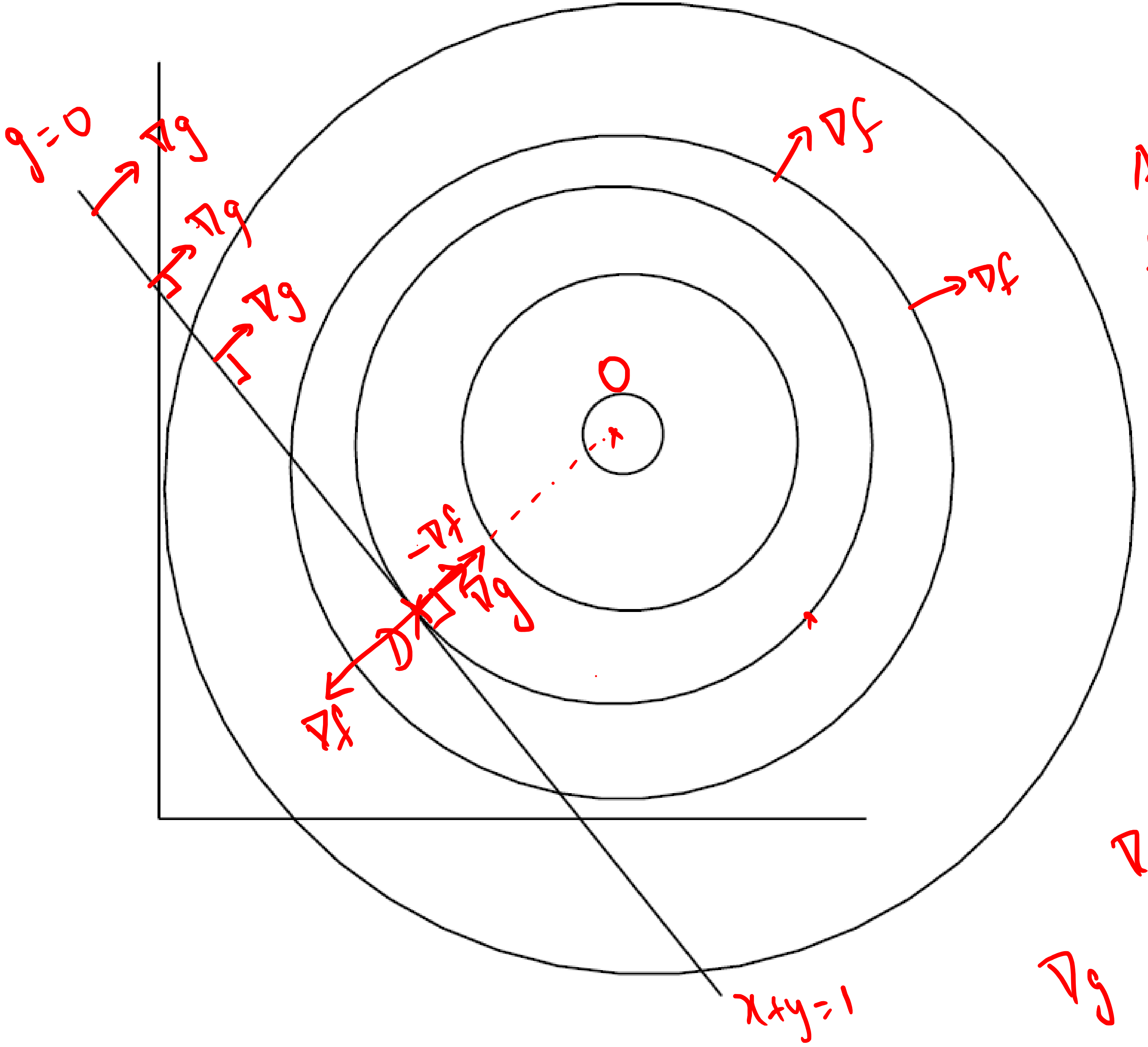
Contour plots: concentric circles

Constraint: $x + y = 1$

which of the points A, B, C, D, E, F, G, ~~X~~ are optimum?

$$\min \quad (x-a)^2+(y-b)^2$$
$$\text{s.t} \quad x+y=1$$





$\min\ (x-a)^2+(y-b)^2$ $\quad f:(x-a)^2+(y-b)^2$

s.t. $x+y=1$

At D, the line $x+y=1$ is tangent to the contour circle!

What vector is ⊥ar the contour circle?

$-\nabla f$ is ⊥ar contour circle

$-\nabla f$ ⊥ar line!

$g = x+y-1 = 0$

$$\nabla g = \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{\partial g}{\partial x}\\ \frac{\partial g}{\partial y}\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

∇g ⊥ar line!

$-\nabla f$ for circle

∇f for line $x+y-1=0$

∇g for line

$-\nabla f \parallel \nabla g$

$-\nabla f = \lambda \nabla g$ (λ is some scalar)

$$\underbrace{-\begin{bmatrix} 2(x-2) \\ 2(y-2) \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$x+y=1$ } (constraint)

unknowns x, y, λ (# 3)

equations 2 (?)

$x = 1/2$

$y = 1/2$

$\lambda = -2(x-2) = -2(1/2-2) = 3$

min $(x-2)^2 + (y-2)^2$.

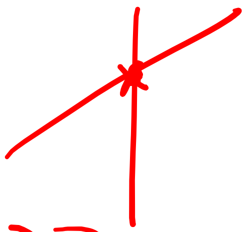
s.t. $x+y=1$.

$$\nabla f = \begin{bmatrix} 2(x-2) \\ 2(y-2) \end{bmatrix} = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\min (x-2)^2+(y-2)^2+(z-2)^2.$$

$$\text{s.t}\left(x+y=1,\quad x+2y=2\right)$$



3 eqn. 3 unknowns

$$\begin{cases} -\nabla f = \lambda \nabla g \\ x+y-1=0 \end{cases}$$

$$\nabla f = \begin{bmatrix} 2(x-2) \\ 2(y-2) \\ 2(z-2) \end{bmatrix}.$$

$$\underbrace{-\nabla f}_{H} = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\begin{cases} g_1 = x+y-1 = 0 \\ g_2 = x+2y-2=0 \\ -2(x-2) = \lambda_1 + \lambda_2 \\ -2(y-2) = \lambda_1 + 2\lambda_2 \\ -2(z-2) = 0 \end{cases}$$

$$\nabla g_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} \partial g_2 / \partial x \\ \partial g_2 / \partial y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

5 unknowns
5 equations

n decision variables

$\min f(x)$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

λ : Lagrange multipliers

m equality
$$\begin{cases} g_1(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$$

$$-\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \dots + \lambda_m \nabla g_m \quad \} \; n \text{ equations}$$

$$\left.\begin{aligned} g_1(x) &= 0 \\ g_2(x) &= 0 \\ &\vdots \\ g_m(x) &= 0 \end{aligned}\right\} \; m \text{ equations}$$

$n+m$ equations

$m + (n)$ variables $(x_1, x_2, \dots x_n)$

$\lambda_1, \lambda_2, \dots \lambda_m$ (Lagrange multipliers)

KKT conditions

Karush Kuhn Tucker

applicable for continuous & differentiable f, g.

$\min f(x)$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ n decision variables

$x+y+1 \geq 0$

$-(x+y+1) \leq 0$

$m \begin{cases} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$ $\quad$ p inequality $\begin{cases} h_1(x) \leq 0 \\ h_2(x) \leq 0 \\ \vdots \\ h_p(x) \leq 0 \end{cases}$

$\left.\begin{array}{l} x+y-1 = 0 \\ -(x+y-1) = 0 \end{array}\right\}$

KKT: $(n)\{ -\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_m \nabla g_m + \mu_1 \nabla h_1 + \mu_2 \nabla h_2 + \cdots + \mu_p \nabla h_p$

$m \begin{cases} g_1(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases}$

λ: sign unrestricted

$\underbrace{\begin{array}{l} \overset{(=0)}{\mu_1 h_1(x) = 0} \\ \mu_2 h_2(x) = 0 \\ \vdots \\ \mu_p h_p(x) = 0 \end{array}}_{(2^p \text{ combinations!})}$

$\left(\begin{array}{ll} \mu_1 \geq 0 & h_1(x) \leq 0 \\ \mu_2 \geq 0 & \vdots \\ \vdots & \\ \mu_p \geq 0 & h_p(x) \leq 0 \end{array}\right)$