MECHANICS

- NEWTON'S LAWS OF MOTION
- PROJECTILE MOTION
- UNIFORM CIRCULAR MOTION
- SIMPLE HARMONIC MOTION
- GRAVITATION
- ROTATION OF RIGID BODIES

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A. Newton's Laws Of Motion

Classical mechanics is the theory of motion based on mass and force. It describes this phenomenon using three Newton's laws of motion which relates object acceleration to its mass and forces acting on it.

Newton's first law of motion;-

"Everybody continues on it state of rest or in a uniform motion in a straight line unless a resultant force act on it"

NOTE;

- The **inertia of a body** is a reluctance to start moving when a body is at rest or reluctance to stop when it is in motion.
- Mass is a measure of inertia of a body

The first law gives a definition of a force as the agency which acting on a body produces a change of state of a body.

The SI unit of force is ***Newton (N)***, where 1N is a force which gives a mass of 1kg an acceleration of $1m/s^2$.

Equilibrant Forces on a Body

If an object is stationary and remains stationary is said to be in static equilibrium.

Necessary conditions for a body to be in equilibrium are;-

(i) The net external force acting on a body must be zero

i.e. $\sum \overline{F} = 0$

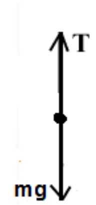
(ii) The net external torque about any point must be zero

i.e. $\sum \overline{\tau} = 0$

Consider examples of a body at static equilibrium;-

(i) A mass m hang on a string;-

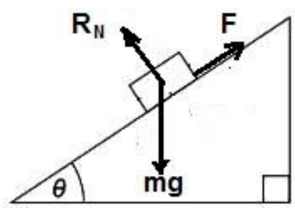
When a stationary mass m hangs on a string forces acting on the mass are weight of the mass (mg) and tension on the string (T)



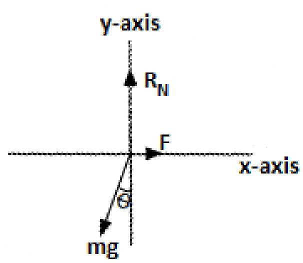
From free body diagram $T = mg$

(ii) A block rest on an inclined plane

When a block rest on an inclined plane forces acting on the body are;-weight (mg), normal reaction (R_N) and friction force (F) which prevent the body to slide down the inclined plane as shown below;-



Thecomponets of the weight are obtained from free body diagram



From the diagram;- $\sum \overline{F} = 0$

In the direction of friction force along the plane (x-axis)

$$F - mg\sin\theta = 0$$

$$\therefore \; F = mg\sin\theta$$

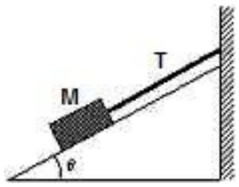
In the direction of normal reaction perpendicular to the plane (y-axis)

$$R_N - mg\cos\theta = 0$$

$$\therefore \; R_N = mg\cos\theta$$

Example

A block of mass M is held motionless on a frictionless inclined plane by means of a string attached to a vertical wall as shown below. If the mass of the block is 5.6kg and the angle of inclination to the horizontal is 36^o, what is the magnitude of the tension?



Solution

Forces acting on the block are;- weight (mg), tension (T) and normal reaction (R).

Along the inclined plane;-

$$\sum \overline{F} = 0$$

$$T - Mg\sin\theta = 0$$

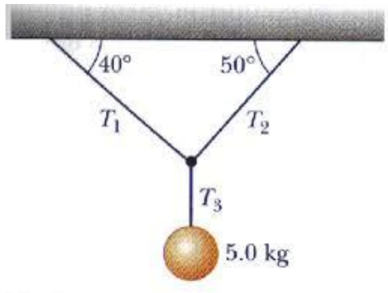
$$\therefore \; T = Mg\sin\theta$$

$$= 5.6kg \times 9.8ms^{-2} \times \sin 36 = 32.26N$$

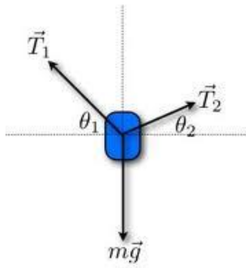
$$\therefore \; T = 32.26N$$

(iii) A mass hangs on the two strings

Consider a mass m = 5.0kg hang on two strings at an angle $\theta_1 = 40^o$ and $\theta_2 = 50^o$ to the horizontal. The value of T_1 and T_2 are determined as follows;-



A free body diagram for above arrengement is given as;-



From the diagram;-

Consider the forces acting in x- direction;-

$$\sum \overline{F}_x = 0$$

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = 0$$

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 \text{ .... (1)}$$

Consider the forces acting in y- direction;-

$$\sum \overline{F}_y = 0$$

$$(T_1 \sin\theta_1 + T_2 \sin\theta_2) - mg = 0$$

$$T_1 \sin\theta_1 + T_2 \sin\theta_2 = mg \text{ .... (2)}$$

From eqn. (1)

$$T_1 \cos 40^o = T_2 \cos 50^o$$

$$T_1 = 0.839T_2$$

Substitute in eqn. (2)

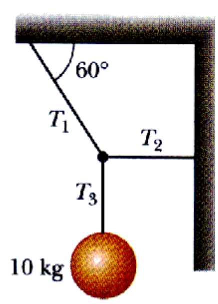
$$(0.839T_2)\sin 40^o + T_2 \sin 50^o = 5.0kg \times 9.8ms^{-2}$$

$$T_2 = \frac{49N}{0.539 + 0.766} = 37.55N$$

$$T_1 = (0.839) \times 37.55N = 31.50N$$

Try this!

Find the tension in each cord for the systems shown in Figure below (Neglect the mass of the cords**!**). (**Answers** $T_1 = 115.5N$, $T_2 = 57.7N$ **and** $T_3 = 100N$)



Newton's second law of motion;-

"The rate of changing momentum is proportional to the resultant force and takes place in a direction of force"

If F is the magnitude of force acting on a body of mass m, then;-

$$F = \frac{mv - mu}{t} = m\left(\frac{v-u}{t}\right) = ma$$

$$F = ma$$

where, a is acceleration on a body due to external force

Relationship of Newton's first and second law;-

From Newton's 2nd law

$$F = m\left(\frac{v-u}{t}\right)$$

If $F = 0$ from the expression of the 2nd law two possible things must be involved

- $v = u = 0$, which means the body is at rest or
- $v = u$, which means the body is moving with uniform velocity in a straight line

The above two observations are the situations for a body in Newton's 1st law of motion.

Consider example of a body in motion with resultant force;-

A body of mass 2kg is moving up an inclined plane with an acceleration of 4ms^{-2}. If the angle of inclination with horizontal is $\theta = 30^o$ and the coefficient of dynamic friction is 0.2, determine the force *F* required to make the body to move up the inclined plane with this acceleration.

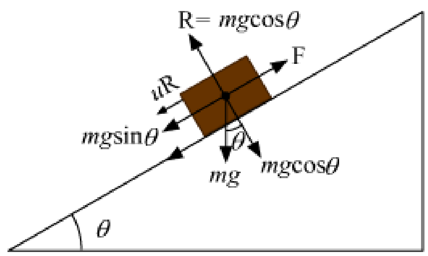
Solution

Given:

Mass of the body *m* = 2 kg

Angle of inclination with the horizontal θ = 30°

Coefficient of dynamic friction μ = 0.2



For the block moving up on the inclined plane; the motion is along the inclined plane, so the force F has to overcome the friction force and the components of weight of the body along the plane.

From the diagram

$$F - (mg\sin\theta + \mu R) = ma$$

$$\therefore\ F - (mg\sin\theta + \mu mg\cos\theta) = ma$$

$$F = (mg\sin\theta + \mu mg\cos\theta) + ma = m[g(\sin\theta + \mu\cos\theta) + a]$$

$$F = 2kg[9.8ms^{-2}(\sin 30^o + 0.2\cos 30^o) + 4ms^{-2}]$$

$$F = 21.2N$$

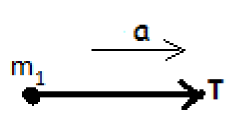
Examples involving tension and acceleration for connected bodies;-

1. Two masses, m_1 and m_2, situated on a frictionless, horizontal surface are connected by a massless string. A force, **F**, is exerted on one of the masses to the right. Determine the acceleration of the system and the tension, T, in the string.

Solution

Both body moves with the same acceleration a

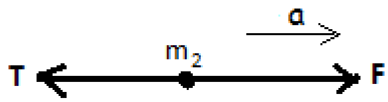
Consider Free body diagram for m_1



From the diagram;- $T = m_1 a$ …………………………………….. (1)

(since the motion is only horizontally)

Free body diagram for m_2



From the diagram;- $F > T$,then;-

$$F - T = m_2 a \text{ ……………………………….. (2)}$$

Substitute T in eqn. (1) into eqn. (2)

$$F - m_1 a = m_2 a$$

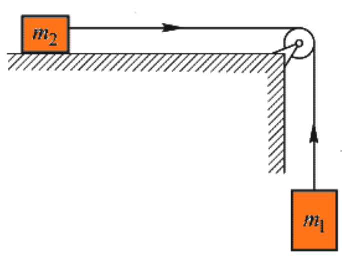
$$\therefore \; a = \frac{F}{m_1 + m_2}$$

Substitute in eqn. (1)

$$T = m_2 \left(\frac{F}{m_1 + m_2} \right)$$

$$\therefore \; T = \left(\frac{m_2}{m_1 + m_2} \right) F$$

2. Two bodies A and B of masses m_1 and m_2 respectively are connected as shown in the diagram below;-



m_2 hangs freely while m_1 is rest on the frictionless surface. If the system is allowed to move, determine;-

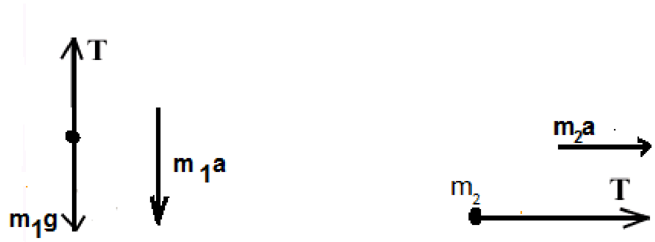
(i) The expression of the acceleration of the system

(ii) The tension on the string

Solution

From the diagram Tension on the string is the same

Consider free body diagram in each body;-



From Free body diagram for m_1;- $m_1 g > T$, then;-

$$m_1 g - T = m_1 a \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

From Free body diagram for m_2;-

$$T = m_2 a \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

(i) Substitute T in eqn. (2) into eqn. (1)

$$m_1 g - m_2 a = m_1 a$$

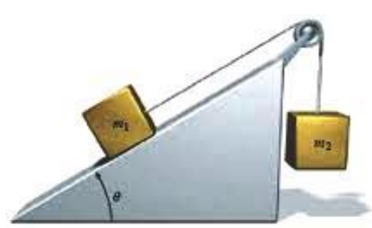
$$\therefore \quad a = \frac{m_1 g}{m_1 + m_2}$$

(ii) From eqn. (2) substitute the value of a

$$T = m_2 \left(\frac{m_1 g}{m_1 + m_2} \right) = \left(\frac{m_1 m_2}{m_1 + m_2} \right) g$$

Try this!

Two objects are connected by a light string that passes over a frictionless pulley. The incline is frictionless,

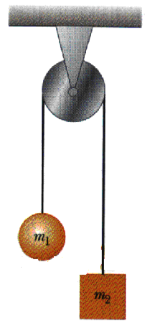


If m_1 = 4.0kg, m_2 =3.0kg, and θ =28^o;-

(a) What will be the direction of the motion of the system?

(b) Determine the value of acceleration of the system and the tension on the string.

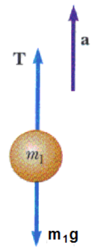
3. Two objects of mass m_1 and m_2 are attached to the ends of a cord which passes over a fixed frictionless pulley as shown below;-



Assuming m_2 is greater than m_1, Determine the acceleration and tension on the cord.

Solution

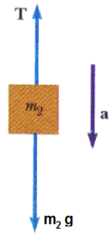
Consider free body diagram in each body;-



From Free body diagram of m_1;- $m_1 g < T$, then;-

$$T - m_1 g = m_1 a \text{ .... (1)}$$

$$\therefore T = m_1 g + m_1 a$$



From Free body diagram of m_2;- $m_2 g > T$

$$m_2 g - T = m_2 a \text{ .... (2)}$$

The acceleration is obtained by substituting T in eqn. (1) into eqn. (2)

$$m_2 g - (m_1 g + m_1 a) = m_2 a$$

$$\therefore m_2 g - m_1 g = m_1 a + m_2 a$$

$$a = \left(\frac{m_2 - m_1}{m_2 + m_1} \right) g$$

The tension is obtained by substituting the value of a in eqn. (1)

$$T = m_1 g + m_1\left(\frac{m_2 - m_1}{m_1 + m_2}\right)g = m_1 g\left[1 + \left(\frac{m_2 - m_1}{m_2 + m_1}\right)\right]$$

$$= m_1 g\left[\left(\frac{m_2 + m_1}{m_2 + m_1}\right) + \left(\frac{m_2 - m_1}{m_2 + m_1}\right)\right] = m_1 g\left(\frac{(m_2 + m_1) + (m_2 - m_1)}{m_2 + m_1}\right)$$

$$= m_1 g\left(\frac{2m_2}{m_2 + m_1}\right)$$

$$\therefore T = \left(\frac{2m_2 m_1}{m_2 + m_1}\right)g$$

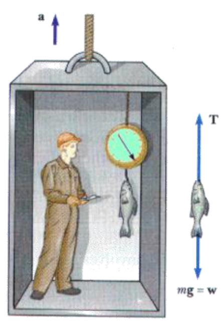
Try this!

A 3kg mass and a 5kg mass are connected over a pulley by a light inextensible string. When the masses are released from rest, what is;-

(i) the acceleration of each mass?

(ii) the tension in the string

4. This particular example is stated in terms of weighing a **fish** in an accelerating elevator. It is also fun to think of weighing **yourself** in an accelerating elevator.
   The diagram below shows an elevator **accelerating up**wards;-



$T > mg$

The forces acting on the fish are shown in the free-body diagram. T is the tension supplied by the scale. This is the value the scale reading. We may call it the **apparent weight** of the fish. The net force on the fish is;-

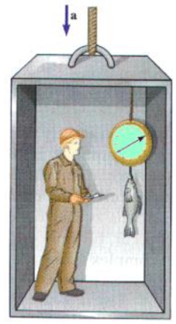
$T - mg = ma$ (Since the elevator is moving upwards)

$T = mg + ma = m(g + a)$

While the elevator accelerates **upward**, the **apparent weight** of the fish is **greater** than its true weight (mg).

Try this!

What happens to the scale reading as the elevator accelerates **down**ward?



Newton's third law of motion;-

"To every action there is equal and opposite reaction"

- The law gives the effects of force. For every force A that one body exerts on a second body, the second body will exert an equal force R on the first body.
- Action and reaction do not neutralize each other, since they act on different objects.

Examples;-

(i) When a man jump from a small boat to shore, the boat moves backwards.

(ii) When a bullet is fired from the gun, the gun recoils due to reaction force.

(iii) When a rocket is fired the fuel burns at the bottom and a stream of gas carries with a great force, the rocket flies up due to reaction force.

Reaction Force due to Fluid flow through a pipe or a hole

Consider a mass m of a fluid of density ρ which has a velocity v and is brought to rest. Its loss of momentum is mv and if it stopped in a time interval t, then the rate of change of momentum $= \frac{mv}{t}$. The force *F* required to stop the moving mass is therefore $F = \frac{mv}{t}$.

Now if this is applied to a jet of fluid with a mass flow rate $\left(\frac{dm}{dt}\right)$ which is equivalent to the volumetric flow rate times the density $\left(\frac{dV}{dt}\rho = Q\rho\right)$ the equivalent force on a flowing fluid is;-

$$F = Q\rho v$$

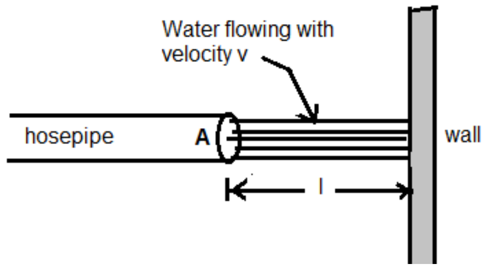
The above equation can be represented in terms of area of the pipe, since $Q = A\frac{dl}{dt} = Av$

Then; $$F = A\rho v^2$$

According to Newton's third law the resulting force of the fluid by a flowing fluid on its surroundings is $(-F)$. Newton's third law states that for every force there is an equal and opposite force.

(i) Reaction force due to water flow through a pipe

The figure below illustrates this principle.



When water flow from a horizontal hose-pipe of cross section area A strikes a wall at right angle, the action force exerted in the wall produces the reaction force.

Suppose the water come to rest on hitting the wall, then;-

Force = Rate of change of momentum

= (Mass per second) x (change in velocity)

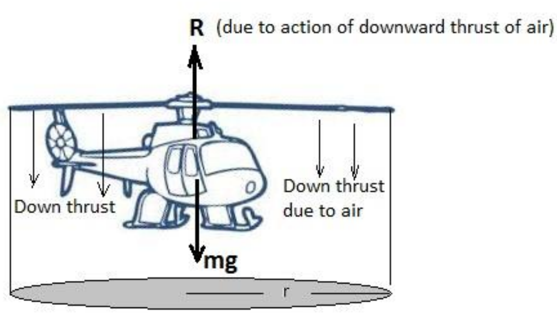
$= \left(\rho A \frac{dl}{dt}\right) \times v$, since $\frac{dl}{dt} = v$ then;

$$F = \rho A v^2$$

The reaction force on the pipe is $-F$ in the horizontal direction as shown.

(ii) Reaction force due to rotating helicopter blades;-

When helicopter blades are rotating, they strike air molecules in downward direction as shown below;-



The rate of change of momentum of the air molecules produce a downward force (Down thrust). By Newton's third law (action and reaction) an equal upward force is exerted to the helicopter (**R**).

This upward force helps to keep the helicopter hovering in the air because it canbalance the downward weight (**mg**) of the helicopter.

If *r* is the radius of the helicopter blades, ρ is the density of air and v is the downward speed of air, therefore the force produced by air molecules is $F = A\rho v^2 = \pi r^2 \rho v^2$.

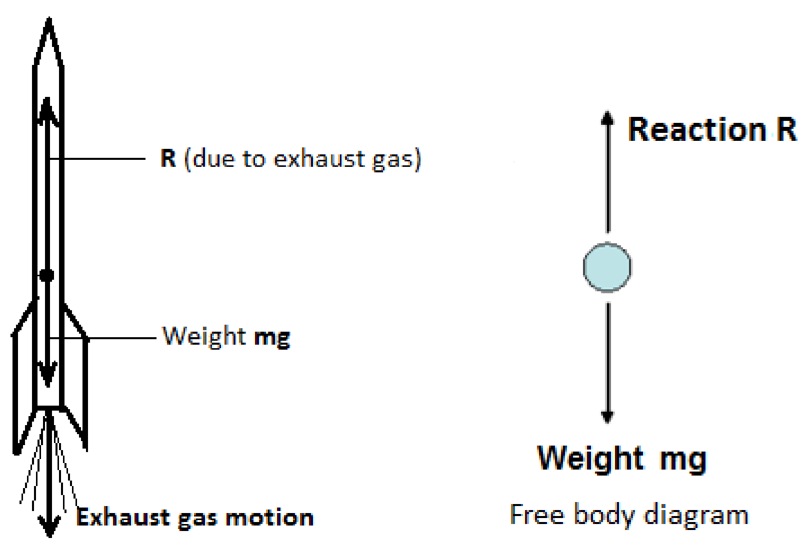
Reaction force on the helicopter is - R in the vertical direction.

Try this!

A helicopter of mass 500kg hovering when it's rotating blades moves through area of $30m^2$ and gives an average speed *v* to the air. Estimate the value of *v* that will make the helicopter just to lift. (Take density of air = $1.3kgm^{-3}$ and g= $10m/s^{-2}$) ***(Answer is 11m/s)***

(iii) Reaction force due to rift off a rocket;-

Gases and fuels are burnt in the combustion chamber of the rocket until the pressure become very high. Hot gases are then allowed to escape through the nozzle at very high speed v which produces Action fore.



Free body diagram

Action Force is due to the force of the exhaust Gas going downwards. If *r* is the radius of the nozzle, ρ is the density of exhaust gas moving with velocity v, therefore the force produced by Exhaust gas motion is;-

$$F = A\rho v^2 = \pi r^2 \rho v^2$$

Reaction force on the rocket is $-F$ in the vertical direction which is equal to F_N in the free body diagram.

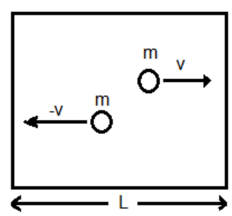
The resultant force $F = F_N - mg = ma$

Try this!

A toy rocket containing water and compressed air initially weigh 80g. When a cover is removed the air pushes the water out of the rocket with a speed of 10m/s. Taking the density of water =1000kg/m^3, estimate the necessary minimum diameter of the nozzle in order the rocket just to lift off. **(Answer 3.16mm)**

Reaction force due to change in direction;-

For a molecule of mass m in a gas strikes a wall of a vessel respectively with a velocity v and rebound the velocity –v, as shown below;-



The momentum change $\Delta P = mv - (-mv) = 2mv$

Suppose the containing vessel is a cube of sides l, the time taken by a molecule to make a collision with the same side repeated is;

$$t = \frac{2l}{v}$$

But number of collisions per unit time is given by;-

$$n = \frac{1}{t} = \frac{v}{2l}$$

The average force on the wall $= n \times \Delta P$

$\therefore$ Force $F = \frac{v}{2l}(2mv) = \frac{mv^2}{l}$

Alternatively;

From force is the rate of change of momentum, then;

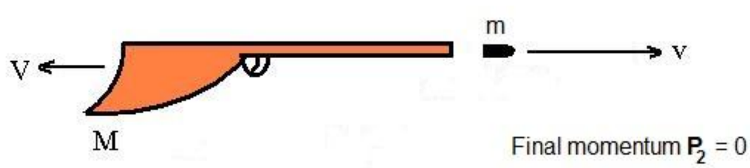
$$\therefore \text{ Force } F = \frac{\Delta P}{t} = \frac{2mv}{\left(2l/v\right)} = \frac{mv^2}{l}$$

Reaction force on firing a gun;-

Consider a rifle of mass M fires a bullet of mass m with a velocity v. This action will cause the reaction on the gun with recoiling velocity V.

By applying principle of conservation of linear momentum, initially the gun and the bullet are at rest so the momentum is zero

After being fired the bullet has momentum mv to the right and the gun has momentum MV to the left. The final momentum of the system is also zero.



$$mv - MV = 0$$

The recoil velocity $V = \dfrac{mv}{M}$

Force and Momentum

If a person of mass 50kg is jumped from a height of 5m will land on the ground with a velocity v given by the expression;-

$$v = \sqrt{2gh} = \sqrt{2 \times 10ms^{-2} \times 5m} = 10ms^{-1}$$

If he does not flex his kneel on landing he will brought to rest quickly say one-tenth second ($^1/_{10}$ s), the force F acting on is given by;-

$$F = m\left(\frac{v-u}{t}\right) = 50kg\left(\frac{10ms^{-1}}{\frac{1}{10}s}\right) = 5000N$$

This large force has severe effects on the body.

Suppose the person flexes his knees and brought to rest slowly say in one second (1s), the force acting on is given by;-

$$F = m\left(\frac{v-u}{t}\right) = 50kg\left(\frac{10ms^{-1}}{1s}\right) = 500N$$

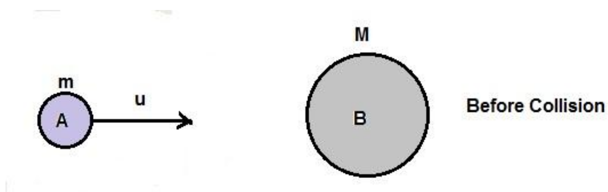
This will cause less effect to the person.

Try this!

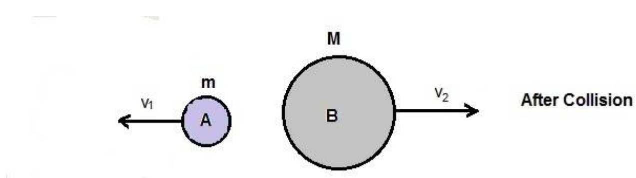
There is a big possibility for an egg to break when dropped on a hard floor but when dropped on mattress from the same height it may not break. Explain!

Principle of Conservation of Linear Momentum

Consider two objects in an isolated system (no external force acts on them), suppose a small steel sphere A of mass m move with a velocity u towards a big sphere B of mass M and has a head collision with it



After collision A turns back with velocity v_1 and B moves forward with a velocity v_2 (the velocity of the spheres are in the same line but in opposite direction)



Total momentum before collision = Total momentum after collision

$$mu + 0 = -mv_1 + Mv_2$$

Principle of conservation of linear momentum states that "***If no net external force acts on the system of several particles, the total linear momentum of the system is constant***"

Elastic and Inelastic Collision

In every collision or interaction, the linear momentum and energy is conserved.

In most ordinary collisions, some kinetic energy is lost in forms of heat, sound etc. and during collision the total kinetic energy usually decreases as the results of collision.

NOTE

Elastic collision - is the collision in which both total momentum and total kinetic energy are conserved.

Inelastic collision- is the collision in which only total momentum is conserved, the total kinetic energy is always not conserved.

Coefficient of Restitution

In practice colliding objects do not stick together and kinetic energy is always lost. If a ball A moving with velocity u_1 collides head on with a ball B moving with a velocity u_2 in the same direction, then B will move faster with velocity v_2 and A may slower with a velocity v_1 in the same direction

The coefficient of restitution (e) between A and B is defined as the ratio of relative velocity of separation to the relative velocity of approach.

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

NOTE

- The value of coefficient of restitution varies from 0 to1
- When the value of coefficient of restitution is zero, the collision is **perfect inelastic.**
- When the coefficient of restitution is 1 the collision is **perfect elastic.**

Example

In a crash test, a car of mass 1.2 x 10^3 kg collides with a wall and rebounds as illustrated below. The initial and final velocities of the car are 12 m.s^{-1} to the left and 2m.s^{-1} to the right respectively. The collision lasts 0.1 s.

Calculate the:

(a) The coefficient of restitution.

(b) Average force exerted on the car

(c) Impulse of the car during the accident

(d) How will the magnitude of the force exerted on the car be affected if the time interval of the collision remains 0.1 s, but the car does not bounce off the wall? Write down only INCREASES, DECREASES or REMAINS THE SAME. Explain your answer.

Solution

(a) From $e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{0-(-2ms^{-1})}{12ms^{-1}-0} = 0.1$

(b) Since force is the rate of change of momentum

$$F = \frac{m(v-u)}{t} = \frac{1.2\times10^3 kg(-2ms^{-1} - 12ms^{-1})}{0.1} = 1.68\times10^4 N$$

(c) Impulse of the car

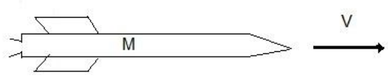
Impulse = change in momentum = $Ft = 1.68\times10^4 \times 0.1s = 1.68\times10^3 Ns$

(d) Try this!

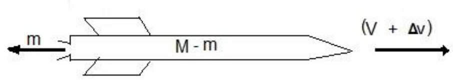
Rocket in Outer space (OPTIONAL)

Consider the rocket moving in outer space where no external forces acting on it.

Suppose its mass is M and velocity V at a particular instant.



When a mass m of fuels is ejected the mass of the rocket become ($M-m$) and its velocity increases to ($V+\Delta v$) (***from principle of conservation of linear momentum***



Suppose the fuels is ejected at constant speed u , the velocity of m relative to the rocket is $\left[V + \frac{\Delta v}{2} - u\right]$, in the direction of the rocket.

NOTE

Initially velocity of the rocket is V and final velocity of rocket is $(V+\Delta v)$ and the average velocity of the rocket is $\left[V+\frac{\Delta v}{2}\right]$

Apply the principle of conservation of linear momentum to the rocket and the fuel

- Initially before the mass m of fuels ejected (Momentum of the rocket) $=MV$
- After the mass m of the fuels ejected;

Momentum of the rocket $=(M-m)(V+\Delta v)$

Momentum of the fuels $=m\left[V+\frac{\Delta v}{2}-u\right]$ (the same direction as rocket)

Total final momentum $=(M-m)(V+\Delta v)+m\left[V+\frac{\Delta v}{2}-u\right]$

So, Total final momentum = total initial momentum

$$MV=(M-m)(V+\Delta v)+m\left[V+\frac{\Delta v}{2}-u\right]$$

$$MV=MV+M\Delta v-mV-m\Delta v+mV+m\frac{\Delta v}{2}-mu$$

$$0=M\Delta v-m\Delta v+m\frac{\Delta v}{2}-mu$$

Neglecting the product $m\Delta v$ which is very small, we get

$$0=M\Delta v-mu$$

$$mu=M\Delta v$$

$$\frac{m}{M}=\frac{\Delta v}{u}$$

Now $m=$ mass of fuel ejected $=-\Delta m$

$$\therefore\ \frac{-\Delta m}{M}=\frac{\Delta v}{u}$$

Integrating between the limits M_o to M and V_o to V

$$\int_{M_o}^{M}-\frac{\Delta m}{M}=\int_{V_o}^{V}\frac{\Delta v}{u}$$

$$-\ln\left[M\right]_{M_o}^{M}=\frac{1}{u}\left[V\right]_{V_o}^{V}$$

$$-\ln\left(\frac{M}{M_o}\right)=\frac{1}{u}(V-V_o)$$

$$V=V_o-u\ln\left(\frac{M}{M_o}\right)$$

When the mass M decreases to $\frac{M}{2}$

$$V=V_o-u\ln\left(\frac{M}{2M}\right)=V_o-n\ln\left(\frac{1}{2}\right)=V_o+u\ln 2$$

Example

A rocket is moving freely in outer space with a velocity of 20m/s. the fuel is ejected at constant speed of 5m/s relative to the rocket. Calculate the speed of the rocket when its mass is reduced to half of its value.

Solution

From $$V=V_o-u\ln\left(\frac{M}{M_o}\right)$$

$$V=V_o-n\ln\left(\frac{1}{2}\right)=V_o+u\ln 2=(20ms^{-1})+(5ms^{-1})\ln 2$$

$$=23.5ms^{-1}$$

Work and Energy

In physic work is said to be done when force moves through a distance *s* in the direction of force i.e. $W=F\times s$

Consider a constant force *F* applied by a woman at an angle θ with the direction of the motion of her suitcase.

The component of F in the direction of s is $F\cos\theta$, then;-

$$W=(F\cos\theta)\times s$$

Work is the scalar quantity.

If the component of the force is in the same direction as displacement, the work is positive otherwise is negative.

If the force is at right angles to the displacement, then work done is zero.

Example

A box is dragged along a horizontal surface by a constant force P= 50N making an angle $\theta = 37^o$ with the direction of the motion. The friction force acting on the box is 15N. If the box move through a distance 20m, find the work done on a box.

Solution

Work done = work done by P – work done by friction

= (Force by P – Friction force) x s

= $(P\cos\theta - F)s$ = $(50\cos 37^o - 15)\times 20$ = 500Nm or 500J

Work done by a varying force

When a spring is stretched the force required to stretch it increases steadily as the spring balance. The total work done is obtained by summing up the small work done in every small interval Δs

$$\therefore W = \int_{s_1}^{s_2} \Delta W = \int_{s_1}^{s_2} F\cos\theta ds$$

Work required in stretching a spring a distance x from its original length

If the force required increases directly proportional to the amount of elongation

i.e. $F \; \alpha \; x$

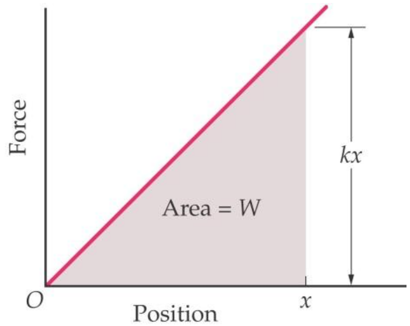
$\therefore \quad F = kx$

k , is constant called force constant of the spring.

The work done required to stretch the spring from *x = 0*, to *x = x* is;-

$$W = \int_0^x F.dx = \int_0^x kx.dx = \frac{1}{2}kx^2$$

The result can be obtained **graphically**; the area of the shaded triangle represents the total work.



$$Area = \frac{1}{2}Fx = \frac{1}{2}kx.x = \frac{1}{2}kx^2$$

Also when a spring is stretched through a distance x, the initial force is zero and the final force is F. Then the spring is stretched through a distance x by the average force;-

$$\overline{F} = \frac{F+0}{2}$$

Work done $W = \overline{F} \times x \quad = \left(\frac{F+0}{2}\right) \times x = \frac{Fx}{2}$

Since $F = kx$ then; Work done $= \frac{1}{2}kx^2$

Work and Kinetic Energy

Consider a body of mass m acted by a force P horizontal and friction force F_R

The resultant force on it is $(P - F_R)$

From Newton's second law

$$F = ma$$

Suppose the speed of the body increases from u to v while the body undergoes a displacement s.

Since a is a constant, then;- $v^2 = u^2 + 2as$

So;- $a = \frac{v^2 - u^2}{2s}$

$$\therefore F = m\left(\frac{v^2 - u^2}{2s}\right)$$

$$F \times s = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Fs, is the work done of the resultant force F and $\frac{1}{2}mv^2$ is the kinetic energy E_k

$$E_k = \frac{1}{2}mv^2$$

NOTE;-

The work of the resultant external force on a body is equal to the change in kinetic energy of the body

i.e. $$W = E_{kf} - E_{ki} = \Delta E_k$$

Gravitational Potential Energy

Suppose a body of mass m moves vertically from height y_1 above given point to a height y_2.

The downward gravitational force on the body is; - $W = mg$ and direction of W is opposite to the upward displacement.

The work done of this is;-

$$W_{grav} = -(mgy_2 - mgy_1) = -mg(y_2 - y_1)$$

The work of gravity force depends on the initial and final elevation and not on the path.

Let $W^{'}$ be the work done by forces other than gravitational force.

Since; - Total work = change in kinetic energy

$$W^{'} - mg(y_2 - y_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W^{'} = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + \left(mgy_2 - mgy_1\right) = \left(\frac{1}{2}mv_2^2 + mgy_2\right) - \left(\frac{1}{2}mv_1^2 + mgy_1\right)$$

The sum of kinetic energy and potential energy of the body is called its **total mechanical energy.**

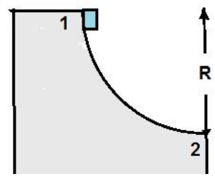
In the case in which the only force on the body is gravitational force, the work $W^{'}$ is zero, the equation above can be written as;-

$$\frac{1}{2}mv_2^2 + mgy_2 = \frac{1}{2}mv_1^2 + mgy_1$$

Under this condition, the total mechanical energy remains constant or is conserved. This is called the ***principle of conservation of mechanical energy***

Example

A body slides down a curved track which is one quadrant of a circle of radius R as shown below;-



The mass of the body is 0.5kg and track radius is 1m. Suppose it starts from rest and there is no friction;-

(i) Find the speed at the bottom of the track

(ii) If its speed at the bottom is only 3m/s. What was the work by the friction on the body?

Solution

(i) Let the position 2 be the reference point.

By conservation of mechanical energy; -

"Total mechanical energy at position 1 = Total mechanical energy at position 2"

But $v_1 = 0$ and $y_1 = R$, $v_2 = ?$, if $y_2 = 0$

$$\therefore\ 0 + mgR = \frac{1}{2}mv_2^2 + 0$$

$$mgR = \frac{1}{2}mv_2^2$$

So;- $v_2 = \sqrt{2gR}$

$$v_2 = \sqrt{2\times 9.8\times 1} = 4.4ms^{-2}$$

(ii) Let $W^{'}$ be work by friction, then;-

$$W^{'} = \left(\frac{1}{2}mv_2^2 + mgy_2\right) - \left(\frac{1}{2}mv_1^2 + mgy_1\right) = \left(\frac{1}{2}mv_2^2 + 0\right) - (0 + mgR)$$

$$W^{'} = \frac{1}{2}mv_2^2 - mgR$$

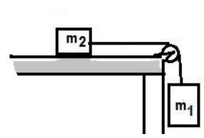
$$W^{'} = \left(\frac{1}{2}\times 0.5\times 3^2\right) - (0.5\times 9.8\times 1) = -2.65J$$

Work by friction is 2.65J

Exercise 2A

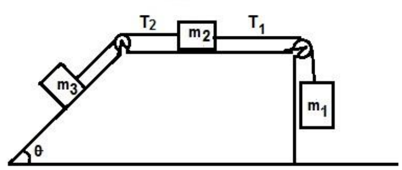
1. (a) If your teacher's car is traveling west with a constant velocity of 55km/h on a straight line highway, what is the net force acting on it?
   (b) State any four applications of Newton's laws of motion in our daily life.
   (c) A lamp hangs vertically from a cord in a descending elevator. The elevator has a deceleration of 2.4ms^{-2} before coming to stop, If the tension in the cord is 89N;-
      (i) what is the mass of the lamp?
      (ii) what will be the tension in the cord when the elevator in (b) (i) above ascends with an upward acceleration of 2.4ms^{-2}?

2. (a) If the mass $m_2 = 10kg$ and coefficient of static friction between m_2 and the table is 0.6, what mass m_1 will just set the system in motion?



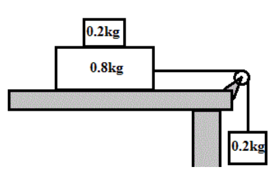
   (b) With mass m_1 as in part (a), after the system moves, what is the acceleration? If the coefficient of dynamic friction is 0.4?

3. The diagram below shows three connected bodies in which m_3 and m_2 rest on frictionless planes and m_1 hangs free.



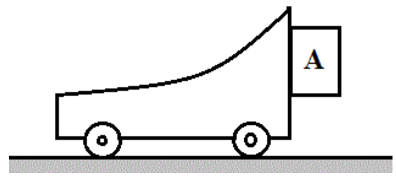
   If m_1= 4kg, m_2 = 5kg, m_3 = 6kg and the value of $\theta = 30^o$, Find;-
      (i) The value of tension T_1 and T_2
      (ii) The acceleration of the system.

4. A block of mass 0.2kg on top of a blockof mass 0.8kg. the combination is dragged along a level surface at constant velocity by a hanging block of mass 0.2kg as shown below;-



   (i) The first 0.2kg block is removed from 0.8kg block and attached to the hanging block. What is now the acceleration of the system?
   (ii) What is the tension in the cord attached to the 0.8kg block in part (b)? **(1.96m/s^2, 3.14N)**

5. If the coefficient of friction between the block and the car is 0.3, what acceleration must the car in figure below have in order the block A will not fall?



($32.7 m/s^2$)

6. (a) (i) State principles of conservations of linear momentum and define coefficient of restitution

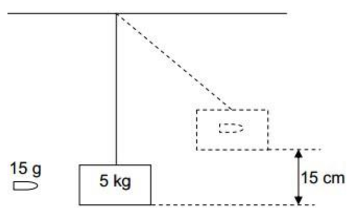
   (ii) Differentiate between elastic and inelastic collision.

   (b) A ball of mass 10g is dropped from a height of 2.0m to a horizontal ground. It is then bounces back to a height of 1.5m. determine;

   (i) the energy lost by the ball when in contact with the ground.

   (ii) the coefficient of restitution of the ball and the ground.

7. During an investigation a police officer fires a bullet of mass 15g into a stationary wooden block, of mass 5kg, suspended from a long, strong cord. The bullet remains stuck in the block and the block-bullet system swings to a height of 15 cm above the equilibrium position, as shown below. (Effects of friction and the mass of the cord may be ignored.)



   (a) State the law of conservation of momentum in words.

   (b) Use energy principles to show that the magnitude of the velocity of the block-bullet system is $2ms^{-1}$ immediately after the bullet struck the block.

   (c) Calculate the magnitude of the velocity of the bullet just before it strikes the block.

   (d) The police officer is pushed slightly backwards by the butt of the rifle, which he is holding against his shoulder, whilst firing the rifle. Use the relevant law of motion to explain why this happens.

B. Projectile Motion

Projectile motion: is the motion of a body travelling freely under the action of gravity and air resistance.

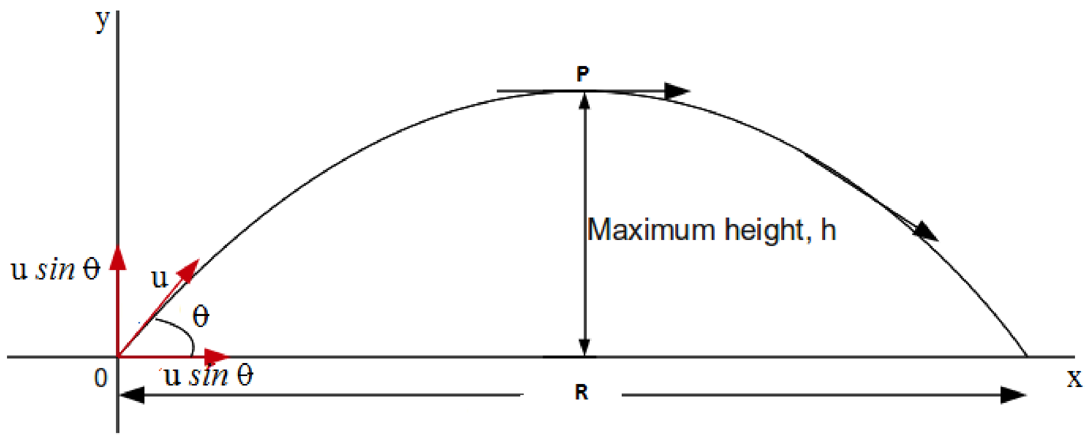
The path which described by a projectile is known as **trajectory**.

The velocity of a projectile is considered to be the resultant of horizontal and vertical component of the velocity.

For any projectile its vertical motion is at constant acceleration g , and its horizontal motion is at constant velocity (air resistance is neglected).

Equations of Projectile Motion

Suppose a projectile is projected with an initial velocity u making an angle θ with the horizontal



If x and y denotes the horizontal and vertical components of displacement formed at time t, then;-

The vertical component of the initial velocity u is $u\sin\theta$, that of horizontal is $u\cos\theta$.

The magnitude of the vertical component of the velocity at any instant is given by;-

$$v_y = u\sin\theta - gt \quad \text{..} \quad (1)$$

where t is the time interval from starting up to that instant of time.

Neglecting air resistance, the horizontal acceleration is zero, hence the magnitude of velocity remain constant at any time t;-

$$v_x = u\cos\theta \quad \text{..} \quad (2)$$

At the highest point P, the projectile is totally horizontal and the y-component of the velocity is zero, x-component has the constant value $v_x = u\cos\theta$.

At P;

$$v_y = u\sin\theta - gt = 0$$

$$u\sin\theta = gt$$

$$t = \frac{u\sin\theta}{g} \quad \text{……………………………………. (3)}$$

The projectile reaches the highest point after time t.

Distance covered in the horizontal direction at any instant t is given by;

$$x = v_x t = (u\cos\theta)t \quad \text{……………………………….. (4)}$$

Distance y covered in the vertical direction at any instant t is given by;

$$y = v_y t - \frac{1}{2}gt^2 = (u\sin\theta)t - \frac{1}{2}gt^2 \quad \text{…………………….. (5)}$$

If the projectile rises to the highest point P after a time t given by equation (3), then the maximum height h reached by a projectile is given by substituting the value of t in equation (5).

$$h = (u\sin\theta)\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$h = \left(\frac{u^2\sin^2\theta}{g}\right) - \left(\frac{u^2\sin^2\theta}{2g}\right)$$

$$h = \frac{u^2\sin^2\theta}{2g} \quad \text{…………………………….. (6)}$$

The magnitude of the velocity of the projectile at any instant is given by;

$$v = \sqrt{v_x^2 - v_y^2} \quad \text{……………………………. (7)}$$

The angle the resultant velocity makes with horizontal is given by;

$$\tan\theta = \frac{v_y}{v_x} \quad \text{……………………………….. (8)}$$

Examples

1. A particle P is projected from point O on horizontal plane with velocity of 20m/s at an angle of 60^o to the horizontal. Find the;-

   (a) Horizontal and vertical speed of p after 1second.

   (b) Maximum vertical height reached by P.

   ### Solution

   Given $u = 20m/s \qquad \theta = 60^o$

   (a) Horizontal speed $v_x = u\cos\theta$

   $$= 20ms^{-1} \times \cos 60^o$$
   $$= 10ms^{-1}$$

   Vertical speed $v_y = u\sin\theta - gt$

   $$= (20ms^{-1} \times \sin 60^o) - (10ms^{-2} \times 1s)$$
   $$= 7.32m/s$$

   (b) Maximum height h

   From maximum height $h = \dfrac{u^2 \sin^2\theta}{2g}$

   $$= \frac{(20ms^{-1})^2(\sin 60^o)^2}{2 \times 10}$$
   $$= 15m$$

2. A ball is projected from the ground with a velocity of 25m/s at an angle of 45^o to the horizontal (ground). For how long the ball is at 15m above the ground?

   ### Solution

   Given; $u = 25m/s \qquad \theta = 45^o$

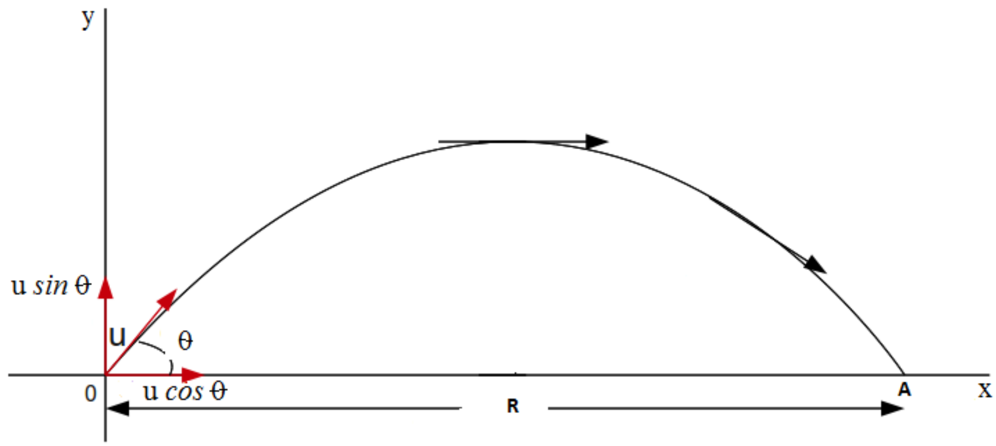
   From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

   $$15m = (25ms^{-1}\sin 45^o)t - \frac{1}{2}(9.8ms^{-2})t^2$$
   $$15 = 17.68t - 4.9t^2$$

   $t = 1.36s$, and $2.24s$

The Range R of a Projectile

Range: is the horizontal distance travelled by a projectile from the point of projection to the point when it reaches the horizontal axis.



O is the point of projection and A is the point where the trajectory intersects the horizontal axis, OA is the horizontal distance called range of the projectile denoted by R.

At A the y-coordinate is zero. If the projectile reaches A after time interval t_R from equation (5);

$$y = (u\sin\theta)t_R - \frac{1}{2}gt_R^2$$

$$0 = (u\sin\theta)t_R - \frac{1}{2}gt_R^2$$

So, $$(u\sin\theta) = \frac{1}{2}gt_R$$

$$t_R = \frac{2u\sin\theta}{g} \quad \text{...} \ (9)$$

NOTE

- t_R is the time of flight which is twice the time taken to reach the maximum height of the trajectory

Since; $R = v_x t_R = (u\cos\theta)t_R$

$$R = (u\cos\theta)\left(\frac{2u\sin\theta}{g}\right) = \frac{2u^2\cos\theta\sin\theta}{g}$$

But $2\cos\theta\sin\theta = \sin 2\theta$

Then;

$$R = \frac{u^2 \sin 2\theta}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

From equation (10);

R is maximum, when $\sin 2\theta = 1$ i.e. when $2\theta = 90^o$, when $\theta = 45^o$

Equation of the trajectory is a Parabola

From the equation of distances covered in both y and x direction;

$$y = (u\sin\theta)t - \frac{1}{2}gt^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$x = (u\cos\theta)t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Then from equation (ii)

$$t = \frac{x}{u\cos\theta}$$

Substitute t in the equation of vertical distance, we get;

$$y = (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = (\tan\theta)x - \frac{gx^2}{2u^2\cos^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(11)}$$

The above equation can be written as;

$$y = px - qx^2$$

p and q are constant representing $\tan\theta$ and $\dfrac{g}{2u^2\cos^2\theta}$ respectively

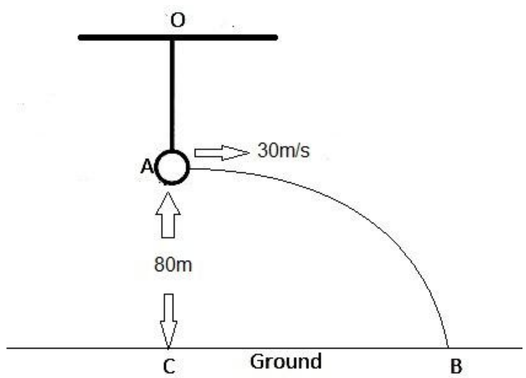
The above is the equation of a parabola. Thus trajectory of a projectile is a parabola.

Applications of Projectile

1. During disaster to the area where communication is problem, food aid are dropped by planes follows the projectile motion.
2. During army range, the bullet fired from the gun follows the projectile motion.
3. During the war, the shells projected from the mortar follows the projectile motion.
4. The football kicked by the player follows the projectile motion.
5. During firefighting especially on tall building the water projected from the pipe follows projectile motion.

Example

1. A small ball A suspended from a string OA is set into oscillation



When the ball passes through the lowest of the motion the string is cut. If the ball is then moving with the velocity of 30m/s at a height of 80m above the ground. Find the horizontal distance CB travelled by the ball.

Solution

From $h = (u\sin\theta)t - \frac{1}{2}gt^2 = \frac{1}{2}gt^2$

$$\therefore\ t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 80m}{10ms^{-2}}} = 4s$$

Since $x = (u\cos\theta)t$

$$x = (30ms^{-1}\cos 0)\times 4s = 120m$$

2. A lump of ice slides 10m down a smooth, sloping roofing making an angle of 30^o with the horizontal. The edge of the roof is 10m above a sidewalk which extend 5m out from the side of building. Will the ice land on the sidewalk or in the street? Give proof.

Solution

From principle conservation of mechanical energy

$$\frac{1}{2}mv^2 = mgh = mg(l\sin\theta)$$

$$v = \sqrt{2gl\sin\theta} = \sqrt{2\times 9.8\times 10\times \sin 30^o} = 9.9ms^{-1}$$

Time taken to reach the ground;-

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$-10 = (9.9\sin 330^o)t - \frac{1}{2}\times 9.8t^2$$, (or $\sin 30^o$ in the 4th quadrant)

$$10 = 4.95t + 4.9t^2$$

$$4.9t^2 + 4.95t - 10 = 0$$

$$t = 1s$$

From, $x = (u\cos\theta)t$

$$= (9.9\times\cos 30^0)\times 1$$

$$= 8.57m$$

Since, $x > 5m$, the ice will land in the street

3. A ball is thrown upwards with an initial velocity of 33m/s from a point 65^o on top of the building 16m above the ground.
   (a) At what distance from the point down the building does the ball strike the ground?
   (b) Calculate the time of flight

Solution

(a) Consider the maximum height reached by the projectile from the building

$$h = \frac{u^2\sin^2\theta}{2g} = \frac{33^2\times(\sin 65)^2}{2\times 9.8} = 45.64m$$

Time to reach max height from the building

$$t_1 = \frac{u\sin\theta}{g} = \frac{33 \times \sin 65^o}{9.8} = 3.05s$$

Time from maximum height to the ground,

$$t_2 = \sqrt{\frac{2 \times (H+h)}{g}} = \sqrt{\frac{2 \times (16 + 45.64)}{9.8}} = 3.55s$$

Distance reached by the projectile from the building to the point the projectile strike the ground $x = u\cos\theta \times t$

$$= 33\cos 65^o \times (6.6) = 92m$$

(b) Time to reach max height from the building

$$t_1 = \frac{u\sin\theta}{g} = \frac{33 \times \sin 65^o}{9.8} = 3.05s$$

Time from maximum height to the ground,

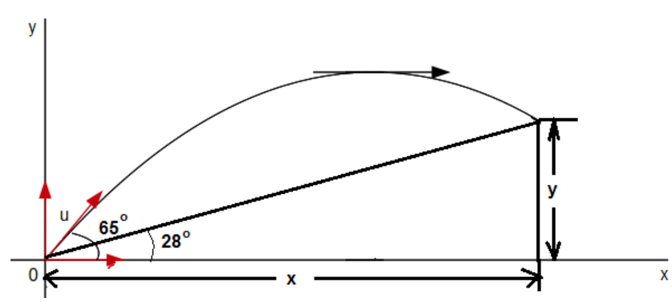
$$t_2 = \sqrt{\frac{2 \times (H+h)}{g}} = \sqrt{\frac{2 \times (16 + 45.64)}{9.8}} = 3.55s$$

Time of flight $= 3.05 + 3.55 = 6.6s$

8. A ball is thrown upwards with an initial velocity of 33m/s from a point 65^o on the side of a hill which slopes upward uniformly at an angle of 28^o. At what distance up the slope does the ball strike?

Solution

Consider sketched diagram



Let $\theta = 65^o$ and $\beta = 28^o$

Then, from $y = (\tan\theta)x - \dfrac{gx^2}{2u^2\cos^2\theta}$

$$\frac{y}{x} = (\tan\theta) - \frac{gx}{2u^2\cos^2\theta}$$

$$\tan\beta = (\tan\theta) - \frac{gx}{2u^2\cos^2\theta}$$

$$x = \frac{2u^2\cos^2\theta(\tan\theta - \tan\beta)}{g}$$

$$= \frac{2\times(33)^2\times\cos^2 65^o(\tan 65^o - \tan 28^o)}{9.8}$$

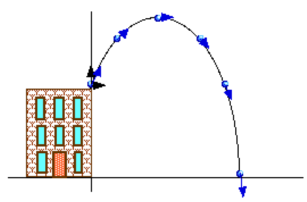
$$= 64m$$

From, $\tan\beta = \dfrac{y}{x}$

$$\therefore\ y = x\tan\beta = 64\times\tan 28^o = 34m$$

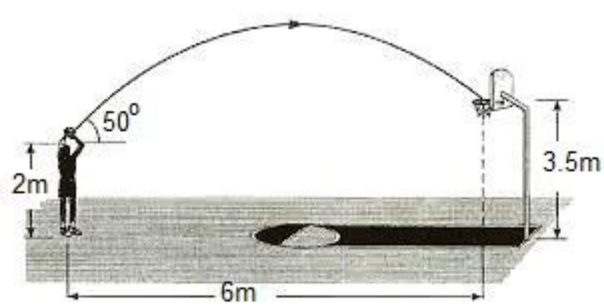
Exercise 2B

1. A projectile is thrown with a speed of 98m/s in a direction of 30^o above the horizontal. Find the time of flight, range and the height to which it rises.
2. One ball was dropped from rest and at the same time, another ball projected horizontally at the same height. Which one will reach the ground first? Explain mathematically.
3. The shells fired from the artillery piece have a muzzle speed of 150m/s, and target is at horizontal distance of 2.0km.
   (i) At what angle relative to the horizontal should the gun be aimed?
   (ii) Could the gun hit the target 3.0km away?
4. A boy standing 4m away from a wall which is 5m high. He throws a ball at $10ms^{-1}$ at an elevation of 40^o above the horizontal from his hand which is 1m above the ground. Will the ball pass over the wall? Give proof.
5. The top of a certain flat roofed building is 40m above the ground. A small solid ball is projected from a point O at the roof of the building with an initial speed of 20m/s at an angle of 30^o to the horizontal as illustrated below;-



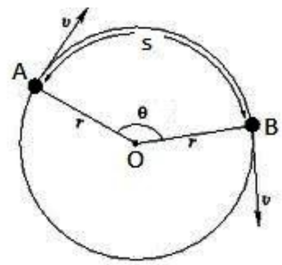
   Calculate;-
   (i) Time taken for the ball to reach the ground,
   (ii) The maximum speed reached by the ball.

6. At what initial speed must the basketball player throw the ball at 50^o above the horizontal, to make the final shot as shown in the diagram below?



C. Uniform Circular Motion

Consider an object of mass m moving with a uniform speed v round a fixed point O as a centre of the circle.



If the object moves from A to B such that the radius OA moves through an angle θ at a time t its **angular velocity** ω about O is defined as the rate of change of angle.

$$\omega = \frac{d\theta}{dt} = \frac{\theta}{t}$$

ω, is expressed in **radian per second** ($rads^{-1}$)

From $\omega = \frac{\theta}{t}$

$$\therefore \theta = \omega t$$

θ, is **angular displacement** in **radian** (rad)

The time T describe the circle once is given by;

$$T = \frac{2\pi}{\omega}$$

T is known as the **period of the motion** and 2π radian is the angle in one revolution.

$$2\pi = 360^o$$

If s is the length of arc AB, then by definition of an angle in radian, then;

$$\theta = \frac{s}{r}$$

$$\therefore s = r\theta$$

Divide by the time t to move from A to B

$$\frac{s}{t} = \frac{r\theta}{t}$$

Since $\frac{s}{t} = v$ and $\frac{\theta}{t} = \omega$

$$\therefore v = r\omega$$

v, is the velocity (speed) of rotating object

Example

A model car moves round a circular track of radius 0.3m at the rate of 2 revolutions per second. What is:-

(a) The angular velocity
(b) The period of motion
(c) The speed of the car
(d) The angular speed of the car if it moves with uniform speed of 2m/s in a circle of radius 0.4m

Solution

(a) Given; 2revolutions per second

$$\therefore \omega = 2(2\pi) rad \sec^{-1}$$

$$= 4\pi rad \sec^{-1}$$

The angular velocity is 4π *rads*$^{-1}$

(b) From period

$$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi rad}{4\pi rad/s}$$

$$= 0.5s$$

The period of the motion is 0.5 second

(c) From velocity

$$v = r\omega$$

$$= 0.3m \times 4(3.14) rad/s$$

$$= 3.8m/s$$

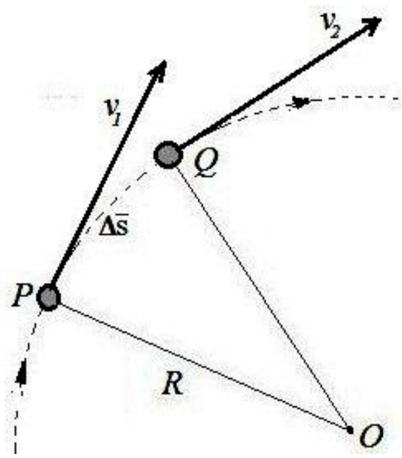
The speed (v) is 3.8m/s

(d) From angular velocity $\omega = \frac{v}{r} = \frac{2m/s}{0.4m} = 5rad/s$

Acceleration in a Circle

The acceleration of a particle moving in a curved path can be resolved into components normal and tangential to the path (For uniform circular motion no tangential component of acceleration since the speed is constant).

There is a simple relation between the normal component of the acceleration, the speed of the particle and the radius of the curvature of the path.



The figure above represents a particle moving in a circular path of radius R with centre at O. Vector v_1 and v_2 represent its velocities at point P and Q. The vector change in velocity Δv is obtained from the triangle OPQ. Since the triangle is isosceles triangle. Their long sides are mutual perpendicular to each other. Hence

$$\frac{\Delta v_{\perp}}{v_1} = \frac{\Delta s}{R} \text{ or } \Delta v_{\perp} = \frac{v_1}{R}\Delta s$$

The magnitude of the average normal acceleration $a_{\perp}$ is therefore;-

$$a_{\perp} = \frac{\Delta v_{\perp}}{\Delta t} = \frac{v_1 \Delta s}{R \Delta t}$$

But $\frac{\Delta s}{\Delta t} = v_1$ at a point P and since point P can be any point of the path, and the subscript can be dropped from v_1 to v the speed at any point, then;-

$$a_{\perp} = \frac{v^2}{R}$$

The magnitude of the instantaneous normal acceleration is therefore equal to the square of the speed divided by the radius. The direction of this acceleration is inward along the radius towards the centre. It is called **centripetal acceleration** or **radial acceleration**

If v is the uniform speed in the circle and r is the radius of the circle, acceleration towards the centre is given by;-

$$\text{Acceleration toward centre} = \frac{v^2}{r}$$

Since $v = r\omega$

$$\text{Acceleration toward centre} = \frac{\omega^2 r^2}{r} = \omega^2 r$$

$$\frac{v^2}{r}\text{, has dimension of } \frac{\left([L][T]^{-1}\right)^2}{[L]} = [L][T]^{-2}$$

Centripetal Force

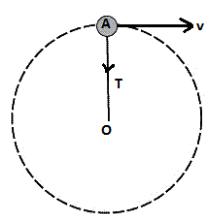
Centripetal force is the force required to keep an object of mass m to move in a circle of radius r, it acts toward the centre of the circle.

$$F = ma = \frac{mv^2}{r}$$

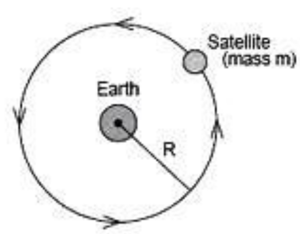
$$\therefore\ F = m\omega^2 r$$

Example of sources of centripetal forces in the system;

1. When a stone A is whirled in a horizontal circle of centre O by means of string, the tension T provide the centripetal force.



2. For racing car moving round a circular track, the friction at the wheels provides a centripetal force.
3. A satellite of mass m moving around the circular orbit round the Earth has centripetal force due to gravitational attraction between the Earth and the satellite.



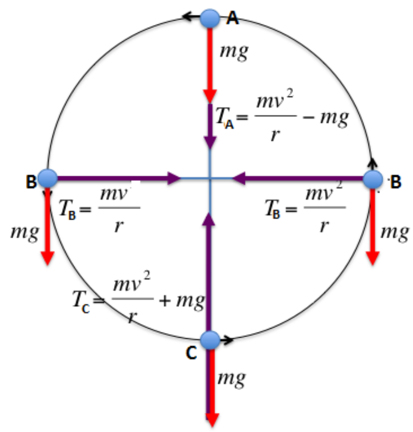
Try this!

What is the speed of the tip of a minute head of the clock, where the hand is of height 7cm?

(Answer 1.22 x 10^{-4}m/s)

Motion in a Vertical Circle

When an object of mass m is whirled with a constant speed v in a vertical circle of the centre O by a string of length r



- **At point A (Top of the Motion)**

  Suppose T_A is the tension in the string since the weight mg acts downwards towards the centre O, then;

  Force towards the centre $F = mg + T_A = \dfrac{mv^2}{r}$

  $$\therefore T_A = \frac{mv^2}{r} - mg \text{ .... (i)}$$

- **At point B**

  Suppose T_B is the tension in the string since the weight mg acts vertically downwards and has no components in the horizontal direction. So the force towards the centre O is;

  Force towards the centre $F = \dfrac{mv^2}{r}$

  $$\therefore T_B = \frac{mv^2}{r} \text{ .... (ii)}$$

- **At point C (The lowest point of the Motion)**

  Suppose T_C is the tension in the string acting upwards towards the centre O and the weight mg acts in opposite direction, then;

  Force towards the centre $F = T_C - mg = \frac{mv^2}{r}$

  $$\therefore T_C = \frac{mv^2}{r} + mg \text{ .... (iii)}$$

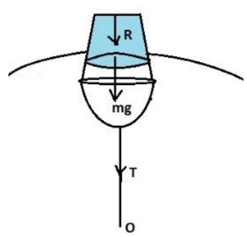
NOTE

Maximum tension occurs at C, the bottom of the circle (at the lowest position). T_C, must be greater than mg by $\frac{mv^2}{r}$ to make the object keep moving in a circular path.

The minimum tension occurs at A, the top of the circle (the highest position) where part of the required centripetal force is provided by the weight mg and the rest by T_A

Bucket whirled in a vertical circle

If some water is placed in bucket attached to the end of a string, the bucket can be whirled in a vertical circle plane without any water falling out, when the bucket is vertical above the point of support, the weight mg of the water is less than the required force $F = \frac{mv^2}{r}$ towards the centre so the water sty in.



$$\frac{mv^2}{r} > mg$$

The reaction R of the bucket base on the water provides the rest of the force. If the bucket is whirled slowly $mg > \frac{mv^2}{r}$, part of the weight provided the force $\frac{mv^2}{r}$ the rest of the weight causes the water to accelerate downward and hence leave the bucket.

Example

1. The radius of the ferns wheel is 5.0m and its makes one revolution in 10sec. find the difference between the apparent weight of a passenger at the highest and lowest point, hence express the answer as ratio of its own weight.

   Solution

   From; $T = \frac{2\pi}{\omega}$

   $$\therefore\ \omega = \frac{2\pi}{T} = \frac{2\pi}{10s} rad = 0.5\pi rads^{-1}$$

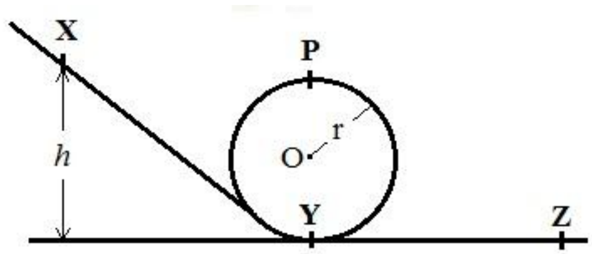
   Consider the apparent weight recorded is F

   - At highest point; $F_{highest} = mg - \frac{mv^2}{r} = mg - mr\omega^2 =$
   - At lowest point; $F_{lowest} = \frac{mv^2}{r} + mg = mr\omega^2 + mg = m(r\omega^2 + g)$

   $\therefore$ The differences $(mr\omega^2 + mg) - (mg - mr\omega^2) = 2mr\omega^2$

   In terms of ratio, the fraction $= \frac{2mr\omega^2}{mg} = \frac{2r\omega^2}{g} = \frac{2 \times 5m \times (0.5\pi rads^{-1})^2}{9.8ms^{-2}} = 0.4$

2. The figure below shows a toy runway. After release from a point such as **X**, a small model car runs down the slope, "loops the loop", and travels on towards **Z**. the radius of the loop is 0.25m. (g = 10m/s^2)



   (i) Ignoring the effect of friction outline the energy changes as the model moves from X to Z.

   (ii) What is the minimum speed with which the car must pass point P at the top of the loop if it is to remain in contact with the runway?

   (iii) What is the minimum value of h which allows the speed calculated in (ii) to be achieved?

Solution

(i) The energy changes

- from X to Z is Potential energy to kinetic energy
- from Y to P Kinetic energy to (potential energy + kinetic energy)
- from P to Z (Potential energy + kinetic energy) to kinetic energy

(ii) From Centripetal force = weight ;-

$$\frac{mv^2}{r} = mg$$

$$v = \sqrt{gr} = \sqrt{10 \times 0.25} = 1.6ms^{-1}$$

(iii) From Principle of conservation of energy

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$$

$u =$ is the speed at point **Y**

$h =$ is the height from **Y** to **P** $= 2r$

So $u = \sqrt{v^2 + 2g(2r)} = \sqrt{v^2 + 4gr} = \sqrt{(1.6)^2 + 4(10 \times 0.25)} = 3.544ms^{-1}$

From $mgh = \frac{1}{2}mu^2$

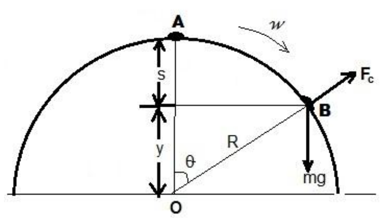
$$h = \frac{u^2}{2g} = \frac{(3.54)^2}{2 \times 10} = 0.63m$$

Example (Necta 2014)

An insect is released from rest at the top of the smooth bowling ball such that it slides over the ball. Prove that it will lose its footing with the ball at an angle of about 48^o with the vertical

Solution

Consider the sketched diagram



At a point B the insect lose its footings with the ball, At this point

$$F_C = \frac{mv^2}{R} = mg\cos\theta$$

$$\cos\theta = \frac{v^2}{Rg} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From principle of conservation of mechanical energy from point A to B

$$\text{PE} = \text{KE}$$

$$mgs = \frac{1}{2}mv^2$$

$$v^2 = 2gs = 2g(R - y) = 2g(R - R\cos\theta) = 2gR(1 - \cos\theta) \ldots \text{(ii)}$$

Substitute equation (ii) into (i)

$$\cos\theta = \frac{2Rg(1 - \cos\theta)}{Rg} = 2(1 - \cos\theta) = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

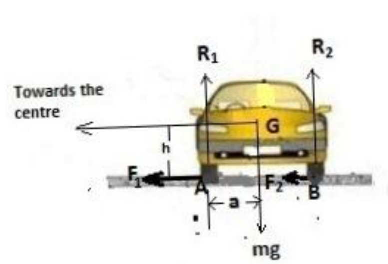
$$\cos\theta = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48^o \text{ Proved}$$

Motion of a Car or Train Round a Circular Track

When a car travels along a circular track the friction force between the tyres and the road provides the centripetal force F toward the centre of the circle.

Suppose a car is moving with a velocity v round horizontal circular track of radius r. R_1, and R_2 are normal reaction at wheel A and B, F_1 and F_2 are the corresponding friction forces.



For Circular motion;

- Horizontal forces $\quad F_1 + F_2 = \frac{mv^2}{r}$ ………………………… (i)
- Vertical forces $\quad R_1 + R_2 = mg$ ………………………… (ii)

Taking moment of forces about G (Assume G is the centre of mass and midway between the wheels)

$$(F_1 + F_2)h + R_1 a = R_2 a \text{ .... (iii)}$$

$$\frac{mv^2}{r}h + (mg - R_2)a = R_2 a$$

$$R_2 = \frac{1}{2}m\left(g + \frac{v^2 h}{ra}\right)$$

From equation (iii)

$$\frac{mv^2}{r}h + R_1 a = (mg - R_i)a$$

$$R_1 = \frac{1}{2}m\left(g - \frac{v^2 h}{ra}\right)$$

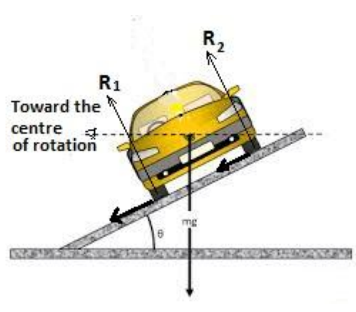
If $v^2 = \frac{\arg}{h}$, $R_1 = 0$ and the car is about to overturn outwards. R_1, is positive if $v^2 < \frac{\arg}{h}$

NOTE

R_2, cannot vanish since it has a positive value.

Motion of a Car or Train Round a Banked Track

Suppose a car is moving round a banked track in a circular path of radius r. If the only force at the wheels A and B are normal reaction R_1 and R_2 respectively i.e. there is no side slip or friction at the wheels. The force towards the centre of the track is the component of the weight along the inclined plane.



If the speed of the car is v, the force acting are given as;-

- Vertically $(R_1 + R_2)\cos\theta = mg$ .... (i)

  i.e. vertical component of the normal reaction force balance the weight

- Horizontally $(R_1 + R_2)\sin\theta = \frac{mv^2}{r}$ .... (ii)

Divide equation (ii) by equation (i), we get

$$\tan\theta = \frac{v^2}{rg}$$

NOTE

- For a given speed v and radius r, the angle of inclination of the track θ for no side slip must be $\tan^{-1}\left(\frac{v^2}{rg}\right)$
- As the speed v increases the angle θ also increases

Application of Banking

(a) A racing track is made saucer shape because at high speed the car can move towards a part of the track which is steeper and sufficient to prevent side-slip.

(b) The outer rail of a curved railway track is raised above the inner rail so that to balance the reaction forces acting on the wheels.

(c) For the same reasons as the racing track is balanced, it is desirable to bank a road at the corner.

Advantage of Banking

(a) Road safety; prevent skidding and overturning.

(b) Reduce wearing or tearing out of the tyres and wheels by reducing friction on them.

(c) Increase life span of the road and railways lines at the corners

Example

1. A curve on a road forming an arc whose radius of curvature is 200m. If the width of the road is 30m and its outer edge is 0.6m higher than the inner edge for what speed it is ideally banked?

   Solution

   From; $\tan\theta = \frac{v^2}{rg}$ since θ is the banking angle

   $\therefore\ v = \sqrt{rg\tan\theta}$

   But, $\tan\theta = \frac{\Delta h}{w}$

Δh, and w are the difference in level of two edge and width of the road.

$$\tan\theta = \frac{0.6m}{30m} = 0.02$$

$$v = \sqrt{200m \times 9.8ms^{-2} \times 0.02} = 6.3ms^{-1}$$

2. A train travelling with a speed of 10m/s round a curve of radius 5 x10^2m. If the distance between the rails is 1.5m, find the required elevation of the outer rail above the inner one so that there may be lateral thrust on the wheels.

   Solution

   From; $\tan\theta = \dfrac{v^2}{rg}$ since θ is the banking angle

   $$\therefore \tan\theta = \frac{(10m/s)^2}{500m \times 9.8ms^{-1}} = 0.0204$$

   But, $\tan\theta = \dfrac{h}{d}$

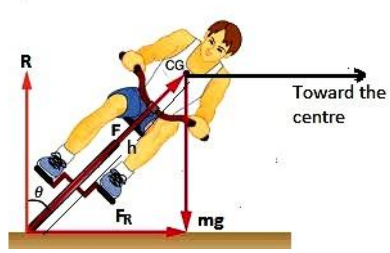
   h = the difference in height between two elevations

   d = distance between rails

   $$\therefore h = d\tan\theta$$
   $$= 1.5m \times 0.0204$$
   $$= 0.03m$$

Motion of Bicycle's Rider (Cyclist) Round Circular Track

When a person on bicycle, ride round a circular racing track the friction force F at the ground provides the centripetal force.



Resolving vertically; $R = mg$

Horizontally; $F_R = \dfrac{mv^2}{r}$

Taking moment about the centre of gravity;

$$Rh\sin\theta = F_R h\cos\theta$$

From above equation;

$$\tan\theta = \frac{F_R}{R} = \frac{F_R}{mg}$$

Since θ is the angle of inclination to the vertical, h is the height of the bicycle to the centre of gravity, from; $F_R = \frac{mv^2}{r}$

Then, $\tan\theta = \frac{v^2}{rg}$

When is greater than the limiting friction force, sliding will occur i.e.

$$F_R > \mu mg$$

$$mg\tan\theta > \mu mg$$

$\therefore\ \tan\theta > \mu$, is the condition for sliding

Example

The curve on a level road has a radius of 75m. A motorbike whose total mass with rider is 120kg has to negotiate the curve at the speed of 54kph. If the coefficient of friction between the tyres and ground is 0.4, will the rider be able to go round without slipping? At what angle and maximum speed should be learn to avoid slipping?

Solution

From the condition for skidding to occur; $\tan\theta > \mu$

But, $\tan\theta = \frac{v^2}{rg}$

$\therefore\ \tan\theta = \frac{(15ms^{-1})^2}{75m \times 9.8ms^{-2}} = 0.31$ (Since 54kph = 15m/s)

Since $\tan\theta < \mu$, then the sliding will not occur and the rider will negotiate the corner safely

The angle to avoid slipping at this corner is given by;

$$\tan\theta = \mu$$

$$\theta = \tan^{-1}\mu = \tan^{-1}0.4 = 22^o$$

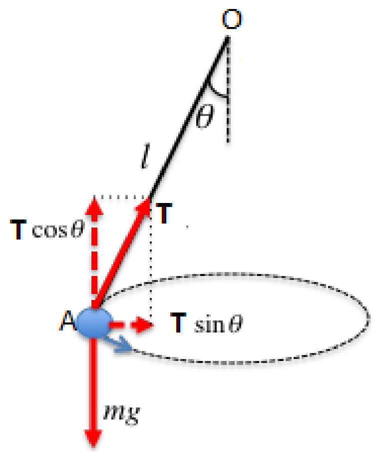
The maximum speed should be;

$$v = \sqrt{rg\tan\theta}$$

$$v = \sqrt{75m \times 9.8ms^{-2} \times 0.4} = 17.15m/s$$

Conical Pendulum

Suppose a small object of mass m is tied to a string OA of length l and whirled round in a horizontal circle of radius r with angle θ fixed directly above the centre B of the circle.



If the circular speed of A is constant the string turns at constant angle θ to the vertical. This is called **conical pendulum**.

As A moves with constant speed v in a circle of radius r, the centripetal force towards the centre is;

$$F = \frac{mv^2}{r}$$

This force is provided by horizontal component of the tension T in the string given by;

$$\frac{mv^2}{r} = T\sin\theta \quad \text{.................................. (i)}$$

The weight mg of the object is counterbalanced by vertical component of the tension given by;

.

$$mg = T\cos\theta \quad \text{.................................. (ii)}$$

Dividing equation (i) by (ii), we get

$$\tan\theta = \frac{v^2}{rg}$$

NOTE

As the speed v increases the angle θ is also increases

Example

A conical pendulum consists of 1m long, inextensible string one end attached to a fixed point and the other end carrying a bob of mass 20g. The bob is moving in horizontal circle of radius 0.3m. Calculate the speed of the bob and the tension on the string.

Solution

- Consider the speed of the bob

From, $$\tan\theta = \frac{v^2}{rg}$$

$$\therefore v = \sqrt{rg\tan\theta}$$

But $$\sin\theta = \frac{r}{l} = \frac{0.3m}{1m} = 0.3$$

$$\therefore \theta = \sin^{-1} 0.3 = 17.46^o$$

So, $$v = \sqrt{0.3m \times 9.8ms^{-1} \times \tan 17.46^o} = 0.96ms^{-1}$$

- Consider the tension on the string

From, $$mg = T\cos\theta$$

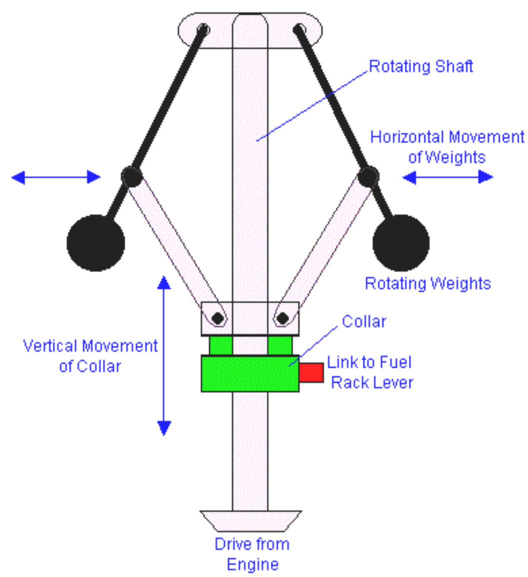
$$T = \frac{mg}{\cos\theta} = \frac{0.02kg \times 9.8ms^{-2}}{\cos 17.46^0}$$

$$T = 0.21N$$

Application of Circular Motion

(i) Speed governor of the steam engine

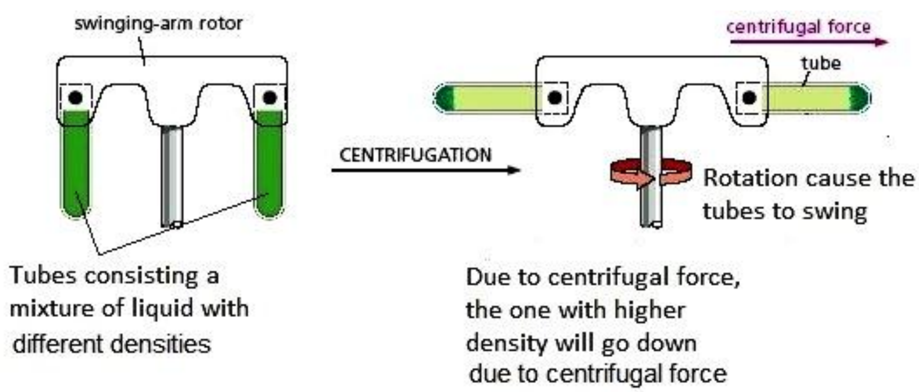
When rotating weights swing round a conical pendulum, causes horizontal movement of the weights away from the rotating shaft. This causes the vertical movement of the collar which link to the fuel.



As the weights move out, the collar rises on the shaft. If the weights move inwards, the collar moves down the shaft. The movement of the collar is used to operate the fuel rack lever controlling the amount of fuel supplied to the engine by the injectors hence controlling the speed of the engine as the governor.

(ii) Centrifuge

A centrifuge is the device used to separate mixture of liquid of different densities



Exercise 2C

1. (a) Define and state its SI unit of the following
   - (i) Angular displacement
   - (ii) Angular velocity

   (b) Calculate the angular speeds in radian per second of the earth;-
   - (i) rotates on its axis and
   - (ii) revolving around the sun

2. (a) Mention the sources of centripetal forces in each of the following cases
   - (i) An object at the end of a string is whirled in horizontal circle
   - (ii) A car traveling round a banked road.
   - (iii) The moon orbiting the earth.
   - (iv) A motorbike traveling round a circular path.

   (b) An object of mass 4kg is whirled in a vertical circle of radius 2m with speed of 5m/s. Calculate the maximum and minimum tension in the string connecting the object and the centre of the circle.

3. (a) What do you understand by the term "*centripetal force"* as applied in mechanics?

   (b) Suppose that you are standing at the equator, traveling in a circle of radius 6400km, going round once in 24 hours.
   - (i) What is the centripetal force required in this motion?
   - (ii) Where does this force comes from?
   - (iii) Explain, why the weight of an object increases when you carry it from the equator to one of the pole?

4. (a) Why roads and railway lines are banked at the corner?

   (b) A car is traveling round a corner of radius 80m at a speed of 120km/h.
   - (i) If the coefficient of dynamic friction is 0.4, will the car pass the corner without skidding?
   - (ii) Find the angle of banking necessary for a corner in order for the car to go without relying on the frictional forces.

D. Simple Harmonic Motion

Any motion that repeats itself after a certain period is known as ***periodic motion*** and since that motion can be represented in terms of sines and cosines, it is called ***harmonic motion***

Simple harmonic motion (S.H.M);

S.M.H is the motion of a body in which its acceleration is proportional to the displacement and is always directed toward the mean position (fixed point).

The equation of simple harmonic motion (S.H.M) can be written as;-

Acceleration $a \propto x$

$$\therefore a = -kx$$

k , is constant and x is the displacement of a body from the fixed point at any time t.

The **maximum displacement** of the body on either side of its central position is called amplitude (r).

The **period of the motion** (T) is the time taken for the body to make one complete oscillation.

The equation for S.H.M is actually written as;

$$a = -\omega^2 x$$

ω , is a constant depending on the particular system of oscillation.

Velocity in Simple Harmonic Motion

(a) Using Calculus Method:

From the equation of simple harmonic motion

$$a = -\omega^2 x$$

$$\therefore \quad \frac{dv}{dt} = -\omega^2 x$$

So, $$v\frac{dv}{dx} = -\omega^2 x$$

$$vdv = -\omega^2 xdx$$

$$\int v\,dv = -\omega^2 \int x\,dx$$

$$\frac{v^2}{2} = -\left(\frac{\omega^2 x^2}{2}\right) + c$$

Since $v = 0$, when $x = r$ and $v = \omega r$, when $x = 0$

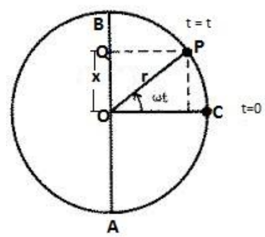
Therefore, $\frac{v^2}{2} = -\left(\frac{\omega^2 x^2}{2}\right) + \frac{\omega^2 r^2}{2}$

$$v^2 = \omega^2 r^2 - \omega^2 x^2 = \omega^2 (r^2 - x^2)$$

$v = \pm\omega\sqrt{r^2 - x^2}$ ………………………………….. (ii)

(b)Using Trigonometric Method:

Consider the a point P through a diameter moving in a circle of radius r



Let the velocity at a point P round the circle be v.

The displacement x of the projection P from O along AB is given by;

$x = r\sin\theta$ (Since $\theta = \omega t$)

$\therefore\ x = r\sin\omega t$ …………………….. (i)

But velocity of a point P is given by;

$$v = \frac{dx}{dt} = \frac{d(r\sin\omega t)}{dt}$$

$v = \omega r\cos\omega t$ …………………………. (ii)

Acceleration is given as;-

$$a = \frac{dv}{dt} = \frac{d(\omega r\cos\omega t)}{dt}$$

$$\therefore\ a = -\omega^2 r\sin\omega t$$

Since $x = r\sin\omega t$

$\therefore\ a = -\omega^2 x$ ……………………………. (iii)

From equation (i) and (ii) we have

$$\sin \omega t = \frac{x}{r}$$

$$cow\, \omega t = \frac{v}{\omega r}$$

Since $\sin^2 \omega t + \cos^2 \omega t = 1$

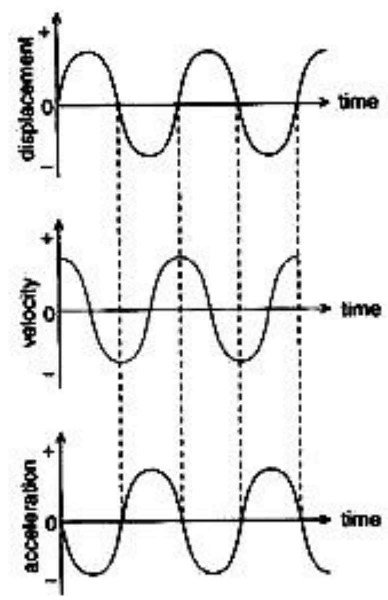
$$\frac{x^2}{r^2} + \frac{v^2}{\omega^2 r^2} = 1$$

Solve for v we get

$$\frac{\omega^2 x^2 + v^2}{\omega^2 r^2} = 1$$

$$v = \pm \omega \sqrt{r^2 - x^2} \text{ .... (iv)}$$

Curves showing the variations of displacement, velocity and acceleration in S.H.M



Example!

1. The displacement in (cm) of a particle from the equilibrium position moving with simple harmonic motion is given by $x = 0.05 \sin 6t$, where t is the time in seconds measured from instant when $x = 0$. Calculate

    (i) Amplitude of oscillations

    (ii) Period of oscillations.

    (iii) Maximum acceleration of the particle.

Solution

(i) Given, $x = 0.05\sin 6t$

From, $x = r\sin \omega t$

$\therefore$ Amplitude $r = 0.05cm$

(ii) From the equation above, $\omega = 6rads^{-1}$

$$T = \frac{2\pi}{\omega} = \frac{2\times 3.14}{6} = 1.05s$$

(iii) Since, $a = -\omega^2 r$

$= -6^2 \times 0.05cm$

$= -1.8cms^{-2}$

2. A body is vibrating with simple harmonic motion of amplitude 15cm and frequency 4Hz. Compute

(i) the maximum acceleration and velocity

(ii) the velocity when the coordinate is 9cm

(iii) the time required to move from equilibrium position to a point 12cm distance from it

Solution

(i) From $w = 2\pi f = 2\times 3.14\times 4 = 25.12rads^{-1}$

Acceleration $a = -w^2 r = (25.12rads^{-1})^2 \times 15\times 10^{-2}m = 94.7ms^{-2}$

Velocity, $v = \pm wr = \pm 25.12\times 15\times 10^{-2} = 3.8ms^{-1}$

(ii) Velocity, $v = \pm w\sqrt{r^2 - x^2} = \pm 25.12\sqrt{(15\times 10^{-2})^2 - (9\times 10^{-2})^2}$

$= 3.0ms^{-1}$

(iii) From $y = r\sin \omega t$

$$wt = \sin^{-1}\left(\frac{y}{r}\right) = \sin^{-1}\left(\frac{12}{15}\right) = 53.13 = 0.927radians$$

$$t = \frac{0.927radians}{25.12rads^{-1}} = 0.0369s$$

Try this!

From the equilibrium position a particle is oscillating in a SHM is displaced by a distance x measured in metre, given by equation $x = 0.08\sin 9t$, where t is the time in second measured from an instant when $x = 0$, determine the period of oscillation and maximum acceleration of the particle.

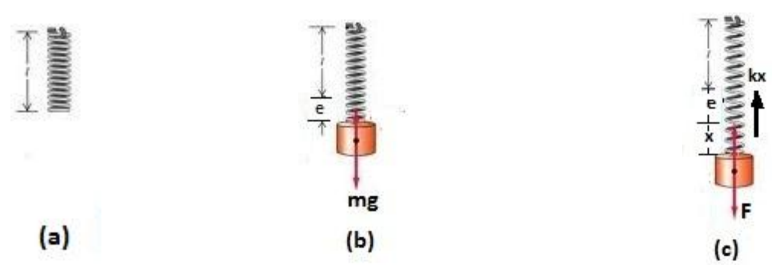
The following are examples of systems undergoing S.H.M

(i) Helical spring
(ii) Cylinder floating in liquid
(iii) Simple pendulum
(iv) Liquid in U-tube
(v) Compound pendulum
(vi) Torsional oscillation
(vii) Bifilar pendulum

(a) Helical Spring

Consider a mass m suspended at rest from a spring (b). If the spring constant is k and e is the extension, then;

$$mg = ke$$



The mass m is then pulled down a small distance x and released; the mass will oscillate due to both, the effect of gravitational attraction and varying force in spring.

When the mass pulled down (c), the restoring force on the spring is due to the force on the spring given as;

$$\textbf{Restoring force} = -kx$$

The restoring force has negative sign, since it opposing the pull of mass, so restoring force balanced the force applied on mass

$$-kx = ma$$

$$\therefore\ a = -\frac{k}{m}x$$

From, $a = -\omega^2 x$

$$\omega^2 = \frac{k}{m} = \frac{g}{e}$$

Since $T = \frac{2\pi}{\omega}$, then; $T^2 = \frac{4\pi^2}{\omega^2}$

$$\therefore\ T^2 = 4\pi^2\frac{m}{k} = 4\pi^2\frac{e}{g}$$

The period of oscillation of helical spring is given as

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{OR} \quad T = 2\pi\sqrt{\frac{e}{g}}$$

NOTE

The equation (i) and (ii) can be used to determine experimentally the value of spring constant k of and acceleration due to gravity g, by varying the masses m in equation (i) and extension e in equation (ii) respectively with the period of oscillation.

Example

A particle P, oscillate with a simple harmonic motion along a line between extremes A and B with O being the centre of oscillation

(a) What is the direction of the acceleration at any instant?

(b) When the acceleration is

(i) maximum and

(ii) Minimum?

(c) When the velocity is

(i) Maximum

(ii) zero

Solution

(a) Acceleration is towards the mean position

(b) (i) At A and B

(ii) At O

(c) (i) At O

(ii) At A and B

Potential and Kinetic Energy in Oscillating Systems

Energy of stretched spring is potential energy. Potential energy (PE) for an extension x is given by;

$$PE = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2}kx^2$$

The energy of the mass m in kinetic energy is

$$KE = \frac{1}{2}mv^2, \quad v \text{ is the velocity}$$

From; $x = r\sin\omega t$

$$v = \omega r\cos\omega t$$

Total energy of the spring is the sum of KE and PE of the system

$$E_T = PE + KE = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$E_T = \frac{1}{2}kr^2\sin^2\omega t + \frac{1}{2}m\omega^2 r^2\cos^2\omega t$$

But for mass spring system, $\omega^2 = \frac{k}{m}$ or $k = m\omega^2$

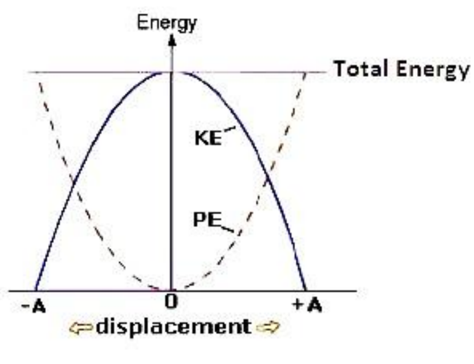
$$E_T = \frac{1}{2}m\omega^2 r^2\sin^2\omega t + \frac{1}{2}m\omega^2 r^2\cos^2\omega t$$

$$E_T = \frac{1}{2}m\omega^2 r^2(\sin^2\omega t + \cos^2\omega t)$$

$$E_T = \frac{1}{2}m\omega^2 r^2 = \text{Constant}$$

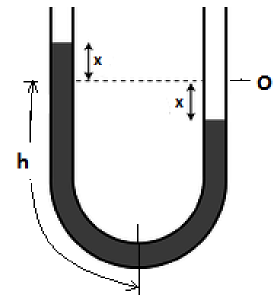
Then the total energy of the system is constant. When KE is maximum (at the centre of oscillation), the PE of the spring is zero.

The diagram below shows the variation of KE, PE and the total energy of oscillating system.



A is amplitude of oscillating system.

(b)Liquid Oscillating in a U-Tube



Density of the liquid is ρ

Consider the liquid of density ρ in a U-tube. If the liquid in one side of the U-tube is depressed by blowing gently down in one side, level of the liquid will oscillate for short time about their respective initial position O (when it was at equilibrium).

Suppose the level of liquid on left side is at D a distance x above the original position O, the excess pressure on the liquid in the left side is given by;

Excess pressure $= 2x\rho g$

Since, **Force = Pressure x Area**

Therefore, the restoring force is given by;

Restoring force =Excess pressure x Area $= -2x\rho gA$

$\therefore$ Restoring force = ma

m, is the mass of liquid in the tube

$$-2x\rho gA = (2A\rho h)a$$

$$-xg = ha$$

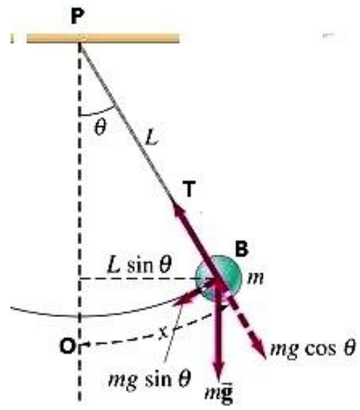
$$\therefore\ a = -\frac{g}{h}x = -\omega^2 x$$

$\omega^2 = \frac{g}{h}$, so the motion of the liquid about O is S.H.M. The period T is given by;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$$

(c)Simple Pendulum

Consider a simple pendulum of small mass m attached to the end of length L of wire.



Suppose that a vibrating mass at B at some instant, where OB $= x$ and angle OPB $= \theta$, At B, the force pulling the mass m towards O (Restoring force) is directed along the tangent at B and equal to $mg\sin\theta$.

The tension T in the wire has no component in the direction of the restoring force towards O, θ is very small i.e. $\sin\theta = \theta$ in radians. So, $\theta = \frac{x}{L}$

Hence, $a = -\frac{g}{L}x = \omega^2 x$

$\omega^2 = \frac{g}{L}$, so the motion of the Simple pendulum about O is S.H.M. The period T is given by;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Example

A simple pendulum has a period of 2.8s. When its length is shortened by 1.0m, the period becomes 2.0s. From this information determine the acceleration g due to gravity and the original length of the pendulum.

Solution

The value of is obtained from;

$$g = 4\pi^2 \frac{\Delta l}{\Delta T^2} = 4\times(3.14)^2\left(\frac{1m}{2.8^2 - 2^2}\right) = 10.27ms^{-2}$$

The value of original length L is obtained from;

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$T_1 = 2\pi\sqrt{\frac{L_1}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$T_2 = 2\pi\sqrt{\frac{L_2}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$\frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}}$$

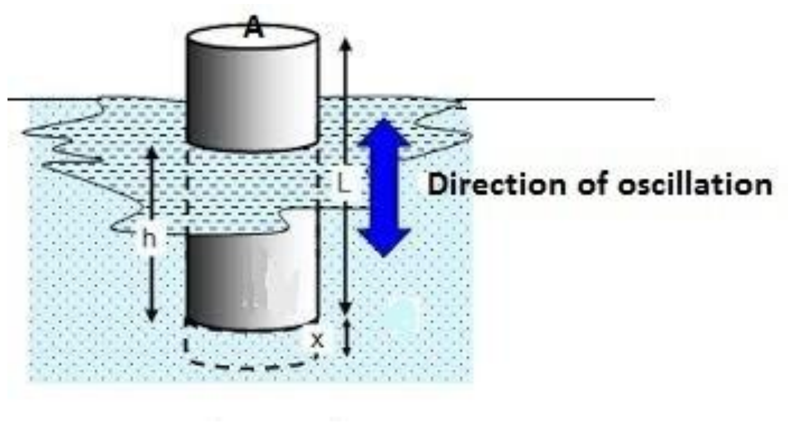
$$\left(\frac{T_1}{T_2}\right)^2 = \frac{L_1}{L_1 - 1m}$$

$$\left(\frac{2.8s}{2s}\right)^2 = \frac{L_1}{L_1 - 1m} = 1.96$$

Solve for L_1 we get 2.04m

(d)The Floating cylinder

A cylinder of length L, cross-section area A and mass m floating with height h immersed in a liquid of density ρ .



At equilibrium, the weight of the cylinder balanced the upthrust of the liquid

$$mg = Ah\rho g$$

$$m = Ah\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

If the cylinder is pushed downward a little distance x, allow it to bob up and down. The force causes the oscillation are gravity and the varying upthrust of a liquid on a cylinder.

Extra upthrust = extra weight of the liquid displaced = Restoring force

$$ma = -A\rho gx \text{ .... (ii)}$$

Substitute the value of m from equation (i) in equation (ii)

$$Ah\rho a = -A\rho gx$$

$$\therefore\ a = -\frac{g}{h}x = -\omega^2 x$$

$\omega^2 = \frac{g}{h}$, so the motion of the cylinder about O is S.H.M. The period T is given by;

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$$

Exercise 2D

1. (a) What are the conditions for a body to undergo simple harmonic motion?
   (b) Prove that a body moving in a circular motion like earth round the sun undergoing simple harmonic motion, hence writes the expression for the period T of revolution
2. (a) Define the amplitude as applied in S.H.M
   (b) Derive an expression of velocity of a body at any position when it is undergo simple harmonic motion, hence find the speed of a particle moving with this motion with a period of 8 seconds and amplitude of 5.0m when it is 3.0m from the centre of its motion
3. (a) State at what point in simple harmonic motion a body has maximum kinetic energy.
   (b) The period of oscillation of the body of mass 2kg under SHM is 1.8seconds, with what kinetic energy of the body when it passes through a point half way between the mean position and maximum displacement, if the maximum displacement is 22cm?
4. (a) Give two practical examples of oscillatory motion, which approximate to simple harmonic motion
   (b) A body of mass M is performing simple harmonic motion whose amplitude is A. Sketch, on the same axes the variation of Potential energy and kinetic energy with displacement.
5. (a) State three applications of simple harmonic motion in daily life.
   (b) The displacement in (m) of a particle from the equilibrium position moving with simple harmonic motion is given by $y = 0.3\sin 10t$, where t is the time in seconds measured from instant when $y = 0$. Calculate
      (i) Period of oscillations (ii) maximum velocity and
      (iii) Maximum acceleration of the particle.
6. (a) If the mass of an object attached on the spring is doubled, what is the ratio of the new period to the old one?
   (b) When a metal cylinder of mass 0.2kg is attached to the lower end of a light helical spring the upper end being fixed the spring extends by 0.16m. The metal cylinder is then pulled down further 0.08m.
      (i) Find the restoring force when pulled 0.08m
      (ii) if the metal cylinder is released, find the period of oscillation
      (iii) The kinetic energy when it is at maximum height

E. Gravitation

Kepler's laws of planetary motion

(a) 1st Kepler's law

This law states "The planets describe ellipses about the sun as one focus"

(b) 2nd Kepler's law

This law states that "The line joining the sun and the planets sweeps out equal areas in equal time"

(c) 3rd Kepler's law

States that "The square of the period of revolution of the planet are directly proportional to the cubes of their means distance from the sun".

$$T^2 \; \alpha \; R^3$$

Newton's Investigation on Planetary Motion

Newton's investigation was based on the motion of planets moving in a circle round the sun as centre.

The force acting on the planet of mass m is;

$$F = \frac{mv^2}{R} = mR\omega^2$$

$R =$ Radius of the circle

$\omega =$ Angular speed of the motion of planet

Since the gravitational force on the planet obeys inversely square law, then;

$$mR\omega^2 = k\frac{m}{R^2}$$

But, $\omega = \dfrac{2\pi}{T}$ $\quad T =$ Period of revolution of the planet

$$\therefore \frac{4\pi^2}{T^2} = k\frac{1}{R^3}$$

$$T^2 = \frac{4\pi^2 R^3}{k} = \left(\frac{4\pi^2}{k}\right)R^3$$

Since 4, k and π are constants, then $T^2 \; \alpha \; R^3$. This agrees with Kepler's 3rd law

Newton's Law of Universal Gravitation

Newton's law of universal gravitation states that "***The force of attraction between two given particles is directly proportional to the product of their masses and inversely proportional to the square of their distance apart***"

$$F \ \alpha \ \frac{mM}{R^2}$$

$$\therefore \ F = G\frac{mM}{R^2}$$

$G =$ Universal gravitational constant, it can be expressed in Nm^2kg^{-2}, measurements shows that G = 6.67 x $10^{-11}Nm^2kg^{-2}$.

Try this!

Using dimensional analysis, find the dimensions of universal gravitational constant G and predict its other possible units.

The Relation between g and G

On the Earth's surface an object of mass m has a gravitational force mg on it (weight).

To find the gravitation force on mass on the earth's surface or outside, the mass of the object and that of the Earth is assumed to be concentrated at the centre and the earth is assumed to be the sphere of radius R_E.

For the mass on the earth's surface, the gravitation force is given as;-

$$F = \frac{GM_E m}{R_E^2}$$

$M_E =$ Mass of the earth

$$\therefore \ mg = \frac{GM_E m}{R_E^2}$$

So, $$g = \frac{GM_E}{R_E^2}$$

$$gR_E^2 = GM_E$$

From above GM_E can be replaced by gR_E^2 in any formula

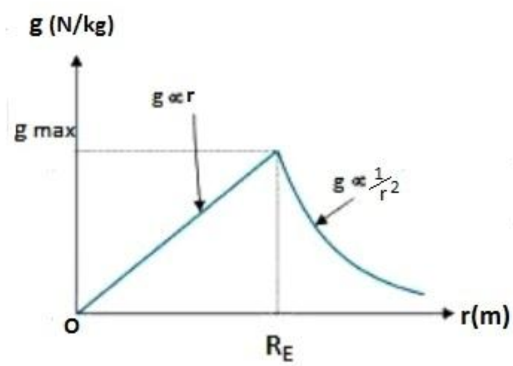
Variation of Acceleration due to Gravity (g)

- **For the point outside the earth**, the gravitational force obeys an inverse-square law. So, the acceleration due to gravity is inversely proportional to the square of the distance from the centre of the earth.

$$g \ \alpha \ \frac{1}{r^2}$$

  The value of g is maximum at the earth's surface where $r = R_E$

- **Inside the earth**, the value of g is not inversely proportional to the square of the distance from the centre.
  By assuming that the earth's density is uniform, the value of g varies linearly proportional with distance from the centre.



The gravitational force on a mass m is given;

$$F = mg$$

$\therefore \ g = \dfrac{F}{m}$, can be expressed in Newton's per kilogram (N/kg)

g , is known as the gravitational force (field strength) which can be defined as the force acting per unit mass in the gravitational field of the earth (planet)

For a given planet, the value of gravitational field strength on its surface will depends on its mass and its radius.

$$g = \frac{GM}{R^2}$$

Example

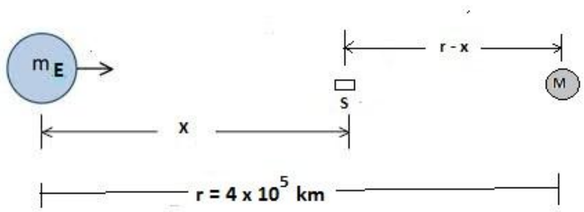
The mass of the earth is 81 times that of the moon and the distance from the centre of the earth to that of the moon is about 4.0 x 10^5km.

(a) Calculate the distance from the centre of the earth where the resultant gravitation force becomes zero when a spacecraft is launched from the earth to the moon.

(b) Draw a sketch showing roughly how the gravitational force on the spacecraft varies in its journey

Solution

(a) Consider the sketched diagram



From sketched diagram;

At point S, the resultant force = 0

$$F_E = F_M$$

$$G\frac{m_s m_E}{x^2} = G\frac{m_s m}{(r-x)^2}$$

$$\frac{m_E}{x^2} = \frac{m}{(4\times10^5 - x)^2}$$

$$\frac{m_E}{m} = \frac{x^2}{(4\times10^5 - x)^2} \text{ ,}$$

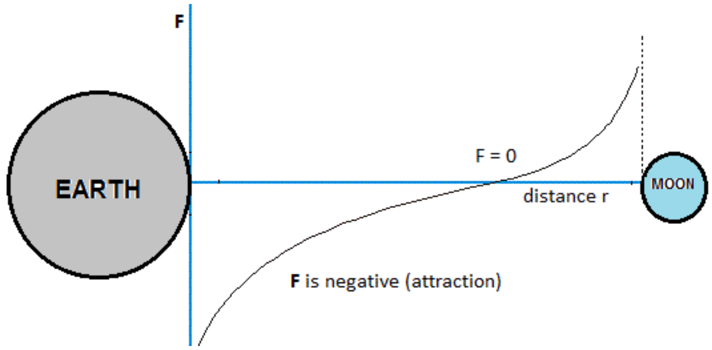
But $\frac{m_E}{m} = 81$

$$\therefore \quad 81 = \frac{x^2}{(4\times10^5 - x)^2}$$

$$9 = \frac{x}{4\times10^5 - x}$$

Solve for x wen get $x = 3.6\times10^5 km$

(b) Sketched diagram



Try this!

1. The mean radii of the planets, earth and mars are 6400km and 3360km respectively. If the mass of the mars is 0.11 times that of earth, determine the gravitation of the mars. (Use g = 9.8N/kg) **(Answer 3.92N/kg)**

2. The ratio R^3/T^2 for the moon-earth system is $1.02 \times 10^{13} m^3 s^{-2}$, R is the radius of orbit and T is the period. Calculate
   (i) the value of the earth gravitational field if the radius of the earth is 6400km
   (ii) the mean radius given that it takes 27.3 days to round the earth.
   (Answer (i) $g = 9.82 ms^{-2}$ (ii) $R = 46 \times 10^6 m$)

3. If the gravitational field on the earth surface is $9.8 ms^{-2}$, the universal gravitational constant is $6.67 \times 10^{-11} Nm^2kg^{-2}$ and the radius of the earth is 64000km, calculate the mass and density of the earth. Give the assumptions made to obtain the answers. **(Mass = 6.0×10^{24}kg, density = $5.5 g/cm^3$)**

Gravitational Potential

The potential V_G at a point due to gravitational field of the earth is defined as the work done per unit mass in taking a mass from infinity to that point.

The gravitational potential at infinity is taken as zero.

Assume the whole mass of the earth is concentrated at its centre. The force of attraction of the mass m outside the earth is given as;

$$F = G\frac{M_E m}{r^2}$$, where r is the distance from the earth's centre

The work done by gravitational force in moving a unit mass a distance dr towards the earth is given by;-

$$\Delta W = Fdr = G\frac{M_E m}{r^2}dr$$

The potential at a point distance "a" from the centre is given as;

$$V_a = \frac{W}{m} = \int_{\alpha}^{a} G\frac{M_E}{r^2}dr = GM_E\int_{\alpha}^{a} r^{-2}dr$$

$$= GM_E\left[-\frac{1}{r}\right]_{\alpha}^{a}$$

$$= -GM_E\left(\frac{1}{a}-\frac{1}{\alpha}\right)$$

Since, $\frac{1}{\alpha} = 0$

$$\therefore V_a = -\frac{GM_E}{a}$$

Negative sign indicate that the potential at infinity is higher than the potential at the earth surface.

On the earth's surface, the radius is R_E; the gravitational potential is obtained as;

$$V_o = -\frac{GM_E}{R_E}$$

For the large distance above the earth, the change in P.E of the mass m is given by using

$$\Delta PE = m\left[-\frac{GM_E}{b} - \left(-\frac{GM_E}{a}\right)\right]$$

$$= m\left[\frac{GM_E}{a} - \frac{GM_E}{b}\right]$$

$$= GM_E m\left(\frac{1}{a}-\frac{1}{b}\right)$$

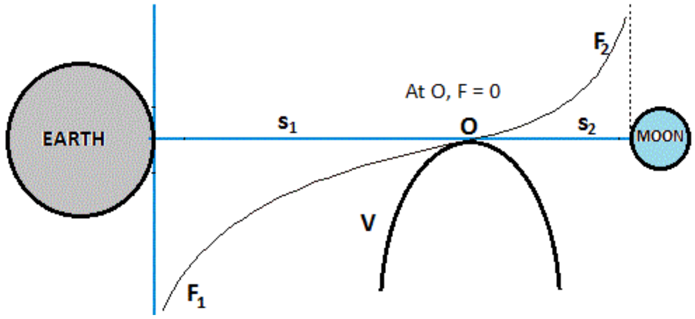
$$= GM_E m\left(\frac{b-a}{ab}\right)$$

For a small distance above the earth, the gravitation potential on the mass is fair constant;

$$\Delta PE = Fds = mgh$$

Variation of Gravitational potential from the Earth to the moon

When a satellite moves from the Earth towards the moon, the direction of force changes from F_1 at s_1, where the gravitational pull is greater than that of the moon to F_2 at s_2 near the moon where the pull of planet is now stronger than that of the earth



At O, the gravitational pull of the earth is balanced by that of the moon. The potential energy V of the satellite is the sum of negative potential value due to the earth and the moon,

The maximum value of V occurs just below O, where the resultant force is zero.

Since $F = \frac{dV}{dr}$, the gradient $\frac{dV}{dr} = 0$

Escape Velocity

Suppose a rocket of mass m is fired from the earth's surface so that it just escapes from the gravitational influence of the earth, then the work done is equal to the mass times the gravitational potential difference between infinity and the earth's surface.

$$W = m\left[-\frac{GM_E}{\alpha} - \left(-\frac{GM_E}{R_E}\right)\right]$$

$$= \frac{GM_E m}{R_E}$$

Since work done = Kinetic energy of the rocket

$$W = \frac{1}{2}mv^2 = \frac{GM_E m}{R_E}$$

$$v = \sqrt{\frac{2GM_E}{R_E}}$$

Since, $GM_E = gR_E^2$

$$\therefore\ v = \sqrt{2gR_E} \qquad v = \text{Velocity of escape}$$

$$v = \sqrt{2 \times 9.8ms^{-2} \times 6.4 \times 10^6 m} = 11 \times 10^3 ms^{-1} = 11kms^{-1}$$

With an initial velocity of 11km/s a rocket will complete escape from the gravitational attraction of the earth.

Example

1. A particle is situated at a height *h* above the surface of a planet with mass *M* and radius *R*.
   (i) Derive an expression for the escape velocity in terms of *h, M, R* and universal gravitational constant *G*.
   (ii) In what ways does the earth's atmosphere lose particles

   Solution

   (i) From work done = change in PE = change in KE

$$W = m\left[-\frac{GM}{\alpha} - \left(-\frac{GM}{R+h}\right)\right] = \frac{1}{2}mv^2$$

$$\therefore \quad \frac{GMm}{R+h} = \frac{1}{2}mv^2$$

$$v^2 = \frac{2GM}{R+h}$$

$$v = \sqrt{\frac{2GM}{R+h}}$$

   (ii) The earth's atmosphere loses particles in a space when the particles gain sufficient energy to overcome the potential between infinity and their position above the earth's surface.

2. **Try this!**

   A rocket is fired from the earth's surface and gains just enough energy to make it to reach the moon. Find the position between the earth and the moon at which the rocket will have maximum gravitational potential. .(Mass of the earth = 5.98 x 10^{24}kg, Mass of the moon = 7.34 x 10^{22}kg and Moon-earth distance = 3.84 x 10^8km)

Earth's Satellites

A satellite is a body which is moving around the planet. Example, the moon is the natural satellite of the earth.

Satellite can be launched from the earth's surface to circle the earth. They are kept in orbit by the gravitational attraction of the earth.

Consider a satellite of mass m which just circle the earth of mass M_E along its surface in an orbit. If R_E is the radius of the earth then;

Centripetal force = Gravitational force

$$\frac{mv^2}{R_E} = \frac{GM_E m}{R_E^2}$$

If g is the acceleration due to gravity at the earth's surface and $GM_E = gR_E^2$, then;

$$v = \sqrt{\frac{GM_E}{R_E}} = \sqrt{gR_E}$$

$$v = \sqrt{9.8ms^2 \times 6.4 \times 10^6 m} = 8 \times 10^3 ms^{-1} = 8kms^{-1}$$

The speed in the orbit is about 8km/s, the period T in the orbit is given by

$$T = \frac{circumference}{v} = \frac{2\pi R_E}{v} = \frac{2 \times 3.14 \times 6.4 \times 10^6 m}{8 \times 10^3 ms^{-1}}$$

$= 5000s$ or $83\min$

Parking Orbits

A satellite is said to be in parking orbit when it remain at the same position above the earth's surface.

This is possible when;

- The direction of revolution of the satellite is the same as that of rotation of the earth on its axis.
- The period of the satellite is the same as that of the earth that is 24hours.

Application of Parking Orbits

Relay satellite for communication are placed in parking orbit so that television programs can be transmitted continuously from one part of the word to another

Height of Parking Orbits

Suppose R is the distance of the orbit from the centre of the earth and v is the speed of the planet on parking orbit, then;

Centripetal force = Gravitational force

$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

But $GM_E = gR_E^2$, where R_E is the radius of the earth

$$\frac{mv^2}{R} = \frac{gR_E^2 m}{R^2}$$

$$v^2 = \frac{gR_E^2}{R}$$

If T is the period of the satellite in its orbit, then; $v = \frac{2\pi R}{T}$

$$\therefore \frac{4\pi^2 R^2}{T^2} = \frac{gR_E^2}{R}$$

$$R^3 = \frac{gR_E^2 T^2}{4\pi^2}$$

Since T = 24hours, the radius R can be found from equation above

$$R = \left(\frac{9.8ms^{-1} \times (6.4\times10^6 m)^2 \times (24\times3600s)^2}{4\times(3.14)^2} \right)^{\frac{1}{3}} = 4.24\times10^7 m$$

The height h above the surface of the parking orbit is given as;

$$h = R - R_E = (4.24\times10^7 - 6.4\times10^6) = 3.6\times10^7 m$$

Assume the orbit is circular; the speed of the satellite will be;

$$v = \frac{2\pi R}{T}$$

$$= \frac{2\times3.14\times4.24\times10^7 m}{24\times3600}$$

$$= 3.1ms^{=1}$$

PE and KE of Satellite

A satellite of mass m in orbit round the earth has both kinetic energy (KE) and potential energy (PE). The kinetic energy is due to the speed of the satellite in the orbit while potential energy is due to the position from the centre of the earth.

From;
$$\frac{mv^2}{R} = \frac{GM_E m}{R^2}$$

$$\frac{1}{2}mv^2 = \frac{GM_E m}{2R}$$

$$\therefore\ KE = \frac{GM_E m}{2R}$$

Since the potential energy at infinity is assumed to be equal to zero, then PE of the satellite in orbit is;

$$PE = -\frac{GM_E m}{R}$$

From above, the PE of the mass m in orbit is twice its KE numerically but of opposite sign. So, total energy of a satellite in the orbit is equal to the sum of KE and PE.

$$E_T = KE + PE$$

$$E_T = \frac{GM_E m}{2R} + \left(-\frac{GM_E m}{R}\right)$$

$\therefore$ Total energy $E_T = -\dfrac{GM_E m}{2R}$

Due to the friction in the earth's atmosphere the energy of a satellite diminished and radius of the orbit decrease to R_1

Total energy in orbit of radius R_1 is given by;

$$E_T = -\frac{GM_E m}{2R_1}$$

This is less than the initial energy. Since $\dfrac{GM_E m}{2R_1} > \dfrac{GM_E m}{2R}$

Therefore the KE of the satellite increases when it falls to an orbit of smaller radius ie the satellite speed up.

Example

A satellite of mass 1tonne moves in a circular orbit of radius 7 x 10^3km round the earth. Assuming the earth is a sphere of radius 6.4 x 10^3km, calculate;

(a) The speed of the satellite

(b) The potential energy of the satellite

(c) The total energy of the satellite.

Solution

(a) From, $\frac{mv^2}{R} = \frac{GM_E m}{R^2}$

$$v = \sqrt{\frac{GM_E}{R}} = \sqrt{\frac{gR_E^2}{R}}$$

$$= \sqrt{\frac{9.8ms^{-2} \times (6.4 \times 10^6 m)^2}{7 \times 10^6 m}}$$

$$= 7.57 \times 10^3 ms^{-1}$$

The velocity of the satellite is 7.57 x 10^3m/s

(b) From, $PE = -\frac{GM_E m}{R} = -\frac{mgR_E^2}{R}$

$$= -\left(\frac{1000kg \times 9.8ms^{-2} \times (6.4 \times 10^6)^2}{7 \times 10^6 m}\right)$$

$$= -5.7344 \times 10^{10} J$$

(c) From, Total energy $E_T = -\frac{GM_E m}{2R} = -\frac{mgR_E^2}{2R}$

$$= -\left(\frac{1000kg \times 9.8ms^{-2} \times (6.4 \times 10^6)^2}{2 \times 7 \times 10^6 m}\right)$$

$$= -2.8672 \times 10^{10} J$$

Exercise 2E

1. (a) (i) State Kepler's laws of planetary motion
   (ii) As you know, the gravitational attraction of the earth on object is proportional to the mass of the object. If two objects of different masses (heavy one and light one) fall freely under the action of gravity, which one or both will reach the ground first? Explain why?

   (b) (i) Explain the variation of acceleration due to gravity, g, inside and outside the Earth
   (ii) Find the gravitational field intensity at a point 1000m above the sea level

   (c) The maximum and minimum distance of a comet from the sun are 1.4×10^{12} m and 2.2×10^{10} m. If its velocity nearest to the sun is 6.0×10^4 ms^{-1}, find its velocity in the furthest position from the sun. State assumptions made in your calculations

2. (a) (i) State Newton's law of universal gravitation.
   (ii) Derive the formulae for mass and density of the earth.
   (b) What do you understand by the term "escape velocity" as applied in gravitation?

   (c) Jupiter has a mass 318 times that of the earth, its radius is 11 times the earth's radius. Use this information to estimate the escape velocity of a body from Jupiter's surface, if the escape velocity from earth's surface is $11.2kms^{-1}$

3. (a) (i) What is gravitational potential
   (ii) Derive the expression for gravitational potential at the earth's surface
   (b) Using Kepler's third law, calculate the height above the earth's surface for a satellite in a parking orbit.
   (c) A 300kg satellite orbits the earth at height of 500km from its surface. Compute its kinetic energy, potential energy and total energy

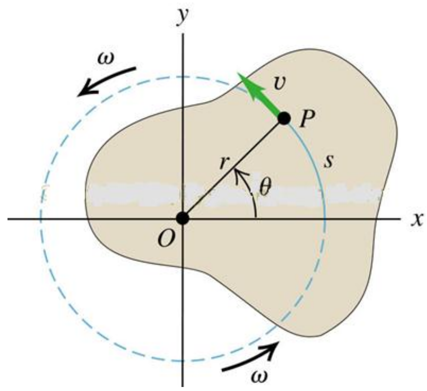
4. (a) What do you understand by the term satellite?
   (b) When a satellite moves in a circular orbit of radius R about the earth, the centripetal force is provided by gravitational attraction. Derive the expression for period of revolution T in terms of R, gravitational constant G and mass of the earth M_E

   (c) A satellite of mass 100kg moves in a circular orbit of radius 7000km around the earth, assumed to be a sphere of radius 6400km. calculate the total energy needed to place the satellite in orbit from the earth, assuming $g=10Nkg^{-1}$

F. Rotation of a Rigid Bodies

A rigid body is the one in which the particles making it keep the same distance from a given axis under rotation.

Suppose a rigid object is rotating about a fixed axis *O*, and a particle *P* of the object makes an angle θ with a fixed line *Ox*.



The angular velocity of a body is $\frac{d\theta}{dt} = \omega$. Every particle about *O* has the same angular velocity.

Velocity of a particle *P* at this instant is given by $v = \omega r$, where r is the distance *OP*

The kinetic energy of $P = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$

Another particle of the body has the kinetic energy equal to $= \frac{1}{2}m_2\omega^2 r_2^2$

Therefore, the kinetic energy of the whole object is given by;

$$K.E = \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 .... + \frac{1}{2}m_n r_n^2\omega^2$$

$$K.E = \frac{1}{2}\omega^2(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 .... + m_n r_n^2)$$

$$\therefore\ K.E = \frac{1}{2}\omega^2 \sum mr^2$$

$\sum mr^2$, represent the sum of the magnitude of "mr^2" for all particles of the object.

The magnitude $\sum mr^2$ is called moment of inertia of the object about the axis concerned and it is denoted I , $\quad \therefore\ K.E = \frac{1}{2}\omega^2 I$

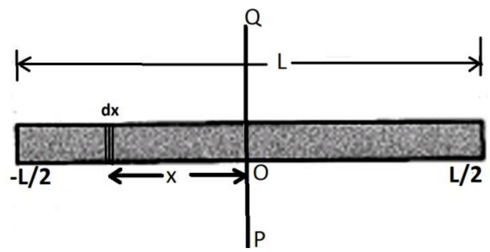
The unit of I are Kgm^2 and the unit of ω is radian per second ($rads^{-1}$)

How to deduce the Moment of Inertia?

1. Moment of inertia of Uniform rod

(a) About axis through its middle

Consider a uniform rod of mass M and length L rotating about axis O through its middle.



Moment of inertia of a small element dx about an axis PQ through its centre O perpendicular to the length is given as;-

$$I_x = m_x x^2 = \left(\frac{dx}{L}M\right)x^2,$$

x, is the distance of small element dx from O.

$\therefore$ Moment of inertia of the rod is given by;-

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{dx}{L}M\right)x^2$$

$$I = \frac{M}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$I = \frac{M}{L}\left[\frac{\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{3}\right)^3}{3}\right]$$

$$I = \frac{M}{L}\left[\frac{L^3}{12}\right]$$

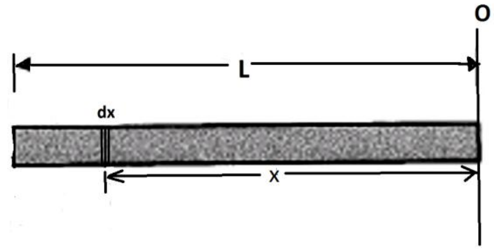
$$\therefore\ I = \frac{1}{12}ML^2$$

For a rod of mass 60g and length 20cm moment of inertia is given by;-

$$I = \frac{1}{12}ML^2 = \frac{1}{12}(0.06kg)\times(0.2m)^2 = 2\times10^{-4}\,kgm^2$$

(b) About axis through its one end

Mass of small element $dx = \left(\frac{dx}{L}\right)M$



Moment of inertia of element dx is given as;-

$$I_x = m_x x^2 = \left(\frac{dx}{L}M\right)x^2$$

$\therefore$ Moment of inertia of the rod given as;-

$$I = \int_0^L \left(\frac{dx}{L}M\right)x^2$$

$$I = \frac{M}{L}\int_0^L x^2 dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L$$

$$I = \frac{M}{L}\left[\frac{(L)^3 - (0)^3}{3}\right]$$

$$I = \frac{M}{L}\left[\frac{L^3}{3}\right]$$

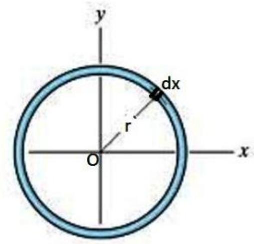
$$\therefore\ I = \frac{1}{3}ML^2$$

For a rod of mass 60g and length 20cm moment of inertia is given by;-

$$I = \frac{1}{3}ML^2 = \frac{1}{3}(0.06kg)\times(0.2m)^2 = 8\times10^{-4}kgm^2$$

2. Moment of inertia of a Ring

Consider a ring of mass M and radius r rotating at the axis through its centre.



Every element of the ring is the same distance r from the centre.

Consider an element dx of mass m along a ring. Its moment of inertia is;

$$I_x = mr^2$$

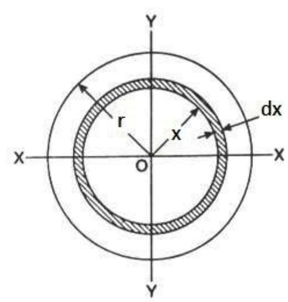
Total moment of inertia of the ring $= \sum mr^2 = r^2 \sum m$

But, $\sum m = M$ (mass of the ring)

$$\therefore\ I = Mr^2$$

3. Moment of inertia of a Circular disc

Moment of inertia of a uniform circular disc about an axis through its centre perpendicular to the plane is obtained by considering a small ring of the disc enclosed between radius x and $x + dx$



The mass of the small ring to its area is $m_x = \left(\frac{2\pi x dx}{\pi r^2}\right)M = \left(\frac{2xdx}{r^2}\right)M$

Moment of inertia of a small ring about the axis through O is

$$I_x = m_x x^2 = \left(\frac{2xdx}{r^2}\right)Mx^2 = \frac{2Mx^3}{r^2}dx$$

Now, moment of inertia of the whole disc will be;-

$$I = \int_0^r \frac{2Mx^2}{r^2}dx = \frac{2M}{r^2}\int_0^r x^3 dx = \frac{2M}{r^2}\left[\frac{r^4}{4}\right]_0^r$$

$$= \frac{2M}{r^2}\left(\frac{r^4}{4}\right) = \frac{1}{2}Mr^2$$

$\therefore$ Moment of inertia of circular disc is $I = \frac{1}{2}Mr^2$

For a disc of mass 60g and radius 10cm,

$$I = \frac{1}{2}Mr^2 = \frac{1}{2}(6\times10^{-2}kg)\times(0.1m)^2 = 3\times10^{-4}kgm^2$$

4. Moment of inertia of a Cylinder

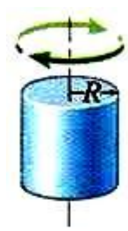
(a) Hollow cylinder

Imagine the cylinder is cut into a ring, the moment of inertia of each ring is $= mR^2$.

If M is the mass of the cylinder of radius R, then the moment of inertia of the hollow cylinder is;- $I = MR^2$

(b) Solid cylinder

Imagine the cylinder is cut into a disc, the moment of inertia of each disc is $= \frac{1}{2}mR^2$.



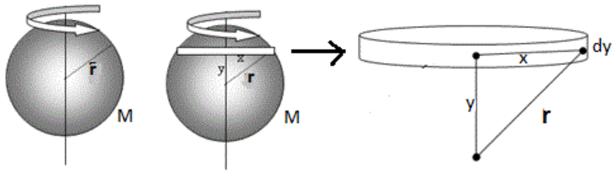
If M is the mass of the cylinder of radius R, then the moment of inertia of the solid cylinder is;- $I = \frac{1}{2}MR^2$

NOTE

For the hollow and solid cylinder of the same mass M, the same radius R and the same height, moment of inertia for hollow cylinder is greater than that of solid cylinder about the axis of symmetry.

5. Moment of inertia of a Sphere

Moment of inertia of a sphere about an axis through its centre can be found by cutting a thin disc of radius x



Volume of the disc of thickness dy and distance y from the centre is;

$$\pi x^2 dy = \pi(r^2 - y^2)dy \text{, since } r^2 = y^2 + x^2$$

Mass of the disc is $m = \dfrac{\pi(r^2 - y^2)dy}{\frac{4}{3}\pi r^3}M = \dfrac{3}{4}M\dfrac{(r^2 - y^2)}{r^3}dy$

M is the mass of the sphere and r is its radius

Moment of inertia of a disc about the axis is given as;-

$$I_x = \frac{1}{2}mx^2$$

$$= \frac{1}{2}\left[\frac{3M}{4r^3}(r^2 - y^2)dy\right]x^2$$

$$= \frac{1}{2}\left[\frac{3M}{4r^3}(r^2 - y^2)dy\right](r^2 - y^2)$$

$$= \frac{3M}{8r^3}(r^4 - 2r^2y^2 + y^4)dy$$

Consider the moment of inertia of the whole sphere is given by;-

$$I = \int_{-r}^{r} \frac{3M}{8r^3}(r^4 - 2r^2y^2 + y^4)dy$$

$$= \frac{3M}{8r^3}\int_{-r}^{r}(r^4 - 2r^2y^2 + y^4)dy$$

$$= \frac{3M}{8r^3}\left[r^4y - \frac{2r^2y^3}{3} + \frac{y^5}{5}\right]_{-r}^{r}$$

$$= \frac{3M}{8r^3}\left[(r^5 - \frac{2r^5}{3} + \frac{r^5}{5}) - (-r^5 + \frac{2r^5}{3} - \frac{r^5}{5})\right]$$

$$= \frac{3M}{8r^3}\left[2r^5 - \frac{4r^5}{3} + \frac{2r^5}{5}\right] = \frac{3M}{8r^3}\left(\frac{30r^5 - 20r^5 + 6r^5}{15}\right)$$

$$= \frac{3M}{8r^3}\left(\frac{16r^5}{15}\right) = \frac{2}{5}Mr^2$$

$\therefore$ Moment of inertia of a solid sphere is $= \frac{2}{5}Mr^2$

Try this!

A uniform rod of mass 50g is rotating with angular speed of 50r.p.m (rotations per minute) when the axis of rotation is through its centre. Find;-

(a) its moment of inertia.
(b) Its rotational kinetic energy
(c) The size of sphere with the same mass and angular speed as uniform rod which could produce the same rotational kinetic energy as that of the rod in part (b).

Radius of Gyration (k)

The moment of inertia of an object about an axis is $\sum mr^2$, sometimes it can be written as mk^2, m is the mass of the body and k is the quantity called *radius of gyration.*

Radius of gyration is the distance from a given axis to the point where all mass of the body appear to act without changing its moment of inertial to that axis.

Example

1. The moment of inertia of uniform rod when the axis of rotation is at one end is $I = \frac{1}{3}ML^2$, its radius of gyration is given as;

$$Mk^2 = \frac{1}{3}ML^2$$

$$k^2 = \frac{1}{3}L^2$$

$$\therefore\ k = \sqrt{\frac{L^2}{3}}$$

2. Determine the radius of gyration of the sphere of radius R, when the axis of rotation is through its centre.

Solution

For the solid sphere, moment of inertia is $I = \frac{2}{5}Mr^2$

But, $Mk^2 = \frac{2}{5}Mr^2$, then $k = \sqrt{\frac{2r^2}{5}}$

3. **Try this!**

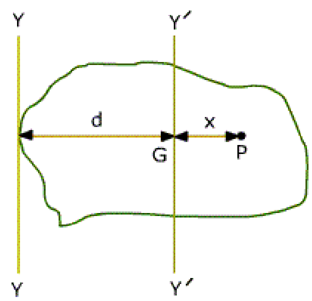
Determine the value of moment of inertia and radius of gyration of;

(i) Circular disc of mass 40g and radius 0.6cm about an axis through its centre perpendicular to the plane.

(ii) Uniform steel rod of mass 56g and length 10cm rotating about axis through its middle

Parallel Axes Theorem

Tis theorem states that "The moment of inertia of a body about an axis is equal to its moment of inertia about a parallel axis through its centre of gravity plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes".



Let 'I' be the moment of inertia of a plane object of mass M about an axis YY. 'G' is the centre of gravity of the object. Y^lY^l is an axis parallel to YY and passing through 'G'. 'I_G' is the moment of inertia of the body about Y^lY^l.

Consider the moment of inertia of a particle P of mass m ;

- About axis Y^lY^l through its centre of gravity $I_G = mx^2$

  Moment of inertia of the whole body about axis Y^lY^l $I_G = \sum mx^2$

- About axis YY at distance d and parallel to Y^lY^l $I = m(x+d)^2$

  Moment of inertia of the whole body about axis YY is

$$I = \sum m(x+d)^2 = \sum (mx^2 + 2mxd + md^2)$$

$$I = \sum mx^2 + \sum 2mxd + \sum md^2$$

Since the body will balance itself about its centre of gravity. So the algebraic sum of the moments of the **weights of constituent particles** about the centre of gravity G should be zero. $\sum mx = 0$, then;

$$I = \sum mx^2 + \sum md^2$$

$$I = I_G + d^2 \sum m = I_G + Md^2$$

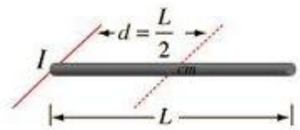
$$\therefore \; I = I_G + Md^2$$

Application

1. The moment of inertia of a uniform rod of length *L* and mass *M* about its centre is given as $I = \frac{1}{12}ML^2$, determine the moment of inertia when the axis of rotation is at one end of the rod.

   Solution

   From the Parallel axis theorem $I = I_G + Md^2$

   Since, $d = \frac{L}{2}$ and $I_G = \frac{1}{12}ML^2$

   $$\therefore \; I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2$$

   $$I = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{4ML^2}{12}$$

   $$\therefore \; I = \frac{1}{3}ML^2$$

2. Determine the Moment of inertia of a disc of radius r and mass m about axis through a point on its circumference.

   Solution

   From the Parallel axis theorem $I = I_G + Md^2$

Since, $d = r$ and $I_G = \frac{1}{2}mr^2$

$$\therefore\ I = \frac{1}{2}mr^2 + mr^2$$

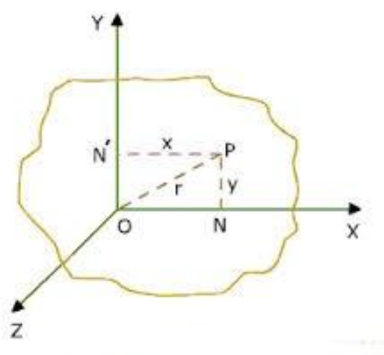
$$I = \frac{3mr^2}{2}$$

Try this!

Find the moment of inertia and radius of gyration of a sphere of radius r and mass m about axis through a point on its circumference.

Perpendicular Axis Theorem

This theorem states that "The moment of inertia of a plane body about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the body about any two mutually perpendicular axes, passing at the point through which the perpendicular axis passes".



Consider a plane body lying in the XOY plane. The body is made up of a large number of particles. Consider a particle of mass '*m*' at P. From P, PN and PN^{l} are drawn perpendicular to X-axis and Y-axis, respectively.

Now PN^{l}= *x*, PN = *y*

Consider the moment of inertia of a particle P

- Moment of inertia of mass *m* about X-axis $= my^2$

  Moment of inertia of the whole body about X-axis $I_X = \sum my^2$

- Moment of inertia of mass *m* about Y-axis $= mx^2$

  Moment of inertia of the whole body about Y-axis $I_Y = \sum mx^2$

- Moment of inertia of mass *m* about Z-axis $= mr^2$

Moment of inertia of the whole body about Y-axis $I_Z = \sum mr^2$

But, $r^2 = x^2 + y^2$

Then; Moment of inertia of the whole body about Y-axis $I_Z = \sum m(x^2 + y^2)$

$$I_Z = \sum mx^2 + \sum my^2$$

$$I_Z = I_Y + I_X$$

Application

1. If the moment of inertia of the ring at the axis through its centre is $I = MR^2$, determine the moment of inertia of the ring about axis through its diameter.

   Solution

   A ring has two perpendicular axes OX, OY in its plane

   From perpendicular axis theorem;

   $$I_Z = I_Y + I_X = MR^2$$

   But; $I_Y = I_X$

   $$2I_X = MR^2$$

   $$I_X = \frac{1}{2}MR^2$$

2. **Try this!**

   Find the moment of inertia and the radius of gyration of the disc of mass m = 2kg and radius r = 0.6m when the axis of rotation is through its diameter.

Torque (τ) and Angular acceleration (α)

In rotating motion a wheel spinning about its centre may increase its angular velocity from ω_o to ω in time t. The angular acceleration α is given by;

$$\alpha = \frac{\omega - \omega_o}{t}$$

$$\omega = \omega_o + \alpha t$$

To make the wheel spin faster, a couple or torque τ is applied to a wheel. The turning effect or torque of a force F applied tangentially to a wheel of radius r spinning about the axis through its centre is given by;-

$$\tau = Fr$$

The SI unit of τ is **Newton metre** (Nm)

A torque applied to a rotating wheel gives its angular acceleration given by;

$$\tau = I\alpha \qquad I\text{, is the moment of inertia}$$

Example

1. A heavy flywheel of moment of inertia 0.3kgm^2 is maintained on a horizontal axle of radius 0.01m with negligible mass compared to the flywheel, neglecting friction force, find;-

(i) The angular acceleration, if the force of 40N is applied tangentially to the axle.

(ii) The angular velocity of the fly wheel after 10seconds from rest.

Solution

(i) From, $\tau = Fr$

$$= 40N \times 0.01m$$

$$= 0.4Nm$$

But, $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{0.4N}{0.3kgm^2}$$

$$= 1.33rads^{-2}$$

(ii) From, $\omega = \omega_o + \alpha t$

$$\omega = 0 + (1.33rads^{-2}) \times 10s$$

$$= 13.3rads^{-1}$$

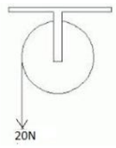
2. The moment of inertia of a solid flywheel about its axis is 0.1kgm^2; it is set in rotating by applying a tangential force of 20N with a rope wound round the circumference, the radius of the wheel being 0.1m.

(a) Calculate the angular acceleration of the flywheel.

(b) What would be the angular acceleration if the mass of 3kg were hung from the end of the rope?

Solution

(a) Consider the diagram



From $\tau = Fr$

$$= 20N \times 0.1m$$

$$= 2.0Nm$$

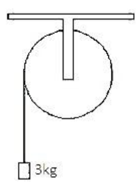
But, $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{2.0N}{0.1kgm^2}$$

$$= 20rads^{-2}$$

The angular acceleration $= 20rads^{-2}$

(b)



For the mass m;

$mg - F = ma$ ……………………… (i)

F is the tension on the string

For the flywheel

$\tau = Fr = I\alpha$ ………………….. (ii)

But $a = r\alpha$

Equation (i) can be written as;-

$mg - F = mr\alpha$ ………………….. (iii)

Multiply equation (iii) by r, we get;

$mgr - Fr = mr^2\alpha$ …………….(iv)

Substitute equation (ii) into equation (iv)

$$mgr - I\alpha = mr^2\alpha$$

$$I\alpha + mr^2\alpha = mgr$$

$$\alpha(I + mr^2) = mgr$$

$$\alpha = \frac{mgr}{I + mr^2}$$

$$= \frac{3kg \times 9.8ms^{-2} \times 0.1m}{0.1kgm^2 + [3kg \times (0.1m)^2]} = 22.6rads^{-2}$$

$\therefore$ The angular acceleration $\alpha = 22.6rads^{-2}$

Angular Momentum (L)

The angular momentum of a particle P on a rotating rigid body is defined as; the moment of linear momentum about its axis.

$$L_m = mvr = mr^2\omega$$

r, is the perpendicular distance from O.

Angular momentum of the second particle of mass m_2 at a distance r_2 from the axis of rotation is given by;

$$L_2 = m_2vr_2 = m_2r_2^2\omega$$

Total angular momentum of the whole body is

$$L = \sum mr^2\omega$$

$$L = \omega\sum mr^2 \text{ , But } I = \sum mr^2$$

$$\therefore \quad L = I\omega$$

A torque τ acting on rotating bod for a time t produce a change in angular momentum given by;

$$\tau \times t = I\omega_2 - I\omega_1$$

$$= I(\omega_2 - \omega_1)$$

I, is the moment of inertia about a given axis ω_1 and ω_2 are initial and final angular velocities produced by torque τ.

Example

The moment of inertia of a wheel is 2.0kgm^2 and the wheel is spinning with an angular velocity of 15rads^{-1}. Calculate the value of a steady breaking torque required to bring it to rest in 5seconds.

Solution

From; $\tau = I\left(\frac{\omega - \omega_o}{t}\right)$

$$= 2.0kgm^2 \times \left(\frac{0 - 15rads^{-1}}{5s}\right) = -6Nm$$

Required torque is $6Nm$

Conservation of Angular Momentum

States that "When the resultant torque on a system is zero, the angular momentum of the system is constant"

$$I_1\omega_1 = I_2\omega_2$$

Applications of Conservation of Angular Momentum

1. When high diver jumps from diving pond his moment of inertia I can be decreased by curling his body more, in this case his angular velocity ω is increased.
2. If a mass is dropped gently on a turntable rotating freely at a steady speed, the conservation of angular momentum leads to decreases in the angular velocity.
3. If the meteorites strike the earth, the effective mass of the earth increases so the earth will slow down.
4. The spinning of the ballet dancer about the vertical axis may be affected by arms outstretched or folded due to conservation of angular momentum.

Example

1. A ballet dancer spins about a vertical axis at 1 revolution per second with arms outstretched, with her arms folded her moment of inertia about vertical axis decrease by 60%. Calculate the new rate of revolution.

   Solution

   Let, I_o = the initial value of moment of inertia

   ω_o = the initial value of angular velocity

   I = the final value of moment of inertia

   ω = the final value of angular velocity

By Conservation of angular momentum;-

$$I_o\omega_o = I\omega$$

$$\therefore\ \omega = \frac{I_o\omega_o}{I}$$

Given $\omega_o = 1revs^{-1}$, $I = 0.4I_o$ (Since initial angular momentum reduced by 60%)

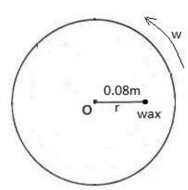
$$\omega = \left(\frac{I_o}{0.4I_o}\right)\times 1revs^{-1}$$

$$\omega = 2.5revs^{-1}$$

New rate of revolution is $2.5revs^{-1}$

2. A disc of moment of inertia 5 $5.0\times10^{-4}kgm^2$ is rotating freely about an axis O through its centre at 40r.p.m. Calculate the new revolution per minute (r.p.m) if some wax of mass 20g is dropped gently to the disc 0.08m from its axis.

Solution



- Initial angular momentum of the disc $I_o\omega_o$
- If the final angular velocity of the disc plus wax is ω_f , then total angular momentum about O of the disc plus wax $= I_o\omega_f + mr^2\omega_f$, where m is the mass of the wax.

From principle of conservation of angular momentum;

$$I_o\omega_o = I_o\omega_f + mr^2\omega_f = \omega_f(I_o + mr^2)$$

$$\omega_f = \frac{I_o\omega_o}{I_o + mr^2}$$

$$= \frac{(5\times10^{-4}kgm^2)\times(40r.p.m)}{(5\times10^{-4}kgm^2)+[0.02kg\times(0.08m)^2]}$$

$$= 32r.p.m$$

$\therefore$ New revolution per minute is 32

Rotational Kinetic Energy

The rotational kinetic energy of the object about an axis is equal to the sum of K.E of all particles making it.

$$K.E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 ....+ \frac{1}{2}m_nv_n^2$$

But, $v = \omega r$

$$\therefore\ K.E = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 ....+ \frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 ....+ m_nr_n^2)$$

$$= \frac{1}{2}\omega^2\sum mr^2$$

Since, $I = \sum mr^2$

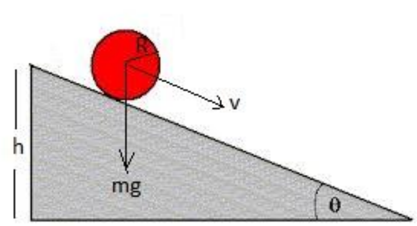
$\therefore$ Rotational $KE = \frac{1}{2}\omega^2 I$

Kinetic Energy of Rolling Object

When an object rolls on a plane, the object has rotational energy and translational energy e.g. a cylinder or a ball.

Consider a cylinder rolling along an inclined plane without slipping. The forces acting on the cylinder are;-

- Its weight (mg)
- The friction force F which prevent slipping.



Since the cylinder is released free to roll down the plane, then from principle of conservation of mechanical energy, the loss in P.E is equal to the gain in K.E;

Potential Energy = Total kinetic Energy

Total kinetic energy = Translational K.E + Rotational K.E

$$=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2$$, where I = moment of inertia of the cylinder

ω = angular velocity.

v = translational velocity

If the cylinder does not slip, then $v=\omega r$

$$\text{Total K.E} = \frac{1}{2}mv^2+\frac{1}{2}I\omega^2$$

$$=\frac{1}{2}mv^2+\frac{1}{2}I\left(\frac{v}{r}\right)^2$$

$$=\frac{1}{2}v^2\left(m+\frac{I}{r^2}\right) \text{................................ (i)}$$

Then,

$$mgh=\frac{1}{2}v^2\left(m+\frac{I}{r^2}\right), \text{ since } h=s\sin\theta$$

$$mgs\sin\theta=\frac{1}{2}v^2\left(m+\frac{I}{r^2}\right)$$

$$v^2=\frac{2mgs\sin\theta}{\left(m+\frac{I}{r^2}\right)} \text{................................ (ii)}$$

But, $v^2=2as$, where a is the linear acceleration down the plane

$$2as=\frac{2mgs\sin\theta}{\left(m+\frac{I}{r^2}\right)}$$

$$a=\frac{mg\sin\theta}{\left(m+\frac{I}{r^2}\right)} \text{................................ (iii)}$$

NOTE

If two cylinders of the same dimensions but one is hollow and the other is solid, they can be compared by determining the value of translational acceleration. The one with greater acceleration will reach down first.

- Consider a **uniform solid cylinder** of mass m and radius R.

$$I=\frac{1}{2}mR^2\text{, then} \qquad \frac{I}{R^2}=\frac{m}{2}$$

Substitute in equation (iii), we get

$$a=\frac{mg\sin\theta}{\left(m+\frac{m}{2}\right)}=\frac{mg\sin\theta}{\frac{3}{2}m}=\frac{2}{3}g\sin\theta$$

- Consider a **uniform hollow cylinder** of mass m and radius R.

$I = mR^2$**,** then $\frac{I}{R^2} = m$

Substitute in equation (iii), we get

$$a = \frac{mg\sin\theta}{(m+m)} = \frac{mg\sin\theta}{2m} = \frac{1}{2}g\sin\theta$$

From above the solid cylinder will reach down first since it has larger acceleration compared to the hollow cylinder.

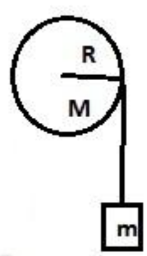
Exercise 2F

1 Is it possible for two objects with the same mass and the same rotational speed to have different value of angular momentum? Explain your answer.

2 A wheel with moment of inertia $1.2kgm^2$ is rotating with an angular speed of $800revmin^{-1}$ on a shaft whose moment of inertia is negligible. A second wheel initial at rest and with moment of inertia of $4.8kgm^2$ is suddenly coupled to the same shaft. What is the angular speed of the resultant combination of the shaft and two wheels?

3 A solid ball with mass m and radius R was released from rest at height h and the same hollow ball start from rest at height h rolled down a slope inclined at an angle θ, compare quantitatively their translational speed when they reached the ground and comment on the differences or similarities.

4 A pulley of mass M and radius *R* mounted on an axle is free to rotate about an axis through its centre and perpendicular to its plane. A light cord is wrapped around the rim of the wheel and mass *m* is suspended from the end of the chord as shown below;-



Find the expression in terms of M, *m*, *R* and g for

(i) the angular acceleration

(ii) the tension in the cord

5. Circular discs are used in automobile clutches and transmissions. When a rotating disc couples with a stationary on through friction, energy from rotating disc is transferred to the stationary one. If a disc rotating at 800 rpm couples with a stationary disc with three times the moment of inertia, what is the angular speed of the combination?

6. A constant torque of 10Nm is applied to a 10kg uniform disc of radius 0,20m, what is its rotational kinetic energy about an axis through the centre after it rotates 2.0 revolutions from rest?

7. A uniform solid cylinder of mass **M** and radius **R** rotates about vertical axis on frictionless bearing. A mass less rapped with many turns round the cylinder passes over a pulley of rotational inertia **I** and radius **r** and then attached to a small mass **m** that is otherwise free to fall under the influence of gravity as shown below. If there is no friction in the pulley axle and the cord does not slip. What is the speed of the small mass after it has fallen a distance **h** from rest