Vector spaces in One shot | All concepts and Examples

$V \to$ non empty set

$F \to$ field

$V \to$ vector space over field F

$V(F)$

\# $u+v \in V(F) \quad \forall\ u,v \in V(F)$

\# $u+(v+w) = (u+v)+w \quad \forall\ u,v,w \in V(F)$

\# $u+v = v+u \quad \forall\ u,v \in V(F)$

\# $\exists\ 0 \in V$ such that $0+u = u \quad \forall\ u \in V(F)$

\# $\exists\ v \in V$ such that $u+v=0$

\# $\alpha u \in V(F) \quad \forall\ u \in V\ \&\ \alpha \in F$

\# $\alpha(u+v) = \alpha u + \alpha v \quad \forall\ \alpha \in F,\ u,v \in V$

\# $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\ \alpha,\beta \in F,\ u \in V$

\# $(\alpha\beta)u = \alpha(\beta u) \quad \forall\ \alpha,\beta \in F\ \ u \in V$

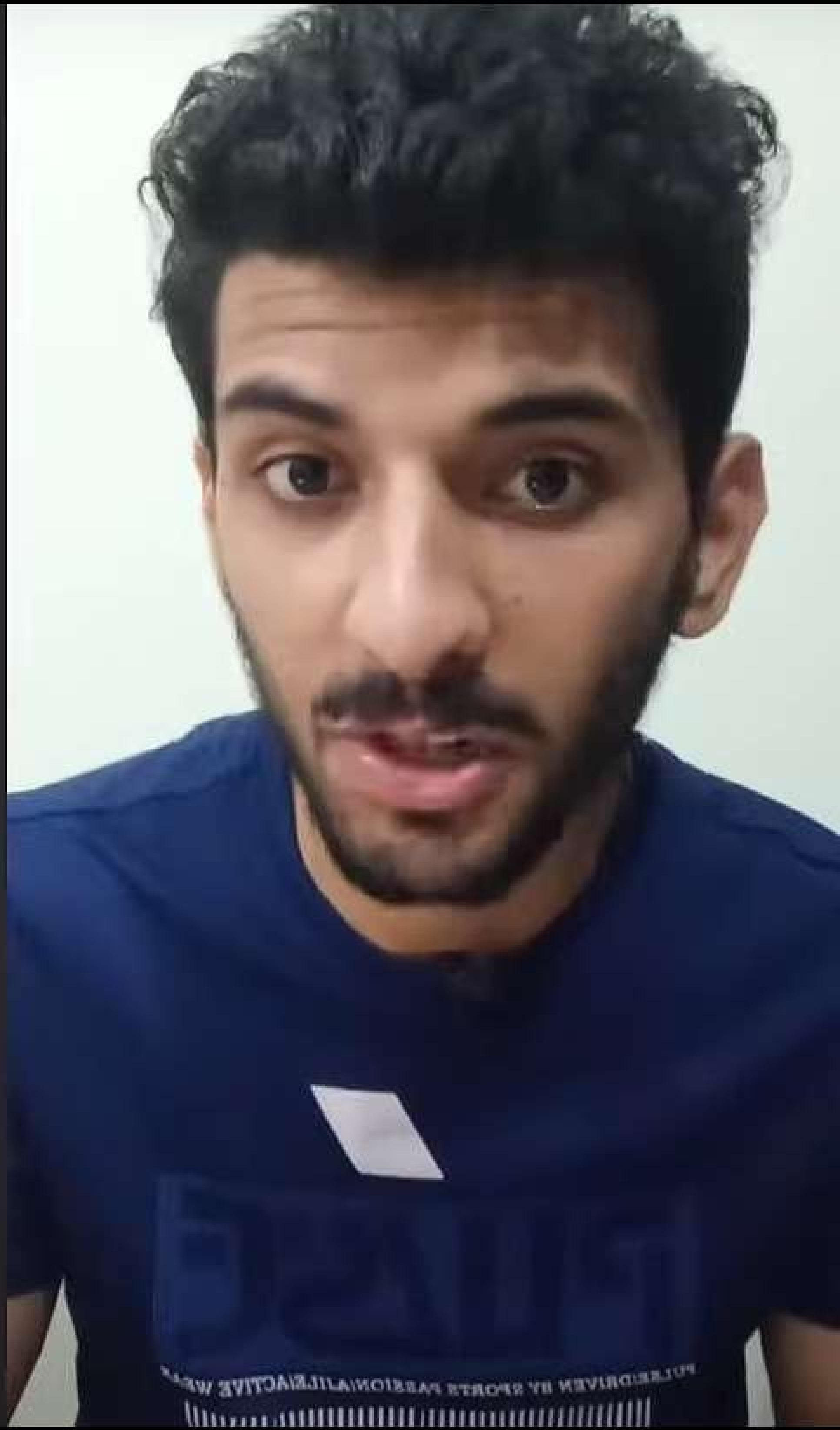
\# $1u = u \quad \forall\ u \in V$

Scalar multiplication $(\cdot)$

$\alpha \in F$

$v \in V$

$\alpha v \in V(F)$



51:54 / 4:12:28 • Vector spaces

Vector spaces in One shot | All concepts and Examples

Verify that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ w.r.t. operation component wise addition and scalar multiplication.

$$V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad V_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad V_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

① $$V_1 + V_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in \mathbb{R}^2$$

② $$\alpha V_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix} \in \mathbb{R}^2$$

③ $$(V_1 + V_2) + V_3 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 + x_3 \\ y_2 + y_3 \end{bmatrix}$$

$$= V_1 + (V_2 + V_3)$$

④ $$e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

e Exist

⑤ $$V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad V_2 = \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⑥ $$\alpha(V_1 + V_2) = \alpha \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

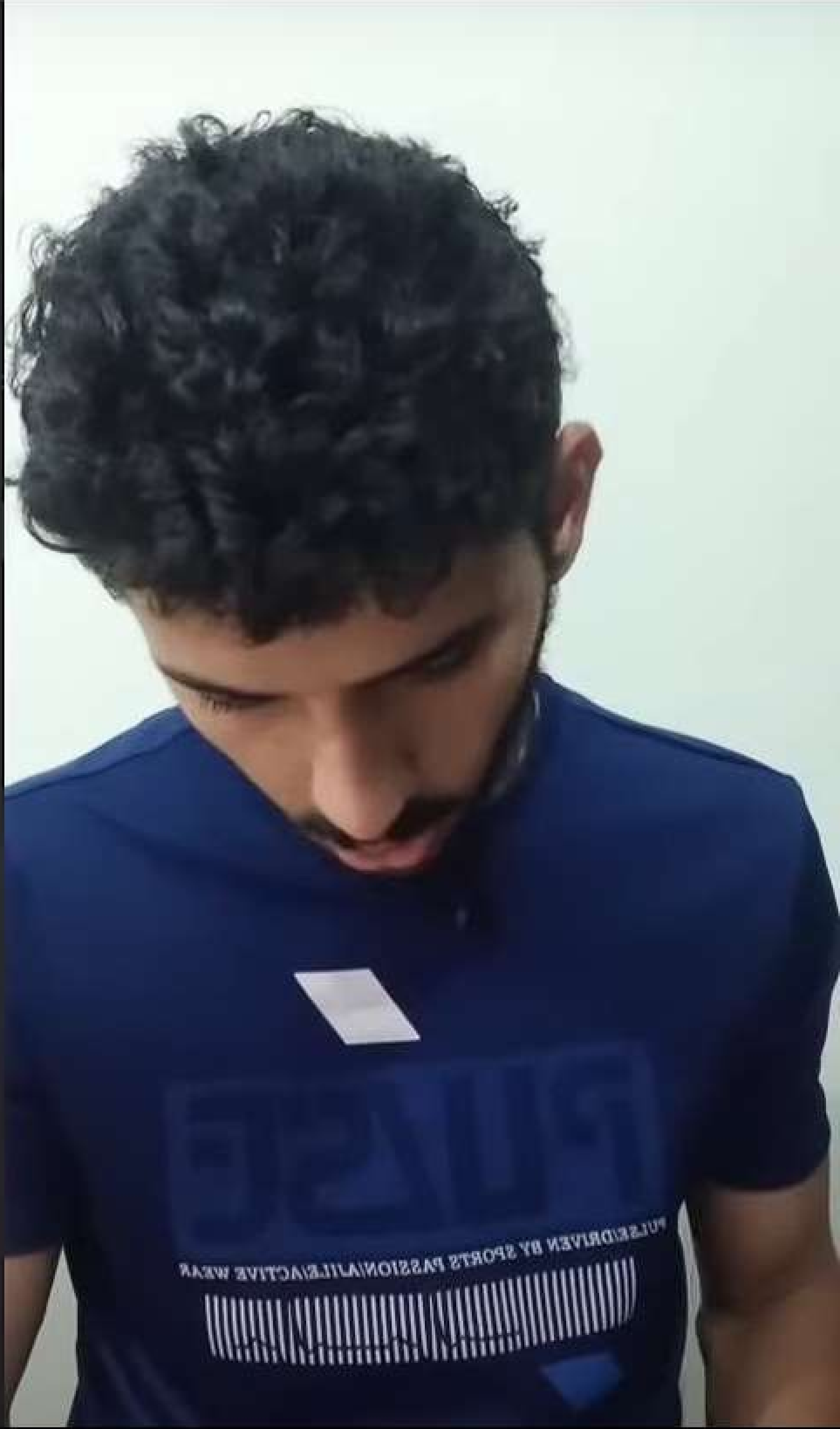
$$= \begin{bmatrix} \alpha x_1 + \alpha x_2 \\ \alpha y_1 + \alpha y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix} + \begin{bmatrix} \alpha x_2 \\ \alpha y_2 \end{bmatrix}$$

$$= \alpha V_1 + \alpha V_2$$

⑦ $$(\alpha + \beta) V_1 = \begin{bmatrix} (\alpha + \beta) x_1 \\ (\alpha + \beta) y_1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta x_1 \\ \alpha y_1 + \beta y_1 \end{bmatrix}$$



(8) $(\alpha\beta)(v_1) = (\alpha\beta)\begin{bmatrix}x_1\\ y_1\end{bmatrix} = \begin{bmatrix}(\alpha\beta)x_1\\ (\alpha\beta)y_1\end{bmatrix} = \begin{bmatrix}\alpha(\beta x_1)\\ \alpha(\beta y_1)\end{bmatrix}$

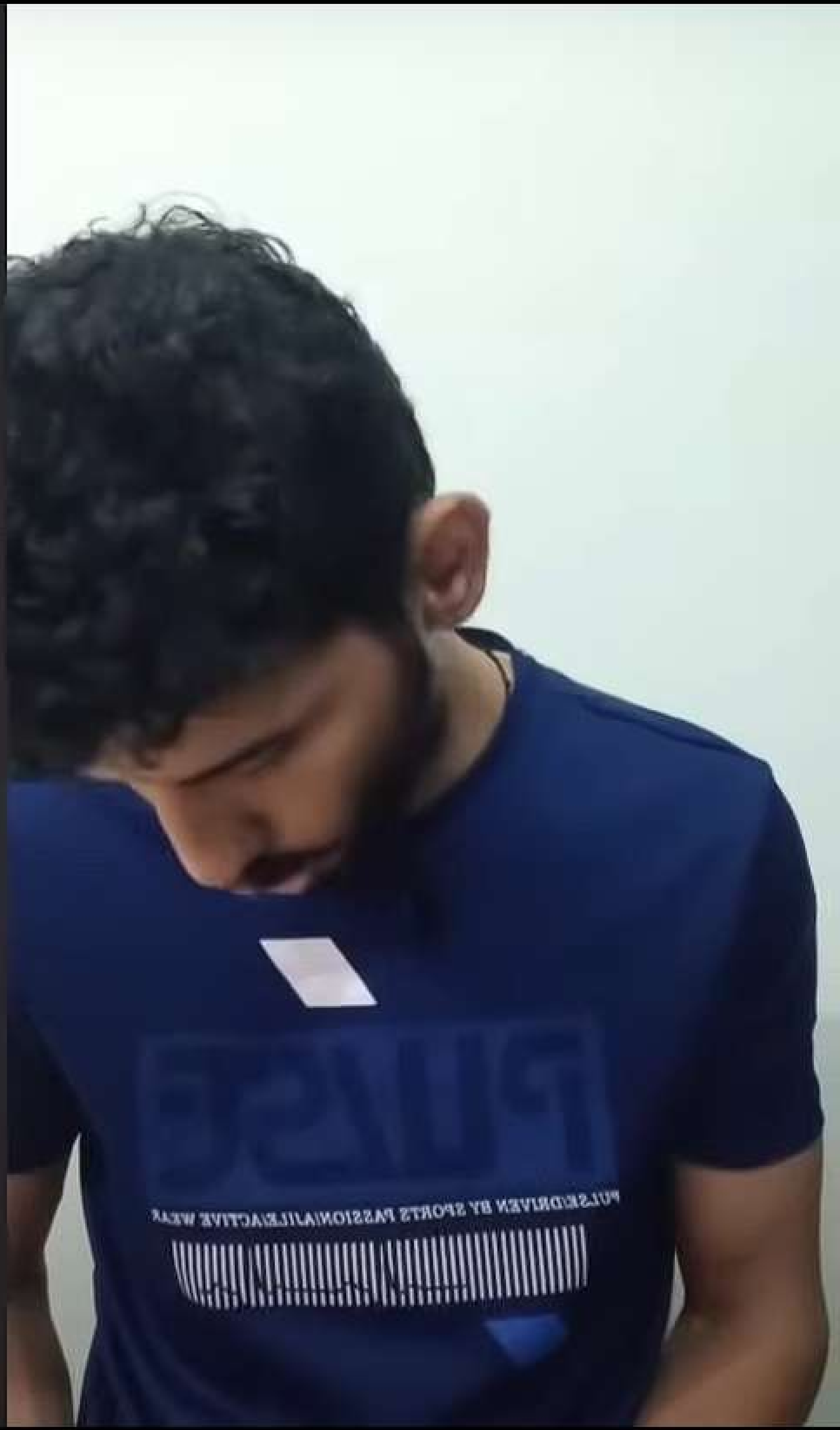
$$= \alpha\begin{bmatrix}\beta x_1\\ \beta y_1\end{bmatrix} = \alpha(\beta v_1)$$

(9) $1 \in \mathbb{R}$

$$1 v_1 = \begin{bmatrix}1x_1\\ 1y_1\end{bmatrix} = \begin{bmatrix}x_1\\ y_1\end{bmatrix} = v_1$$

(10) $v_1 + v_2 = \begin{bmatrix}x_1+x_2\\ y_1+y_2\end{bmatrix} = \begin{bmatrix}x_2+x_1\\ y_2+y_1\end{bmatrix} = \begin{bmatrix}x_2\\ y_2\end{bmatrix} + \begin{bmatrix}x_1\\ y_1\end{bmatrix} = v_2 + v_1$

$\mathbb{R}^2 \to$ vector space



Let $V = \mathbb{R}^2$. Define

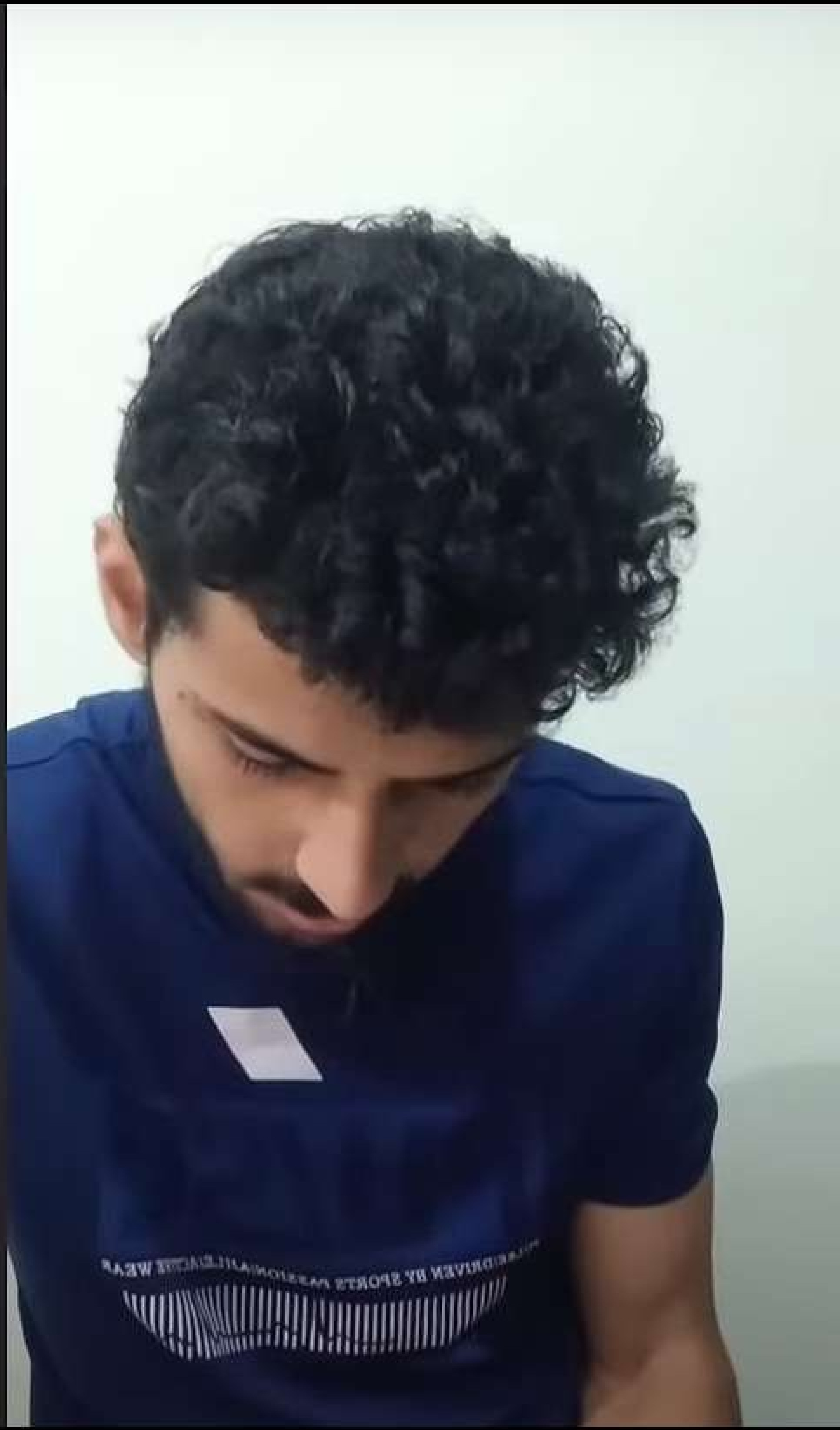
$$(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$c(x_1, x_2) = (cx_1, 0)$$

Is V a vector space.

$$1u = u$$

$$1(x_1, x_2) = (x_1, 0) \neq (x_1, x_2)$$

Not a vector



Vector spaces in One shot | All concepts and Examples

Vector space

$V \to$ non empty set

$F \to$ field

$V \to$ vector space over field F

$V(F)$

addition $(+)$

Scalar multiplication $(\cdot)$

$\alpha \in F$

$v \in V$

$\alpha v \in V(F)$

\# $u+v \in V(F) \quad \forall\ u, v \in V(F)$

\# $u+(v+w) = (u+v)+w \quad \forall\ u, v, w \in V(F)$

\# $u+v = v+u \quad \forall\ u, v \in V(F)$

\# $\exists\ 0 \in V$ such that $0+u = u \quad \forall\ u \in V(F)$

\# $\exists\ v \in V$ such that $u+v = 0$

\# $\alpha u \in V(F) \quad \forall\ u \in V$ & $\alpha \in F$

\# $\alpha(u+v) = \alpha u + \alpha v \quad \forall\ \alpha \in F,\ u, v \in V$

\# $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\ \alpha, \beta \in F,\ u \in V$

Vector spaces in One shot | All concepts and Examples

Subspace

$V(F) \rightarrow$ Vector Space

W is subspace of $V(F)$ iff

(i) $0 \in W$

(ii) $w_1 + w_2 \in W$

(iii) $\alpha w \in W \quad \forall \alpha \in F, w \in W$

WVImp

NOTE:

\# Subspaces in $\mathbb{R}$

(i) $\{0\}$ & $\mathbb{R}$

trivial subspaces

\# $\mathbb{R}^2$

(i) $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

(ii) $\mathbb{R}^2$ itself

& Z

(iii) Any line passing through origin

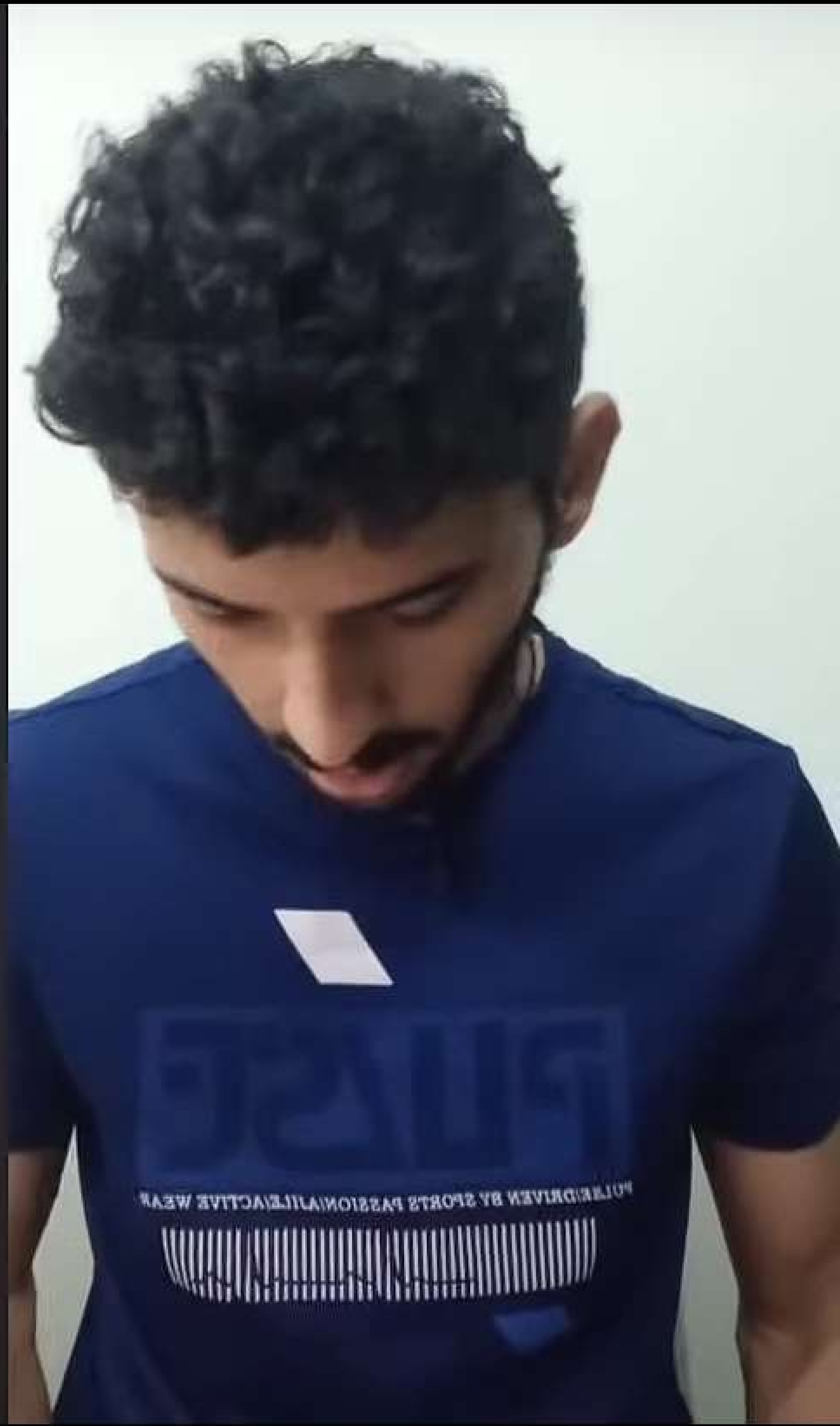
\# $\mathbb{R}^3$

(i) $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(ii) Z & $\mathbb{R}^3$ itself

(iii) lines passing through origin

(iv) planes passing through origin



1:17:38 / 4:12:28 • Subspace ›

Vector spaces in One shot | All concepts and Examples

linear span of a subset

linear Combination

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \qquad S = \{v_1, v_2, \dots v_n\}$$

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad \mathbb{R}^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

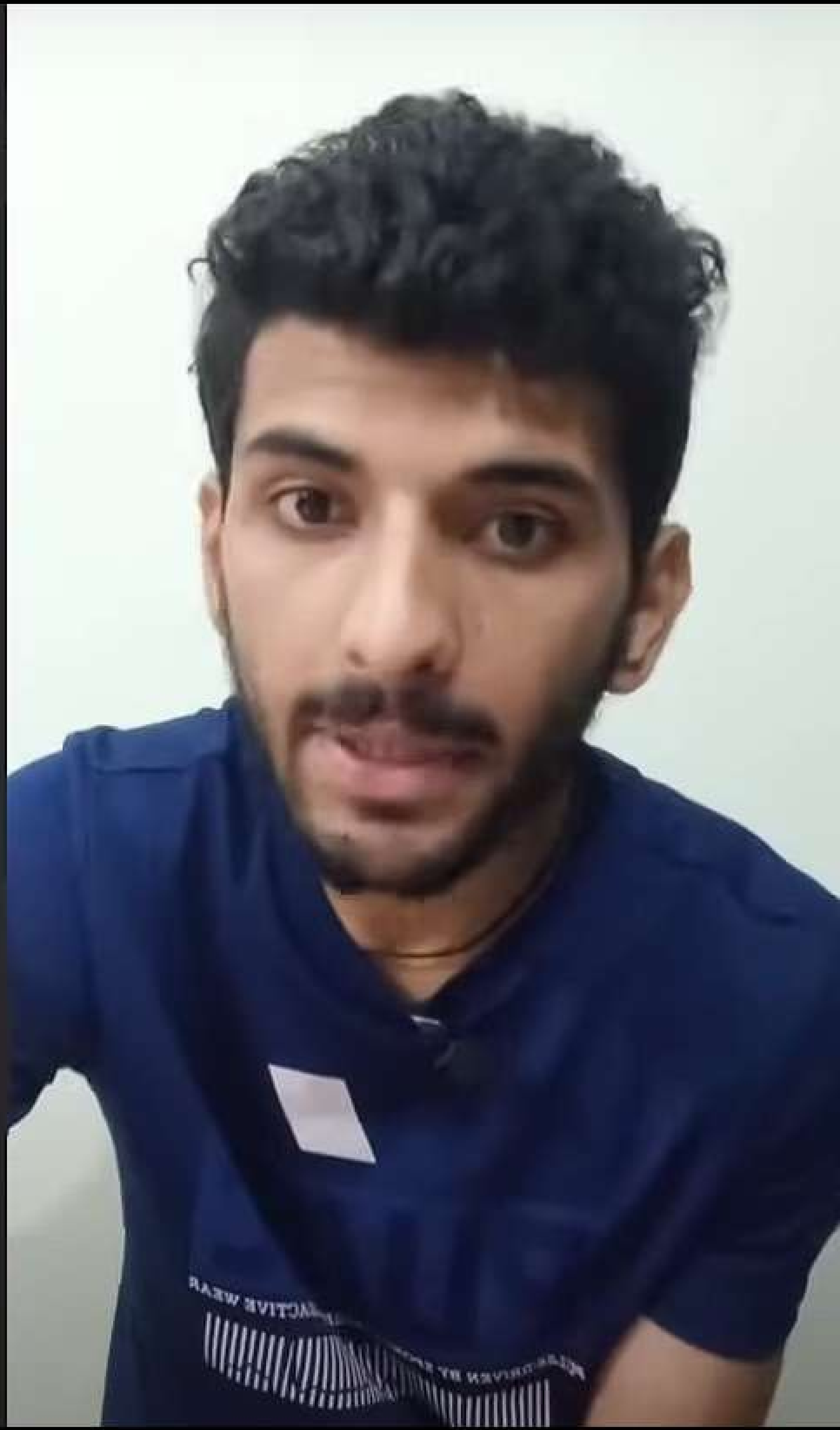
$\boxed{\text{finitely generated vector space (FDVS)}}$ if S is finite then V(F)

linear span of Null set is $\{0\}$

linear Independent :

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

iff c_1, c_2 ____ are all zero



1:32:34 / 4:12:28 • Linear span ›

Vector spaces in One shot | All concepts and Examples

Basis of a Vector space

Set S is called basis of $V(F)$ if

(i) S spans V

(ii) S is linearly Independent.

NOTE: ① no. of Vectors in basis = $\dim(V)$

VImp ② The zero vector is LD.

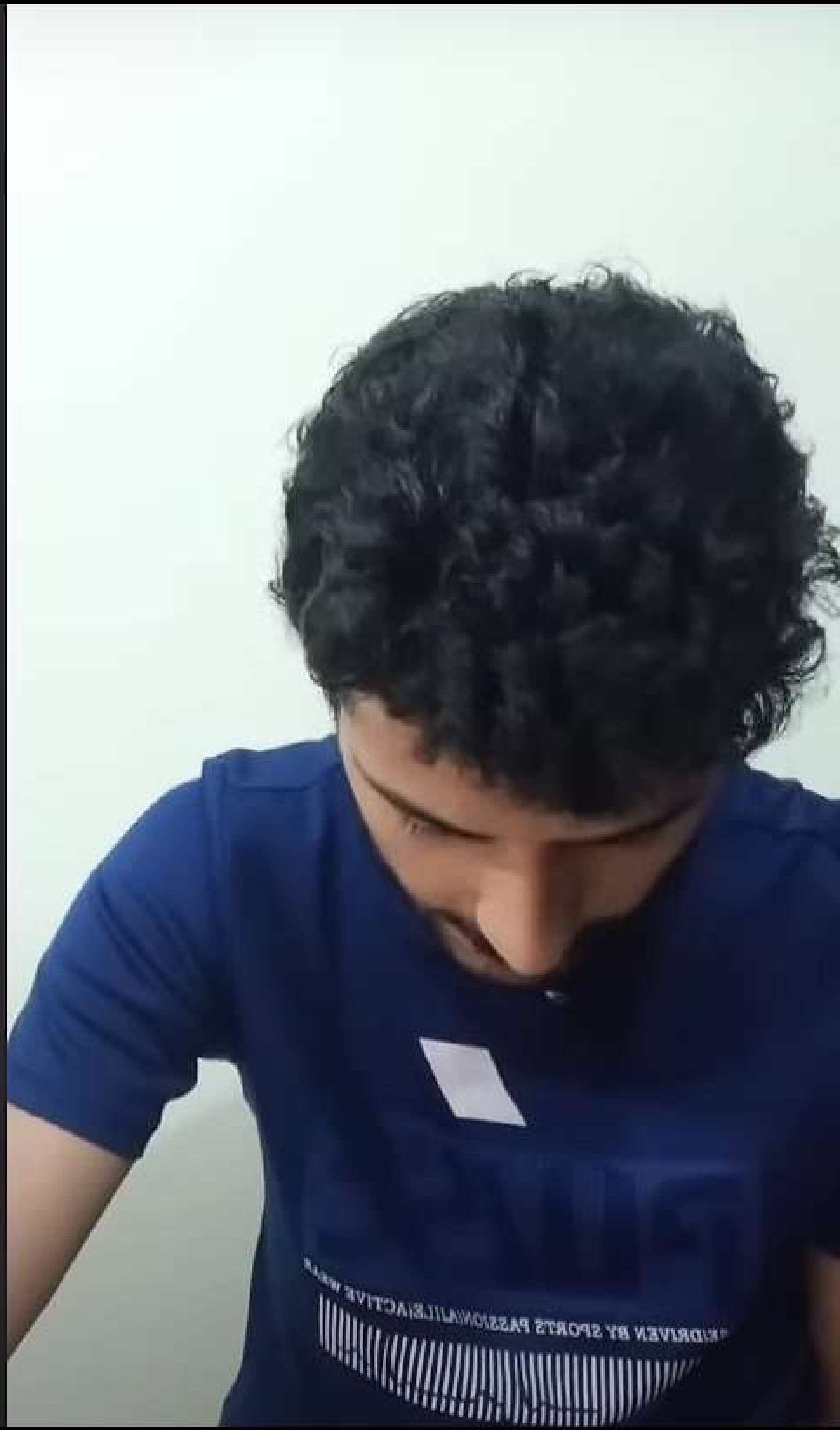
$$\alpha 0 = 0 \qquad \alpha \neq 0$$

③ Any non zero vector is LI.

④ Every superset of LD set is LD

⑤ Every Subset of LI set is LI

⑥ Any set Containing zero vector i



1:41:32 / 4:12:28 • Basis of a vector space ›

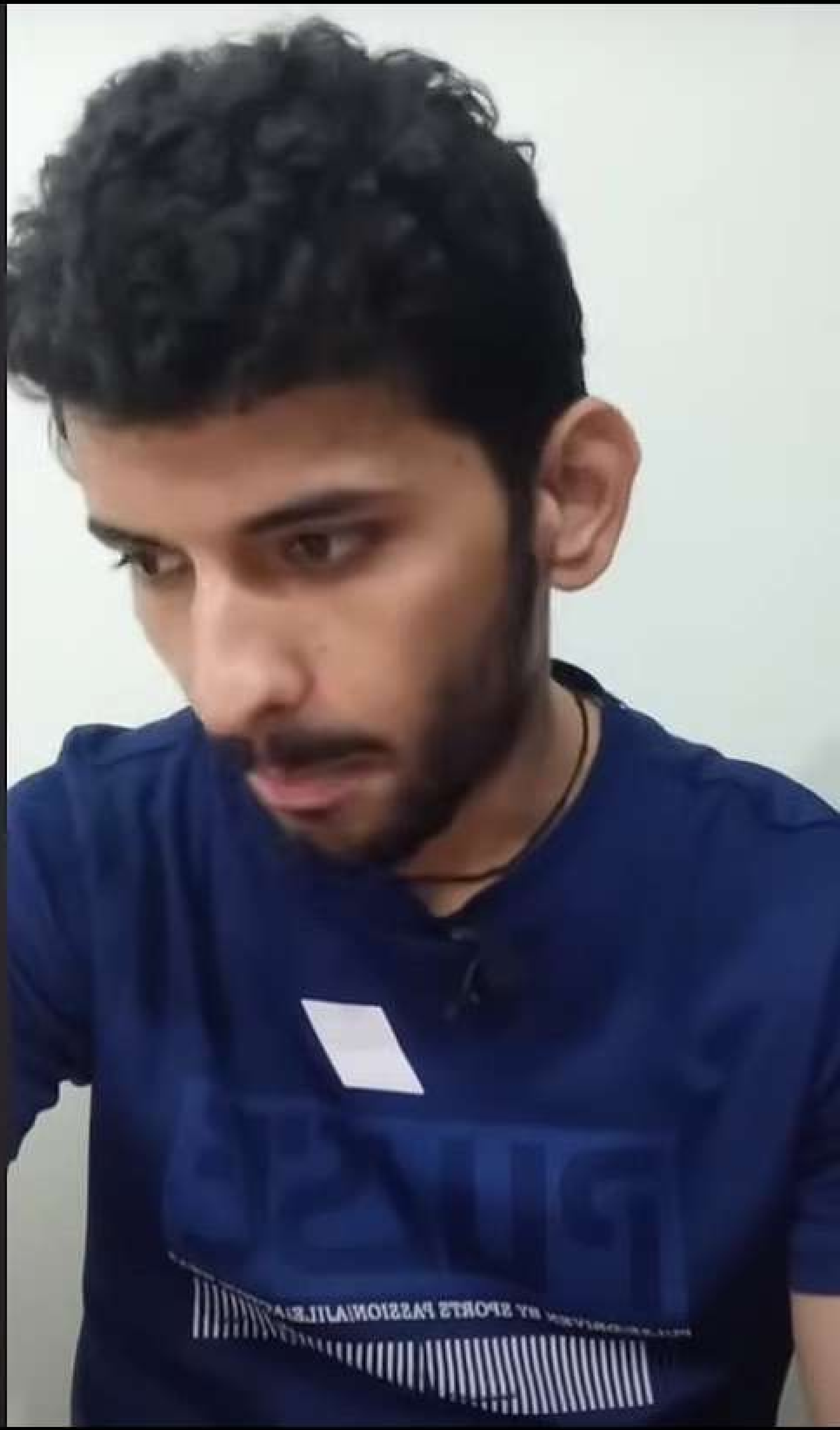
Vector spaces in One shot | All concepts and Examples

Subspaces Associated with matrix

$[A_{m\times n}]$

$C(A) = \text{Span}(C_1, C_2, \ldots C_n)$

1. Column space $C(A)$
2. Row space $R(A)$
3. Null space $N(A)$
4. Null space of A^T, $N(A^T)$



1:43:34 / 4:12:28 • Subspace associated with matrix ›

Vector spaces in One shot | All concepts and Examples

Find the row space and column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

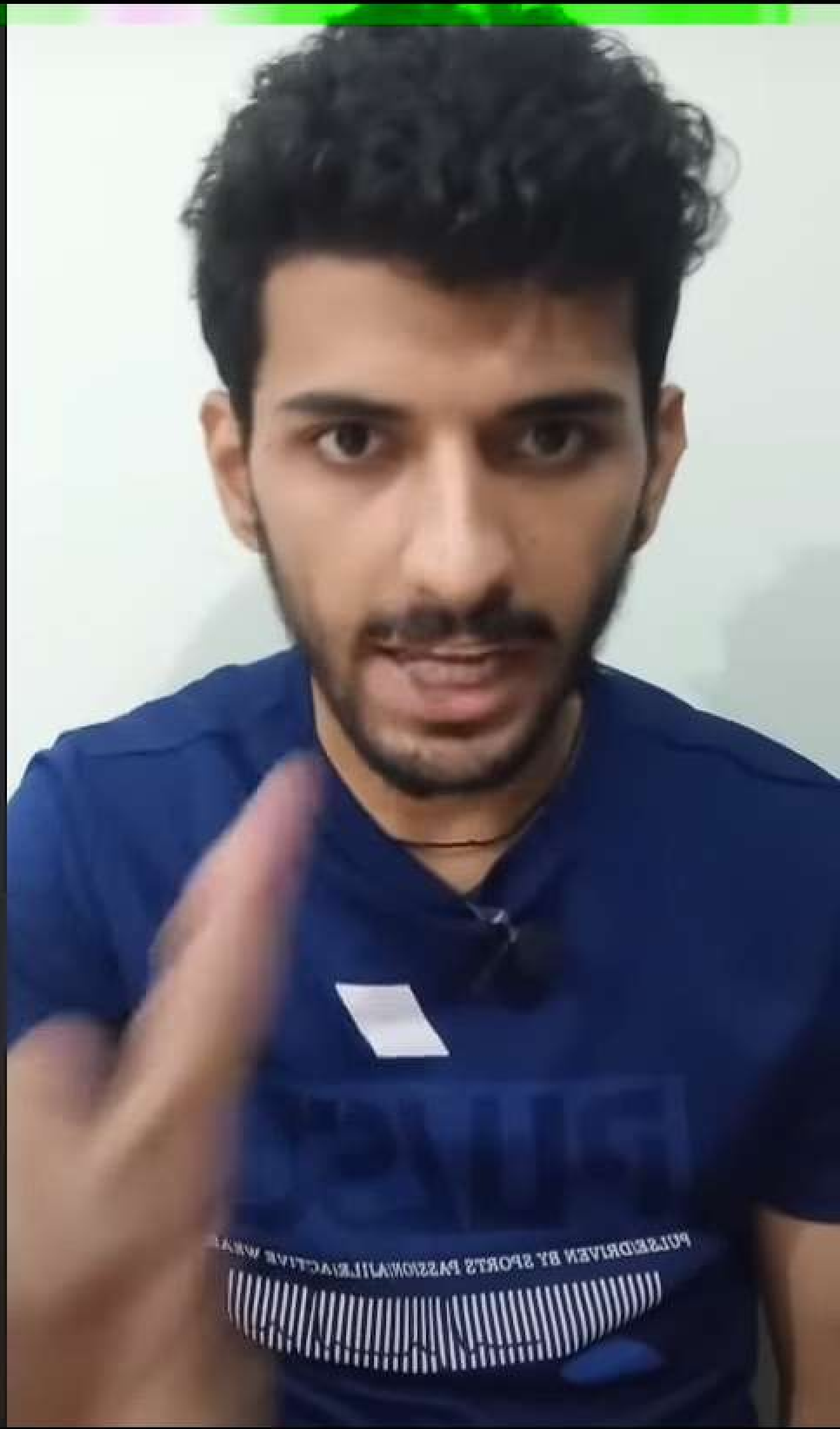
Basis of $R(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \end{bmatrix} \right\}$

Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$

dim of $R(A) = 1$

Rank of $A = 1$

$$A = \begin{bmatrix} c & 3c & 4c & 3c \end{bmatrix}$$



Vector spaces in One shot | All concepts and Examples

Find all solution of the following system of equations.

$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

$R_4 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{1}{3} R_2$

$R_3 \to R_3 - 5R_1$

$R_4 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -1 & -8 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$x + 3y + 13z + w = 0$$
$$-y - 8z - 3w = 0$$

$z = c_1$

$w = c_2$

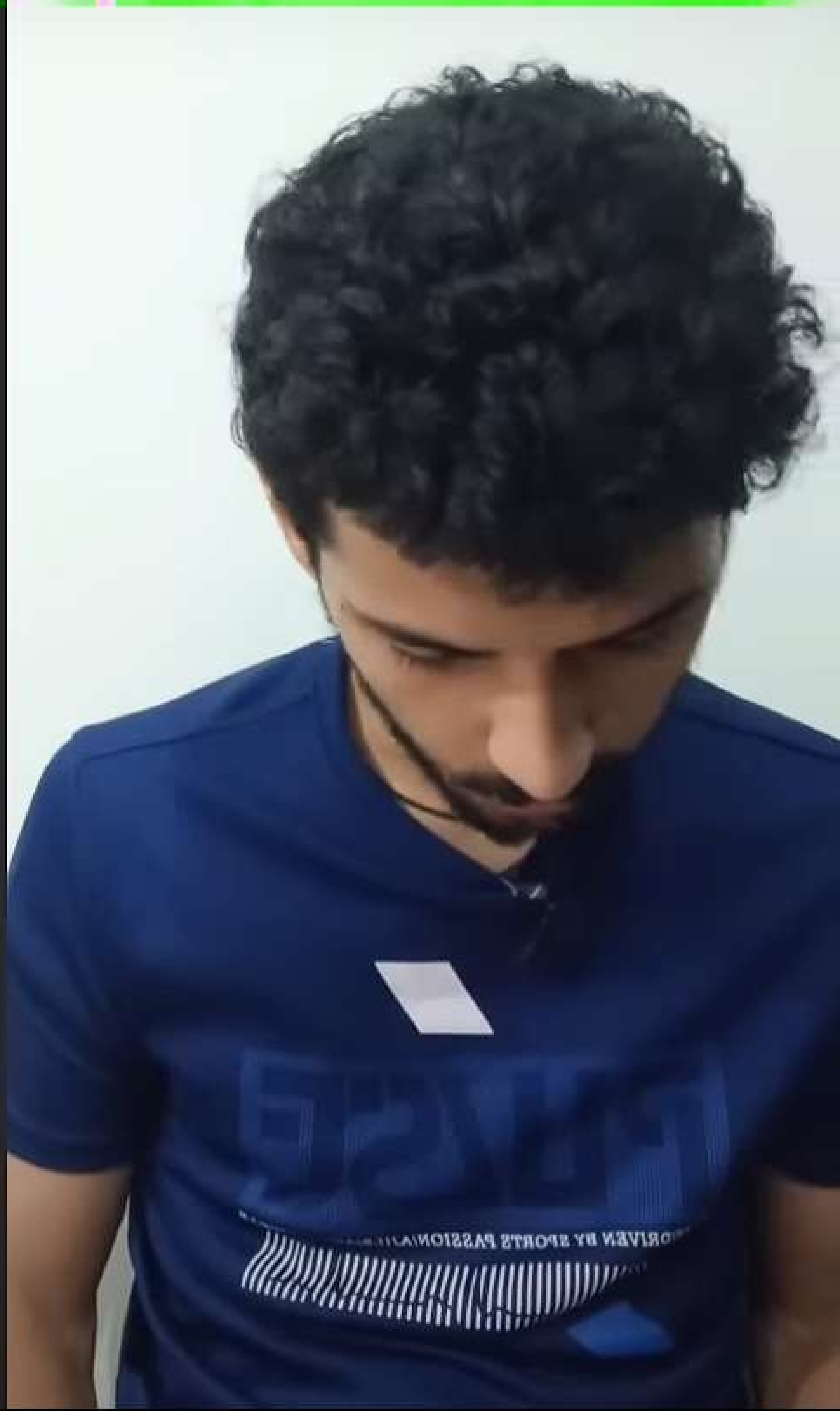
$y = -8c_1 - 3c_2$

$x + 3(-8c_1 - 3c_2) + 13c_1 + c_2 = 0$

$x = 11c_1 + 8c_2$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11c_1 + 8c_2 \\ -8c_1 - 3c_2 \\ c_1 + 0 \\ 0 + c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 11c_1 \\ -8c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 8c_2 \\ -3c_2 \end{bmatrix}$$



2:18:34 / 4:12:28 • Null space and dhasu questions ›

Find all solution of the following system of equations.

$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y + 13z + 3w = 0$$

$R_4 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{1}{3} R_2$

$R_3 \to R_3 - 5R_1$

$R_4 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -1 & -8 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$x + 3y + 13z + w = 0$$
$$-y - 8z - 3w = 0$$

$$z = c_1$$
$$w = c_2$$

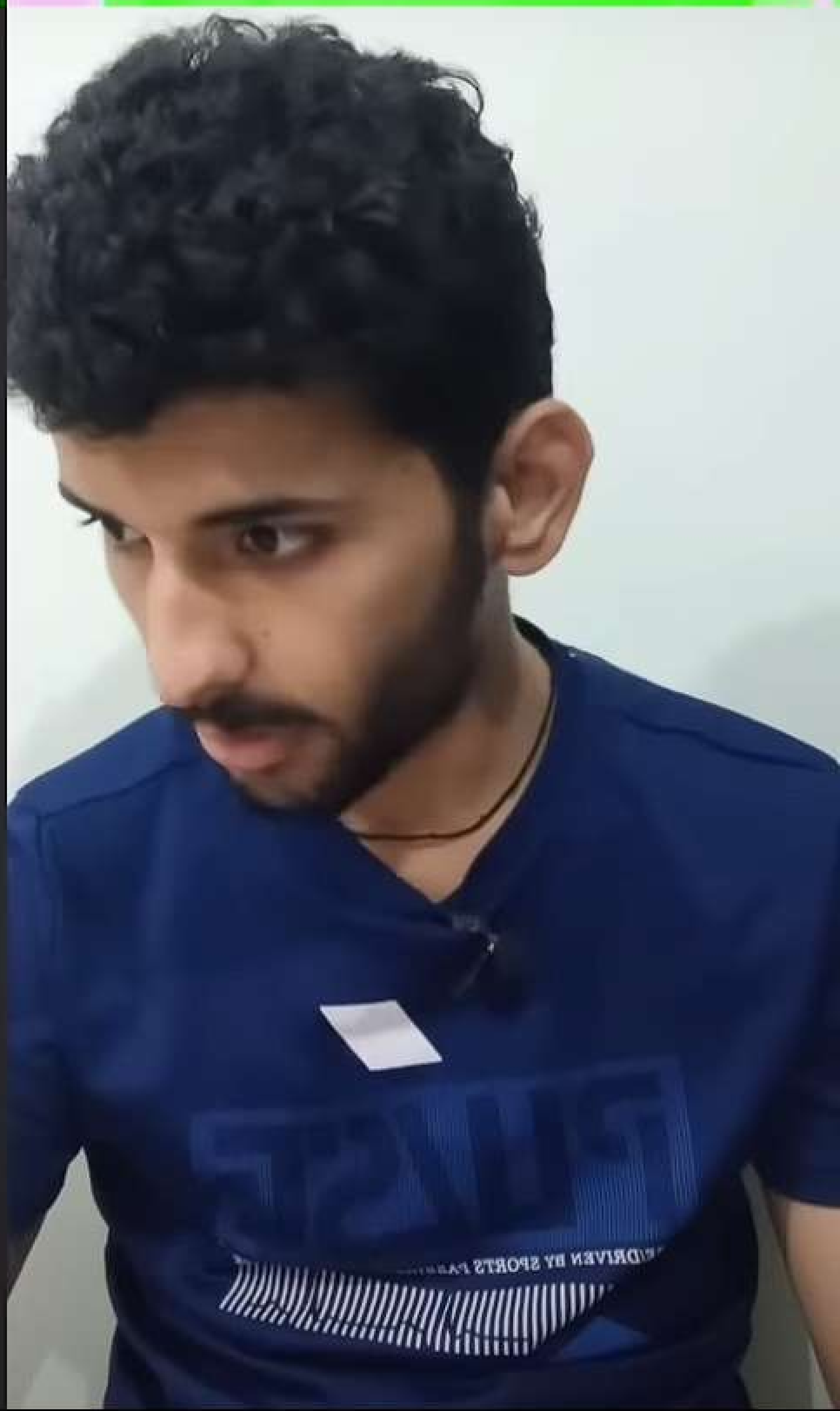
$$y = -8c_1 - 3c_2$$

$$x + 3(-8c_1 - 3c_2) + 13c_1 + c_2 = 0$$

$$x = 11c_1 + 8c_2$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11c_1 + 8c_2 \\ -8c_1 - 3c_2 \\ c_1 + 0 \\ 0 + c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 11c_1 \\ -8c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 8c_2 \\ -3c_2 \end{bmatrix}$$



Vector spaces in One shot | All concepts and Examples

To exit full screen, press Esc

Rank of a matrix

$\Downarrow$

No. of Non-zero Rows

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\text{Rank}(A) = 2$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\text{Rank}(B) = 1$

\# $|A| = 0$ $\quad O(A) = n$

$\text{Rank}(A) < n$

\# $|A| \neq 0$ $\quad O(A) = n$

$\text{Rank}(A) = n$

\# $A \neq$ null matrix

$\text{Rank}(A) \geq 1$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

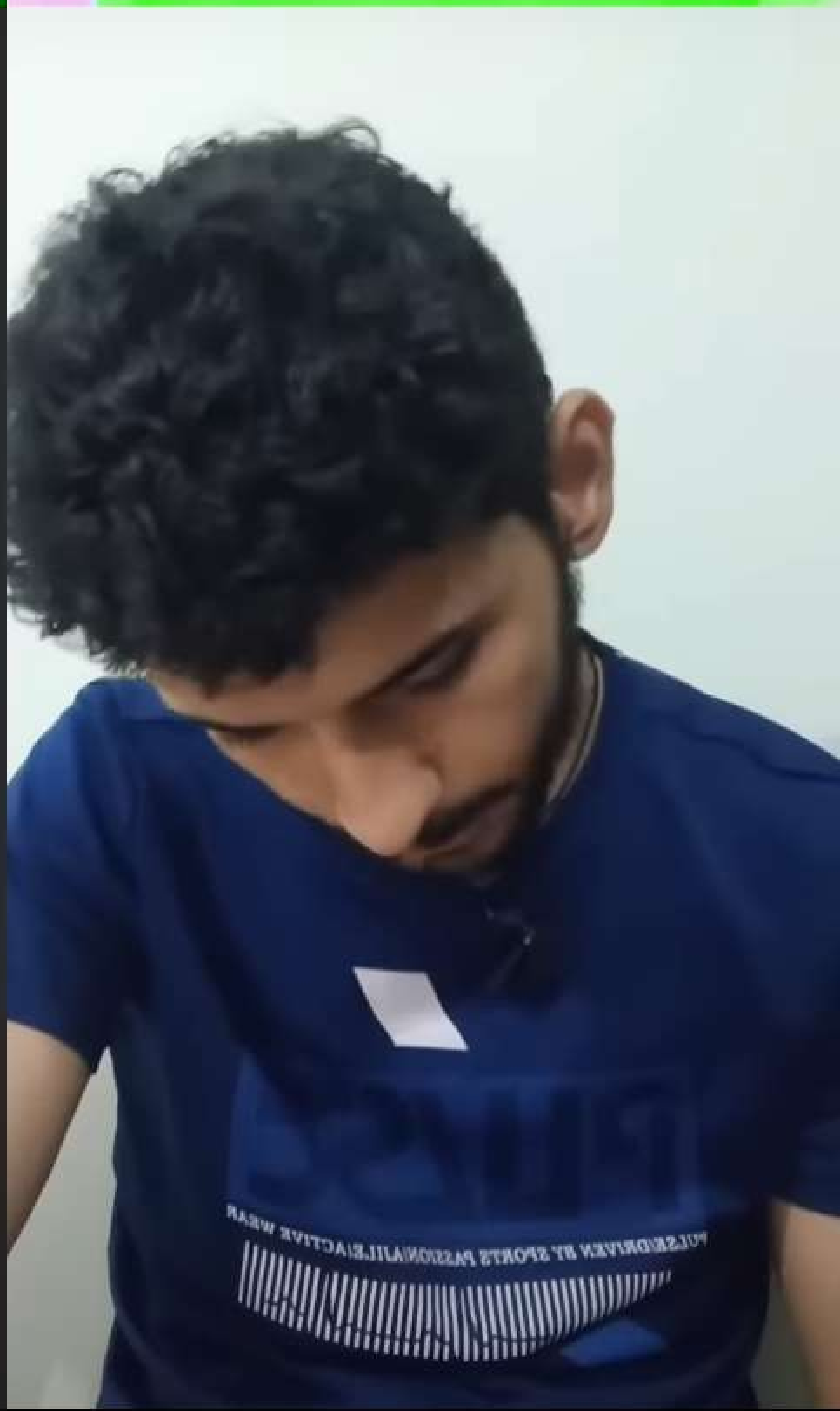
$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & -2 & -4 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$\text{Rank}(B) = 3$



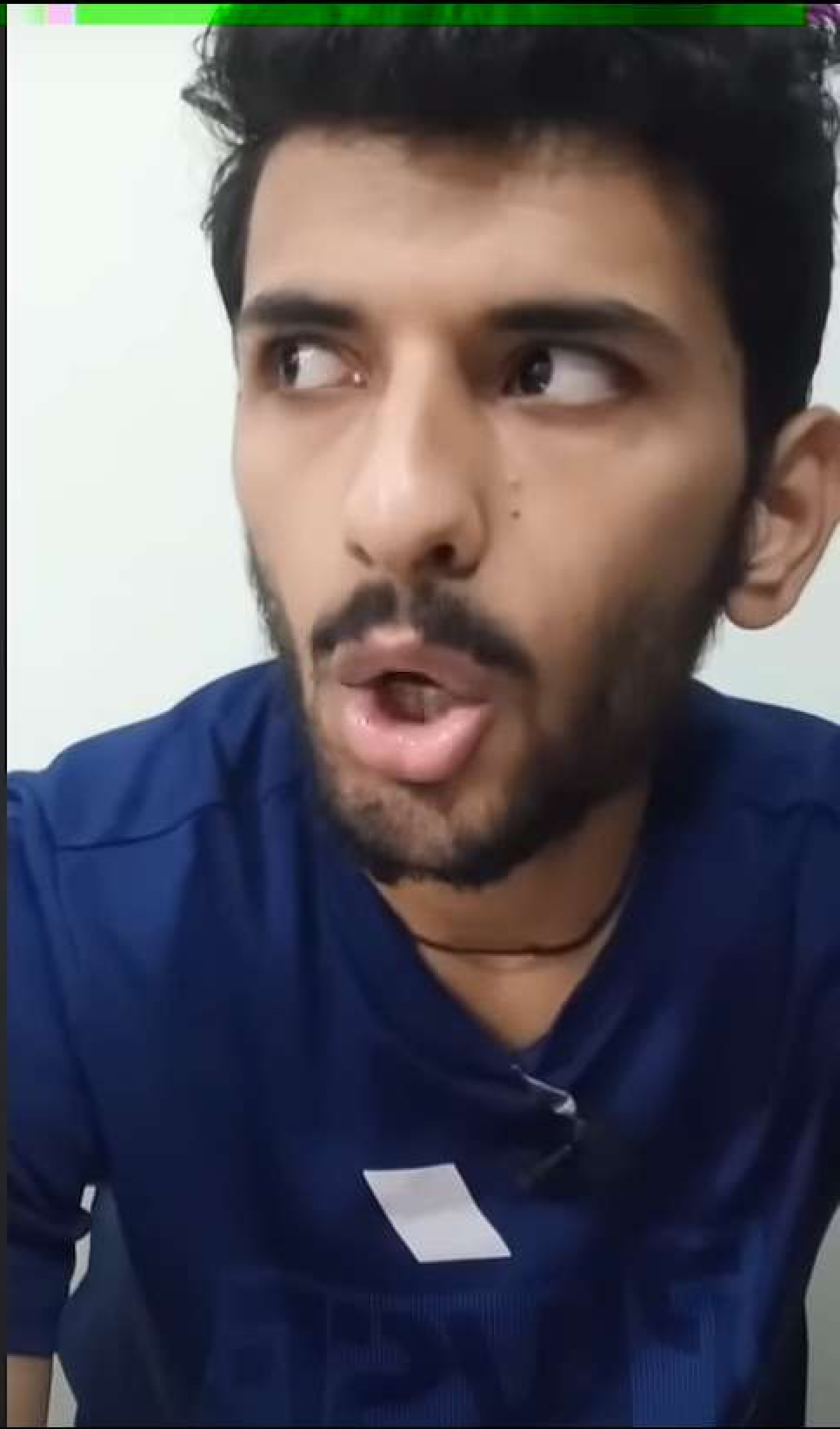
2:23:17 / 4:12:28 • Null space and dhasu questions ›

Vector spaces in One shot | All concepts and Examples

Condition of Consistancy theorem

$\boxed{\text{Rank}[A:B] = \text{Rank}[A]}$ Consistent

$\text{Rank}[A:B] > \text{Rank}[A]$ Inconsistent



2:32:53 / 4:12:28 • System of non-homogeneous equations ›

Show that the equation

$$x+y+z=-3$$
$$3x+y-2z=-2$$
$$2x+4y+7z=7$$

and not consistent.

$$\text{Rank}[A:B] > \text{Rank}[A]$$

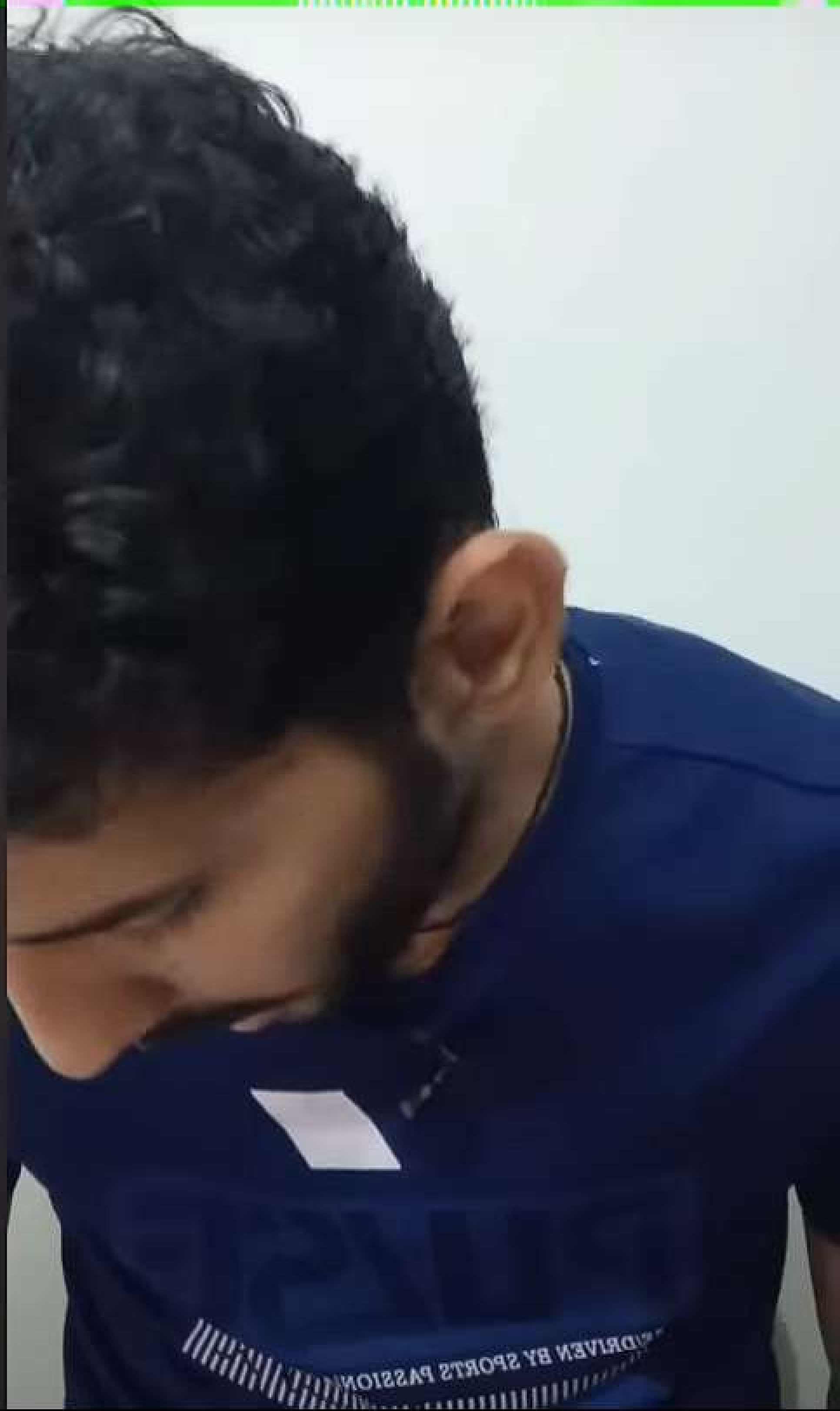
$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_3 \to R_3 - 2R_1 \qquad R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$



Vector spaces in One shot | All concepts and Examples

Solve completely the following system of linear equations

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1 \qquad R_3 \to R_3 + 3R_1$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{array}\right] \qquad R_3 \to 3R_3 - 7R_2$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & \cancel{-38} & -76 \end{array}\right] \qquad R_3 \to -\frac{1}{38} R_3$$

$1 \qquad 2$

$z = 2$

$3y + 5z = 16$

$3y + 10 = 16$

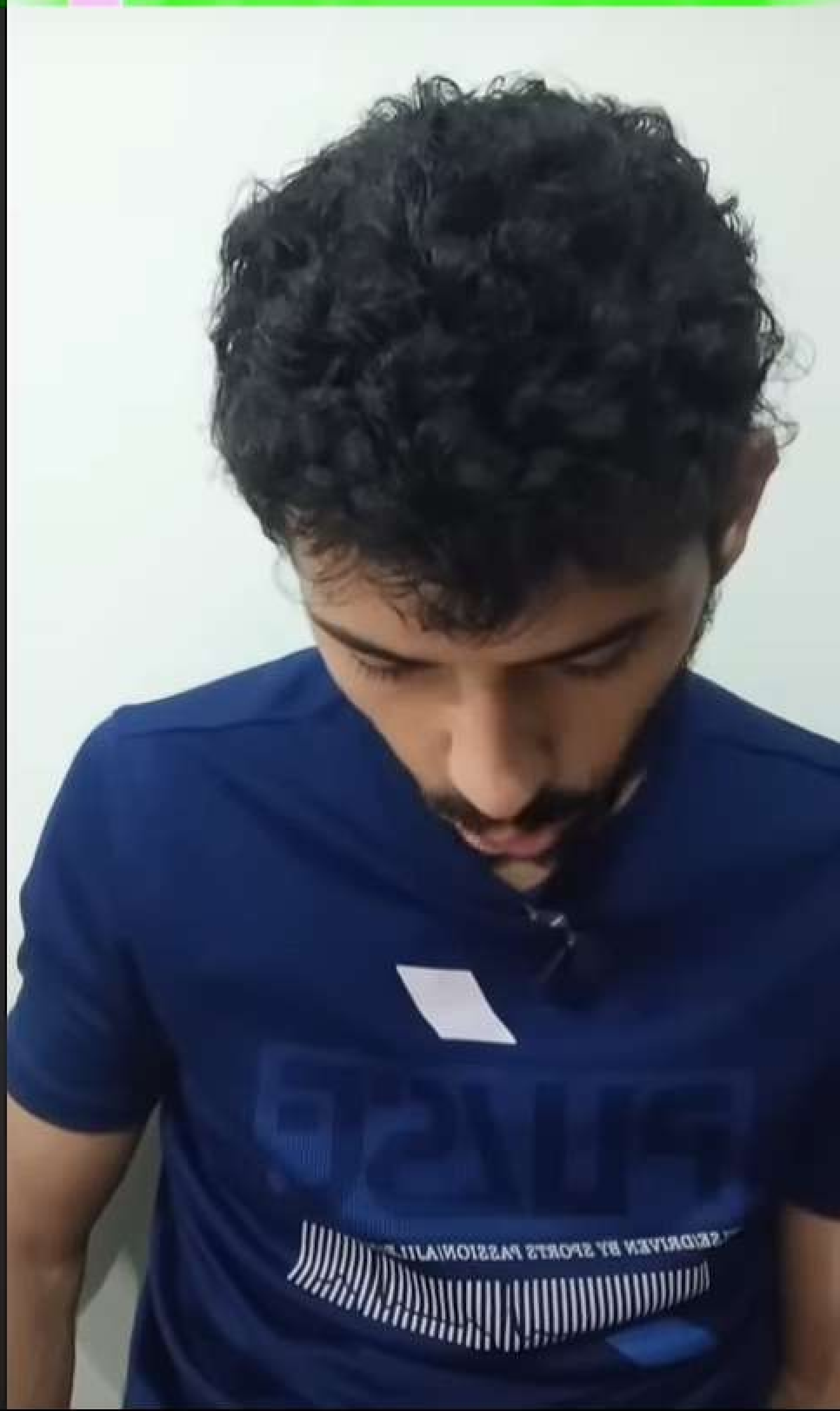
$3y = 6$

$y = 2$

$-x + 2y + z = 4$

$-x + \cancel{4} + 2 = \cancel{4}$

$x = 2$



2:40:24 / 4:12:28 • Questions ›

Vector spaces in One shot | All concepts and Examples

Investigate for what value of λ, μ the simultaneous equations have

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution (iii) an infinite number of solution.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 16 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_3 \to R_3 - R_2 \qquad R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) $\lambda = 3 \qquad \mu \neq 10$

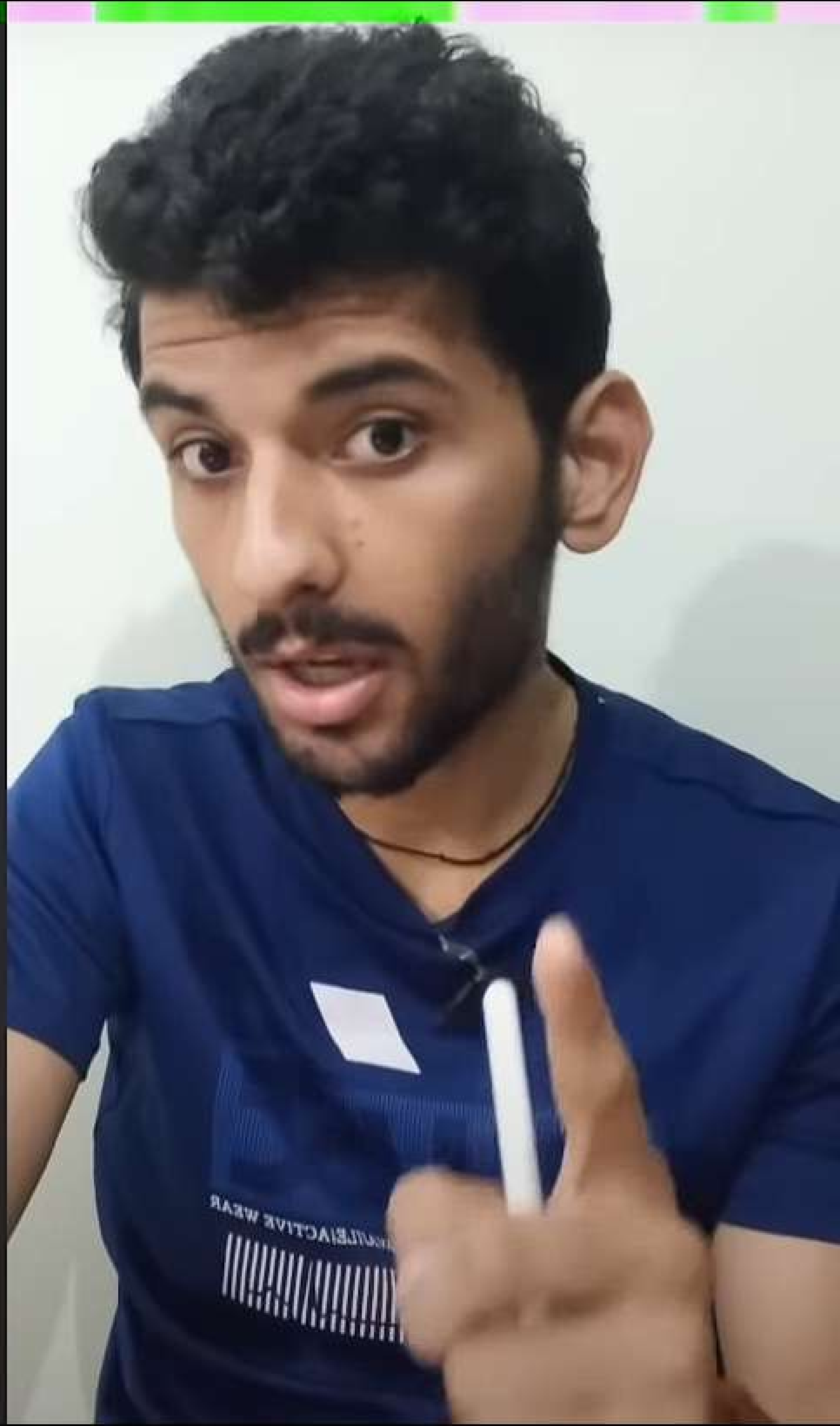
Rank $[A:B] >$ Rank $[A]$

No Solution

(ii) $\lambda \neq 3$

Unique Solution

(iii) $\lambda = 3 \qquad \mu = 10$



In $V(\mathbb{R})$ where $V = P_3(x)$. Let

$$v_1 = 1 + x + x^3$$
$$v_2 = 1 + x^2 - x^3$$
$$v_3 = x + x^2 + x^3$$
$$v_4 = 1 + 2x + 3x^3$$

Prove that v_1, v_2, v_3, v_4 are linearly independent.

$$\{1, x, x^2, x^3\} = V(\mathbb{R})$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

$R_4 \to R_4 - R_1 \quad R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & 2 \end{bmatrix}$$

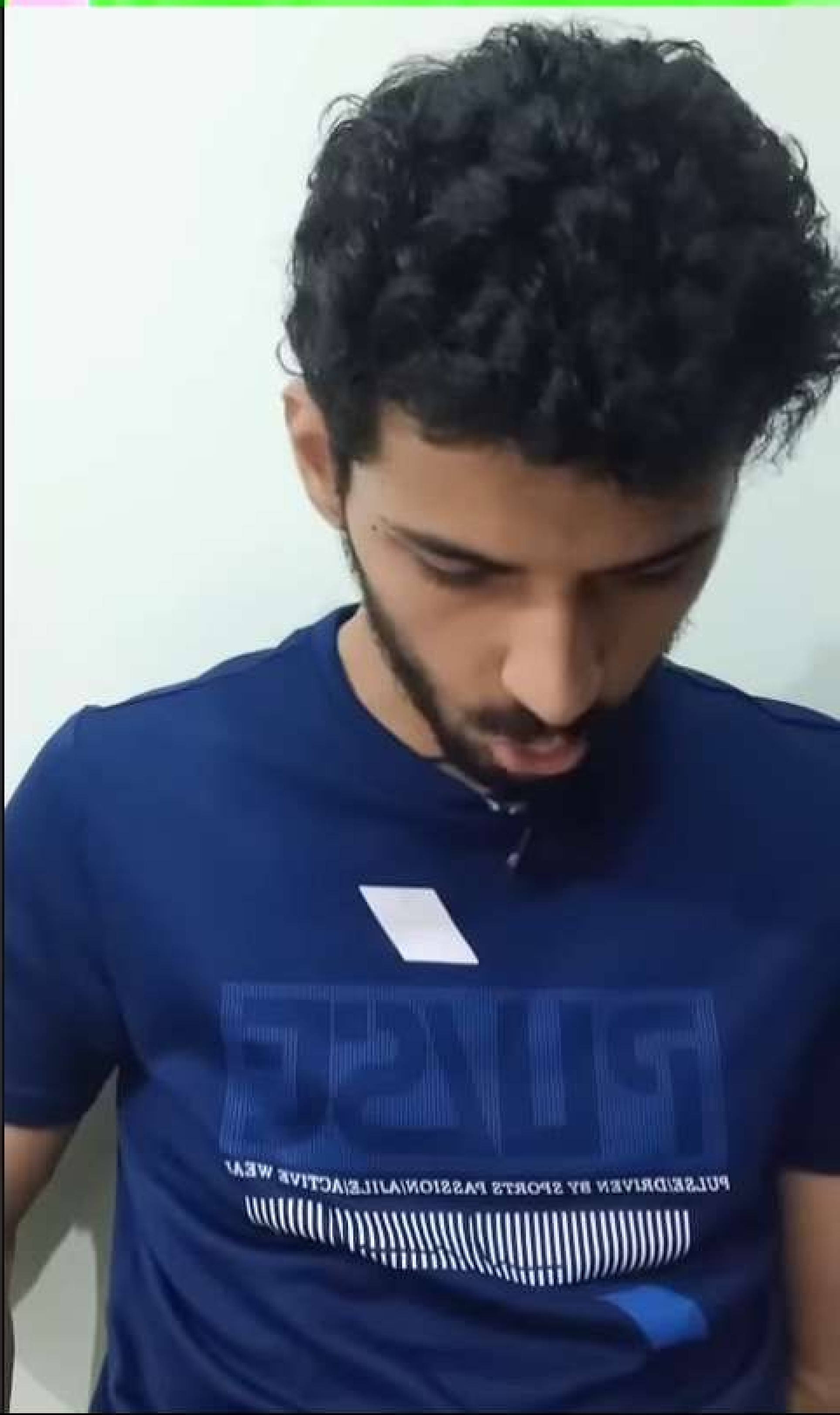
$R_4 \to R_4 + 2R_3 \quad R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to 2R_4 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & & & \end{bmatrix}$$



Vector spaces in One shot | All concepts and Examples

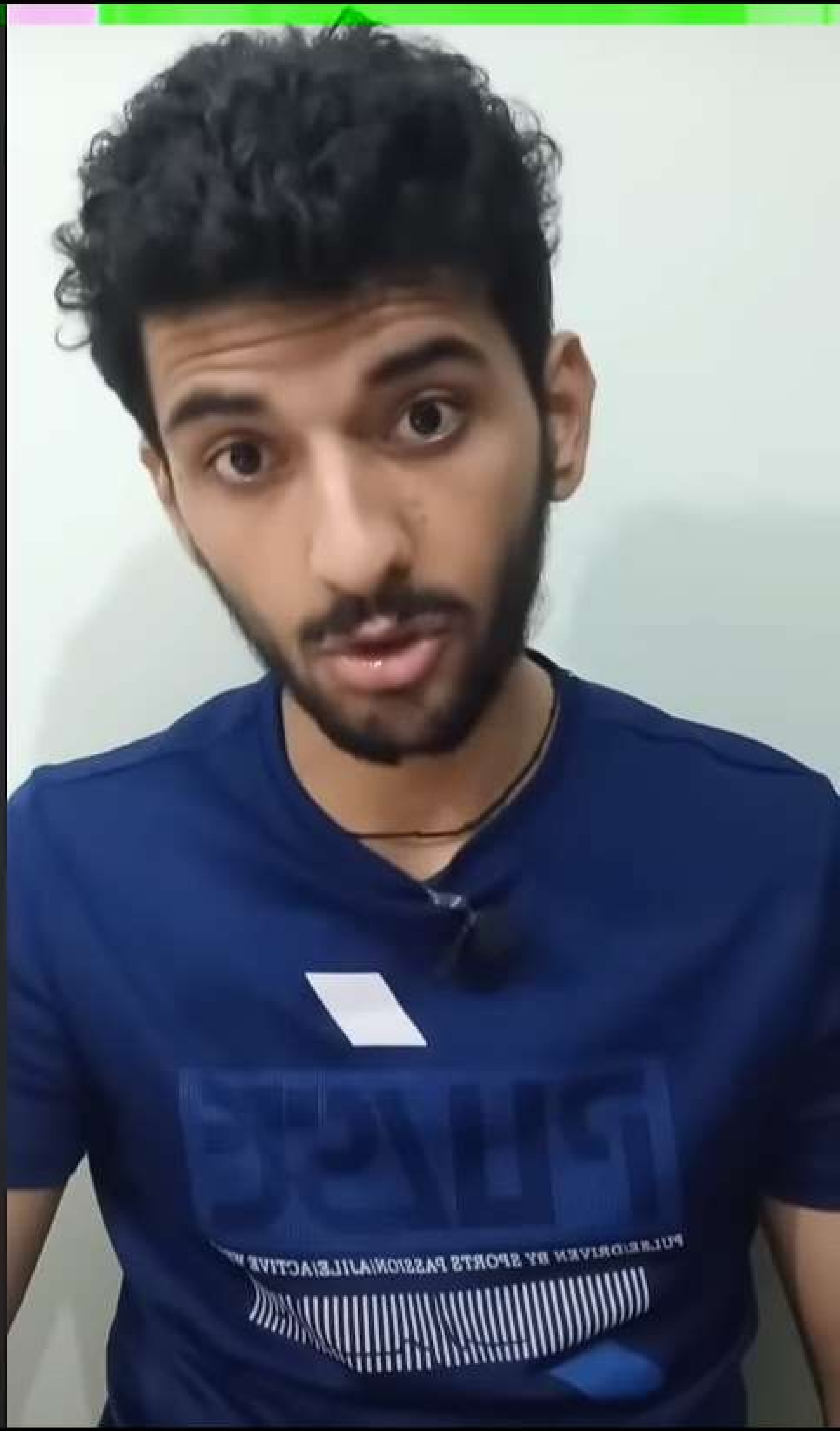
For what value of a in the set $\{(1, 1, 1+a), (2, 2+a, 2+a), (3+a, 3+a, 3+a)\}$ linearly independent.

$$\begin{bmatrix} 1 & 2 & 3+a \\ 1 & 2+a & 3+a \\ 1+a & 2+a & 3+a \end{bmatrix}$$

$R_3 \to R_3 - (1+a)R_1 \qquad R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3+a \\ 0 & a & 0 \\ 0 & -a & -a(3+a) \end{bmatrix}$$

LI if $a \neq 0$ & $a \neq -3$



Vector spaces in One shot | All concepts and Examples

Check for the Linear independence the polynomials $i + x + x^2$, $-(1+i)-2x+2ix^2$, $x-x^2$ over $\mathbb{C}$.

$$\begin{bmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}$$

$$\alpha_1(i+x+x^2) + \alpha_2(-(1+i)-2x) + \alpha_3(x-x^2) = 0$$

$\alpha_1, \alpha_2, \alpha_3$

$R_3 \to R_3 + iR_1 \qquad R_2 \to R_2 + iR_1$

$$\begin{bmatrix} i & -1-i & 0 \\ 0 & -1-i & 1 \\ 0 & i+1 & -1 \end{bmatrix}$$

LD

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} i & -1-i & 0 \\ 0 & -1-i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$