Sogic and Let Theory

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1 Propositional Logic

Let P be a set of ***primitive propositions***, i.e. P is a set of symbols with $(,),\bot,\implies\notin P$. Unless stated otherwise (i.e. that P is uncountable), we may assume that $P=\{p_1,p_2,\ldots\}$.

The set of ***propositions***, denoted by $L(P)$ or simply just L, is defined inductively as follows:

1. $P\subset L$
2. $\bot\in L$, called FALSE
3. if $p,q\in L$, then $(p\implies q)\in L$

Each proposition is a string of symbols from $P\cup\{(,),\bot,\implies\}$, for instance we have the propositions $p_1,(p_1\implies p_1),((p_1\implies p_2)\implies(p_2\implies(\bot\implies p_3)))$. For readability, we often draw symbols $(,)$ in different ways, for instance as $[,(,\Big($.

Sometimes we omit the outside pair of parentheses when writing down propositions, for instance $p_1\implies p_2$ is shorthand for $(p_1\implies p_2)$.

Also we use some abbreviations, e.g.:

NOT: $\neg p$ to mean $(p\implies\bot)$

OR: $p\vee q$ to mean $(\neg p\implies q)$

AND: $p\wedge q$ to mean $\neg(\neg p\vee\neg q)$

What do we mean by L "defined inductively"? Define $L_0=P\cup\{\bot\}$. Then, given L_n, we can define $L_{n+1}=L_{n-1}\cup\{(p\implies q):p,q\in L_{n-1}\}$. Then we set $L=\bigcup_{n=0}^{\infty}L_n$. Note: if $p\in L\setminus(P\cup\{\bot\})$, then it is easy to show that there are ***unique*** $q,r\in L$ with $p=(q\implies r)$.

1.1 Semantic Entailment

A ***valuation*** is a function $v:L\to\{0,1\}$ satisfying:

1. $v(\bot)=0$
2. For all $p,q\in L,v(p\implies q)=\begin{cases}0 & v(p)=1,v(q)=0\\ 1 & \text{otherwise}\end{cases}$.

If $p \in L$ and $v(p) = 1$ for every valuation, we say that p is a ***tautology***, and write $\vDash p$.

Examples:

1. $\vDash (p \implies p)$

$v(p)$	$v(p \implies p)$
0	1
1	1

   So this is a tautology.

2. $\vDash (p \implies (q \implies p))$

p	q	$q \implies p$	$p \implies (q \implies p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

   So this is a tautology.

3. Is $\vDash (p \implies (q \implies r)) \implies ((p \implies q) \implies (p \implies r))$?

   Suppose not. Then for some p, q, r and valuation v we have:

   $$v(p \implies (q \implies r)) = 1$$
   $$v((p \implies q) \implies (p \implies r)) = 0.$$

   So $v(p \implies q) = 1, v(p \implies r) = 0$. Hence $v(p) = 1, v(r) = 0, v(q) = 1$. But then $v(q \implies r) = 0$, and so $v(p \implies (q \implies r)) = 0 \lightning$.

4. $\vDash ((p \implies \bot) \implies \bot) \implies p$, i.e. $\neg\neg p \implies p$, i.e. $(\neg p \vee p)$. This is the Law of the Excluded Middle, and is also a tautology.