Sogic and Let Theory

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1 Propositional Logic

Let P be a set of ***primitive propositions***, i.e. P is a set of symbols with $(,), \bot, \implies \notin P$. Unless stated otherwise (i.e. that P is uncountable), we may assume that $P = \{p_1, p_2, \ldots\}$.

The set of ***propositions***, denoted by $L(P)$ or simply just L, is defined inductively as follows:

1. $P \subset L$
2. $\bot \in L$, called FALSE
3. if $p, q \in L$, then $(p \implies q) \in L$

Each proposition is a string of symbols from $P \cup \{(,), \bot, \implies\}$, for instance we have the propositions $p_1, (p_1 \implies p_1), ((p_1 \implies p_2) \implies (p_2 \implies (\bot \implies p_3)))$. For readability, we often draw symbols $(,)$ in different ways, for instance as $[, (, \Big($.

Sometimes we omit the outside pair of parentheses when writing down propositions, for instance $p_1 \implies p_2$ is shorthand for $(p_1 \implies p_2)$.

Also we use some abbreviations, e.g.:

NOT: $\neg p$ to mean $(p \implies \bot)$

OR: $p \vee q$ to mean $(\neg p \implies q)$

AND: $p \wedge q$ to mean $\neg(\neg p \vee \neg q)$

What do we mean by L "defined inductively"? Define $L_0 = P \cup \{\bot\}$. Then, given L_n, we can define $L_{n+1} = L_{n-1} \cup \{(p \implies q) : p, q \in L_{n-1}\}$. Then we set $L = \bigcup_{n=0}^{\infty} L_n$. Note: if $p \in L \setminus (P \cup \{\bot\})$, then it is easy to show that there are ***unique*** $q, r \in L$ with $p = (q \implies r)$.

1.1 Semantic Entailment

A ***valuation*** is a function $v : L \to \{0, 1\}$ satisfying:

1. $v(\bot) = 0$
2. For all $p, q \in L, v(p \implies q) = \begin{cases} 0 & v(p) = 1, v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$.

If $p \in L$ and $v(p) = 1$ for every valuation, we say that p is a ***tautology***, and write $\vDash p$.

Examples:

1. $\vDash (p \implies p)$

$v(p)$	$v(p \implies p)$
0	1
1	1

   So this is a tautology.

2. $\vDash (p \implies (q \implies p))$

p	q	$q \implies p$	$p \implies (q \implies p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

   So this is a tautology.

3. Is $\vDash (p \implies (q \implies r)) \implies ((p \implies q) \implies (p \implies r))$?

   Suppose not. Then for some p, q, r and valuation v we have:

$$v(p \implies (q \implies r)) = 1$$
$$v((p \implies q) \implies (p \implies r)) = 0.$$

   So $v(p \implies q) = 1, v(p \implies r) = 0$. Hence $v(p) = 1, v(r) = 0, v(q) = 1$. But then $v(q \implies r) = 0$, and so $v(p \implies (q \implies r)) = 0 \lightning$.

4. $\vDash ((p \implies \perp) \implies \perp) \implies p$, i.e. $\neg\neg p \implies p$, i.e. $(\neg p \vee p)$. This is the Law of the Excluded Middle, and is also a tautology.

Note that a valuation is entirely determined by its values on the primitive propositions.

Proposition 1.1.

1. *Let $v, w : L \to \{0, 1\}$ be valuations with $v|_P = w|_P$. Then $v = w$.*
2. *Let $f : P \to \{0, 1\}$. Then there is a valuation $v : L \to \{0, 1\}$ with $v|_P = f$.*

Proof.

1. We prove this by induction on n, so that $v|_{L_n} = w|_{L_n}$. For the base case of $n = 0$, $v|_P = w|_P$, and $v(\perp) = 0 = w(\perp)$. Then for the induction step, $v|_{L_{n-1}} = w|_{L_{n-1}}$. Let $p \in L_n \setminus L_{n-1}$. Then $p = (q \implies r)$ for some $q, r \in L_{n-1}$. We know that $v(q) = w(q), v(r) = w(r)$, and so $v(p) = w(p)$.
2. We define v successively on $L_0, L_1, L_2, \ldots$

   L_0: Let $v|_P = f$ and let $v(\perp) = 0$

   L_n: If $p \in L_n \setminus L_{n-1}$, then $p = (q \implies r)$, and so set $v(p)$ to be 0 if $v(q) = 1, v(r) = 0$, and 1 otherwise. Since propositions are built up in a unique way, this is indeed a valuation.

$\square$

Let $S \subset L$. We say that v is a ***model*** of S if v is a valuation with $v(x) = 1$ for all $x \in S$. If $S = \{p\}$, we say that v is a model of p. If every model of $S \subset L$ is a model of $p \in L$, we say that S ***semantically entails*** p, and write $S \models p$. Note that $\emptyset \models p$ is exactly the same as $\models p$.

For example, $\{p, p \implies q\} \models q$.

1.2 Syntactic Entailment (Provability)

Our proof system will have axioms as follows for all $p, q, r \in L$:

A1 $p \implies (q \implies p)$

A2 $(p \implies (q \implies r)) \implies ((p \implies q) \implies (p \implies r))$

A3 $((p \implies \perp) \implies \perp) \implies p$

Our proof system also has a ***deduction rule*** known as ***modus ponens*** (MP): for all $p, q \in L$, from p and $(p \implies q)$ we can deduce q.

Note that each axiom is a tautology. For MP, see the last example of **§1.1**

Let $S \subset L$ and $p \in L$. A ***proof*** of p from S is a sequence $t_1, t_2, \ldots, t_n \in L$ of finite length with $t_n = p$ such that, for each i, either t_i is an axiom, or $t_i \in S$ (a ***hypothesis***), or there exist $j, k < i$ with $t_k = (t_j \implies t_i)$.

If there exists a proof of p from S, we say that S ***syntactically entails*** p, or S ***proves*** p, and we write $S \vdash p$. If $S = \emptyset$, we say p is a ***theorem*** and write $\vdash p$.

Example: $\vdash (p \implies p)$

Use A2, with $r = p$, to get $(p \implies (q \implies p)) \implies ((p \implies q) \implies (p \implies p))$. Now the first bracket is a theorem by A1, and if we take $q = (p \implies p)$ in the second, we can use modus ponens twice with A1 to deduce the final bracket, that $(p \implies p)$. We will write this formally:

Lemma 1.2. *For all* $p \in L, \vdash (p \implies p)$

Proof.

1. $(p \implies ((p \implies p) \implies p)) \implies ((p \implies (p \implies p)) \implies (p \implies p))$ (A2)
2. $p \implies ((p \implies p) \implies p)$ (A1)
3. $(p \implies (p \implies p)) \implies (p \implies p)$ (MP on 1, 2)
4. $p \implies (p \implies p)$ (A1)
5. $p \implies p$ (MP on 3, 4)

□

Proposition 1.3 (The Deduction Theorem)**.** *Let* $S \subset L$ *and* $p, q \in L$*. Then* $S \vdash (p \implies q)$ *if and only if* $S \cup \{p\} \vdash q$.

Proof. Suppose $t_1, \ldots, t_n$ is a proof of $p \implies q$ from S. Then $t_1, \ldots, t_n, p, q$ is a proof of q from $S \cup \{p\}$. Suppose that $t_1, \ldots, t_n$ instead is a proof of q from $S \cup \{p\}$. We show by induction on i that $S \vdash (p \implies t_i)$ for each i, and then we will be done since $t_n = q$.

1. If $t_i \in S$:

- $t_i \implies (p \implies t_i)$ (A1)
- t_i (hypothesis)
- $(p \implies t_i)$ (MP)

2. If $t_i = p$, use Lemma **1.2**
3. If $t_j = (t_j \implies t_i)$ for some $j, k < i$, then write down proofs of $(p \implies t_j), (p \implies t_k)$ from S. Then append:
   - $(p \implies (t_j \implies t_i)) \implies ((p \implies t_j) \implies (p \implies t_i))$ (A2)
   - $(p \implies t_j) \implies (p \implies t_i)$ (MP)
   - $p \implies t_i$ (MP)

□

1.3 The Completeness Theorem and Applications

The key result of this section will be that $\vDash$ and $\vdash$ coincide. There will be two directions to prove:

1. ***Soundness:*** If $S \vdash p$ then $S \vDash p$.
2. ***Adequacy:*** If $S \vDash p$ then $S \vdash p$.

Proposition 1.4 (Soundness Theorem)**.** *Let $S \subset L$ and $p \in L$ with $S \vdash p$. Then $S \vDash p$.*

Proof. Let $t_1, \ldots, t_n$ be a proof of p from S. Let v be a model of S. We show by induction on i that $v(t_i) = 1$ for $1 \leq i \leq n$.

If $t_i \in S$ then $v(t_i) = 1$. If t_i is an axiom then $\vDash t_i$ so $v(t_i) = 1$. Otherwise, $t_k = (t_j \implies t_i)$ for some $j, k < i$. By the induction hypothesis, $v(t_j) = v(t_j \implies t_i) = 1$, so $v(t_i) = 1$. □

For adequacy, first consider the special case $p = \bot$, i.e. "If $S \vDash \bot$ them $S \vdash \bot$". We will prove the contrapositive: "If $S \nvdash \bot$ then $S \nvDash \bot$". If $S \nvdash \bot$ we say that S is ***consistent***. '$S \vDash \bot$" means "if v is a model of S then $v(\bot) = 1$". But $v(\bot) = 0$ for every valuation v, so this says "S has no model." Hence "$S \nvDash \bot$" says "S has a model".

Theorem 1.5 (Model Existence Lemma)**.** *Let $S \subset L$ be consistent. Then S has a model.*

Proof in the case P is countable. L is countable, as each $p \in L$ is a finite string of symbols from $P \cup \{(,), \bot, \implies\}$.

We write $L = \{x_1, x_2, \ldots\}$. We shall recursively construct sets $S_n \subset L$ with $S = S_0 \subset S_1 \subset \ldots$ and S_n consistent.

The base case is trivial, as $S_0 = S$ is consistent by hypothesis. Then for $n > 0$, we have S_{n-1} consistent. If $S_{n-1} \cup \{\neg x_n\}$ is consistent, let $S_n = S_n \cup \{\neg x_n\}$. Otherwise, $S_{n-1} \cup \{\neg x_n\} \vdash \bot$, and by the deduction theorem, $S_{n-1} \vdash (\neg x_n \implies \bot)$, i.e. that $S_{n-1} \vdash \neg\neg x_n$. But $S_{n-1} \vdash (\neg\neg x_n \implies x_n)$ by (A3), and so $S_{n-1} \vdash x_n$ by (MP). But S_{n-1} is consistent, so let $S_n = S_{n-1} \cup \{x_n\}$.

Then let $\bar{S} = \bigcup_{n=1}^{\infty} S_n$. Firstly, S_n is consistent - suppose $t_1, \ldots, t_n$ is a proof of $\bot$ from $\bar{S}$. Then there is some collection $i_1, \ldots, i_m \in \mathbb{N}$ such that the hypotheses used in the proof come from $S_{i_1}, \dots, S_{i_m}$. Let $I = \max\{i_1, \dots, i_m\}$. Then every hypothesis comes from S_I, and so $t_1, \dots, t_n$ is a proof of $\perp$ from S_I. $\lightning$

Also, for every $p \in L$ we have $p \in \bar{S}$ or $\neg p \in \bar{S}$. Moreover, $\bar{S}$ is ***deductively closed*** (d.c): if $\bar{S} \vdash p$ then $p \in \bar{S}$. Indeed, suppose that $\bar{S} \vdash p$ but $p \notin \bar{S}$. Then $\neg p \in \bar{S}$. Now $\bar{S} \vdash p$ and $\bar{S} \vdash \neg p$, i.e. $\bar{S} \vdash (p \implies \perp)$. So by (MP), $\bar{S} \vdash \perp \lightning$.

Now let $v : L \to \{0, 1\}$ be the indicator function of $\bar{S}$. We must check that v is a valuation. As $\bar{S}$ is consistent, it is certainly true that $\perp \notin \bar{S}$, and so $v(\perp) = 0$.

Let $p, q \in L$. We want to think about $(p \implies q)$:

Case 1. Suppose $v(q) = 1$. Then $q \in \bar{S}$, so $\bar{S} \vdash (p \implies q)$, but $\bar{S}$ is deductively closed, and so $(p \implies q) \in \bar{S}$, and $v(p \implies q) = 1$.

Case 2. Suppose $v(p) = 0$. Again, we must show that $v(p \implies q) = 1$, i.e. that $\bar{S} \vdash (p \implies q)$. By the Deduction Theorem, this is equivalent to $S \cup \{p\} \vdash q$, and $p \notin S$, so $\neg p \in S$ and it will be enough to show that $\{p, \neg p\} \vdash q$. We have:

1. $(p \implies \perp)$ (hyp)
2. p (hyp)
3. $\perp$ (MP on 1,2)
4. $((q \implies \perp) \implies \perp) \implies q$ (A3)
5. $\perp \implies ((q \implies \perp) \implies \perp)$ (A1)
6. $(q \implies \perp) \implies \perp$ (MP on 3,5)
7. q (MP on 4,6)

Case 3. $v(p) = 1, v(q) = 0$. We want to show that $v(p \implies q) = 0$. Suppose instead that $v(p \implies q) = 1$, so that $(p \implies q) \in \bar{S}, p \in \bar{S}$. But then by (MP) $\bar{S} \vdash q$, so $q \in \bar{S}$, so $v(q) = 1$. $\lightning$

We have now shown that v is a valuation. Moreover, $S \subset \bar{S}$ so $v(p) = 1$ for all $p \in S$. Hence v is a model of S. $\square$