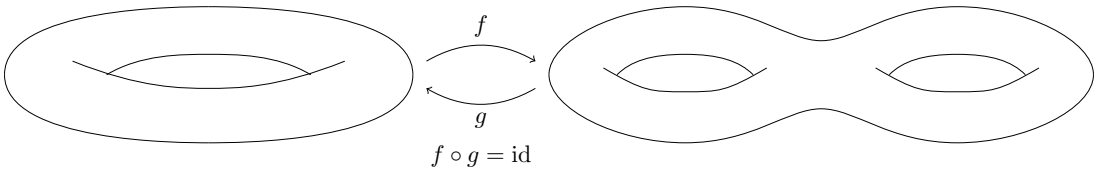
# Algebraic Topology

October 25, 2019

## 1 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions $f, g$ going from a topological space $X$ to another space $Y$ and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.




If such $f, g$ continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space $X$ a group $G(X)$, and for every continuous function $f : X \to Y$ a group homomorphism $G(f) : G(X) \to G(Y)$ with $G(\mathrm{id}) = \mathrm{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \to Y$ is a homeomorphism with inverse $g : Y \to X$, then $G(g) \circ G(f) = \mathrm{id}, G(f) \circ G(g) = \mathrm{id}$, so $G(f)$ is an is an isomorphism.

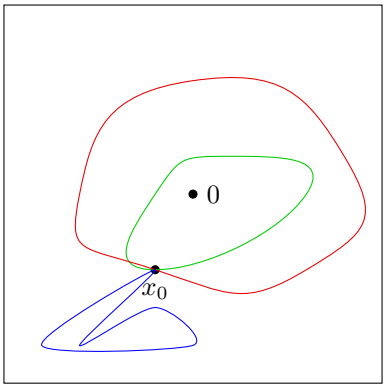
<u>Extension problem:</u> Let $X$ be a topological space, $A \subseteq X$ a subspace, and $f : A \to Y$ a continuous function. Does there exist a continuous function $F : X \to Y$ with $F|_A = f$

$$\begin{tikzcd} X \arrow[r, dashed, "\exists\,?"'] & Y \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

**Theorem 1.1.** *There is no continuous function*

$$f : D^n \to S^{n-1} \text{ with } f|_{S^{n-1}} = \mathrm{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct $G$ with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \to D^n \to S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \to 0 \to \mathbb{Z}$ being the identity.




Here, the green and red loops are the "same" loop, whilst the blue one is distinct

### Conventions

A topological space will be referred to as a ***space***
A continuous function $f : X \to Y$ will be called a ***map***

## 2 The Fundamental Group

The idea here is that, if $X$ is a space, $x_0 \in X$ a fixed point, called the ***basepoint***, we consider loops based at $x_0$, i.e. maps $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the ***fundamental group*** $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at $x_0$ modulo "deforming loops". Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing $\gamma_1$ and then $\gamma_2$. But what do we mean by "deforming" a loop?

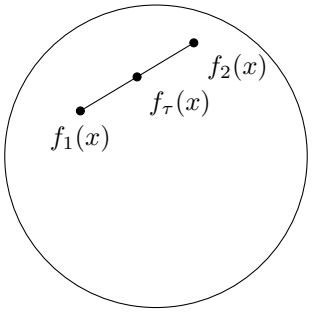
Let $f_0, f_1 : X \to Y$ be maps. A ***homotopy*** between $f_0$ and $f_1$ is a map

$$\begin{aligned} F : X \times I \to Y \text{ where } I = [0,1] \text{ and} \\ F(x,0) = f_0(x) \text{ and} \\ F(x,1) = f_1(x) \end{aligned}$$

We often write $f_\tau(x) = F(x,\tau), f_\tau : X \to Y$.

If such $F$ exists, we say $f_0$ and $f_1$ are ***homotopic***.

<u>Example:</u> Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \to Y$ are homotopic, via $F(x,t) = tf_1(x) + (1-t)f_0(x) \in Y$ by convexity.




If $f_0$ is homotopic to $f_1$, we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \to Y$. If $Z \subseteq X$ and $f_0(z) = F(z,t) = f_1(z) \forall z \in Z, t \in I$, then we say $f_0$ is homotopic to $f_1$ ***relative to*** $Z$.

**Lemma 2.1.** *Let $Z \subseteq X, Y$ be spaces. Then $\simeq$ relative to $Z$ is an equivalence relation on the set of maps $X \to Y$.*

*Proof.*

- Reflexive: $f_0 \simeq f_0$ via $F(x,t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x,t) = f(x, 1-t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x,t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t-1) & t \geq 1/2 \end{cases}$$

  All homotopies are relative to $Z$.

□

A ***homotopy equivalence*** $f : X \to Y$ is a map with a ***homotopy inverse*** $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$. We then write $X \simeq Y$.

<u>Remark:</u> Most (all?) invariants in the course are ***homotopy invariants***

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \to *$ be the constant map, and let $g : * \to \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \mathrm{id}_*$, and $g \circ f(x) = 0 \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(x,t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map $x$ to the intersection of $\overrightarrow{\mathbf{0}x}$ with $S^{n-1}$) . Then $g \circ f = \mathrm{id}_{S^{n-1}}$ and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x,t) = (1-t)x + t \cdot \frac{x}{|x|}$

If $X \simeq *$, then we say $X$ is ***contractible***.
Let $f : X \to Y, g : Y \to X$ be maps. If $g \circ f = \mathrm{id}_X$, then we say $X$ is a ***retract*** of $Y$, and $g$ is a ***retraction***. If in addition $f \circ g \simeq \mathrm{id}_Y$ relative to $f(X)$, then we say $X$ is a ***deformation retract*** of $Y$. Hence, in example 2, we see that $S^{n-1}$ is a deformation retract of $\mathbb{R}^n$.

**Lemma 2.2.** *Homotopy equivalence of spaces is an equivalence relation.*

*Proof.* Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

We want to show $f' \circ f$, $g \circ g'$ induces a homotopy equivalence

$$\begin{tikzcd} X \arrow[r, bend left, "f' \circ f"] & Z \arrow[l, bend left, "g \circ g'"] \end{tikzcd}$$

Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \mathrm{id}_Y$, and so:

$$(x,t) \mapsto g(F'(f(x),t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y$, $g \circ f \simeq \mathrm{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \mathrm{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \mathrm{id}_Z$ $\square$

## Loops and $\pi_1$

If $X$ is a space, a ***path*** in $X$ is a map $\gamma : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say $\gamma$ is a path ***from*** $x_0$ ***to*** $x_1$.
We say $\gamma_1$ and $\gamma_2$ are ***homotopic*** if $\gamma_1 \simeq \gamma_2$ relative to $\{0,1\}$, and we write $[\gamma]$ for the homotopy equivalence class of $\gamma$.

If $X$ is a space with points $x, y, z \in X$, and $\gamma_1$ is a path from $x$ to $y$, $\gamma_2$ is a path from $y$ to $z$, then:

- The ***concatenation*** of $\gamma_1$ and $\gamma_2$ is the path from $x$ to $z$ given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

- The ***constant path*** at $x$ is the path $c_x(s) = x \forall s \in I$
- The ***inverse of*** $\gamma_1$ is $\overline{\gamma_1}(s) = \gamma_1(1-s)$, a path from $y$ to $x$.

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend left, "\gamma_1 \cdot \gamma_2"] & y \arrow[l, bend left, "\overline{\gamma_1}"] \arrow[r, bend right, "\gamma_2"'] & z \end{tikzcd}$$

**Theorem 2.3.** *Let $X$ be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in $X$ with endpoint $x_0$ (we say they are* **based** *at $x_0$). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity $c_{x_0}$ and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.*

*This group is called the* **fundamental group** *of $X$ (based at $x_0$).*

To prove this, we will need the following lemmas:

**Lemma 2.4.** *If $\gamma_0 \simeq \gamma_1$ to $y$ and $\delta_0 \simeq \delta_1$ from $y$, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$*

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\gamma_0"] \arrow[r, bend right, "\gamma_1"'] & y \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] & \bullet \end{tikzcd}$$

*Proof.* Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq 1/2 \\ G(2s-1,t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s,t) = F(1-s,t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. □

**Lemma 2.5.** *Let $\alpha, \beta, \gamma$ be paths from $w$ to $x$ to $y$ to $z$ in $X$.*

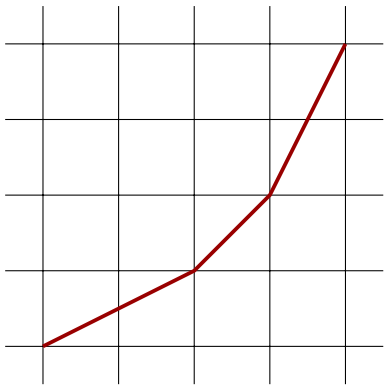
$$\begin{tikzcd} w \arrow[r, bend left, "\alpha"] & x \arrow[r, bend right, "\beta"'] & y \arrow[r, bend left, "\gamma"] & z \end{tikzcd}$$

*Then:*

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha\beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

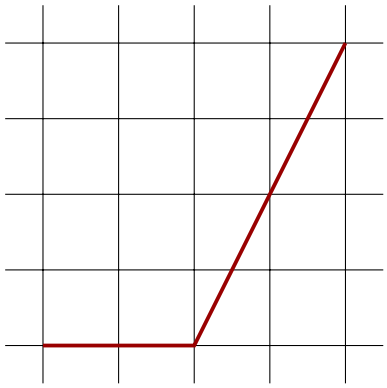
*Proof.* First, given a path $\delta : I \to X$, a ***reparametrization*** of $\delta$ is a path $\delta \circ \phi$ where $\phi : I \to I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that $\phi$ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s,t) = \delta(t\phi(s) + (1-t)s)$, and this homotopy is relative to $\{0,1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function $\phi$ whose plot is:

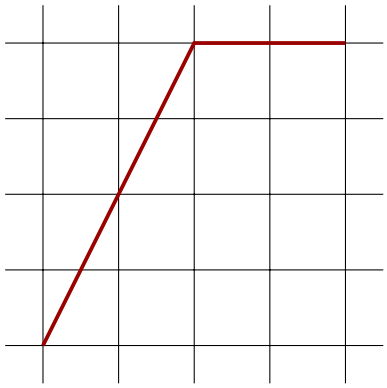


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize $\alpha$ via:



i.e. do $c_w$ for the first half of the time, then do $\alpha$, so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s,t) = \begin{cases} \alpha(2s) & 0 \leq s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \alpha(2-2s) & 1 - t/2 \leq s \leq 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have $c_w$ at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

$\square$

Then theorem **1.3** giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If $\gamma$ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

## Formal Properties of $\pi_1$

**Lemma 2.6.** *Let $f : X \to Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

*Furthermore:*

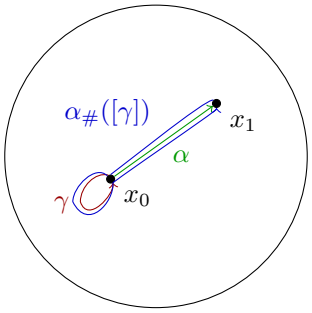
1. *If $f \simeq f'$ relative to $x_0$, then $f'_* = f_*$.*
2. *If $g : Y \to Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$*
3. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$

*Proof.* $f_*$ is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to $x_0$, then for $\gamma$ a loop based at $x_0$, $(s,t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

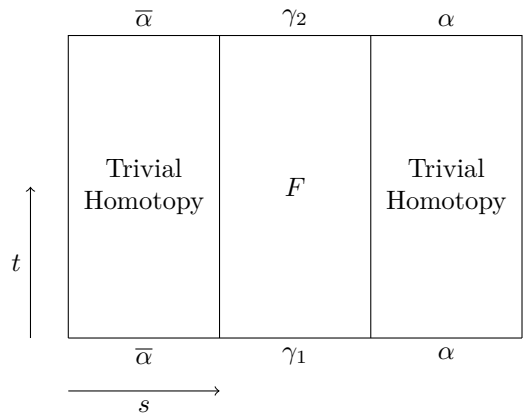
**Lemma 2.7.** *let $X$ be a space, $x_0, x_1 \in X$ and $\alpha$ a path from $x_0$ to $x_1$. Then there is a group isomorphism $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.*




*Furthermore,*

1. *If $\alpha \simeq \alpha'$ relative to $\{0, 1\}$, then $\alpha_\# = \alpha'_\#$.*
2. $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$
3. *If $\beta$ is a path from $x_2$ to $x_2$, then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$*
4. *If $f : X \to Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.*

*Proof.* Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

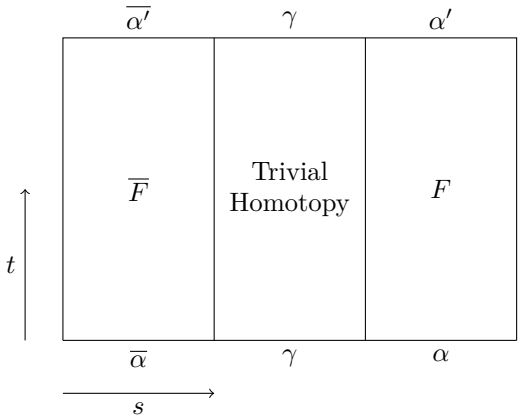



This is indeed a group homomorphism: for loops $\gamma, \delta$ based at $x_0$,

$$\begin{aligned}
\bar{\alpha} \cdot \gamma \cdot \alpha) \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\
&\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha
\end{aligned}$$

Thus $\alpha_\#(\gamma \cdot \delta) = \alpha_\#(\gamma) \cdot \alpha_\#(\delta)$. Also $\bar{\alpha_\#} = (\alpha_\#)^{-1}$ - this is easy to check. Thus $\alpha_\#$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$




   gives $\alpha_\#(\gamma) \simeq \alpha'_\#(\gamma)$

2. Immediate since $c_{x_0}$ is the identity in $\pi_1(X, x_0)$.

3. 

$$\begin{aligned}
(\alpha \cdot \beta)_\#(\gamma) &= \alpha \bar{\cdot} \beta \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot \alpha_\#(\gamma) \cdot \beta \\
&= \beta_\#(\alpha_\#(\gamma))
\end{aligned}$$

4.

$$\begin{aligned}((f\circ\alpha)_\# \cdot f_*)(\gamma) &= (f\circ\alpha)_\#(f\cdot\gamma)\\ &= (f\circ\alpha)_\#(f\cdot\gamma)\\ &= \overline{f\cdot\alpha}\cdot(f\circ\gamma)\cdot(f\circ\alpha)\\ &= f\circ(\bar{\alpha}\cdot\gamma\cdot\alpha)\\ &= f_*(\alpha_\#(\gamma))\end{aligned}$$

□

A path connected space $X$ is ***simply connected*** if $\pi_1(x, x_0) = 0$ for any, and hence all, $x_0 \in X$.

Our aim here is to prove that $\pi_1$ is a ***homotopy invariant***, i.e. that homotopy equivalent spaces have the same fundamental group. We will start with the following lemma:

**Lemma 2.8.** *Let $x_0 \in X$ and $f, g : X \to Y$ with $f \simeq_F g$. Set $x(t) = F(x_0, t)$ so that $\alpha(0) = f(x_0)$ and $\alpha(1) = g(x_0$. Then the diagram:*

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[dd, "\alpha_\#"] \\ \pi_1(x, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_1(Y, g(x_0)) \end{tikzcd}$$

*commutes, i.e. we have $\alpha_\# \circ f_* = g_*$.*

*Proof.* We need to check that, for a loop $\gamma$ based at $x_0$, $\overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha \simeq g\circ\gamma$.

Let $G : I \times I \to Y$ defined by $G(s, t) = F(\gamma(s), t)$. For $t = 0$, this is $f \circ \gamma$, and for $t = 1$, this is $g \circ \gamma$. Now consider two paths in $I \times I$:

$$\begin{gathered} a(t) = (t, 1); b = b_1 \cdot b_2 \cdot b_3 \text{ where:} \\ b_1(t) = (0, 1-t), b_2(t) = (t, 0), b_3(t) = (1, t) \end{gathered}$$

Then $(G \circ a)(s) = G(s, 1) = g \circ \gamma(s)$, whilst $G \circ b = \overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha$.

Now, since $I \times I$ is convex, we have that $a \simeq_H b$, and so $G \circ H$ is the desired homotopy between $g \circ \gamma$ and $\overline{\alpha}\cdot(f\circ\gamma)\cdot\alpha$. □

**Theorem 2.9.** *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a homomorphism for any $x_0 \in X$.*

*Proof.* We'll show that $f_*$ is a bijection:

Let $g : Y \to X$ be a homotopic inverse to $f$, with $\mathrm{id}_X \simeq_F g \circ f$. Let $\alpha : I \to X$ given by $\alpha(t) = F(x_0, t)$.

Note that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)); g : \pi_1(Y, f(x_0)) \to \pi_1(X, g(f(x_0)))$

Then $g_* \circ f_* = (g \circ f)_* = \alpha_\# \circ (\mathrm{id}_X)_* = \alpha_\#$. $\alpha_\#$ is an isomorphism, and so $f_*$ is injective.

If $\mathrm{id}_Y \simeq_G f \circ g$ let $\beta(t) = G(f(x_0), t)$ Then $f_* \circ g_* = (g \circ f)_* = \beta_\# \circ (\mathrm{id}_Y)_* = \beta_\#$, an isomorphism, and hence $f_*$ is surjective. $\square$

**Corollary 2.10.** *Contractible spaces are simple connected.*

*Proof.* If $X$ is contractible, there exists some $x_0 \in X$ and a homotopy $F$ between $\mathrm{id}_X$ and $X \to \{x_0\}$. So $F(x, \cdot)$ is a path from any $x \in X$ to $x_0$, so $X$ is path connected. Since $X$ is homotopic to $\{x_0\}$, $\pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0) = 0$. $\square$

## Covering Spaces

Let $p : \widehat{X} \to X$ be a map. An open set $U \subseteq X$ is ***evenly covered*** if there exists a set $\Delta_U$ with the discrete topology and there is a homeomorphism:

$$p^{-1}(U) \xrightarrow{\cong} U \times \Delta_U$$

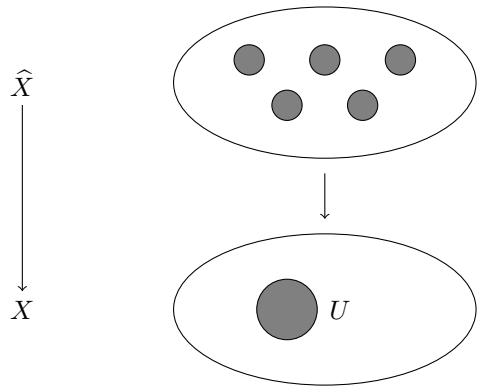
such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Delta_U \arrow[dl, "(x,\delta) \mapsto x"] \\ & U & \end{tikzcd}$$

We write, for $\delta \in \Delta_0$, $U_\delta = U \times \{\delta\}$ and $p_\delta = p|_{U_\delta}$. So $p_\delta : U_\delta \to U$ is a homeomorphism.

Note that we can canonically identify $\Delta_U$ with $p^{-1}(x)$ for any $x \in U$, Note also that $p^{-1}(U) \cong \coprod_{\delta \in \Delta_U} U_\delta$, where $\coprod$ denotes disjoint union.

If every point of $X$ has an open neighbourhood which is evenly covered, then we say that $p$ is a ***covering map*** and $\widehat{X}$ is a ***covering space*** of $X$.




<u>Examples:</u>

1. $\widehat{X} = X \times \Delta$ for $\Delta$ a set with the discrete topology, e.g. $\widehat{I} = I \times \{1, 2, 3\}$. Then $\widehat{X}$ is a covering space of $X$, the identity map on the first element is a covering map.
2. $\widehat{X} = \mathbb{R}, X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p : \mathbb{R} \to S^1$ and $p(t) = \exp(2\pi \mathrm{i} \cdot t)$. Them $p$ is a covering map:

let $U = S \setminus \{p\}$. We can define a branch of the logarithm $\log : \mathbb{C} \setminus \{rp : r \geq 0\} \to \mathbb{C}$. Then every point $\widehat{z} \in p^{-1}(U)$ can be written uniquely as $\widehat{z} = k + \frac{\log z}{2\pi \mathrm{i}}$ for some $k \in \mathbb{Z}$.

Thus $p^{-1}(U) \cong U \times \mathbb{Z}$, via $\widehat{z} \mapsto \left(\frac{\log z}{2\pi \mathrm{i}}, k\right)$, and so each proper subset of $S^1$ is evenly covered, however $S^1$ as a whole is not evenly covered, since $p^{-1}(S^1)$ is not a union of copies of $S^1$.

3. $\widehat{X} = X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p(z) = z^n$.

   $p$ is a covering map by choosing a branch of the $n$th root on proper open subsets of $S^1$

4. Let $\widehat{X} = S^2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $S^2$ by the antipodal map $z \mapsto -z$. Then let $X = \widehat{X}/G = \widehat{X}/\sim$, where $x \sim y \iff x = \pm y$.

   Then $X$ is $\mathbb{RP}^2$, the real projective plane. If $x \in X$, let $U$ be an open neighbourhood of $x$ disjoint from its negation. Then the image of $U$ in $X$ is evenly covered.

We say a covering map $p : \widehat{X} \to X$ is ***n-sheeted*** if $\#p^{-1}(x) = n$ for all $x \in X$, and call $n$ the ***degree*** of $p$.

## Lifting Properties

Let $p : \widehat{X} \to X$ be a covering map, and $f : Y \to X$ b a mp. A ***lift*** of $f$ to $\widehat{X}$ is a map $\widehat{f} : Y \to \widehat{X}$ such that the following diagram commutes:

$$\begin{tikzcd} & \widehat{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widehat{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

A space $X$ is ***locally path connected*** if for every $x \in X$ and $U \subseteq X$ open neighbourhood of $x$, there exists a neighbourhood $V \subseteq U$ of $x$ which is path connected.

**Lemma 2.11** (Uniqueness of Lifting)**.** *Let $p : \widehat{X} \to X$ be a covering map and $\widehat{f}_1, \widehat{f}_2 : Y \to \widehat{X}$ be two lifts of $f : Y \to X$ with $Y$ connected and locally path connected.*

*If there exists some $x_0 \in Y$ with $\widehat{f}_1(x_0) = \widehat{f}_2(x_0)$, then $\widehat{f}_1 = \widehat{f}_2$.*

*Proof.* We will show that the set $S := \{y \in Y : \widehat{f}_1(y) = \widehat{f}_2(y)\}$ is both open and closed. By assumption we have $x_0 \in S$, so $S \neq \emptyset$. Since $Y$ is connected, we must have then that $S = Y$ as otherwise $S$ and $Y \setminus S$ would disconnect $Y$.

Let $y_1 \in Y$ be an arbitrary point, and let $U \subseteq X$ be an open neighbourhood of $f(y_1)$ which is evenly covered by $p$. Let $V \subseteq f^{-1}(U)$ be an open neighbourhood of $y_1$ which is path connected. We then want to show that, if $y_1 \in S$ then all of $V \subseteq S$, and otherwise $V \subseteq Y \setminus S$.

Let $y \in V$ be arbitrary and let $\alpha$ be a path from $y_1 \to y$. Then $\widehat{f}_i \circ \alpha$ is a path from $\widehat{f}_i(y_1) \to \widehat{f}_i(y)$ for $i = 1, 2$.

Note that $p \circ \widehat{f}_1 \circ \alpha(t) = f(\alpha(t)) \in U$, and so $\widehat{f}_i(y)$ and $\widehat{f}_i(y_1)$ lie in the same component of $p^{-1}(U)$, say $U_{\delta_i}$

If $y_1 \in S$, then $\widehat{f}_1(y_1) = \widehat{f}_2(y_1)$, so $\delta_1 = \delta_2$, and so $\widehat{f}_1(y) = p_{\delta_1}^{-1}(f(y)) = p_{\delta_2}^{-1}(f(y)) = \widehat{f}_2(y)$, so $y \in S$, and hence all of $V \subseteq S$.

Otherwise $y_1 \notin S$ , then $\widehat{f}_1(y_1) \neq \widehat{f}_2(y_1)$. Each $U_{\delta_i}$ contains a unique point of $p^{-1}(\{f(y_1)\})$, and we must have $\delta_1 \neq \delta_2$.
So $\widehat{f}_1(y) \neq \widehat{f}_2(y)$, so $y \notin S$, and in general $V \subseteq Y \setminus S$.

Hence $S$ is open, $Y \setminus S$ is open, and we are done $\square$

Let $\gamma : I \to X$ be a path from $x_0 \in X$ and $p : \widehat{X} \to X$ be a covering map. A lift of $\gamma$ at (or from) $\widehat{x_0}$ is a lift $\widehat{\gamma}$ of $\gamma$ with $\widehat{x_0} = \widehat{\gamma}(0)$. In particular, $p(\widehat{x_0}) = p(\widehat{\gamma}(0)) = \gamma(0) = x_0$.

**Lemma 2.12** (Path Lifting Lemma)**.** *Let $p : \widehat{X} \to X$ be a covering map, and let $\gamma : I \to X$ be a path from $x_0$. Then for any choice of $\widehat{x_0} \in p^{-1}(x_0)$, there exists a unique lift $\widehat{\gamma}$ of $\gamma$ from $\widehat{x_0}$.*

*Proof.* Uniqueness follows from the previous lemma showing uniqueness of lifts. For existence, let $S = \{t \in I | \gamma|_{[0,t]}$ lifts to path from $\widehat{x_0}$ in $\widehat{X}\}$. Note $o \in S$. If we show that $S$ is open and closed, then since $I$ is connected, $S = I$. Note that if $t \in S$, then $[0, t] \subseteq S$.

Let $t_0 \in I$, and let $U$ be an evenly covered neighbourhood of $\gamma(t_0)$. Let $V \subseteq \gamma^{-1}(U)$ be an open interval containing $t_0$. Let $t \in V$ and suppose first that $t_0 \in S$. If $t \leq t_0$, then $t \in S$, so instead assume that $t > t_0$. Since $\gamma|_{[0,t_0]}$ has a lift $\widehat{\gamma} : [0, t_0] \to \widehat{X}$, and we have $\widehat{\gamma}(t_0) \in U_\delta$ for some $\delta \in \Delta_U$.

Recall that we have a homeomorphism $p_\delta : U_\delta \to U$ where $p_\delta = p|_{U_\delta}$. Hence the path:

$$s \mapsto \begin{cases} \widehat{\gamma}(s) & 0 \leq s \leq t_0 \\ p_\delta^{-1} \circ \gamma & t_0 \leq s \leq t \end{cases}$$

is a lift of $\gamma|_{[0,t]}$. Hence $t \in S$, and so $V \subseteq S$, and so $S$ is open.

If $t_0 \notin S$, $t \in V$, $t \geq t_0$ and $t \in S$, contradicting $t_0 \notin S$. If $t < t_0$ by the previous argument above we have a contradiction as then $t_0 \in S$. So $V \subseteq I \setminus S$, and hence $S$ must also be closed. $\square$

**Corollary 2.13.** *Let $p : \widehat{X} \to X$ be a covering map with $X$ path connected. Then $p$ is $n$-sheeted for some $n \in \mathbb{N} \cup \{\infty\}$. In fact, $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality for all pairs $x, y \in X$.*

*Proof.* Let $\gamma$ be a path from $x$ to $y$ in $X$. If $\widehat{x} \in p^{-1}(x)$, let $\widehat{\gamma}_{\widehat{x}}$ be the lift of $\gamma$ from $x$. Then map $\widehat{x}$ to $\widehat{\gamma}_{\widehat{x}}(1)$. The path $\overline{\gamma}$ similarly gives a map $p^{-1}(y) \to p^{-1}(x)$, inverse to the first map.

For example to show that the composition $p^{-1}(x) \to p^{-1}(y) \to p^{-1}(x)$ is the identity, we need to show that, for $\widehat{x} \in p^{-1}(x)$, $\widehat{(\overline{\gamma})}_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \widehat{x}$. But $\widehat{\gamma}_{\widehat{x}} \cdot \widehat{(\overline{\gamma})}_{\overline{\gamma}_{\widehat{x}}(1)}$ is a lift of $\gamma \cdot \overline{\gamma}$, and $\widehat{\gamma}_{\widehat{x}} \cdot \overline{(\widehat{\gamma}_{\widehat{x}})}$ is also a lift of $\gamma \cdot \overline{\gamma}$, and so by uniqueness, $\widehat{(\overline{\gamma})}_{\overline{\gamma}_{\widehat{x}}(1)} = \overline{\widehat{\gamma}_{\widehat{x}}}$. Hence $\widehat{(\overline{\gamma})}_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \overline{\widehat{\gamma}}_{\widehat{x}}(1) = \widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$. $\square$

**Lemma 2.14** (Homotopy Lifting Lemma)**.** *Let $p : \widehat{X} \to X$ be a covering map and $g_0 : Y \to X$ a map with $Y$ locally path connected. Let $F : Y \times I \to X$ be a homotopy with $F(y, 0) = f_0(y)$ for all $y \in Y$. Let $\widehat{f}_0 : Y \to \widehat{X}$ be a lift of $Y$. Then there is a unique lift $\widehat{F}$ of $F$ to $\widehat{X}$ so that $\widehat{F}(y, 0) = \widehat{f}_0(y)$.*

*Proof.* For each $y \in Y$, we obtain a path $\gamma_y$ given by $\gamma_y(t) = F(y, t)$ from $f_0(y)$. By the path lifting lemma, each $\gamma_y$ lifts uniquely to a path $\widehat{\gamma}_y$ from $\widehat{f}_0(y)$. Now define:

$$\widehat{F}(y, t) = \widehat{\gamma}_y(t)$$

This clearly is a lift of $F$ in the sense that

$$(p \circ \widehat{F})(y,t) = p(\widehat{\gamma}_y(t)) = \gamma_y(t) = F(y,t)$$

but is $\widehat{F}$ continuous?

We will construct a different map $\widetilde{F} : Y \times I \to \widehat{X}$ which is continuous by construction, and then we will show that $\widehat{F} = \widetilde{F}$.

Fix $y_0 \in Y$. The for each $t \in I$ we have an evenly covered neighbourhood $U_t$ of $F(y_0,t) \in X$. Then $F^{-1}(U_t) \subseteq Y \times I$ is an open neighbourhood of $(y_0,t)$. We can find an open neighbourhood of $(y_0,t)$ in $F^{-1}(U_t)$ of the form $V_t \times (t-\epsilon_t, t+\epsilon_t)$ with $V_t$ path connected.

Note that these neighbourhoods cover $Y \times I$, and as $\{y_0\} \times I$ is compact, there is a finite subcover $\{J_i\}$ of $\{(t-\epsilon_t, t+\epsilon_t) | t \in I\}$. Then, if $J_i = (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$, we can find a path connected subset $V \subseteq \cap_i V_{t_i}$ containing $y_0$. Hence we may assume there is a path-connected neighbourhood $V$ of $y_0$ in $Y$, and a finite number of intervals $J_i$ covering $I$ such that $F(V \times J_i)$ is contained in an evenly covered neighbourhood $U$ of $X$.

Let $\delta_i \in \Delta_U$ be the unique index such that:

$$\widehat{F}(\{y_0\} \times J_i) \subseteq U_{\delta_i}$$

Now for $(y,t) \in V \times J_i$, we can define $\widetilde{F}(y,t) \coloneqq p_{\delta_i}^{-1} \circ F(y,t)$ for $(y,t) \in V \times J_i$.

These maps agree on overlaps, i.e. when $(V \times J_i) \cap (V \times J_j) = V \times (J_i \cap J_j) \neq V \times \emptyset$ - to see this, suppose that $t \in J_i \cap J_j$, and let $\alpha$ be a path in $V$ from $y_0$ to $y$ in $V$, and let $\alpha_t(s) = F(\alpha(s), t)$.

Then $p_{\delta_i}^{-1} \circ \alpha_t$ is a lift of $\alpha_t$ from $p_{\delta_i}^{-1} \circ \alpha_t(0)$, and likewise for $p_{\delta_j}^{-1} \circ \alpha$.

But $p_{delta_i}^{-1} \circ \alpha_t(0) = p_{\delta_i}^{-1} \circ F(y_0,t) = \widehat{F}(y_0,t) = \widehat{\gamma}_{y_0}(t)$ as defined on $J_i$, and $p_{\delta_j}^{-1} \circ \alpha_t(0) = p_{\delta_j}^{-1} \circ F(y_0,t) = \widetilde{F}(y_0,t) = \widehat{\gamma}_{y_0}(t)$ as defined on $J_j$. Hence $p_{\delta_i}^{-1} \circ \alpha_t$ and $p_{\delta_j}^{-1} \circ \alpha_t$ have the same initial end point, and they are both lifts of $\alpha_t$, and hence they must agree by **2.11** uniqueness of lifting.

Hence $p_{\delta_j}^{-1} \circ F(y,t) = p_{\delta_j}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ F(y,t)$, and so the two definitions of $\widetilde{F}$ on $V \times J_i$ and $V \times J_j$ agree on the overlap $V \times (J_1 \cap J_j)$.

Thus we have a well-defined continuous lifting:

$$\widetilde{F} : V \times I \to \widehat{X} \text{ of } F_{V \times I} \to X$$

But by construction, $\widetilde{F}(y_0,0) = \widehat{f}_0(y_0)$, and so $\widetilde{F}(y,0)$ is a lift of $f_0(y)$ for all $y \in V$, and $\widetilde{F}(y_0,0) = \widehat{f}_0(y_0)$. Hence by **2.11** uniqueness of lifting, $\widetilde{F}(\cdot,0)$ is $\widehat{f}_0$ on $V$.

For each $y \in V$, $\widetilde{F}(y,\cdot)$ is a lift of $\gamma_y$ from $\widehat{f}_0(y)$. By uniqueness of lifts of paths, we must have $\widetilde{F}(y,t) = \widehat{\gamma}_y(t)$, and $\widetilde{F}(y,t)$ hence agrees with $\widehat{F}(y,t)$. □

**Corollary 2.15.** *Let $p : \widehat{X} \to X$ be a covering map and let $F : I \times I \to X$ be a homotopy of paths. Then any lift $\widehat{F}$ of $F$ to $\widehat{X}$ is also a homotopy of paths.*

*Proof.* Let $\widehat{F}$ be a lift of $F$, so that $\widehat{F} : I \times I \to \widehat{X}$. We need to check that $\widehat{F}$ is a homotopy relative to $\{0,1\}$. But $\widehat{F}(0,\cdot)$ and $\widehat{F}(1,\cdot)$ are paths in $\widehat{X}$. Since $F$ is a homotopy of paths, $F(0,\cdot)$

and $F(1,\cdot)$ are constant. Thus $\widehat{F}(0,\cdot)$ and $\widehat{F}(1,\cdot)$ are paths in $p^{-1}(x_0$ and $p^{-1}(x_1)$ respectively, with $x_0 = F(0,\cdot), x_1 = F(1,\cdot)$ and hence $\widehat{F}(0,\cdot), \widehat{F}(1,\cdot)$ are constant since $p^{-1}(x_0), p^{-1}(x_1)$ are discrete. $\square$

## Applications to $\pi_1$

**Lemma 2.16.** *Let $p : \widehat{X} \to X$ be a covering map with $p(\widehat{X}) = X$. Then the induced map:*

$$p_* : \pi_1(\widehat{X}, \widehat{x}) \to \pi_1(X, x)$$

*is injective.*

*Proof.* Suppose $[\widehat{\gamma}]$ is in the kernel of $p_*$, i.e. $p \circ \widehat{\gamma} = \gamma \simeq_F c_x$. But then there is a lift $\widehat{F}$ of $F$ to $\widehat{X}$ with the property that $\widehat{\gamma} \simeq_{\widehat{F}} \widehat{c_x} = c_{\widehat{x}}$. Hence $\ker p_* = \{[c_{\widehat{x}}]\} = \{\mathbb{1}_{\pi_1(\widehat{X},\widehat{x})}\}$, and so $p_*$ is injective. $\square$

Observe that, if, $[\gamma] \in \pi_1(X, x)$, we get a map $p^{-1}(X) \to p^{-1}(X)$ via $\widehat{x} \mapsto \widehat{\gamma}_{\widehat{x}}(1)$, where $\widehat{\gamma}_{\widehat{x}}$ is a lift of $\gamma$ with $\widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$.

For example, let $p : \mathbb{R} \to S^1$, with $p(t) = e^{2\pi \mathrm{i} t}$

This gives an action of $\pi_1(X, x)$ on $p^{-1}(X)$. This is a ***right action***, i.e. for $\widehat{x} \in p^{-1}(X), [\gamma] \in \pi_1(X, x)$, and we write $x \cdot \gamma = \widehat{\gamma}_{\widehat{x}}(1)$, and then $x \cdot \gamma) \cdot \delta = x \cdot (\gamma \cdot \delta)$.

**Lemma 2.17.** *Let $p : \widehat{X} \to X$ be a covering map, and suppose $\widehat{X}$ is path connected. Let $x \in X$. Then the map:*

$$p_*(\pi_1(\widehat{X}, \widehat{x}))\backslash\pi_1(X, x) \to p^{-1}(x)$$

*where $G\backslash H$ is the set of right cosets of $H$ in $G$, given by $(p_*(\pi_1(\widehat{X}, \widehat{x}))) \cdot [\gamma] \mapsto \widehat{x} \cdot \gamma$, is a bijection for any choice of $\widehat{x} \in p^{-1}(x)$.*

*Furthermore, this bijection satisfies:*

$$p_*(\pi_1(\widehat{X}, \widehat{x})) \cdot ([\gamma] \cdot [\delta]) \mapsto \widehat{x} \cdot (\gamma, \delta)$$

<u>Example:</u> $p : \mathbb{R} \to S^1$. We know $\pi_1(\mathbb{R}, \widehat{x}) = 0$ since $\mathbb{R}$ is contractible, so the lemma gives a bijection $\pi_1(S^1, x) \to p^{-1}(x)$.