Algebraic Geometry

January 20, 2020

0 Introduction

What is algebraic geometry? Broadly speaking, it is the study of the geometry of solutions to systems of polynomial equations. For example, in $\mathbb{R}^2$, if we have the set X of solutions to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then we know that this set forms a circle, and we know lots of geometric facts about circles. If we take a more complicated function, such as $y^2 = x^3 - x$, we get something that looks like:

If we instead think about complex solutions, we get something of the form of a torus minus a single point, with another rich geometric structure.

In $\mathbb{C}^3$, if $X = \{(x, y, z) \in \mathbb{C}^3 : x^3 + y^3 + z^3 = 1\}$, then X contains 27 lines: $x = -\xi^m y, z = \xi^n$ for $i, j \in \{0, 1, 2\}$ gives 9 of them, and the other 18 come by rotating x, y, z in this linear system.

In $\mathbb{R}^3$, consider the equation $1 + x^3 + y^3 + z^3 = (1 + x + y + z)^3$.

1 Basic Setup

Fix a field K. We define an ***affine n-space over K*** to be $\mathbb{A}^n := K^n$. Let $A := K[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables over K, and let $S \subseteq A$ be a subset of A. We then define $Z(S)$, the ***zero set of S*** to be the set of all n-tuples $(a_1, \dots, a_n) \in \mathbb{A}^n$ where $f(a_1, \dots, a_n) = 0$ for all $f \in S$.

Proposition 1.1.

1. $Z(\{0\}) = \mathbb{A}^n$
2. $Z(A) = \emptyset$
3. $Z(S_1 \cdot S_2) = Z(S_1) \cup Z(S_2)$, *where* $S_1 \cdot S_2 = \{f_1 \cdot f_2 : f_1 \in S_1, f_2 \in S_2\}$.
4. *Let* I *be an index set,* $S_i \subseteq A$ *for each* $i \in I$*. Then* $\bigcap_{i \in I} Z(S_i) = Z(\bigcup_{i \in I} S_i)$

Proof. 1., 2. are obvious

1. If $p \in Z(S_1) \cup Z(S_2)$, then either $p \in Z(S_1)$ or $p \in Z(S_2)$. If $p \in Z(S_1)$, then $f_1(p) = 0$ for all $f_1 \in S_1$, and so $f_1(p) \cdot f_2(p) = 0$ for all $f_1 \in S_1, f_2 \in S_2$, so $p \in Z(S_1 \cdot S_2)$, and similarly for if $p \in Z(S_2)$.

Conversely, suppose that $p \in Z(S_1 \cdot S_2)$, and $p \notin Z(S_1)$. Then there is some $f_1 \in S_1$ with $f_1(p) \neq 0$. But $f_1(p) \cdot f_2(p) = 0$ for all $f_2 \in S_2$, and so $f_2(p) = 0$ for all $f_2 \in S_2$), so $p \in Z(S_2)$.

2. If $p \in Z(S_i)$ for all $i \in I$, then $f_i(p) = 0$ for all $f_i \in S_i$, and so for all $f \in \bigcup_i S_i$, so $p \in Z(\bigcup_{i \in I} S_i)$.

   Conversely, if $p \in Z(\bigcup_i S_i)$, then $f(p) = 0$ for all the polynomials in $\bigcup_i S_i$, and so $p \in \bigcap_i S_i$.

□

These four properties should remind you of the four axioms for a topology.

A subset of $\mathbb{A}^n$ is ***algebraic*** if it of the form $Z(S)$ for some $S \subseteq A$. A ***Zariski open set*** in $\mathbb{A}^n$ is a set of the form $\mathbb{A}^n \setminus Z(S)$ for some $S \subseteq \mathbb{A}^n$. This proposition tells us that the Zariski open sets define a topology on $\mathbb{A}^n$, called the ***Zariski topology***.

Examples:

1. $K = \mathbb{C}$. The Zariski open (or closed) subsets of $\mathbb{C}^n = \mathbb{A}^n$ are in particular open (or closed) in the usual Euclidean sense, but not vice versa.
2. For any K, consider $\mathbb{A}^1, A = K[x], S \subseteq K[x]$. If S has a non-zero element, then $Z(S)$ is finite. Thus the closed sets are the finite subsets of $\mathbb{A}^1$, and all of $\mathbb{A}^1$. The open sets are $\emptyset$ and all the co-finite sets (i.e. sets with finite complement).

Recall that, if A is any commutative ring with $S \subseteq A$ a subset, then the ***ideal generated by S*** is the ideal $A \supseteq \langle S \rangle = \{\sum_{i=1}^q f_i g_i : q \geq 0, f_i \in S, g_i \in A\}$, or the smallest ideal of A containing S.

Lemma 1.2. *Let $S \subseteq A = K[x_1, \dots, x_n]$. Then $Z(S) = Z(\langle S \rangle)$.*

Proof. If $p \in Z(S)$, then for $f_1, \dots, f_q \in S; g_1, \dots, g_q \in A$ we have:

$$\left(\sum_{i=1}^q f_i g_i\right)(p) = \sum_{i=1}^q f_i(p) g_i(p) = \sum_{i=1}^p 0 \cdot g_i(p) = 0$$

So $p \in Z(\langle S \rangle)$, and so $Z(S) \subseteq Z(\langle S \rangle)$.

The other inclusion follows from the fact that $S \subseteq \langle S \rangle$, we must have $Z(\langle S \rangle) \subseteq Z(S)$. □

Let $X \subseteq \mathbb{A}^n$ be a subset. Define $I(X) := \{f \in A : f(p) = 0 \ \forall p \in X\}$, the ***ideal of X***. Note that $I(X)$ is indeed an ideal, since if $f, g \in I(X)$ then $f + g \in I(X)$, and if $f \in I(X), g \in A$, then $f \cdot g \in I(X)$.

Note that if $S_1 \subseteq S_2 \subseteq A_1$, then $Z(S_2) \subseteq Z(S_1)$, and if $X_1 \subseteq X_2$, then $I(X_2) \subseteq I(X_1)$.