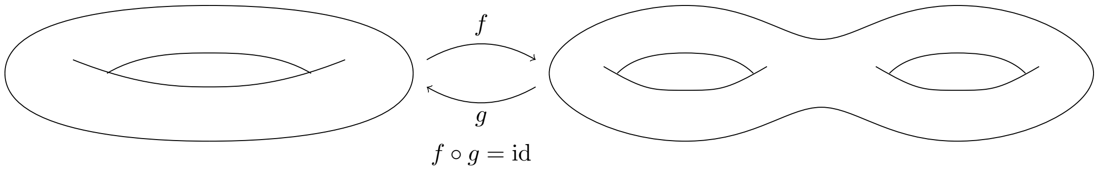
Algebraic Topology

October 23, 2019

1 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions f, g going from a topological space X to another space Y and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



If such f, g continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space X a group $G(X)$, and for every continuous function $f : X \to Y$ a group homomorphism $G(f) : G(X) \to G(Y)$ with $G(\mathrm{id}) = \mathrm{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \to Y$ is a homeomorphism with inverse $g : Y \to X$, then $G(g) \circ G(f) = \mathrm{id}, G(f) \circ G(g) = \mathrm{id}$, so $G(f)$ is an is an isomorphism.

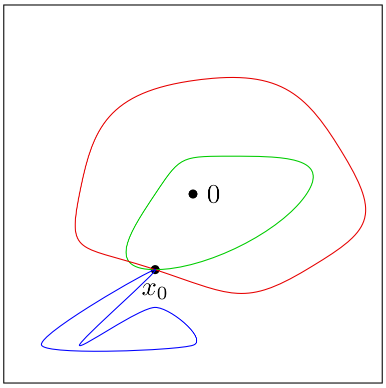
Extension problem: Let X be a topological space, $A \subseteq X$ a subspace, and $f : A \to Y$ a continuous function. Does there exist a continuous function $F : X \to Y$ with $F|_A = f$

$$\begin{tikzcd} X \arrow[r, dashed, "\exists ?"'] & Y \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

Theorem 1.1. *There is no continuous function*

$$f : D^n \to S^{n-1} \text{ with } f|_{S^{n-1}} = \mathrm{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct G with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \to D^n \to S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \to 0 \to \mathbb{Z}$ being the identity.



Here, the green and red loops are the "same" loop, whilst the blue one is distinct

Conventions

A topological space will be referred to as a ***space***
A continuous function $f : X \to Y$ will be called a ***map***

2 The Fundamental Group

The idea here is that, if X is a space, $x_0 \in X$ a fixed point, called the ***basepoint***, we consider loops based at x_0, i.e. maps $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the ***fundamental group*** $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at x_0 modulo "deforming loops". Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing γ_1 and then γ_2. But what do we mean by "deforming" a loop?

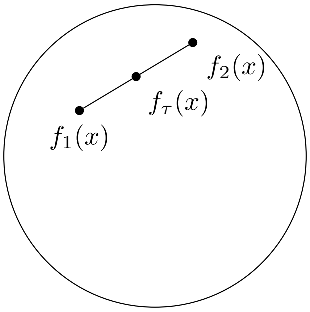
Let $f_0, f_1 : X \to Y$ be maps. A ***homotopy*** between f_0 and f_1 is a map

$$\begin{aligned} &F : X \times I \to Y \text{ where } I = [0,1] \text{ and} \\ &F(x,0) = f_0(x) \text{ and} \\ &F(x,1) = f_1(x) \end{aligned}$$

We often write $f_\tau(x) = F(x,\tau), f_\tau : X \to Y$.

If such F exists, we say f_0 and f_1 are ***homotopic***.

Example: Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \to Y$ are homotopic, via $F(x,t) = tf_1(x) + (1-t)f_0(x) \in Y$ by convexity.



If f_0 is homotopic to f_1, we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \to Y$. If $Z \subseteq X$ and $f_0(z) = F(z,t) = f_1(z) \forall z \in Z, t \in I$, then we say f_0 is homotopic to f_1 ***relative to*** Z.

Lemma 2.1. *Let $Z \subseteq X, Y$ be spaces. Then $\simeq$ relative to Z is an equivalence relation on the set of maps $X \to Y$.*

Proof.

- Reflexive: $f_0 \simeq f_0$ via $F(x,t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x,t) = f(x, 1-t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x,t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t-1) & t \geq 1/2 \end{cases}$$

  All homotopies are relative to Z.

□

A ***homotopy equivalence*** $f : X \to Y$ is a map with a ***homotopy inverse*** $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$. We then write $X \simeq Y$.

Remark: Most (all?) invariants in the course are ***homotopy invariants***

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \to *$ be the constant map, and let $g : * \to \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \mathrm{id}_*$, and $g \circ f(x) = 0 \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(x,t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map x to the intersection of $\overrightarrow{\mathbf{0}x}$ with S^{n-1}) . Then $g \circ f = \mathrm{id}_{S^{n-1}}$ and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x,t) = (1-t)x + t \cdot \frac{x}{|x|}$

If $X \simeq *$, then we say X is ***contractible***.
Let $f : X \to Y, g : Y \to X$ be maps. If $g \circ f = \mathrm{id}_X$, then we say X is a ***retract*** of Y, and g is a ***retraction***. If in addition $f \circ g \simeq \mathrm{id}_Y$ relative to $f(X)$, then we say X is a ***deformation retract*** of Y. Hence, in example 2, we see that S^{n-1} is a deformation retract of $\mathbb{R}^n$.

Lemma 2.2. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

We want to show $f' \circ f$, $g \circ g'$ induces a homotopy equivalence

$$\begin{tikzcd} X \arrow[r, bend left, "f' \circ f"] & Z \arrow[l, bend left, "g \circ g'"] \end{tikzcd}$$

Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \mathrm{id}_Y$, and so:

$$(x,t) \mapsto g(F'(f(x),t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y$, $g \circ f \simeq \mathrm{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \mathrm{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \mathrm{id}_Z$ □

Loops and π_1

If X is a space, a ***path*** in X is a map $\gamma : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say γ is a path ***from*** x_0 ***to*** x_1.
We say γ_1 and γ_2 are ***homotopic*** if $\gamma_1 \simeq \gamma_2$ relative to $\{0,1\}$, and we write $[\gamma]$ for the homotopy equivalence class of γ.

If X is a space with points $x, y, z \in X$, and γ_1 is a path from x to y, γ_2 is a path from y to z, then:

- The ***concatenation*** of γ_1 and γ_2 is the path from x to z given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

- The ***constant path*** at x is the path $c_x(s) = x \forall s \in I$
- The ***inverse of*** γ_1 is $\overline{\gamma_1}(s) = \gamma_1(1-s)$, a path from y to x.

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend left=50, "\gamma_1 \cdot \gamma_2"] & y \arrow[l, bend left, "\overline{\gamma_1}"] \arrow[r, bend right, "\gamma_2"'] & z \end{tikzcd}$$

Theorem 2.3. *Let X be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X with endpoint x_0 (we say they are* **based** *at x_0). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity c_{x_0} and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.*

This group is called the **fundamental group** *of X (based at x_0).*

To prove this, we will need the following lemmas:

Lemma 2.4. *If $\gamma_0 \simeq \gamma_1$ to y and $\delta_0 \simeq \delta_1$ from y, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$*

$$\begin{tikzcd} {} \arrow[r, bend left, "\gamma_0"] \arrow[r, bend right, "\gamma_1"'] & y \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] & {} \end{tikzcd}$$

Proof. Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s,t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s-1, t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s,t) = F(1-s, t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. □

Lemma 2.5. *Let α, β, γ be paths from w to x to y to z in X.*

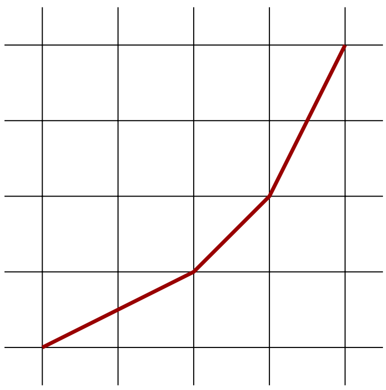
$$\begin{tikzcd} w \arrow[r, bend left, "\alpha"] & x \arrow[r, bend right, "\beta"] & y \arrow[r, bend left, "\gamma"] & z \end{tikzcd}$$

Then:

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha\beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

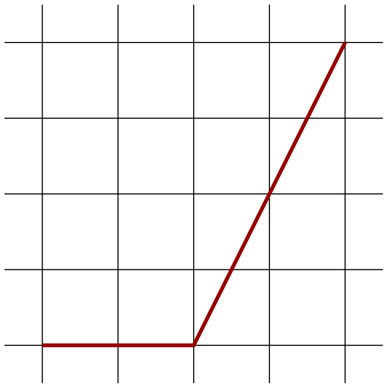
Proof. First, given a path $\delta : I \to X$, a ***reparametrization*** of δ is a path $\delta \circ \phi$ where $\phi : I \to I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that ϕ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s,t) = \delta(t\phi(s) + (1-t)s)$, and this homotopy is relative to $\{0, 1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function ϕ whose plot is:

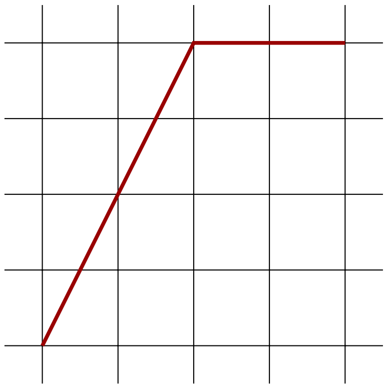


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize α via:



i.e. do c_w for the first half of the time, then do α, so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s,t) = \begin{cases} \alpha(2s) & 0 \le s \le t/2 \\ \alpha(t) & t/2 \le s \le 1 - t/2 \\ \alpha(2-2s) & 1 - t/2 \le s \le 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have c_w at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

□

Then theorem **1.3** giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If γ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

Formal Properties of π_1

Lemma 2.6. *Let $f : X \to Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

Furthermore:

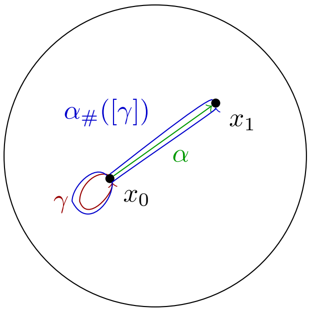
1. *If $f \simeq f'$ relative to x_0, then $f'_* = f_*$.*
2. *If $g : Y \to Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$*
3. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$

Proof. f_* is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to x_0, then for γ a loop based at x_0, $(s,t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

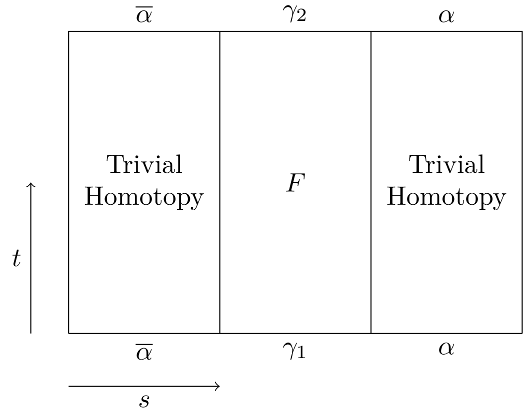
Lemma 2.7. *let X be a space, $x_0, x_1 \in X$ and α a path from x_0 to x_1. Then there is a group isomorphism $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.*



Furthermore,

1. *If $\alpha \simeq \alpha'$ relative to $\{0, 1\}$, then $\alpha_\# = \alpha'_\#$.*
2. $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$
3. *If β is a path from x_2 to x_2, then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$*
4. *If $f : X \to Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.*

Proof. Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

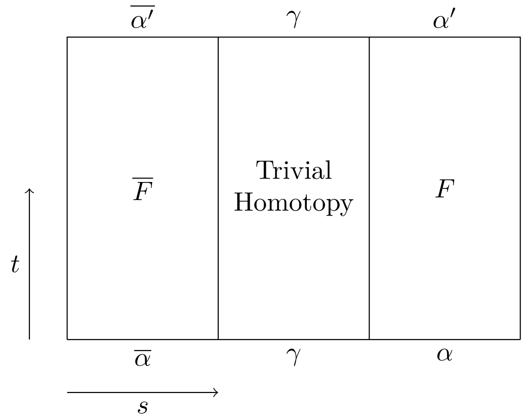


This is indeed a group homomorphism: for loops γ, δ based at x_0,

$$\begin{aligned}\bar{\alpha} \cdot \gamma \cdot \alpha) \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\ &\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\ &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\ &\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha\end{aligned}$$

Thus $\alpha_{\#}(\gamma \cdot \delta) = \alpha_{\#}(\gamma) \cdot \alpha_{\#}(\delta)$. Also $\bar{\alpha}_{\#} = (\alpha_{\#})^{-1}$ - this is easy to check. Thus $\alpha_{\#}$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$



   gives $\alpha_{\#}(\gamma) \simeq \alpha'_{\#}(\gamma)$

2. Immediate since c_{x_0} is the identity in $\pi_1(X, x_0)$.

3.

$$\begin{aligned}(\alpha \cdot \beta)_{\#}(\gamma) &= \alpha \bar{\cdot} \beta \cdot \gamma \cdot \alpha \cdot \beta \\ &= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta \\ &= \bar{\beta} \cdot \alpha_{\#}(\gamma) \cdot \beta \\ &= \beta_{\#}(\alpha_{\#}(\gamma))\end{aligned}$$

4.

$$
\begin{aligned}
((f\circ\alpha)_\# \cdot f_*)(\gamma) &= (f\circ\alpha)_\#(f\cdot\gamma)\\
&= (f\circ\alpha)_\#(f\cdot\gamma)\\
&= \overline{f\cdot\alpha}\cdot(f\circ\gamma)\cdot(f\circ\alpha)\\
&= f\circ(\bar{\alpha}\cdot\gamma\cdot\alpha)\\
&= f_*(\alpha_\#(\gamma))
\end{aligned}
$$

$\square$

A path connected space X is ***simply connected*** if $\pi_1(x,x_0)=0$ for any, and hence all, $x_0\in X$.

Our aim here is to prove that π_1 is a ***homotopy invariant***, i.e. that homotopy equivalent spaces have the same fundamental group. We will start with the following lemma:

Lemma 2.8. *Let $x_0\in X$ and $f,g:X\to Y$ with $f\simeq_F g$. Set $x(t)=F(x_0,t)$ so that $\alpha(0)=f(x_0)$ and $\alpha(1)=g(x_0$. Then the diagram:*

$$
\begin{tikzcd}
 & \pi_1(Y,f(x_0)) \arrow[dd, "\alpha_\#"] \\
\pi_1(x,x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y,g(x_0))
\end{tikzcd}
$$

commutes, i.e. we have $\alpha_\#\circ f_*=g_*$.

Proof. We need to check that, for a loop γ based at x_0, $\overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha\simeq g\circ\gamma$.

Let $G:I\times I\to Y$ defined by $G(s,t)=F(\gamma(s),t)$. For $t=0$, this is $f\circ\gamma$, and for $t=1$, this is $g\circ\gamma$. Now consider two paths in $I\times I$:

$$
\begin{gathered}
a(t)=(t,1); b=b_1\cdot b_2\cdot b_3 \text{ where:}\\
b_1(t)=(0,1-t), b_2(t)=(t,0), b_3(t)=(1,t)
\end{gathered}
$$

Then $(G\circ a)(s)=G(s,1)=g\circ\gamma(s)$, whilst $G\circ b=\overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha$.

Now, since $I\times I$ is convex, we have that $a\simeq_H b$, and so $G\circ H$ is the desired homotopy between $g\circ\gamma$ and $\overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha$. $\square$

Theorem 2.9. *If $f:X\to Y$ is a homotopy equivalence, then $f_*:\pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is a homomorphism for any $x_0\in X$.*

Proof. We'll show that f_* is a bijection:

Let $g:Y\to X$ be a homotopic inverse to f, with $\mathrm{id}_X\simeq_F g\circ f$. Let $\alpha:I\to X$ given by $\alpha(t)=F(x_0,t)$.

Note that $f_*:\pi_1(X,x_0)\to\pi_1(Y,f(x_0)); g:\pi_1(Y,f(x_0))\to\pi_1(X,g(f(x_0)))$

Then $g_*\circ f_*=(g\circ f)_*=\alpha_\#\circ(\mathrm{id}_X)_*=\alpha_\#$. $\alpha_\#$ is an isomorphism, and so f_* is injective.

If $\mathrm{id}_Y \simeq_G f \circ g$ let $\beta(t) = G(f(x_0), t)$ Then $f_* \circ g_* = (g \circ f)_* = \beta_\# \circ (\mathrm{id}_Y)_* = \beta_\#$, an isomorphism, and hence f_* is surjective. $\square$

Corollary 2.10. *Contractible spaces are simple connected.*

Proof. If X is contractible, there exists some $x_0 \in X$ and a homotopy F between id_X and $X \to \{x_0\}$. So $F(x, \cdot)$ is a path from any $x \in X$ to x_0, so X is path connected. Since X is homotopic to $\{x_0\}$, $\pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0) = 0$. $\square$

Covering Spaces

Let $p : \hat{X} \to X$ be a map. An open set $U \subseteq X$ is ***evenly covered*** if there exists a set Δ_U with the discrete topology and there is a homeomorphism:

$$p^{-1}(U) \xrightarrow{\cong} U \times \Delta_U$$

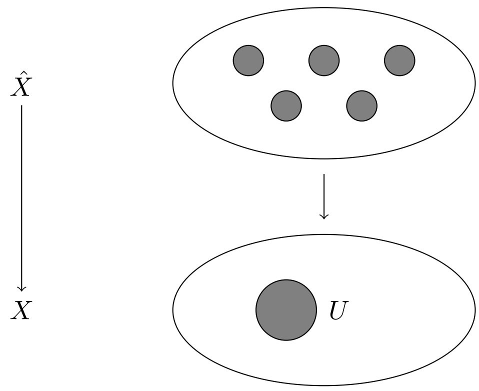
such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Delta_U \arrow[dl, "(x,\delta) \mapsto x"] \\ & U & \end{tikzcd}$$

We write, for $\delta \in \Delta_0$, $U_\delta = U \times \{\delta\}$ and $p_\delta = p|_{U_\delta}$. So $p_\delta : U_\delta \to U$ is a homeomorphism.

Note that we can canonically identify Δ_U with $p^{-1}(x)$ for any $x \in U$, Note also that $p^{-1}(U) \cong \coprod_{\delta \in \Delta_U} U_\delta$, where $\coprod$ denotes disjoint union.

If every point of X has an open neighbourhood which is evenly covered, then we say that p is a ***covering map*** and $\hat{X}$ is a ***covering space*** of X.



<u>Examples:</u>

1. $\hat{X} = X \times \Delta$ for Δ a set with the discrete topology, e.g. $\hat{I} = I \times \{1, 2, 3\}$. Then $\hat{X}$ is a covering space of X, the identity map on the first element is a covering map.
2. $\hat{X} = \mathbb{R}, X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p : \mathbb{R} \to S^1$ and $p(t) = \exp(2\pi\mathfrak{i} \cdot t)$. Them p is a covering map:

let $U = S \setminus \{p\}$. We can define a branch of the logarithm $\log : \mathbb{C} \setminus \{rp : r \geq 0\} \to \mathbb{C}$. Then every point $\hat{z} \in p^{-1}(U)$ can be written uniquely as $\hat{z} = k + \frac{\log z}{2\pi \mathrm{i}}$ for some $k \in \mathbb{Z}$.

Thus $p^{-1}(U) \cong U \times \mathbb{Z}$, via $\hat{z} \mapsto \left(\frac{\log z}{2\pi \mathrm{i}}, k\right)$, and so each proper subset of S^1 is evenly covered, however S^1 as a whole is not evenly covered, since $p^{-1}(S^1)$ is not a union of copies of S^1.

3. $\hat{X} = X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p(z) = z^n$.

   p is a covering map by choosing a branch of the nth root on proper open subsets of S^1

4. Let $\hat{X} = S^2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on S^2 by the antipodal map $z \mapsto -z$. Then let $X = \hat{X}/G = \hat{X}/\sim$, where $x \sim y \iff x = \pm y$.

   Then X is $\mathbb{RP}^2$, the real projective plane. If $x \in X$, let U be an open neighbourhood of x disjoint from its negation. Then the image of U in X is evenly covered.

We say a covering map $p : \hat{X} \to X$ is ***n*-sheeted** if $\#p^{-1}(x) = n$ for all $x \in X$, and call n the ***degree*** of p.

Lifting Properties

Let $p : \hat{X} \to X$ be a covering map, and $f : Y \to X$ b a mp. A ***lift*** of f to $\hat{X}$ is a map $\hat{f} : Y \to \hat{X}$ such that the following diagram commutes:

$$\begin{tikzcd} & \hat{X} \arrow[d, "p"] \\ Y \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

A space X is ***locally path connected*** if for every $x \in X$ and $U \subseteq X$ open neighbourhood of x, there exists a neighbourhood $V \subseteq U$ of x which is path connected.

Lemma 2.11 (Uniqueness of Lifting)**.** *Let $p : \hat{X} \to X$ be a covering map and $\hat{f}_1, \hat{f}_2 : Y \to \hat{X}$ be two lifts of $f : Y \to X$ with Y connected and locally path connected.*

If there exists some $x_0 \in Y$ with $\hat{f}_1(x_0) = \hat{f}_2(x_0)$, then $\hat{f}_1 = \hat{f}_2$.

Proof. We will show that the set $S := \{y \in Y : \hat{f}_1(y) = \hat{f}_2(y)\}$ is both open and closed. By assumption we have $x_0 \in S$, so $S \neq \emptyset$. Since Y is connected, we must have then that $S = Y$ as otherwise S and $Y \setminus S$ would disconnect Y.

Let $y_1 \in Y$ be an arbitrary point, and let $U \subseteq X$ be an open neighbourhood of $f(y_1)$ which is evenly covered by p. Let $V \subseteq f^{-1}(U)$ be an open neighbourhood of y_1 which is path connected. We then want to show that, if $y_1 \in S$ then all of $V \subseteq S$, and otherwise $V \subseteq Y \setminus S$.

Let $y \in V$ be arbitrary and let α be a path from $y_1 \to y$. Then $\hat{f}_i \circ \alpha$ is a path from $\hat{f}_i(y_1) \to \hat{f}_i(y)$ for $i = 1, 2$.

Note that $p \circ \hat{f}_1 \circ \alpha(t) = f(\alpha(t)) \in U$, and so $\hat{f}_i(y)$ and $\hat{f}_i(y_1)$ lie in the same component of $p^{-1}(U)$, say U_{δ_i}

If $y_1 \in S$, then $\hat{f}_1(y_1) = \hat{f}_2(y_1)$, so $\delta_1 = \delta_2$, and so $\hat{f}_1(y) = p_{\delta_1}^{-1}(f(y)) = p_{\delta_2}^{-1}(f(y)) = \hat{f}_2(y)$, so $y \in S$, and hence all of $V \subseteq S$.

Otherwise $y_1 \notin S$, then $\hat{f}_1(y_1) \neq \hat{f}_2(y_1)$. Each U_{δ_i} contains a unique point of $p^{-1}(\{f(y_1)\})$, and we must have $\delta_1 \neq \delta_2$.
So $\hat{f}_1(y) \neq \hat{f}_2(y)$, so $y \notin S$, and in general $V \subseteq Y \setminus S$.

Hence S is open, $Y \setminus S$ is open, and we are done $\square$

Let $\gamma : I \to X$ be a path from $x_0 \in X$ and $p : \hat{X} \to X$ be a covering map. A lift of γ at (or from) $\hat{x_0}$ is a lift $\hat{\gamma}$ of γ with $\hat{x_0} = \hat{\gamma}(0)$. In particular, $p(\hat{x_0}) = p(\hat{\gamma}(0)) = \gamma(0) = x_0$.

Lemma 2.12 (Path Lifting Lemma)**.** *Let $p : \hat{X} \to X$ be a covering map, and let $\gamma : I \to X$ be a path from x_0. Then for any choice of $\hat{x_0} \in p^{-1}(x_0)$, there exists a unique lift $\hat{\gamma}$ of γ from $\hat{x_0}$.*

Proof. Uniqueness follows from the previous lemma showing uniqueness of lifts. For existence, let $S = \{t \in I | \gamma|_{[0,t]}$ lifts to path from $\hat{x_0}$ in $\hat{X}\}$. Note $o \in S$. If we show that S is open and closed, then since I is connected, $S = I$. Note that if $t \in S$, then $[0, t] \subseteq S$.

Let $t_0 \in I$, and let U be an evenly covered neighbourhood of $\gamma(t_0)$. Let $V \subseteq \gamma^{-1}(U)$ be an open interval containing t_0. Let $t \in V$ and suppose first that $t_0 \in S$. If $t \leq t_0$, then $t \in S$, so instead assume that $t > t_0$. Since $\gamma|_{[0,t_0]}$ has a lift $\hat{\gamma} : [0, t_0] \to \hat{X}$, and we have $\hat{\gamma}(t_0) \in U_\delta$ for some $\delta \in \Delta_U$.

Recall that we have a homeomorphism $p_\delta : U_\delta \to U$ where $p_\delta = p|_{U_\delta}$. Hence the path:

$$s \mapsto \begin{cases} \hat{\gamma}(s) & 0 \leq s \leq t_0 \\ p_\delta^{-1} \circ \gamma & t_0 \leq s \leq t \end{cases}$$

is a lift of $\gamma|_{[0,t]}$. Hence $t \in S$, and so $V \subseteq S$, and so S is open.

If $t_0 \notin S$, $t \in V$, $t \geq t_0$ and $t \in S$, contradicting $t_0 \notin S$. If $t < t_0$ by the previous argument above we have a contradiction as then $t_0 \in S$. So $V \subseteq I \setminus S$, and hence S must also be closed. $\square$

Corollary 2.13. *Let $p : \hat{X} \to X$ be a covering map with X path connected. Then p is n-sheeted for some $n \in \mathbb{N} \cup \{\infty\}$. In fact, $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality for all pairs $x, y \in X$.*

Proof. Let γ be a path from x to y in X. If $\hat{x} \in p^{-1}(x)$, let $\hat{\gamma}_{\hat{x}}$ be the lift of γ from x. Then map $\hat{x}$ to $\hat{\gamma}_{\hat{x}}(1)$. The path $\overline{\gamma}$ similarly gives a map $p^{-1}(y) \to p^{-1}(x)$, inverse to the first map.

For example to show that the composition $p^{-1}(x) \to p^{-1}(y) \to p^{-1}(x)$ is the identity, we need to show that, for $\hat{x} \in p^{-1}(x)$, $(\hat{\overline{\gamma}})_{\hat{\gamma}_{\hat{x}}(1)}(1) = \hat{x}$. But $\hat{\gamma}_{\hat{x}} \cdot (\hat{\overline{\gamma}})_{\overline{\gamma}_{\hat{x}}(1)}$ is a lift of $\gamma \cdot \overline{\gamma}$, and $\hat{\gamma}_{\hat{x}} \cdot \overline{(\hat{\gamma}_{\hat{x}})}$ is also a lift of $\gamma \cdot \overline{\gamma}$, and so by uniqueness, $(\hat{\overline{\gamma}})_{\overline{\gamma}_{\hat{x}}(1)} = \overline{\hat{\gamma}_{\hat{x}}}$. Hence $(\hat{\overline{\gamma}})_{\hat{\gamma}_{\hat{x}}(1)}(1) = \overline{\hat{\gamma}}_{\hat{x}}(1) = \hat{\gamma}_{\hat{x}}(0) = \hat{x}$. $\square$

Lemma 2.14 (Homotopy Lifting Lemma)**.** *Let $p : \hat{X} \to X$ be a covering map and $g_0 : Y \to X$ a map with Y locally path connected. Let $F : Y \times I \to X$ be a homotopy with $F(y, 0) = f_0(y)$ for all $y \in Y$. Let $\hat{f}_0 : Y \to \hat{X}$ be a lift of Y. Then there is a unique lit $\hat{F}$ of F to $\hat{X}$ so that $\hat{F}(y, 0) = \hat{f}_0(y)$.*

Proof. For each $y \in Y$, we obtain a path γ_y given by $\gamma_y(t) = F(y, t)$ from $f_0(y)$. By the path lifting lemma, each γ_y lifts uniquely to a path $\hat{\gamma}_y$ from $\hat{f}_0(y)$. Now define:

$$\hat{F}(y, t) = \hat{\gamma}_y(t)$$

This clearly is a lift of F in the sense that

$$(p \circ \hat{F})(y,t) = p(\hat{\gamma}_y(t)) = \gamma_y(t) = F(y,t)$$

but is $\hat{F}$ continuous.

We will construct a different map $\tilde{F} : Y \times I \to \hat{X}$ which is continuous by construction, and then we will show that $\hat{F} = \tilde{F}$.

Fix $y_0 \in Y$. The for each $t \in I$ we have an evenly covered neighbourhood U_t of $F(y_0, t) \in X$. Then $F^{-1}(U_t) \subseteq Y \times I$ is an open neighbourhood of (y_0, t). We can find an open neighbourhood of (y_0, t) in $F^{-1}(U_t)$ of the form $V_t \times (t - \epsilon_t, t + \epsilon_t)$ with V_t path connected.

Note that these neighbourhoods cover $Y \times I$, and as $\{y_0\} \times I$ is compact, there is a finite subcover $\{J_i\}$ of $\{(t - \epsilon_t, t + \epsilon_t) | t \in I\}$. Then, if $J_i = (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$, we can find a path connected subset $V \subseteq \cap_i V_{t_i}$ containing y_0. Hence we may assume there is a path-connected neighbourhood V of y_0 in Y, and a finite number of intervals J_i covering I such that $F(V \times J_i)$ is contained in an evenly covered neighbourhood U_i of X. $\square$