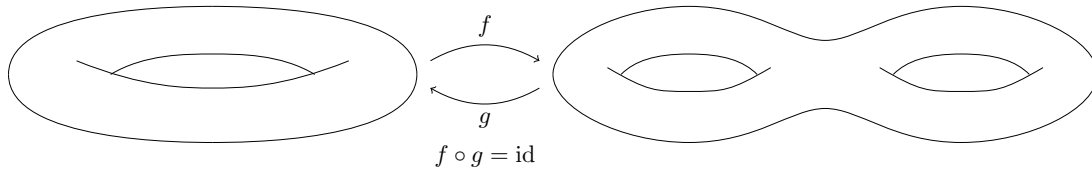


Algebraic Topology

November 1, 2019

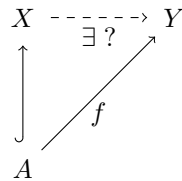
1 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions f, g going from a topological space X to another space Y and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



If such f, g continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space X a group $G(X)$, and for every continuous function $f : X \rightarrow Y$ a group homomorphism $G(f) : G(X) \rightarrow G(Y)$ with $G(\text{id}) = \text{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \rightarrow Y$ is a homeomorphism with inverse $g : Y \rightarrow X$, then $G(g) \circ G(f) = \text{id}, G(f) \circ G(g) = \text{id}$, so $G(f)$ is an isomorphism.

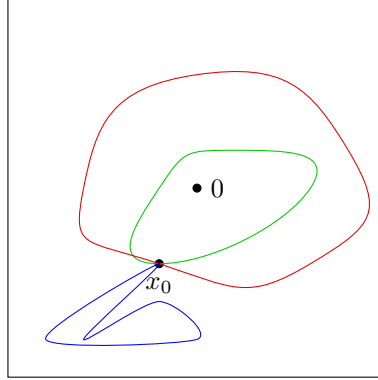
Extension problem: Let X be a topological space, $A \subseteq X$ a subspace, and $f : A \rightarrow Y$ a continuous function. Does there exist a continuous function $F : X \rightarrow Y$ with $F|_A = f$



Theorem 1.1. *There is no continuous function*

$$f : D^n \rightarrow S^{n-1} \text{ with } f|_{S^{n-1}} = \text{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct G with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \rightarrow D^n \rightarrow S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}$ being the identity.



Here, the green and red loops are the “same” loop, whilst the blue one is distinct

Conventions

A topological space will be referred to as a *space*

A continuous function $f : X \rightarrow Y$ will be called a *map*

2 The Fundamental Group

The idea here is that, if X is a space, $x_0 \in X$ a fixed point, called the *basepoint*, we consider loops based at x_0 , i.e. maps $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the *fundamental group* $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at x_0 modulo “deforming loops”. Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing γ_1 and then γ_2 . But what do we mean by “deforming” a loop?

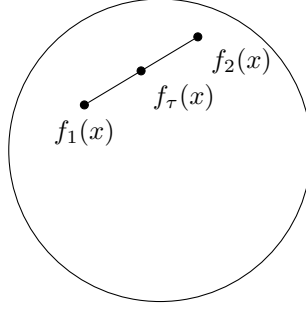
Let $f_0, f_1 : X \rightarrow Y$ be maps. A *homotopy* between f_0 and f_1 is a map

$$F : X \times I \rightarrow Y \text{ where } I = [0, 1] \text{ and} \\ F(x, 0) = f_0(x) \text{ and} \\ F(x, 1) = f_1(x)$$

We often write $f_\tau(x) = F(x, \tau)$, $f_\tau : X \rightarrow Y$.

If such F exists, we say f_0 and f_1 are *homotopic*.

Example: Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \rightarrow Y$ are homotopic, via $F(x, t) = tf_1(x) + (1 - t)f_0(x) \in Y$ by convexity.



If f_0 is homotopic to f_1 , we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \rightarrow Y$. If $Z \subseteq X$ and $f_0(z) = F(z, t) = f_1(z) \forall z \in Z, t \in I$, then we say f_0 is homotopic to f_1 **relative to** Z .

Lemma 2.1. *Let $Z \subseteq X, Y$ be spaces. Then \simeq relative to Z is an equivalence relation on the set of maps $X \rightarrow Y$.*

Proof.

- Reflexive: $f_0 \simeq f_0$ via $F(x, t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x, t) = f(x, 1 - t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f_1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x, t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t - 1) & t \geq 1/2 \end{cases}$$

All homotopies are relative to Z .

□

A **homotopy equivalence** $f : X \rightarrow Y$ is a map with a **homotopy inverse** $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y, g \circ f = \text{id}_X$. We then write $X \simeq Y$.

Remark: Most (all?) invariants in the course are **homotopy invariants**

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \rightarrow *$ be the constant map, and let $g : * \rightarrow \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \text{id}_*$, and $g \circ f(x) = 0 \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \text{id}_{\mathbb{R}^n}$ via $F(x, t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map x to the intersection of $\overrightarrow{\mathbf{0}x}$ with S^{n-1}). Then $g \circ f = \text{id}_{S^{n-1}}$ and $f \circ g \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x, t) = (1 - t)x + t \cdot \frac{x}{|x|}$

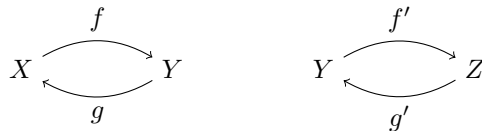
If $X \simeq *$, then we say X is **contractible**.

Let $f : X \rightarrow Y, g : Y \rightarrow X$ be maps. If $g \circ f = \text{id}_X$, then we say X is a **retract** of Y , and g is a **retraction**. If in addition $f \circ g \simeq \text{id}_Y$ relative to $f(X)$, then we say X is a **deformation retract** of Y . Hence, in example 2, we see that S^{n-1} is a deformation retract of \mathbb{R}^n .

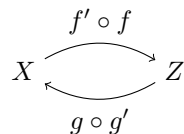
Lemma 2.2. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:



We want to show $f' \circ f, g \circ g'$ induces a homotopy equivalence



Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \text{id}_Y$, and so:

$$(x, t) \mapsto g(F'(f(x), t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y, g \circ f \simeq \text{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \text{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \text{id}_Z$ \square

Loops and π_1

If X is a space, a **path** in X is a map $\gamma : I \rightarrow X$, where $I = [0, 1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say γ is a path **from** x_0 **to** x_1 .

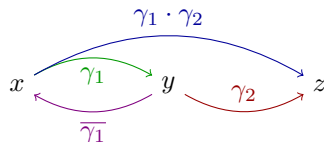
We say γ_1 and γ_2 are **homotopic** if $\gamma_1 \simeq \gamma_2$ relative to $\{0, 1\}$, and we write $[\gamma]$ for the homotopy equivalence class of γ .

If X is a space with points $x, y, z \in X$, and γ_1 is a path from x to y , γ_2 is a path from y to z , then:

- The **concatenation** of γ_1 and γ_2 is the path from x to z given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s - 1) & 1/2 \leq s \leq 1 \end{cases}$$

- The **constant path** at x is the path $c_x(s) = x \forall s \in I$
- The **inverse of** γ_1 is $\overline{\gamma_1}(s) = \gamma_1(1 - s)$, a path from y to x .

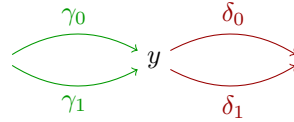


Theorem 2.3. Let X be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X with endpoint x_0 (we say they are **based** at x_0). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity c_{x_0} and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.

This group is called the **fundamental group** of X (based at x_0).

To prove this, we will need the following lemmas:

Lemma 2.4. If $\gamma_0 \simeq \gamma_1$ to y and $\delta_0 \simeq \delta_1$ from y , then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$



Proof. Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s, t) = F(1 - s, t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. □

Lemma 2.5. Let α, β, γ be paths from w to x to y to z in X .

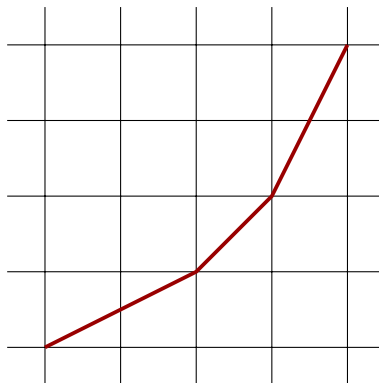


Then:

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

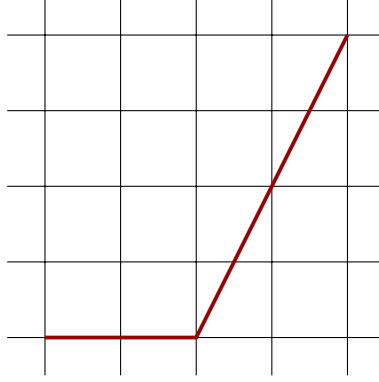
Proof. First, given a path $\delta : I \rightarrow X$, a **reparametrization** of δ is a path $\delta \circ \phi$ where $\phi : I \rightarrow I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that ϕ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s, t) = \delta(t\phi(s) + (1 - t)s)$, and this homotopy is relative to $\{0, 1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function ϕ whose plot is:

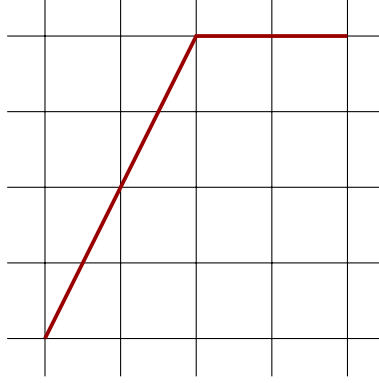


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize α via:



i.e. do c_w for the first half of the time, then do α , so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s, t) = \begin{cases} \alpha(2s) & 0 \leq s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \alpha(2 - 2s) & 1 - t/2 \leq s \leq 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have c_w at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

□

Then theorem 1.3 giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If γ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

Formal Properties of π_1

Lemma 2.6. *Let $f : X \rightarrow Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

Furthermore:

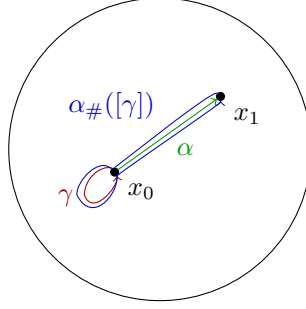
1. If $f \simeq f'$ relative to x_0 , then $f'_* = f_*$.
2. If $g : Y \rightarrow Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$
3. $(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}$

Proof. f_* is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to x_0 , then for γ a loop based at x_0 , $(s, t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

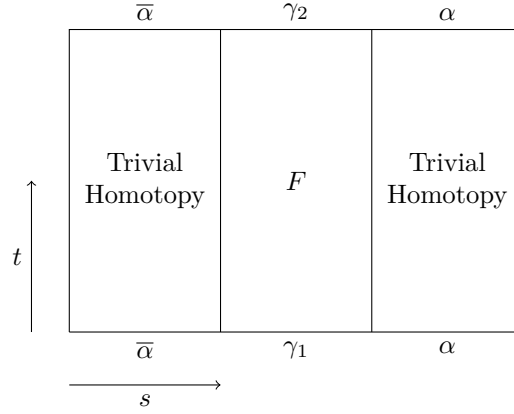
Lemma 2.7. let X be a space, $x_0, x_1 \in X$ and α a path from x_0 to x_1 . Then there is a group isomorphism $\alpha_\# : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.



Furthermore,

1. If $\alpha \simeq \alpha'$ relative to $\{0, 1\}$, then $\alpha_\# = \alpha'_\#$.
2. $(c_{x_0})_\# = \text{id}_{\pi_1(X, x_0)}$
3. If β is a path from x_2 to x_1 , then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$
4. If $f : X \rightarrow Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.

Proof. Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

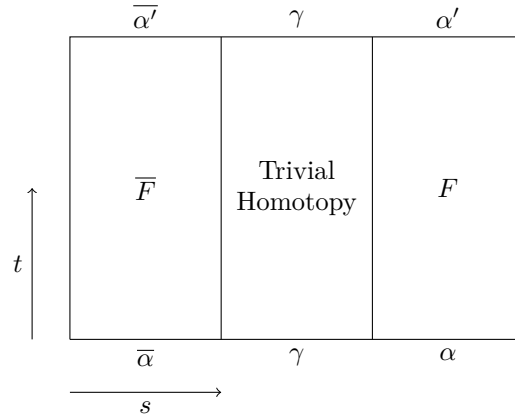


This is indeed a group homomorphism: for loops γ, δ based at x_0 ,

$$\begin{aligned}
 \bar{\alpha} \cdot \gamma \cdot \alpha \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\
 &\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\
 &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\
 &\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha
 \end{aligned}$$

Thus $\alpha_{\#}(\gamma \cdot \delta) = \alpha_{\#}(\gamma) \cdot \alpha_{\#}(\delta)$. Also $\alpha_{\#}^{-1} = (\alpha_{\#})^{-1}$ - this is easy to check. Thus $\alpha_{\#}$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$



gives $\alpha_{\#}(\gamma) \simeq \alpha'_{\#}(\gamma)$

2. Immediate since c_{x_0} is the identity in $\pi_1(X, x_0)$.
- 3.

$$\begin{aligned}
 (\alpha \cdot \beta)_{\#}(\gamma) &= \alpha \cdot \bar{\beta} \cdot \gamma \cdot \alpha \cdot \beta \\
 &= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta) \\
 &= \bar{\beta} \cdot \alpha_{\#}(\gamma) \cdot \beta \\
 &= \beta_{\#}(\alpha_{\#}(\gamma))
 \end{aligned}$$

4.

$$\begin{aligned}
((f \circ \alpha)_\# \cdot f_*)(\gamma) &= (f \circ \alpha)_\#(f \cdot \gamma) \\
&= (f \circ \alpha)_\#(f \cdot \gamma) \\
&= \overline{f \cdot \alpha} \cdot (f \circ \gamma) \cdot (f \circ \alpha) \\
&= f \circ (\bar{\alpha} \cdot \gamma \cdot \alpha) \\
&= f_*(\alpha_\#(\gamma))
\end{aligned}$$

□

A path connected space X is **simply connected** if $\pi_1(x, x_0) = 0$ for any, and hence all, $x_0 \in X$.

Our aim here is to prove that π_1 is a **homotopy invariant**, i.e. that homotopy equivalent spaces have the same fundamental group. We will start with the following lemma:

Lemma 2.8. *Let $x_0 \in X$ and $f, g : X \rightarrow Y$ with $f \simeq_F g$. Set $x(t) = F(x_0, t)$ so that $\alpha(0) = f(x_0)$ and $\alpha(1) = g(x_0)$. Then the diagram:*

$$\begin{array}{ccc}
& & \pi_1(Y, f(x_0)) \\
& \nearrow f_* & \downarrow \alpha_\# \\
\pi_1(x, x_0) & & \\
& \searrow g_* & \downarrow \\
& & \pi_1(Y, g(x_0))
\end{array}$$

commutes, i.e. we have $\alpha_\# \circ f_* = g_*$.

Proof. We need to check that, for a loop γ based at x_0 , $\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha \simeq g \circ \gamma$.

Let $G : I \times I \rightarrow Y$ defined by $G(s, t) = F(\gamma(s), t)$. For $t = 0$, this is $f \circ \gamma$, and for $t = 1$, this is $g \circ \gamma$. Now consider two paths in $I \times I$:

$$\begin{aligned}
a(t) &= (t, 1); b = b_1 \cdot b_2 \cdot b_3 \text{ where:} \\
b_1(t) &= (0, 1 - t), b_2(t) = (t, 0), b_3(t) = (1, t)
\end{aligned}$$

Then $(G \circ a)(s) = G(s, 1) = g \circ \gamma(s)$, whilst $G \circ b = \bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$.

Now, since $I \times I$ is convex, we have that $a \simeq_H b$, and so $G \circ H$ is the desired homotopy between $g \circ \gamma$ and $\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$. □

Theorem 2.9. *If $f : X \rightarrow Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is a homomorphism for any $x_0 \in X$.*

Proof. We'll show that f_* is a bijection:

Let $g : Y \rightarrow X$ be a homotopic inverse to f , with $\text{id}_X \simeq_F g \circ f$. Let $\alpha : I \rightarrow X$ given by $\alpha(t) = F(x_0, t)$.

Note that $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0)); g : \pi_1(Y, f(x_0)) \rightarrow \pi_1(X, g(f(x_0)))$

Then $g_* \circ f_* = (g \circ f)_* = \alpha_\# \circ (\text{id}_X)_* = \alpha_\#$. $\alpha_\#$ is an isomorphism, and so f_* is injective.

If $\text{id}_Y \simeq_G f \circ g$ let $\beta(t) = G(f(x_0), t)$. Then $f_* \circ g_* = (g \circ f)_* = \beta_{\#} \circ (\text{id}_Y)_* = \beta_{\#}$, an isomorphism, and hence f_* is surjective. \square

Corollary 2.10. *Contractible spaces are simply connected.*

Proof. If X is contractible, there exists some $x_0 \in X$ and a homotopy F between id_X and $X \rightarrow \{x_0\}$. So $F(x, \cdot)$ is a path from any $x \in X$ to x_0 , so X is path connected. Since X is homotopic to $\{x_0\}$, $\pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0) = 0$. \square

Covering Spaces

Let $p : \hat{X} \rightarrow X$ be a map. An open set $U \subseteq X$ is **evenly covered** if there exists a set Δ_U with the discrete topology and there is a homeomorphism:

$$p^{-1}(U) \xrightarrow{\cong} U \times \Delta_U$$

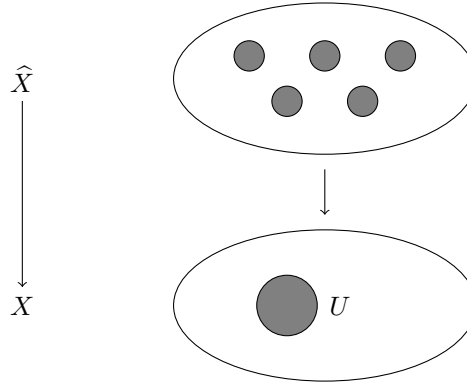
such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \Delta_U \\ & \searrow p & \swarrow (x, \delta) \mapsto x \\ & U & \end{array}$$

We write, for $\delta \in \Delta_U$, $U_{\delta} = U \times \{\delta\}$ and $p_{\delta} = p|_{U_{\delta}}$. So $p_{\delta} : U_{\delta} \rightarrow U$ is a homeomorphism.

Note that we can canonically identify Δ_U with $p^{-1}(x)$ for any $x \in U$. Note also that $p^{-1}(U) \cong \coprod_{\delta \in \Delta_U} U_{\delta}$, where \coprod denotes disjoint union.

If every point of X has an open neighbourhood which is evenly covered, then we say that p is a **covering map** and \hat{X} is a **covering space** of X .



Examples:

1. $\hat{X} = X \times \Delta$ for Δ a set with the discrete topology, e.g. $\hat{I} = I \times \{1, 2, 3\}$. Then \hat{X} is a covering space of X , the identity map on the first element is a covering map.
2. $\hat{X} = \mathbb{R}, X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p : \mathbb{R} \rightarrow S^1$ and $p(t) = \exp(2\pi i \cdot t)$. Then p is a covering map:

let $U = S \setminus \{p\}$. We can define a branch of the logarithm $\log : \mathbb{C} \setminus \{rp : r \geq 0\} \rightarrow \mathbb{C}$. Then every point $\hat{z} \in p^{-1}(U)$ can be written uniquely as $\hat{z} = k + \frac{\log z}{2\pi i}$ for some $k \in \mathbb{Z}$.

Thus $p^{-1}(U) \cong U \times \mathbb{Z}$, via $\hat{z} \mapsto \left(\frac{\log z}{2\pi i}, k\right)$, and so each proper subset of S^1 is evenly covered, however S^1 as a whole is not evenly covered, since $p^{-1}(S^1)$ is not a union of copies of S^1 .

3. $\hat{X} = X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p(z) = z^n$.

p is a covering map by choosing a branch of the n th root on proper open subsets of S^1

4. Let $\hat{X} = S^2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on S^2 by the antipodal map $z \mapsto -z$. Then let $X = \hat{X}/G = \hat{X}/\sim$, where $x \sim y \iff x = \pm y$.

Then X is \mathbb{RP}^2 , the real projective plane. If $x \in X$, let U be an open neighbourhood of x disjoint from its negation. Then the image of U in X is evenly covered.

We say a covering map $p : \hat{X} \rightarrow X$ is **n -sheeted** if $\#p^{-1}(x) = n$ for all $x \in X$, and call n the **degree** of p .

Lifting Properties

Let $p : \hat{X} \rightarrow X$ be a covering map, and $f : Y \rightarrow X$ be a map. A **lift** of f to \hat{X} is a map $\hat{f} : Y \rightarrow \hat{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \hat{X} \\ & \nearrow \hat{f} & \downarrow p \\ Y & & X \\ & \searrow f & \end{array}$$

A space X is **locally path connected** if for every $x \in X$ and $U \subseteq X$ open neighbourhood of x , there exists a neighbourhood $V \subseteq U$ of x which is path connected.

Lemma 2.11 (Uniqueness of Lifting). *Let $p : \hat{X} \rightarrow X$ be a covering map and $\hat{f}_1, \hat{f}_2 : Y \rightarrow \hat{X}$ be two lifts of $f : Y \rightarrow X$ with Y connected and locally path connected.*

If there exists some $x_0 \in Y$ with $\hat{f}_1(x_0) = \hat{f}_2(x_0)$, then $\hat{f}_1 = \hat{f}_2$.

Proof. We will show that the set $S := \{y \in Y : \hat{f}_1(y) = \hat{f}_2(y)\}$ is both open and closed. By assumption we have $x_0 \in S$, so $S \neq \emptyset$. Since Y is connected, we must have then that $S = Y$ as otherwise S and $Y \setminus S$ would disconnect Y .

Let $y_1 \in Y$ be an arbitrary point, and let $U \subseteq X$ be an open neighbourhood of $f(y_1)$ which is evenly covered by p . Let $V \subseteq f^{-1}(U)$ be an open neighbourhood of y_1 which is path connected. We then want to show that, if $y_1 \in S$ then all of $V \subseteq S$, and otherwise $V \subseteq Y \setminus S$.

Let $y \in V$ be arbitrary and let α be a path from $y_1 \rightarrow y$. Then $\hat{f}_i \circ \alpha$ is a path from $\hat{f}_i(y_1) \rightarrow \hat{f}_i(y)$ for $i = 1, 2$.

Note that $p \circ \hat{f}_1 \circ \alpha(t) = f(\alpha(t)) \in U$, and so $\hat{f}_1(y)$ and $\hat{f}_1(y_1)$ lie in the same component of $p^{-1}(U)$, say U_{δ_1}

If $y_1 \in S$, then $\hat{f}_1(y_1) = \hat{f}_2(y_1)$, so $\delta_1 = \delta_2$, and so $\hat{f}_1(y) = p_{\delta_1}^{-1}(f(y)) = p_{\delta_2}^{-1}(f(y)) = \hat{f}_2(y)$, so $y \in S$, and hence all of $V \subseteq S$.

Otherwise $y_1 \notin S$, then $\widehat{f}_1(y_1) \neq \widehat{f}_2(y_1)$. Each U_{δ_i} contains a unique point of $p^{-1}(\{f(y_1)\})$, and we must have $\delta_1 \neq \delta_2$.

So $\widehat{f}_1(y) \neq \widehat{f}_2(y)$, so $y \notin S$, and in general $V \subseteq Y \setminus S$.

Hence S is open, $Y \setminus S$ is open, and we are done \square

Let $\gamma : I \rightarrow X$ be a path from $x_0 \in X$ and $p : \widehat{X} \rightarrow X$ be a covering map. A lift of γ at (or from) \widehat{x}_0 is a lift $\widehat{\gamma}$ of γ with $\widehat{x}_0 = \widehat{\gamma}(0)$. In particular, $p(\widehat{x}_0) = p(\widehat{\gamma}(0)) = \gamma(0) = x_0$.

Lemma 2.12 (Path Lifting Lemma). *Let $p : \widehat{X} \rightarrow X$ be a covering map, and let $\gamma : I \rightarrow X$ be a path from x_0 . Then for any choice of $\widehat{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\widehat{\gamma}$ of γ from \widehat{x}_0 .*

Proof. Uniqueness follows from the previous lemma showing uniqueness of lifts. For existence, let $S = \{t \in I \mid \gamma|_{[0,t]} \text{ lifts to path from } \widehat{x}_0 \text{ in } \widehat{X}\}$. Note $0 \in S$. If we show that S is open and closed, then since I is connected, $S = I$. Note that if $t \in S$, then $[0, t] \subseteq S$.

Let $t_0 \in I$, and let U be an evenly covered neighbourhood of $\gamma(t_0)$. Let $V \subseteq \gamma^{-1}(U)$ be an open interval containing t_0 . Let $t \in V$ and suppose first that $t_0 \in S$. If $t \leq t_0$, then $t \in S$, so instead assume that $t > t_0$. Since $\gamma|_{[0,t_0]}$ has a lift $\widehat{\gamma} : [0, t_0] \rightarrow \widehat{X}$, and we have $\widehat{\gamma}(t_0) \in U_\delta$ for some $\delta \in \Delta_U$.

Recall that we have a homeomorphism $p_\delta : U_\delta \rightarrow U$ where $p_\delta = p|_{U_\delta}$. Hence the path:

$$s \mapsto \begin{cases} \widehat{\gamma}(s) & 0 \leq s \leq t_0 \\ p_\delta^{-1} \circ \gamma & t_0 \leq s \leq t \end{cases}$$

is a lift of $\gamma|_{[0,t]}$. Hence $t \in S$, and so $V \subseteq S$, and so S is open.

If $t_0 \notin S$, $t \in V$, $t \geq t_0$ and $t \in S$, contradicting $t_0 \notin S$. If $t < t_0$ by the previous argument above we have a contradiction as then $t_0 \in S$. So $V \subseteq I \setminus S$, and hence S must also be closed. \square

Corollary 2.13. *Let $p : \widehat{X} \rightarrow X$ be a covering map with X path connected. Then p is n -sheeted for some $n \in \mathbb{N} \cup \{\infty\}$. In fact, $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality for all pairs $x, y \in X$.*

Proof. Let γ be a path from x to y in X . If $\widehat{x} \in p^{-1}(x)$, let $\widehat{\gamma}_{\widehat{x}}$ be the lift of γ from x . Then map \widehat{x} to $\widehat{\gamma}_{\widehat{x}}(1)$. The path $\overline{\gamma}$ similarly gives a map $p^{-1}(y) \rightarrow p^{-1}(x)$, inverse to the first map.

For example to show that the composition $p^{-1}(x) \rightarrow p^{-1}(y) \rightarrow p^{-1}(x)$ is the identity, we need to show that, for $\widehat{x} \in p^{-1}(x)$, $(\widehat{\gamma})_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \widehat{x}$. But $\widehat{\gamma}_{\widehat{x}} \cdot (\widehat{\gamma})_{\widehat{\gamma}_{\widehat{x}}(1)}$ is a lift of $\gamma \cdot \overline{\gamma}$, and $\widehat{\gamma}_{\widehat{x}} \cdot (\widehat{\gamma}_{\widehat{x}})$ is also a lift of $\gamma \cdot \overline{\gamma}$, and so by uniqueness, $(\widehat{\gamma})_{\widehat{\gamma}_{\widehat{x}}(1)} = \widehat{\gamma}_{\widehat{x}}$. Hence $(\widehat{\gamma})_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \widehat{\gamma}_{\widehat{x}}(1) = \widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$. \square

Lemma 2.14 (Homotopy Lifting Lemma). *Let $p : \widehat{X} \rightarrow X$ be a covering map and $g_0 : Y \rightarrow X$ a map with Y locally path connected. Let $F : Y \times I \rightarrow X$ be a homotopy with $F(y, 0) = g_0(y)$ for all $y \in Y$. Let $\widehat{f}_0 : Y \rightarrow \widehat{X}$ be a lift of g_0 . Then there is a unique lift \widehat{F} of F to \widehat{X} so that $\widehat{F}(y, 0) = \widehat{f}_0(y)$.*

Proof. For each $y \in Y$, we obtain a path γ_y given by $\gamma_y(t) = F(y, t)$ from $g_0(y)$. By the path lifting lemma, each γ_y lifts uniquely to a path $\widehat{\gamma}_y$ from $\widehat{f}_0(y)$. Now define:

$$\widehat{F}(y, t) = \widehat{\gamma}_y(t)$$

This clearly is a lift of F in the sense that

$$(p \circ \widehat{F})(y, t) = p(\widehat{\gamma}_y(t)) = \gamma_y(t) = F(y, t)$$

but is \widehat{F} continuous?

We will construct a different map $\widetilde{F} : Y \times I \rightarrow \widehat{X}$ which is continuous by construction, and then we will show that $\widehat{F} = \widetilde{F}$.

Fix $y_0 \in Y$. The for each $t \in I$ we have an evenly covered neighbourhood U_t of $F(y_0, t) \in X$. Then $F^{-1}(U_t) \subseteq Y \times I$ is an open neighbourhood of (y_0, t) . We can find an open neighbourhood of (y_0, t) in $F^{-1}(U_t)$ of the form $V_t \times (t - \epsilon_t, t + \epsilon_t)$ with V_t path connected.

Note that these neighbourhoods cover $Y \times I$, and as $\{y_0\} \times I$ is compact, there is a finite subcover $\{J_i\}$ of $\{(t - \epsilon_t, t + \epsilon_t) | t \in I\}$. Then, if $J_i = (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$, we can find a path connected subset $V \subseteq \cap_i V_{t_i}$ containing y_0 . Hence we may assume there is a path-connected neighbourhood V of y_0 in Y , and a finite number of intervals J_i covering I such that $F(V \times J_i)$ is contained in an evenly covered neighbourhood U of X .

Let $\delta_i \in \Delta_U$ be the unique index such that:

$$\widehat{F}(\{y_0\} \times J_i) \subseteq U_{\delta_i}$$

Now for $(y, t) \in V \times J_i$, we can define $\widetilde{F}(y, t) := p_{\delta_i}^{-1} \circ F(y, t)$ for $(y, t) \in V \times J_i$.

These maps agree on overlaps, i.e. when $(V \times J_i) \cap (V \times J_j) = V \times (J_i \cap J_j) \neq V \times \emptyset$ - to see this, suppose that $t \in J_i \cap J_j$, and let α be a path in V from y_0 to y in V , and let $\alpha_t(s) = F(\alpha(s), t)$.

Then $p_{\delta_i}^{-1} \circ \alpha_t$ is a lift of α_t from $p_{\delta_i}^{-1} \circ \alpha_t(0)$, and likewise for $p_{\delta_j}^{-1} \circ \alpha$.

But $p_{\delta_i}^{-1} \circ \alpha_t(0) = p_{\delta_i}^{-1} \circ F(y_0, t) = \widehat{F}(y_0, t) = \widehat{\gamma}_{y_0}(t)$ as defined on J_i , and $p_{\delta_j}^{-1} \circ \alpha_t(0) = p_{\delta_j}^{-1} \circ F(y_0, t) = \widetilde{F}(y_0, t) = \widehat{\gamma}_{y_0}(t)$ as defined on J_j . Hence $p_{\delta_i}^{-1} \circ \alpha_t$ and $p_{\delta_j}^{-1} \circ \alpha_t$ have the same initial end point, and they are both lifts of α_t , and hence they must agree by **2.11** uniqueness of lifting.

Hence $p_{\delta_j}^{-1} \circ F(y, t) = p_{\delta_j}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ F(y, t)$, and so the two definitions of \widetilde{F} on $V \times J_i$ and $V \times J_j$ agree on the overlap $V \times (J_i \cap J_j)$.

Thus we have a well-defined continuous lifting:

$$\widetilde{F} : V \times I \rightarrow \widehat{X} \text{ of } F_{V \times I} \rightarrow X$$

But by construction, $\widetilde{F}(y_0, 0) = \widehat{f}_0(y_0)$, and so $\widetilde{F}(y, 0)$ is a lift of $f_0(y)$ for all $y \in V$, and $\widetilde{F}(y_0, 0) = \widehat{f}_0(y_0)$. Hence by **2.11** uniqueness of lifting, $\widetilde{F}(\cdot, 0)$ is \widehat{f}_0 on V .

For each $y \in V$, $\widetilde{F}(y, \cdot)$ is a lift of γ_y from $\widehat{f}_0(y)$. By uniqueness of lifts of paths, we must have $\widetilde{F}(y, t) = \widehat{\gamma}_y(t)$, and $\widetilde{F}(y, t)$ hence agrees with $\widehat{F}(y, t)$. \square

Corollary 2.15. *Let $p : \widehat{X} \rightarrow X$ be a covering map and let $F : I \times I \rightarrow X$ be a homotopy of paths. Then any lift \widehat{F} of F to \widehat{X} is also a homotopy of paths.*

Proof. Let \widehat{F} be a lift of F , so that $\widehat{F} : I \times I \rightarrow \widehat{X}$. We need to check that \widehat{F} is a homotopy relative to $\{0, 1\}$. But $\widehat{F}(0, \cdot)$ and $\widehat{F}(1, \cdot)$ are paths in \widehat{X} . Since F is a homotopy of paths, $F(0, \cdot)$

and $F(1, \cdot)$ are constant. Thus $\widehat{F}(0, \cdot)$ and $\widehat{F}(1, \cdot)$ are paths in $p^{-1}(x_0)$ and $p^{-1}(x_1)$ respectively, with $x_0 = F(0, \cdot)$, $x_1 = F(1, \cdot)$ and hence $\widehat{F}(0, \cdot), \widehat{F}(1, \cdot)$ are constant since $p^{-1}(x_0), p^{-1}(x_1)$ are discrete. \square

Applications to π_1

Lemma 2.16. *Let $p : \widehat{X} \rightarrow X$ be a covering map with $p(\widehat{X}) = X$. Then the induced map:*

$$p_* : \pi_1(\widehat{X}, \widehat{x}) \rightarrow \pi_1(X, x)$$

is injective.

Proof. Suppose $[\widehat{\gamma}]$ is in the kernel of p_* , i.e. $p \circ \widehat{\gamma} = \gamma \simeq_F c_x$. But then there is a lift \widehat{F} of F to \widehat{X} with the property that $\widehat{\gamma} \simeq_{\widehat{F}} \widehat{c}_x = c_{\widehat{x}}$. Hence $\ker p_* = \{[c_{\widehat{x}}]\} = \{1_{\pi_1(\widehat{X}, \widehat{x})}\}$, and so p_* is injective. \square

Observe that, if, $[\gamma] \in \pi_1(X, x)$, we get a map $p^{-1}(X) \rightarrow p^{-1}(X)$ via $\widehat{x} \mapsto \widehat{\gamma}_{\widehat{x}}(1)$, where $\widehat{\gamma}_{\widehat{x}}$ is a lift of γ with $\widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$.

For example, let $p : \mathbb{R} \rightarrow S^1$, with $p(t) = e^{2\pi i t}$

This gives an action of $\pi_1(X, x)$ on $p^{-1}(X)$. This is a **right action**, i.e. for $\widehat{x} \in p^{-1}(X)$, $[\gamma] \in \pi_1(X, x)$, and we write $x \cdot \gamma = \widehat{\gamma}_{\widehat{x}}(1)$, and then $x \cdot \gamma \cdot \delta = x \cdot (\gamma \cdot \delta)$.

Lemma 2.17. *Let $p : \widehat{X} \rightarrow X$ be a covering map, and suppose \widehat{X} is path connected. Let $x \in X$. Then the map:*

$$p_*(\pi_1(\widehat{X}, \widehat{x})) \backslash \pi_1(X, x) \rightarrow p^{-1}(x)$$

where $G \backslash H$ is the set of right cosets of H in G , given by $p_(\pi_1(\widehat{X}, \widehat{x})) \cdot [\gamma] \mapsto \widehat{x} \cdot \gamma$, is a bijection for any choice of $\widehat{x} \in p^{-1}(x)$.*

Furthermore, this bijection satisfies:

$$p_*(\pi_1(\widehat{X}, \widehat{x})) \cdot ([\gamma] \cdot [\delta]) \mapsto \widehat{x} \cdot (\gamma \circ \delta)$$

Example: $p : \mathbb{R} \rightarrow S^1$. We know $\pi_1(\mathbb{R}, \widehat{x}) = 0$ since \mathbb{R} is contractible, so the lemma gives a bijection $\pi_1(S^1, x) \rightarrow p^{-1}(x)$.

Proof. We want to apply the orbit stabilizer theorem to the right action of $\pi_1(X, x)$ on $p^{-1}(x)$.

The stabilizer of \widehat{x} is the set of loops $[\gamma]$ based at x such that $\widehat{\gamma}_{\widehat{x}}(1) = \widehat{x}$, i.e. the set of loops based at \widehat{x} , i.e. we have that $[\widehat{\gamma}_{\widehat{x}}] \in \pi_1(\widehat{X}, \widehat{x})$. Thus the stabilizer is precisely $p_*(\pi_1(\widehat{X}, \widehat{x}))$. Hence we just need to show this action is transitive, but this follows from path connectedness:

If $\widehat{x}, \widehat{x}' \in p^{-1}(x)$, we have a path $\widehat{\gamma}$ from \widehat{x} to \widehat{x}' in \widehat{X} . We let $\gamma = p \circ \widehat{\gamma}$, so that γ is a loop based at x and $\widehat{x} \cdot \gamma = \widehat{x}'$ since $\widehat{\gamma}_{\widehat{x}} = \widehat{\gamma}$.

So the orbit stabilizer theorem gives this bijection. \square

Note that the degree of $p : \widehat{X} \rightarrow X$ is just the index of the subgroup $p_*(\pi_1(\widehat{X}, \widehat{x}))$ in $\pi_1(X, x)$, i.e.

$$\deg p = [\pi_1(X, x) : p_*\pi_1(\widehat{X}, \widehat{x})]$$

Thus, since we have covers $\mathbb{R} \rightarrow S^1$ of degree ∞ and $S^1 \rightarrow S^1$ of degree n for all $n > 0$, and hence $\pi_1(S^1, 1)$ must be an infinite group with subgroups of every possible index.

If $\widehat{X} \rightarrow X$ is a covering map with X path connected and \widehat{X} simply connected, then we say \widehat{X} is a **universal cover** of X .

Corollary 2.18. *If $p : \widehat{X} \rightarrow X$ is a universal cover, then any choice of base point $\widehat{x} \in p^{-1}(x)$ defines a bijection from $\pi_1(X, x) \rightarrow p^{-1}(x)$, and the group structure on $\pi_1(X, x)$ is determined by $\widehat{x} \cdot (\gamma \cdot \delta) = (\widehat{x} \cdot \gamma) \cdot \delta$.*

Example: $p : \mathbb{R} \rightarrow S^1; t \mapsto e^{2\pi i t}$ is a universal cover, and so we get a bijection $\pi_1(S^1, 1) \rightarrow p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$. (Note that \mathbb{Z} here means the *set*, not group).

For $n \in \mathbb{Z}$, we can define $\widetilde{\gamma}_n(t) = nt$, a path from 0 to n in \mathbb{R} , and let $\gamma_n = p \circ \widetilde{\gamma}_n$, i.e. an n times wrapping around S^1 in the anticlockwise direction.

By the **2.18**, any loop in S^1 based at 1 must be homotopic to one of the γ_n , otherwise we would not have an injective map. In particular, the bijection of **2.18** is given by:

$$\begin{aligned} \pi_1(S^1, 1) &\rightarrow p^{-1}(1) = \mathbb{Z} \\ [\gamma_n] &\mapsto n = (\widehat{\gamma_n})_0(1) = \widehat{\gamma_n}(1) \end{aligned}$$

Note that for any $m \in \mathbb{Z}$, $m + \widehat{\gamma}_n$ is a path from m to $m + n$. So $0 \cdot (\gamma_m \cdot \gamma_n) = (0 \cdot \gamma_m) \cdot \gamma_n = m \cdot \gamma_n = (m + \widehat{\gamma}_n)(1) = m + n$, and so $\gamma_m \cdot \gamma_n = \gamma_{m+n}$ in $\pi_1(S^1, 1)$, and hence $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Theorem 2.19. *The identity map*

$$\text{id}_{S^1} : S^1 \rightarrow S^1$$

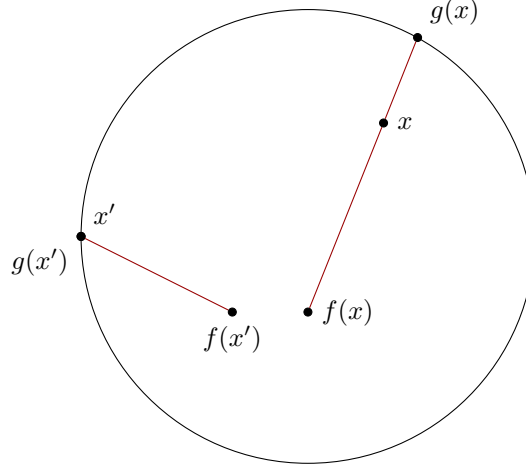
does not extend to a map from the disc D^2 , i.e. there is no map $f : D^2 \rightarrow S^1$ with $f|_{S^1} = \text{id}_{S^1}$, and in particular S^1 is not a retract of D^2 .

Proof. Suppose f does in fact exist, and let $\iota : S^1 \hookrightarrow D^2$ be the inclusion map. Then $(\text{id}_{S^1})_* = f_* \circ \iota_* : \pi_1(S^1, 1) \rightarrow \pi_1(S^1, 1)$.

But $\pi_1(D^2, 1) = 0$ as D^2 contractible so we have a contradiction, as otherwise $f_* : 0 \rightarrow \mathbb{Z}$. \square

Theorem 2.20 (Brouwer's Fixed Point Theorem). *Every map $f : D^2 \rightarrow D^2$ has a fixed point, i.e. some point $x \in D^2$ with $f(x) = x$.*

Proof. Suppose not, and let $g : D^2 \rightarrow S^1$ be defined by projecting $f(x)$ through x to the boundary of D^2 - note that this is well defined precisely because $f(x) \neq x$ for all $x \in D^2$.



Then g is continuous and $g(x) = x$ for all $x \in S^1$, as $f(S^1) = S^1$. But this contradicts **2.19** \square

Theorem 2.21 (The Fundamental Theorem of Algebra). *If f is a non-constant polynomial with coefficients in \mathbb{C} , then there is some $x \in \mathbb{C}$ with $f(x) = 0$.*

Proof (Sketch). Let $r : \mathbb{C} \setminus \{0\} \rightarrow S^1$ be given by $r(z) = \frac{z}{|z|}$, and let $\lambda_R : S^1 \rightarrow \mathbb{C}; z \mapsto R \cdot z$ for some $R \in \mathbb{R}$.

Suppose f has no root. Then define:

$$f_R := r \circ f \circ \lambda_R; S^1 \rightarrow S^1$$

to be a map for any $R \geq 0$. Note that the straight line homotopy is a homotopy between λ_{R_1} and λ_{R_2} , and so f_{R_1}, f_{R_2} are homotopic. Then we have a well defined map:

$$(f_R)_* = g : \pi_1(S^1, 1) \rightarrow \pi_1(S^1, 1)$$

which is necessarily multiplication by some integer d . For $R = 0$, we have that $(f_R)_* = 0$. For R very large, the top degree term z^d dominates and g is given by multiplication by d . Hence $d = 0$ since g is the same in both cases, and so f is constant. \square

We say a space X is **locally simply connected** if for all $x \in X$ and $U \subseteq X$ an open neighbourhood of x , there is some open neighbourhood $V \subseteq U$ of x with V simply connected.

Theorem 2.22 (Existence of Universal Covers). *Let X be a path connected and locally simply connected. Then there exists a universal cover $p : \hat{X} \rightarrow X$.*

For example, if $X = \cup_{n=1}^{\infty} \{(x, y) : (x - \frac{1}{\sqrt{n}})^2 + y^2 = 1/n\} \subseteq \mathbb{R}^2$, the ‘‘Hawaiian Earring’’. Then X is not locally simply connected at $(0, 0)$.

Sketch proof, non-examinable. Fix $x_0 \in X$ and let $\chi = \{\gamma : I \rightarrow X | \gamma \text{ a path from } x_0\}$.

We then define $\hat{X} = \chi / \simeq$, identifying paths which are homotopic, and then we have a covering map $[\gamma] \mapsto \gamma(1)$. \square

The Galois Correspondence

The goal of this course is to classify all covering spaces of a space X using subgroups of $\pi_1(X, x)$.

Let X be a path connected space and $p_1 : \hat{X}_1 \rightarrow X, p_2 : \hat{X}_2 \rightarrow X$ be covering spaces. Then an **isomorphism of covering spaces** is a homeomorphism $\phi : \hat{X}_1 \rightarrow \hat{X}_2$ with $p_2 \circ \phi = p_1$, i.e.:

$$\begin{array}{ccc} \hat{X}_1 & \xrightarrow{\phi} & \hat{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

Note that ϕ^{-1} is also an isomorphism of covering spaces. If \hat{X}_i is equipped with a basepoint \hat{x}_i and $\phi(\hat{x}_1) = \hat{x}_2$, then we say ϕ is **based**.

Note that ϕ is a lift p_1 to \hat{X}_2 , and so if \hat{X}_1 is path connected then ϕ is uniquely determined by $\phi(\hat{x}_1)$ by uniqueness of lifting.

Theorem 2.23 (Galois Correspondence with Basepoints). *Let X be path connected and locally simply connected, with basepoint x_0 . The map which associates a covering map $p : \hat{X} \rightarrow X$ with basepoint $\hat{x}_0 \in p^{-1}(x_0)$ to the subgroup $p_*(\pi_1(\hat{X}, \hat{x}_0)) \leq \pi_1(X, x_0)$ induces a bijection between based isomorphism classes of path connected covering spaces and subgroups of $\pi_1(X, x_0)$.*

Proof. Omitted and non-examinable. □

Example: as $\pi_1(S^1, 1) = \mathbb{Z}$ has subgroups $n\mathbb{Z}$ for n a non-negative integer, we have 1 cover for each n . $n = 0$ gives the universal cover. For $n > 0$ we have $p_n : S^1 \rightarrow S^1, p_n(z) = z^n$. Hence every based covering space is based isomorphic to p or p_n .

Corollary 2.24. *Let X be path connected and locally simply connected. Any two universal covers are (based) isomorphic.*

Proof. Let $p_1 : \hat{X}_1 \rightarrow X, p_2 : \hat{X}_2 \rightarrow X$ be two universal covers. Pick $x \in X, \hat{x}_1 \in p_1^{-1}(x), \hat{x}_2 \in p_2^{-1}(x)$. Since $\pi_1(\hat{X}_i, \hat{x}_i) = 0$, these correspond to the 0 group in $\pi_1(X, x)$, and so by the Galois correspondence these two covering spaces are based isomorphic. □

Corollary 2.25 (Galois Correspondence without Base Points). *Let X be path connected and locally simply connected, with basepoint $x_0 \in X$. The map that sends a covering space $p : \hat{X} \rightarrow X$ with a basepoint $\hat{x}_0 \in p^{-1}(x_0)$ to the subgroup $p_*\pi_1(\hat{X}, \hat{x}_0) \subseteq \pi_1(X, x_0)$ induces a bijection between isomorphism classes of path connected covering spaces without a base point and conjugate classes of subgroups of $\pi_1(X, x_0)$.*

Proof. The map is surjective by the Galois correspondence with basepoints.

To see that this map is injective, we need to show that if, given $p_1 : \hat{X}_1 \rightarrow X, p_2 : \hat{X}_2 \rightarrow X, \hat{x}_i \in p_i^{-1}(x_0)$ and if $p_{1*}\pi_1(\hat{X}_1, \hat{x}_1)$ is conjugate to $p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)$, then p_1, p_2 are isomorphic covering spaces.

So suppose $p_{1*}\pi_1(\widehat{X}_1, \widehat{x}_1) = [\gamma](p_{2*}\pi_1(\widehat{X}_2, \widehat{x}_2))[\bar{\gamma}]$ for some $[\gamma] \in \pi_1(X, x_0)$.

Then let $\widehat{\gamma}$ be the lift of $\bar{\gamma}$ from \widehat{x}_2 . In particular, $\widehat{\gamma}$ is a path in \widehat{X}_2 . Let $\widehat{x}'_2 = \widehat{\gamma}(1)$, so that $p_2(\widehat{x}'_2) = x_0$.

Then $[\gamma](p_{2*}\pi_1(\widehat{X}_2, \widehat{x}_2))[\bar{\gamma}] = g\bar{a}\bar{m}m\bar{a}_{\#} = p_{2*}(\widehat{\gamma}_{\#}(\pi_1(\widehat{X}_2, \widehat{x}_2))) = p_{2*}(\pi_1(\widehat{X}_2, \widehat{x}'_2))$.

And so, by the Galois correspondence with base points there is a based isomorphism between \widehat{X}_1 and \widehat{X}_2 with basepoints $\widehat{x}_1, \widehat{x}'_2$. And so $\widehat{X}_1, \widehat{X}_2$ are isomorphic as covering spaces. \square

3 The Seifert - van Kampen Theorem

Presentation of a Group

Let D_{2n} be the dihedral group of order $2n$. We can represent $D_{2n} = \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1} \rangle$.

Let A be a set and $F(A)$ a group, with $A \rightarrow F(A)$ a map of sets. We say that $F(A)$ is **the free group on A** if it satisfies the following **universal property**:

For any group G and any set map $A \rightarrow G$ there exists a unique group homomorphism $f : F(A) \rightarrow G$, such that the following diagram commutes:

$$\begin{array}{ccc} & F(A) & \\ \uparrow & \searrow f & \\ A & \longrightarrow & G \end{array}$$

f is called the **canonical homomorphism** induced by $A \rightarrow G$

Example: Let $A = \{\alpha\}$, and $A \rightarrow \mathbb{Z}$ given by $\alpha \mapsto 1$. Given a map $A \rightarrow G; \alpha \mapsto g$ we can define a map $\mathbb{Z} \rightarrow G$ by $n \mapsto g^n$, and this is the unique such homomorphism that makes the diagram commute, and so \mathbb{Z} is the free group on $\{\alpha\}$.

Note that the universal property guarantees that $A \rightarrow F(A)$ is unique “up to unique isomorphism” if it exists. To see this, suppose $\phi : A \rightarrow F(A), \phi' : A \rightarrow F'(A)$ both satisfy the universal property. Taking $A \rightarrow G$ to be ϕ or ϕ' , we end up with a function $f : F(A) \rightarrow F'(A)$ and $g : F'(A) \rightarrow F(A)$ homomorphisms. Then we get a diagram:

$$\begin{array}{ccc} & F'(A) & \\ \uparrow & \searrow f \circ g & \\ A & \longrightarrow & F'(A) \end{array}$$

But note that this diagram also commutes with $f \circ g$ replaced by $\text{id}_{F'(A)}$, and so by the uniqueness part of the definition, $g \circ f = \text{id}_{F(A)}$, and so $f : F(A) \rightarrow F'(A)$ is a unique isomorphism such that this diagram commutes.

We don't know yet that $F(A)$ exists, but if it does (and we'll see that it does) and $|A| = r$, we say that $F(A) = F_r$, the **free group of rank r** .

We can write **words** in A as strings of elements and their inverses, for instance $a, b \in A$, $abba^{-1}ba^{-1}b^{-1}b^{-1}$ is a word in $A = \{a, b\}$. These words then give an element in $F(A)$ by applying *phi* to each symbol then multiplying those. Let G be the subgroup of $F(A)$ generated by all words in A . This in fact the set of all elements of $F(A)$ describable as words in $F(A)$, and so we have a map $\phi : A \rightarrow G$. We can check that ϕ also satisfies the universal property, and hence $G = F(A)$.

A **presentation** of a group is a set A and a subset of relations $R \subseteq F(A)$. It **presents** the group $\langle A|R \rangle := F(A)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ denotes the **normal closure** of R , i.e. the subgroup of $F(A)$ generated by $\{srs^{-1} : r \in R, s \in F(A)\}$. The presentation is finite if A and R are both finite sets.

Lemma 3.1 (Universal Property of Presentations). *Let $q : F(A) \rightarrow \langle A|R \rangle$ be the quotient map. Whenever $f : F(A) \rightarrow G$ is a group homomorphism such that $R \subseteq \ker f$ then there exists a unique homomorphism $g : \langle A|R \rangle \rightarrow G$ making the following diagram commute:*

$$\begin{array}{ccc} & \langle A|R \rangle & \\ \uparrow q & \searrow g & \\ F(A) & \xrightarrow{f} & G \end{array}$$

Proposition 3.2. *As $\langle\langle R \rangle\rangle$ is generated by srs^{-1} and $f(srs^{-1}) = f(s)f(r)f(s)^{-1} = f(s)f(s)^{-1} = 1 \in G$, since $r \in \ker f$.*

Hence $\langle\langle R \rangle\rangle \subseteq \ker f$, and so we obtain a well defined $q : F(A)/\langle\langle R \rangle\rangle \rightarrow G$, with $q(a\langle\langle R \rangle\rangle) = g(a)$.

Examples:

1. $\langle a|a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$
2. $\langle r, s|r^n, s^2, rsrs \rangle \rightarrow D_{2n}$. This homomorphism exists by the universal property of the lemma, and is surjective. One can show that every element on the LHS can be written as $1, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}$, which is $2n$ elements, and so the map is also injective, and hence an isomorphism.
3. Every group has a presentation. The identity set map $G \rightarrow G$ gives a group homomorphism $F(G) \rightarrow G$ with kernel R . Then $G \cong \langle G|R \rangle$.

Consider a commutative square of group homomorphisms:

$$\begin{array}{ccc} C & \xrightarrow{i} & A \\ j \downarrow & & \downarrow k \\ R & \xrightarrow{\ell} & \Gamma \end{array}$$

This diagram is called a **pushout** if for every commutative square of groups

$$\begin{array}{ccc}
C & \xrightarrow{i} & A \\
j \downarrow & & \downarrow f \\
R & \xrightarrow{g} & G
\end{array}$$

there is a unique map $\phi : \Gamma \rightarrow G$ making the diagram:

$$\begin{array}{ccccc}
C & \xrightarrow{i} & A & & \\
j \downarrow & & \downarrow k & \searrow f & \\
R & \xrightarrow{\ell} & \Gamma & \xrightarrow{\phi} & G \\
& \searrow g & & & \\
& & & & G
\end{array}$$

commute, i.e. $g = \phi \circ \ell$, $f = \phi \circ k$. If a pushout exists, we write $\Gamma = A \amalg_C B$, and it is unique up to unique isomorphism.

If $C = \{1\}$ then $A \amalg_C B$ is written as $A * B$ and is called the **free product** of A and B . If i, j are injective we write $A \amalg_C B = A *_C B$, the **free product with amalgamation**.