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1 Introduction

1.1 Classical Motivation: Newton's Method

Newton's method (also called Newton-Raphson) is an algorithm for finding roots of a polynomial. If $p(z)$ is a polynomial, then we construct the auxiliary function $f(z) = z - \frac{p(z)}{p'(z)}$. We then pick some value z_0 and we iterate, so that $z_{i+1} = f(z_i)$. Sometimes, depending on p and z_0 , this sequence converges to some z , so that $z = z - \frac{p(z)}{p'(z)}$, i.e. $p(z) = 0$. We can view this graphically for real values of z as follows:

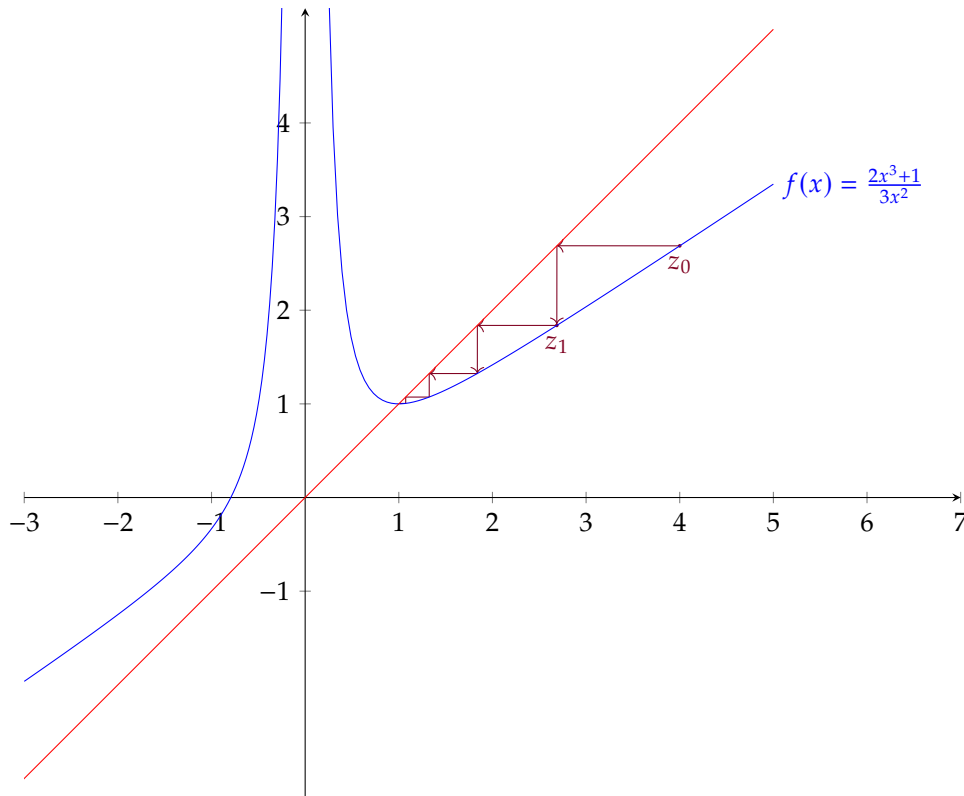


Figure 1: Newton's method for $p(z) = z^3 - 1$

It's not too hard to see that, no matter our starting choice of z_0 , this process will always eventually converge towards $z = 1$, and that $p(1) = 0$.

If instead we consider this iteration for **complex** values of z_0 , we might converge to different roots of $p(z)$, since p also has two complex roots. It turns out that the question of deciding which root we will converge to is not easy, and in fact we can see which root will be converged to in the following convergence diagram¹

¹Courtesy of usefuljs.net/fractals/docs/newtonian_fractals.html

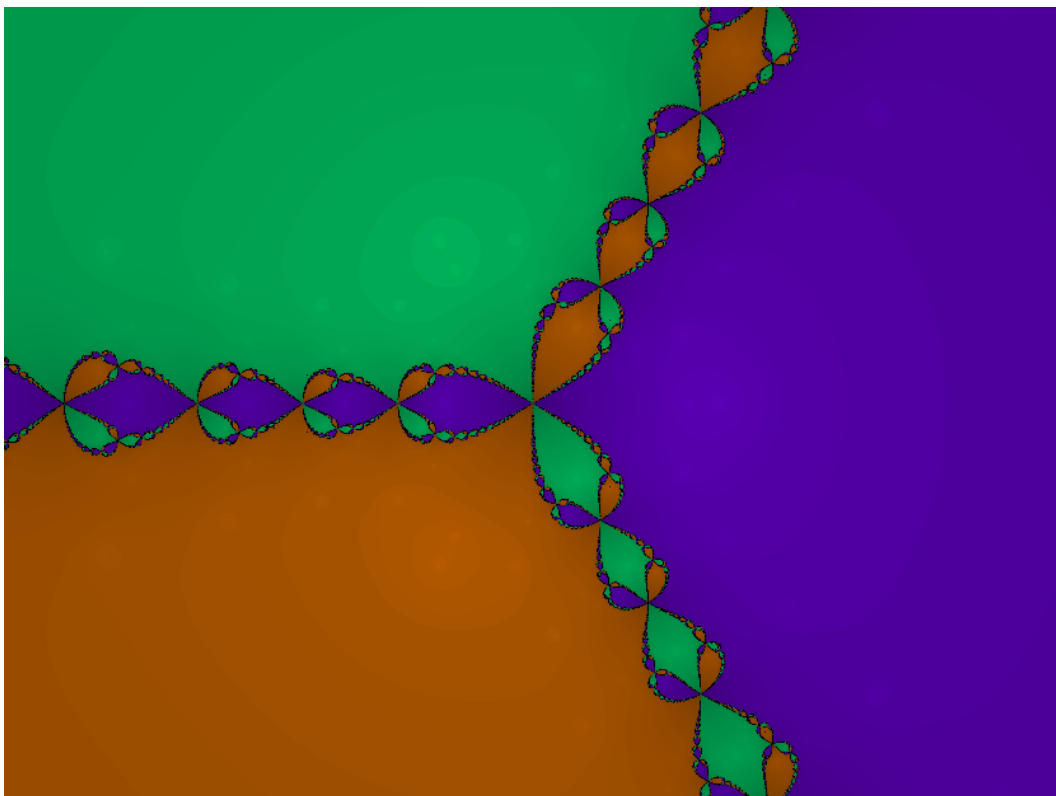


Figure 2: Convergence diagram for $z^3 - 1$

We have many different stable behaviours of this dynamical system, but the boundary is very chaotic. For example, if we consider $p(z) = z^4 - 2z^2 + 9$, note that if $z \in \mathbb{R}$, then $f(z) \in \mathbb{R}$. However, the four roots of p are all complex, and so Newton's method will fail to work for $z_0 \in \mathbb{R}$, and this can be seen in the following image:

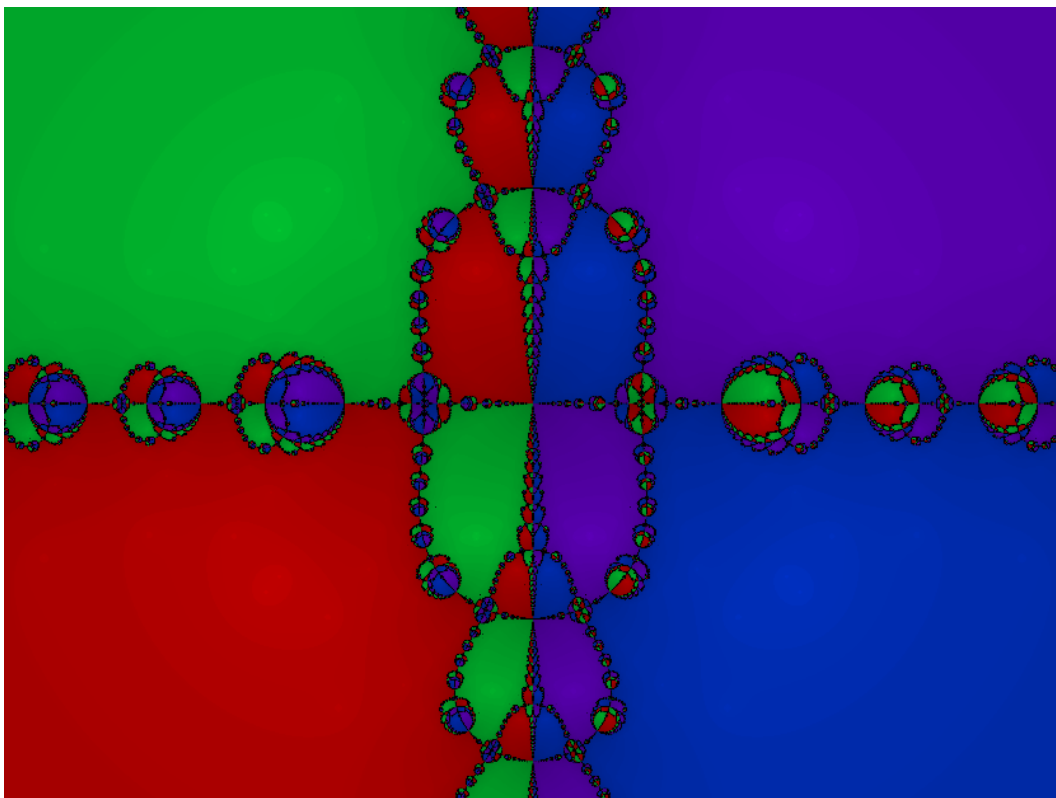


Figure 3: Convergence diagram for $z^4 - 2z^2 + 9$

1.2 Applications Motivation: Fixed Point Problems

Gravitational Lensing: Suppose we have n point masses in a plane perpendicular to the line of sight, and we are observing an object at the origin of this plane, identified with the complex plane. Call the points z_1, \dots, z_n , with masses $\sigma_1, \dots, \sigma_n$ respectively. Then the images of the object at the origin are the solutions to the *lens equation* $z = \sum_{j=1}^n \frac{\sigma_j}{\bar{z} - \bar{z}_j}$. We see this for instance in the following image, courtesy of Nasa

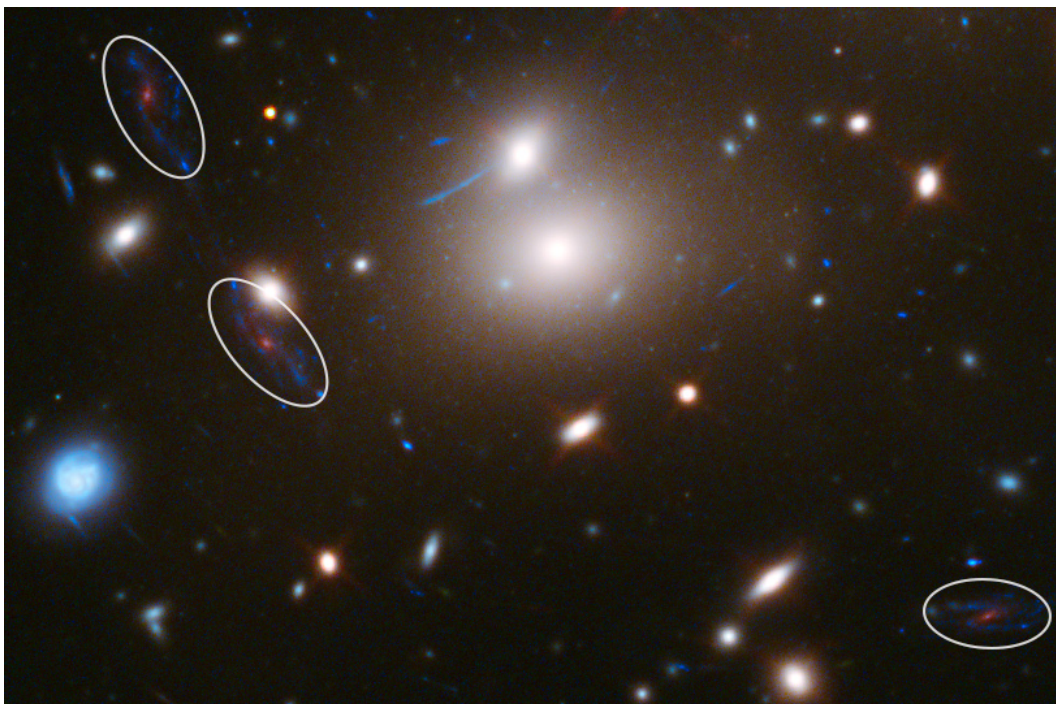


Figure 4: The same galaxy appearing 3 times in an image captured by Hubble.

For example, if we have a point mass at the origin $n = 1, z_1 = 0$. The lens equation gives us images of it at $z = \frac{\sigma_1}{z}$, i.e. $|z|^2 = \sigma_1$. The solution is a circle, known as an *Einstein ring*.

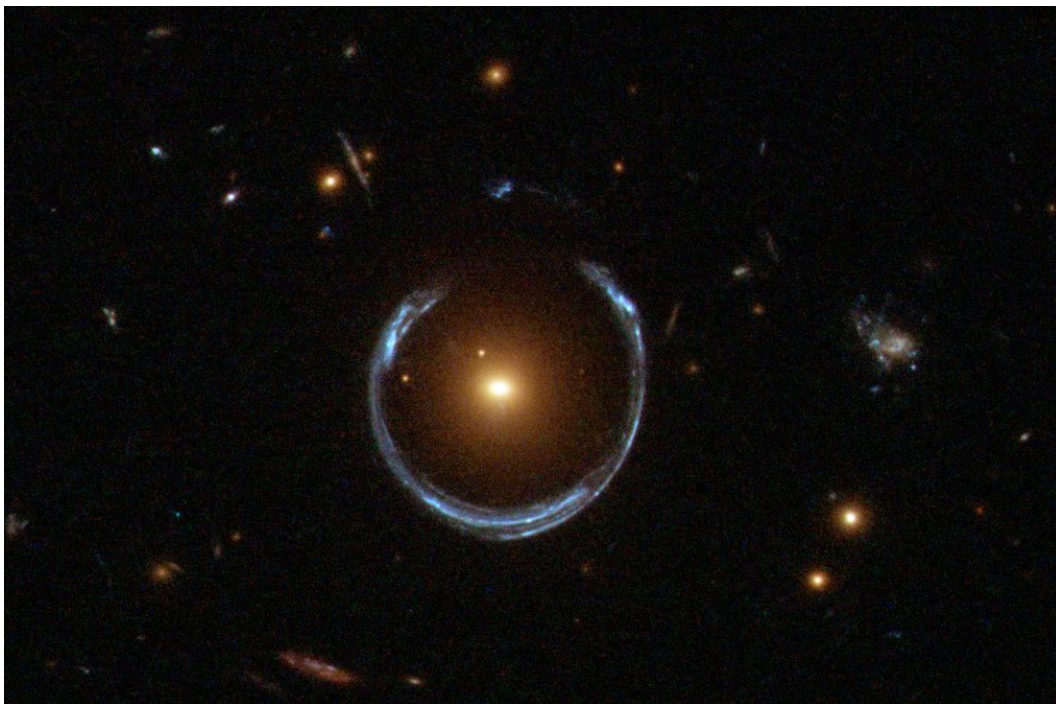


Figure 5: An Einstein ring, captured by Hubble.

In general, we might want to ask how many images we will see for a give configuration, or perhaps a given number, of point masses.

Define $r(z) := \sum_{j=1}^n \frac{\sigma_j}{z-z_j}$, and we are looking for solutions to $z = \overline{r(z)}$, i.e. fixed points of the (non-meromorphic) function $z \mapsto \overline{r(z)}$. These will be among the solutions of the (meromorphic) function $z \mapsto \overline{r(\overline{r(z)})} = r(r(z))$, of degree n^2 . One can show that it suffices to bound the number of *attracting* fixed points, which will be done in one of the main theorems of this course:

Theorem 1.1. *Let $f(z)$ be a rational map of degree $d \geq 2$. Then f has at most $2d - 2$ attracting fixed points.*

There are still open questions related to this application; for example, the following conjecture is still open despite its simplicity:

Conjecture 1.2 ((Lee, Lerario, Lundberg)). *If p is a polynomial of degree n , q a polynomial of degree m , and $n > m$, then the number of solutions to $p(z) = \overline{q(z)}$ is bounded above by $2m(n - 1) + n$.*

1.3 Where We're Heading

Define $f_c(z) := z^2 + c$, where $c \in \mathbb{C}$. **The Mandelbrot Set** $M := \{c \in \mathbb{C} : |f_c^n(0)| \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$.

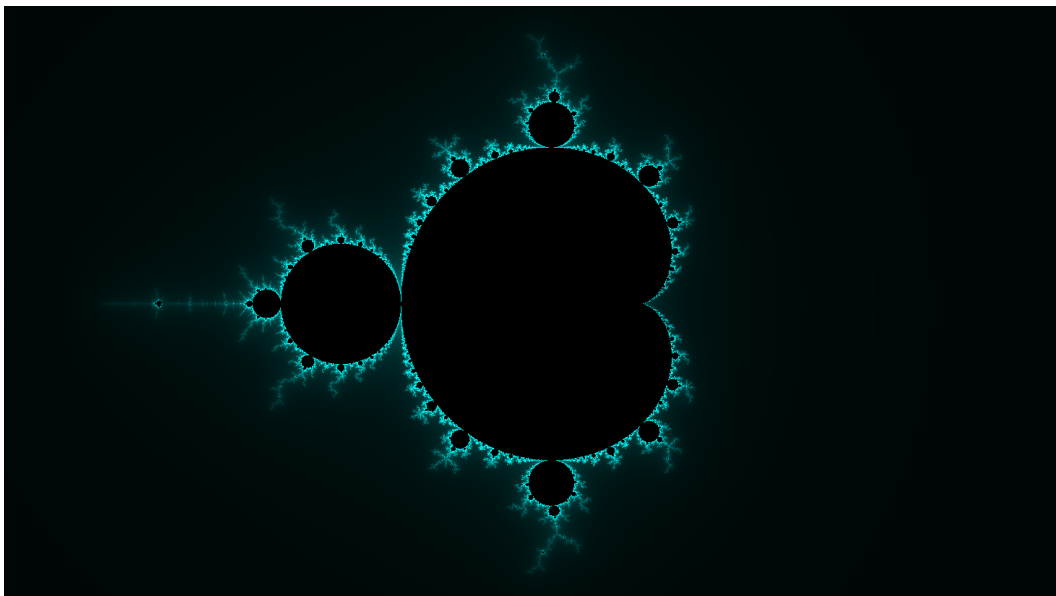


Figure 6: The Mandelbrot set, generated by me.

2 Complex Basics

2.1 The Riemann Sphere

The *Riemann Sphere* $\widehat{\mathbb{C}}$ is the one-point compactification of \mathbb{C} , given by adding a point at ∞ . The basis of this topology around ∞ are the complements of the closed discs in \mathbb{C} .

Definition 2.1. A **domain** in the Riemann sphere is any open connected subset of $\widehat{\mathbb{C}}$.

We can then think about *maps of the Riemann sphere*, which are rational maps of the form $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials with no common roots. We define the *degree* of f to be the maximum of $\deg p, \deg q$.

Definition 2.2. We say that $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is **holomorphic at ∞** if $f(1/z)$ is meromorphic on a neighbourhood of 0.

Lemma 2.3 (Identity Principle). Let $f : D \rightarrow D$ be a holomorphic, non-constant map on a domain D in $\widehat{\mathbb{C}}$. Then for any $w \in D$, $f^{-1}(w)$ is discrete. In particular, if $D = \widehat{\mathbb{C}}$, then $f^{-1}(w)$ is finite.

Proof. First, composing with a Möbius map, we may assume $w = 0$. By the principle of isolated zeros, $\forall z \in \widehat{\mathbb{C}}$, there is some disc $D_z \ni z$ such that either $f \equiv 0$ on D_z or f is nonzero on $D_z \setminus \{z\}$.

Let $V := \bigcup_{f=0 \text{ on } D_z} D_z$, $W := \bigcup_{f \neq 0 \text{ on } D_z} D_z$

Then $V \cup W = D$, these sets are open, and $V \cap W = \emptyset$. Since D is connected, V and W cannot disconnect D , so one of V and W is empty. f is nonconstant, so $V = \emptyset$, and so the zeros are discrete. As $\widehat{\mathbb{C}}$ is compact, any discrete subset is finite. \square

Proposition 2.4. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a holomorphic, non-constant map. Then f is a non-constant rational function; that is, there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{C}$, and $c \in \mathbb{C}^\times$ such that:

$$f(z) = \frac{c(z - a_1) \dots (z - a_m)}{(z - b_1) \dots (z - b_n)}$$

Proof. Replace by $\frac{1}{f}$ if necessary to assume $f(\infty) \in \mathbb{C}$. By the identity principle, $f^{-1}(\infty)$ is a finite subset of $\widehat{\mathbb{C}}$. Call it $\{b_1, \dots, b_n\}$. Near each b_j , we have a local expansion of f as $f(z) = \sum_{i=-k}^{\infty} a_{j,i}(z - b_j)^i$. Set $Q_j(z) = \sum_{i=-k}^{-1} a_{j,i}(z - b_j)^i$, each Q_j is a rational function and $g := f - (Q_1 + \dots + Q_n)$ has only removable singularities. So $g(\widehat{\mathbb{C}}) \subseteq \mathbb{C}$. Since $\widehat{\mathbb{C}}$ is compact, $g(\widehat{\mathbb{C}})$ is a compact, proper, open subset of $\mathbb{C} \setminus \{0\}$ (by Heine-Borel). Therefore g is constant, and f is rational. \square

2.2 The Derivative at Infinity

Caution: there is not a globally well defined notion of a derivative of $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, since we can change coordinates in many different ways. However, if $f(\infty) = \infty$, then we do, as the local derivative is independent of choice of coordinates. In this case, we define $f'(\infty)$ to be the derivative at 0 of $\frac{1}{f(1/z)}$.

2.3 Uniformizations

We say a group Γ acts *properly discontinuously* on a set if all points p have a neighbourhood $N \ni p$ with $g(N) \cap h(N) = \emptyset$ for all $g \neq h \in \Gamma$

Theorem 2.5 (Uniformization in $\widehat{\mathbb{C}}$). Let U be a domain in $\widehat{\mathbb{C}}$. Then the universal cover \widetilde{U} of U is either $\widehat{\mathbb{C}}$, \mathbb{C} , or \mathbb{D} , and $U \simeq \widetilde{U}/\Gamma$ for some subgroup $\Gamma \leq \text{Aut}(\widetilde{U})$ which acts properly discontinuously.

We now consider the different cases:

- Universal Cover $\widehat{\mathbb{C}}$: $\text{Aut}(\widehat{\mathbb{C}}) = \{\text{Möbius transformations}\}$

All Möbius transformations have at least one fixed point. So if Γ acts properly discontinuously on $\widehat{\mathbb{C}}$, Γ must be $\{\text{id}\}$, and as such $U = \widetilde{U} = \widehat{\mathbb{C}}$.

- Universal Cover \mathbb{C} : $\text{Aut}(\mathbb{C}) = \{\text{Möbius transformations fixing } \infty\}$

These transformations are the linear maps, i.e. $f(z) = az + b, a \neq 0$. If a non-identity map lies in Γ , then it has no fixed point, and hence $az + b = z$ has no solutions, i.e. $a = 1, b \neq 0$. So Γ consists of the translations, and we can identify $g \in \Gamma$ with $g(0) \in \mathbb{C}$.

$\Gamma \leq \mathbb{C}$ consists of isolated points, so $\Gamma = \{\text{id}\}, \langle \omega \rangle$ for some $\omega \in \mathbb{C}$, or a lattice Λ .

These correspond respectively to $U \simeq \mathbb{C}, U \simeq \mathbb{C}^\times$, and a complex torus (which is not a subdomain of \mathbb{C}).

- Universal Cover \mathbb{D} : $\text{Aut}(\mathbb{D}) = \{e^{i\theta} \frac{z - \omega}{1 - \bar{\omega}z} : \theta \in \mathbb{R}, \omega \in \mathbb{D}\}$.

All the rest! Any domain which omits at least three points of $\widehat{\mathbb{C}}$ has the disc as its universal cover. This gives a lot of different subgroups of $\text{Aut}(\mathbb{D})$ - we won't study which ones are possible in this course.

Definition 2.6. We say a domain $U \subseteq \widehat{\mathbb{C}}$ is **hyperbolic** if it has universal cover \mathbb{D} , or equivalently if it omits at least 3 points.

If U is a hyperbolic domain, then the covering maps $\widetilde{U} \rightarrow U$ are holomorphic. So if $f : U \rightarrow V$ is a holomorphic map between hyperbolic domains, then its lifted map g is also holomorphic. g is also unique up to a choice of an element of Γ_V

Theorem 2.7 (Picard Theorem). *If U is a hyperbolic domain, and $f : \mathbb{C} \rightarrow U$ is holomorphic, then f is constant.*

Proof. The following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g} & \mathbb{D} \\ \downarrow \text{id} & & \downarrow \pi_U \\ \mathbb{C} & \xrightarrow{f} & U \end{array}$$

So g is a holomorphic function from \mathbb{C} to \mathbb{D} , so bounded, so constant by Liouville's theorem. Hence f must also be constant. \square

In fact, there are no non-constant holomorphic maps from $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$ to any hyperbolic domain.

Recall that the hyperbolic (or Poincaré) distance between any two points on the disc is the infimum along smooth curves $\gamma : [a, b] \rightarrow \mathbb{D}$ which connect the two points of the length value:

$$\ell(\gamma) = \int_a^b \frac{2}{1 - |\gamma(t)|^2} |\gamma'(t)| dt$$

Using the origin and automorphisms, one computes that the distance function is given by

$$d(x, y) = \frac{\log(1 + R)}{\log(1 - R)}$$

where $R = \left| \frac{y-x}{1-\bar{x}y} \right|$.

Definition 2.8 (Hyperbolic/Poincaré metric of a hyperbolic domain). *Taking the covering map $\pi : \mathbb{D} \rightarrow U$ for U a hyperbolic domain to be a local isometry, we define a unique metric on U known as the hyperbolic/Poincaré metric on U .*

Theorem 2.9 (Pick Theorem). *The $f : U \rightarrow V$ be a holomorphic map of hyperbolic domains $U, V \subset \widehat{\mathbb{C}}$ with Poincaré metrics ρ_U, ρ_V respectively. Then for all $x, y \in U$:*

$$\rho_V(f(x), f(y)) \leq \rho_U(x, y)$$

with strict inequality unless f lifts to an automorphism of \mathbb{D} .

Proof. By definition of ρ_U, ρ_V , it suffices to prove this for $f : \mathbb{D} \rightarrow \mathbb{D}$. We want to show that

$$\frac{\log(1 + R')}{\log(1 - R')} \leq \frac{\log(1 + R)}{\log(1 - R)}$$

where

$$R = \left| \frac{y - x}{1 - \bar{x}y} \right|, \quad R' = \left| \frac{f(y) - f(x)}{1 - \overline{f(x)}f(y)} \right|$$

As $\frac{\log(1+x)}{\log(1-x)}$ is strictly increasing, it is sufficient to show $R' \leq R$. Define:

$$\sigma_1(z) = \frac{y - z}{1 - \bar{y}z}, \quad \sigma_2(z) = \frac{f(y) - f(z)}{1 - \overline{f(y)}f(z)}$$

Then $\sigma_2 \circ f \circ \sigma_1^{-1} : \mathbb{D} \rightarrow \mathbb{D}$ fixes 0. So by Schwarz's lemma, $|\sigma_2 \circ f \circ \sigma_1^{-1}(z)| \leq |z|$, i.e.

$$\left| \frac{f(y) - f(x)}{1 - \overline{f(y)}f(x)} \right| \leq \left| \frac{y - x}{1 - \bar{y}x} \right|$$

by setting $\sigma_1^{-1}(z) = x$. The statement with strict inequality follows similarly. \square

Thus all maps of hyperbolic domains are (weakly) contracting with respect to the hyperbolic metric. This is in contrast to the non-hyperbolic case, where we can have things like $f : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto 2z$ which is expanding, or $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}, z \mapsto z + 1$, which has expansion and contraction at the same time in different regions.

3 What is a reasonable notion of stable/predictable behaviour under iteration?

For example, take $f(z) = z^2$, so that $f^n(z) = z^{2^n}$. If $|z| > 1$, then $|f^n(z)| = |z|^{2^n} \rightarrow \infty$. On the other hand, if we start with a point inside the unit disc, then $|f^n(z)| \rightarrow 0$. This is a stable property - if z goes off to ∞ or 0 under iteration of f , then $z + \epsilon$ for $|\epsilon|$ small also goes off towards ∞ or 0 respectively.

In contrast, if $|z| = 1$, then we do not have this stable behaviour.

Definition 3.1. Let S, T be metric spaces and $f_n : S \rightarrow T$ a sequence of continuous maps. We say that $\{f_n\}$ **converges locally uniformly** if for all compact $K \subset S$ and all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup_{z \in K} d_T(f_n(z), f_m(z)) < \epsilon$$

for all $n, m \geq N$.

We say that $\{f_n\}$ **diverges locally uniformly** if for all compact $K \subset S$ and compact $K' \subset T$, we have $N \in \mathbb{N}$ such that $f_n(K) \cap K' = \emptyset$ for all $n \geq N$.

Example: Let $p(z) = a_d z^d + \dots + a_0 \in \mathbb{C}[z]$ a polynomial, with sequence of iterates $\{p^n(z) : n \in \mathbb{N}\}$. If $|z| \gg \max\{|a_d|, \dots, |a_0|\}$, then $|p(z)| \approx |a_d z^d|$. So $|p^n(z)| \rightarrow \infty$ as $n \rightarrow \infty$. Considering $p : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, the sequence $\{p^n(z)\}$ converges locally uniformly on a neighbourhood of ∞ .

Note that the definition of S, T matter - if we consider $p : \mathbb{C} \rightarrow \mathbb{C}$ instead then the sequence of iterates $\{p^n(z)\}$ diverges locally uniformly on the complement of some large disc.

Definition 3.2. A family \mathcal{F} of continuous functions $f : S \rightarrow T$ is **normal** if every sequence in \mathcal{F} has a subsequence which is either locally uniformly convergent or locally uniformly divergent.

Lemma 3.3. Let U, V be domains and \mathcal{F} a normal family of holomorphic maps $U \rightarrow V$. Then any sequence $\{f_n\}$ in \mathcal{F} has a subsequence which converges to a holomorphic limit $g : U \rightarrow \bar{V}$, where \bar{V} is the closure of V in the Riemann sphere.

Proof. \mathcal{F} is normal, so pass to a subsequence of holomorphic maps to assume locally uniform convergence or divergence.

If $\{f_n\}$ converges locally uniformly, then \bar{V} compact implies we have a limit function $g : U \rightarrow \bar{V}$, by taking $K = \{\text{pt}\}$ in the definition. By the Weierstrass uniform convergence theorem, this is holomorphic.

If $\{f_n\}$ diverges locally uniformly, then for any compact $K \subset U$, we have some convergent subsequence $f_{n_k}(z) \rightarrow z_0 \in \partial V$ for any choice of $z \in K$. If U, V are hyperbolic, then by Pick's theorem, all elements of K converge to z_0 , and so we have local uniform convergence to the constant function at z_0 . If U, V are not hyperbolic, we may shrink U slightly to still contain K , but be hyperbolic U_0 with compact closure in U . By local divergence, we can remove points from V and initial elements of the sequence $\{f_n\}$ to assume V is hyperbolic. \square

Example: $f(z) = z^2$. $\mathcal{F} := \{f^n(z) : n \in \mathbb{N}\}$.

$\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$ is a hyperbolic domain, and $f : \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$, and the iterates $\{f^n(z)|_{\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}}\}$ converge locally uniformly to the constant map at ∞ . Similarly, $\{f^n(z)|_{\mathbb{D}}\}$ converge locally uniformly to the constant map at 0.

These sets are maximal for these behaviours, i.e. $\{f^n\}$ is a normal family on \mathbb{D} and on $\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$ but on no larger set. If $|z_0| = 1$, then we have for any neighbourhood of z_0 , some points go to infinity and some go to zero, and so there can be no holomorphic limit of iterates on any neighbourhood of z_0 .

4 A More Useful Criterion for Normality

Definition. A family \mathcal{F} of continuous functions $S \rightarrow T$ is *equicontinuous* if, for all $\epsilon > 0$ there is $\delta > 0$ such that, for all $f \in \mathcal{F}$ and all $s, t \in S$:

$$d_S(s, t) < \delta \implies d_T(f(s), f(t)) < \epsilon$$

Theorem 4.1 (Arzelà-Ascoli in $\widehat{\mathbb{C}}$). A family $\mathcal{F} : S \rightarrow T$ of continuous maps of domains $S, T \subset \widehat{\mathbb{C}}$ has the property that all sequences have a locally uniformly convergent subsequence if and only if the following are both satisfied:

- \mathcal{F} is equicontinuous on any compact subset of S , and
- For all $z \in S$, $\{f(z) : f \in \mathcal{F}\}$ lies in a compact subset of T .

Proof. The forwards direction is left as an exercise.

For the reverse direction, since $S, T \subseteq \widehat{\mathbb{C}}$, we can fix a countable dense subset $\{z_k\}$ of S . Fix a sequence $\{f_n\}$ of elements of \mathcal{F} . We can find a set of indices $\{n_{ij}\}$ such that:

$$\begin{aligned} n_{11} &< n_{12} < n_{13} < \dots < n_{1j} < \dots \\ n_{21} &< n_{22} < n_{23} < \dots < n_{2j} < \dots \\ &\vdots \\ n_{i1} &< n_{i2} < n_{i3} < \dots < n_{ij} < \dots \end{aligned}$$

so that each row is contained in the previous as a subsequence, and $\lim_{j \rightarrow \infty} f_{n_{kj}}(z_k)$ exists for each k .

The diagonal sequence $g_k := f_{n_{kk}}$ is strictly increasing in index, and for each k , eventually contained in the k^{th} row. So g_k converges at each z_k .

Consider now K , a compact subset of S . Given $\epsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in K$ and all g_k , $d_S(x, y) < \delta \implies d_T(g_k(x), g_k(y)) < \epsilon/3$.

Cover K by finitely many $\delta/2$ diameter neighbourhoods, selecting some z_k for each of these neighbourhoods.

Given $z \in K$, we have some z_k with δ , and there are only finitely many z_k in this set, so in fact we can find N such that $n, m > N$ implies $d_T(g_n(z_k), g_m(z_k)) < \frac{\epsilon}{3}$.

So we have:

$$\begin{aligned} d_T(g_n(z), g_m(z)) &\leq d_T(g_n(z), g_m(z_k)) + d_T(g_n(z_k), g_m(z_k)) + d_T(g_m(z_k), g_m(z)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

Since this holds for all $z \in K$, we have locally uniform convergence as claimed. \square

Corollary 4.1. *If T is compact, a family of continuous maps from S to T is normal if and only if it is equicontinuous on compact subsets.*

This brings us to the critical takeaway applicable to dynamics:

Theorem 4.2 (Montel's theorem). *Let U, V be domains with V hyperbolic. Then the family of holomorphic maps from U to V forms a normal family.*

Proof. Suppose first that U is hyperbolic. By Pick's theorem, the family is equicontinuous.

If $\exists x \in U$ such that $\{f(x) : f \in \mathcal{F}\}$ is in some compact $K \subseteq V$, then again by Pick's theorem, the same is true for all points in U . So by Arzelà-Ascoli, we have a normal family.

Otherwise, we can fix $x \in U, y \in V$, and an infinite sequence $\{f_n\} \subseteq \mathcal{F}$ such that $\rho_V(f_n(x), y) \rightarrow \infty$ as $n \rightarrow \infty$. Fix compact $K \subseteq U, K' \subseteq V$, noting that $\rho_U(x, K), \rho_V(y, K')$ are finite, we have by Pick's theorem that $f_n(K) \cap K' = \emptyset$ for all n sufficiently large. So this sequence diverges locally uniformly, and again we have a normal family.

In the case where U is not hyperbolic, then \mathcal{F} consists of constant maps.

If there is a subsequence with images in a compact set, then we have a locally uniformly convergent subsequence.

If not, then only finitely many of these maps have image in any compact set - we have locally uniform divergence. \square

Example: $f(z) = z^2$. We've seen that $\{f^n(z) : n \in \mathbb{N}\}$ form a normal family on \mathbb{D} and $\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$, by manual computation. But since $f^n(\mathbb{D}) \subseteq \mathbb{D}$, $f^n(\widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}) \subseteq \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$, the iterates of f must form a normal family on these domains because the image spaces are hyperbolic.

More generally, if $U \subseteq \widehat{\mathbb{C}}$ is a hyperbolic domain, either $\{f^n(U) : n \in \mathbb{N}\}$ covers all but at most 2 points of $\widehat{\mathbb{C}}$, or the iterates of f form a normal family on U .

5 Degree and Ramification

Definition 5.1. A continuous map $f : U \rightarrow V$ of topological spaces is **proper** if for all $K \subset V$, $f^{-1}(K)$ is compact in U .

Lemma 5.1. Let $f : U \rightarrow V$ be continuous. If U is compact and V is Hausdorff, then f is proper.

Proof. Let $K \subseteq V$ be compact. V is Hausdorff $\implies K$ is closed, so $V \setminus K$ is open $\implies f^{-1}(V \setminus K)$ is open (as f is continuous). So $f^{-1}(K) = U \setminus f^{-1}(V \setminus K)$ is closed, and so compact as it is a closed subset of a compact set. So f is proper. \square

Recall from complex analysis the concept of local degree/ramification: if f is holomorphic on a neighbourhood of z_0 , with $f(z_0) = w_0$, then locally we have an expansion

$$f(z) = w_0 + (z - z_0)^e g(z - z_0)$$

where $g(z - z_0)$ is nonzero at z_0 , and holomorphic on a neighbourhood of z_0 . Then e is the local degree/ramification index at z_0 . If $e > 1$, then z_0 is said to be critical.

Let U, V be domains in $\widehat{\mathbb{C}}$.

Proposition 5.1. If $f : U \rightarrow V$ is a proper holomorphic map, then f has a well defined degree on U , equal to

$$\sum_{z \in U: f(z)=w} e_z$$

for any $w \in V$, where e_z is the ramification index of f at z .

Note that this is independent of w , and by Lemma 5.1, all rational maps have well-defined degree!

Theorem 5.1 (Riemann-Hurwitz). Let $f : U \rightarrow V$ be proper of degree d . Then

$$\chi(U) = d\chi(V) - \sum_{p \in V} (e_p - 1)$$

where e_p is the ramification index of f at p and χ is the Euler characteristic.

We will take both of these theorems for granted - they follow immediately from the corresponding statements for compact sets.

Corollary 5.2. Any degree d rational function f has $2d - 2$ critical points, counting multiplicity.

Proof. $\chi(\widehat{\mathbb{C}}) = 2$, and use Riemann-Hurwitz. \square

Corollary 5.3. If U, V have $\chi(U) = \chi(V) = 0$, (e.g. annuli, punctured discs), then any proper holomorphic $f : U \rightarrow V$ is unramified.

6 Julia Sets

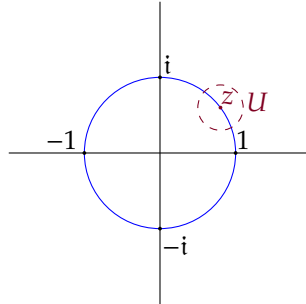
Definition 6.1 (Fatou Set, Julia Set). Let f be a nonconstant rational map. The **Fatou set** of f is:

$$F(f) := \{x \in \widehat{\mathbb{C}} : \exists \text{ a neighbourhood } U \ni x \text{ on which } \{f^n|_U(z)\} \text{ form a normal family}\}$$

The **Julia set** of f is $J(f) := \widehat{\mathbb{C}} \setminus F(f)$.

Intuitively, $F(f)$ is the stable locus of f , and $J(f)$ is the chaotic or unstable locus of f .

For example, take $f(z) = \lambda z$ for $\lambda = e^{2\pi i \theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\{f^n(z)\}$ forms a normal family on any disc centered at 0, because $\{f^n|_{\text{disc}}\}$ is a family of self-maps of hyperbolic domains, so we can use Montel's theorem. Note that $f^n(z)$ doesn't converge to any point, but it does have convergent subsequences.



From now on in the course, we will usually only consider rational maps of degree at least 2. In the case of Möbius transformations, the Fatou and Julia sets can be explicitly calculated - see example sheet 1.

6.1 Basic Properties of $J(f)$

It immediately follows from the definition that $F(f)$ is open, and so $J(f)$ is closed and compact in $\widehat{\mathbb{C}}$.

Lemma 6.2. The Fatou and Julia sets of f are totally invariant under f . That is, $f(J(f)) = J(f) = f^{-1}(J(f))$, and similarly for $F(f)$.

Proof. It suffices to show that $f^{-1}(F(f)) = F(f)$, and so to prove that, for any open $U \subseteq \widehat{\mathbb{C}}$, a sequence of iterates $f^{n_k}(z)$ converges uniformly on compact subsets of U if and only if $f^{n_k+1}(z)$ converges uniformly on compact subsets of $f^{-1}(U)$.

We have:

$$\sup_{z \in K} d(f^{n_j+1}(z), f^{n_k+1}(z)) = \sup_{w \in f(K)} d(f^{n_j}(w), f^{n_k}(w))$$

Since f is proper and continuous, preimages and images of compact sets are compact, and so we have the desired equivalence. \square

This tells us why we get self similarity in Julia sets: if $z \in J(f)$ and $f'(z) \neq 0$, then there is a neighbourhood N of z such that $f : N \rightarrow f(N)$ is a conformal equivalence. Since $N \cap J(f) \xrightarrow{f} f(N) \cap J(f)$, we have self-similarity.

Lemma 6.3. $J(f) = J(f^n)$ for any iterate f^n of f , and similarly for the Fatou set.

Lemma 6.4. Let f be a rational map and μ a Möbius transformation, we define $g := \mu \circ f \circ \mu^{-1}$. Then $\mu(J(f)) = J(g)$.

The proofs of these are exercises on example sheet 1.

6.2 Conjugation and Dynamics

In general, if μ is a Möbius map, and f is rational, with $g := \mu \circ f \circ \mu^{-1}$, then $g^n = \mu \circ f^n \circ \mu^{-1}$.

As an example, let f be a rational map of degree 2 with two fixed critical points. If ∞ is fixed, and ∞ maps to itself with degree 2, then we know ∞ has no other preimages, and we have a polynomial of degree 2. Let's conjugate by μ , a Möbius map taking one of the fixed critical points to 0, and the other to ∞ .²

So we are conjugate to a degree 2 polynomial with 0 fixed and critical, i.e. $f(z) = az^2$. Conjugating again by $\mu(z) = az$, which doesn't change 0 or ∞ , we have a map conjugate to $g(z) = z^2$. This has the unit circle as its Julia set.

Therefore f 's Julia set is a Möbius image of the unit circle, i.e. a circle.

Theorem 6.5. Let $z \in J(f)$ and U be an open neighbourhood of z . Then the union $V := \bigcup f^n(U)$ of forward images of U contains all but at most two points of the Riemann sphere, both of which are critical points in the Fatou set.

Proof. If $\{f^n|_U\}$ omit at least 3 points of $\widehat{\mathbb{C}}$, then Montel's theorem tells us we have a normal family on U , and so $U \subseteq F(f)$. But $U \cap J(f) \neq \emptyset$.

If $\bigcup f^n(U)$ misses any points, then we want to show they are Fatou and critical. Without loss of generality, assume these points are in $\{0, \infty\}$. Since $f(V) \subseteq V$, no preimage of 0 or ∞ lies in V .

So the only possible behaviours are:

1. $0 \mapsto 0, \infty \mapsto \infty$, both with degree $d = \deg f \geq 2$
2. $0 \mapsto \infty, \infty \mapsto 0$, both with degree d
3. $\infty \mapsto \infty$ with degree d

²Notice that being fixed and critical are conjugation invariant

4. No points in this set.

Hence all points in the set are critical, as they are images of ramification degree $d > 1$.

In case 1, $f(z) = az^d$, conjugate to z^d , with $0, \infty$ Fatou.

In case 2, $f(z) = \frac{a}{z^d}$, conjugate to z^{-d} , with Julia set a circle again, so $0, \infty$ Fatou.

In case 3, $f(z)$ is a polynomial map, and so ∞ is Fatou. \square

Corollary 6.6. *If the Julia set contains an interior point, then $J(f) = \widehat{\mathbb{C}}$.*

Proof. By the previous theorem and forward invariance of the Julia set, if U is open and contained in $J(f)$, then $\bigcup f^n(U) \subseteq J(f)$ contains all but at most two points of $\widehat{\mathbb{C}}$. But $J(f)$ is closed, hence $J(f) = \widehat{\mathbb{C}}$. \square

The Julia set is the smallest closed totally invariant set in $\widehat{\mathbb{C}}$ containing at least 3 points.

Note that $J(f) = \widehat{\mathbb{C}}$ can happen, but not with polynomial maps. For instance, take $f(z) = \frac{1}{1-2z^2}$ - by the end of the course we'll be able to prove this function is chaotic everywhere.

On the other hand, we will show soon that $J(f) \neq \emptyset$.

7 Cycles and Multipliers

Definition 7.1. *Let $z_0 \in \widehat{\mathbb{C}}$. We say that z_0 is **periodic** or in a **periodic cycle** for f if $f^m(z_0) = z_0$ for some $m \in \mathbb{N}$. The minimal such m is the period of the cycle. If z_0 has period m , then the derivative*

$$\lambda := (f^m)'(z_0) = \prod_{i=0}^{m-1} f'(f^i(z_0))$$

*is called the **multiplier** of the cycle.*

Example: Take $p(z) = a_d z^d + \dots + a_0$ a polynomial, so that $p(\infty) = \infty$. Then ∞ is in a cycle of period 1. By example sheet 1, $p'(\infty) = \lim_{z \rightarrow \infty} \frac{1}{p'(z)} = 0$. Hence ∞ is a fixed point of multiplier zero for any polynomial.

We classify periodic cycles according to their multiplier:

Definition 7.2. *Let z_0 be a point of period m for f with multiplier λ . Then we say z_0 is:*

- **superattracting** if $\lambda = 0$
- **attracting** if $|\lambda| < 1$
- **indifferent** if $|\lambda| = 1$
- **repelling** if $|\lambda| > 1$

Note that superattracting \implies attracting.

If $g = \mu \circ f \circ \mu^{-1}$ for μ a Möbius map, and z_0 has period m for f with multiplier λ , then $\mu(z_0)$ has period m for g , again with multiplier λ .

Definition 7.3. Suppose $C := \{z_0, \dots, z_{m-1}\}$ is an attracting orbit of period m . Then the **basin of attraction** for the cycle C is the set:

$$\mathcal{A} := \{z \in \widehat{\mathbb{C}} : \lim_{n \rightarrow \infty} f^{nm}(z) \in C\}$$

For example, take $f(z) = z^2$, $f'(z) = 2z$. Then if z is a periodic point with period m , then $z = z^{2^m}$. These points are:

- 0
- ∞
- Roots of unity of the form $z^{2^m-1} = 1$

These points respectively have multipliers:

- $\lambda = 2 \cdot 0 = 0$, and the basin of attraction is $|z| < 1$.
- $\lambda = \lim_{z \rightarrow \infty} \frac{1}{2z} = 0$, and the basin of attraction is $|z| > 1$.
- $|\lambda| = 2|z| = 2$, so these are repelling.

Theorem 7.4. Every attracting basin is contained in the Fatou set. Any repelling periodic point is contained in the Julia set.

Proof. Since $J(f) = J(f^m)$ for $m \in \mathbb{N}$, we may assume that z_0 is a fixed point of f with multiplier λ . Suppose that $|\lambda| < 1$. Then locally, by Taylor's theorem, we have $|f(z) - z_0| \leq c|z - z_0|$ for any constant $|\lambda| < c < 1$. So we have uniform convergence of the iterates on compact subsets on a neighbourhood of z_0 to the constant function $z \mapsto z_0$. So $z_0 \in F(f)$ as claimed.

If $z \in J(f)$, with $\lim_{n \rightarrow \infty} f^n(z) = z_1$ say, then $z_1 \in J(f)$. But by above, $z_1 \in F(f)$. Hence the attracting basin of an attracting point is also Fatou.

Suppose $|\lambda| > 1$. The derivative at z_0 of the iterates $f^k(z)$ is $\lambda^k \rightarrow \infty$ as $k \rightarrow \infty$. So there is no equicontinuity of $\{f^k\}$ on any neighbourhood of z_0 , and so by Arzelà-Ascoli, $\{f^k\}$ is not normal on a neighbourhood of z_0 . Hence $z_0 \in J(f)$. \square

The derivative at fixed points is a local property, but it turns out there is still some global constraints on the multipliers of fixed points.

Definition 7.5. Let z_0 be a fixed point of a rational map f of degree d . The **residue index** of f at z_0 is

$$i_f(z_0) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - f(z)}$$

where γ is any small positively oriented loop around z_0 .

Note: a degree d map has $d + 1$ fixed points.

Lemma 7.6. Let λ be the multiplier of a fixed point z_0 of f . If $\lambda \neq 1$, then

$$i_f(z_0) = \frac{1}{1 - \lambda}$$

Proof. Without loss of generality, assume that $z_0 = 0$ or $z_0 = \infty$, after noticing that both the residue and the multiplier are invariant under conjugation by a translation.

If $z_0 = 0$, then $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ near $z_0 = 0$. Since $\lambda \neq 1$, we have

$$\begin{aligned} z - f(z) &= (1 - \lambda)z(1 + b_1 z + b_2 z^2 + \dots) \\ \implies \frac{1}{z - f(z)} &= \frac{1}{(1 - \lambda)z} \frac{1}{1 + b_1 z + b_2 z^2 + \dots} \\ &= \frac{1}{(1 - \lambda)z} (1 + c_1 z + c_2 z^2 + \dots) \\ &= \frac{1}{(1 - \lambda)z} + g(z) \end{aligned}$$

where $g(z)$ is holomorphic. Integrating both sides causes the $g(z)$ to disappear, and so by the residue theorem we have the result.

If $z_0 = \infty$, a similar computation works. □

On example sheet 1, there is an exercise to show that the residue index is conjugation invariant by any Möbius map. Note that this lemma deals with every case except for when $\lambda = 1$.

Theorem 7.7 (Holomorphic Lefschetz on the Riemann Sphere). *Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree at least 2. Then*

$$\sum_{z=f(z)} i_f(z) = 1$$

Proof. Assume without loss of generality that ∞ is not a fixed point. Let C_R denote the circle of radius R about 0, positively oriented, and suppose that it is large enough to strictly contain all the fixed points of f . Then with $w = \frac{1}{z}$:

$$\int_{C_R} \frac{dz}{z - f(z)} = \int_{-C_{1/R}} \frac{-dw}{w^2 \left(\frac{1}{w} - f\left(\frac{1}{w}\right) \right)} = \int_{C_{1/R}} \frac{dw}{w \left(1 - w f\left(\frac{1}{w}\right) \right)}$$

The nonzero poles of the integral are reciprocals of the fixed points of f , which by construction lie outside $C_{1/R}$. So, by the residue theorem, :

$$\sum_{z=f(z)} i_f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{dz}{z - f(z)} = \operatorname{Res}_{w=0} \left[\frac{1}{w \left(1 - w f\left(\frac{1}{w}\right) \right)} \right] = 1$$

□

7.1 Structure of the Julia Set

We have shown already that $J(f)$ is compact, and has no interior points unless $J(f) = \widehat{\mathbb{C}}$. In this section we will show some other structural properties, starting with the following corollary of 7.7

Corollary 7.8. *A rational map f of degree $d \geq 2$ has a nonempty Julia set.*

Proof. Suppose that $J(f) = \emptyset$, and f is of degree $d \geq 2$, so that f has $d + 1$ fixed points, counting with multiplicity.

Since any repelling fixed point lies in $J(f)$, we can assume that $|\lambda| \leq 1$ for all fixed points of f . Suppose first that no fixed point has multiplier 1. Then the derivative of $f(z) - z$ is $f'(z) - 1$, so we have all of the $d + 1$ fixed points of f distinct, as $f'(z) \neq 1$. Then all multipliers lie in \mathbb{D} , and the map $x \mapsto \frac{1}{1-x}$, so that $\lambda(z) \mapsto i_f(z)$, is a Möbius transformation taking \mathbb{D} to $\{z : \Re(z) \geq \frac{1}{2}\}$. So the sum of the $i_f(z)$ has real part $\geq \frac{d+1}{2} > 1$, violating 7.7.

Suppose now that $\lambda = 1$ for some fixed point $f(z_0) = z_0$ - wlog, move it to 0. Then locally, $f(z) = z + a_k z^k + \dots$, $a_k \neq 0$, and so:

$$f^n(z) = z + n a_k z^k + \dots$$

Hence the k^{th} derivative of f^n at z_0 tends to ∞ as $n \rightarrow \infty$, and so we don't have normality on any neighbourhood of z_0 , and so $z_0 \in J(f)$. \square

In fact, we have proved something slightly stronger. We've proved that:

1. Any periodic cycle with $\lambda^k = 1$ for some $k \in \mathbb{N}$ is contained in $J(f)$.

If λ is a multiplier with $\lambda^k = 1$ for some $k \in \mathbb{N}$, f^k has a periodic cycle with $\lambda = 1$, and so the cycle is contained in $J(f^k)$. Since $J(f) = J(f^k)$, any such cycle is contained in $J(f)$.

2. $J(f)$ is infinite.

We produced a fixed point z_0 with $|\lambda| \geq 1$ and $z_0 \in J(f)$. Consider $S = \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(z_0) \subseteq J(f)$. If this is finite, then since all elements' preimages are still in the set, f must act as a bijection on S . Since $f(z_0) = z_0$, z_0 is its only preimage, so $f'(z_0) = 0$, \nless .

Example: $f(z) = z^2 + \frac{1}{4}$. Consider the fixed points: $f(\infty) = \infty$, $f(1/2) = 1/2$. ∞ has multiplier 0, and is mapped to itself with degree 2. $\frac{1}{2}$ has multiplier 1, so lies in $J(f)$, as do its infinitely many iterated preimages.

Proposition 7.9. *If $\mathcal{A} \subset \widehat{\mathbb{C}}$ is the attracting basin of a periodic orbit for f . Then the Julia set $J(f)$ is the topological boundary $\partial \mathcal{A}$ of \mathcal{A} .*

Proof. Let U be an open neighbourhood intersecting $J(f)$. Then we have already seen that $\bigcup_{n \in \mathbb{N}} f^n(U)$ contains all but at most 2 points of $\widehat{\mathbb{C}}$, and so intersects \mathcal{A} , i.e., $f^n(U) \cap \mathcal{A} = \emptyset$ for some $n \in \mathbb{N}$.

U was arbitrary, and hence $J(f) \subseteq \bar{\mathcal{A}}$. Then $J(f) \cap \mathcal{A} = \emptyset \implies J(f) \subseteq \partial \mathcal{A}$.

Now suppose that $z \in \partial \mathcal{A}$. If U is a neighbourhood of z on which the iterates form a normal family (i.e. $z \in F(f)$), then pick a convergent subsequence of $\{f^k\}$ converging to g , holomorphic on U . Then $g|_U$ on $U \cap \mathcal{A}$ takes one of a finite number of values, namely the elements of the cycle with \mathcal{A} is attracting to. So g must be constant as it is holomorphic. Hence $U \subseteq \mathcal{A}$, contradicting that $z \in \partial \mathcal{A}$. Therefore $z \notin F(f)$, i.e. $z \in J(f)$. Hence $\partial \mathcal{A} \subseteq J(f)$. \square

Example: Consider again $f(z) = z^2$. Then, given $\alpha = \exp(2\pi i \theta) \in J(f)$, the preimages of α are all the numbers which are a root of $z^{2^m} - \alpha = 0$ for some $m \in \mathbb{N}$, i.e. $\{\exp(2\pi i \frac{\theta+k}{2^m}) : m \in \mathbb{N}, 0 \leq k < 2^m\}$. These points provide a dense subset of $J(f)$, the unit circle.

Corollary 7.10. *For any rational map f and $z_0 \in J(f)$, the set of preimages*

$$\{z : f^n(z) = z_0 \text{ for some } n \geq 0\}$$

form a dense subset of $J(f)$.

Proof. If we fix $z_0, z_1 \in J(f)$, if z_1 is not approximated by preimages of z_0 , then there is some neighbourhood U of z_1 whose forward images don't contain z_0 . But these forward images contain all but at most 2 Fatou points of $\widehat{\mathbb{C}}$, and in particular contain the Julia set, and hence $z_0 \notin U$. \square

Definition 7.11. *A set is **perfect** if it is closed with no isolated points, or equivalently is equal to its set of limit points.*

$J(f)$ is thus a perfect set: it is closed and infinite, so it contains at least one non-isolated point z_0 . The preimages of z_0 form a dense non-isolated subset of the Julia set, so no point of $J(f)$ is isolated.