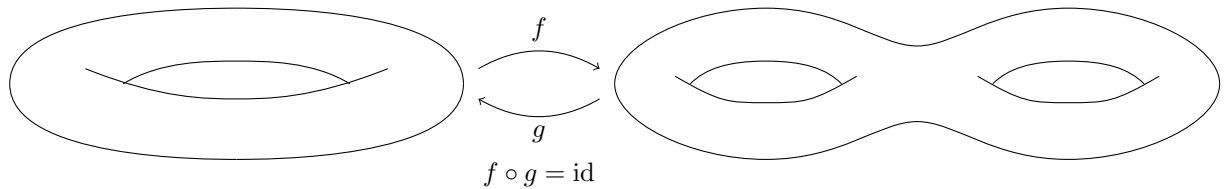


# Algebraic Topology

October 14, 2019

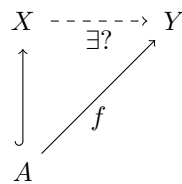
## 0 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions  $f, g$  going from a topological space  $X$  to another space  $Y$  and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



If such  $f, g$  continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space  $X$  a group  $G(X)$ , and for every continuous function  $f : X \rightarrow Y$  a group homomorphism  $G(f) : G(X) \rightarrow G(Y)$  with  $G(\text{id}) = \text{id}$  and  $G(f \circ g) = G(f) \circ G(g)$ . Thus if  $f : X \rightarrow Y$  is a homeomorphism with inverse  $g : Y \rightarrow X$ , then  $G(g) \circ G(f) = \text{id}, G(f) \circ G(g) = \text{id}$ , so  $G(f)$  is an isomorphism.

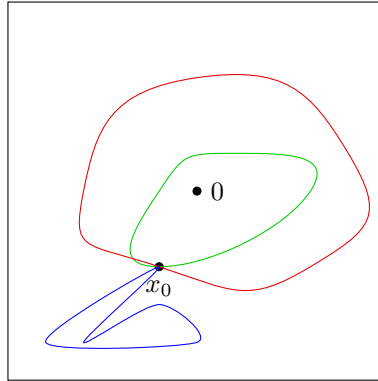
Extension problem: Let  $X$  be a topological space,  $A \subseteq X$  a subspace, and  $f : A \rightarrow Y$  a continuous function. Does there exist a continuous function  $F : X \rightarrow Y$  with  $F|_A = f$



**Theorem 0.1.** *There is no continuous function*

$$f : D^n \rightarrow S^{n-1} \text{ with } f|_{S^{n-1}} = \text{id}$$

By hand, we can see why this fails for e.g.  $n = 1, 2$ , but it gets hard to generalise. Eventually, we will construct  $G$  with  $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$ . Then, if we have  $S^{n-1} \rightarrow D^n \rightarrow S^{n-1}$  with composition being the identity, then we have maps  $\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}$  being the identity.



Here, the green and red loops are the “same” loop, whilst the blue one is distinct

## Conventions

A topological space will be referred to as a *space*

A continuous function  $f : X \rightarrow Y$  will be called a *map*

## 1 The Fundamental Group

The idea here is that, if  $X$  is a space,  $x_0 \in X$  a fixed point, called the *basepoint*, we consider loops based at  $x_0$ , i.e. maps  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = \gamma(1) = x_0$ .

For example, if we let our space  $X = \mathbb{R}^2 \setminus \{0\}$

Then the *fundamental group*  $\pi_1(X) = \pi_1(X, x_0)$  is defined to be the set of loops based at  $x_0$  modulo “deforming loops”. Multiplication in this group  $\gamma_1 \cdot \gamma_2$  is given by first traversing  $\gamma_1$  and then  $\gamma_2$ . But what do we mean by “deforming” a loop?

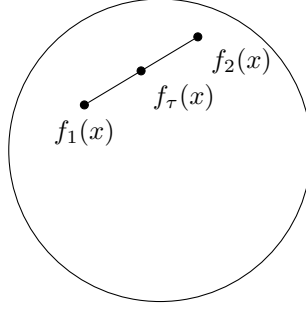
Let  $f_0, f_1 : X \rightarrow Y$  be maps. A *homotopy* between  $f_0$  and  $f_1$  is a map

$$F : X \times I \rightarrow Y \text{ where } I = [0, 1] \text{ and} \\ F(x, 0) = f_0(x) \text{ and} \\ F(x, 1) = f_1(x)$$

We often write  $f_\tau(x) = F(x, \tau)$ ,  $f_\tau : X \rightarrow Y$ .

If such  $F$  exists, we say  $f_0$  and  $f_1$  are *homotopic*.

Example: Let  $Y \subseteq \mathbb{R}^2$  be a convex set. Then any  $f_0, f_1 : X \rightarrow Y$  are homotopic, via  $F(x, t) = tf_1(x) + (1 - t)f_0(x) \in Y$  by convexity.



If  $f_0$  is homotopic to  $f_1$ , we write  $f_0 \simeq f_1$ , or  $f_0 \simeq_F f_1$  if we want to be explicit about the homotopy we are using.

Suppose  $f_0 \simeq_F f_1$ , both functions  $X \rightarrow Y$ . If  $Z \subseteq X$  and  $f_0(z) = F(z, t) = f_1(z) \forall z \in Z, t \in I$ , then we say  $f_0$  is homotopic to  $f_1$  **relative to**  $Z$ .

**Lemma 1.1.** *Let  $Z \subseteq X, Y$  be spaces. Then  $\simeq$  relative to  $Z$  is an equivalence relation on the set of maps  $X \rightarrow Y$ .*

*Proof.*

- Reflexive:  $f_0 \simeq f_0$  via  $F(x, t) = f_0(x) \forall x, t$
- Symmetric: Given  $f_0 \simeq_F f_1$ , then  $f_1 \simeq f_0$  via  $F'(x, t) = f(x, 1 - t)$
- Transitive: If  $f_0 \simeq_{F_0} f_1, f_1 \simeq_{F_1} f_2$ , then  $f_0 \simeq_F f_2$  with:

$$F(x, t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t - 1) & t \geq 1/2 \end{cases}$$

All homotopies are relative to  $Z$ .

□

A **homotopy equivalence**  $f : X \rightarrow Y$  is a map with a **homotopy inverse**  $g : Y \rightarrow X$  such that  $f \circ g = \text{id}_Y, g \circ f = \text{id}_X$ . We then write  $X \simeq Y$ .

Remark: Most (all?) invariants in the course are **homotopy invariants**

Examples:

1. Let  $*$  be the one point space,  $f : \mathbb{R}^n \rightarrow *$  be the constant map, and let  $g : * \rightarrow \mathbb{R}^n; x \mapsto \mathbf{0}$ . Then  $f \circ g = \text{id}_*$ , and  $g \circ f(x) = \mathbf{0} \forall x \in \mathbb{R}^n$ . Now  $g \circ f \simeq \text{id}_{\mathbb{R}^n}$  via  $F(x, t) = tx$ .
2. Let  $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$  be the inclusion map, and  $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}; x \mapsto \frac{x}{|x|}$  (i.e. map  $x$  to the intersection of  $\overrightarrow{\mathbf{0}x}$  with  $S^{n-1}$ ). Then  $g \circ f = \text{id}_{S^{n-1}}$  and  $f \circ g \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$  via  $F(x, t) = (1 - t)x + t \cdot \frac{x}{|x|}$

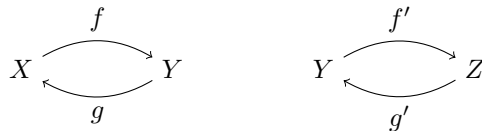
If  $X \simeq *$ , then we say  $X$  is **contractible**.

Let  $f : X \rightarrow Y, g : Y \rightarrow X$  be maps. If  $g \circ f = \text{id}_X$ , then we say  $X$  is a **retract** of  $Y$ , and  $g$  is a **retraction**. If in addition  $f \circ g \simeq \text{id}_Y$  relative to  $f(X)$ , then we say  $X$  is a **deformation retract** of  $Y$ . Hence, in example 2, we see that  $S^{n-1}$  is a deformation retract of  $\mathbb{R}^n$ .

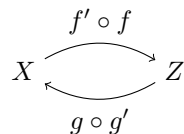
**Lemma 1.2.** *Homotopy equivalence of spaces is an equivalence relation.*

*Proof.* Reflexivity and symmetry are trivial from the definition.

Suppose  $X \simeq Y, Y \simeq Z$  via:



We want to show  $f' \circ f, g \circ g'$  induces a homotopy equivalence



Now  $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$ . We know already that  $g' \circ f' \simeq_{F'} \text{id}_Y$ , and so:

$$(x, t) \mapsto g(F'(f(x), t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as  $g \circ (g' \circ f') \circ f \simeq g \circ f$ , and since  $X \simeq Y, g \circ f \simeq \text{id}_X$ . Hence  $(g \circ g') \circ (f' \circ f) \simeq \text{id}_X$  via transitivity of homotopy equivalence for maps. Similarly  $(f' \circ f) \circ (g \circ g') \simeq \text{id}_Z$   $\square$

## Loops and $\pi_1$

If  $X$  is a space, a **path** in  $X$  is a map  $\gamma : I \rightarrow X$ , where  $I = [0, 1] \subseteq \mathbb{R}$ . If  $\gamma(0) = x_0, \gamma(1) = x_1$  then we say  $\gamma$  is a path **from**  $x_0$  **to**  $x_1$ .

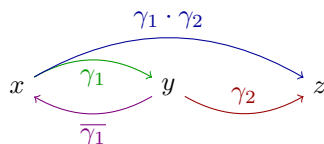
We say  $\gamma_1$  and  $\gamma_2$  are **homotopic** if  $\gamma_1 \simeq \gamma_2$  relative to  $\{0, 1\}$ , and we write  $[\gamma]$  for the homotopy equivalence class of  $\gamma$ .

If  $X$  is a space with points  $x, y, z \in X$ , and  $\gamma_1$  is a path from  $x$  to  $y$ ,  $\gamma_2$  is a path from  $y$  to  $z$ , then:

- The **concatenation** of  $\gamma_1$  and  $\gamma_2$  is the path from  $x$  to  $z$  given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s - 1) & 1/2 \leq s \leq 1 \end{cases}$$

- The **constant path** at  $x$  is the path  $c_x(s) = x \forall s \in I$
- The **inverse of**  $\gamma_1$  is  $\overline{\gamma_1}(s) = \gamma_1(1 - s)$ , a path from  $y$  to  $x$ .

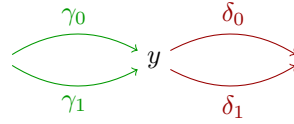


**Theorem 1.3.** Let  $X$  be space, and  $x_0 \in X$ . Let  $\pi_1(X, x_0)$  be the set of homotopy classes of loops in  $X$  with endpoint  $x_0$  (we say they are **based** at  $x_0$ ). Then  $\pi_1(X, x_0)$  forms a group under the product  $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$ , with identity  $c_{x_0}$  and inverses  $[\gamma_1]^{-1} = [\overline{\gamma_1}]$ .

This group is called the **fundamental group** of  $X$  (based at  $x_0$ ).

To prove this, we will need the following lemmas:

**Lemma 1.4.** If  $\gamma_0 \simeq \gamma_1$  to  $y$  and  $\delta_0 \simeq \delta_1$  from  $y$ , then  $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$  and  $\overline{\gamma_0} \simeq \overline{\gamma_1}$



*Proof.* Suppose  $\gamma_0 \simeq_F \gamma_1$ , and  $\delta_0 \simeq_G \delta_1$ . Set:

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

Then  $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let  $F'(s, t) = F(1 - s, t)$ . Then  $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$ . □

**Lemma 1.5.** Let  $\alpha, \beta, \gamma$  be paths from  $w$  to  $x$  to  $y$  to  $z$  in  $X$ .



Then:

1.  $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \beta \cdot \gamma$
2.  $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3.  $\alpha \cdot \overline{\alpha} \simeq c_w$