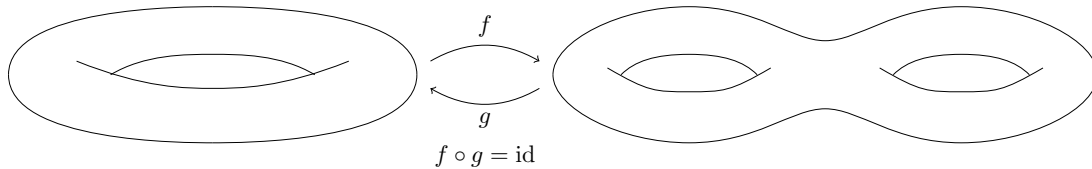


Algebraic Topology

October 21, 2019

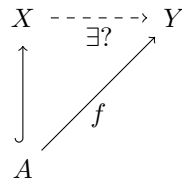
0 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions f, g going from a topological space X to another space Y and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



If such f, g continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space X a group $G(X)$, and for every continuous function $f : X \rightarrow Y$ a group homomorphism $G(f) : G(X) \rightarrow G(Y)$ with $G(\text{id}) = \text{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \rightarrow Y$ is a homeomorphism with inverse $g : Y \rightarrow X$, then $G(g) \circ G(f) = \text{id}, G(f) \circ G(g) = \text{id}$, so $G(f)$ is an isomorphism.

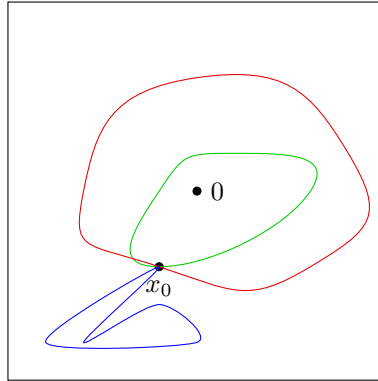
Extension problem: Let X be a topological space, $A \subseteq X$ a subspace, and $f : A \rightarrow Y$ a continuous function. Does there exist a continuous function $F : X \rightarrow Y$ with $F|_A = f$



Theorem 0.1. *There is no continuous function*

$$f : D^n \rightarrow S^{n-1} \text{ with } f|_{S^{n-1}} = \text{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct G with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \rightarrow D^n \rightarrow S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}$ being the identity.



Here, the green and red loops are the “same” loop, whilst the blue one is distinct

Conventions

A topological space will be referred to as a *space*

A continuous function $f : X \rightarrow Y$ will be called a *map*

1 The Fundamental Group

The idea here is that, if X is a space, $x_0 \in X$ a fixed point, called the *basepoint*, we consider loops based at x_0 , i.e. maps $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the *fundamental group* $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at x_0 modulo “deforming loops”. Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing γ_1 and then γ_2 . But what do we mean by “deforming” a loop?

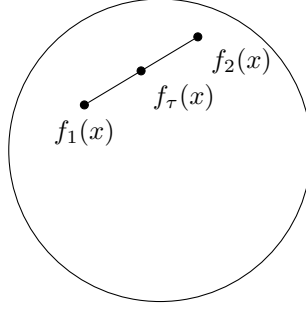
Let $f_0, f_1 : X \rightarrow Y$ be maps. A *homotopy* between f_0 and f_1 is a map

$$F : X \times I \rightarrow Y \text{ where } I = [0, 1] \text{ and} \\ F(x, 0) = f_0(x) \text{ and} \\ F(x, 1) = f_1(x)$$

We often write $f_\tau(x) = F(x, \tau)$, $f_\tau : X \rightarrow Y$.

If such F exists, we say f_0 and f_1 are *homotopic*.

Example: Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \rightarrow Y$ are homotopic, via $F(x, t) = tf_1(x) + (1 - t)f_0(x) \in Y$ by convexity.



If f_0 is homotopic to f_1 , we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \rightarrow Y$. If $Z \subseteq X$ and $f_0(z) = F(z, t) = f_1(z) \forall z \in Z, t \in I$, then we say f_0 is homotopic to f_1 **relative to** Z .

Lemma 1.1. *Let $Z \subseteq X, Y$ be spaces. Then \simeq relative to Z is an equivalence relation on the set of maps $X \rightarrow Y$.*

Proof.

- Reflexive: $f_0 \simeq f_0$ via $F(x, t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x, t) = f(x, 1 - t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f_1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x, t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t - 1) & t \geq 1/2 \end{cases}$$

All homotopies are relative to Z .

□

A **homotopy equivalence** $f : X \rightarrow Y$ is a map with a **homotopy inverse** $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y, g \circ f = \text{id}_X$. We then write $X \simeq Y$.

Remark: Most (all?) invariants in the course are **homotopy invariants**

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \rightarrow *$ be the constant map, and let $g : * \rightarrow \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \text{id}_*$, and $g \circ f(x) = \mathbf{0} \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \text{id}_{\mathbb{R}^n}$ via $F(x, t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map x to the intersection of $\overrightarrow{\mathbf{0}x}$ with S^{n-1}). Then $g \circ f = \text{id}_{S^{n-1}}$ and $f \circ g \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x, t) = (1 - t)x + t \cdot \frac{x}{|x|}$

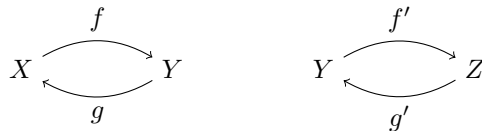
If $X \simeq *$, then we say X is **contractible**.

Let $f : X \rightarrow Y, g : Y \rightarrow X$ be maps. If $g \circ f = \text{id}_X$, then we say X is a **retract** of Y , and g is a **retraction**. If in addition $f \circ g \simeq \text{id}_Y$ relative to $f(X)$, then we say X is a **deformation retract** of Y . Hence, in example 2, we see that S^{n-1} is a deformation retract of \mathbb{R}^n .

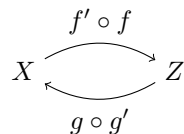
Lemma 1.2. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:



We want to show $f' \circ f, g \circ g'$ induces a homotopy equivalence



Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \text{id}_Y$, and so:

$$(x, t) \mapsto g(F'(f(x), t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y, g \circ f \simeq \text{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \text{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \text{id}_Z$ \square

Loops and π_1

If X is a space, a **path** in X is a map $\gamma : I \rightarrow X$, where $I = [0, 1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say γ is a path **from** x_0 **to** x_1 .

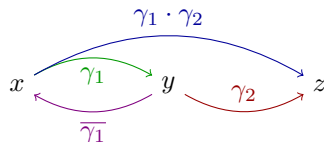
We say γ_1 and γ_2 are **homotopic** if $\gamma_1 \simeq \gamma_2$ relative to $\{0, 1\}$, and we write $[\gamma]$ for the homotopy equivalence class of γ .

If X is a space with points $x, y, z \in X$, and γ_1 is a path from x to y , γ_2 is a path from y to z , then:

- The **concatenation** of γ_1 and γ_2 is the path from x to z given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s - 1) & 1/2 \leq s \leq 1 \end{cases}$$

- The **constant path** at x is the path $c_x(s) = x \forall s \in I$
- The **inverse of** γ_1 is $\overline{\gamma_1}(s) = \gamma_1(1 - s)$, a path from y to x .

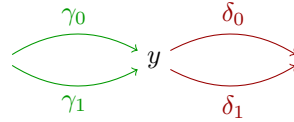


Theorem 1.3. Let X be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X with endpoint x_0 (we say they are **based** at x_0). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity c_{x_0} and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.

This group is called the **fundamental group** of X (based at x_0).

To prove this, we will need the following lemmas:

Lemma 1.4. If $\gamma_0 \simeq \gamma_1$ to y and $\delta_0 \simeq \delta_1$ from y , then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$



Proof. Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s, t) = F(1 - s, t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. □

Lemma 1.5. Let α, β, γ be paths from w to x to y to z in X .

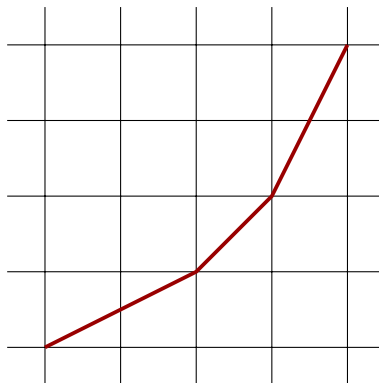


Then:

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

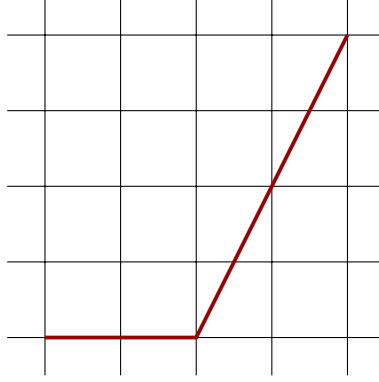
Proof. First, given a path $\delta : I \rightarrow X$, a **reparametrization** of δ is a path $\delta \circ \phi$ where $\phi : I \rightarrow I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that ϕ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s, t) = \delta(t\phi(s) + (1 - t)s)$, and this homotopy is relative to $\{0, 1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function ϕ whose plot is:

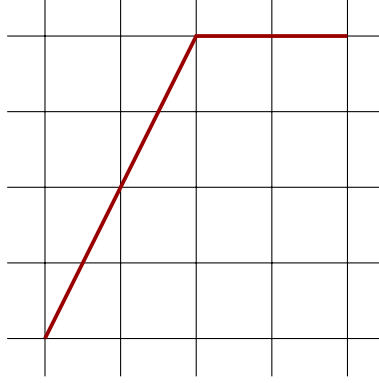


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize α via:



i.e. do c_w for the first half of the time, then do α , so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s, t) = \begin{cases} \alpha(2s) & 0 \leq s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \alpha(2 - 2s) & 1 - t/2 \leq s \leq 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have c_w at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

□

Then theorem 1.3 giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If γ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

Formal Properties of π_1

Lemma 1.6. *Let $f : X \rightarrow Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

Furthermore:

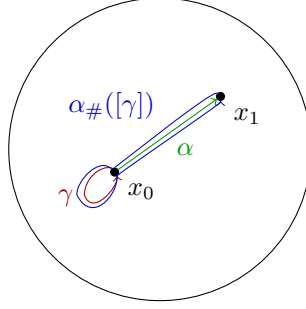
1. If $f \simeq f'$ relative to x_0 , then $f'_* = f_*$.
2. If $g : Y \rightarrow Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$
3. $(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}$

Proof. f_* is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to x_0 , then for γ a loop based at x_0 , $(s, t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

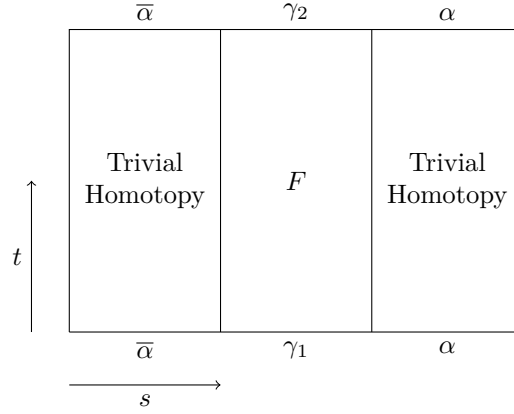
Lemma 1.7. let X be a space, $x_0, x_1 \in X$ and α a path from x_0 to x_1 . Then there is a group isomorphism $\alpha_\# : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.



Furthermore,

1. If $\alpha \simeq \alpha'$ relative to $\{0, 1\}$, then $\alpha_\# = \alpha'_\#$.
2. $(c_{x_0})_\# = \text{id}_{\pi_1(X, x_0)}$
3. If β is a path from x_2 to x_1 , then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$
4. If $f : X \rightarrow Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.

Proof. Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

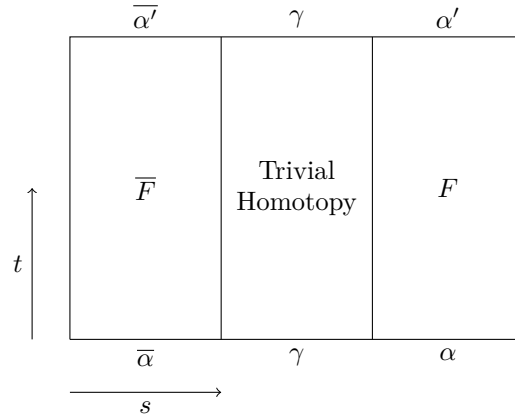


This is indeed a group homomorphism: for loops γ, δ based at x_0 ,

$$\begin{aligned}
 \bar{\alpha} \cdot \gamma \cdot \alpha \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\
 &\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\
 &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\
 &\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha
 \end{aligned}$$

Thus $\alpha_{\#}(\gamma \cdot \delta) = \alpha_{\#}(\gamma) \cdot \alpha_{\#}(\delta)$. Also $\alpha_{\#}^{-1} = (\alpha_{\#})^{-1}$ - this is easy to check. Thus $\alpha_{\#}$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$



gives $\alpha_{\#}(\gamma) \simeq \alpha'_{\#}(\gamma)$

2. Immediate since c_{x_0} is the identity in $\pi_1(X, x_0)$.
- 3.

$$\begin{aligned}
 (\alpha \cdot \beta)_{\#}(\gamma) &= \alpha \cdot \bar{\beta} \cdot \gamma \cdot \alpha \cdot \beta \\
 &= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta) \\
 &= \bar{\beta} \cdot \alpha_{\#}(\gamma) \cdot \beta \\
 &= \beta_{\#}(\alpha_{\#}(\gamma))
 \end{aligned}$$

4.

$$\begin{aligned}
((f \circ \alpha)_\# \cdot f_*)(\gamma) &= (f \circ \alpha)_\#(f \cdot \gamma) \\
&= (f \circ \alpha)_\#(f \cdot \gamma) \\
&= \overline{f \cdot \alpha} \cdot (f \circ \gamma) \cdot (f \circ \alpha) \\
&= f \circ (\bar{\alpha} \cdot \gamma \cdot \alpha) \\
&= f_*(\alpha_\#(\gamma))
\end{aligned}$$

□

A path connected space X is **simply connected** if $\pi_1(x, x_0) = 0$ for any, and hence all, $x_0 \in X$.

Our aim here is to prove that π_1 is a **homotopy invariant**, i.e. that homotopy equivalent spaces have the same fundamental group. We will start with the following lemma:

Lemma 1.8. *Let $x_0 \in X$ and $f, g : X \rightarrow Y$ with $f \simeq_F g$. Set $x(t) = F(x_0, t)$ so that $\alpha(0) = f(x_0)$ and $\alpha(1) = g(x_0)$. Then the diagram:*

$$\begin{array}{ccc}
& & \pi_1(Y, f(x_0)) \\
& \nearrow f_* & \downarrow \alpha_\# \\
\pi_1(x, x_0) & & \\
& \searrow g_* & \downarrow \\
& & \pi_1(Y, g(x_0))
\end{array}$$

commutes, i.e. we have $\alpha_\# \circ f_ = g_*$.*

Proof. We need to check that, for a loop γ based at x_0 , $\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha \simeq g \circ \gamma$.

Let $G : I \times I \rightarrow Y$ defined by $G(s, t) = F(\gamma(s), t)$. For $t = 0$, this is $f \circ \gamma$, and for $t = 1$, this is $g \circ \gamma$. Now consider two paths in $I \times I$:

$$\begin{aligned}
a(t) &= (t, 1); b = b_1 \cdot b_2 \cdot b_3 \text{ where:} \\
b_1(t) &= (0, 1 - t), b_2(t) = (t, 0), b_3(t) = (1, t)
\end{aligned}$$

Then $(G \circ a)(s) = G(s, 1) = g \circ \gamma(s)$, whilst $G \circ b = \bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$.

Now, since $I \times I$ is convex, we have that $a \simeq_H b$, and so $G \circ H$ is the desired homotopy between $g \circ \gamma$ and $\bar{\alpha} \cdot (f \circ \gamma) \cdot \alpha$. □

Theorem 1.9. *If $f : X \rightarrow Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is a homomorphism for any $x_0 \in X$.*

Proof. We'll show that f_* is a bijection:

Let $g : Y \rightarrow X$ be a homotopic inverse to f , with $\text{id}_X \simeq_F g \circ f$. Let $\alpha : I \rightarrow X$ given by $\alpha(t) = F(x_0, t)$.

Note that $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0)); g_* : \pi_1(Y, f(x_0)) \rightarrow \pi_1(X, g(f(x_0)))$

Then $g_* \circ f_* = (g \circ f)_* = \alpha_\# \circ (\text{id}_X)_* = \alpha_\#$. $\alpha_\#$ is an isomorphism, and so f_* is injective.

If $\text{id}_Y \simeq_G f \circ g$ let $\beta(t) = G(f(x_0), t)$. Then $f_* \circ g_* = (g \circ f)_* = \beta_{\#} \circ (\text{id}_Y)_* = \beta_{\#}$, an isomorphism, and hence f_* is surjective. \square

Corollary 1.10. *Contractible spaces are simply connected.*

Proof. If X is contractible, there exists some $x_0 \in X$ and a homotopy F between id_X and $X \rightarrow \{x_0\}$. So $F(x, \cdot)$ is a path from any $x \in X$ to x_0 , so X is path connected. Since X is homotopic to $\{x_0\}$, $\pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0) = 0$. \square

Covering Spaces

Let $p : \hat{X} \rightarrow X$ be a map. An open set $U \subseteq X$ is **evenly covered** if there exists a set Δ_U with the discrete topology and there is a homeomorphism:

$$p^{-1}(U) \xrightarrow{\cong} U \times \Delta_U$$

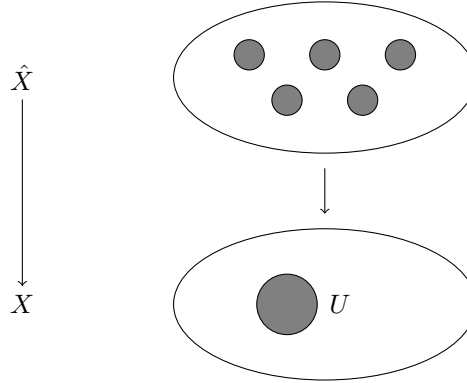
such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \Delta_U \\ & \searrow p & \swarrow (x, \delta) \mapsto x \\ & U & \end{array}$$

We write, for $\delta \in \Delta_U$, $U_{\delta} = U \times \{\delta\}$ and $p_{\delta} = p|_{U_{\delta}}$. So $p_{\delta} : U_{\delta} \rightarrow U$ is a homeomorphism.

Note that we can canonically identify Δ_U with $p^{-1}(x)$ for any $x \in U$. Note also that $p^{-1}(U) \cong \coprod_{\delta \in \Delta_U} U_{\delta}$, where \coprod denotes disjoint union.

If every point of X has an open neighbourhood which is evenly covered, then we say that p is a **covering map** and \hat{X} is a **covering space** of X .



Examples:

1. $\hat{X} = X \times \Delta$ for Δ a set with the discrete topology, e.g. $\hat{I} = I \times \{1, 2, 3\}$. Then \hat{X} is a covering space of X , the identity map on the first element is a covering map.
2. $\hat{X} = \mathbb{R}$, $X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p : \mathbb{R} \rightarrow S^1$ and $p(t) = \exp(2\pi i \cdot t)$. Then p is a covering map:

let $U = S \setminus \{p\}$. We can define a branch of the logarithm $\log : \mathbb{C} \setminus \{rp : r \geq 0\} \rightarrow \mathbb{C}$. Then every point $\hat{z} \in p^{-1}(U)$ can be written uniquely as $\hat{z} = k + \frac{\log z}{2\pi i}$ for some $k \in \mathbb{Z}$.

Thus $p^{-1}(U) \cong U \times \mathbb{Z}$, via $\hat{z} \mapsto \left(\frac{\log z}{2\pi i}, k\right)$, and so each proper subset of S^1 is evenly covered, however S^1 as a whole is not evenly covered, since $p^{-1}(S^1)$ is not a union of copies of S^1 .

3. $\hat{X} = X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p(z) = z^n$.

p is a covering map by choosing a branch of the n th root on proper open subsets of S^1

4. Let $\hat{X} = S^2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on S^2 by the antipodal map $z \mapsto -z$. Then let $X = \hat{X}/G = \hat{X}/\sim$, where $x \sim y \iff x = \pm y$.

Then X is \mathbb{RP}^2 , the real projective plane. If $x \in X$, let U be an open neighbourhood of x disjoint from its negation. Then the image of U in X is evenly covered.

We say a covering map $p : \hat{X} \rightarrow X$ is **n -sheeted** if $\#p^{-1}(x) = n$ for all $x \in X$, and call n the **degree** of p .

Lifting Properties

Let $p : \hat{X} \rightarrow X$ be a covering map, and $f : Y \rightarrow X$ be a map. A **lift** of f to \hat{X} is a map $\hat{f} : Y \rightarrow \hat{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \hat{X} \\ & \nearrow \hat{f} & \downarrow p \\ Y & & X \\ & \searrow f & \end{array}$$

A space X is **locally path connected** if for every $x \in X$ and $U \subseteq X$ open neighbourhood of x , there exists a neighbourhood $V \subseteq U$ of x which is path connected.

Lemma 1.11 (Uniqueness of Lifting). *Let $p : \hat{X} \rightarrow X$ be a covering map and $\hat{f}_1, \hat{f}_2 : Y \rightarrow \hat{X}$ be two lifts of $f : Y \rightarrow X$ with Y connected and locally path connected.*

If there exists some $x_0 \in Y$ with $\hat{f}_1(x_0) = \hat{f}_2(x_0)$, then $\hat{f}_1 = \hat{f}_2$.

Proof. We will show that the set $S := \{y \in Y : \hat{f}_1(y) = \hat{f}_2(y)\}$ is both open and closed. By assumption we have $x_0 \in S$, so $S \neq \emptyset$. Since Y is connected, we must have then that $S = Y$ as otherwise S and $Y \setminus S$ would disconnect Y .

Let $y_1 \in Y$ be an arbitrary point, and let $U \subseteq X$ be an open neighbourhood of $f(y_1)$ which is evenly covered by p . Let $V \subseteq f^{-1}(U)$ be an open neighbourhood of y_1 which is path connected. We then want to show that, if $y_1 \in S$ then all of $V \subseteq S$, and otherwise $V \subseteq Y \setminus S$.

Let $y \in V$ be arbitrary and let α be a path from $y_1 \rightarrow y$. Then $\hat{f}_i \circ \alpha$ is a path from $\hat{f}_i(y_1) \rightarrow \hat{f}_i(y)$ for $i = 1, 2$.

Note that $p \circ \hat{f}_1 \circ \alpha(t) = f(\alpha(t)) \in U$, and so $\hat{f}_1(y)$ and $\hat{f}_1(y_1)$ lie in the same component of $p^{-1}(U)$, say U_{δ_1}

If $y_1 \in S$, then $\hat{f}_1(y_1) = \hat{f}_2(y_1)$, so $\delta_1 = \delta_2$, and so $\hat{f}_1(y) = p_{\delta_1}^{-1}(f(y)) = p_{\delta_2}^{-1}(f(y)) = \hat{f}_2(y)$, so $y \in S$, and hence all of $V \subseteq S$.

Otherwise $y_1 \notin S$, then $\hat{f}_1(y_1) \neq \hat{f}_2(y_1)$. Each U_{δ_i} contains a unique point of $p^{-1}(\{f(y_1)\})$, and we must have $\delta_1 \neq \delta_2$.

So $\hat{f}_1(y) \neq \hat{f}_2(y)$, so $y \notin S$, and in general $V \subseteq Y \setminus S$.

Hence S is open, $Y \setminus S$ is open, and we are done \square

Let $\gamma : I \rightarrow X$ be a path from $x_0 \in X$ and $p : \hat{X} \rightarrow X$ be a covering map. A lift of γ at (or from) \hat{x}_0 is a lift $\hat{\gamma}$ of γ with $\hat{x}_0 = \hat{\gamma}(0)$. In particular, $p(\hat{x}_0) = p(\hat{\gamma}(0)) = \gamma(0) = x_0$.