Graph Theory

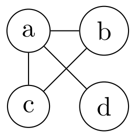
October 14, 2019

1 Introduction

Definitions

Formally, we define a ***graph*** to be an ordered pair $G = (V, E)$ where $E \subseteq V^{(2)} := \{Y \subseteq V : |Y| = 2\}$

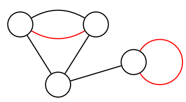
e.g. $V = \{a, b, c, d\}, E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}\}$



The set $V(G)$ is the ***vertex set*** of G and $E(G)$ is the ***edge set***.
The ***order*** of G is $|V(G)|$, often written $|G|$.
The ***size*** of G is $|E(G)|$, often written $e(G)$.
Note that for the purposes of this course, all graphs will be finite.

This definition that we use precludes:

- Having more than one edge between the same two vertices
- Having an edge with both ends at the same vertex



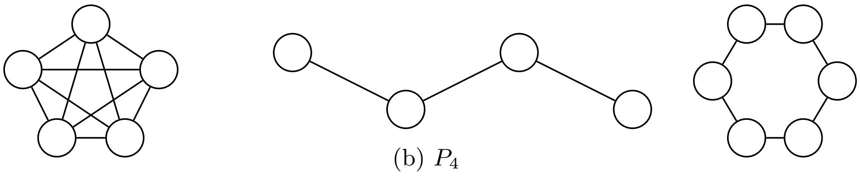
These 2 red edges are not allowed

Some authors allow graphs to have these features, and call what we're studying ***simple graphs***.

It's easier to write the edge $\{a, b\}$ as ab.
If $ab \in E(G)$, we say that "a is ***adjacent*** to b", and vice versa. We might write $ab \in G$.

Edges containing a vertex x are said to be ***incident*** to x and to each other.

Two graphs G, H are said to be ***isomorphic*** if there exists a bijection
$f : V(G) \to V(H)$ s.t. $uv \in E(G) \iff f(u)f(v) \in E(H)$.



(a) K_5 (b) P_4 (c) C_6

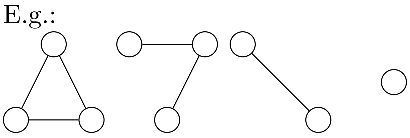
There are some canonical graphs:

Name	Notation	Order	Size
Empty graph	E_n	n	0
Complete graph	K_n	n	$\binom{n}{2}$
Path	P_n	n	$n-1$
Circuit	C_n	n	n

A ***subgraph*** of G is another graph H with $V(H) \subseteq V(G), E(H) \subseteq E(G)$. (Ever graph of order $\leq n$ is a subgraph of K_n).
The subgraph of G ***induced by*** $W \subseteq V(G)$, written $G[W]$ is the graph $(W, E(G) \cap W^{(2)})$, i.e. the vertex set W and all edges of G lying inside W.

A graph is ***connected*** if there is a $u-v$ path (a path from u to v in G for every pair $u, v \in V(G)$. The ***components*** of G are the maximal connected subgraphs of G (induced by the equivalence classes of the relation $u \leftrightarrow v \iff u = v$ or there is a $u - v$ path).

E.g.:



has 4 components.

A ***forest*** is a graph containing no circuit.
A ***tree*** is a connected forest, i.e. the components of a forest are trees.
Equivalently, a tree is a connected circuit-free graph. This bring us on to our first theorem of the course:

Theorem 1.1. *The following are equivalent:*

1. *G is a tree*
2. *G is a minimal connected (connected, but the removal of any edge kills connectivity)*
3. *G is maximal circuit-free (the addition of any edge creates a circuit)*

Proof. $(a) \implies (b)$

Let G be connected and circuit free. Suppose $uv \in G$ and $G - uv$ is connected. Then there is a $u - v$ path P in $G - uv$, so G contains the circuit $P + uv$ $\lightning$ (G circuit free)
$\therefore G$ is minimal connected.

$(b) \implies (a)$

Let G be minimal connected, but suppose G has a circuit C, with $xy \in C$. Let $u, v \in V(G)$.
There is a $u - v$ path P in G.
If $xy \notin P$, then P joins u to v in $G - xy$.
If $xy \in P$, then $(P \cup C) - xy$ contains a $u - v$ path in $G - xy$.
Hence $G - xy$ is connected $\lightning$ (G ***minimal*** connected).
$\therefore G$ is circuit free, and connected by hypothesis, hence a tree.

$(a) \implies (c)$

Let G be a tree. If $uv \notin E(G)$ there is a path P in G from u to v, so $P + uv$ is circuit in $G + uv$, so G is maximal circuit free.

$(c) \implies (a)$

Let G be maximal circuit free. Let $u, v \in V(G)$. Either $uv \in E(G)$ or $G + uv$ has a circuit C, so $C - uv$ is a $u - v$ path in G, and so G is connected, and hence a tree. □

A subgraph T of a graph G is said to be ***spanning*** if $V(T) = V(G)$, i.e. the subgraph touches every vertex of G

Corollary 1.2. *A graph is connected if and only if it has a spanning tree.*

Proof.

- $\implies$ T is connected as it is a tree, and is spanning, so all vertices in G are connected via T.
- $\impliedby$ Remove edges from G to obtain a minimal connected spanning subgraph T. By **1.1** T is a tree.

□

The set of ***neighbours*** of a vertex v is $\Gamma(v) = \{w : vw \in E(G)\}$.
The ***degree*** $d(v)$ is the number of neighbours of v, $|\Gamma(v)|$
The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$.
If $\delta(G) = \Delta(G) = k$, i.e. $d(v) = k \forall v$ we say G is k***-regular***. A ***regular*** graph is one that is k-regular for some k. If a graph is 3-regular, it is called ***cubic***.
The degrees of the graph written in some order form a ***degree sequence*** of G.

Lemma 1.3 (Handshaking Lemma)**.**

$$\sum_{v \in G} d(v) = 2e(G)$$

Proof. Each edge has 2 vertices at the end, so contributes 2 to the sum of all degrees. □

A ***leaf*** is a vertex of degree 1.

Theorem 1.4. *Every tree T with $|T| \geq 2$ has at least two leaves.*

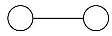
Proof. We note T is connected so $\delta(T) \geq 1$. Let x_1 be a vertex (a leaf if there is one). Let $x_1 x_2 \dots x_k$ be a maximal path from x_1. T is circuit-free so none of $x_1, \dots, x_{k-2}$ is in $\Gamma(x_k)$, and x_k has no other neighbours except for x_{k-1}, otherwise this path can be extended, hence $\Gamma(x_k) = \{x_{k-1}\}$, so x_k is a leaf. Repeat the proof starting at x_k for a second leaf. $\square$

Corollary 1.5. *A tree of order n has size $n-1$*

Proof.

This is true for $n = 1, 2$ by inspection, as there is only 1 tree:

(a) $n = 1$



(b) $n = 2$

Let $|T| \geq 3$. By **1.3** T has a leaf v. Then $T - v$ is certainly circuit free and connected, because the $x - y$ path in T uses v if and only if x or y is v. Hence the corollary follows by induction. $\square$

Corollary 1.6. *The following are equivalent for a graph G of order n:*

1. *G is a tree*
2. *G is connected and $e(G) = n - 1$*
3. *G is acyclic and $e(G) = n - 1$*

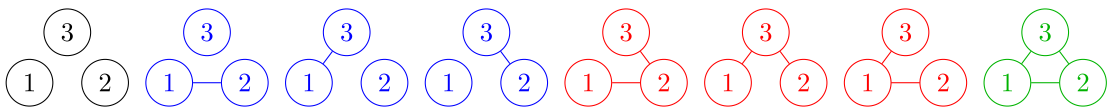
Proof.

- $(1) \implies (2), (3)$: True by definition and **1.5**.
- $(2) \implies (1)$: By **1.2** G contains a spanning tree T. By **1.5** $e(T) = e(G)$ so $T = G$
- $(3) \implies (1)$: Add edges to G to get a graph G' maximal acyclic. By **1.1** G' is a tree, so by **1.5** $e(G') = n - 1 = e(G)$, so $G' = G$.

$\square$

How many graphs of order n are there?

Let the vertex set be $[n] \coloneqq \{1, 2, \dots, n\}$. E.g. for $n = 3$ we have:



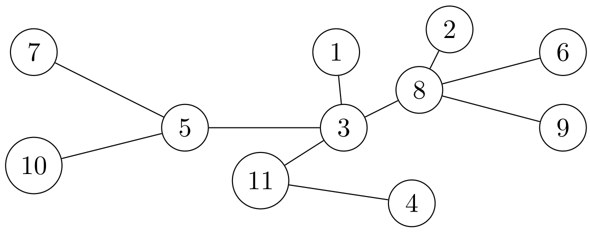
There are $2^{\binom{n}{2}}$ ***labelled*** graphs of order n - $\binom{n}{2}$ potentially edges, each of which we can independently choose to include or not.
However, there are only 4 unlabelled graphs of order 3 - all the graphs of the same colour are essentially the same, just with different labels given to the vertices. In general we must count the number of orbits amongst labelled graphs under the action of a group, using Burnside's Lemma. In fact, it turns out this number is asymptotic to $2^{\binom{n}{2}}/n!$

How many trees of order n are there?

Theorem 1.7 (Cayley)**.** *There are n^{n-2} labelled tress of order n.*

Proof. (Prüfer)
We construct a bijection between trees strings length $n-2$ over the alphabet n.

- Tree $\to$ string: Select the lowest leaf, write down its neighbour, remove that leaf, and repeat until only one edge is left.

  Here, we would get the string $3, 8, 11, 8, 5, 8, 3, 5, 10$. Each vertex is written down $d(n) - 1$ times

- String $\to$ tree: Mark all vertices "unused". Then repeatedly choose the smallest unused vertex not in the string. Join it to the first vertex in the string. Mark it as used. Delete first vertex in the string.

$\square$