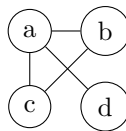# Graph Theory

October 15, 2019

1 Introduction

Definitions

Formally, we define a **graph** to be an ordered pair $G = (V, E)$ where $E \subseteq V^{(2)} := \{Y \subseteq V : |Y| = 2\}$

e.g. $V = \{a, b, c, d\}, E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}\}$

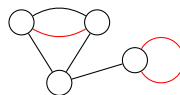The set $V(G)$ is the **vertex set** of G and $E(G)$ is the **edge set**.
The **order** of G is $|V(G)|$, often written $|G|$.
The **size** of G is $|E(G)|$, often written $e(G)$.
Note that for the purposes of this course, all graphs will be finite.

This definition that we use precludes:

- Having more than one edge between the same two vertices

- Having an edge with both ends at the same vertex


These 2 red edges are not allowed

Some authors allow graphs to have these features, and call what we're studying **simple graphs**.

It's easier to write the edge $\{a, b\}$ as ab.
If $ab \in E(G)$, we say that "a is **adjacent** to b", and vice versa. We might write $ab \in G$.

Edges containing a vertex x are said to be **incident** to x and to each other.

Two graphs G, H are said to be **isomorphic** if there exists a bijection
$f : V(G) \to V(H)$ s.t. $uv \in E(G) \iff f(u)f(v) \in E(H)$.

(a) K_5 (b) P_4 (c) C_6
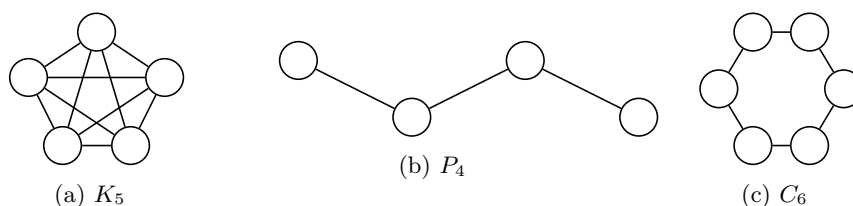
There are some canonical graphs:

Name	Notation	Order	Size
Empty graph	E_n	n	0
Complete graph	K_n	n	$\binom{n}{2}$
Path	P_n	n	$n-1$
Circuit	C_n	n	n

A **subgraph** of G is another graph H with $V(H) \subseteq V(G), E(H) \subseteq E(G)$. (Ever graph of order $\leq n$ is a subgraph of K_n).
The subgraph of G **induced by** $W \subseteq V(G)$, written $G[W]$ is the graph $(W, E(G) \cap W^{(2)})$, i.e. the vertex set W and all edges of G lying inside W.

A graph is **connected** if there is a $u-v$ path (a path from u to v in G for every pair $u, v \in V(G)$. The **components** of G are the maximal connected subgraphs of G (induced by the equivalence classes of the relation $u \leftrightarrow v \iff u = v$ or there is a $u - v$ path).

E.g.:
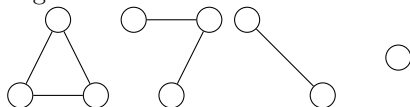
has 4 components.

A **forest** is a graph containing no circuit.
A **tree** is a connected forest, i.e. the components of a forest are trees.
Equivalently, a tree is a connected circuit-free graph. This bring us on to our first theorem of the course:

Theorem 1.1. *The following are equivalent:*

  1. G is a tree

  2. G is a minimal connected (connected, but the removal of any edge kills connectivity)

  3. G is maximal circuit-free (the addition of any edge creates a circuit)

Proof. $(a) \implies (b)$

Let G be connected and circuit free. Suppose $uv \in G$ and $G - uv$ is connected. Then there is a $u - v$ path P in $G - uv$, so G contains the circuit $P + uv \nleqq (G$ circuit free)
$\therefore G$ is minimal connected.

$(b) \implies (a)$

Let G be minimal connected, but suppose G has a circuit C, with $xy \in C$. Let $u, v \in V(G)$.
There is a $u - v$ path P in G.
If $xy \notin P$, then P joins u to v in $G - xy$.
If $xy \in P$, then $(P \cup C) - xy$ contains a $u - v$ path in $G - xy$.
Hence $G - xy$ is connected $\nleq$ (G **minimal** connected).
$\therefore G$ is circuit free, and connected by hypothesis, hence a tree.

$(a) \implies (c)$

Let G be a tree. If $uv \notin E(G)$ there is a path P in G from u to v, so $P + uv$ is circuit in $G + uv$, so G is maximal circuit free.

$(c) \implies (a)$

Let G be maximal circuit free. Let $u, v \in V(G)$. Either $uv \in E(G)$ or $G + uv$ has a circuit C, so $C - uv$ is a $u - v$ path in G, and so G is connected, and hence a tree. $\qquad \square$

A subgraph T of a graph G is said to be **spanning** if $V(T) = V(G)$, i.e. the subgraph touches every vertex of G

Corollary 1.2. *A graph is connected if and only if it has a spanning tree.*

Proof.

- $\implies$ T is connected as it is a tree, and is spanning, so all vertices in G are connected via T.

- $\impliedby$ Remove edges from G to obtain a minimal connected spanning subgraph T. By **1.1** T is a tree.

$\qquad \square$

The set of **neighbours** of a vertex v is $\Gamma(v) = \{w : vw \in E(G)\}$.
The **degree** $d(v)$ is the number of neighbours of v, $|\Gamma(v)|$
The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$.
If $\delta(G) = \Delta(G) = k$, i.e. $d(v) = k \forall v$ we say G is k-**regular**. A **regular** graph is one that is k-regular for some k. If a graph is 3-regular, it is called **cubic**.
The degrees of the graph written in some order form a **degree sequence** of G.

Lemma 1.3 (Handshaking Lemma)**.**

$$\sum_{v \in G} d(v) = 2e(G)$$

Proof. Each edge has 2 vertices at the end, so contributes 2 to the sum of all degrees. $\qquad \square$

A **leaf** is a vertex of degree 1.

Theorem 1.4. *Every tree T with $|T| \geq 2$ has at least two leaves.*

Proof. We note T is connected so $\delta(T) \geq 1$. Let x_1 be a vertex (a leaf if there is one). Let $x_1 x_2 \ldots x_k$ be a maximal path from x_1. T is circuit-free so none of $x_1, \ldots, x_{k-2}$ is in $\Gamma(x_k)$, and x_k has no other neighbours except for x_{k-1}, otherwise this path can be extended, hence $\Gamma(x_k) = \{x_{k-1}\}$, so x_k is a leaf. Repeat the proof starting at x_k for a second leaf. $\square$

Corollary 1.5. *A tree of order n has size $n - 1$*

Proof.

This is true for $n = 1, 2$ by inspection, as there is only 1 tree:

(a) $n = 1$ (b) $n = 2$
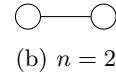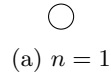
Let $|T| \geq 3$. By **1.3** T has a leaf v. Then $T - v$ is certainly circuit free and connected, because the $x - y$ path in T uses v if and only if x or y is v. Hence the corollary follows by induction. $\square$

Corollary 1.6. *The following are equivalent for a graph G of order n:*

1. *G is a tree*

2. *G is connected and $e(G) = n - 1$*

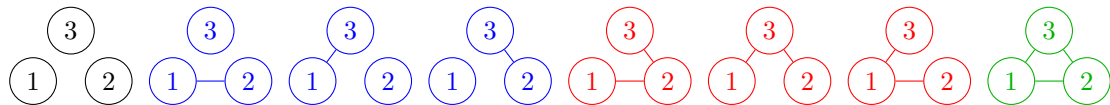3. *G is acyclic and $e(G) = n - 1$*

Proof.

- (1) $\implies$ (2), (3): True by definition and **1.5**.

- (2) $\implies$ (1): By **1.2** G contains a spanning tree T. By **1.5** $e(T) = e(G)$ so $T = G$

- (3) $\implies$ (1): Add edges to G to get a graph G' maximal acyclic. By **1.1** G' is a tree, so by **1.5** $e(G') = n - 1 = e(G)$, so $G' = G$.

$\square$

How many graphs of order n are there?

Let the vertex set be $[n] := \{1, 2, \ldots, n\}$. E.g. for $n = 3$ we have:

There are $2^{\binom{n}{2}}$ **labelled** graphs of order n - $\binom{n}{2}$ potentially edges, each of which we can independently choose to include or not.

However, there are only 4 unlabelled graphs of order 3 - all the graphs of the same colour are essentially the same, just with different labels given to the vertices. In general we must count the number of orbits amongst labelled graphs under the action of a group, using Burnside's Lemma.

In fact, it turns out this number is asymptotic to $2^{\binom{n}{2}}/n!$

How many trees of order n are there?

Theorem 1.7 (Cayley)**.** *There are n^{n-2} labelled tress of order n.*

Proof. (Prüfer, Non-examinable)
We construct a bijection between trees strings length $n-2$ over the alphabet n.

- Tree $\rightarrow$ string: Select the lowest leaf, write down its neighbour, remove that leaf, and repeat until only one edge is left.
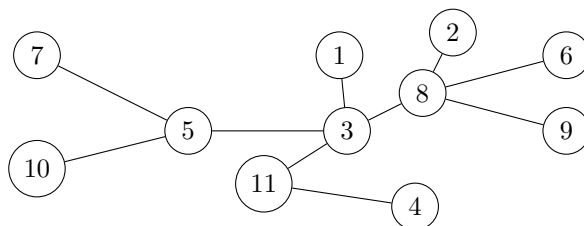
  Here, we would get the string $3, 8, 11, 8, 5, 8, 3, 5, 10$. Each vertex is written down $d(n) - 1$ times

- String $\rightarrow$ tree: Mark all vertices "unused". Then repeatedly choose the smallest unused vertex not in the string. Join it to the first vertex in the string. Mark it as used. Delete first vertex in the string.

$\square$

A graph is r-***partite*** if its vertices can be partitioned into r classes so that no edge lies within a class. In the special case where $r = 2$ we call the graph ***bipartite***. Remarkably, we can characterise all bipartite graphs.

Theorem 1.8. *A graph is bipartite if and only if it contains no odd circuit.*

Proof. For the "only if" part, in a bipartite graph the vertices of a circuit alternate between the two classes, otherwise an edge would lie in a class. Hence any circuit must be of even length.

For the "if" part, take some graph G, pick a vertex v_0 and let $V_i = \{v : d(v_0, v) = i\}$ where $d(v_0, v)$ is the ***distance*** of v from v_0, i.e. the length of a shortest $v_0 - v$ path.
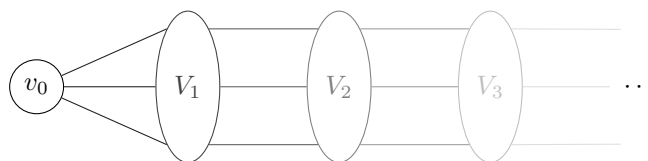
If G is connected, $V_0, V_1, V_2, \ldots$ partition $V(G)$. (Note $V_0 = \{v_0\}$). There is no edge between V_i and V_{i+k} if $k \geq 2$ by construction. Suppose G has no odd circuit. Then G cannot have an edge inside one of the V_i: if $u, v \in V_i$ and $uv \in G$, then there is a way of getting from v_0 to itself in an odd number of steps. $v_0 \xrightarrow{i} u \xrightarrow{1} v \xrightarrow{i} v_0$. This must contain an odd circuit: let the first path be $v_0 = a_0, a_1, a_2, \ldots, a_{i-1}, u$ and the second be $v_0 = b_0, b_1, b_2, \ldots, b_{i-1}, v$, and let $k = \max\{j : a_j = b_j\}$, so $0 \leq k < i$. Then $a_k, a_{k+1}, \ldots, u, v, b_{i-1}, \ldots, b_k = a_k$ is an odd cycle. Hence we can let our two classes be $\cup_i V_{2i}$ and $\cup_i V_{2i+1}$, and G is bipartite. $\square$
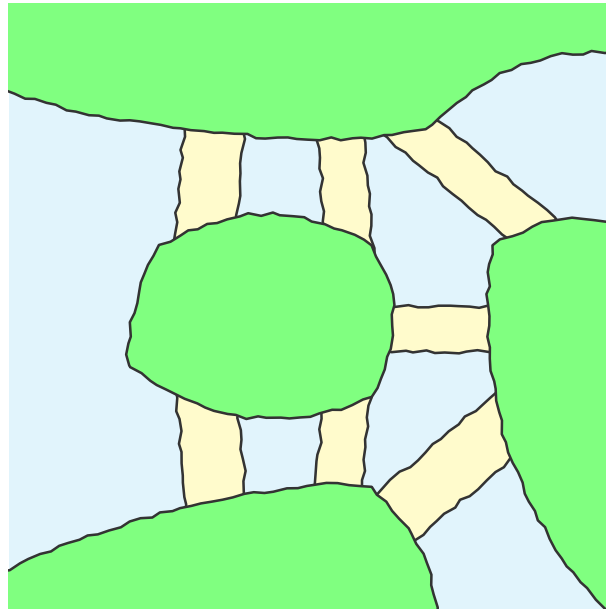
The bridges of Königsberg, which inspired this problem

An **_Euler cycle_** as a "walk" around all the edges of a graph returning to the starting point so that every vertex is visited and every edge is used precisely once.

A graph is **_Eulerian_** if it is a single vertex or has an Euler cycle.

Theorem 1.9. *A graph is Eulerian if and only if it is connected and every vertex has even degree*

Proof. The forwards direction is clear: an Eulerian graph must be connected, and for every time we enter a vertex along an edge we must also leave that vertex, so each vertex has even degree.

We do the opposite direction by induction on $e(G)$. Since $\delta(G) \geq 2$ by **1.4** G is not a tree and so has a circuit C. Every vertex of $G - E(C)$ has even degree, so by induction each component of $G - E(C)$ is Eulerian. Now set off walking round C, taking the time to traverse any Euler cycle of $G - E(C)$ when first encountered. Since G is connected, this gives an Euler cycle of G. $\qquad\square$

A graph that can be drawn in the plane without edges crossing is **_planar_**. A graph drawn in the plane is a **_plane graph_**. A **_face_** of a plane graph is a connected region of $\mathbb{R}^2 \setminus G$. We define the **_size_** of a face to be the number of edges it has.
E.g.:
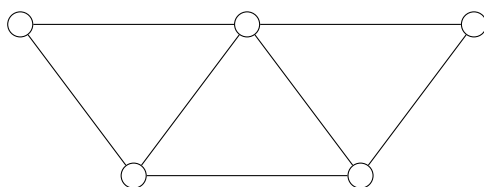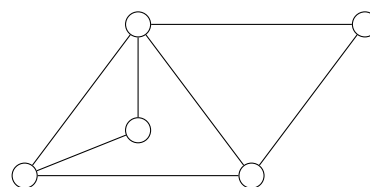
(a) Face sizes $3, 3, 3, 5$ (b) Face sizes $3, 3, 4, 4$

These two graphs are the same planar graph, but are different plane graphs. Note that we include the outer infinite face. There are no analytical/topological difficulties involved. Any planar graph can be drawn with piecewise linear edges, and in fact with straight line edge (see example sheet 1), so any issues are purely combinatorial.

Theorem 1.10 (Euler)**.** *Let G be a connected plane graph with n vertices, m edges, and f faces. Then $n - m + f = 2$.*