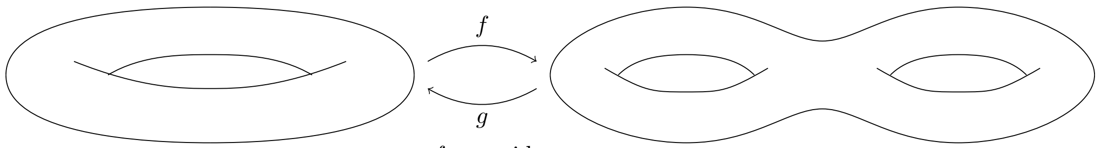
Algebraic Topology

November 6, 2019

1 Introduction

The fundamental problem of topology is to establish whether or not there exist continuous functions f, g going from a topological space X to another space Y and back again. For example, in the case of this torus and double-torus, we know from Geometry that such functions cannot exist by considering the Euler characteristic, but in general this is a hard problem.



$f \circ g = \mathrm{id}$

If such f, g continuous functions exist, then we say the two spaces are homeomorphic. Basic idea of algebraic topology is that we want to associate to any topological space X a group $G(X)$, and for every continuous function $f : X \to Y$ a group homomorphism $G(f) : G(X) \to G(Y)$ with $G(\mathrm{id}) = \mathrm{id}$ and $G(f \circ g) = G(f) \circ G(g)$. Thus if $f : X \to Y$ is a homeomorphism with inverse $g : Y \to X$, then $G(g) \circ G(f) = \mathrm{id}, G(f) \circ G(g) = \mathrm{id}$, so $G(f)$ is an is an isomorphism.

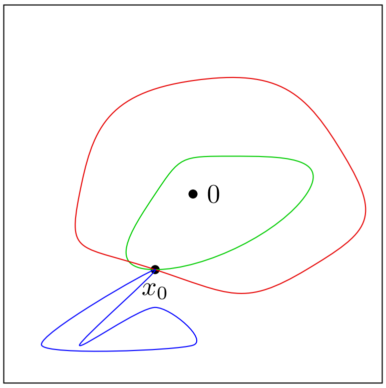
Extension problem: Let X be a topological space, $A \subseteq X$ a subspace, and $f : A \to Y$ a continuous function. Does there exist a continuous function $F : X \to Y$ with $F|_A = f$

$$\begin{tikzcd} X \arrow[r, dashed, "\exists\,?"'] & Y \\ A \arrow[u, hook] \arrow[ur, "f"'] & \end{tikzcd}$$

Theorem 1.1. *There is no continuous function*

$$f : D^n \to S^{n-1} \text{ with } f|_{S^{n-1}} = \mathrm{id}$$

By hand, we can see why this fails for e.g. $n = 1, 2$, but it gets hard to generalise. Eventually, we will construct G with $G(D^n) = 0, G(S^{n-1}) = \mathbb{Z}$. Then, if we have $S^{n-1} \to D^n \to S^{n-1}$ with composition being the identity, then we have maps $\mathbb{Z} \to 0 \to \mathbb{Z}$ being the identity.



Here, the green and red loops are the "same" loop, whilst the blue one is distinct

Conventions

A topological space will be referred to as a ***space***
A continuous function $f : X \to Y$ will be called a ***map***

2 The Fundamental Group

The idea here is that, if X is a space, $x_0 \in X$ a fixed point, called the ***basepoint***, we consider loops based at x_0, i.e. maps $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

For example, if we let our space $X = \mathbb{R}^2 \setminus \{0\}$

Then the ***fundamental group*** $\pi_1(X) = \pi_1(X, x_0)$ is defined to be the set of loops based at x_0 modulo "deforming loops". Multiplication in this group $\gamma_1 \cdot \gamma_2$ is given by first traversing γ_1 and then γ_2. But what do we mean by "deforming" a loop?

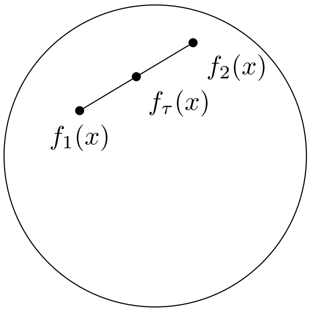
Let $f_0, f_1 : X \to Y$ be maps. A ***homotopy*** between f_0 and f_1 is a map

$$\begin{aligned} F : X \times I \to Y & \text{ where } I = [0,1] \text{ and} \\ F(x,0) = f_0(x) & \text{ and} \\ F(x,1) = f_1(x) & \end{aligned}$$

We often write $f_\tau(x) = F(x, \tau), f_\tau : X \to Y$.

If such F exists, we say f_0 and f_1 are ***homotopic***.

<u>Example:</u> Let $Y \subseteq \mathbb{R}^2$ be a convex set. Then any $f_0, f_1 : X \to Y$ are homotopic, via $F(x,t) = tf_1(x) + (1-t)f_0(x) \in Y$ by convexity.



If f_0 is homotopic to f_1, we write $f_0 \simeq f_1$, or $f_0 \simeq_F f_1$ if we want to be explicit about the homotopy we are using.

Suppose $f_0 \simeq_F f_1$, both functions $X \to Y$. If $Z \subseteq X$ and $f_0(z) = F(z,t) = f_1(z) \forall z \in Z, t \in I$, then we say f_0 is homotopic to f_1 ***relative to*** Z.

Lemma 2.1. *Let $Z \subseteq X, Y$ be spaces. Then $\simeq$ relative to Z is an equivalence relation on the set of maps $X \to Y$.*

Proof.

- Reflexive: $f_0 \simeq f_0$ via $F(x,t) = f_0(x) \forall x, t$
- Symmetric: Given $f_0 \simeq_F f_1$, then $f_1 \simeq f_0$ via $F'(x,t) = f(x, 1-t)$
- Transitive: If $f_0 \simeq_{F_0} f_1, f1 \simeq_{F_1} f_2$, then $f_0 \simeq_F f_2$ with:

$$F(x,t) = \begin{cases} F_0(x, 2t) & t \leq 1/2 \\ F_1(x, 2t-1) & t \geq 1/2 \end{cases}$$

All homotopies are relative to Z.

□

A ***homotopy equivalence*** $f : X \to Y$ is a map with a ***homotopy inverse*** $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$. We then write $X \simeq Y$.

Remark: Most (all?) invariants in the course are ***homotopy invariants***

Examples:

1. Let $*$ be the one point space, $f : \mathbb{R}^n \to *$ be the constant map, and let $g : * \to \mathbb{R}^n; x \mapsto \mathbf{0}$. Then $f \circ g = \mathrm{id}_*$, and $g \circ f(x) = 0 \forall x \in \mathbb{R}^n$. Now $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(x,t) = tx$.
2. Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map, and $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}; x \mapsto \frac{x}{|x|}$ (i.e. map x to the intersection of $\overrightarrow{\mathbf{0}x}$ with S^{n-1}) . Then $g \circ f = \mathrm{id}_{S^{n-1}}$ and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(x,t) = (1-t)x + t \cdot \frac{x}{|x|}$

If $X \simeq *$, then we say X is ***contractible***.
Let $f : X \to Y, g : Y \to X$ be maps. If $g \circ f = \mathrm{id}_X$, then we say X is a ***retract*** of Y, and g is a ***retraction***. If in addition $f \circ g \simeq \mathrm{id}_Y$ relative to $f(X)$, then we say X is a ***deformation retract*** of Y. Hence, in example 2, we see that S^{n-1} is a deformation retract of $\mathbb{R}^n$.

Lemma 2.2. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial from the definition.

Suppose $X \simeq Y, Y \simeq Z$ via:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

We want to show $f' \circ f$, $g \circ g'$ induces a homotopy equivalence

$$\begin{tikzcd} X \arrow[r, bend left, "f' \circ f"] & Z \arrow[l, bend left, "g \circ g'"] \end{tikzcd}$$

Now $(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f$. We know already that $g' \circ f' \simeq_{F'} \mathrm{id}_Y$, and so:

$$(x,t) \mapsto g(F'(f(x),t)) = \begin{cases} g(g'(f'(f(x)))) & t = 0 \\ g(f(x)) & t = 1 \end{cases}$$

is a homotopy, as $g \circ (g' \circ f') \circ f \simeq g \circ f$, and since $X \simeq Y$, $g \circ f \simeq \mathrm{id}_X$. Hence $(g \circ g') \circ (f' \circ f) \simeq \mathrm{id}_X$ via transitivity of homotopy equivalence for maps. Similarly $(f' \circ f) \circ (g \circ g') \simeq \mathrm{id}_Z$ $\square$

Loops and π_1

If X is a space, a ***path*** in X is a map $\gamma : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$. If $\gamma(0) = x_0, \gamma(1) = x_1$ then we say γ is a path ***from*** x_0 ***to*** x_1.
We say γ_1 and γ_2 are ***homotopic*** if $\gamma_1 \simeq \gamma_2$ relative to $\{0,1\}$, and we write $[\gamma]$ for the homotopy equivalence class of γ.

If X is a space with points $x, y, z \in X$, and γ_1 is a path from x to y, γ_2 is a path from y to z, then:

- The ***concatenation*** of γ_1 and γ_2 is the path from x to z given by

$$(\gamma_1 \cdot \gamma_2)(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

- The ***constant path*** at x is the path $c_x(s) = x \forall s \in I$
- The ***inverse of*** γ_1 is $\overline{\gamma_1}(s) = \gamma_1(1-s)$, a path from y to x.

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[rr, bend left=40, "\gamma_1 \cdot \gamma_2"] & y \arrow[l, bend left, "\overline{\gamma_1}"] \arrow[r, bend right, "\gamma_2"'] & z \end{tikzcd}$$

Theorem 2.3. *Let X be space, and $x_0 \in X$. Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X with endpoint x_0 (we say they are* **based** *at x_0). Then $\pi_1(X, x_0)$ forms a group under the product $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$, with identity c_{x_0} and inverses $[\gamma_1]^{-1} = [\overline{\gamma_1}]$.*

This group is called the **fundamental group** *of X (based at x_0).*

To prove this, we will need the following lemmas:

Lemma 2.4. *If $\gamma_0 \simeq \gamma_1$ to y and $\delta_0 \simeq \delta_1$ from y, then $\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1$ and $\overline{\gamma_0} \simeq \overline{\gamma_1}$*

$$\begin{tikzcd} {} \arrow[r, bend left, "\gamma_0"] \arrow[r, bend right, "\gamma_1"'] & y \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] & {} \end{tikzcd}$$

Proof. Suppose $\gamma_0 \simeq_F \gamma_1$, and $\delta_0 \simeq_G \delta_1$. Set:

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq 1/2 \\ G(2s-1,t) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\gamma_0 \cdot \delta_0 \simeq_H \gamma_1 \cdot \delta_1$

Let $F'(s,t) = F(1-s,t)$. Then $\overline{\gamma_0} \simeq_{F'} \overline{\gamma_1}$. □

Lemma 2.5. *Let α, β, γ be paths from w to x to y to z in X.*

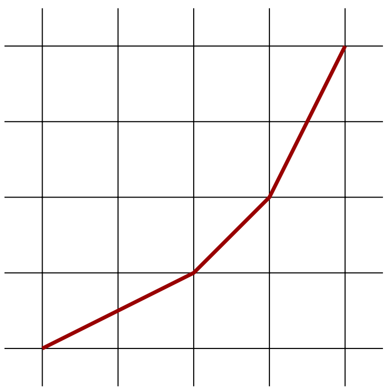
$$\begin{tikzcd} w \arrow[r, bend left, "\alpha"] & x \arrow[r, bend right, "\beta"'] & y \arrow[r, bend left, "\gamma"] & z \end{tikzcd}$$

Then:

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha\beta \cdot \gamma$
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$
3. $\alpha \cdot \overline{\alpha} \simeq c_w$

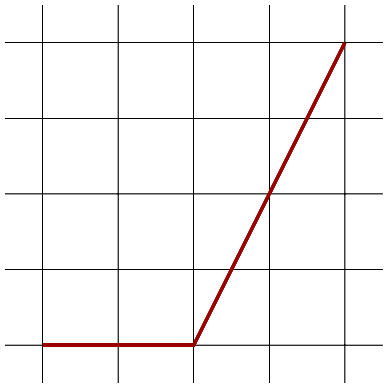
Proof. First, given a path $\delta : I \to X$, a ***reparametrization*** of δ is a path $\delta \circ \phi$ where $\phi : I \to I$ is a map with $\phi(0) = 0, \phi(1) = 1$. Note that ϕ needn't be monotonic, and that $\delta \simeq \delta \circ \phi$ via $F(s,t) = \delta(t\phi(s) + (1-t)s)$, and this homotopy is relative to $\{0,1\}$.

1. Now we reparametrize $(\alpha \cdot \beta) \cdot \gamma$ via the function ϕ whose plot is:

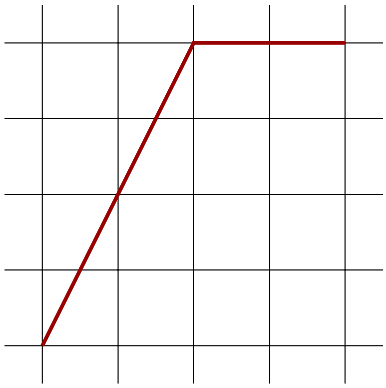


Note that $((\alpha \cdot \beta) \cdot \gamma) \circ \phi = \alpha \cdot (\beta \cdot \gamma)$, so $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.

2. Reparametrize α via:



i.e. do c_w for the first half of the time, then do α, so $\alpha \simeq c_w \cdot \alpha$. Likewise, we can get $\alpha \simeq \alpha \cdot c_x$ using the reparametrization



3. use the homotopy:

$$F(s,t) = \begin{cases} \alpha(2s) & 0 \leq s \leq t/2 \\ \alpha(t) & t/2 \leq s \leq 1 - t/2 \\ \alpha(2-2s) & 1 - t/2 \leq s \leq 1 \end{cases}$$

So $c_w \simeq \alpha \cdot \bar{\alpha}$, as we have c_w at $t = 0$ and $\alpha \cdot \bar{\alpha}$ at $t = 1$.

□

Then theorem **1.3** giving the existence of $\pi_1(X, x_0)$ follows from the previous two lemmas.

Example: $X = \mathbb{R}^n, x_0 = 0$. If γ is a loop based at 0, then $\gamma \simeq c_0$ via the straight line homotopy, and so $\pi_1(\mathbb{R}^n, 0) = 0$.

Formal Properties of π_1

Lemma 2.6. *Let $f : X \to Y$ be a map with $f(x_0) = y_0$. Then there is a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$.*

Furthermore:

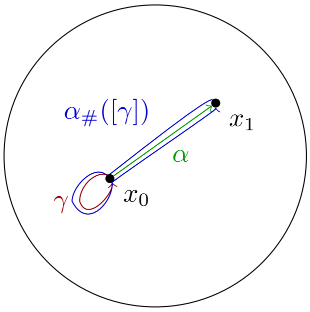
1. *If $f \simeq f'$ relative to x_0, then $f'_* = f_*$.*
2. *If $g : Y \to Z$ with $g(y_0) = z_0$, then $g_* \circ f_* = (g \circ f)_*$*
3. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$

Proof. f_* is well-defined: if $\gamma_1 \simeq_F \gamma_2$, then $f \circ \gamma_1 \simeq_{f \circ F} f \circ \gamma_2$. Then $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by definition, and so we have a group homomorphism.

1. If $f \simeq_F f'$ relative to x_0, then for γ a loop based at x_0, $(s,t) \mapsto F(\gamma(s), t)$ is a homotopy between $f \circ \gamma$ and $f' \circ \gamma$.
2. and 3. are immediate by definition.

□

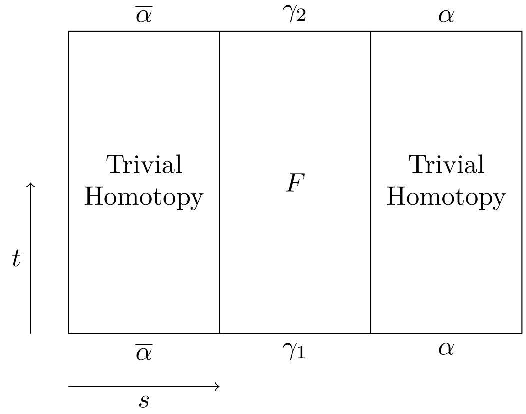
Lemma 2.7. *let X be a space, $x_0, x_1 \in X$ and α a path from x_0 to x_1. Then there is a group isomorphism $\alpha_\# : \pi_1(X,x_0) \to \pi_1(X,x_1)$ via $\alpha_\#([\gamma]) = [\bar{\alpha} \cdot \gamma \cdot \alpha]$.*



Furthermore,

1. *If $\alpha \simeq \alpha'$ relative to $\{0,1\}$, then $\alpha_\# = \alpha'_\#$.*
2. $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$
3. *If β is a path from x_2 to x_2, then $(\alpha \cdot \beta)_\# = \beta_\# \circ \alpha_\#$*
4. *If $f : X \to Y$ and $y_1 = f(x_1)$, then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.*

Proof. Well-defined: If $\gamma_1 \simeq_F \gamma_2$ then $\bar{\alpha} \cdot \gamma_1 \cdot \alpha \simeq \bar{\alpha} \cdot \gamma_2 \cdot \alpha$ via:

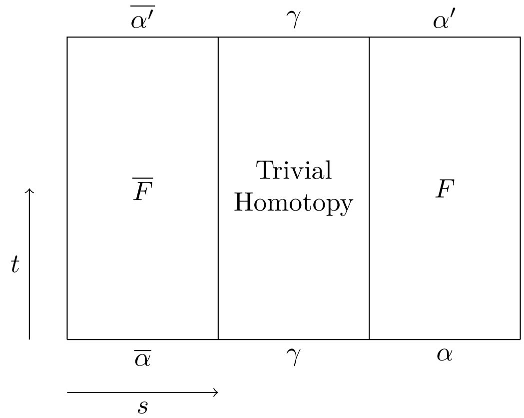


This is indeed a group homomorphism: for loops γ, δ based at x_0,

$$\begin{aligned}
\bar{\alpha} \cdot \gamma \cdot \alpha) \cdot (\bar{\alpha} \cdot \delta \cdot \alpha) &\simeq (\bar{\alpha} \cdot \gamma) \cdot (\alpha \cdot \bar{\alpha}) \cdot (\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma)(c_{x_0})(\delta \cdot \alpha) \\
&\simeq (\bar{\alpha} \cdot \gamma) \cdot (\delta \cdot \alpha) \\
&\simeq \bar{\alpha} \cdot (\gamma \cdot \delta) \cdot \alpha
\end{aligned}$$

Thus $\alpha_\#(\gamma \cdot \delta) = \alpha_\#(\gamma) \cdot \alpha_\#(\delta)$. Also $\bar{\alpha_\#} = (\alpha_\#)^{-1}$ - this is easy to check. Thus $\alpha_\#$ is a group isomorphism.

1. If $\alpha \simeq_F \alpha'$



   gives $\alpha_\#(\gamma) \simeq \alpha'_\#(\gamma)$

2. Immediate since c_{x_0} is the identity in $\pi_1(X, x_0)$.

3.

$$\begin{aligned}
(\alpha \cdot \beta)_\#(\gamma) &= \alpha \bar{\cdot} \beta \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot (\bar{\alpha} \cdot \gamma \cdot \alpha \cdot \beta \\
&= \bar{\beta} \cdot \alpha_\#(\gamma) \cdot \beta \\
&= \beta_\#(\alpha_\#(\gamma))
\end{aligned}$$

4.

$$\begin{aligned}
((f \circ \alpha)_{\#} \cdot f_*)(\gamma) &= (f \circ \alpha)_{\#}(f \cdot \gamma) \\
&= (f \circ \alpha)_{\#}(f \cdot \gamma) \\
&= \overline{f \cdot \alpha} \cdot (f \circ \gamma) \cdot (f \circ \alpha) \\
&= f \circ (\bar{\alpha} \cdot \gamma \cdot \alpha) \\
&= f_*(\alpha_{\#}(\gamma))
\end{aligned}$$

$\square$

A path connected space X is ***simply connected*** if $\pi_1(x, x_0) = 0$ for any, and hence all, $x_0 \in X$.

Our aim here is to prove that π_1 is a ***homotopy invariant***, i.e. that homotopy equivalent spaces have the same fundamental group. We will start with the following lemma:

Lemma 2.8. *Let $x_0 \in X$ and $f, g : X \to Y$ with $f \simeq_F g$. Set $x(t) = F(x_0, t)$ so that $\alpha(0) = f(x_0)$ and $\alpha(1) = g(x_0$. Then the diagram:*

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[d, "\alpha_{\#}"] \\
\pi_1(x, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

commutes, i.e. we have $\alpha_{\#} \circ f_* = g_*$.

Proof. We need to check that, for a loop γ based at x_0, $\overline{\alpha} \cdot (f \circ \gamma) \cdot \alpha \simeq g \circ \gamma$.

Let $G : I \times I \to Y$ defined by $G(s, t) = F(\gamma(s), t)$. For $t = 0$, this is $f \circ \gamma$, and for $t = 1$, this is $g \circ \gamma$. Now consider two paths in $I \times I$:

$$\begin{gathered}
a(t) = (t, 1); b = b_1 \cdot b_2 \cdot b_3 \text{ where:} \\
b_1(t) = (0, 1 - t), b_2(t) = (t, 0), b_3(t) = (1, t)
\end{gathered}$$

Then $(G \circ a)(s) = G(s, 1) = g \circ \gamma(s)$, whilst $G \circ b = \overline{\alpha} \cdot (f \circ \gamma) \cdot \alpha$.

Now, since $I \times I$ is convex, we have that $a \simeq_H b$, and so $G \circ H$ is the desired homotopy between $g \circ \gamma$ and $\overline{\alpha} \cdot (f \circ \gamma) \cdot \alpha$. $\square$

Theorem 2.9. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a homomorphism for any $x_0 \in X$.*

Proof. We'll show that f_* is a bijection:

Let $g : Y \to X$ be a homotopic inverse to f, with $\mathrm{id}_X \simeq_F g \circ f$. Let $\alpha : I \to X$ given by $\alpha(t) = F(x_0, t)$.

Note that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)); g : \pi_1(Y, f(x_0)) \to \pi_1(X, g(f(x_0)))$

Then $g_* \circ f_* = (g \circ f)_* = \alpha_{\#} \circ (\mathrm{id}_X)_* = \alpha_{\#}$. $\alpha_{\#}$ is an isomorphism, and so f_* is injective.

If $\mathrm{id}_Y \simeq_G f \circ g$ let $\beta(t) = G(f(x_0), t)$ Then $f_* \circ g_* = (g \circ f)_* = \beta_\# \circ (\mathrm{id}_Y)_* = \beta_\#$, an isomorphism, and hence f_* is surjective. ☐

Corollary 2.10. *Contractible spaces are simple connected.*

Proof. If X is contractible, there exists some $x_0 \in X$ and a homotopy F between id_X and $X \to \{x_0\}$. So $F(x, \cdot)$ is a path from any $x \in X$ to x_0, so X is path connected. Since X is homotopic to $\{x_0\}$, $\pi_1(X, x_0) \cong \pi_1(\{x_0\}, x_0) = 0$. ☐

Covering Spaces

Let $p : \widehat{X} \to X$ be a map. An open set $U \subseteq X$ is ***evenly covered*** if there exists a set Δ_U with the discrete topology and there is a homeomorphism:

$$p^{-1}(U) \xrightarrow{\cong} U \times \Delta_U$$

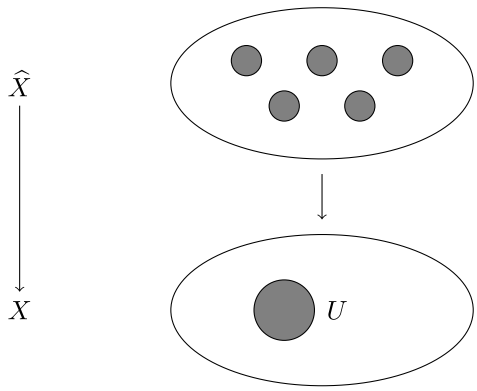
such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Delta_U \arrow[dl, "(x,\delta) \mapsto x"] \\
& U &
\end{tikzcd}$$

We write, for $\delta \in \Delta_0$, $U_\delta = U \times \{\delta\}$ and $p_\delta = p|_{U_\delta}$. So $p_\delta : U_\delta \to U$ is a homeomorphism.

Note that we can canonically identify Δ_U with $p^{-1}(x)$ for any $x \in U$, Note also that $p^{-1}(U) \cong \coprod_{\delta \in \Delta_U} U_\delta$, where $\coprod$ denotes disjoint union.

If every point of X has an open neighbourhood which is evenly covered, then we say that p is a ***covering map*** and $\widehat{X}$ is a ***covering space*** of X.



Examples:

1. $\widehat{X} = X \times \Delta$ for Δ a set with the discrete topology, e.g. $\widehat{I} = I \times \{1, 2, 3\}$. Then $\widehat{X}$ is a covering space of X, the identity map on the first element is a covering map.
2. $\widehat{X} = \mathbb{R}, X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p : \mathbb{R} \to S^1$ and $p(t) = \exp(2\pi\mathrm{i} \cdot t)$. Them p is a covering map:

let $U = S \setminus \{p\}$. We can define a branch of the logarithm $\log : \mathbb{C} \setminus \{rp : r \geq 0\} \to \mathbb{C}$. Then every point $\widehat{z} \in p^{-1}(U)$ can be written uniquely as $\widehat{z} = k + \frac{\log z}{2\pi \mathrm{i}}$ for some $k \in \mathbb{Z}$.

Thus $p^{-1}(U) \cong U \times \mathbb{Z}$, via $\widehat{z} \mapsto \left(\frac{\log z}{2\pi \mathrm{i}}, k\right)$, and so each proper subset of S^1 is evenly covered, however S^1 as a whole is not evenly covered, since $p^{-1}(S^1)$ is not a union of copies of S^1.

3. $\widehat{X} = X = S^1 \subseteq \mathbb{C}$, the unit circle, with $p(z) = z^n$.

   p is a covering map by choosing a branch of the nth root on proper open subsets of S^1

4. Let $\widehat{X} = S^2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on S^2 by the antipodal map $z \mapsto -z$. Then let $X = \widehat{X}/G = \widehat{X}/\sim$, where $x \sim y \iff x = \pm y$.

   Then X is $\mathbb{RP}^2$, the real projective plane. If $x \in X$, let U be an open neighbourhood of x disjoint from its negation. Then the image of U in X is evenly covered.

We say a covering map $p : \widehat{X} \to X$ is ***n-sheeted*** if $\#p^{-1}(x) = n$ for all $x \in X$, and call n the ***degree*** of p.

Lifting Properties

Let $p : \widehat{X} \to X$ be a covering map, and $f : Y \to X$ b a mp. A ***lift*** of f to $\widehat{X}$ is a map $\widehat{f} : Y \to \widehat{X}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \widehat{X} \arrow[d, "p"] \\
Y \arrow[ur, "\widehat{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

A space X is ***locally path connected*** if for every $x \in X$ and $U \subseteq X$ open neighbourhood of x, there exists a neighbourhood $V \subseteq U$ of x which is path connected.

Lemma 2.11 (Uniqueness of Lifting)**.** *Let $p : \widehat{X} \to X$ be a covering map and $\widehat{f_1}, \widehat{f_2} : Y \to \widehat{X}$ be two lifts of $f : Y \to X$ with Y connected and locally path connected.*

If there exists some $x_0 \in Y$ with $\widehat{f_1}(x_0) = \widehat{f_2}(x_0)$, then $\widehat{f_1} = \widehat{f_2}$.

Proof. We will show that the set $S \coloneqq \{y \in Y : \widehat{f_1}(y) = \widehat{f_2}(y)\}$ is both open and closed. By assumption we have $x_0 \in S$, so $S \neq \emptyset$. Since Y is connected, we must have then that $S = Y$ as otherwise S and $Y \setminus S$ would disconnect Y.

Let $y_1 \in Y$ be an arbitrary point, and let $U \subseteq X$ be an open neighbourhood of $f(y_1)$ which is evenly covered by p. Let $V \subseteq f^{-1}(U)$ be an open neighbourhood of y_1 which is path connected. We then want to show that, if $y_1 \in S$ then all of $V \subseteq S$, and otherwise $V \subseteq Y \setminus S$.

Let $y \in V$ be arbitrary and let α be a path from $y_1 \to y$. Then $\widehat{f_i} \circ \alpha$ is a path from $\widehat{f_i}(y_1) \to \widehat{f_i}(y)$ for $i = 1, 2$.

Note that $p \circ \widehat{f_1} \circ \alpha(t) = f(\alpha(t)) \in U$, and so $\widehat{f_i}(y)$ and $\widehat{f_i}(y_1)$ lie in the same component of $p^{-1}(U)$, say U_{δ_i}

If $y_1 \in S$, then $\widehat{f_1}(y_1) = \widehat{f_2}(y_1)$, so $\delta_1 = \delta_2$, and so $\widehat{f_1}(y) = p_{\delta_1}^{-1}(f(y)) = p_{\delta_2}^{-1}(f(y)) = \widehat{f_2}(y)$, so $y \in S$, and hence all of $V \subseteq S$.

Otherwise $y_1 \notin S$, then $\widehat{f_1}(y_1) \neq \widehat{f_2}(y_1)$. Each U_{δ_i} contains a unique point of $p^{-1}(\{f(y_1)\})$, and we must have $\delta_1 \neq \delta_2$.
So $\widehat{f_1}(y) \neq \widehat{f_2}(y)$, so $y \notin S$, and in general $V \subseteq Y \setminus S$.

Hence S is open, $Y \setminus S$ is open, and we are done $\square$

Let $\gamma : I \to X$ be a path from $x_0 \in X$ and $p : \widehat{X} \to X$ be a covering map. A lift of γ at (or from) $\widehat{x_0}$ is a lift $\widehat{\gamma}$ of γ with $\widehat{x_0} = \widehat{\gamma}(0)$. In particular, $p(\widehat{x_0}) = p(\widehat{\gamma}(0)) = \gamma(0) = x_0$.

Lemma 2.12 (Path Lifting Lemma). *Let $p : \widehat{X} \to X$ be a covering map, and let $\gamma : I \to X$ be a path from x_0. Then for any choice of $\widehat{x_0} \in p^{-1}(x_0)$, there exists a unique lift $\widehat{\gamma}$ of γ from $\widehat{x_0}$.*

Proof. Uniqueness follows from the previous lemma showing uniqueness of lifts. For existence, let $S = \{t \in I | \gamma|_{[0,t]}$ lifts to path from $\widehat{x_0}$ in $\widehat{X}\}$. Note $o \in S$. If we show that S is open and closed, then since I is connected, $S = I$. Note that if $t \in S$, then $[0, t] \subseteq S$.

Let $t_0 \in I$, and let U be an evenly covered neighbourhood of $\gamma(t_0)$. Let $V \subseteq \gamma^{-1}(U)$ be an open interval containing t_0. Let $t \in V$ and suppose first that $t_0 \in S$. If $t \leq t_0$, then $t \in S$, so instead assume that $t > t_0$. Since $\gamma|_{[0,t_0]}$ has a lift $\widehat{\gamma} : [0, t_0] \to \widehat{X}$, and we have $\widehat{\gamma}(t_0) \in U_\delta$ for some $\delta \in \Delta_U$.

Recall that we have a homeomorphism $p_\delta : U_\delta \to U$ where $p_\delta = p|_{U_\delta}$. Hence the path:

$$s \mapsto \begin{cases} \widehat{\gamma}(s) & 0 \leq s \leq t_0 \\ p_\delta^{-1} \circ \gamma & t_0 \leq s \leq t \end{cases}$$

is a lift of $\gamma|_{[0,t]}$. Hence $t \in S$, and so $V \subseteq S$, and so S is open.

If $t_0 \notin S$, $t \in V$, $t \geq t_0$ and $t \in S$, contradicting $t_0 \notin S$. If $t < t_0$ by the previous argument above we have a contradiction as then $t_0 \in S$. So $V \subseteq I \setminus S$, and hence S must also be closed. $\square$

Corollary 2.13. *Let $p : \widehat{X} \to X$ be a covering map with X path connected. Then p is n-sheeted for some $n \in \mathbb{N} \cup \{\infty\}$. In fact, $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality for all pairs $x, y \in X$.*

Proof. Let γ be a path from x to y in X. If $\widehat{x} \in p^{-1}(x)$, let $\widehat{\gamma}_{\widehat{x}}$ be the lift of γ from x. Then map $\widehat{x}$ to $\widehat{\gamma}_{\widehat{x}}(1)$. The path $\overline{\gamma}$ similarly gives a map $p^{-1}(y) \to p^{-1}(x)$, inverse to the first map.

For example to show that the composition $p^{-1}(x) \to p^{-1}(y) \to p^{-1}(x)$ is the identity, we need to show that, for $\widehat{x} \in p^{-1}(x)$, $\widehat{(\overline{\gamma})}_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \widehat{x}$. But $\widehat{\gamma}_{\widehat{x}} \cdot \widehat{(\overline{\gamma})}_{\overline{\gamma}_{\widehat{x}}(1)}$ is a lift of $\gamma \cdot \overline{\gamma}$, and $\widehat{\gamma}_{\widehat{x}} \cdot \overline{(\widehat{\gamma}_{\widehat{x}})}$ is also a lift of $\gamma \cdot \overline{\gamma}$, and so by uniqueness, $\widehat{(\overline{\gamma})}_{\overline{\gamma}_{\widehat{x}}(1)} = \overline{\widehat{\gamma}_{\widehat{x}}}$. Hence $\widehat{(\overline{\gamma})}_{\widehat{\gamma}_{\widehat{x}}(1)}(1) = \overline{\widehat{\gamma}}_{\widehat{x}}(1) = \widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$. $\square$

Lemma 2.14 (Homotopy Lifting Lemma). *Let $p : \widehat{X} \to X$ be a covering map and $g_0 : Y \to X$ a map with Y locally path connected. Let $F : Y \times I \to X$ be a homotopy with $F(y, 0) = f_0(y)$ for all $y \in Y$. Let $\widehat{f_0} : Y \to \widehat{X}$ be a lift of Y. Then there is a unique lift $\widehat{F}$ of F to $\widehat{X}$ so that $\widehat{F}(y, 0) = \widehat{f_0}(y)$.*

Proof. For each $y \in Y$, we obtain a path γ_y given by $\gamma_y(t) = F(y, t)$ from $f_0(y)$. By the path lifting lemma, each γ_y lifts uniquely to a path $\widehat{\gamma}_y$ from $\widehat{f_0}(y)$. Now define:

$$\widehat{F}(y, t) = \widehat{\gamma}_y(t)$$

This clearly is a lift of F in the sense that

$$(p \circ \widehat{F})(y,t) = p(\widehat{\gamma}_y(t)) = \gamma_y(t) = F(y,t)$$

but is $\widehat{F}$ continuous?

We will construct a different map $\widetilde{F} : Y \times I \to \widehat{X}$ which is continuous by construction, and then we will show that $\widehat{F} = \widetilde{F}$.

Fix $y_0 \in Y$. The for each $t \in I$ we have an evenly covered neighbourhood U_t of $F(y_0,t) \in X$. Then $F^{-1}(U_t) \subseteq Y \times I$ is an open neighbourhood of (y_0,t). We can find an open neighbourhood of (y_0,t) in $F^{-1}(U_t)$ of the form $V_t \times (t - \epsilon_t, t + \epsilon_t)$ with V_t path connected.

Note that these neighbourhoods cover $Y \times I$, and as $\{y_0\} \times I$ is compact, there is a finite subcover $\{J_i\}$ of $\{(t-\epsilon_t, t+\epsilon_t) | t \in I\}$. Then, if $J_i = (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$, we can find a path connected subset $V \subseteq \cap_i V_{t_i}$ containing y_0. Hence we may assume there is a path-connected neighbourhood V of y_0 in Y, and a finite number of intervals J_i covering I such that $F(V \times J_i)$ is contained in an evenly covered neighbourhood U of X.

Let $\delta_i \in \Delta_U$ be the unique index such that:

$$\widehat{F}(\{y_0\} \times J_i) \subseteq U_{\delta_i}$$

Now for $(y,t) \in V \times J_i$, we can define $\widetilde{F}(y,t) := p_{\delta_i}^{-1} \circ F(y,t)$ for $(y,t) \in V \times J_i$.

These maps agree on overlaps, i.e. when $(V \times J_i) \cap (V \times J_j) = V \times (J_i \cap J_j) \neq V \times \emptyset$ - to see this, suppose that $t \in J_i \cap J_j$, and let α be a path in V from y_0 to y in V, and let $\alpha_t(s) = F(\alpha(s), t)$. Then $p_{\delta_i}^{-1} \circ \alpha_t$ is a lift of α_t from $p_{\delta_i}^{-1} \circ \alpha_t(0)$, and likewise for $p_{\delta_j}^{-1} \circ \alpha$.

But $p_{delta_i}^{-1} \circ \alpha_t(0) = p_{\delta_i}^{-1} \circ F(y_0,t) = \widehat{F}(y_0,t) = \widehat{\gamma}_{y_0}(t)$ as defined on J_i, and $p_{\delta_j}^{-1} \circ \alpha_t(0) = p_{\delta_j}^{-1} \circ F(y_0,t) = \widehat{F}(y_0,t) = \widehat{\gamma}_{y_0}(t)$ as defined on J_j. Hence $p_{\delta_i}^{-1} \circ \alpha_t$ and $p_{\delta_j}^{-1} \circ \alpha_t$ have the same initial end point, and they are both lifts of α_t, and hence they must agree by **2.11** uniqueness of lifting.

Hence $p_{\delta_j}^{-1} \circ F(y,t) = p_{\delta_j}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ \alpha_t(1) = p_{\delta_i}^{-1} \circ F(y,t)$, and so the two definitions of $\widetilde{F}$ on $V \times J_i$ and $V \times J_j$ agree on the overlap $V \times (J_1 \cap J_j)$.

Thus we have a well-defined continuous lifting:

$$\widetilde{F} : V \times I \to \widehat{X} \text{ of } F_{V \times I} \to X$$

But by construction, $\widetilde{F}(y_0, 0) = \widehat{f}_0(y_0)$, and so $\widetilde{F}(y,0)$ is a lift of $f_0(y)$ for all $y \in V$, and $\widetilde{F}(y_0,0) = \widehat{f}_0(y_0)$. Hence by **2.11** uniqueness of lifting, $\widetilde{F}(\cdot, 0)$ is $\widehat{f}_0$ on V.

For each $y \in V$, $\widetilde{F}(y, \cdot)$ is a lift of γ_y from $\widehat{f}_0(y)$. By uniqueness of lifts of paths, we must have $\widetilde{F}(y,t) = \widehat{\gamma}_y(t)$, and $\widetilde{F}(y,t)$ hence agrees with $\widehat{F}(y,t)$. □

Corollary 2.15. *Let $p : \widehat{X} \to X$ be a covering map and let $F : I \times I \to X$ be a homotopy of paths. Then any lift $\widehat{F}$ of F to $\widehat{X}$ is also a homotopy of paths.*

Proof. Let $\widehat{F}$ be a lift of F, so that $\widehat{F} : I \times I \to \widehat{X}$. We need to check that $\widehat{F}$ is a homotopy relative to $\{0,1\}$. But $\widehat{F}(0,\cdot)$ and $\widehat{F}(1,\cdot)$ are paths in $\widehat{X}$. Since F is a homotopy of paths, $F(0,\cdot)$

and $F(1, \cdot)$ are constant. Thus $\widehat{F}(0, \cdot)$ and $\widehat{F}(1, \cdot)$ are paths in $p^{-1}(x_0$ and $p^{-1}(x_1)$ respectively, with $x_0 = F(0, \cdot), x_1 = F(1, \cdot)$ and hence $\widehat{F}(0, \cdot), \widehat{F}(1, \cdot)$ are constant since $p^{-1}(x_0), p^{-1}(x_1)$ are discrete. ☐

Applications to π_1

Lemma 2.16. *Let $p : \widehat{X} \to X$ be a covering map with $p(\widehat{X}) = X$. Then the induced map:*

$$p_* : \pi_1(\widehat{X}, \widehat{x}) \to \pi_1(X, x)$$

is injective.

Proof. Suppose $[\widehat{\gamma}]$ is in the kernel of p_*, i.e. $p \circ \widehat{\gamma} = \gamma \simeq_F c_x$. But then there is a lift $\widehat{F}$ of F to $\widehat{X}$ with the property that $\widehat{\gamma} \simeq_{\widehat{F}} \widehat{c_x} = c_{\widehat{x}}$. Hence $\ker p_* = \{[c_{\widehat{x}}]\} = \{\mathbb{1}_{\pi_1(\widehat{X}, \widehat{x})}\}$, and so p_* is injective. ☐

Observe that, if, $[\gamma] \in \pi_1(X, x)$, we get a map $p^{-1}(X) \to p^{-1}(X)$ via $\widehat{x} \mapsto \widehat{\gamma}_{\widehat{x}}(1)$, where $\widehat{\gamma}_{\widehat{x}}$ is a lift of γ with $\widehat{\gamma}_{\widehat{x}}(0) = \widehat{x}$.

For example, let $p : \mathbb{R} \to S^1$, with $p(t) = e^{2\pi i t}$

This gives an action of $\pi_1(X, x)$ on $p^{-1}(X)$. This is a ***right action***, i.e. for $\widehat{x} \in p^{-1}(X), [\gamma] \in \pi_1(X, x)$, and we write $x \cdot \gamma = \widehat{\gamma}_{\widehat{x}}(1)$, and then $x \cdot \gamma) \cdot \delta = x \cdot (\gamma \cdot \delta)$.

Lemma 2.17. *Let $p : \widehat{X} \to X$ be a covering map, and suppose $\widehat{X}$ is path connected. Let $x \in X$. Then the map:*

$$p_*(\pi_1(\widehat{X}, \widehat{x})) \backslash \pi_1(X, x) \to p^{-1}(x)$$

where $G \backslash H$ is the set of right cosets of H in G, given by $p_*(\pi_1(\widehat{X}, \widehat{x})) \cdot [\gamma] \mapsto \widehat{x} \cdot \gamma$, is a bijection for any choice of $\widehat{x} \in p^{-1}(x)$.

Furthermore, this bijection satisfies:

$$p_*(\pi_1(\widehat{X}, \widehat{x})) \cdot ([\gamma] \cdot [\delta]) \mapsto \widehat{x} \cdot (\gamma \circ \delta)$$

Example: $p : \mathbb{R} \to S^1$. We know $\pi_1(\mathbb{R}, \widehat{x}) = 0$ since $\mathbb{R}$ is contractible, so the lemma gives a bijection $\pi_1(S^1, x) \to p^{-1}(x)$.

Proof. We want to apply the orbit stablizer theorem to the right action of $\pi_1(X, x)$ on $p^{-1}(x)$.

The stabilizer of $\widehat{x}$ is the set of loops $[\gamma]$ based at x such that $\widehat{\gamma}_{\widehat{x}}(1) = \widehat{x}$, i.e. the set of loops based at $\widehat{x}$, i.e. we have that $[\widehat{\gamma}_{\widehat{x}}] \in \pi_1(\widehat{X}, \widehat{x})$. Thus the stabilizer is precisely $p_*(\pi_1(\widehat{X}, \widehat{x}))$. Hence we just need to show this action is transitive, but this follows from path connectedness:

If $\widehat{x}, \widehat{x}' \in p^{-1}(x)$, we have a path $\widehat{\gamma}$ from $\widehat{x}$ to $\widehat{x}'$ in $\widehat{X}$. We let $\gamma = p \circ \widehat{\gamma}$, so that γ is a loop based at x and $\widehat{x} \cdot \gamma = \widehat{x}'$ since $\widehat{\gamma}_{\widehat{x}} = \widehat{\gamma}$.

So the orbit stabilizer theorem gives this bijection. ☐

Note that the degree of $p : \widehat{X} \to X$ is just the index of the subgroup $p_*(\pi_1(\widehat{X}, \widehat{x})$ in $\pi_1(X, x)$, i.e.

$$\deg p = [\pi_1(X, x) : p_* \pi_1(\widehat{X}, \widehat{x})]$$

Thus, since we have covers $\mathbb{R} \to S^1$ of degree ∞ and $S^1 \to S^1$ of degree n for all $n > 0$, and hence $\pi_1(S^1, 1)$ must be an infinite group with subgroups of every possible index.

If $\widehat{X} \to X$ is a covering map with X path connected and $\widehat{X}$ simply connected, then we say $\widehat{X}$ is a ***universal cover*** of X.

Corollary 2.18. *If $p : \widehat{X} \to X$ is a universal cover, then any choice of base point $\widehat{x} \in p^{-1}(x)$ defines a bijection from $\pi_1(X, x) \to p^{-1}(x)$, and the group structure on $\pi_1(X, x)$ is determined by $\widehat{x} \cdot (\gamma \cdot \delta) = (\widehat{x} \cdot \gamma) \cdot \delta$.*

Example: $p : \mathbb{R} \to S^1; t \mapsto e^{2\pi i t}$ is a universal cover, and so we get a bijection $\pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$. (Note that $\mathbb{Z}$ here means the *set*, not group).

For $n \in \mathbb{Z}$, we can define $\widetilde{\gamma}_n(t) = nt$, a path from 0 to n in $\mathbb{R}$, and let $\gamma_n = p \circ \widetilde{\gamma}_n$, i.e. an n times wrapping around S^1 in the anticlockwise direction.

By the **2.18**, any loop in S^1 based at 1 must be homotopic to one of the γ_n, otherwise we would not have an injective map. In particular, the bijection of **2.18** is given by:

$$\begin{aligned} \pi_1(S^1, 1) &\to p^{-1}(1) = \mathbb{Z} \\ [\gamma_n] &\mapsto n = \widehat{(\gamma_n)}_0(1) = \widehat{\gamma}_n(1) \end{aligned}$$

Note that for any $m \in \mathbb{Z}$, $m + \widehat{\gamma}_n$ is a path from m to $m+n$. So $0 \cdot (\gamma_m \cdot \gamma_n) = (0 \cdot \gamma_m) \cdot \gamma_n = m \cdot \gamma_n = (m + \widehat{\gamma}_n)(1) = m + n$, and so $\gamma_m \cdot \gamma_n = \gamma_{m+n}$ in $\pi_1(S^1, 1)$, and hence $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Theorem 2.19. *The identity map*

$$\mathrm{id}_{S^1} : S^1 \to S^1$$

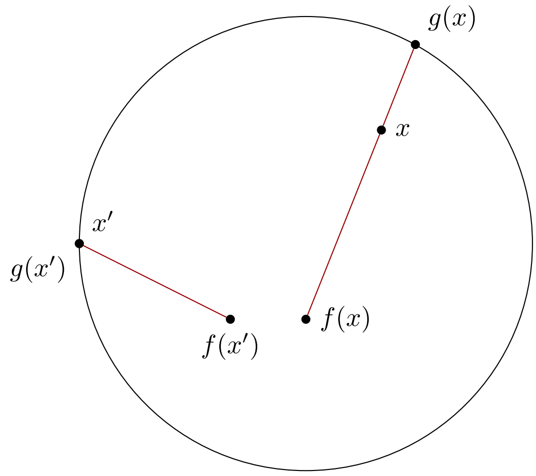
does not extend to a map from the disc D^2, i.e. there is no map $f : D^2 \to S^1$ with $f|_{S^1} = \mathrm{id}_{S^1}$, and in particular S^1 is not a retract of D^2.

Proof. Suppose f does in fact exist, and let $\iota : S^1 \hookrightarrow D^2$ be the inclusion map. Then $(\mathrm{id}_{S^1})_* = f_* \circ \iota_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$.

But $\pi_1(D^2, 1) = 0$ as D^2 contractible so we have a contradiction, as otherwise $f_* : 0 \twoheadrightarrow \mathbb{Z}$. $\square$

Theorem 2.20 (Brouwer's Fixed Point Theorem)**.** *Every map $f : D^2 \to D^2$ has a fixed point, i.e. some point $x \in D^2$ with $f(x) = x$.*

Proof. Suppose not, and let $g : D^2 \to S^1$ be defined by projecting $f(x)$ through x to the boundary of D^2 - note that this is well defined precisely because $f(x) \neq x$ for all $x \in D^2$.



Then g is continuous and $g(x) = x$ for all $x \in S^1$, as $f(S^1) = S^1$. But this contradicts **2.19** $\square$

Theorem 2.21 (The Fundamental Theorem of Algebra)**.** *If f is a non-constant polynomial with coefficients in $\mathbb{C}$, then there is some $x \in \mathbb{C}$ with $f(x) = 0$.*

Proof (Sketch). Let $r : \mathbb{C} \setminus \{0\} \to S^1$ be given by $r(z) = \frac{z}{|z|}$, and let $\lambda_R : S^1 \to \mathbb{C}; z \mapsto R \cdot z$ for some $R \in \mathbb{R}$.

Suppose f has no root. Then define:

$$f_R \coloneqq r \circ f \circ \lambda_R; S^1 \to S^1$$

to be a map for any $R \geq 0$. Note that the straight line homotopy is a homotopy between λ_{R_1} and λ_{R_2}, and so f_{R_1}, f_{R_2} are homotopic. Then we have a well defined map:

$$(f_R)_* = g : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$$

which is necessarily multiplication by some integer d. For $R = 0$, we have that $(f_R)_* = 0$. For R very large, the top degree term z^d dominates and g is given by multiplication by d. Hence $d = 0$ since g is the same in both cases, and so f is constant. $\square$

We say a space X is ***locally simply connected*** if for all $x \in X$ and $U \subseteq X$ an open neighbourhood of x, there is some open neighbourhood $V \subseteq U$ of x with V simply connected.

Theorem 2.22 (Existence of Universal Covers)**.** *Let X be a path connected and locally simply connected. Then there exists a universal cover $p : \widehat{X} \to X$.*

For example, if $X = \cup_{n=1}^{\infty}\{(x, y) : (x - \frac{1}{\sqrt{n}})^2 + y^2 = 1/n\} \subseteq \mathbb{R}^2$, the "Hawaiian Earring". Then X is not locally simply connected at $(0, 0)$.

Sketch proof, non-examinable. Fix $x_0 \in X$ and let $\chi = \{\gamma : I \to X | \gamma \text{ a path from } x_0\}$.

We then define $\widehat{X} = \chi/\simeq$, identifying paths which are homotopic, and then we have a covering map $[\gamma] \mapsto \gamma(1)$. $\square$

The Galois Correspondence

The goal of this course is to classify all covering spaces of a space X using subgroups of $\pi_1(X, x)$.

Let X be a path connected space and $p_1 : \widehat{X}_1 \to X, p_2 : \widehat{X}_2 \to X$ be covering spaces. Then an ***isomorphism of covering spaces*** is a homeomorphism $\phi : \widehat{X}_1 \to \widehat{X}_2$ with $p_{"} \circ \phi = p$, i.e.:

$$\begin{tikzcd} \widehat{X}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & \widehat{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Note that ϕ^{-1} is also an isomorphism of covering spaces. If $\widehat{X}_i$ is equipped with a basepoint $\widehat{x}_i$ and $\phi(\widehat{x_1}) = \widehat{x_2}$, then we say ϕ is ***based***.

Note that ϕ is a lift p_1 to $\widehat{X}_2$, and so if $\widehat{X}_1$ is path connected then ϕ is uniquely determined by $\phi(\widehat{x}_1)$ by uniqueness of lifting.

Theorem 2.23 (Galois Correspondence with Basepoints)**.** *Let X be path connected and locally simply connected, with basepoint x_0. The map which associates a covering map $p : \widehat{X} \to X$ with basepoint $\widehat{x}_0 \in p^{-1}(x_0)$ to the subgroup $p_*(\pi_1(\widehat{X}, \widehat{x}_0)) \leq \pi_1(X, x_0)$ induces a bijection between based isomorphism classes of path connected covering spaces and subgroups of $pi_1(X, x_0)$.*

Proof. Omitted and non-examinable. $\square$

<u>Example</u>: as $\pi_1(S^1, 1) = \mathbb{Z}$ has subgroups $n\mathbb{Z}$ for n a non-negative integer, we 1 cover for each n. $n = 0$ gives the universal cover. For $n > 0$ we have $p_n : S^1 \to S^1, p_n(z) = z^n$. Hence every based covering space is based isomorphic to p or p_n.

Corollary 2.24. *Let X be path connected and locally simply connected. Any two universal covers are (based) isomorphic.*

Proof. Let $p_1 : \widehat{X}_1 \to X, p_2 : \widehat{X}_2 \to X$ be two universal covers. Pick $x \in X, \widehat{x}_1 \in p_1^{-1}(x), \widehat{x}_2 \in p_2^{-1}(x)$. Since $\pi_1(\widehat{X}_i, \widehat{x}_i) = 0$, these correspond to the 0 group in $\pi_1(X, x)$, and so by the Galois correspondence these two covering spaces are based isomorphic. $\square$

Corollary 2.25 (Galois Correspondence without Base Points)**.** *Let X be path connected and locally simply connected, with basepoint $x_0 \in X$. The map that sends a covering space $p : \widehat{X} \to X$ with a basepoint $\widehat{x}_0 \in p^{-1}(x_0)$ to the subgroup $p_*\pi_1(\widehat{X}, \widehat{x}_0) \subseteq \pi_1(X, x_0)$ induces a bijection between isomorphism classes os path connected covering spaces without a base point and conjugace classes of subgroups of $\pi_1(X, x_0)$.*

Proof. The map is surjective by the Galois correspondence with basepoints.

To see that this map is injective, we need to show that if, given $p_1 : \widehat{X}_1 \to X, p_2 : \widehat{X}_2 \to X, \widehat{x}_i \in p^{-1}(x_0)$ and if $p_{1*}\pi_1(\widehat{X}_1, \widehat{x}_1)$ is conjugate to $p_{2*}\pi_1(\widehat{X}_2, \widehat{x}_2)$, then p_1, p_2 are isomorphic covering spaces.

So suppose $p_{1*}\pi_1(\widehat{X}_1,\widehat{x}_1) = [\gamma](p_{2*}\pi_1(\widehat{X}_2,\widehat{x}_2))[\bar{\gamma}]$ for some $[\gamma] \in \pi_1(X,x_0)$.

Then let $\widehat{\bar{\gamma}}$ be the lift of $\bar{\gamma}$ from $\widehat{x}_2$. In particular, $\widehat{\bar{\gamma}}$ is a path in $\widehat{X}_2$. Let $\widehat{x}'_2 = \widehat{\bar{\gamma}}(1)$, so that $p_2(\widehat{x}'_2) = x_0$.

Then $[\gamma](p_{2*}\pi_1(\widehat{X}_2,\widehat{x}_2))[\bar{\gamma}] = gam\bar{m}a_{\#} = p_{2*}(\widehat{\bar{\gamma}}_{\#}(\pi_1(\widehat{X}^2,\widehat{x}^2)) = p_{2*}(\pi_1(\widehat{X}_2,\widehat{x}'_2))$.

And so, by the Galois correspondence with base points there is a based isomorphism between $\widehat{X}_1$ and $\widehat{X}_2$ with basepoints $\widehat{x}_1, \widehat{x}'_2$. And so $\widehat{X}_1, \widehat{X}_2$ are isomorphic as covering spaces. $\square$

3 The Seifert - van Kampen Theorem

Presentation of a Group

Let D_{2n} be the dihedral group of order $2n$. We can represent $D_{2n} = \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1}\rangle$.

Let A be a set and $F(A)$ a group, with $A \to F(A)$ a map of sets. We say that $F(A)$ is ***the free group on A*** if it satisfies the following ***universal property:***

For any group G and any set map $A \to G$ there exists a unique group homomorphism $f : F(A) \to G$, such that the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[dr, "f"] & \\ A \arrow[u] \arrow[r] & G \end{tikzcd}$$

f is called the ***canonical homomorphism*** induced by $A \to G$

Example: Let $A = \{\alpha\}$, and $A \to \mathbb{Z}$ given by $\alpha \mapsto 1$. Given a map $A \to G; \alpha \mapsto g$ we can define a map $\mathbb{Z} \to G$ by $n \mapsto g^n$, and this is the unique such homomorphism that makes the diagram commute, and so $\mathbb{Z}$ is the free group on $\{\alpha\}$.

Note that the universal property guarantees that $A \to F(A)$ is unique "up to unique isomorphism" if it exists. To see this, suppose $\phi : A \to F(A), \phi' : A \to F'(A)$ both satisfy the universal property. Taking $A \to G$ to be ϕ or ϕ', we end up with a function $f : F(A) \to F'(A)$ and $g : F'(A) \to F(A)$ homomorphisms. Then we get a diagram:

$$\begin{tikzcd} F'(A) \arrow[dr, "f \circ g"] & \\ A \arrow[u] \arrow[r] & F'(A) \end{tikzcd}$$

But note that this diagram also commutes with $f \circ g$ replaced by $\mathrm{id}_{F'(A)}$, and so by the uniqueness part of the definition, $g \circ f = \mathrm{id}_{F(A)}$, and so $f : F(A) \to F'(A)$ is a unique isomorphism such that this diagram commutes.

We don't know yet that $F(A)$ exists, but if it does (and we'll see that it does) and $|A| = r$, we say that $F(A) = F_r$, the ***free group of rank r***.

We can write ***words*** in A as strings of elements and their inverses, for instance $a, b \in A$, $abba^{-1}ba^{-1}b^{-1}b^{-1}$ is a word in $A = \{a, b\}$. These words then give an element in $F(A)$ by applying *phi* to each symbol then multiplying those. Let G be the subgroup of $F(A)$ generated by all words in A. This in fact the set of all elements of $F(A)$ describable as words in $F(A)$, and so we have a map $\phi : A \to G$. We can check that ϕ also satisfies the universal property, and hence $G = F(A)$.

A ***presentation*** of a group is a set A and a subset of relations $R \subseteq F(A)$. It ***presents*** the group $\langle A|R\rangle \coloneqq F(A)/\langle\langle R\rangle\rangle$, where $\langle\langle R\rangle\rangle$ denotes the ***normal closure*** of R, i.e. the subgroup of $F(A)$ generated by $\{srs^{-1} : r \in R, s \in F(A)\}$. The presentation is finite if A and R are both finite sets.

Lemma 3.1 (Universal Property of Presentations)**.** *Let $q : F(A) \to \langle A|R\rangle$ be the quotient map. Whenever $f : F(A) \to G$ is a group homomorphism such that $R \subseteq \ker f$ then there exists a unique homomorphism $g : \langle A|R\rangle \to G$ making the following diagram commute:*

$$\begin{tikzcd}
\langle A|R\rangle \arrow[dr, "g"] & \\
F(A) \arrow[u, "q"] \arrow[r, "f"'] & G
\end{tikzcd}$$

Proposition 3.2. *As $\langle\langle R\rangle\rangle$ is generated by srs^{-1} and $f(srs^{-1}) = f(s)f(r)f(s)^{-1} = f(s)f(s)^{-1} = 1 \in G$, since $r \in \ker f$.*

Hence $\langle\langle R\rangle\rangle \subseteq \ker f$, and so we obtain a well defined $q : F(A)/\langle\langle R\rangle\rangle \to G$, with $q(a\langle\langle R\rangle\rangle) = g(a)$.

Examples:

1. $\langle a|a^n\rangle \cong \mathbb{Z}/n\mathbb{Z}$
2. $\langle r, s|r^n, s^2, rsrs\rangle \to D_{2n}$. This homomorphism exists by the universal property of the lemma, and is surjective. One can show that every element on the LHS can be written as $1, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}$, which is $2n$ elements, and so the map is also injective, and hence an isomorphism.
3. Every group has a presentation. The identity set map $G \to G$ gives a group homomorphism $F(G) \to G$ with kernel R. Then $G \cong \langle G|R\rangle$.

Consider a commutative square of group homomorphisms:

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "k"] \\
B \arrow[r, "\ell"'] & \Gamma
\end{tikzcd}$$

This diagram is called a ***pushout*** if for every commutative square of groups

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & G \end{tikzcd}$$

there is a unique map $\phi : \Gamma \to G$ making the diagram:

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "k"] \arrow[ddr, bend left, "f"] & \\ B \arrow[r, "\ell"'] \arrow[drr, bend right, "g"'] & \Gamma \arrow[dr, "\phi"] & \\ & & G \end{tikzcd}$$

commute, i.e. $g = \phi \circ \ell, f = \phi \circ k$. If a pushout exists, we write $\Gamma = A \coprod_C B$, and it unique up to unique isomorphism.

If $C = \{1\}$ then $A \coprod_C B$ is written as $A * B$ and is called the ***free product*** of A and B. If i, j are injective we write $A \coprod_C B = A *_C B$, the ***free product with amalgamation***.

Lemma 3.3. *For $i : C \to A$, $j : C \to \{1\} = B$, then $A \coprod_C B = A/\langle\langle i(C)\rangle\rangle$*

Proof. Take q to be the quotient map. Since $f \circ i = g \circ j$, then necessarily $f(i(C)) = \{1\}$, and so $i(C) \subseteq \ker f$, Thus $\langle\langle i(C)\rangle\rangle \subseteq \ker f$ since $\ker f$ is normal. Thus we get a unique factorisation of f as $q : A \to A/\langle\langle i(C)\rangle\rangle$ composed with a map $A/\langle\langle i(C)\rangle\rangle \to C$, and so $A/\langle\langle i(C)\rangle\rangle$ satisfies the universal property. $\square$

Lemma 3.4. *Let $A = \langle S_1|R_1\rangle, B = \langle S_2|R_2\rangle$, and T is a generating set for C. Let $\widetilde{i} : T \to F(S_1)$ be a lift of $i : T \to A$, and $\widetilde{j} : T \to F(S_2)$ be a lift of $j : T \to B$. Then:*

$$\Gamma = \langle S_1 \coprod S_2 | R_1 \cup R_2 \cup \{\widetilde{i}^{-1}\widetilde{j}(t) : t \in T\}\rangle$$

is a presentation for $A \coprod_C B$.

Proof.

$$\begin{tikzcd}
G & & & \\
& \Gamma \arrow[ul, dashed, "\exists\,!"'] & B \arrow[l, "\ell"] \arrow[ull, bend right, "g"'] & F(S_2) \arrow[l] \arrow[ulll, bend right, "\widetilde{g}"'] \\
& A \arrow[u, "k"'] \arrow[uul, bend left, "f"] & C \arrow[l, "i"'] \arrow[u, "j"] \arrow[dr, leftrightarrow] & \\
& F(S_1) \arrow[u] \arrow[uuul, bend left, "\widetilde{f}"] & & T \arrow[ll, "\widetilde{i}"] \arrow[uu, "\widetilde{j}"']
\end{tikzcd}$$

This diagram is commutative. Note that $\widetilde{f}(R_1) = \{1\} = \widetilde{g}(R_1)$, and $\widetilde{f} \circ \widetilde{i}(t) = \widetilde{g} \circ \widetilde{j}(t)$ for $t \in T$ using the big outer square.

We have a map $S_1 \coprod S_2 \to G$ via $s_1 \in S_1 \mapsto \widetilde{f}(s_1); s_2 \in S_2 \mapsto \widetilde{g}(s_2)$. So it is sufficient to check that all relations for Γ map to 1 in G. R_1, R_2 map to 1 since $\widetilde{f}(R_1) = \{1\} = \widetilde{g}(R_2)$. Hence if we call the map $\Gamma \to G$ ϕ say, then $\phi(\widetilde{i}(t)^{-1}\widetilde{j}(1)) = \widetilde{f}(\widetilde{i}(t))^{-1}\widetilde{g}(\widetilde{j}(t)) = 1$.

Hence by the universal property of presentations, we get a unique map $\Gamma \to G$. $\square$

Seifert - vn Kampen theorem for Wedges

Let X, Y be spaces with $x_0 \in X, y_0 \in Y$ basepoints. Then the ***wedge*** of X and Y is $(X \coprod Y)/\sim$, where $\sim$ is the smallest equivalence relation such that $x_0 \sim y_0$. We write this as $X \vee Y$, and call the equivalence class $\{x_0, y_0\}$ the wedge point.

Theorem 3.5. *Suppose $X = Y_1 \vee Y_2$ with x_0 the wedge point, and y_i the basepoint in Y_i.*

Then $\pi_1(X, x_0) = \pi_1(Y_1, y_1) * \pi_1(Y_2, y_2)$

Sketch proof (non-examinable): We need to demonstrate the universal property. Suppose we have $f_i : pi_1(Y_i, y_i) \to G$. WE have inclusions $i_j : Y_j \hookrightarrow X$ with $i_j(y_j) = x_0$. This induces $i_{j*} : \pi_1(Y_j, y_j) \to \pi_1(X, x_0)$. Let γ be a lööp in X based at x_0. We can write γ as a concatenation $\alpha_1\beta_1\alpha_2\beta_2\ldots$ with the α_i loops in Y_1, and β_i loops in Y_2.

Then we have no choice but to define $h : \pi_1(X, x_0) \to G$ by $h(\gamma) = f(\alpha_1)g(\beta_1)f(\alpha_2)g(\beta_2)\ldots$.

The technically difficult part is to show that this is well defined on homotopy classes. $\square$

Examples:

1. $\pi_1(S_1 \vee S_1) = \mathbb{Z} * \mathbb{Z} = F(\{a, a'\}) = F_2$
2. $\pi_1(\vee_{[n]} S_1) = F([n])$, the free group on n symbols.

Theorem 3.6 (Generalised Seifert - van Kampen Theorem)**.** *Suppose $Y_1, Y_2 \subseteq X$ are open subsets with $X = Y_1 \vee Y_2$ and $Z = Y_1 \cap Y_2$ nonempty, with Y_1, Y_2, Z all path connected. Let $x_0 \in Z, i_k : Z \hookrightarrow Y_k, j_k : Y_k \to X$ be the inclusion maps. Then:*

$$\begin{tikzcd}
\pi_1(X, x_0) & \pi_1(Y_2, x_0) \arrow[l, "j_{2*}"'] \\
\pi_1(Y_1, x_0) \arrow[u, "j_{1*}"] & \pi_1(Z, x_0) \arrow[l, "i_{1*}"] \arrow[u, "i_{2*}"']
\end{tikzcd}$$

is a pushout diagram

Proof omitted. □

Example: Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere, and $x_\pm = (\pm 1, 0, 0, \ldots, 0) \in S^n$. Then let $U_\pm = S^n \setminus \{x_\mp\}$, and $V = U_+ \cap U_- = S^n \setminus \{x_\pm\}$. Then $S^n = U_+ \cup U_-$. We know already that U_+, U_- are homeomorphic to $\mathbb{R}^n$ via stereographic projection, and $V \cong (-1, 1) \times S^{n-1}$.

Then we have a map $(x_0, x_n) \mapsto (x_0, (x_1, \ldots, x_n)/|(x_1, \ldots, x_n)|)$. Then everything is the trivial group in the pushout diagram since $\mathbb{R}^n$ is contractible, and so $\pi_1(S^n) = \mathbb{1}$.

We call a subset $Y \subseteq X$ a ***neighbourhood retract*** if $\exists\ V \subseteq X$ open with $Y \subseteq V$ and Y a deformation retract of V.

Theorem 3.7 (Siefert-van Kampen for Closed Sets)**.** *Suppose that $Y_1, Y_2 \subseteq X$ are closed subsets with $X = Y_1 \cup Y_2, Z = Y_1 \cap Y_2$. If Y_1, Y_2, Z are path connected and Z is a neighbourhood retract in both Y_1, Y_2, then*

$$\begin{tikzcd}
\pi_1(X, x_0) & \pi_1(Y_2, x_0) \arrow[l] \\
\pi_1(Y_1, x_0) \arrow[u] & \pi_1(Z, x_0) \arrow[l] \arrow[u]
\end{tikzcd}$$

is a pushout diagram

Attaching Cells

Let X be a space and $\alpha : S^{n-1} \to X$ be a map. Define $X \cup_\alpha D^n$ to be the space $(X \coprod D^n)/\sim$, where $\sim$ is the smallest equivalence relation containing $\alpha(x) \sim x$ for all $x \in S^{n-1} = \partial D^n$. We ask the question: "How does attaching an n-cell affect π_1?"

Lemma 3.8. *If $n \geq 3$, then $i : X \hookrightarrow X \cup_\alpha D^n$ induces an isomorphism $i_* : \pi_1(X) \to \pi_1(X \cup_\alpha D^n)$*

Proof. The ***mapping cylinder*** of α is the space $M_\alpha := (X \coprod (S^{n-1} \times I))/\sim$, where $(\theta, 0) \in S^{n-1} \times \{0\} \subseteq S^{n-1} \times I$ is equivalent to $\alpha(\theta)$ for all $\theta \in S^{n-1}$.

We now identify the other end of the cylinder, $S^{n-1} \times \{1\}$ with S^{n-1}. Then we can see that S^{n-1} is a neighbourhood retract in both M_α and D^n, and hence $X \cup_\alpha D^n \cong M_\alpha \cup_{\mathrm{id}: S^{n-1} \to S^{n-1} \times \{1\}} D^n$.

So we can take $Y_1 = M_\alpha, Y_2 = D^n, Z = S^{n-1} \times \{1\}$. Then $S \vee K$ gives a pushout:

$$\begin{array}{ccc} \pi_1(X \cup_\alpha D^n) & \longleftarrow & \pi_1(M_\alpha) \\ \uparrow & & \uparrow \\ \pi_1(D^n) & \longleftarrow & \pi_1(S^{n-1}) \end{array}$$

Note that $\mathbb{1} = \pi_1(D^n) = \pi_1(S^{n-1})$. Hence $\pi_1(X \cup_\alpha D^n) \cong \pi_1(M_\alpha) \cong \pi_1(X)$, as X is a deformation retract of M_α. ☐

Lemma 3.9. *Let $\alpha : S^1 \to X$ be a map with $x_0 = \alpha(1)$. Then $\pi_1(X \cup_\alpha D^2, x_0) \cong \pi_1(X, x_0)/\langle\langle\alpha\rangle\rangle$, and the inclusion map $i : X \hookrightarrow X \cup_\alpha D^2$ induces the quotient map $\pi_1(X) \to \pi_1(X)/\langle\langle\alpha\rangle\rangle$*

Proof. As before, we have the pushout

$$\begin{array}{ccc} \pi_1(X \cup_\alpha D^n) & \longleftarrow & \pi_1(M_\alpha) \\ \uparrow & & \uparrow \\ \pi_1(D^n) & \longleftarrow & \pi_1(S^{n-1}) \end{array}$$

Once more, $\pi_1(D^n) = \mathbb{1}$, but now $\pi_1(S^{n-1}) = \mathbb{Z}$. We can compute the map from $\pi_1(S^1) \to \pi_1(M_\alpha)$ as $1 \mapsto [\alpha]$, and so $\pi_1(X \cup_\alpha D^2) \cong \pi_1(X)/\langle\langle\alpha\rangle\rangle$ as required, and $\pi_1(X) \to \pi_1(X \cup_\alpha D^2)$ is the quotient map. ☐

This theorem gives us a nice way to build up topological spaces to have desired fundamental groups:

Theorem 3.10. *For any finitely presented group G there is a compact space with $\pi_1(X, x_0) = G$.*

Proof. Let $G = \langle A|B\rangle$, and let $Y = \vee_A S^1$. Then $\pi_1(Y) = F(A)$. For any $r \in R$ let $\gamma_r : S^1 \to Y$ represent r. Then repeatedly attach D^2 to Y using all the elements of R.

So then let $X = ((Y \cup_{\gamma_{r_1}} D^2) \cup_{\gamma_{r_2}} D^2) \ldots$.

Then $\pi_1(X) = F(A)/\langle\langle r_1, \ldots, r_n\rangle\rangle = G$. ☐

Classification of Spaces

An ***n*-dimensional manifold** is a Hausdorff space M such that every point $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$. A two dimensional manifold is called a surface. For instance, a figure 8 is not a manifold, whilst S^1 is a manifold.

Examples:

1. Let $\alpha : S^1 \to *$. Then $* \cup_\alpha D^2 \cong S^2$.

2. Let $g \in \mathbb{Z}, g > 0$. Set $\Gamma_{2g} = \vee_{i=1}^{2g} S_i^1$ with S_i^1 being distinct circles. Now let $\alpha_i : I \to S_i^1; \beta_i : I \to S_{i+g}^1$ be loops generating $\pi_1(S^1)$.

   Then let the loop $\rho_g = \alpha_1 \beta_1 \bar{\alpha_1} \bar{\beta_1} \ldots \alpha_g \beta_g \bar{\alpha_g} \bar{\beta_g} : S^2 \to \Gamma_{2g}$.

   Let $\Sigma_g = \Gamma_{2g} \cup_{\rho_g} D^2$. We claim then that Σ_{2g} is a compact surface.

   Proof. Compactness is immediate. Now if $x \in D^2 \setminus S^1$, then $D^2 \setminus S^1$ is an open neighbourhood of x in Σ_{2g} homeomorphic to an open subset of $\mathbb{R}^2$. If x is on S_i^1 for some i, but is not the edge point, $\square$