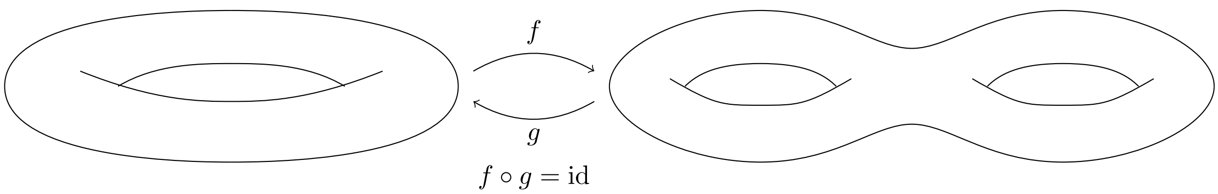
Algebraic Topology

October 17, 2019

0 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions f, g going from a topological space X to another space Y and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



If such f, g continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space X a group $G(X)$, and for every continuous function $f : X \to Y$ a group homomorphism $G(f) : G(X) \to G(Y)$ with $G(\mathrm{id}) = \mathrm{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \to Y$ is a homeomorphism with inverse $g : Y \to X$, then $G(g) \circ G(f) = \mathrm{id}, G(f) \circ G(g) = \mathrm{id}$, so $G(f)$ is an is an isomorphism.

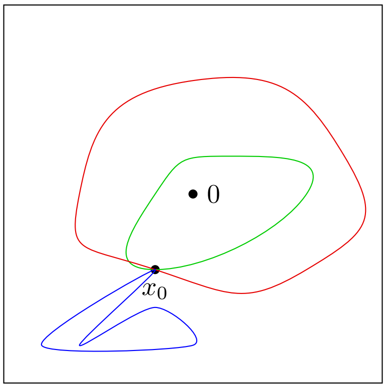
Extension problem: Let X be a topological space, $A \subseteq X$ a subspace, and $f : A \to Y$ a continuous function. Does there exist a continuous function $F : X \to Y$ with $F|_A = f$

$$\begin{tikzcd} X \arrow[r, dashed, "\exists?"'] & Y \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

Theorem 0.1. *There is no continuous function*

$$f : D^n \to S^{n-1} \text{ with } f|_{S^{n-1}} = \mathrm{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct G with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \to D^n \to S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \to 0 \to \mathbb{Z}$ being the identity.



Here, the green and red loops are the "same" loop, whilst the blue one is distinct

Conventions

A topological space will be referred to as a ***space***
A continuous function $f : X \to Y$ will be called a ***map***

1 The Fundamental Group

The idea here is that, if X is a space, $x_0 \in X$ a fixed point, called the ***basepoint***, we consider loops based at x_0, i.e. maps $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the ***fundamental group*** $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at x_0 modulo "deforming loops". Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing γ_1 and then γ_2. But what do we mean by "deforming" a loop?

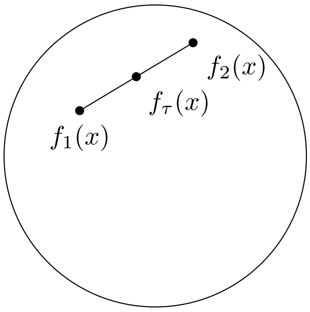
Let $f_0, f_1 : X \to Y$ be maps. A ***homotopy*** between f_0 and f_1 is a map

$$\begin{aligned} F &: X \times I \to Y \text{ where } I = [0, 1] \text{ and} \\ F&(x, 0) = f_0(x) \text{ and} \\ F&(x, 1) = f_1(x) \end{aligned}$$

We often write $f_\tau(x) = F(x, \tau), f_\tau : X \to Y$.

If such F exists, we say f_0 and f_1 are ***homotopic***.

Example: Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \to Y$ are homotopic, via $F(x, t) = tf_1(x) + (1 - t)f_0(x) \in Y$ by convexity.



If f_0 is homotopic to f_1, we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \to Y$. If $Z \subseteq X$ and $f_0(z) = F(z,t) = f_1(z) \forall z \in Z, t \in I$, then we say f_0 is homotopic to f_1 ***relative to*** Z.

Lemma 1.1. *Let $Z \subseteq X, Y$ be spaces. Then $\simeq$ relative to Z is an equivalence relation on the set of maps $X \to Y$.*

Proof.

- Reflexive: $f_0 \simeq f_0$ via $F(x,t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x,t) = f(x, 1-t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x,t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t-1) & t \geq 1/2 \end{cases}$$

  All homotopies are relative to Z.

□

A ***homotopy equivalence*** $f : X \to Y$ is a map with a ***homotopy inverse*** $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$. We then write $X \simeq Y$.

<u>Remark:</u> Most (all?) invariants in the course are ***homotopy invariants***

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \to *$ be the constant map, and let $g : * \to \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \mathrm{id}_*$, and $g \circ f(x) = 0 \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(x,t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map x to the intersection of $\overrightarrow{\mathbf{0}x}$ with S^{n-1}) . Then $g \circ f = \mathrm{id}_{S^{n-1}}$ and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x,t) = (1-t)x + t \cdot \frac{x}{|x|}$

If $X \simeq *$, then we say X is ***contractible***.
Let $f : X \to Y, g : Y \to X$ be maps. If $g \circ f = \mathrm{id}_X$, then we say X is a ***retract*** of Y, and g is a ***retraction***. If in addition $f \circ g \simeq \mathrm{id}_Y$ relative to $f(X)$, then we say X is a ***deformation retract*** of Y. Hence, in example 2, we see that S^{n-1} is a deformation retract of $\mathbb{R}^n$.

Lemma 1.2. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

We want to show $f' \circ f$, $g \circ g'$ induces a homotopy equivalence

$$\begin{tikzcd} X \arrow[r, bend left, "f' \circ f"] & Z \arrow[l, bend left, "g \circ g'"] \end{tikzcd}$$

Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \mathrm{id}_Y$, and so:

$$(x,t) \mapsto g(F'(f(x),t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y$, $g \circ f \simeq \mathrm{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \mathrm{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \mathrm{id}_Z$ □

Loops and π_1

If X is a space, a ***path*** in X is a map $\gamma : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say γ is a path ***from*** x_0 ***to*** x_1.
We say γ_1 and γ_2 are ***homotopic*** if $\gamma_1 \simeq \gamma_2$ relative to $\{0,1\}$, and we write $[\gamma]$ for the homotopy equivalence class of γ.

If X is a space with points $x, y, z \in X$, and γ_1 is a path from x to y, γ_2 is a path from y to z, then:

- The ***concatenation*** of γ_1 and γ_2 is the path from x to z given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

- The ***constant path*** at x is the path $c_x(s) = x \forall s \in I$
- The ***inverse of*** γ_1 is $\overline{\gamma_1}(s) = \gamma_1(1-s)$, a path from y to x.

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend left=40, "\gamma_1 \cdot \gamma_2"] & y \arrow[l, bend left, "\overline{\gamma_1}"] \arrow[r, bend right, "\gamma_2"'] & z \end{tikzcd}$$

Theorem 1.3. *Let X be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X with endpoint x_0 (we say they are* **based** *at x_0). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity c_{x_0} and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.*

This group is called the **fundamental group** *of X (based at x_0).*

To prove this, we will need the following lemmas:

Lemma 1.4. *If $\gamma_0 \simeq \gamma_1$ to y and $\delta_0 \simeq \delta_1$ from y, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$*

$$\begin{tikzcd} {} \arrow[r, bend left, "\gamma_0"] \arrow[r, bend right, "\gamma_1"'] & y \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] & {} \end{tikzcd}$$

Proof. Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq 1/2 \\ G(2s-1,t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s,t) = F(1-s,t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. $\square$

Lemma 1.5. *Let α, β, γ be paths from w to x to y to z in X.*

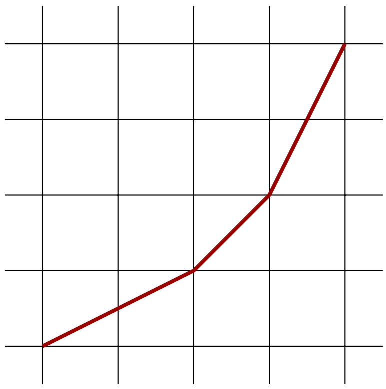
$$\begin{tikzcd} w \arrow[r, bend left, "\alpha"] & x \arrow[r, bend right, "\beta"'] & y \arrow[r, bend left, "\gamma"] & z \end{tikzcd}$$

Then:

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha\beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

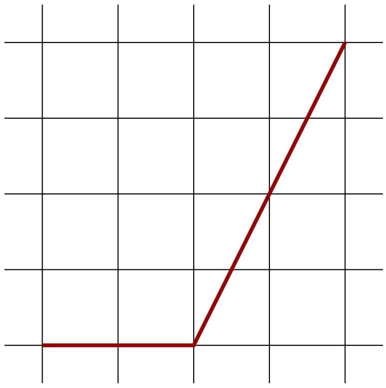
Proof. First, given a path $\delta : I \to X$, a ***reparametrization*** of δ is a path $\delta \circ \phi$ where $\phi : I \to I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that ϕ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s,t) = \delta(t\phi(s) + (1-t)s)$, and this homotopy is relative to $\{0, 1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function ϕ whose plot is:

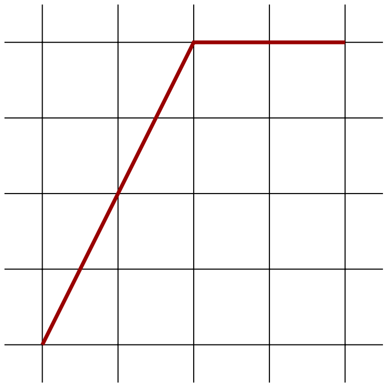


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize α via:



i.e. do c_w for the first half of the time, then do α, so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s,t) = \begin{cases} \alpha(2s) & 0 \leq s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \alpha(2-2s) & 1 - t/2 \leq s \leq 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have c_w at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

□

Then theorem **1.3** giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If γ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

Formal Properties of π_1

Lemma 1.6. *Let $f : X \to Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

Furthermore:

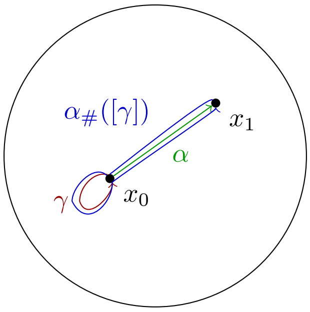
1. *If $f \simeq f'$ relative to x_0, then $f'_* = f_*$.*
2. *If $g : Y \to Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$*
3. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$

Proof. f_* is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to x_0, then for γ a loop based at x_0, $(s,t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

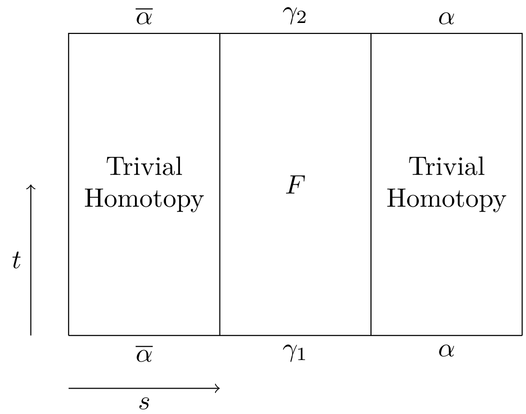
Lemma 1.7. *let X be a space, $x_0, x_1 \in X$ and α a path from x_0 to x_1. Then there is a group isomorphism $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.*



Furthermore,

1. *If $\alpha \simeq \alpha'$ relative to $\{0, 1\}$, then $\alpha_\# = \alpha'_\#$.*
2. $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$
3. *If β is a path from x_2 to x_2, then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$*
4. *If $f : X \to Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.*

Proof. Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

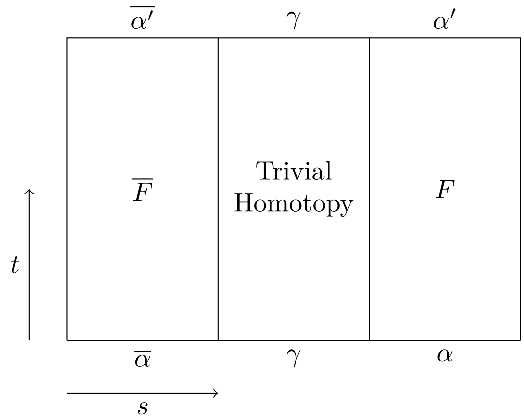


This is indeed a group homomorphism: for loops γ, δ based at x_0,

$$
\begin{aligned}
\bar{\alpha} \cdot \gamma \cdot \alpha) \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\
&\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha
\end{aligned}
$$

Thus $\alpha_\#(\gamma \cdot \delta) = \alpha_\#(\gamma) \cdot \alpha_\#(\delta)$. Also $\bar{\alpha_\#} = (\alpha_\#)^{-1}$ - this is easy to check. Thus $\alpha_\#$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$



   gives $\alpha_\#(\gamma) \simeq \alpha'_\#(\gamma)$
2. Immediate since c_{x_0} is the identity in $\pi_1(X, x_0)$.
3.

$$
\begin{aligned}
(\alpha \cdot \beta)_\#(\gamma) &= \alpha \bar{\cdot} \beta \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot \alpha_\#(\gamma) \cdot \beta \\
&= \beta_\#(\alpha_\#(\gamma))
\end{aligned}
$$

4.

$$
\begin{aligned}
((f \circ \alpha)_{\#} \cdot f_*)(\gamma) &= (f \circ \alpha)_{\#}(f \cdot \gamma) \\
&= (f \circ \alpha)_{\#}(f \cdot \gamma) \\
&= \overline{f \cdot \alpha} \cdot (f \circ \gamma) \cdot (f \circ \alpha) \\
&= f \circ (\bar{\alpha} \cdot \gamma \cdot \alpha) \\
&= f_*(\alpha_{\#}(\gamma))
\end{aligned}
$$

□