Ramsey Theory

Harry Armitage

January 21, 2021

Contents

1 Monochromatic Systems

Here, we let $\mathbb{N} = \{1, 2, 3, \dots\}$, and write $[n] = \{1, 2, \dots, n\}$. For a set X and $r \in \mathbb{N}$, we write $X^{(r)} = \{A \subset X : |A| = r\}$, the collection of all r-sets in X.

Suppose we are given a 2-colouring of $\mathbb{N}^{(2)}$, i.e. a function $C : \mathbb{N}^{(2)} \to \{1, 2\}$. We can think of this being a colouring of the edges of the complete graph on $\mathbb{N}$. Can we find an infinite monochromatic M, i.e. a set $M \subset \mathbb{N}$ such that C is constant on $M^{(2)}$.

Examples

1. Colour ij (shorthand for the set i, j) red if $i + j$ is even, and blue if $i + j$ is odd. Here, the answer is yes - take $M = 2\mathbb{N} = \{2, 4, 6, \dots\}$.
2. Colour ij red if $\max\{n : 2^n | i + j\}$ is even, and blue if it is odd. The answer is yes - $M = \{4^0, 4^1, 4^2, 4^3, \dots\}$.
3. Colour ij red if $i + j$ has an even number of distinct prime factors, and blue if odd. This is more difficult. To save some time, we shall use the following theorem to answer every question of this form:

Theorem 1.1 (Ramsey's Theorem). *Let $C : \mathbb{N}^{(2)} \to \{1, 2\}$ be a 2-colouring of $\mathbb{N}^{(2)}$. Then there exists an infinite monochromatic subset of $\mathbb{N}$.*

Proof. Pick $a_1 \in \mathbb{N}$. Then there are infinitely many edges out of a_1, so infinitely many have the same colour - say all edges from a_1 to the infinite set B_1 have colour c_1.

Now pick $a_2 \in B_1$. There must be some infinite set $B_2 \subseteq B_1 \setminus \{a_2\}$ with all edges a_2 to B_2 are the same colour, say c_2, and repeat inductively.

We then obtain $a_1, a_2, a_3, \dots$ and colours $c_1, c_2, c_3, \dots$ such that $a_i a_j$ for $i < j$ has colour c_i. Now infinitely many of the c_i must be the same colour, say c. Then we may take $M = \{a_i : c_i = c\}$. $\square$

Remarks

1. This is sometimes called a 2-pass proof - we went through all the numbers to build the sequence $a_1, a_2, \dots$.
2. In example 3, no explicit example is known.
3. The exact same proof works for n colours. Alternatively, we could deduce this from Ramsey's theorem + induction - view the colours as '1' and '2 or 3 or ...'. If the infinite set is coloured 1, we are done, otherwise repeat with the $n - 1$ colours remaining.
4. An infinite monochromatic set is more than having arbitrarily large finite monochromatic sets. For example, make $\{1\}, \{2, 3\}, \{4, 5, 6\}, \dots$ all monochromatic blue sets, but make all edges between them red. Then there is no infinite monochromatic blue set (there is however an infinite red set - $\{1, 3, 6, \dots\}$).

Example Any sequence $x_1, x_2, \dots$ in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence. This was seen in Analysis I, where the proof worked by fiddling around with lim sups and lim infs. Instead, just colour ij 'up' if $x_i \leq x_j$, and 'down' if $x_i > x_j$. Then apply Ramsey's theorem.