Mediated Information Design with Money Burning for Commitment Power*

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Abstract

This paper explores information design problems with the money-burning tactic for commitment in mediated communication. In our model, the sender possesses state-independent preferences and is required to design a communication mechanism that incorporates both a signaling scheme and a money-burning scheme. We analyze the sender's maximum equilibrium payoff, with and without budget constraints, which has clear geometric interpretations and links to two types of robust Bayesian persuasion. The communication model applies directly to Web 3.0 communities, showing that the presence of commitment value is equivalent to its presence in traditional societies but strictly reduced in almost all scenarios.

Keywords: Information Design, Mechanism Design, Mediator, Cheap Talk, Bayesian Persuasion, Commitment;

JEL Classification: D82, D83.

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1 Introduction

Information design problems arise in asymmetric information environments, where one party (call him Sender) has more information than the other party (call her Receiver). Sender can use this advantage to persuade Receiver by strategically designing the information transmission. Although Bayesian persuasion (BP) Kamenica and Gentzkow [2011] addresses the optimal information design problem, it assumes that Sender possesses full commitment power and can commit ex ante to any information structure for Receiver. In the limited commitment environment, one of the predominant protocols is mediated communication (MD) Myerson [1982], Forges [1986] wherein Sender designs information that is transmitted via intermediary communication devices. In the context of MD, Sender only has to communicate with the mediator who, although unaware of the Sender's private information, assists in committing to a communication mechanism that transmits a message to the Receiver based on the Sender's report following a predetermined rule. Consequently, when he communicates with the mediator, Sender achieves commitment through the strategic design of incentives, meticulously crafting the expected interim payoff induced by the Receiver's actions. In practical application, Sender can further gain the credibility by employing costly messages, which is known as the money-burning tactic Austen-Smith and Banks [2000]. Therefore, it is natural to ask what is the Sender's optimal payoff and the corresponding optimal design when employing a money-burning tactic to enhance the mediated information design. Can the employment of a money-burning tactic enhance the Sender's ex ante payoff?

As a pragmatic illustration, analogous design challenges emerge in the communication dynamics within the Web 3.0 community. In this decentralized, anonymized, and tokenized ecosystem, where reliability is not guaranteed, users interact and communicate through smart contracts within the Blockchain framework. This communication paradigm, as outlined by Drakopoulos et al. [2023], diverges from traditional societal interactions, since smart contracts operate as transparent (and potentially randomized) algorithms, open to scrutiny by all participants. Furthermore, users incur a gas fee for utilizing these smart contracts. Since the Web 3.0 community is also tokenized, Sender has the option to design the gas fee, as proposed by Drakopoulos et al. [2023], or to directly subsidize Receiver with cryptocurrencies through smart contracts to establish credibility. Consequently, for Sender, this constitutes a problem of mediated information design that incorporates money-burning tactics.

In this paper, we study *mediated communication with money-burning mechanism* (or MDMB for short), where in addition to the mediator committing the message-generating process, Sender guarantees the truthfulness of the information sent to the mediator by money-burning mechanism. Sender predetermines the MDMB, which constitutes a set of rules for the mediator to determine the message and the extent of money-burning based on the Sender's report. The message and money-burning account generated by the mediator will be sent to Receiver. We mainly investigate our proposed communication protocol under a substantive assumption: Sender has has state-independent preferences over the Receiver's actions. This transparent-motives assumption simplifies the analysis while retaining substantial real-world economic applications.

Although our model appears to be just an extension of the model of Myerson [1982], Forges [1986] with the addition of a monetary transfer, we cannot use the traditional revelation principle as in auctions to simplify this problem. This is because the money-burning account also contains information about Sender's type which in turn influences the Receiver's actions, an element beyond Sender's control. Consequently, our model embodies the characteristics of mechanism design with limited commitment. To streamline our analysis, we generalize the revelation principle provided by Doval and Skreta [2022] to align with our settings. We demonstrate that the joint design of information and money-burning mechanisms can be decomposed into a sequential process, wherein the message and money-burning account are designed consecutively, contingent upon the message that has been realized. By employing the belief-based approach to this simplified problem, we convert the initial non-convex optimization and equilibrium selection problem into an optimization problem subject to incentive-compatible and Bayes-plausible constraints, thereby leading to the following conclusions.

First, we determine Sender's maximum payoff when employing MBMDs, which is referred to as the value of MBMD. Theorem 1 provides a min-max and max-min characterization of this value, illustrating geometrically that the value of MDMB is the minimum value over all concavification values of Sender's subjective payoff functions[1]. This result has two significant implications. It indicates that the value of MDMB is equivalent to the payoff of cautious Sender with full commitment power.[2] Additionally, it suggests that

[1]The convex combination of truth-adjust welfare functions Doval and Smolin [2024].

[2]Cautious Sender is one who cares about his minimum possible payoff/welfare Doval and Smolin

the value of MDMB aligns with the payoff of Sender facing the worst subjective prior in BP with heterogeneous beliefs.[3] Consequently, these implications provide a justification for robust BP from a theoretical perspective. Moreover, utilizing the equality of min-max and max-min values, we delineate the optimal design of MDMBs.

Moreover, we extend our model to account for a pragmatic constraint wherein Sender possesses a finite budget for money burning. In this generalized model, we demonstrate that the fundamental approach employed in Theorem 1 can be broadly applied. Theorem 2 provides a min-max and max-min characterization of the value of MDMB under a budgetary constraint, illustrating geometrically that this value represents the concavification value of the Sender's generalized subjective payoff function under worst generalized subjective prior, which permits negative probabilities. The structure of the generalized subjective payoff function in Theorem 2 confirms a previous finding that there is an optimal MDMB such that the messages sent by Sender consist of two components: the first component corresponds to the case where Sender employs a costless message for persuasion; the second component corresponds to the case where Sender employs a costly message, the expense of which is maximized to the extent feasible by the constraints of the game for establishing commitment.

Furthermore, by applying our characterization, we investigate the refined value of commitment within Web 3.0 communities, which is defined as the discrepancy between the value of BP and that of MDMB. Additionally, in order to analyze the effect of money-burning tactic to mediator and to elucidate whether commitment value is diminished in Web 3.0 communities relative to traditional societies, we examine the improved value of mediator, which is the difference between the value of MDMB and that of MD. Proposition 6 demonstrates that the refined value of commitment in Web 3.0 communities is positive if and only if the conventional value of commitment is positive, which is the difference between the value of BP and that of cheap talk (CT) Crawford and Sobel [1982], Lipnowski and Ravid [2020]. Moreover, we identify a topological *generic* property of the payoff function set, wherein any action that is Receiver-optimal under certain beliefs is uniquely optimal under beliefs with the same support. Theorem 3 indicates that if the Receiver's payoff function possesses the *generic* property, the improved value of mediator

[2021].

[3]The model of Bayesian persuasion with heterogeneous beliefs is introduced by Alonso and Câmara [2016].

is zero if and only if the refined value of commitment is the same as the conventional value of commitment. Consequently, utilizing Theorem 1 and Theorem 3, we establish that under generic settings, for almost all prior beliefs, there is either no value of commitment or a positive improved value of mediator, which establishes a new benchmark for unreliable Sender communication. (See Proposition 7 and Corollary 5.)

1.1 Related Literature

This paper proposes a novel communication protocol where adopting the money-burning mechanism design enhances the commitment power in a limited commitment environment. Our work contributes to the literature that studies communication protocols under various degrees of commitment power. In addition to the literature mentioned earlier, Min [2021], Lipnowski et al. [2022], which concentrates on the case where Sender's commitment power has a Bernoulli distribution on full commitment and cheap talk; and Lin and Liu [2024], which examines the situation where Sender cannot commit to the message-generating process but he can commit to the marginal distribution of types and messages. Bergemann and Morris [2019] integrates information design problems involving persuasion and mediation. Furthermore, our paper is closely related to studies that analyze the effect of commitment on information design under various communication protocols. Fréchette et al. [2022] investigates the effect of communication with different levels of commitment power through experimental methods. Additionally, Corrao and Dai [2023] comparatively analyzes different communication protocols at various levels of commitment power, but they do not take money burning into account.

The main contribution of this paper is to extend the domain of mediated information design problems. Previous research, such as Salamanca [2021], illustrates the value of Sender's achievements through mediated communication without burning money. Drakopoulos et al. [2023] establish a blockchain system as a mediator, demonstrating that designing costly messages can improve mediated information design under transparent motives, but they do not identify the optimal Sender's communication efficiency in general as we do. Additionally, several studies, including Goltsman et al. [2009], Ivanov [2014], have identified the optimal mediation plan for Receiver. Furthermore, Ivanov [2014] compares the outcomes of mediated communication and cheap talk.

Our paper is also related to the literature on communication with transfers. Some studies discuss cheap talk involving monetary expenditure, Austen-Smith and Banks [2000], Karamychev and Visser [2017], noting that Sender seldom employs a money-burning strategy in such settings, and chooses not to burn money even in state-independent Sender preference environments. However, in our work, money-burning mechanism plays a key role in enhancing Sender's commitment power and thus obtaining better communication efficiency. Kolotilin and Li [2021] investigate the application of monetary transfers in repeated cheap talk settings, while Sadakane [2023] examines a model featuring repeated cheap talk games with monetary transfers from Receiver to Sender. This latter study observes that the equilibrium set in such settings is larger than that of the original long-term cheap talk setting. Corrao [2023] analyzes the mediation market and characterizes the information and market outcomes of the revenue-maximizing mediator and the Sender-optimal mediator. Additionally, several studies focus on Bayesian persuasion involving transferable utility and the cost of information, such as Li and Shi [2017], Bergemann et al. [2018]. Dughmi et al. [2019] explore the case where Sender can enter into contracts prior to persuasion, while Perez-Richet and Skreta [2022] investigate the Receiver-optimal experiment under the condition that Sender can costly falsify his private type.

Another important category of literature related to us is about mechanism design with limited commitment. Liu and Wu [2023] examine the implementation problem in general outcome-contingent settings, which is a more generalized context than ours. Bester and Strausz [2001] show that the revelation principle fails to hold in a limited commitment environment, where the principal cannot fully commit to the outcome induced by the mechanism. Doval and Skreta [2022] provide the general revelation principle for limited commitment mechanism design, where the joint design of information and mechanism can be separated into two steps: first, design the information, and second, design the mechanism based on the information.

2 Model

In Section 2.1, we develop the basic model of the Sender-Receiver game, and in Section 2.2, we introduce the methodology for simplifying the Sender's programming problem.

2.1 Basic Setup

Primitives. We consider a basic game with two players: Sender (he) and Receiver (she). Sender possesses private information θ, which denotes his type and belongs to a finite set Θ. The type θ is drawn according to a prior distribution $\mu_0 \in \Delta(\Theta)$, which is common knowledge between both players. Receiver can choose an action from a finite set A. The payoffs of both players depend on the Receiver's action $a \in A$. Additionally, the Receiver's payoff depends on the Sender's type $\theta \in \Theta$. The Sender's value function of Receiver's action is $v(\cdot) : A \to \mathbb{R}$ and the Receiver's value function is $u(\cdot, \cdot) : A \times \Theta \to \mathbb{R}$. Both players are risk-neutral and fully rational.

Communication with money-burning mechanisms. Before the game, Sender commits a communication with money-burning mechanism (or MDMB) which consists of an input set M, an output message set S, and a mapping $\phi : M \to \Delta(S \times \mathbb{R}_{\geq 0})$. The mechanism prescribes how Sender designs the message and determines the money-burning amount according to his private input. The MDMB is designed by Sender and implemented by a mediator that is trusted by Receiver. The mediator will and is trusted to act fully according to MDMB. When the game starts, the mediator will receive one input message $m \in M$ (and the Sender's money account) from Sender and produce an output message $s \in S$ and a money-burning account $t \geq 0$. The output message s and money t are sent to Receiver and the mediator burns t-value money from the Sender's account. Based on everything she knew, Receiver will select an action. We formally define the MDMB as follows.

Definition 1 (Mediated communication with money-burning mechanisms)**.** *A mediated communication with money-burning mechanism (or MDMB) consists of an input set M, an output set S, and a corresponding mechanism $\phi : M \to \Delta(S \times \mathbb{R}_{\geq 0})$.*

In order to avoid getting involved in tedious mathematical rigor discussions, we restrict attention to the case that M, S, and the support set of ϕ are all finite.

The Sender-Receiver game. Note, the MDMB is the ex ante commitment of Sender through a trusted mediator. The ensuing delineation encapsulates the timeline of the extensive-form game:

Stage 1. Sender commits to the MDMB (M, S, ϕ) with a mediator.

Stage 2. The Sender's type θ is revealed to him according to the prior distribution μ_0.

Stage 3. Sender sends a message $m \in M$ to the mediator, who sends an output message $s \in S$ to the Receiver and burns $t \geq 0$ money from the Sender's account, with probability $\phi(s,t|m)$.

Stage 4. Receiver receives the money burnt by Sender and the message m, updates her belief, and chooses an action $a \in A$.

Stage 5. The Receiver's ex-post payoff is $u(a,\theta) + t$ and the Sender's ex-post payoff is $v(a) - t$.

Remark: In our model, money is not burned, instead, it is transferred from Sender to Receiver. However, this doesn't compromise the model's extensive applicability. Since we consider the Sender's optimal MDMB and Receiver takes the action after being informed, the fundamental nature of monetary transfer is that Receiver is informed about both the output message and the money-burning account regardless of the value of the money burning to Receiver.

To analyze the optimal MDMB for Sender, we apply backward induction to examine the sub-game spanning stage 2 through stage 5 at first, subsequently advancing to the design of the MDMB in stage 1. Let $\mathcal{G}_{(M,S,\phi)}(\mu_0)$ denote the sub-game spanning Stage 2 through Stage 5 under the MDMB (M,S,ϕ). The solution construct for the game $\mathcal{G}_{(M,S,\phi)}(\mu_0)$ is the Perfect Bayesian equilibrium. Therefore, we should introduce the beliefs and strategies before formally introducing the equilibrium.

Beliefs and strategies. The Sender's strategy in $\mathcal{G}_{(M,S,\phi)}(\mu_0)$ prescribes a transition probability $\sigma : \Theta \to \Delta(M)$ where $\sigma(\theta)$ denotes the probability distribution of input messages when the Sender's type is θ. Receiver is only informed about the output message and the subsidy amount. Hence, the output message s and the subsidy amount t form the information set of Receiver. For each information set (s,t), the Receiver's strategy prescribes a transition probability $\alpha : S \times \mathbb{R}_{\geq 0} \to \Delta(A)$. Here, $\alpha(s,t)$ denotes the probability distribution of the actions responding to (s,t). In addition, at each information set (s,t), Receiver must form a belief $\mu : S \times \mathbb{R}_{\geq 0} \to \Delta(\Theta)$, where $\mu(s,t)$ denotes the probability distribution of Sender's types given information set (s,t). We call the triple (σ, α, μ) an assessment.

Equilibrium. In this paper, we use Perfect Bayesian equilibrium (henceforth, PBE) as the solution concept of game $\mathcal{G}_{(M,S,\phi)}(\mu_0)$. An assessment (σ, α, μ) is a PBE if it is sequentially rational and the belief μ satisfies Bayes' rule where possible. We denote the set of PBE of game $\mathcal{G}_{(M,S,\phi)}(\mu_0)$ as $\mathcal{E}[\mathcal{G}_{(M,S,\phi)}(\mu_0)]$. Formally, an assessment $(\sigma^*, \alpha^*, \mu^*)$ is a PBE, $(\sigma^*, \alpha^*, \mu^*) \in \mathcal{E}[\mathcal{G}_{(M,S,\phi)}(\mu_0)]$, if it satisfies following three conditions:

Sender's optimality: for any $\theta \in \Theta$,

$$\sigma^*(\theta) \in \arg \max_{\sigma(\theta)\in\Delta(M)} \sum_{m\in M, s\in S, t\geq 0, a\in A} \sigma(m|\theta)\phi(s,t|m)\alpha^*(a|s,t)(v(a) - t). \tag{1}$$

Receiver's optimality: for any $s \in S, t \geq 0$,

$$\alpha^*(s,t) \in \arg \max_{\alpha(s,t)\in\Delta(a)} \sum_{\theta\in\Theta, a\in A} \mu^*(\theta|s,t)\alpha(a|s,t)(u(a,\theta) + t). \tag{2}$$

Bayes updating: for any $s \in S, t \geq 0$ and $\theta \in \Theta$,

$$\mu^*(\theta|s,t) \sum_{\theta'\in\Theta, m\in M} \mu_0(\theta')\sigma^*(m|\theta')\phi(s,t|m) = \mu_0(\theta) \sum_{m\in M} \sigma^*(m|\theta)\phi(s,t|m). \tag{3}$$

Communication efficiency. Previously, we have elucidated the solution concept for the game spanning stages 2 through 5. We now turn our attention to the optimal MDMB design for Sender in stage 1. Given that Sender seeks to maximize his ex ante expected payoff across all possible PBEs, we can formulate the Sender's optimization problem in the context of MDMB as follows:

$$\begin{aligned} &\sup_{M,S,\phi} \sum_{\theta\in\Theta} \mu_0(\theta) \sum_{m\in M, s\in S, t\geq 0, a\in A} \sigma^*(m|\theta)\phi(s,t|m)\alpha^*(a|s,t)(v(a) - t) \\ &s.t. (\sigma^*, \alpha^*, \mu^*) \in \mathcal{E}[\mathcal{G}_{(M,S,\phi)}(\mu_0)]. \end{aligned} \tag{4}$$

The value of Equation 4 is referred to as the communication efficiency of MDMB, or alternatively, *the value of MDMB*, denoted as $\mathcal{V}^*(\mu_0)$. Therefore, we can compare the values of MDMB and MD Salamanca [2021] to evaluate the improved value of mediator through the money-burning tactic. We can also calibrate the refined value of commitment by comparing the values of MDMB and BP. In Section 5, we further discuss the values of various communication protocols in detail.

2.2 Simplifying the Problem

In this section, we simplify the problem presented in Equation 4, which is originally a complex issue involving non-convex optimization and equilibrium selection. To achieve this, we first explain how to consider our MDMB design problem as a mechanism design problem with imperfect commitment, and then generalize the revelation principle from Doval and Skreta [2022] to our context. We then use our generalized revelation principle and belief-based approach to convert the original complex problem into an optimization problem under incentive-compatible and Bayes-plausible constraints.

Although our communication protocol has a similar structure to the mediated communication proposed by Myerson [1982], Forges [1986], we cannot directly apply their revelation principle, the traditional revelation principle of mechanism design, to simplify our MDMB design problem. This means that when analyzing our model, we cannot directly add report-contingent payments to the incentive-compatible constraints[4] of traditional mediated communication problems. This is because the subsidy also contains information about the Sender's type that Receiver is informed of before the Receiver's final decision-making. Hence, our model is more like a mechanism design problem in a limited commitment environment Bester and Strausz [2001], where the mechanism designer can only control the outcome of the payment part, while the Receiver controls the outcome of the action part.

To manage the mechanism design with limited commitment problems, we have to develop new techniques to simplify the optimization problem Equation 4. Inspired by canonical mechanisms and canonical assessments that simplify mechanism design problems where the principal cannot commit to future mechanisms in Doval and Skreta [2022], we can generalize their revelation principle to our environment. Subsequently, we formally define the canonical MDMBs and canonical assessments under which we can calculate the value of MDMB without loss of generality.

Definition 2 (Canonical MDMBs)**.** *An MDMB is canonical if $M = \Theta$, $S = \Delta(\Theta)$, and there exists a signaling scheme $\pi : \Theta \to \Delta(\Delta(\Theta))$ and a deterministic function $x : \Delta(\Theta) \to \mathbb{R}_{\geq 0}$ such that $\phi(u, x(u)|\theta) = \pi(u|\theta)$ for all $\theta \in \Theta$ and $u \in \boldsymbol{supp}\{\pi(\theta)\}$*[5].

[4] See Salamanca [2021] for the incentive-compatible constraints of transitional mediated communication problems.

[5] Previously, we have assume that ϕ has a finite support, so there $\pi(\theta)$ is also assumed to have a finite support.

In canonical MDMBs depicted in Figure 1, the input sets are the type sets, and the output sets are the sets of distributions of types. The output message contains *all* information transmitted to the Receiver, and the Receiver's subsidy is independent and doesn't provide any additional information about the Sender's type, given the output message. Hence, ϕ in a canonical MDMB can be decomposed into two parts. The first part is a *signaling scheme* π, and the second part is a *money-burning scheme* x, which is contingent on the output message. Henceforth, we will briefly use (π, x) to refer to a canonical MDMB. This decomposition structure is the same as that of the revelation principle in Doval and Skreta [2022].

In a canonical MDMB, the canonical assessment ensures that the Sender's strategy is truthful-telling and the Receiver's posterior belief coincides with the output message.

Definition 3 (Canonical assessments)**.** *An assessment (σ, α, μ) corresponding to a canonical MDMB is canonical if $\sigma(\theta|\theta) = 1$ and $\mu(u, x(u)) = u$ for any $u \in \boldsymbol{supp}\{\pi(\theta)\}$.*

As depicted in Figure 1, to calculate the value of MDMB $\mathcal{V}^*(\mu_0)$, it is without loss of generality to consider canonical MDMBs and the corresponding canonical assessments that are PBEs.

Proposition 1. *For any MDMB (M, S, ϕ) and $(\sigma, \alpha, \mu) \in \mathcal{E}[\mathcal{G}_{M,S,\phi}(\mu_0)]$, there exists a canonical MDMB (π, x) and a canonical assessment $(\sigma^*, \alpha^*, \mu^*) \in \mathcal{E}[\mathcal{G}_{(\pi,x)}(\mu_0)]$ such that the expected payoffs of Sender in both assessments are the same.*

$$\underset{\theta}{\text{Sender}}\ M \xrightarrow{\phi(\cdot|m)} S \times \mathbb{R}_{\geq 0}\ \text{Receiver} \qquad \Leftrightarrow \qquad \underset{\theta}{\text{Sender}}\ \Theta \xrightarrow{\pi(\cdot|\theta)} \Delta(\Theta) \xrightarrow{x(\cdot)} \mathbb{R}_{\geq 0}\ \text{Receiver}$$

Figure 1: Revelation principle

Proposition 1 helps us simplify the equilibrium selection problem and convert the Sender's optimality constraints into incentive-compatible constraints. Moreover, thanks to the belief-based approach Kamenica and Gentzkow [2011], we can convert the problem of analyzing the Bayes updating constraint and the Receiver's optimality constraint into the problem of analyzing the Receiver's posterior belief with a Bayes plausible constraint and the associated impacts on Sender's payoff.

In the subsequent analysis, we concentrate on canonical MDMBs capable of inducing a PBE canonical assessment.

Next, we will go through the steps of belief-based approach and relevant notations. Given any Receiver's posterior belief $\mu \in \Delta(\Theta)$, we can summarize the Receiver's best responses according to μ as the Receiver-optimal set,

$$RO(\mu) \triangleq \big\{\alpha \in \Delta(A)\big|\mathbf{supp}\{\alpha\} \subset \arg\max_{a' \in A} \sum_{\theta \in \Theta} \mu(\theta) u(a', \theta)\big\}.$$

Hence, the associated impact of the Receiver's action on the Sender's payoff can be summarized as a *belief-value correspondence*, $\mathbb{V} : \Delta(\Theta) \rightrightarrows \mathbb{R}$, where $\mathbb{V}$ is the collection of all possible ex-post signaling payoffs for Sender and

$$\mathbb{V}(\mu) \triangleq \big\{q\big|\exists \alpha \in RO(\mu), q = \sum_{a \in A} \alpha(a) v(a)\big\}.$$

Furthermore, suppose $p \in \Delta(\Delta(\Theta))$ denotes the distribution over the Receiver's posterior belief induced by a PBE canonical assessment. Kamenica and Gentzkow [2011] show that the Bayes updating condition is equivalent to a Bayesian plausible condition, which means that p belongs to the following Bayesian plausible set:

$$BP(\mu_0) \triangleq \big\{p \in \Delta(\Delta(\Theta))\big| \int_{\mu} \mu dp(\mu) = \mu_0\big\}.$$

Hence, we can transform the equilibrium constraints in the equilibrium selection problem of Equation 4 into incentive compatibility, obedience, and Bayesian plausibility constraints.

Proposition 2. *A distribution over the Receiver's posterior belief p and the Sender's ex-post signaling payoff $V : \Delta(\Theta) \to \mathbb{R}$ in terms of the posterior belief can be induced by a PBE canonical assessment of a canonical MDMB if and only if p and V fulfill the following conditions:*

Incentive-compatibility: for any $\theta, \theta' \in \Theta$,

$$\int_{\mu} (\frac{\mu(\theta)}{\mu_0(\theta)} - \frac{\mu(\theta')}{\mu_0(\theta')})(V(\mu) - x(\mu))dp(\mu) = 0. \tag{5}$$

Obedience: for any $\mu \in \boldsymbol{supp}\{p\}$,

$$V(\mu) \in \mathbb{V}(\mu). \tag{6}$$

Bayesian Plausibility:

$$p \in BP(\mu_0). \tag{7}$$

By identifying simpler PBEs that yield the same equilibrium outcome, Proposition 2 reduces the equilibrium constraints in Equation 4 to three types of constraints. Furthermore, when we attempt to maximize the programming problem of Equation 4, we can further streamline the obedience constraints.

Corollary 1. *$\mathcal{V}^*(\mu_0)$ can be calculated by following optimization problem.*

$$\begin{aligned}
&\sup_{p \in BP(\mu_0), x} \int_\mu (V(\mu) - x(\mu)) dp(\mu) \\
&\qquad s.t. V(\mu) = \max \mathbb{V}(\mu) \\
&\qquad\qquad \int_\mu \left(\frac{\mu(\theta)}{\mu_0(\theta)} - \frac{\mu(\theta')}{\mu_0(\theta')}\right)(V(\mu) - x(\mu)) dp(\mu) = 0, \forall \theta, \theta' \in \Theta.
\end{aligned} \tag{8}$$

Henceforth, unless specified otherwise, we employ $V(\mu)$ to denote $\max \mathbb{V}(\mu)$.

3 Binary-Type Example

In this section, we employ a straightforward binary-type example to elucidate the primary insight underlying the determination of the MDMB value and its corresponding optimal canonical MDMB.

We consider an advertising problem between a consumer (Receiver) and a salesman (Sender). The consumer faces a binary choice of whether to purchase a product, whose quality is either high (θ^H) or low (θ^L). The salesman has private information about the true quality of the product, while the consumer has a prior belief that the product is high-quality with probability $0 < \mu_0 < \frac{1}{2}$. We assume that the market price of the product is fixed at 5. The consumer's payoff depends on the quality of the product: if she purchases a high-quality product, she receives a feedback of 10; if she purchases a low-quality product, she receives a feedback of 0. The salesman's payoff is determined by the consumer's decision: he receives a payoff of 1 from the producer as his commission for selling the product and receives nothing otherwise.

Hiring a salesman to promote the producer's product is a prevalent practice in online platforms. For instance, in Tiktok, consumers may encounter some advertisements where a vlogger conducts a trial for a certain product to demonstrate its quality. However, consumers don't know whether the product they purchase will have the same quality as the one in the trial. Sometimes, at the end of the vlog, there is a coupon for the audience. In the process of communication between salesman and consumers, consumers can only verify the message-generating process, i.e. observe the whole trial process, but the salesman can still alter the input, i.e. the quality of the product that the consumer buys may differ from the quality of the product in the trial. Hence, we regard the trial and coupon joint design as a kind of MDMB design of the salesman.

In this example, we define the Sender's *interim signaling payoff* of type θ as

$$V_\pi(\theta) \triangleq \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta) V(\mu),$$

or equivalently, suppose π induce distribution of Receiver's posteriors p, $V_p(\theta) \triangleq \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu) dp(\mu)$. Suppose two types θ, θ' of Sender have different interim payoffs, i.e., $V_\pi(\theta) \neq V_\pi(\theta')$. Then, the type of Sender with the lower interim payoff has an incentive to deviate and input the other type. This deviation undermines the trust between Sender and Receiver and results in a loss of benefits from information disclosure of Sender. One way to mitigate this problem is to use burning money, which is a voluntary sacrifice of some benefits by Sender to gain more credibility and can be easily verified by Receiver.

In order to obtain the value of MDMB, we conduct a two-step analysis. The first step is to identify the upper bound of the salesman's expected payoff. The second step is to construct a sequence of MDMBs such that the salesman's payoff in an equilibrium under these mechanisms approaches the upper bound.

The upper bound. We begin by deriving the upper bound of the value of MDMB. Consider a mechanism (π, x) that satisfies the incentive-compatible constraint. In this mechanism, the salesman's payoff is $V_\pi(\theta^H) - \sum_{\mu \in \mathbf{supp}\{\pi(\theta^H)\}} \pi(\mu|\theta^H) x(\mu)$, which is equal to $V_\pi(\theta^L) - \sum_{\mu \in \mathbf{supp}\{\pi(\theta^L)\}} \pi(\mu|\theta^L) x(\mu)$. Since $x(\mu) \geq 0$, it follows that

$$\mathcal{V}^*(\mu_0) \leq \max_\pi \{\min\{V_\pi(\theta^H), V_\pi(\theta^L)\}\} \leq \min\{\max_\pi V_\pi(\theta^H), \max_\pi V_\pi(\theta^L)\}.$$

According to the method of Kamenica and Gentzkow [2011], we can geometrically characterize $\max_\pi V_\pi(\theta)$. To this end, we define type θ's *share of ex-post payoff* $V(\mu)$

given the posterior belief μ as

$$\hat{V}_\theta(\mu) \triangleq \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu),$$

which implies that each type in the support of μ receives a share $\frac{\mu(\theta)}{\mu_0(\theta)}$ of the ex-post payoff $V(\mu)$ given the posterior belief μ. If we denote the concave envelope of function f by $cav(f)$, we can succinctly write that $\max_\pi V_\pi(\theta) = cav(\hat{V}_\theta)(\mu_0)$. The geometric illustrations of $\max_\pi V_\pi(\theta^H), \max_\pi V_\pi(\theta^L)$ are given in Figure 2. Hence, we obtain $\mathcal{V}^*(\mu_0) \leq cav(\hat{V}_{\theta^L})(\mu_0) = \frac{\mu_0}{1-\mu_0}$.

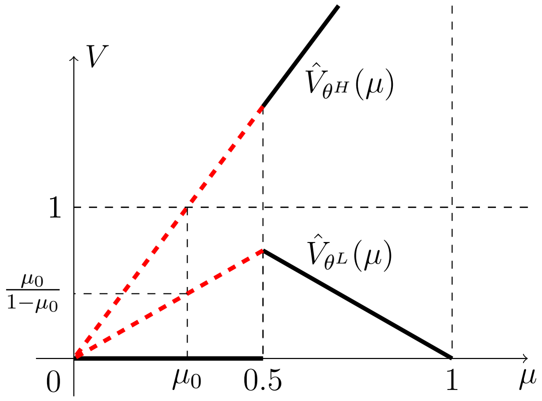

Figure 2: $\hat{V}_{\theta^H}(\mu)$ and $\hat{V}_{\theta^L}(\mu)$.

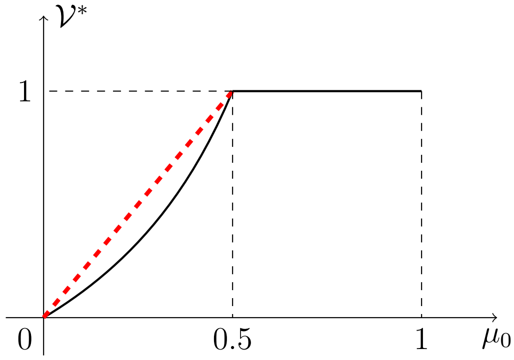

Figure 3: $\mathcal{V}^*(\mu)$.

Construction of MDMB. We construct a canonical MDMB that attains the upper bound. The signaling scheme depicted in Figure 2 is characterized by $M = \{\frac{1}{2}, 0\}$ and $\pi(\frac{1}{2}|\theta^H) = 1, \pi(\frac{1}{2}|\theta^L) = \frac{\mu_0}{1-\mu_0}, \pi(0|\theta^L) = \frac{1-2\mu_0}{1-\mu_0}$. Subsequently, we introduce a money burning message "1". For any $\delta > 0$, the modified signaling scheme is defined as $M^* = \{1, \frac{1}{2}, 0\}$, $\pi^*(\frac{1}{2}|\theta^H) = 1 - \delta, \pi^*(1|\theta^H) = \delta$ and $\pi^*(\frac{1}{2}|\theta^L) = (1-\delta)\pi(\frac{1}{2}|\theta^L), \pi^*(0|\theta^L) = 1 - \pi^*(\frac{1}{2}|\theta^L)$. The associated money-burning scheme is $x(1) = \frac{1-(2-\delta)\mu_0}{\delta(1-\mu_0)}, x(\frac{1}{2}) = t(0) = 0$.This MDMB remains canonical and incentive-compatible for any $\delta > 0$. The salesman's expected payoff is $\frac{\mu_0}{1-\mu_0}(1-\delta)$. As $\delta \to 0^+$, the salesman's payoff converges to the upper bound of the value of MDMB. We can illustrate the value of MDMB $\mathcal{V}^*(\mu_0)$ for different priors μ_0 in Figure 3.

The characterization of the value of MDMB for binary type space extends beyond this example. We articulate the general findings of binary type space in the subsequent proposition.

Proposition 3. *Suppose Sender has a binary type set $\Theta = \{\theta_1, \theta_2\}$. Then, for any prior distribution $\mu_0 \in \Delta(\Theta)$, we have $\mathcal{V}^*(\mu_0) = \min\{cav(\hat{V}_{\theta_1})(\mu_0), cav(\hat{V}_{\theta_2})(\mu_0)\}$.*

Proof. This proof is relegated to Appendix B. $\square$

4 General Results

This section expands the analysis of the binary case to encompass a finite type space. The foundational concept mirrors the illustration provided in Section 3. We delineate the MDMB value in Section 4.1 and discuss two pertinent implications concerning its correlation with existing literature. Furthermore, we delineate the optimal MDMB in Section 4.2. Analogous to Section 3, our optimal MDMB may entail the burning of infinite funds at a specific message with negligible probability. In Section 4.3, we delineate the optimal mechanism when Sender operates within a constrained budget for money burning.

4.1 The Value of MDMB

Given a canonical MDMB (π, x), we define the *interim signaling payoff* of type θ under signaling scheme π as follows,

$$V_\pi(\theta) \triangleq \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta) V(\mu). \tag{9}$$

Moreover, we define the type θ's share of ex-post payoff $V(\mu)$ given the posterior μ as $\hat{V}_\theta(\mu) = \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu)$.[6] Based on this adjusted ex-post payoff, we introduce the Sender's *subjective payoff function* under the posterior μ and the subjective prior $\lambda \in \Delta(\Theta)$ as

$$\hat{V}_\lambda(\mu) \triangleq \mathbb{E}_{\theta\sim\lambda}\{\hat{V}_\theta(\mu)\} = \sum_{\theta\in\Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu).^{7} \tag{10}$$

We denote the concave envelope of function f by $cav(f)$. Now, we can derive the value of MDMB.

Theorem 1. $\mathcal{V}^*(\mu_0) = \max_{\pi:\theta\to\Delta(M)} \min_{\theta\in\Theta} V_\pi(\theta) = \min_{\lambda\in\Delta(\Theta)} cav(\hat{V}_\lambda)(\mu_0)$.

[6] This follows from the fact that $\mathbb{E}_{\theta\sim\mu_0}\{\hat{V}_\theta(\mu)\} = V(\mu)$. The notation $\hat{V}_\theta$ is also known as truth-adjust welfare function introduced by Doval and Smolin [2024].

[7] Note that when $\lambda = \mu_0$ the subjective payoff function under the posterior μ becomes $V(\mu)$.

Proof. This proof is relegated to Appendix B. □

We have elucidated the underlying intuition of the proof of Theorem 1 in Section 3. Formally, our proof of Theorem 1 is structured into three distinct components. Initially, we leverage the non-negativity of money burning to establish the upper bound for $\mathcal{V}^*(\mu_0)$. Subsequently, for any signaling scheme $\pi : \Theta \to \Delta(\Theta)$, we present a construction of the MDMB, encapsulated in Proposition 8, demonstrating that Sender can secure at least the minimum interim signaling payoff associated with π. Ultimately, the min-max and max-min equality is demonstrated through the application of Sion's minimax theorem.

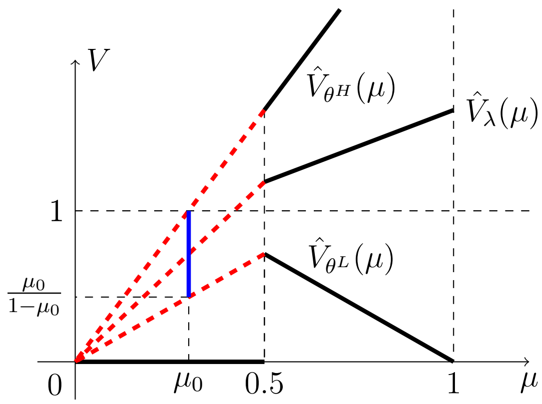

Figure 4: The geometric interpretation.

According to Theorem 1, we can geometrically characterize the value of MDMB and explain how the money burning mechanism reshapes the value function. As depicted in Figure 4, Theorem 1 characterizes the value of the MDMB for the example in Section 3. We first plot the ex-post payoff shares of type θ^H and type θ^L, denoted by $\hat{V}_{\theta^H}(\mu)$ and $\hat{V}_{\theta^L}(\mu)$, respectively, which are both exogenously given. Then we compute $\hat{V}_\lambda$, which is the convex combination or the expected value of $\hat{V}_{\theta^H}$ and $\hat{V}_{\theta^L}$ under λ. The value of the MDMB is the minimum of $cav(\hat{V}_\lambda)(\mu_0)$ over all possible λ, as shown by the blue line in Figure 4. Therefore, the money burning mechanism transforms the belief-payoff function from $V(\mu)$ to $\hat{V}_{\theta^L}(\mu)$ in this example.

In addition to the geometric property of $\mathcal{V}^*(\mu_0)$, Theorem 1 also yields two important implications, linking the value of MDMB to two varieties of robust Bayesian persuasion problems, which provide a theoretical justification for robust BP.

The first equation of Theorem 1 indicates that the value of MDMB is the same as the value of Sender who has full commitment power and opts for a signaling scheme that maximize his minimum interim payoff. The model of Sender opting for such a signaling scheme is named cautious Bayesian persuasion in which Sender only focuses on his lowest possible welfare, Doval and Smolin [2021, 2024]. In cautious Bayesian persuasion setting, Sender has full commitment power but acts robustly to the type realization. The following corollary constitutes our first implication of Theorem 1.

Corollary 2. *The value of MDMB equals the payoff of a Sender with full commitment power but who is cautious.*

The second equation of Theorem 1 relates the value of MDMB to the Sender's payoff in robust Bayesian persuasion with heterogeneous beliefs. To elucidate this, we call a subjective distribution $\lambda \in \Delta(\Theta)$ as the *worst Sender's subjective prior* if it minimizes $cav(\hat{V}_\lambda)(\mu_0)$ which we refer to as *the worst Sender's subjective expected payoff.* Based on the model introduced by Alonso and Câmara [2016] in which the Sender's and the Receiver's subjective priors are heterogeneous, the second equation of Theorem 1 shows that the value of the MDMB coincides with the worst Sender's subjective expected payoff in heterogeneous belief Bayesian persuasion. Hence, the following corollary is our second implication of Theorem 1.

Corollary 3. *The value of MDMB $\mathcal{V}^*(\mu_0)$ equals to the payoff of Sender under Bayesian persuasion with heterogeneous priors, in which Sender holds the worst subjective prior and Receiver has prior μ_0.*

Proposition 3 implies that in the context of a binary type space, the worst subjective prior is attained at the extreme point of $\Delta(\Theta)$. However, we will provide an example at the end of Section 4.2 to demonstrate that this result does not hold in general.

4.2 The Optimal MDMB

Prompted by Theorem 1, the key of constructing the optimal MDMB lies in identifying a signaling scheme that maximizes the minimum interim signaling payoff, which we refer to as *the optimal signaling scheme* in our model. Given this optimal signaling scheme, we can apply Proposition 8 to construct the optimal MDMB. Therefore, in this section, our primary objective is to determine the optimal signaling scheme based on the min-max and max-min equivalence.

The intuition is simple: the min-max and max-min equality implies a zero-sum game where the optimal signaling scheme and the worst Sender's subjective prior form a Nash equilibrium. Let $\lambda \in \Delta(\Theta)$ be a subjective prior and $p \in BP(\mu_0)$ be a Bayes-plausible distribution. We define the "payoff" of the "zero-sum game" as

$$\mathcal{L}(\lambda, p) \triangleq \int_{\mu} \hat{V}_{\lambda}(\mu) dp(\mu). \tag{11}$$

This payoff function represents the Sender's subjective expected payoff under the signaling scheme p and subjective prior λ. The "zero-sum game" is that the Sender chooses a signaling scheme to counter the type realization from nature. We can derive the "indifference condition" for the "Nash-equilibrium" of this "zero-sum game". The following propositions prove these conditions and characterize the worst Sender's subjective prior and the corresponding optimal signaling scheme.

Proposition 4. *A subjective prior $\lambda^* \in \Delta(\Theta)$ is the worst Sender's subjective prior if and only if there exists $p^* \in BP(\mu_0)$ such that $\mathcal{L}(\lambda^*, p^*) = cav(\hat{V}_{\lambda^*})(\mu_0)$, and for any $\theta \in \boldsymbol{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\mu_\theta, p^*)$, and for any $\theta \notin \boldsymbol{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) \leq \mathcal{L}(\mu_\theta, p^*)$.*[8]

Proposition 5. *A signaling scheme $p^* \in BP(\mu_0)$ is optimal if and only if there exists $\lambda^* \in \Delta(\Theta)$ such that $\mathcal{L}(\lambda^*, p^*) = cav(\hat{V}_{\lambda^*})(\mu_0)$, and for any $\theta \in \boldsymbol{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\mu_\theta, p^*)$, and for any $\theta \notin \boldsymbol{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) \leq \mathcal{L}(\mu_\theta, p^*)$.*

Proof. Those proofs are relegated to Appendix B. □

These two propositions jointly characterize the worst subjective prior and the optimal signaling scheme. The condition, "$\mathcal{L}(\lambda^*, p^*) = cav(\hat{V}_{\lambda^*})(\mu_0)$, and for any $\theta \in \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\mu_\theta, p^*)$, and for any $\theta \notin \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) \leq \mathcal{L}(\mu_\theta, p^*)$", implies that λ^* and p^* form a Nash equilibrium in the "zero-sum game", where λ^* is the best response to p^* and vice versa. We apply them to characterize the signaling scheme for general cases in the following example. This example also shows that the worst subjective prior is not necessarily an extreme point of $\Delta(\Theta)$ as in binary-type cases.

Example 1. *We consider an example with three parties: a seller, a buyer, and an influencer. The seller wants to sell a zero-cost product to the buyer. The buyer's valuation of the product is v. The seller only knows that v is distributed uniformly in $\{1, 2, 3\}$. The buyer is a fan of the influencer, who wants to help the buyer reduce the price of the product by disclosing information about the buyer's type and subsidizing the seller.*

[8] μ_θ is the distribution in $\Delta(\Theta)$ with a singleton support $\{\theta\}$.

The influencer acts as Sender who uses our MDMB to influence the seller's action as Receiver. To fit our model, we let the type set be $\Theta = \{v_1 = 1, v_2 = 2, v_3 = 3\}$, the prior distribution be $\mu_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the seller's action set be $A = \{p_1 = 1, p_2 = 2, p_3 = 3\}$.

We assume that the influencer's objective is to minimize the price of the product. If the seller charges a price p_i to the buyer, the influencer's valuation function is $v(p_i) = 4 - p_i$.

We use Theorem 1 to examine the extreme point subjective priors of $\Delta(\Theta)$ at first. We then use Proposition 4 to find the worst Sender's subjective prior and the corresponding maximum payoff of the influencer achieved by the MDMB. We also derive the optimal MDMB backward.

We consider three extreme point subjective priors $\lambda_i \in \Delta(\Theta), i = 1, 2, 3$, where $\lambda_1 = (1, 0, 0), \lambda_2 = (0, 1, 0), \lambda_3 = (0, 0, 1)$. For any $\lambda \in \Delta(\Theta)$, to find the concavification value of $\hat{V}_\lambda(\mu)$ at μ_0, we can assume without loss of generality that we only need to find the distribution of posterior $\tau \in BP(\mu_0)$ that induces different actions of the Receiver.[9] Then finding $cav(\hat{V}_\lambda)(\mu_0)$ becomes a linear programming problem. We obtain that $cav(\hat{V}_{\lambda_1})(\mu_0) = 3$, where $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ forms the support of Receiver's posterior distribution and they are realized with equal probability; $cav(\hat{V}_{\lambda_2})(\mu_0) = 3$, where $(\frac{1}{2}, \frac{1}{2}, 0), (0, 0, 1)$ forms the support of Receiver's posterior distribution and they are realized with probability $\frac{2}{3}, \frac{1}{3}$ respectively; and $cav(\hat{V}_{\lambda_3})(\mu_0) = \frac{8}{3}$. So we can conclude that in this example $\mathcal{V}^*(\mu_0) \leq \frac{8}{3}$.

However, λ_3 is not the worst subjective prior in this case, even though it minimizes $cav(\hat{V}_\lambda)(\mu_0)$ among the extreme points of $\Delta(\Theta)$. Next we show that $\lambda^* = (0, \frac{1}{2}, \frac{1}{2})$ is the worst subjective prior by Proposition 4. We first calculate that $cav(\hat{V}_{\lambda^*}) = \frac{5}{2}$ and the process of concavification splits μ_0 into $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (0, \frac{1}{2}, \frac{1}{2})$ with probability $\frac{2}{3}, \frac{1}{3}$ respectively. We denote this distribution over posterior as τ^*. We verify that $\mathcal{L}(\lambda_2, \tau^*) = \mathcal{L}(\lambda_3, \tau^*) = \frac{5}{2} < 3 = \mathcal{L}(\lambda_1, \tau^*)$. So by Proposition 4, λ^* is the worst subjective prior and by Theorem 1 we have $\mathcal{V}^*(\mu_0) = \frac{5}{2}$. Moreover, by Proposition 5, we know that τ^* is the optimal signaling scheme. We can use Proposition 8 to construct the optimal MDMB that approaches $\mathcal{V}^*(\mu_0)$.

[9] This is true because if two posteriors μ_1, μ_2 in the support of τ lead to the same action of the Receiver, we can merge them as posterior $\frac{\tau(\mu_1)}{\tau(\mu_1)+\tau(\mu_2)}\mu_1 + \frac{\tau(\mu_2)}{\tau(\mu_1)+\tau(\mu_2)}\mu_2$ with probability $\tau(\mu_1) + \tau(\mu_2)$.

4.3 Bounded Credit for Money Burning

This section addresses the scenario wherein the Sender is subject to a budgetary constraint for the practice of money burning. Contrary to the preceding setting, where no limitations were imposed on the quantity of money to be burnt, we postulate that Sender plays under a budget constraint denoted by a threshold $C \geq 0$. Formally, the MDMB with a budget constraint C for money burning is characterized by an input set M, an output set S, and a mapping $\phi : M \to \Delta(S \times [0, C])$. The Sender-Receiver game remains identical to that delineated in Section 2.1. The case of $C = 0$ aligns with the classical mediated communication problem studied by Salamanca [2021] and Corrao and Dai [2023]. Analogous to the definition of the value of MDMB, for any given value of C, we use $\mathcal{V}_C^*(\mu_0)$ to denote the value of MDMB with a budget constraint C for money burning.

We introduce the *generalized subjective payoff function* for a posterior $\mu \in \Delta(\Theta)$ and a generalized subjective prior $\lambda : \Theta \to \mathbb{R}$ that satisfies $\sum_\theta \lambda(\theta) = 1$. The function is defined as

$$\hat{V}_{\lambda,C}(\mu) \triangleq \max\{\sum_{\theta\in\Theta} \lambda(\theta)\frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu), \sum_{\theta\in\Theta} \lambda(\theta)\frac{\mu(\theta)}{\mu_0(\theta)}(\min \mathbb{V}(\mu) - C)\}. \tag{12}$$

Given $p \in BP(\mu_0)$, we also define the *generalized subjective expected payoff function* for a signaling scheme p as

$$\mathcal{L}_C(\lambda, p) \triangleq \int_\mu \hat{V}_{\lambda,C}(\mu) dp(\mu). \tag{13}$$

We obtain results analogous to the ones in the unbounded budget setting.

Theorem 2.

$$\mathcal{V}_C^*(\mu_0) = \max_{p\in BP(\mu_0)} \min_{\sum_\theta \lambda(\theta)=1} \mathcal{L}_C(\lambda, p) = \min_{\sum_\theta \lambda(\theta)=1} \max_{p\in BP(\mu_0)} \mathcal{L}_C(\lambda, p).$$

Moreover, $\mathcal{V}_C^*(\mu_0) = \min_{\sum_\theta \lambda(\theta)=1} cav(\hat{V}_{\lambda,C})(\mu_0)$.

Proof. This proof is relegated to Appendix B. □

To establish Theorem 2, we initially generalize the revelation principle (Proposition 1) to demonstrate that computing $\mathcal{V}_C^*(\mu_0)$ in the bounded credit environment necessitates the incorporation of an additional constraint, $x(\mu) \leq C$ for all μ, into the Sender's optimization problem (Equation 8). Subsequently, we employ a two-step optimization approach, commencing with the determination of a money burning scheme that maximizes the Sender's payoff for any given signaling scheme. This process facilitates the identification of the optimal choice of obedience constraints from the Sender's perspective. This stage culminates in a max-min characterization. The subsequent step involves maximizing the Sender's payoff across signaling schemes, which is achieved through the application of Sion's minimax theorem.

The structure of $\hat{V}_{\lambda,C}$ reveals that the messages transmitted by Sender consist of two components: the first component corresponds to the case where $\sum_\theta \lambda^*(\theta)\frac{\mu(\theta)}{\mu_0(\theta)} \geq 0$, in which Sender employs a costless message for the purpose of persuasion; the second component corresponds to the case where $\sum_\theta \lambda^*(\theta)\frac{\mu(\theta)}{\mu_0(\theta)} < 0$, in which Sender employs a costly message for establishing commitment. Consequently, a conceptualization of one possible optimal MDMB (with bounded credit or not) involves the allocation of messages for two fundamental objectives, persuasion and commitment. This rationale can be leveraged to formulate the optimal MDMB. When the message serves the purpose of persuasion, Sender desires Receiver to take the best response that favors him; when the message serves the purpose of commitment, Sender desires Receiver to take the best response that disfavors him and burns C money at this message. Hence, this theorem reveals an intriguing insight. When Sender has bounded budget for money burning, Receiver can enhance the Sender's payoff by adopting a suitable best response that deliberately chooses the best response that either favors or disfavors Sender at different posteriors.

At the end of this section, we apply Theorem 2 to compare the MDMBs with different bounded credit for money burning, Bayesian persuasion Kamenica and Gentzkow [2011], mediated communication Salamanca [2021], and cheap talk Lipnowski and Ravid [2020] in the illustrative example in Section 3.

We begin by characterizing $\mathcal{V}_C^*(\mu_0)$. By Theorem 2, we need to consider four lines for a given parameter $\lambda \in \mathbb{R}$, namely $l_1(\mu) = 0, l_2(\mu) = -C(\frac{\lambda\mu}{\mu_0} + \frac{(1-\lambda)(1-\mu)}{1-\mu_0}), l_3(\mu) = \frac{\lambda\mu}{\mu_0} + \frac{(1-\lambda)(1-\mu)}{1-\mu_0}$ and $l_4(\mu) = (1-C)(\frac{\lambda\mu}{\mu_0} + \frac{(1-\lambda)(1-\mu)}{1-\mu_0})$. Correspondingly, we have that for $\mu \in [0,1]$,

$$\hat{V}_{\lambda,C}(\mu) = \begin{cases} \max\{l_1(\mu), l_2(\mu)\} & \mu < \frac{1}{2} \\ \max\{l_2(\mu), l_3(\mu)\} & \mu = \frac{1}{2} \\ \max\{l_3(\mu), l_4(\mu)\} & \mu > \frac{1}{2} \end{cases}.$$

Since $\hat{V}_{\lambda,C}$ is convex and upper semi-continuous on $\mu \in [0, \frac{1}{2})$ and $\mu \in (\frac{1}{2}, 1]$, to compute $cav(\hat{V}_{\lambda,C})(\mu_0)$ we only need to evaluate $\hat{V}_{\lambda,C}(0) = \max\{0, -C\frac{1-\lambda}{1-\mu_0}\}$, $\hat{V}_{\lambda,C}(\frac{1}{2}) = \max\{\frac{-C}{2}(\frac{\lambda}{\mu_0} + \frac{1-\lambda}{1-\mu_0}), \frac{1}{2}(\frac{\lambda}{\mu_0} + \frac{1-\lambda}{1-\mu_0})\}$ and $\hat{V}_{\lambda,C}(1) = \max\{\frac{\lambda}{\mu_0}, (1-C)\frac{\lambda}{\mu_0}\}$. Assuming $\mu_0 < \frac{1}{2}$, we can only partition μ_0 into $0, \frac{1}{2}$ or $0, 1$. Since $\hat{V}_{\lambda,C}(0), \hat{V}_{\lambda,C}(\frac{1}{2}), \hat{V}_{\lambda,C}(1)$ are all decreasing in λ for $\lambda \geq 0$, to find the minimum concavification value, we only need to consider the case of $\lambda \leq 0$. We then divide this case into two subcases: $\lambda \in [-\frac{\mu_0}{1-2\mu_0}, 0]$ and $\lambda \in (-\infty, -\frac{\mu_0}{1-2\mu_0})$. We can solve for the result and obtain that for $\mu_0 < \frac{1}{2}$,

$$\mathcal{V}_C^*(\mu_0) = \begin{cases} 0 & C \leq 1 \\ \frac{(C-1)\mu_0}{C(1-\mu_0)-\mu_0} & C > 1 \end{cases}.$$

For $\mu_0 \geq \frac{1}{2}$, $\mathcal{V}_C^*(\mu_0) = 1$.

If we apply different communication protocols to this example, we can obtain the optimal payoff corresponding to the prior μ_0, which is shown in Figure 5. The red line is the concave envelope of $\max \mathbb{V}(\mu)$, which is the result of Bayesian persuasion. The black line is the result of the MDMB with bound $C = +\infty$, i.e. $\mathcal{V}^*(\mu_0)$. The blue line is the result of $\mathcal{V}_2^*(\mu_0)$. Finally, we can see that regardless of what we use among the MDMB with bounded credit $C \leq 1$, cheap talk or classical mediated communication, we can only get the results as the green line, which cannot benefit from those protocols. More results about the comparison can be found in Section 5.1.

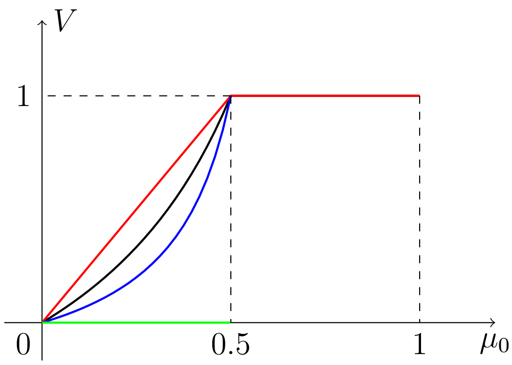

Figure 5: This figure compares the optimal payoffs of different protocols. The red line represents the payoff achieved by Bayesian persuasion. The black line shows $\mathcal{V}^*(\mu_0)$. The blue line indicates $\mathcal{V}_2^*(\mu_0)$. The green line corresponds to $\mathcal{V}_C^*(\mu_0)$ for $C \leq 1$, cheap talk, or classical mediated communication.

5 Applications

This section covers two important applications emanating from the findings of this paper. In Section 5.1, we discuss the refined value of commitment in the Web 3.0 community by MDMB. In Section 5.2, we discuss the improved value of mediator by the money-burning tactic.

In the previous section, we illustrated an example to compare our MDMB with different communication protocols, namely cheap talk, mediated communication, and Bayesian persuasion. In this section, we will conduct a general comparison of our communication protocol with the three other communication protocols. Let $\mathcal{V}^*_{CT}(\mu_0)$, $\mathcal{V}^*_0(\mu_0)$, and $\mathcal{V}^*_{BP}(\mu_0)$ denote the optimal payoff of Sender under cheap talk, mediator without burning money, and Bayesian persuasion, respectively. According to Kamenica and Gentzkow [2011] and Lipnowski and Ravid [2020], $\mathcal{V}^*_{BP}(\mu_0) = cav(\mathbb{V})(\mu_0)$ and $\mathcal{V}^*_{CT}(\mu_0) = qcav(\mathbb{V})(\mu_0)$ where $qcav$ denote the quasi-concave envelope. The discrepancy $\mathcal{V}^*_{BP}(\mu_0) - \mathcal{V}^*_{CT}(\mu_0)$ is recognized as the value of commitment. Between these two values, we have the following fact.

Fact 1. $\mathcal{V}^*_{CT}(\mu_0) \leq \mathcal{V}^*_0(\mu_0) \leq \mathcal{V}^*(\mu_0) \leq \mathcal{V}^*_{BP}(\mu_0)$.

5.1 The Refined Value of Commitment

In conventional societies, the paradigm of communication without commitment is epitomized by the cheap talk model. Consequently, $\mathcal{V}^*_{BP}(\mu_0) - \mathcal{V}^*_{CT}(\mu_0)$ quantifies the value of commitment inherent in communication within conventional societies. However, as delineated in Drakopoulos et al. [2023], the paradigm of communication in Web 3.0 communities, facilitated by Blockchain systems and smart contracts, presents a radically altered landscape. In Web 3.0 communities, users are characterized by full decentralization and a potential for high unreliability. Consequently, a viable approach to facilitating communication among these users is through the deployment of smart contracts, which serve as transparent algorithms. Sender leverage smart contracts to integrate money-burning mechanisms via subsidies and gas fees. Therefore, the communication milieu of Web 3.0 communities is not amenable to modeling as cheap talk but rather as MDMB. Hence, we denote $\mathcal{V}_{BP}(\mu_0) - \mathcal{V}^*(\mu_0)$ as the *refined value of commitment* in Web 3.0 communities.

Our first result establishes the condition under which the refined value of commitment does not exist.

Proposition 6. *If* $\mathcal{V}^*(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$ *then* $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_0(\mu_0) = \mathcal{V}^*(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$*.*

Proof. This proof is relegated to Appendix B. □

Intuitively, we can derive this proposition from the result of Corrao and Dai [2023]. When $\mathcal{V}^*(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$, there is no money burning. Hence, $\mathcal{V}^*_0(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$, which leads to $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$ by Corrao and Dai [2023]. By Proposition 6, we deduce that if $\mathcal{V}^*_{CT}(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$, then $\mathcal{V}^*(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$. Consequently, we promptly arrive at the subsequent corollary, which juxtaposes the value of commitment with the refined value of commitment.

Corollary 4. *There is a positive value of commitment, i.e.* $\mathcal{V}^*_{CT}(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$*, if and only if there is a positive refined value of commitment in the Web 3.0 community, i.e.* $\mathcal{V}^*(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$*.*

Corollary 4 has an important implication that if there is the value of commitment in conventional societies, commitment is still valuable in Web 3.0 communities. In addition, according to Corollary 2 in Lipnowski and Ravid [2020], the refined value of commitment is strictly positive under almost all prior beliefs as long as the value of commitment is strictly positive. Furthermore, building on Corollary 2 and Corollary 3, we obtain two characterizations of the refined value of commitment. Corollary 2 indicates that the refined value of commitment is given by the difference between BP and cautious BP. This difference can be assessed by comparing the expected payoff of the Sender and the lowest interim payoff of the Sender under full commitment. Corollary 3 implies that the refined value of commitment is also given by the difference between BP and heterogeneous BP under the Sender's worst subjective prior. This difference can be evaluated by considering the Receiver's prior μ_0 and the Sender's worst subjective prior λ^*.

Although the existence of the refined value of commitment is contingent upon the presence of the value of commitment, in the subsequent section, we demonstrate that the refined value of commitment within the Web 3.0 community is reduced in comparison to the value of commitment.

5.2 The Improved Value of Mediator

The distinguishing feature of our model vis-à-vis classical mediated communication lies in the incorporation of the money-burning component. Consequently, the trade-off between the adoption of money-burning and the attainment of commitment is encapsulated within what we term the *improved value of mediator* through the money-burning tactic, denoted as $\mathcal{V}^*(\mu_0) - \mathcal{V}_0^*(\mu_0)$. In this section, by characterizing the improved value of mediator, we will contrast the efficacy of MDMB with that of CT and MD. Because the presence of a positive improved value of mediator suggests that the commitment value is diminished in Web 3.0 communities in contrast to traditional societies, thereby providing a more profound understanding of the refined value of commitment.

Initially, we elucidate a topological generic property of the Receiver's payoff function set. When the Receiver's payoff function exhibits this generic property, our communication protocol can improve the Sender's payoff under almost all Receiver's beliefs.

Definition 4. *Recalling the definition of the Receiver's best response set under the belief μ, $RO(\mu)$, we call the setting of A, u and Θ generic if, for any belief μ and any $a \in RO(\mu)$, there exists μ' such that $\boldsymbol{supp}\{\mu'\} = \boldsymbol{supp}\{\mu\}$ and $RO(\mu') = \{a\}$.*

This condition is also present in Lipnowski et al. [2024], where the authors employ it as a generic sufficient criterion for the uniqueness of the Sender's payoff under perfect Bayesian equilibrium in the Bayesian persuasion game. We have an intuitive way to understand the definition of generic is as follows. We define the degenerate setting Θ', A, u that is obtained by restricting the beliefs to a subset of types Θ'. Then, the generic condition for A, u, Θ is equivalent to the condition that satisfies that for any degenerate setting Θ', A, u, if we eliminates all strictly dominated actions from A, there will be no weakly dominated actions left in the remaining action set. Therefore, from this perspective, the condition of Definition 4 is unequivocally generic.

Theorem 3. *Under generic settings, if $\mathcal{V}^*(\mu_0) = \mathcal{V}_0^*(\mu_0)$ then $\mathcal{V}_{CT}^*(\mu_0) = \mathcal{V}_0^*(\mu_0) = \mathcal{V}^*(\mu_0)$.*

Proof. This proof is relegated to Appendix B. □

The proof of this theorem is notably more intricate than the proposition outlined in the preceding section. We establish that in generic circumstances, the continuity of

$\mathcal{V}^*(\mu_0)$ is manifested, which is a pivotal attribute for the derivation of Theorem 3 (cf. Proposition 9 in the appendix).

Technically, we provide two illustrative examples in Appendix A. Example 2 shows that the relation $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_0(\mu_0) = \mathcal{V}^*(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$ can hold even under generic settings, which refers to the case that the refined value of commitment is positive and the same as the value of commitment. Example 3 shows that when the generic condition is not satisfied, it is possible that $\mathcal{V}^*_{CT}(\mu_0) < \mathcal{V}^*_0(\mu_0) = \mathcal{V}^*(\mu_0)$, which demonstrates the necessity of generic settings of Theorem 3.

According to Theorem 3, it is ascertained that within generic settings, the absence of an improved value of mediator necessitates the equivalence between the refined value of commitment and the value of commitment, implying that in Web 3.0 communities, the refined commitment value remains unaltered. Although this outcome might be perceived as unfavorable, we demonstrate that such a scenario is improbable. By Theorem 3, we observe that if $\mathcal{V}^*(\mu_0) > \mathcal{V}^*_{CT}(\mu_0)$, there must be a positive improved value of mediator. Hence, based on this observation, we elucidate a sufficient condition under which the implementation of a money-burning mechanism can enhance the efficacy of mediated information design.

Proposition 7. *Under generic settings, if* $qcav(\mathbb{V})(\mu_0) \neq cav(\mathbb{V})(\mu_0)$ *and there is a sufficiently small* $\varepsilon > 0$ *such that* $qcav(\mathbb{V})(\mu_0+\varepsilon(\mu-\mu_0)) = qcav(\mathbb{V})(\mu_0)$ *for all* $\mu \in \Delta(\Theta)$*, it follows that* $\mathcal{V}^*_0(\mu_0) < \mathcal{V}^*(\mu_0)$*.*

Proof. This proof is relegated to Appendix B. □

We can use Proposition 7 to identify when there is a positive improved value of mediator. The condition $qcav(\mathbb{V})(\mu_0) \neq cav(\mathbb{V})(\mu_0)$ rules out the possibility that CT achieves the same communication efficiency as BP. The condition that there exists a sufficiently small $\varepsilon > 0$ such that $qcav(\mathbb{V})(\mu_0+\varepsilon(\mu-\mu_0)) = qcav(\mathbb{V})(\mu_0)$ for all $\mu \in \Delta(\Theta)$ implies that μ_0 is either an interior point of the quasi-concavification distribution of the posterior of $\mathbb{V}(\mu)$ at point μ_0.

Remark that, under the assumption that A is a finite set, $\max \mathbb{V}(\mu)$ is a piecewise constant function, and thus $qcav(\mathbb{V})(\mu)$ is also a piecewise constant function with finite values. Hence, following a similar argument as Corollary 2 in Lipnowski and Ravid [2020], we can deduce that *almost all* beliefs μ_0 are either interior points of some quasi-concavification distribution of posteriors or $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_{BP}(\mu_0) = \max_{\mu \in \Delta(\Theta)} V(\mu)$. By Proposition 7, for the former μ_0, we have that $\mathcal{V}^*_{CT}(\mu_0) \leq \mathcal{V}^*_0(\mu_0) < \mathcal{V}^*(\mu_0)$. Consequently, we have a subsequent implication.

Corollary 5. *Under generic settings, for almost all prior beliefs μ_0, either there is no value of commitment or there is a positive improved value of mediator.*

6 Conclusion

In this paper, we introduce and investigate a novel communication protocol called mediated communication with money-burning mechanism (MDMB). In our communication protocol, Sender not only employs a trusted mediator to convey the message but also voluntarily gives up some benefits as compensation for Receiver to gain more commitment power. By generalizing the revelation principle of mechanism design with limited commitment, we characterize the communication efficiency of our communication protocol under the transparent motives assumption. We demonstrate that the value of MDMB aligns with that of Cautious Bayesian persuasion, which is equivalent to the concavification value under the worst Sender's subject prior. Moreover, we extend our analyses to scenarios where Sender possesses bounded credits for money-burning.

Our paper has two important applications. The first application pertains to the characterization of the refined value of commitment within Web 3.0 communities. We elucidate that the presence of the refined value of commitment is synonymous with the existence of the traditional value of commitment. Moreover, our result indicates that in the trade-off between money burning and enhancing commitment power, Sender can benefit from burning money in almost all generic non-trivial cases. This also establishes a new benchmark for the unreliable Sender's communication efficiency. It also implies that for almost all prior beliefs under the generic property, there is either no value of commitment at all or a positive improved value of mediator by money-burning tactic, indicating that in Web 3.0 community the refined value of commitment is strictly reduced compared to the value of commitment.

It is imperative to acknowledge that our conclusions are contingent upon the significant assumption of state-independent preferences of Sender. In a more general framework, the ramifications of money-burning warrant further in-depth exploration.

References

Ricardo Alonso and Odilon Câmara. Bayesian persuasion with heterogeneous priors. *Journal of Economic Theory*, 165:672–706, 2016.

Ivan Arandjelović. An extension of the sion's minimax theorem. *Zbornik radova Filozofskog fakulteta u Nišu. Serija Matematik*, pages 1–3, 1992.

David Austen-Smith and Jeffrey S Banks. Cheap talk and burned money. *Journal of Economic Theory*, 91(1):1–16, 2000.

Dirk Bergemann and Stephen Morris. Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95, 2019.

Dirk Bergemann, Alessandro Bonatti, and Alex Smolin. The design and price of information. *American Economic Review*, 108(1):1–48, 2018.

Helmut Bester and Roland Strausz. Contracting with imperfect commitment and the revelation principle: the single agent case. *Econometrica*, 69(4):1077–1098, 2001.

Roberto Corrao. Mediation markets: The case of soft information. *Working Paper*, 2023.

Roberto Corrao and Yifan Dai. The bounds of mediated communication. *Working Paper*, 2023.

Vincent P Crawford and Joel Sobel. Strategic information transmission. *Econometrica: Journal of the Econometric Society*, pages 1431–1451, 1982.

Laura Doval and Vasiliki Skreta. Mechanism design with limited commitment. *Econometrica*, 90(4):1463–1500, 2022.

Laura Doval and Alex Smolin. Information payoffs: An interim perspective. *arXiv preprint arXiv:2109.03061*, 2021.

Laura Doval and Alex Smolin. Persuasion and welfare. *Journal of Political Economy*, 132(7):000–000, 2024.

Kimon Drakopoulos, Irene Lo, and Justin Mulvany. Blockchain mediated persuasion. In *Proceedings of the 24th ACM Conference on Economics and Computation*, 2023.

Shaddin Dughmi, Rad Niazadeh, Alexandros Psomas, and S Matthew Weinberg. Persuasion and incentives through the lens of duality. In *Web and Internet Economics: 15th International Conference, WINE 2019, New York, NY, USA, December 10–12, 2019, Proceedings 15*, pages 142–155. Springer, 2019.

Francoise Forges. An approach to communication equilibria. *Econometrica: Journal of the Econometric Society*, pages 1375–1385, 1986.

Guillaume R Fréchette, Alessandro Lizzeri, and Jacopo Perego. Rules and commitment in communication: An experimental analysis. *Econometrica*, 90(5):2283–2318, 2022.

Maria Goltsman, Johannes Hörner, Gregory Pavlov, and Francesco Squintani. Mediation, arbitration and negotiation. *Journal of Economic Theory*, 144(4):1397–1420, 2009.

Maxim Ivanov. Beneficial mediated communication in cheap talk. *Journal of Mathematical Economics*, 55:129–135, 2014.

Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

Vladimir Karamychev and Bauke Visser. Optimal signaling with cheap talk and money burning. *International Journal of Game Theory*, 46(3):813–850, 2017.

Anton Kolotilin and Hongyi Li. Relational communication. *Theoretical Economics*, 16 (4):1391–1430, 2021.

Hao Li and Xianwen Shi. Discriminatory information disclosure. *American Economic Review*, 107(11):3363–3385, 2017.

Xiao Lin and Ce Liu. Credible persuasion. *Journal of Political Economy*, 132(7):000–000, 2024.

Elliot Lipnowski and Doron Ravid. Cheap talk with transparent motives. *Econometrica*, 88(4):1631–1660, 2020.

Elliot Lipnowski, Doron Ravid, and Denis Shishkin. Persuasion via weak institutions. *Journal of Political Economy*, 130(10):2705–2730, 2022.

Elliot Lipnowski, Doron Ravid, and Denis Shishkin. Perfect bayesian persuasion. *arXiv preprint arXiv:2402.06765*, 2024.

Yi Liu and Fan Wu. Implementing randomized allocation rules with outcome-contingent transfers. *Working Paper*, 2023.

Daehong Min. Bayesian persuasion under partial commitment. *Economic Theory*, 72(3): 743–764, 2021.

Roger B Myerson. Optimal coordination mechanisms in generalized principal–agent problems. *Journal of Mathematical Economics*, 10(1):67–81, 1982.

Eduardo Perez-Richet and Vasiliki Skreta. Test design under falsification. *Econometrica*, 90(3):1109–1142, 2022.

Hitoshi Sadakane. Multistage information transmission with voluntary monetary transfers. *Theoretical Economics*, 18(1):267–301, 2023.

Andrés Salamanca. The value of mediated communication. *Journal of Economic Theory*, 192:105191, 2021.

A Omitted Examples

Example 2. *Receiver has three possible actions a_1, a_2, a_3 and Sender has two possible types H, L. The prior belief assigns probability μ_0 to the Sender's type being H. The Sender's values for the actions are $v(a_1) = 0, v(a_2) = \frac{1}{4}, v(a_3) = 1$. We summarize the Receiver's payoffs in Table 1.*

$u(a, \theta)$	H	L
a_1	-4	1
a_2	0	0
a_3	1	-2

Table 1: Receiver's payoff matrix.

In this example, the belief-value correspondence is

$$\mathbb{V}(\mu) = \begin{cases} 1 & \mu \in (\frac{2}{3}, 1] \\ [\frac{1}{4}, 1] & \mu = \frac{2}{3} \\ \frac{1}{4} & \mu \in (\frac{1}{5}, \frac{2}{3}) \\ [0, \frac{1}{4}] & \mu = 1/5 \\ 0 & \mu < \frac{1}{5} \end{cases}.$$

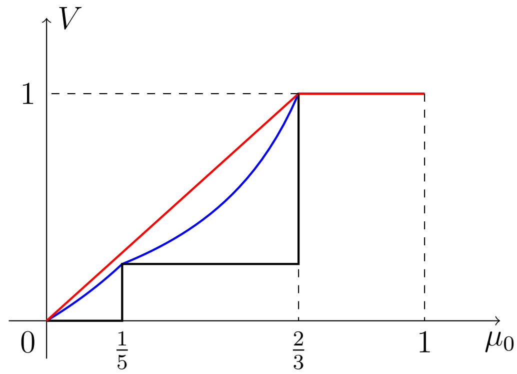

Figure 6: Results of Example 2.

We depict $\mathbb{V}(\mu)$ on Figure 6 as the black line, which corresponds to the outcome under a mediator without money burning and cheap talk, following the results of Salamanca [2021]. Based on Proposition 3, we display the result of $\mathcal{V}^*$ on Figure 6 as the blue line and $\mathcal{V}^*_{BP}$ as the red line, with the procedure omitted. We observe that, when $\mu_0 = \frac{1}{5}$, the case of $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_0(\mu_0) = \mathcal{V}^*(\mu_0) < \mathcal{V}^*_{BP}(\mu_0)$ arises under the generic setting.

Example 3. *We present an abstract setting in this example, where we only specify the belief-value function and ensure the existence of the basic settings of A, u, v, Θ by imposing the upper-semi continuity of the belief-value function.*

We assume that there are three distinct types θ_1, θ_2 and θ_3. The maximum of belief-value correspondence is

$$V(\mu) = \begin{cases} \frac{7}{3} & \mu(\theta_1) = 1 \\ 2 & \mu(\theta_1) = 0, \mu(\theta_2) \in [0, \frac{1}{2}) \\ 3 & \mu(\theta_1) = 0, \mu(\theta_2) \in [\frac{1}{2}, \frac{3}{4}] \\ 1 & \mu(\theta_1) = 0, \mu(\theta_2) \in (\frac{3}{4}, 1] \\ 0 & \text{otherwise} \end{cases}.$$

By restricting the support to $\{\theta_2, \theta_3\}$, the value function of Example 3 coincides with Example 3 or Figure 7 in Salamanca [2021]. For $\mu_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, since $\mathcal{V}_0^*((0, \frac{1}{3}, \frac{2}{3})) = \frac{7}{3}$ as shown by Salamanca [2021], splitting μ_0 into $(1, 0, 0)$ and $(0, \frac{1}{4}, \frac{3}{4})$ yields $\mathcal{V}_0^*(\mu_0) = \frac{7}{3}$. Furthermore, the interim payoff of θ_1 cannot exceed $\frac{7}{3}$, implying that $\mathcal{V}^*(\mu_0) \leq \frac{7}{3}$. Hence, we obtain $\mathcal{V}^*(\mu_0) = \mathcal{V}_0^*(\mu_0) = \frac{7}{3}$. However, to find a cheap talk equilibrium with $\frac{7}{3}$ as the Sender's payoff, we need to split μ_0 into $(1, 0, 0)$ and $(0, \frac{1}{3}, \frac{2}{3})$ and keep $(1, 0, 0)$ unchanged. Since $\mathcal{V}_{CT}^*((0, \frac{1}{3}, \frac{2}{3})) = 2 < \frac{7}{3} = V((1, 0, 0))$, no cheap talk equilibrium achieves $\frac{7}{3}$ for the Sender, and thus $\mathcal{V}_{CT}^*(\mu_0) < \mathcal{V}_0^*(\mu_0) = \mathcal{V}^*(\mu_0)$ for $\mu_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$.

B Omitted Proofs

Appendix B collects all the proofs from the main body of this paper.

Proof of Proposition 1. For any MDMB (M, S, ϕ) and a corresponding PBE assessment $(\sigma, \alpha, \mu) \in \mathcal{E}[\mathcal{G}_{M,S,\phi}(\mu_0)]$, we will directly construct a canonical MDMB (π, x) and a corresponding PBE canonical assessment $(\sigma^*, \alpha^*, \mu^*) \in \mathcal{E}[\mathcal{G}_{(\pi,x)}(\mu_0)]$ such that the expected payoffs of Sender in both assessments are the same.

The canonical MDMB we constructed is as follows: for any $\mu \in \Delta(\Theta), \theta \in \Theta$,

$$\pi(\mu|\theta) = \sum_{s \in S, t \geq 0, \mu(s,t) = \mu, m \in M} \phi(s, t|m)\sigma(m|\theta), \tag{14}$$

and for any $\mu \in \Delta(\Theta)$

$$x(\mu) = \begin{cases} \frac{\sum_{s \in S, t \geq 0, \mu(s,t) = \mu} \phi(s,t|m)\sigma(m|\theta)t}{\sum_{s \in S, t \geq 0, \mu(s,t) = \mu} \phi(s,t|m)\sigma(m|\theta)} & \sum_{s \in S, t \geq 0, \mu(s,t) = \mu} \phi(s, t|m)\sigma(m|\theta) \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

Note that the above canonical MDMB is well-defined since the support of $\phi(m)$ is finite.

The canonical assessment $(\sigma^*, \alpha^*, \mu^*)$ we constructed is as follows: for all θ, $\sigma^*(\theta|\theta) = 1$, for all $\mu \in \Delta(\Theta)$, $\mu^*(\mu) = \mu$, for all $\mu \in \mathbf{supp}\{\pi(\theta)\}$ for some $\theta \in \Theta$

$$\alpha^*(\mu) = \sum_{s \in S, t \geq 0, \mu(s,t) = \mu} \frac{\sum_{\theta \in \Theta, m \in M} \mu_0(\theta)\sigma(m|\theta)\phi(s, t|m)}{\sum_{s' \in S, t' \geq 0, \mu(s',t') = \mu} \sum_{\theta \in \Theta, m \in M} \mu_0(\theta)\sigma(m|\theta)\phi(s', t'|m)} \alpha(s, t).$$

and for $\mu \notin \mathbf{supp}\{\pi(\theta)\}$ for any $\theta \in \Theta$ $\alpha^*(\mu)$ is any best response given posterior belief μ.

Subsequently, we will verify that the canonical assessments $(\sigma^*, \alpha^*, \mu^*) \in \mathcal{E}[\mathcal{G}_{\pi,x}(\mu_0)]$ and the payoffs of Sender in that canonical assessments and original assessments are the same.

Before the verification, we prove the following lemma.

Lemma 1. *Suppose $\mu = \mu(s,t) \in \textbf{\textit{supp}}\{\pi(\hat{\theta})\}$ for some $\hat{\theta}$, then*

$$\frac{\sum_{\theta\in\Theta,m\in M}\mu_0(\theta)\sigma(m|\theta)\phi(s,t|m)}{\sum_{\mu(s',t')=\mu}\sum_{\theta\in\Theta,m\in M}\mu_0(\theta)\sigma(m|\theta)\phi(s',t'|m)} = \frac{\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s,t|m)}{\sum_{\mu(s',t')=\mu}\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s',t'|m)}.$$

Proof of Lemma 1. It is sufficed to show that for any $\hat{\theta}, \bar{\theta} \in \textbf{supp}\{\mu\}$, we have

$$\frac{\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s,t|m)}{\sum_{\mu(s',t')=\mu}\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s',t'|m)} = \frac{\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s,t|m)}{\sum_{\mu(s',t')=\mu}\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s',t'|m)}. \tag{16}$$

According to Bayes updating, for any $s'' \in S, t'' \geq 0$ such that $\mu(s,t) = \mu(s'',t'')$, we have that

$$\frac{\mu_0(\hat{\theta})\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s,t|m)}{\mu_0(\bar{\theta})\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s,t|m)} = \frac{\mu(\hat{\theta}|s,t)}{\mu(\bar{\theta}|s,t)} = \frac{\mu(\hat{\theta}|s'',t'')}{\mu(\bar{\theta}|s'',t'')} = \frac{\mu_0(\hat{\theta})}{\mu_0(\bar{\theta})\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s'',t''|m)}.$$

Thus,

$$\frac{\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s,t|m)}{\sum_{m\in M}\sigma(m|\hat{\theta})\phi(s'',t''|m)} = \frac{\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s,t|m)}{\sum_{m\in M}\sigma(m|\bar{\theta})\phi(s'',t''|m)}.$$

Since s'', t'' can be any one satisfying that $\mu(s'',t'') = \mu$, Equation 16 holds. □

Sender's optmality and payoff equivalence: To show Sender's optimality and the payoff equivalence, it is sufficient to show that the expected payoffs of type θ Sender under both assessments are the same. The expected payoff of type θ Sender under the assessment (σ, α, μ) is

$$\sum_{m\in M,s\in S,t\geq 0,a\in A}\sigma(m|\theta)\phi(s,t|m)\alpha(a|s,t)(v(a)-t).$$

The expected payoff of type θ Sender under the assessment $(\sigma^*, \alpha^*, \mu^*)$ is

$$\sum_{\mu\in\textbf{supp}\{\pi(\theta)\},a\in A}\pi(\mu|\theta)\alpha^*(a|\mu)(v(a)-x(\mu)).$$

Those two expected payoff are the same, since by Lemma 1, for any $a \in A, s \in S, t \geq 0$ the coefficient of $\alpha(a|s,t)$ where $\mu(s,t) = \mu \in \mathbf{supp}\{\pi(\theta)\}$ in the expression

$$\sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}, a \in A} \pi(\mu|\theta)\alpha^*(a|\mu)(v(a) - x(\mu))$$

is

$$\sum_{s',t',m,\mu(s',t')=\mu} \sigma(m|\theta)\phi(s',t'|m) \frac{\sum_{\theta' \in \Theta, m \in M} \mu_0(\theta')\sigma(m|\theta')\phi(s,t|m)}{\sum_{s' \in S, t' \geq 0, \mu(s',t')=\mu} \sum_{\theta' \in \Theta, m \in M} \mu_0(\theta')\sigma(m|\theta')\phi(s',t'|m)} v(a)$$
$$= \sum_{m} \sigma(m|\theta)\phi(s,t|m)v(a),$$

and expected money burning of type θ Sender of $(\sigma^*, \alpha^*, \mu^*)$ is

$$\sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}, a \in A} \pi(\mu|\theta)\alpha^*(a|\mu)x(\mu) = \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta)x(\mu) = \sum_{m \in M, s \in S, t \geq 0} \sigma(m|\theta)\phi(s,t|m)t.$$

Receiver's optmality: Observe that

$$\arg\max_{\alpha(s,t) \in \Delta(a)} \sum_{\theta \in \Theta, a \in A} \mu(\theta|s,t)\alpha(a|s,t)(u(a,\theta)+t) = \arg\max_{\alpha(s,t) \in \Delta(a)} \sum_{\theta \in \Theta, a \in A} \mu(\theta|s,t)\alpha(a|s,t)u(a,\theta).$$

Since $\alpha(s,t)$ is the best response under the belief $\mu(s,t)$ and $\alpha^*(\mu)$ is a convex combination of some $\alpha(s',t')$ where $\mu(s',t') = \mu$, by the convexity of best response set, $\alpha^*(\mu)$ must satisfies the Receiver's optimality condition.

Bayes updating: Given $\mu \in \Delta(\Theta)$, for any s,t such that $\mu(s,t) = \mu = \mu^*(\mu)$, by Bayes updating, we have that for any $\theta \in \Theta$

$$\mu(\theta|s,t) \sum_{\theta' \in \Theta, m \in M} \mu_0(\theta')\sigma(m|\theta')\phi(s,t|m) = \mu_0(\theta) \sum_{m \in M} \sigma(m|\theta)\phi(s,t|m).$$

Hence,

$$\mu(\theta) \sum_{s,t,\mu(s,t)=\mu} \sum_{\theta' \in \Theta, m \in M} \mu_0(\theta')\sigma(m|\theta')\phi(s,t|m) = \sum_{s,t,\mu(s,t)=\mu} \mu_0(\theta) \sum_{m \in M} \sigma(m|\theta)\phi(s,t|m).$$

That is

$$\mu^*(\theta|\mu) \sum_{\theta'} \mu_0(\theta')\pi(\mu|\theta') = \mu_0(\theta)\pi(\mu|\theta).$$

□

Proof of Proposition 2. According to belief-based approach, $p \in \Delta(\Delta(\Theta))$ and ex-post payoff V is induced by a canonical MDMB, if and only if they satisfy that $V(\mu) \in \mathbb{V}(\mu)$, $p \in BP(\mu_0)$ and for any $\theta, \theta' \in \Theta$,

$$\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)}(V(\mu) - x(\mu))dp(\mu) \geq \int_\mu \frac{\mu(\theta')}{\mu_0(\theta')}(V(\mu) - x(\mu))dp(\mu).$$

By swap θ, θ' in above inequality, we can get

$$\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)}(V(\mu) - x(\mu))dp(\mu) = \int_\mu \frac{\mu(\theta')}{\mu_0(\theta')}(V(\mu) - x(\mu))dp(\mu).$$

□

Proof of Corollary 1. According to Proposition 2, we only need to show that, to calculate $\mathcal{V}^*(\mu_0)$, it is without loss of generality to assume $V(\mu) = \max \mathbb{V}(\mu)$.

Suppose there is $p \in BP(\mu_0), x, V : \Delta(\Theta) \to \mathbb{R}$ such that there exists $\mu \in \mathbf{supp}\{p\}$ satisfying that $V(\mu) \neq \max \mathbb{V}(\mu)$, then we construct V', x' such that at posterior μ, $V'(\mu) = \max \mathbb{V}(\mu)$ and $x'(\mu) = x(\mu) + V'(\mu) - V(\mu)$. Now p, x', V' also satisfies the constraints of Equation 8 without reducing the Sender's payoff. □

Proof of Proposition 3. This proof is based on general characterization in Section 4.

By Proposition 4, it suffices to show that there exists $\theta_i \in \{\theta_1, \theta_2\}$ such that for any $p \in BP(\mu_0)$ with $\mathcal{L}(\theta_i, p) = cav(\hat{V}_{\theta_i})(\mu_0)$, we have $\mathcal{L}(\theta_i, p) \leq \mathcal{L}(\theta_{3-i}, p)$.

Let $U = \{\mu | V(\mu) = \max_{x\in[0,1]} V(x)\}$ denote the range of posteriors that yield the maximum value for Sender, for $\mu \in [0, 1]$. Since U is convex and $V(\cdot)$ is upper semi-continuous, U can be expressed as the union of closed intervals. We assume that $l = \min U$ and $r = \max U$. If $l \leq \mu_0 \leq r$, then it is clear that $V^*(\mu_0) = \max_{x\in[0,1]} V(x)$ and for any θ_i, $cav(\hat{V}_{\theta_i})(\mu_0) = V^*(\mu_0)$, which is a constant. Hence, our statement holds trivially. In the following proof, we consider $\mu_0 > r$ or $\mu_0 < l$.

Without loss of generality, we assume that $\mu_0 > r \geq 0$ by symmetry. We focus on θ_1. We prove by contradiction that if there exists $p \in BP(\mu_0)$ with $\mathcal{L}(\theta_1, p) = cav(\hat{V}_{\theta_1})(\mu_0)$ and $\mathcal{L}(\theta_1, p) > \mathcal{L}(\theta_2, p)$, then we reach a contradiction. Since $p \in BP(\mu_0)$ performs the concavification of the function $\hat{V}_{\theta_1}$ at point μ_0, by Proposition 9 of the working paper version of Kamenica and Gentzkow [2011], we have that the points $(\mu, \hat{V}_{\theta_1}(\mu))$ for

$\mu \in \mathbf{supp}\{p\}$ are collinear. This means that there exist parameters k, b such that for any $\mu \in \mathbf{supp}\{p\}$,

$$\frac{\mu}{\mu_0} V(\mu) = k\mu + b.$$

Since $\mathcal{L}(\theta_1, p) > \mathcal{L}(\theta_2, p)$, we have

$$\int_\mu (\mu - \mu_0) V(\mu) dp(\mu) > 0.$$

Substituting $V(\mu)$ with $\mu_0(k + \frac{b}{\mu})$ and using $\int_\mu (\mu - \mu_0) dp(\mu) = 0$, we obtain

$$b(1 - \int_\mu \frac{\mu_0}{\mu} dp(\mu)) = \int_\mu (\mu - \mu_0) \frac{b}{\mu} dp(\mu) > 0.$$

By Cauchy's inequality,

$$\int_\mu \frac{\mu_0}{\mu} dp(\mu) = \int_\mu \frac{\mu}{\mu_0} dp(\mu) \int_\mu \frac{\mu_0}{\mu} dp(\mu) \geq (\int_\mu dp(\mu))^2 = 1.$$

Therefore, we must have $b < 0$. However, $k\mu + b$ is the concavification line of $\hat{V}_{\theta_1}(\cdot)$ at μ_0. Thus, it must satisfy that for any $\mu \in [0, 1]$, $\hat{V}_{\theta_1}(\mu) \leq k\mu + b$. Choosing $\mu = 0$, we get $b \geq 0$. This is a contradiction.

□

Proof of Theorem 1. Before delving into the main body of the proof, we first prove a lemma of upper bound of $\mathcal{V}^*(\mu_0)$ and show our construction method.

Lemma 2. $\mathcal{V}^*(\mu_0) \leq \max_\pi \min_{\theta \in \Theta} V_\pi(\theta)$.

Proof of Lemma 2. For any canonical MDMB (π, t), by incentive-compatible constraint, the Sender's payoff under this mechanism is

$$V_\pi(\theta) - \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta) x(\mu),$$

and by incentive-compatible constraints, for any θ, θ',

$$V_\pi(\theta) - \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta) x(\mu) = V_\pi(\theta') - \sum_{\mu \in \mathbf{supp}\{\pi(\theta)\}} \pi(\mu|\theta') x(\mu).$$

Then by $x(m) \geq 0$, we know that the Sender's payoff is no larger than $\min_{\theta\in\Theta} V_\pi(\theta)$. Hence, $\mathcal{V}^*(\mu_0) \leq \max_\pi \min_{\theta\in\Theta} V_\pi(\theta)$. $\square$

In order to achieve the upper bound, we use the method implied by the following proposition to obtain the minimum interim signaling payoff for any given signaling scheme π.

Proposition 8. *Given any signaling scheme $\pi : \Theta \to \Delta(\Delta(\Theta))$, we construct an MDMB $(\bar{\pi}, \bar{x})$,*

$$\bar{\pi}(\mu|\theta) = \begin{cases} (1-\delta)\pi(\mu|\theta) & \mu(\theta|\theta) \neq 1 \\ \delta + (1-\delta)\pi(\mu|\theta) & \mu(\theta|\theta) = 1 \end{cases}, x(\cdot) = \begin{cases} 0 & \mu(\theta|\theta) \neq 1 \\ \frac{1}{\bar{\pi}(\mu|\theta)}(V_{\bar{\pi}}(\theta) - \min_{\theta\in\Theta} V_{\bar{\pi}}(\theta)) & \mu(\theta|\theta) = 1 \end{cases}.$$

The mechanism $(\bar{\pi}, \bar{x})$ is incentive-compatible for $\delta \in (0,1)$ and the Sender's payoff under this mechanism converges to $\min_{\theta\in\Theta} V_\pi(\theta)$ as $\delta \to 0^+$.

Proof of Proposition 8. Given that $V_{\bar{\pi}}(\theta) - \min_{\theta\in\Theta} V_{\bar{\pi}}(\theta) \geq 0$, it follows that $x(\mu) \geq 0$ for all $\mu \in \Delta(\Theta)$. Therefore, for any $\theta \in \Theta$, the Sender's payoff of type θ under the mechanism $(\bar{\pi}, x)$ is equal to

$$V_{\bar{\pi}}(\theta) - \sum_{\mu\in\mathbf{supp}\{\pi(\theta)\}} \bar{\pi}(\mu|\theta)x(\mu) = \min_{\theta\in\Theta} V_{\bar{\pi}}(\theta).$$

This implies that the mechanism is incentive-compatible. Moreover, the Sender's expected payoff is $\min_{\theta\in\Theta} V_{\bar{\pi}}(\theta)$.

Next, we show that the corresponding canonical assessments also satisfies Bayes updating condition under this MDMB mechanism. Let $\mu_\theta \in \Delta(\Theta)$ be such that $\mu_\theta(\theta) = 1$ and $\mu_\theta(\theta') = 0$ for any $\theta' \neq \theta$. Let $\mu^*_\pi(\mu)$ and $\mu^*_{\bar{\pi}}(\mu)$ denote the posterior beliefs under the signaling schemes π and $\bar{\pi}$ for any $\mu \in \Delta(\Theta)$, respectively. For any $\mu \in \Delta(\Theta)$, we have that

$$\mu^*_{\bar{\pi}}(\theta|\mu) = \frac{\bar{\pi}(\mu|\theta)\mu_0(\theta)}{\sum_{\theta'\in\Theta} \bar{\pi}(\mu|\theta')\mu_0(\theta')} = \frac{\pi(\mu|\theta)\mu_0(\theta)}{\sum_{\theta'\in\Theta} \pi(\mu|\theta')\mu_0(\theta')} = \mu^*_\pi(\theta|\mu).$$

Thus, $\mu^*_\pi(\mu) = \mu$ if and only if $\mu^*_{\bar{\pi}}(\mu) = \mu$. Hence, $V_{\bar{\pi}}(\theta) = \sum_{\mu\in\mathbf{supp}\{\pi(\theta)\}} \bar{\pi}(\mu|\theta)V(\mu) = (1-\delta)V_\pi(\theta) + \delta V(\mu_\theta)$. It follows that

$$\lim_{\delta\to 0^+} \min_{\theta\in\Theta}\{V_{\bar{\pi}}(\theta)\} = \lim_{\delta\to 0^+} \min_{\theta\in\Theta}\{(1-\delta)V_\pi(\theta) + \delta V(\mu_\theta)\} = \min_{\theta\in\Theta} V_\pi(\theta).$$

$\square$

We now return to the problem of attaining the upper bound of the value of the MDMB derived by Lemma 2. As per the construction in Proposition 8, if the Sender adopts the signaling scheme that maximizes the worst-case interim signaling payoff, the Sender's payoff will achieve $\max_\pi \min_{\theta\in\Theta} V_\pi(\theta)$. Hence, we obtain $\mathcal{V}^*(\mu_0) = \max_\pi \min_{\theta\in\Theta} V_\pi(\theta)$. So, we obtain the upper bound and get that $\mathcal{V}^*(\mu_0) = \max_\pi \min_{\theta\in\Theta} V_\pi(\theta)$. In addition, based on our construction in Proposition 8, we demonstrate that in an optimal MDMB, messages are divided into two functions: one is for persuasion and the other one is for money burning to obtain commitment power. Thus, we can decompose the design of MDMB into two steps (the information design step and the mechanism design step): the first step is to identify an optimal signaling scheme that attains the maximum minimum interim signaling payoff; the second step is to devise the additional messages solely for commitment.

For $\lambda \in \Delta(\Theta), p \in BP(\mu_0)$, let

$$\mathcal{L}(\lambda, p) \triangleq \int_\mu \hat{V}_\lambda(\mu) dp(\mu).$$

We have $\Delta(\Theta), BP(\mu_0)$ are both convex and compact. Further by $\hat{V}(\mu)$ is upper semi-continuous [10], then $\mathcal{L}(\lambda, p)$ is continuous and linear in λ, and also it is upper semi-continuous and linear in p. Thus according to Sion's minimax theorem, we have that

$$\max_{p\in BP(\mu_0)} \min_{\lambda\in\Delta(\Theta)} \mathcal{L}(\lambda, p) = \min_{\lambda\in\Delta(\Theta)} \max_{p\in BP(\mu_0)} \mathcal{L}(\lambda, p).$$

Next by Kamenica and Gentzkow [2011], we know that for any given λ,

$$\max_{p\in BP(\mu_0)} \mathcal{L}(\lambda, p) = cav(\hat{V}_\lambda)(\mu_0).$$

Since $\mathcal{L}(\lambda, p)$ in linear in λ, then

$$\max_{p\in BP(\mu_0)} \min_{\lambda\in\Delta(\Theta)} \mathcal{L}(\lambda, p) = \max_\pi \min_{\theta\in\Theta} V_\pi(\theta) = \mathcal{V}^*(\mu_0).$$

So $\mathcal{V}^*(\mu_0) = cav(\hat{V}_\lambda)(\mu_0)$.

[10]See Lipnowski and Ravid [2020] footnote 13

$\square$

Proof of Proposition 4. Since $\Delta(\Theta)$ and $BP(\mu_0)$ are compact sets, by extensions of Sion's minmax theorem Arandjelović [1992], there exists a saddle point (λ_0, p_0) such that

$$\max_{p \in BP(\mu_0)} \min_{\lambda \in \Delta(\Theta)} \mathcal{L}(\lambda, p) = \min_{\lambda \in \Delta(\Theta)} \max_{p \in BP(\mu_0)} \mathcal{L}(\lambda, p) = \mathcal{L}(\lambda_0, p_0).$$

Next, we show that for any saddle point (λ_1, p_1) of $\mathcal{L}$, we have $\mathcal{L}(\lambda_0, p_0) = \mathcal{L}(\lambda_1, p_1)$. We prove by contradiction and assume, without loss of generality, that $\mathcal{L}(\lambda_0, p_0) < \mathcal{L}(\lambda_1, p_1)$. Then, by the property of saddle point, we have

$$\mathcal{L}(\lambda_0, p_1) \leq \mathcal{L}(\lambda_0, p_0) < \mathcal{L}(\lambda_1, p_1).$$

The inequality $\mathcal{L}(\lambda_0, p_1) < \mathcal{L}(\lambda_1, p_1)$ contradicts the fact that (λ_1, p_1) is a saddle point.

Suppose that λ^* satisfies the following conditions: there exists $p^* \in BP(\mu_0)$ such that $\mathcal{L}(\lambda^*, p^*) = cav(\hat{V}_{\lambda^*})(\mu_0)$, and for any $\theta \in \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\theta, p^*)$ and for any $\theta \notin \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) \leq \mathcal{L}(\theta, p^*)$. Then, by definition, we have that (λ^*, p^*) is a saddle point and $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\lambda_0, p_0)$. Hence, λ^* is the worst subjective prior.

If λ^* is the worst subjective prior, then choosing p^* is optimal and we can obtain that

$$\mathcal{L}(\lambda_0, p_0) = \min_{\lambda} \max_{p} \mathcal{L}(\lambda, p) \geq \mathcal{L}(\lambda^*, p^*) \geq \max_{p} \min_{\lambda} \mathcal{L}(\lambda, p) = \mathcal{L}(\lambda_0, p_0).$$

Therefore, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\lambda_0, p_0)$ is the mini-max or max-min value. It follows that for any $\lambda \in \Delta(\Theta)$, $\mathcal{L}(\lambda, p^*) \geq \mathcal{L}(\lambda^*, p^*)$ and for any $p \in BP(\mu_0)$, $\mathcal{L}(\lambda^*, p) \leq \mathcal{L}(\lambda^*, p^*)$. Thus, (λ^*, p^*) is a saddle point and it must satisfy that $\mathcal{L}(\lambda^*, p^*) = cav(\hat{V}_{\lambda^*})(\mu_0)$, and for any $\theta \in \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) = \mathcal{L}(\theta, p^*)$ and for any $\theta \notin \mathbf{supp}(\lambda^*)$, $\mathcal{L}(\lambda^*, p^*) \leq \mathcal{L}(\theta, p^*)$. $\square$

Proof of Proposition 5. The proof is analogous to the proof of Proposition 4. $\square$

Proof of Theorem 2. In this proof, $V(\mu)$ is no longer simply represents $\max \mathbb{V}(\mu)$. Back to the proof of Proposition 1, our construction of x in Equation 15 satisfies that if it is only possible that $t \leq C$, then $x(\mu) \leq C$ for all μ. Hence, the revelation principle can directly be generalized to this case by adding a new constraint that $x(\mu) \leq C$ for all $\mu \in \Delta(\Theta)$.

According to Proposition 2, Corollary 1 and previous discussions, we begin by characterizing $\mathcal{V}_C^*(\mu_0)$ as following optimization problem.

$$\begin{aligned}
&\max \quad k && && (17)\\
&\text{s.t} \quad k = \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)}(V(\mu) - x(\mu))dp(\mu) && \forall \theta \in \Theta && \text{(IC)}\\
&\qquad p \in BP(\mu_0) && && \text{(Bayes plausible)}\\
&\qquad V(\mu) \in \mathbb{V}(\mu) && \forall \mu \in \Delta(\Theta) && \text{(Obedience)}\\
&\qquad 0 \le x(\mu) \le C && \forall \mu \in \Delta(\Theta) &&
\end{aligned}$$

We adopt a two-step optimization approach. First, we fix the signaling scheme $p \in BP(\mu_0)$ and obedience condition $V(\mu)$ and then we find the optimal burning scheme $x(\mu)$ where $0 \le x(\mu) \le C$. Thus, now it is a linear programming problem. By the fundamental theorem of linear programming, we can also obtain $\mathcal{V}_C^*(\mu_0)$ from the following max-min problem.

$$\begin{aligned}
&\max_{p,V} \quad \min_{\lambda} \quad \int_\mu (\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu) + C \max\{0, -\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)}\}) dp(\mu) && (18)\\
&\text{s.t} \quad p \in BP(\mu_0) &&\\
&\qquad V(\mu) \in \mathbb{V}(\mu) && \forall \mu \in \Delta(\Theta)\\
&\qquad \sum_{\theta \in \Theta} \lambda(\theta) = 1 &&
\end{aligned}$$

Given any signaling scheme $p \in BP(\mu_0)$, since $\mathbb{V}(\mu)$ is a compact and convex set, and $\int_\mu (\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} V(\mu) + C \max\{0, -\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)}\}) dp(\mu)$ is convex in λ, and linear in $V(\mu)$, by Sion's minimax theorem we can interchange the $\max_V$ and $\min_\lambda$. Hence, we can obtain that $\mathcal{V}_C^*(\mu_0)$ can be solved by

$$\max_{p \in BP(\mu_0)} \min_{\sum_\theta \lambda(\theta) = 1} \int_\mu \max\{\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu), \sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} (\min \mathbb{V}(\mu) - C)\} dp(\mu). \tag{19}$$

This implies that we choose $V(\mu) = \max \mathbb{V}(\mu(m))$ if $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} > 0$ and $V(\mu) = \min \mathbb{V}(\mu(m))$ if $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} < 0$, which determines the best response selection rule.

Thus, we have shown that

$$\mathcal{V}^*(\mu_0) = \max_{p \in BP(\mu_0)} \min_{\sum_\theta \lambda(\theta)=1} \mathcal{L}_C(\lambda, p).$$

The rest of the proof relies on Sion's minimax theorem as well. It is easy to verify that $BP(\mu_0)$ is a compact and convex set, and $\{\lambda | \sum_\theta \lambda(\theta) = 1\}$ is a convex set. Moreover, $\mathcal{L}_C(\lambda, p)$ is linear in p and convex in λ since it is the maximum of two linear functions. Furthermore, $\mathcal{L}_C(\lambda, p)$ is continuous in λ. Since $\max \mathbb{V}(\mu)$ and $\min \mathbb{V}(\mu) - C$ are upper and lower semi-continuous, respectively, we have that $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu)$ is upper semi-continuous when $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} > 0$ and $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} (\min \mathbb{V}(\mu) - C)$ is upper semi-continuous when $\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} < 0$. Therefore,

$$\hat{V}_{\lambda,C}(\mu) = \max\{\sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu), \sum_{\theta \in \Theta} \lambda(\theta) \frac{\mu(\theta)}{\mu_0(\theta)} (\min \mathbb{V}(\mu) - C)\}$$

is upper semi-continuous and so is $\mathcal{L}_C(\lambda, p)$ in p. Hence, we can apply Sion's minimax theorem directly and complete the proof. $\square$

Proof of Proposition 6. By Fact 1, it suffices to show that $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$. Since $\mathcal{V}^*(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$, by Theorem 1, let $\lambda^* \in \Delta(\Theta)$ be the worst subjective prior and $p^* \in BP(\mu_0)$ be the optimal distribution of posterior that maximizes the minimum interim payoff. Then we have $\mathcal{L}(\lambda^*, p^*) = \mathcal{V}^*_{BP}(\mu_0)$. Hence, for any θ, we must have

$$\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) dp^*(\mu) = \mathcal{V}^*_{BP}(\mu_0),$$

Otherwise, $\mathcal{L}(\lambda^*, p^*) = \min_\theta \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) dp^*(\mu) < \mathcal{V}^*_{BP}(\mu_0)$, since $\mathcal{L}(\lambda^*, p^*)$ is the minimum interim payoff and $\mathcal{V}^*_{BP}(\mu_0)$ is the expected interim payoff. Therefore,

$$\mathcal{V}^*_{BP}(\mu_0) = \sum_\theta \mu_0(\theta) \mathcal{V}^*_{BP}(\mu_0) = \int_\mu \max \mathbb{V}(\mu) dp^*(\mu).$$

Thus, p^* is also the concavification of $\max \mathbb{V}(\cdot)$ at point μ_0. By the same technique of proposition 9 of the working paper version of Kamenica and Gentzkow [2011], we deduce that $(\mu, \max \mathbb{V}(\mu))$ for $\mu \in \mathbf{supp}\{p^*\}$ are coplanar. This means that there exist parameters A_θ for $\theta \in \Theta$ such that for $\mu \in \mathbf{supp}\{p^*\}$,

$$\max \mathbb{V}(\mu) = \sum_\theta A_\theta \mu(\theta).$$

Then, for any θ, we have

$$\int_\mu \mu(\theta) \sum_{\theta'} A_{\theta'} \mu(\theta') dp^*(\mu) = \mu_0(\theta) \sum_\theta A_\theta \mu_0(\theta),$$

Multiplying by A_θ and summing over θ, we obtain

$$\int_\mu (\sum_\theta A_\theta \mu(\theta))^2 dp^*(\mu) = (\sum_\theta A_\theta \mu_0(\theta))^2.$$

By Cauchy inequality, we have

$$(\sum_\theta A_\theta \mu_0(\theta))^2 = \int_\mu dp^*(\mu) \int_\mu (\sum_\theta A_\theta \mu(\theta))^2 dp^*(\mu) \geq (\int_\mu A_\theta \mu(\theta) dp^*(\mu))^2 = (\sum_\theta A_\theta \mu_0(\theta))^2.$$

Therefore, we get $\sum_\theta A_\theta \mu(\theta) = \sum_\theta A_\theta \mu'(\theta)$ for all $\mu, \mu' \in \mathbf{supp}\{p^*\}$, which implies that $\max \mathbb{V}(\mu) = \max \mathbb{V}(\mu')$. This means that all the posterior beliefs induce the same value for Sender, so p^* is also a cheap talk equilibrium, i.e., Sender cannot find a more profitable message. Hence, $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_{BP}(\mu_0)$. □

Proof of Theorem 3. To facilitate the whole proof, we first establish two lemmas that reveal some useful properties of $\mathcal{V}^*(\mu)$ and one important proposition.

Lemma 3. *For any signaling scheme $p \in BP(\mu_0)$, we have that*

$$\mathcal{V}^*(\mu_0) \geq \min_{\theta \in \boldsymbol{supp}\{\mu_0\}} \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \mathcal{V}^*(\mu) dp(\mu).$$

Proof of Lemma 3. Let p_μ denote the optimal signaling scheme that maximizes the minimum interim payoff under any prior $\mu \in \mathbf{supp}\{p\}$. We consider a signaling scheme for μ_0 that consists of two stages: first, it splits μ_0 according to p, and then, for each $\mu \in \mathbf{supp}\{p\}$, it further splits μ according to p_μ. By applying this scheme, we obtain the following inequality:

$$\mathcal{V}^*(\mu_0) \geq \min_{\theta \in \mathbf{supp}\{\mu_0\}} \{\int_\mu \int_{\mu'} \frac{\mu'(\theta)}{\mu_0(\theta)} V(\mu') dp(\mu) dp_\mu(\mu')\} = \min_{\theta \in \mathbf{supp}\{\mu_0\}} \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \mathcal{V}^*(\mu) dp(\mu).$$

$\square$

Lemma 4. *$\mathcal{V}^*(\mu)$ is continuous at any full-support point μ, i.e. $\mu(\theta) > 0$ for all $\theta \in \Theta$.*

Proof of Lemma 4. We begin by recalling that, by Theorem 1, $\mathcal{V}^*(\mu)$ is the infimum of a family of continuous functions, namely the concave closures. Hence, $\mathcal{V}^*(\mu)$ is upper semi-continuous. Without loss of generality, we assume that $v(a) \geq 0$ for all $a \in A$. Suppose that $\mathcal{V}^*(\mu)$ is discontinuous at some point μ with full support. Then there exist two priors $\mu_1, \mu_2 \in \Delta(\Theta)$ and a positive constant D such that, for some sufficiently small constant $B(D, \mu_1, \mu_2)$ (denoted by B), we have $\mathcal{V}^*(\mu) > \mathcal{V}^*(\mu + \varepsilon(\mu_1 - \mu_2)) + D$ for all $\varepsilon \in (0, B)$. Let $\mu_\varepsilon = \mu + \varepsilon(\mu_1 - \mu_2)$ for any $\varepsilon > 0$ such that $\varepsilon < B$ and $\varepsilon^2 < B$. We can rewrite μ_ε as $\mu_\varepsilon = (1 - \sqrt{\varepsilon})\mu + \sqrt{\varepsilon}(\mu + \sqrt{\varepsilon}(\mu_1 - \mu_2))$. Applying Lemma 3, we get

$$
\begin{aligned}
\mathcal{V}^*(\mu_\varepsilon) &\geq \min_\theta\{(1 - \sqrt{\varepsilon})\frac{\mu(\theta)}{\mu_\varepsilon(\theta)}\mathcal{V}^*(\mu) + \sqrt{\varepsilon}\frac{\mu(\theta) + \sqrt{\varepsilon}(\mu_1(\theta) - \mu_2(\theta))}{\mu_\varepsilon(\theta)}\mathcal{V}^*(\mu + \sqrt{\varepsilon}(\mu_1 - \mu_2))\} \\
&\geq \min_\theta\{(1 - \sqrt{\varepsilon})\frac{\mu(\theta)}{\mu_\varepsilon(\theta)}\mathcal{V}^*(\mu)\}.
\end{aligned}
$$

Because $\mu(\theta) > 0$ for any θ, $\lim_{\varepsilon \to 0^+}(1 - \sqrt{\varepsilon})\frac{\mu(\theta)}{\mu_\varepsilon(\theta)} = 1$. Therefore it follows that $\lim_{\varepsilon \to 0^+} \mathcal{V}^*(\mu_\varepsilon) \geq \mathcal{V}^*(\mu)$, which contradicts the assumption that $\mathcal{V}^*(\mu_\varepsilon) < \mathcal{V}^*(\mu) - D$. $\square$

Proposition 9. *Assuming that $\mathcal{V}^*(\mu)$ is continuous, we obtain $\mathcal{V}^*_{CT}(\mu_0) = \mathcal{V}^*_0(\mu_0) = \mathcal{V}^*(\mu_0)$ when $\mathcal{V}^*(\mu_0) = \mathcal{V}^*_0(\mu_0)$.*

Proof of Proposition 9. If $\mathcal{V}^*$ is continuous at the boundary, then $\mathcal{V}^*$ is continuous at any point in $\Delta(\Theta)$. To this end, we adopt a proof by contradiction. Let μ_0 be a belief with the smallest support such that $\mathcal{V}^*_{CT}(\mu_0) < \mathcal{V}^*(\mu_0) = \mathcal{V}^*_0(\mu_0)$. Then, applying Theorem 1 and Theorem 2, we infer that there is a posterior distribution p^* that yields the same interim payoff for every θ under p^*. By Proposition 5, we obtain that there is a worst subjective prior $\lambda^* \in \Delta(\Theta)$ such that for any θ,

$$
\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) dp^*(\mu) = \int_\mu \sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) dp^*(\mu),
$$

and p^* is the concavification of $\sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu)$ at point μ_0. Then we have that there exist parameters A_θ for $\theta \in \Theta$ and for any $\mu \in \mathbf{supp}\{p^*\}$,

$$\sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) = \sum_\theta A_\theta \mu(\theta).$$

Let $\Theta' = \{\theta \big| \lambda^*(\theta) = 0\}$, $U_2 = \{\mu \in \mathbf{supp}\{p^*\} \big| \mathbf{supp}\{\mu\} \subseteq \Theta'\}$, and $U_1 = \mathbf{supp}\{p^*\}/U_2$.

Hence, we obtain that

$$\int_{\mu \in U_1} \sum_\theta A_\theta \mu(\theta) \frac{\sum_\theta A_\theta \mu(\theta)}{\sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)}} dp^*(\mu) = (\sum_\theta A_\theta \mu_0(\theta))^2.$$

Since $\int_{\mu \in U_1} \sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)} dp^*(\mu) = 1$, by Cauchy's inequality and the above equation, we also have that for any $\mu, \mu' \in U_1$,

$$\frac{\sum_\theta A_\theta \mu(\theta)}{\sum_\theta \frac{\lambda^*(\theta)\mu(\theta)}{\mu_0(\theta)}} = \frac{\sum_\theta A_\theta \mu'(\theta)}{\sum_\theta \frac{\lambda^*(\theta)\mu'(\theta)}{\mu_0(\theta)}}.$$

This implies that $\max \mathbb{V}(\mu) = \max \mathbb{V}(\mu')$ for $\mu, \mu' \in U_1$. To simplify the notation, we set this value to be R, which coincides with $\mathcal{V}^*(\mu_0)$ and $\mathcal{V}_0^*(\mu_0)$.

If $U_2 = \emptyset$, then it is also a cheap talk equilibrium and we obtain that $\mathcal{V}^*(\mu_0) = \mathcal{V}_{CT}^*(\mu_0)$. Otherwise, let $p_1 = \int_{\mu \in U_1} dp^*(\mu), p_2 = \int_{\mu \in U_2} dp^*(\mu)$ and $\mu_1 = \int_{\mu \in U_1} \mu dp^*(\mu)/p_1, \mu_2 = \int_{\mu \in U_2} \mu dp^*(\mu)/p_2$. Since $\mathbf{supp}\{\mu_2\} < \mathbf{supp}\{\mu_0\}$, we must have $\mathcal{V}^*(\mu_2) = \mathcal{V}_0^*(\mu_2)$ if and only if $\mathcal{V}^*(\mu_2) = \mathcal{V}_{CT}^*(\mu_2)$. Then for any $\theta \in \mathbf{supp}\{\mu_2\}$, we have that

$$R = \int_\mu \frac{\mu(\theta)}{\mu_0(\theta)} \max \mathbb{V}(\mu) dp^*(\mu) = p_1 \frac{\mu_1(\theta)}{\mu_0(\theta)} R + p_2 \frac{\mu_2(\theta)}{\mu_0(\theta)} \int_{\mu \in U_2} \frac{\mu(\theta)}{\mu_2(\theta)} \max \mathbb{V}(\mu) d\frac{p^*(\mu)}{p_2}.$$

Hence,

$$\int_{\mu \in U_2} \frac{\mu(\theta)}{\mu_2(\theta)} \max \mathbb{V}(\mu) d\frac{p^*(\mu)}{p_2} = R.$$

This implies that $\mathcal{V}_0(\mu_2) \geq R$. Next we will divide our final proof into two cases.

Case 1: If $\mathcal{V}^*(\mu_2) = R$, we have $\mathcal{V}^*(\mu_2) = \mathcal{V}_0(\mu_2)$, which implies that $\mathcal{V}_{CT}(\mu_2) = R$. Furthermore, since $\max \mathbb{V}(\mu) = R$ for all $\mu \in U_1$, we can deduce that there exists a cheap talk equilibrium for μ_0 where Sender achieves payoff R. This means that we have shown that $\mathcal{V}_{CT}(\mu_0) = R$, which contradicts our assumption.

Case 2: Suppose that $\mathcal{V}^*(\mu_2) > R$. Given the continuity of $\mathcal{V}^*(\mu)$, we can find a positive constant $\varepsilon > 0$, such that $\mathcal{V}^*(\frac{\varepsilon\mu_1 + p_2\mu_2}{\varepsilon + p_2}) > R$ and $\mu_0 = (p_1 - \varepsilon)\mu_1 + (\varepsilon + p_2)\frac{\varepsilon\mu_1 + p_2\mu_2}{\varepsilon + p_2}$.

Hence by Lemma 3, we have that

$$\mathcal{V}^*(\mu_0) \geq \min_{\theta \in \mathbf{supp}\{\mu_0\}} \{(p_1 - \varepsilon)\frac{\mu_1(\theta)}{\mu_0(\theta)}\mathcal{V}^*(\mu_1) + \frac{\varepsilon\mu_1(\theta) + p_2\mu_2(\theta)}{\mu_0(\theta)}\mathcal{V}^*(\frac{\varepsilon\mu_1 + p_2\mu_2}{\varepsilon + p_2})\}.$$

Since $\mathcal{V}^*(\mu_1) \geq R$, $\mathcal{V}^*(\frac{\varepsilon\mu_1+p_2\mu_2}{\varepsilon+p_2}) > R$ and for $\theta \in \mathbf{supp}\{\mu_0\}$, $\varepsilon\mu_1(\theta) + p_2\mu_2(\theta) > 0$, we will get that

$$\mathcal{V}^*(\mu_0) > R,$$

which is also a contradiction. □

Back to the proof of Theorem 3, without loss of generality, we assume that $v(a) \geq 0$ for all $a \in A$. According to Proposition 9 and Lemma 4, it is suffice to show that under generic settings $\mathcal{V}^*$ is continuous at the boundary of $\Delta(\Theta)$. To prove this, we firstly prove the following lemma.

Lemma 5. *For any belief μ at the boundary and any direction $\mu_1 \in \Delta(\Theta)$, we have*

$$\lim_{\varepsilon \to 0^+} \mathcal{V}^*((1-\varepsilon)\mu + \varepsilon\mu_1) \geq V(\mu).$$

proof of Lemma 5. If $\mathbf{supp}\{\mu_1\} \subseteq \mathbf{supp}\{\mu\}$, then by Lemma 4, we can get

$$\lim_{\varepsilon \to 0^+} \mathcal{V}^*((1-\varepsilon)\mu + \varepsilon\mu_1) = \mathcal{V}^*(\mu) \geq V(\mu).$$

Consider the case where $\mathbf{supp}\{\mu_1\} \nsubseteq \mathbf{supp}\{\mu\}$. By the genericity of the setting, we can find a belief μ_2 such that $\mathbf{supp}\{\mu_2\} = \mathbf{supp}\{\mu\}$ and $V(\mu_2) = V(\mu)$, and moreover, the singleton $RO(\mu_2) \subseteq RO(\mu)$. It follows that for any $\gamma \in (0,1)$, we have $RO((1-\gamma)\mu + \gamma\mu_2) = RO(\mu_2)$ and $V((1-\gamma)\mu + \gamma\mu_2) = V(\mu)$. Let $\mu_\gamma = (1-\gamma)\mu + \gamma\mu_2$. Since $RO(\mu_\gamma)$ is a singleton, implying that the action in this singleton dominates all other actions, we can identify the largest ε_γ such that $0 < \varepsilon_\gamma < 1$ and $RO(\mu_2) \subseteq RO((1-\varepsilon_\gamma)\mu_\gamma + \varepsilon_\gamma\mu_1)$. We then proceed to examine two cases.

Case 1: $\lim_{\gamma \to 0^+} \varepsilon_\gamma \neq 0$, which means that $\lim_{\varepsilon \to 0^+} V(\varepsilon\mu_1 + (1-\varepsilon)\mu) = V(\mu)$. Thus,

$$\lim_{\varepsilon \to 0^+} \mathcal{V}^*((1-\varepsilon)\mu + \varepsilon\mu_1) \geq \lim_{\varepsilon \to 0^+} V((1-\varepsilon)\mu + \varepsilon\mu_1) = V(\mu).$$

Case 2: $\lim_{\gamma \to 0^+} \varepsilon_\gamma = 0$. Then consider following splitting scheme of $(1-\varepsilon_\gamma)\mu_0 + \varepsilon_\gamma\mu_1$

that

$$(1-\varepsilon_\gamma)\mu_0+\varepsilon_\gamma\mu_1=\min_\theta\frac{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}((1-\varepsilon_\gamma)\mu_\gamma+\varepsilon_\gamma\mu_1)+P_2\mu'$$

where $P_2=1-\min_\theta\frac{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}$ and $\mu'\in\Delta(\Theta)$. By Lemma 3 we have that

$$\begin{aligned}\lim_{\varepsilon\to0^+}\mathcal{V}^*((1-\varepsilon)\mu+\varepsilon\mu_1)&=\lim_{\gamma\to0^+}\mathcal{V}^*((1-\varepsilon_\gamma)\mu+\varepsilon_\gamma\mu_1)\\&\geq\lim_{\gamma\to0^+}\min_\theta\frac{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}\min_\theta\frac{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}V(\mu).\end{aligned}$$

For any $\theta\in\mathbf{supp}\{\mu_0\}$,

$$\lim_{\gamma\to0^+}\frac{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}=\lim_{\gamma\to0^+}\frac{\mu_0(\theta)}{\mu_\gamma(\theta)}=1,$$

and for any $\theta\in\mathbf{supp}\{\mu_1\}/\mathbf{supp}\{\mu_0\}$,

$$\lim_{\gamma\to0^+}\frac{(1-\varepsilon_\gamma)\mu_0(\theta)+\varepsilon_\gamma\mu_1(\theta)}{(1-\varepsilon_\gamma)\mu_\gamma(\theta)+\varepsilon_\gamma\mu_1(\theta)}=1.$$

So we obtain that for any $\mu_1\in\Delta(\Theta)$,

$$\lim_{\varepsilon\to0^+}\mathcal{V}^*((1-\varepsilon)\mu+\varepsilon\mu_1)\geq V(\mu).$$

$\square$

Building on Lemma 5, we demonstrate the continuity of $\mathcal{V}^*$ at any point μ. Let p^* be the optimal signaling scheme of μ_0 that attains the minimum interim payoff, that is, $\mathcal{V}^*(\mu_0)=\min_{\theta\in\mathbf{supp}\{\mu_0\}}\int_\mu\frac{\mu(\theta)}{\mu_0(\theta)}V(\mu)dp^*(\mu)$. Then by Lemma 3, for any $\mu_1\in\Delta(\Theta)$, we have that

$$\begin{aligned}\lim_{\varepsilon\to0^+}\mathcal{V}^*((1-\varepsilon)\mu_0+\varepsilon\mu_1)&\geq\lim_{\varepsilon\to0^+}\min_\theta\int_\mu\frac{(1-\varepsilon)\mu(\theta)+\varepsilon\mu_1(\theta)}{(1-\varepsilon)\mu_0(\theta)+\varepsilon\mu_1(\theta)}\mathcal{V}^*((1-\varepsilon)\mu+\varepsilon\mu_1)dp^*(\mu)\\&\geq\mathcal{V}^*(\mu_0).\end{aligned}$$

The last inequality holds because

$$\lim_{\varepsilon\to0^+}\min_{\theta\in\mathbf{supp}\{\mu_0\}}\int_\mu\frac{(1-\varepsilon)\mu(\theta)+\varepsilon\mu_1(\theta)}{(1-\varepsilon)\mu_0(\theta)+\varepsilon\mu_1(\theta)}\mathcal{V}^*((1-\varepsilon)\mu+\varepsilon\mu_1)dp^*(\mu)\geq\min_{\theta\in\mathbf{supp}\{\mu_0\}}\int_\mu\frac{\mu(\theta)}{\mu_0(\theta)}V(\mu)dp^*(\mu),$$

and for $\theta \in \mathbf{supp}\{\mu_1\}/\mathbf{supp}\{\mu_0\}$

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\mu \frac{(1-\varepsilon)\mu(\theta)+\varepsilon\mu_1(\theta)}{(1-\varepsilon)\mu_0(\theta)+\varepsilon\mu_1(\theta)}\mathcal{V}^*((1-\varepsilon)\mu+\varepsilon\mu_1)dp^*(\mu) &\geq \int_\mu V(\mu)dp^*(\mu) \\ &= \sum_\theta \mu_0(\theta)\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)}V(\mu)dp^*(\mu) \\ &\geq \min_{\theta\in\mathbf{supp}\{\mu_0\}}\int_\mu \frac{\mu(\theta)}{\mu_0(\theta)}V(\mu)dp^*(\mu).\end{aligned}$$

Using the fact that $\mathcal{V}^*$ is upper-semi continuous, we establish the continuity of $\mathcal{V}^*$. □

Proof of Proposition 7. To show that $\mathcal{V}_0^*(\mu_0) < \mathcal{V}^*(\mu_0)$, it suffices to show that $\mathcal{V}_{CT}^*(\mu_0) < \mathcal{V}^*(\mu_0)$. By Corollary 2, we only need to construct a posterior distribution $p \in BP(\mu_0)$ such that the minimum interim payoff under this distribution is greater than $\mathcal{V}_{CT}^*(\mu_0)$.

Since the setting is generic, then there must be a full-support $\hat{\mu}$ such that $\mathbb{V}(\hat{\mu}) = \max_{a\in A} v(a)$. We define $\mu(x) = \frac{\mu_0 - x\hat{\mu}}{1-x}$ and, since μ_0 has full support, there exists a small enough $\varepsilon > 0$ such that $\mu(\varepsilon) \in \Delta(\Theta)$ and $qcav(\max\mathbb{V})(\mu(\varepsilon)) = qcav(\max\mathbb{V})(\mu_0)$. Let τ be the distribution of posterior that is the quasi-concavification of $\max V$ at point $\mu(\varepsilon)$. Then we construct the distribution of posterior τ^* that acts as τ with probability $1-\varepsilon$ and induces the posterior $\hat{\mu}$ with probability ε. Then the interim payoff of type θ under τ^* is

$$qcav(\max\mathbb{V})(\mu_0) + \varepsilon\frac{\hat{\mu}(\theta)}{\mu_0(\theta)}(\max\mathbb{V}(\hat{\mu}) - qcav(\max\mathbb{V})(\mu_0))$$

Since $qcav(\max\mathbb{V})(\mu_0) \neq cav(\max\mathbb{V})(\mu_0)$, we have that $\max\mathbb{V}(\hat{\mu}) = \max_{a\in A} v(a) > qcav(\max\mathbb{V})(\mu_0)$. Hence, we obtain that $\mathcal{V}^*(\mu_0) > \mathcal{V}_{CT}^*(\mu_0)$.

□