Elliptic Curves

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December 8, 2020

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1 Fermat's Method of Infinite Descent

Suppose we have a right-angled triangle Δ with side lengths a, b, c, so that by Pythagoras we have $a^2 + b^2 = c^2$, and $\text{area}(\Delta) = \frac{1}{2}ab$.

Definition 1.1. *Δ is* **rational** *if $a, b, c \in \mathbb{Q}$, and* **primitive** *if $a, b, c \in \mathbb{Z}$ coprime.*

Lemma 1.2. *Every primitive triangle is of the form $a = u^2 - v^2, b = 2uv, c = u^2 + v^2$ for coprime integers $u > v > 0$.*

Proof. If a, b were both odd, then $a^2 + b^2 \equiv 2 \mod 4$, and we have no solutions for c. If a, b both even, then they are not coprime. So we may assume a is odd, b is even, c is odd.

Then $(\frac{b}{2})^2 = \frac{c+a}{2}\frac{c-a}{2}$, and the right hand side is a product of coprime positive integers. So by unique prime factorisation in the integers, $\frac{c+a}{2} = u^2$, $\frac{c-a}{2} = v^2$ for some coprime integers u, v. Rearranging, we have the lemma. □

Definition 1.3. *$D \in \mathbb{Q}_{>0}$ is a* **congruent number** *if it is the area of a rational triangle.*

Note that, by scaling the triangle, it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

For example, $D = 5, 6$ are congruent numbers. $6 = \frac{1}{2} \cdot 3 \cdot 4$, and $3^2 + 4^2 = 5^2$, and 5 is left as an exercise.

Lemma 1.4. *$D \in \mathbb{Q}_{>0}$ is congruent if and only if $Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}$, $y \neq 0$.*

Proof. Lemma **1.2** shows that D is congruent if and only if $Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}$, $w \neq 0$.

Setting $x = \frac{u}{v}, y = \frac{w}{v^2}$ finishes the proof. □

Fermat showed that 1 is not a congruent number.

Theorem 1.5. *There is no solution to*

$$w^2 = uv(u + v)(u - v) \tag{$*$}$$

in integers u, v, w with $w \neq 0$.

Proof. Without loss of generality, u, v are coprime with $u > 0, w > 0$. If $v < 0$ then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2})$. So we may assume that all of $u, v, u + v, u - v$ are coprime positive integers whose product is a square, and hence are all squares, say a^2, b^2, c^2, d^2 respectively, where $a, b, c, d \in \mathbb{Z}_{>0}$.

Since $u \not\equiv v \mod 2$, both c, d are odd. Consider the right angled triangle with side lengths, $\frac{c+d}{2}, \frac{c-d}{2}, a$. This is a primitive triangle, and it has area $\frac{c^2-d^2}{8} = \frac{v}{4} = (\frac{b}{2})^2$.

Let $w_1 = \frac{b}{2}$. Then lemma **1.2** gives $w_1^2 = u_1 v_1 (u_1^2 - v_1^2)$ for some $u_1, v_1 \in \mathbb{Z}$, giving a new solution to $(*)$. But $4w_1^2 = b^2 = v | w^2$, and so $w_1 \leq \frac{1}{2}w$.

So by Fermat's method of infinite descent, if there were a solution we would have a strictly decreasing infinite sequence of positive integers $\lightning$. Hence there is no solution to $(*)$. □

1.1 A Variant for Polynomials

Here, K is a field with char $K \neq 2$. The algebraic closure of K will be $\overline{K}$.

Lemma 1.6. *Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for four distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.*

Proof. Without loss of generality we may assume $K = \overline{K}$, as that doesn't change the degree of polynomials, and every square is still a square.

Changing coordinates on $\mathbb{P}^1$, we may assume the ratios $\alpha : \beta$ are $(1 : 0), (0 : 1), (1 : -1), (1 : -\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$, with $\mu = \sqrt{\lambda}$.

Then $u = a^2, v = b^2, u - v = (a + b)(a - b), u - \lambda v = (a + \mu b)(a - \mu b)$ are all squares. They are also coprime, and so by unique factorisation in $K[t]$, $(a + b), (a - b), (a + \mu b), (a - \mu b)$ are all squares.

But $\max\{\deg a, \deg b\} \leq \frac{1}{2} \max\{\deg u, \deg v\}$. So by Fermat's method of infinite descent, we get that the original $u, v \in K$. □

Now we have some important definitions:

Definition 1.7.

1. *An* **elliptic curve** *E over a field K is the projective closure of the affine curve $y^2 = f(x)$ where $f \in K[x]$ is a monic cubic polynomial with distinct roots.*
2. *For L/K any field extension, $E(L) = \{(x, y) \in L^2 : y^2 = f(x)\} \cup \{0\}$. 0 is called the* **point at infinity**.

We call the point at infinity 0 because we will see that $E(L)$ is naturally an abelian group under an operation we will denote by $+$, and 0 will be the identity for that group. In this course we will study $E(L)$ for L a finite field, a local field, and a number field.

Lemma **1.4** and theorem **1.5** together imply that, if E is given by $y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$, which we will see is the group $C_2 \times C_2$.

Corollary 1.8. *Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.*

Proof. Without loss of generality, $K = \overline{K}$. By a change of coordinates we may assume $E : y^2 = x(x - 1)(x - \lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ with $u, v \in K[t]$ coprime. Then $w^2 = uv(u - v)(u - \lambda v)$ for some $w \in K[t]$.

Unique factorisation in $K[t]$ gives $u, v, u - v, u - \lambda v$ are all squares, and so by lemma **1.6**, $u, v \in K$, and so $x, y \in K$. □

2 Some Remarks on Algebraic Curves

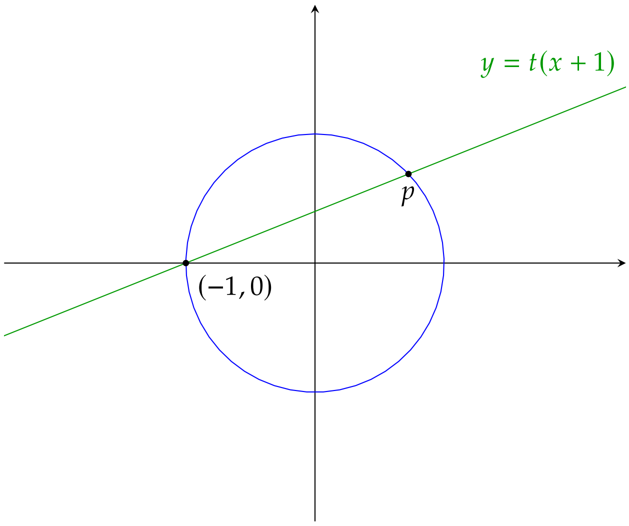
We will be working over an algebraically closed field K.

Definition 2.1. *An (irreducible) plane algebraic curve $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$ is* **rational** *if it has a rational parametrization, i.e. there are $\phi, \psi \in K(t)$ such that:*

1. *$\mathbb{A}^1 \to \mathbb{A}^2; t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{$finite set$\}$.*
2. *$f(\phi(t), \psi(t)) = 0$.*

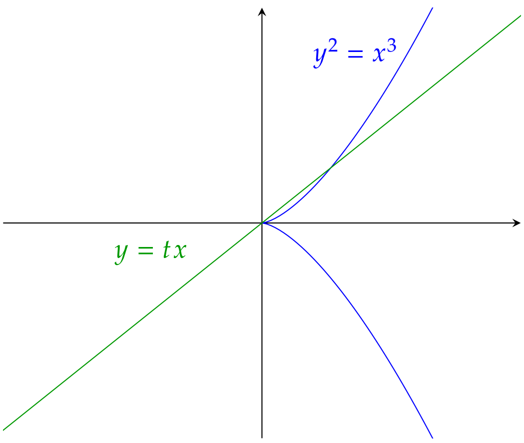
Examples 2.2.

1. Any nonsingular plane conic is rational. For example, take a circle $x^2 + y^2 = 1$. Pick a point on it, $(-1, 0)$. Now draw a line through it with slope t, and solve for the points of intersection between the curve and the line.

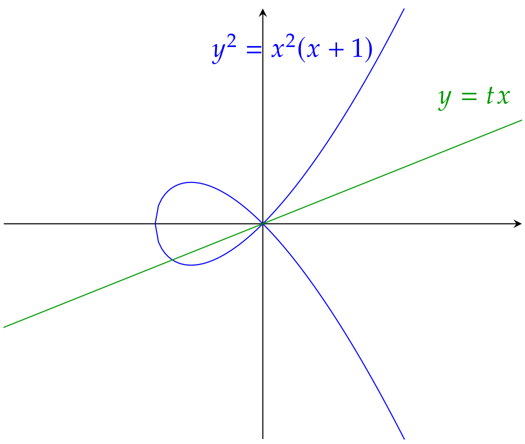


Solving for the coordinates of p, we get the quadratic $x^2 + t^2(x+1)^2 = 1$, i.e. $x = -1$ or $\frac{1-t^2}{1+t^2}$. So we have the rational parametrization $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

2. Any singular plane cubic is rational.



(a) Rational Parametrization $(x, y) = (t^2, t^3)$



(b) Left as an example on the first sheet

3. Corollary **1.8** shows that elliptic curves are *not* rational.

Definition 2.3. *The* **genus** $g(C) \in \mathbb{Z}_{\geq 0}$ *is an invariant of a smooth projective curve.*

- *If* $K = \mathbb{C}$*, then* $g(C) =$ *genus of the Riemann surface* C*.*

- *A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.*

Proposition 2.4. *Let C be a smooth projective curve over K, an algebraically closed field. Then:*

1. *C is rational $\iff$ $g(C) = 0$.*
2. *C is an elliptic curve $\iff$ $g(C) = 1$.*

Proof. A proof of *1* is omitted from this course. For 2, we check (on the first example sheet) that elliptic curves are smooth plane curves. Then they have degree 3, so genus $\frac{2\cdot 1}{2} = 1$. For the other direction, see later on in the course. □

2.1 Order of Vanishing

C will be an algebraic curve, and $K(C)$ its function field, with $P \in C$ a smooth point. Write $\mathrm{ord}_P(f)$ to mean the order of vanishing of $f \in K(C)$ at P (negative if f has a pole).

Fact: $\mathrm{ord}_P : K(C)^\times \to \mathbb{Z}$ is a discrete valuation, i.e. $\mathrm{ord}_P(f_1 f_2) = \mathrm{ord}_P(f_1) + \mathrm{ord}_P(f_2)$ and $\mathrm{ord}_P(f_1 + f_2) \geq \min\{\mathrm{ord}_P(f_1), \mathrm{ord}_P(f_2)\}$.

We say $t \in K(C)^\times$ is a ***uniformizer*** at the point P if $\mathrm{ord}_P(t) = 1$.

Example 2.5. Let $C = \{g(x, y) = 0\} \subseteq \mathbb{A}^2$, where $g \in K[x, y]$ is irreducible. Then $K(C) = \mathrm{Frac}\, \frac{K[x,y]}{(g)}$, with $g = g_0 + g_1(x, y) + g_2(x, y) + \ldots$, g_i homogeneous of degree i.

Suppose $P = (0, 0) \in C$ is a smooth point, i.e. $g_0 = 0, g_1(x, y) = \alpha x + \beta y$ with α, β not both zero.

Let $\gamma, \delta \in K$. It is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\frac{\gamma}{\delta} \neq \frac{\alpha}{\beta}$, i.e. $\alpha\delta - \beta\gamma \neq 0$.

Example 2.6. $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2, \lambda \neq 0, 1$. We take the projective closure, i.e. homogenize the equation as $\{Y^2 Z = X(X - Z)(X - \lambda Z)\} \subset \mathbb{P}^2$ by setting $x = X/Z, y = Y/Z$.

Have we got new points by taking projective closure? We only get these when $Z = 0$, i.e. $0 = X^3 \implies X = 0, Y \neq 0$. Since we're in projective space, this is just one point: $P = (0 : 1 : 0)$. We compute $\mathrm{ord}_P(x)$ and $\mathrm{ord}_P(y)$. Put $t = X/Y, w = Z/Y$ (since we can't return to the original affine piece, as it doesn't contain $Z = 0$). Then we get $w = t(t - w)(t - \lambda w)$. Now P is the point $(t, w) = (0, 0)$. This is a smooth point, as there are linear terms at that point (namely w). So $\mathrm{ord}_P(t) = \mathrm{ord}_P(t - 2) = \mathrm{ord}_P(t - \lambda w) = 1$, and $\mathrm{ord}_P(w) = 1 + 1 + 1 = 3$.

Then:

$$\begin{aligned}
\mathrm{ord}_P(x) &= \mathrm{ord}_P(X/Z) = \mathrm{ord}_P(t/w) = 1 - 3 = -2 \\
\mathrm{ord}_P(y) &= \mathrm{ord}_P(Y/Z) = \mathrm{ord}_P(1/w) = -3
\end{aligned}$$

2.2 Riemann Roch Spaces

Let C be a smooth projective curve. Then a ***divisor*** is a formal sum of points on C, say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$, and only finitely many n_P are nonzero, and let $\deg D = \sum_{P \in C} n_P$. These divisors form a group under addition, denoted $\mathrm{Div}(C)$.

D is said to be ***effective***, written $D \geq 0$ if $n_p \geq 0$ for all $P \in C$.

If $f \in K(C)^\times$, we write $\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) P$.

The Riemann Roch space of $D \in \mathrm{Div}(C)$ is:

$$\mathscr{L}(D) = \{f \in K(C) : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

i.e. the K-vector space of rational functions on C with "poles no worse than specified by D."

Theorem 2.7 (Riemann Roch for genus 1)**.**

$$\dim \mathscr{L}(D) = \begin{cases} 0 & \deg D < 0 \\ 0 \text{ } or \text{ } 1 & \deg D = 0 \\ \deg D & \deg D > 0 \end{cases}$$

Example 2.6 (revisited). Our curve is $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$, together with $P = (0:1:0)$, the point at infinity. Recall $\mathrm{ord}_P(x) = -2, \mathrm{ord}_P(x) = -3$.

We thus deduce that $\mathscr{L}(2P) = \langle 1, x \rangle, \mathscr{L}(3P) = \langle 1, x, y \rangle$.

Proposition 2.8. *Let K be an algebraically closed field not of characteristic 2. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic, and that $P \in C$ is a point of inflection. Then we may change coordinates such that:*

$$\begin{aligned} C : Y^2Z &= X(X-Z)(X-\lambda Z), \quad \lambda \neq 0, 1 \\ P &= (0:1:0) \end{aligned}$$

Proof. We make a change of coordinates such that $P = (0:1:0)$ and the tangent line to C at P, $T_P(C) = \{Z = 0\}$. Now let $C = \{F(X,Y,Z) = 0\}$.

Since $P \in C$ is a point of inflection, $F(t,1,0)$ has a triple root at $t = 0$. But F is degree 3, so we have $F(t,1,0) = kt^3$ for k some constant. I.e., there are no terms in F of the form X^2Y, XY^2, Y^3.

So $F \in \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3 \rangle$. The coefficient of Y^2Z is nonzero, as otherwise P would be singular. The coefficient of X^3 is also nonzero, as C is irreducible and otherwise $\{Z = 0\} \subset C$.

We are free to rescale X, Y, Z, F, and so WLOG C is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

We call this Weierstrass form.

Since our field doesn't have characteristic 2, we may complete the square by substituting $Y = Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$.

Now $C : Y^2Z = Z^3f(X/Z)$, where f is a monic cubic polynomial. Since C is smooth, f has distinct roots, which are WLOG $0, 1, \lambda$. So

$$C : Y^2Z = X(X-Z)(X-\lambda Z)$$

which we call the Legendre form. □

It may be shown that the points of inflection on $C = \{F = 0\} \subset \mathbb{P}^2$ are given by $F = \det\left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right) = 0$

2.3 The Degree of a Morphism

Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Let $\phi^* : K(C_2) \to K(C_1), f \mapsto f \circ \phi$.

Definition.

1. $\deg \phi = [K(C_1) : \phi^* K(C_2)]$
2. *ϕ is separable if $K(C_1)/\phi^* K(C_2)$ is a separable field extension (which by Galois theory is automatic if* $\operatorname{char} K = 0$*)*

Suppose $P \in C_1, Q \in C_2, \phi : P \to Q$. Let $t \in K(C_2)$ be a uniformizer at Q. We then define $e_\phi(p) = \operatorname{ord}_P(\phi^* t)$, which is always ≥ 1, and independent of t. $e_\phi(P)$ is called the ***ramification index*** of ϕ at p.

Theorem 2.9. *Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then*

$$\sum_{p \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi$$

for any point $Q \in C_2$. Moreover, if ϕ is separable then $e_\phi(P) = 1$ with at most finitely many exceptions.

In particular:

1. *ϕ is surjective*
2. *If ϕ is separable, $\#\phi^{-1}(Q) \leq \deg \phi$, with equality for all but finitely many choices of Q.*

Remark 2.10. Let C be an algebraic curve. A rational map is given by $\phi : C \dashrightarrow \mathbb{P}^n, P \mapsto (f_0(P) : \ldots : f_n(P))$, where $f_0, \ldots, f_n \in K(C)$ are not all zero. If C is smooth then ϕ is a morphism.

3 Weierstrass Equations

In this section, K is a perfect field (so that all finite extensions of K are separable), with algebraic closure $\bar{K}$.

Definition. An elliptic curve E over K is a smooth projective curve of genus 1 defined over K with a specified K-rational point O_E.

Example: Take $\{X^3 + pY^3 + p^2Z^3 = 0\} \subset \mathbb{P}^2$ for p prime. This is not an elliptic curve over $\mathbb{Q}$ since there is no $\mathbb{Q}$-points.

Theorem 3.1. *Every elliptic curve E is isomorphic over K to a curve in Weierstrass form via an isomorphism taking O_E to* $(0 : 1 : 0)$*.*

Proposition **2.8** treated the special case where E is a smooth plane cubic and O_E is a point of inflection.

If $D \in \operatorname{Div}(E)$ is defined over K (i.e. fixed by the natural action of $\operatorname{Gal}(\bar{K}/K)$, then $\mathscr{L}(D)$ has a basis in $K(E)$, not just in $\bar{K}(E)$).

Proof. Note that

$$\mathscr{L}(2O_E) \subset \mathscr{L}(3O_E)$$

Pick bases of these spaces, say $\{1, x\}$ and $\{1, x, y\}$.

Note that $\mathrm{ord}_{O_E}(x) = -2, \mathrm{ord}_{O_E}(y) = -3$. The 7 elements $\{1, x, y, x^2, xy, x^3, y^2\}$ are rational functions with no pole except at O_E, where they have poles of degree at most 6, so they all lie in $\mathscr{L}(6O_E)$. Riemann-Roch tells us this space has dimension 6, so there is a dependence relation between these elements.

Leaving out x^3 or y^2 gives a basis for $\mathscr{L}(6O_E)$ since each term has a different order pole at O_E, so they are independent.

Therefore this dependence relation *must* involve both x^3 and y^2. Rescaling x, y we get

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Let E' be the curve defined by this equation (or rather its projective closure).

There is a morphism

$$\begin{aligned} \phi : E &\to E' \\ P &\mapsto (x(P) : y(P) : 1) = \left(\frac{x}{y}(P) : 1 : \frac{1}{y}(P)\right) \\ O_E &\mapsto (0 : 1 : 0) \end{aligned}$$

$[K(E) : K(x)] = \deg(E \xrightarrow{x} \mathbb{P}^1) = \mathrm{ord}_{O_E}\left(\frac{1}{x}\right) = 2$
$[K(E) : K(y)] = \deg(E \xrightarrow{y} \mathbb{P}^1) = \mathrm{ord}_{O_E}\left(\frac{1}{y}\right) = 3$

This gives us a diagram of field extensions

$$\begin{tikzcd}[arrows=dash] & K(E) \arrow[dl, "2"'] \arrow[d] \arrow[dr, "3"] & \\ & K(x,y) \arrow[dl] \arrow[dr] & \\ K(x) & & K(y) \end{tikzcd}$$

So $[K(E) : K(x, y)]$ divides both 2 and 3 by the tower law, and hence $K(E) = K(x, y)$, and hence $\deg(E \xrightarrow{\phi} E') = 1$, and ϕ is birational. If E' is singular, then it is rational, and so E is also rational $\lightning$. So E' is not singular and hence smooth, and we may use remark **2.10** to ϕ^{-1} to see that ϕ^{-1} is a morphism, and hence ϕ is an isomorphism. $\square$

Proposition 3.2. *Let E, E' be elliptic curves over K in Weierstrass form. Then $E \cong E'$ over K if and only if the Weierstrass equations are related by a change of variables of the form*

$$\begin{aligned} x &= u^2x' + r \\ y &= u^3y' + u^2sx' + t \end{aligned}$$

for $u, r, s, t \in K, u \neq 0$.

Proof. Using the notation of the previous proof,

$$\begin{aligned}
\langle 1, x\rangle =& \mathscr{L}(2O_E) = \langle 1, x'\rangle \\
\langle 1, x, y\rangle =& \mathscr{L}(3O_E) = \langle 1, x', y'\rangle \\
\implies & \begin{cases} x = \lambda x' + r & \lambda_1 r \in K, \lambda \neq 0 \\ y = \mu y' + \sigma x' + t & \mu, \sigma, t \in K, \mu \neq 0 \end{cases}
\end{aligned}$$

Looking at the coefficients of x^3 and y^2, $\lambda^3 = \mu^2 \implies (\lambda, \mu) = (u^2, u^3)$ for $u \in K^\times$.

Put $s = \sigma/u^2$ □

The effect of this transformation on the coefficients a_i is on the formula sheet for this course. A Weierstrass equation defines an elliptic curve if and only if defines a smooth curve, if and only if $\Delta(a_1, \dots, a_6) \neq 0$ where Δ is as follows:

$$\begin{aligned}
b_2 &:= a_1^2 + 4a_2 \\
b_4 &:= 2a_4 + a_1 a_3 \\
b_6 &:= a_3^2 + 4a_6 \\
b_8 &:= a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2 \\
\Delta &:= -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6
\end{aligned}$$

If char $K \neq 2, 3$, then we can reduce to the case

$$\begin{aligned}
E : y^2 &= x^3 + ax + b \\
\Delta &= -16(4a^3 + 26b^2)
\end{aligned}$$

Corollary 3.3. *Assume* char $K \neq 2, 3$. *If we have two elliptic curves*

$$\begin{aligned}
E : y^2 &= x^3 + ax + b \\
E' : y^2 &= x^3 + a'x + b'
\end{aligned}$$

then they are isomorphic over K if and only if

$$\begin{aligned}
a' &= u^4 a \\
b' &= u^6 b
\end{aligned}$$

for some $u \in K^\times$.

Proof. E and E' are related as in **3.2** with $r = s = t = 0$. □

Definition. The ***j-invariant*** is $j(E) = \frac{1728(4a^3)}{4a^3+27b^2}$. Note that the denominator is nonzero since the discriminant is nonzero.

Corollary 3.4. $E \cong E' \implies j(E) = j(E')$, *and the converse holds if* $K = \bar{K}$.

Proof.

$$\begin{aligned} E \cong E' &\iff a' = u^4 a; b' = u^6 b \text{ for some } u \in K^\times \\ &\implies (a^3 : b^2) = ((a')^3 : (b')^2) \\ &\iff j(E) = j(E') \end{aligned}$$

and the reverse implication holds in the second line if $K = \bar{K}$. □

4 Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic, and $O_E \in E(K)$. Since E is of degree 3, it meets each line in 3 points counted with multiplicity. Hence, given two points P, Q on E, the line $\overline{PQ}$ meets E at a third point S. Then the line $\overline{O_E S}$ meets E at a third point R. We then define $P \oplus Q = R$.

If $P = Q$, then we take the tangent line at P, likewise if $S = O_E$. We can view this diagrammatically as follows:

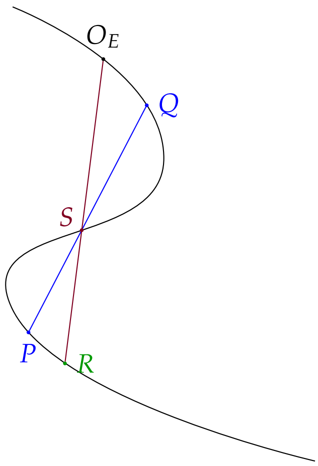


Figure 2: Illustration of the group operation on an elliptic curve

We call this the "chord and tangent process".

Theorem 4.1. $(E, \oplus)$ *is an abelian group.*

Proof.

(i) $P \oplus Q = Q \oplus P$ by construction.

(ii) O_E is the identity.

(iii) For inverses, let S be the third point of intersection of T_{O_E} and E, and Q be the third point of intersection of $\overline{PS}$ and E. Then $P \oplus Q = O_E$.

(iv) Associativity is much harder.

□

Definition. $D_1, D_2 \in Div(E)$ are ***linearly equivalent*** (written $D_1 \sim D_2$) if there is $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) = D_1 - D_2$. Then we will let $[D] = \{D' : D' \sim D\}$.

Definition. The ***Picard group of E***, $\operatorname{Pic}(E) = \operatorname{Div}(E)/\sim$. We write $\operatorname{Div}^0(E) := \ker\left(\operatorname{Div}(E) \xrightarrow{\deg} \mathbb{Z}\right)$ for the group of degree 0 divisors on E, and then $\operatorname{Pic}^0(E) = \operatorname{Div}^0(E)/\sim$. Sometimes Pic^0 is called the Jacobian.

Proposition 4.2. *Let* $\psi : E \to \operatorname{Pic}^0(E); P \mapsto [(P) - (O_E)]$. *Then:*

1. $\psi(P \oplus Q) = \psi(P) + \psi(Q)$
2. ψ *is a bijection*

Proof.

1. Referring back to Fig. 2, let $\{\ell = 0\}$ be the line $\overline{PQ}$, and $\{m = 0\}$ be the line $\overline{O_E R}$. Then:

$$\begin{aligned} \operatorname{div}(\ell/m) &= (P) + (S) + (Q) - (R) - (S) - (O_E) \\ &= (P) + (Q) - (O_E) - (P \oplus Q) \\ \implies & (P \oplus Q) + (O_E) \sim (P) + (Q) \\ \implies & (P \oplus Q) - (O_E) \sim (P) - (O_E) + (Q) - (O_E) \\ \implies & \psi(P \oplus Q) = \psi(P) + \psi(Q) \end{aligned}$$

2. For injectivity, suppose $\psi(P) = \psi(Q)$. Then there is $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) = P - Q$. Then $\deg\left(E \xrightarrow{f} \mathbb{P}^1\right) = \operatorname{ord}_P(f) = 1$. But then f is a birational morphism, so an isomorphism, and $E \cong \mathbb{P}^1 ↯$.

   For surjectivity, let $[D] \in \operatorname{Pic}^0(E)$. Then $D + (O_E)$ has degree 1 (as D had degree 0). Then Riemann-Roch tells us $\dim \mathscr{L}(D + (O_E)) = 1$, and so there exists some $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) + D + (O_E) \geq 0$. Since f is rational, $\deg \operatorname{div}(f) = 0$, and $\deg D = 0$. So the coefficients of $\operatorname{div}(f) + D + (O_E)$ are non-negative and sum to 1, hence one of them is 1 and the rest are 0. So $\operatorname{div}(f) + D + (O_E) = (P)$ for some $P \in E$. But then $(P) - (O_E) \sim D$, i.e. $\psi(P) = [D]$.

□

So ψ is a bijection respecting the group law, and so we deduce that $\oplus$ is associative, and then $(E, \oplus) \overset{\psi}{\cong} (\operatorname{Pic}^0 E, +)$.

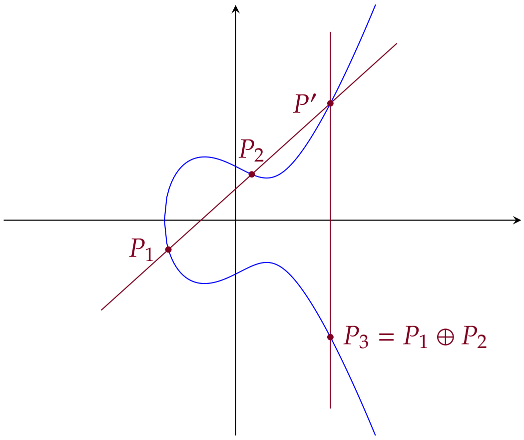
4.1 Explicit Formulae for the Group Law

We consider E in Weierstrass form, with O_E the point at infinity:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{$*$}$$

Note that O_E is a point of inflection. Now $P_1 \oplus P_2 \oplus P_3 = O_E \iff P_1, P_2, P_3$ are collinear.

We will use the following notation:



and put $P_i = (x_i, y_i)$, $P' = (x', y')$.

Now $\ominus P_1 = (x_1, -(a_1x_1 + a_3) - y_1)$, just by setting $y = -y_1$ in $(*)$.

The line through P_1, P_2 has equation say $y = \lambda x + \nu$. Substituting into $(*)$ and looking at the coefficient of x^2, we get:

$$\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x'$$

Since $x_3 = x'$, we have:

$$\begin{aligned} x_3 &= \lambda^2 + a_1\lambda - a_2 - x_1 - x_2 \\ y_3 &= -(a_1x' + a_3) - y' \\ &= -(\lambda + a_1)x_3 - \nu - a_3 \end{aligned}$$

It remains to find λ and ν. There are 3 cases:

1. $x_1 = x_2, P_1 \neq P_2$.

   Then $P_1 \oplus P_2 = O_E$.

2. $x_1 \neq x_2$.

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \quad \nu = y_1 - \lambda x_1 = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

3. $P_1 = P_2$.

   Here we have to compute the equation of the tangent line etc. The solutions are:

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}, \quad \nu = \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}$$

Corollary 4.3. *$E(K)$ is an abelian group.*

Proof. It is a subgroup of E $(= E(\bar{K}))$.

Identity: $O_E \in E(K)$ by definition.

Closure: See formulae above.

Inverses: See formulae above.

Associativity: Inherited from $E(\bar{K})$.

Commutativity: Inherited from $E(\bar{K})$.

□

If there is no ambiguity (i.e. we are not also adding numbers at the same time), the circles will be dropped from the group operation.

Theorem 4.4. *Elliptic curves are group varieties.*

i.e., $[-1] : E \to E; P \mapsto -P$ *and* $+ : E \times E \to E; (P, Q) \mapsto P + Q$ *are morphisms of algebraic varieties.*

Proof. The above formulae show that $[-1]$ and $+$ are rational maps. We know immediately that $[-1]$ is a morphism, as it is a rational map from a smooth curve to a projective variety.

The formulae also show that $+$ is regular on the set

$$U = \{(P, Q) \in E \times E \mid P, Q, P + Q, P - Q \neq O_E\}$$

For $P \in E$, let $\tau_P : E \to E; X \mapsto P + X$ be the "translation by P" map.

Then τ_P is a rational map from a smooth curve to a projective variety, so is a morphism.

We factor $+$ as:

$$E \times E \xrightarrow[\tau_{-A} \times \tau_{-B}]{} E \times E \xrightarrow[\tau_{A+B}]{} E \xrightarrow[\tau_{A+B}]{} E$$

Now $+$ is regular on $(\tau_A \times \tau_B)(U)$ for all $A, B \in E$, and so $+$ is regular on $E \times E$. □

Definition. For any $n \in \mathbb{Z}_{>0}$, let $[n] : E \to E; P \mapsto P + \ldots + P$, n times, and $[-n] = [-1] \circ [n]$, $[0] : P \mapsto O_E$ (i.e., the standard way of turning an abelian group into $\mathbb{Z}$ module).

Definition. The ***n*-torsion** subgroup of E is $E[n] = \ker([n] : E \to E)$.

Lemma 4.5. *If* $\mathrm{char}(K) \neq 2$, *and* $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$.

Then $E[2] = (0, (e_1, 0), (e_2, 0), (e_3 0)) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. Let $P = (x, y) \in E$. Then $[2]P = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. □

4.2 Elliptic Curves over $\mathbb{C}$

Let $\Lambda = \{a\omega_1 + b\omega_2 : a, b \in \mathbb{Z}\}$, where ω_1, ω_2 form a basis for $\mathbb{C}$ over $\mathbb{R}$.

Then the meromorphic functions on the Riemann surface (or lattice) $\mathbb{C}/\Lambda$ are the same as the Λ-invariant meromorphic functions on $\mathbb{C}$ (i.e. $f(z) = f(z + \lambda)$ for $\lambda \in \Lambda$).

This set of functions is a field, and is generated by $\wp(z)$ and $\wp'(z)$, where:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

They satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, for some $g_1, g_3 \in \mathbb{C}$ depending on λ. We call $\wp$ the ***Weierstrass p-function***.

One can show that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where E is the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. This is an isomorphism, not only of Riemann surfaces, but moreover of groups

Theorem 4.6 (Uniformisation Theorem). *Every elliptic curve over $\mathbb{C}$ arises in this way.*

Thus, for elliptic curves $E/\mathbb{C}$, we have:

1. $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$

2. $\deg[n] = n^2$

We will show that 2 holds over any field K, and 1 holds if $\operatorname{char} K \nmid n$.

Summary of Results (N.B. the isomorphisms in 1, 2, 4 respect the relevant topologies)

1. $K = \mathbb{C}$ $\quad E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$
2. $K = \mathbb{R}$ $\quad E(\mathbb{R}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \Delta < 0 \end{cases}$
3. $K = \mathbb{F}_q$ $\quad |\#E(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}$
4. $[K : \mathbb{Q}_p] < \infty$ $\quad E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$
5. $[K : \mathbb{Q}] < \infty$ $\quad E(K)$ is a finitely generated abelian group.

5 Isogenies

Let E_1, E_2 be elliptic curves.

Definition. An ***isogeny*** $\phi : E_1 \to E_2$ is a non-constant morphism taking O_{E_1} to O_{E_2}, and we say E_1 and E_2 are ***isogenous*** if there is an isogeny $E_1 \to E_2$.

Definition. $\operatorname{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \to E_2\} \cup \{0\}$. This is a group under $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$ are isogenies, then $\psi\phi$ is an isogeny. The tower law tells us that $\deg(\psi\phi) = \deg(\phi)\deg(\psi)$.

Lemma 5.1. *If $0 \neq n \in \mathbb{Z}$, then $[n] : E \to E$ is an isogeny.*

Proof. Theorem **4.4** tells us that $[n]$ is a morphism. We must show that $[n] \neq 0$.

Assume $\operatorname{char} K \neq 2$, then we can use Lemma **4.5**. If $n = 2$, then $\#E[2] = 4$, and so $[2] \neq 0$.

If n is odd, then there is $0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n]$ is not the zero map.

Now $[m][n] = [m] \circ [n]$, and any $n = 2^k m$ for m odd, so $[n]$ is not the zero map for any $n \neq 0$.

If $\operatorname{char} K = 2$, then replace **4.5** with a lemma computing $E[3]$. □

Corollary. $\operatorname{Hom}(E_1, E_2)$ *is torsion-free as a $\mathbb{Z}$-module.*

Lemma 5.2. *Let $\phi : E_1 \to E_2$ be an isogeny. Then $\phi(P + Q) = \phi(P) + \phi(Q)$ for all $P, Q \in E_1$.*

Sketch proof. ϕ induces a map $\phi_* : \mathrm{Div}^0(E_1) \to \mathrm{Div}^0(E_2)$ given by $\sum\limits_{P \in E_1} n_p P \mapsto \sum\limits_{P \in E_1} n_p \phi(P)$.

Recall that, via a pullback, $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

If $f \in K(E_1)^*$, then $\phi_*(\operatorname{div} f) = \operatorname{div}(N_{K(E_1)/K(E_2)} f)$ - this is a fact that we'll take for granted.

So ϕ_* takes principal divisors to principal divisors. Since $\phi(O_{E_1}) = O_{E_2}$, the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "\psi_1"] & E_2 \arrow[d, "\psi_2"] \\ \mathrm{Pic}^0(E_1) \arrow[r, "\phi_*"] & \mathrm{Pic}^0(E_2) \end{tikzcd}$$

,where $\psi_1 : P \mapsto [(P) - (O_{E_1})]$, $\psi_2 : Q \mapsto [(Q) - (O_{E_2})]$.

Since ϕ_* is a group homomorphism, ϕ is also a group homomorphism. □

Lemma 5.3. *Let $\phi : E_1 \to E_2$ be an isogeny. Then there is a morphism ξ making the following diagram commute:*

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "x_1"] & E_2 \arrow[d, "x_2"] \\ \mathbb{P}^1 \arrow[r, "\xi"] & \mathbb{P}^1 \end{tikzcd}$$

where x_i is the x-coordinate in a Weierstrass equation for E_i.

Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ for $r, s \in K[t]$ coprime, then $\deg \phi = \deg \xi = \max(\deg r, \deg s)$.

Proof. For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 extension, since the extension is given by adjoining y_i, which satisfies a quadratic (see the Weierstrass equation). Moreover, it is Galois, as $[-1]^*$ is a non-trivial automorphism of $K(E_i)$ fixing $K(x_i)$.

Since ϕ is a group homomorphism, we have that $\phi(-P) = -\phi(P)$, i.e. $\phi \circ [-1] = [-1] \circ \phi$.

If $f \in K(x_2)$, then $[-1]^* f = f$, and $[-1]^*(\phi^* f) = \phi^*([-1]^* f) = \phi^* f$. Hence $\phi^* f$ is fixed by $[-1]$, so is in $K(x_1)$, and $K(x_2) \leq K(x_1)$.

Taking $f = x_2$, then $\phi^* x_2 \in K(x_1)$, say $\xi(x_1)$ for some rational function ξ. Then ξ is as required.

Since $[K(E_1) : K(x_1)] = [K(E_2) : K(x_2)] = 2$, we have the following diagram of field extensions:

$$\begin{tikzcd} & K(E_1) \arrow[dl, dash, "2"'] \arrow[dr, dash, "\deg \phi"] & \\ K(x_1) \arrow[dr, dash, "\deg \xi"'] & & K(x_2) \arrow[dl, dash, "2"] \\ & K(x_2) & \end{tikzcd}$$

Using the tower law, $\deg \phi = \deg \xi$. Now, $K(x_2) \hookrightarrow K(x_1)$ via $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_2)}$ for $r, s \in K[t]$ coprime.

The minimal polynomial of x_1 over $K(x_2)$ is $f(t) = r(t) - s(t)x_2 \in K(x_2)[t]$ - this is clearly a polynomial for x_1, but we need to check it's irreducible.

f is irreducible in $K[t][x_2] = K[x_2][t]$ as it is of degree 1 in x_2, so one of the factors must be constant in x_2, so divide both r and s which are coprime. Then we can use Gauss's lemma, and it is irreducible in $K(x_2)[t]$.

Hence $\deg \phi = \deg \xi = [K(x_1) : K(x_2)] = \deg(r(t) - s(t)x_2) = \max(\deg r, \deg s)$. □

Lemma 5.4. $\deg[2] = 4$

Proof. Assume char $K \neq 2, 3$. Then $E : y^2 = x^3 + ax + b = f(x)$.

If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2-8xf(x)}{4f(x)} = \frac{x^4+\dots}{4f(x)}$.

The numerator and denominator are coprime - suppose there was a common factor. Then $\exists\ \theta \in \bar{K}$ with $f(\theta) = (3\theta^2 + a)^2 = f'(\theta) = 0$, and so f has a multiple root. But E is an elliptic curve so f doesn't have multiple roots.

Hence $\deg[2] = \max(\deg x^4 + \dots, \deg 4f(x)) = \max(4, 3) = 4$. □

Definition. Let A be an abelian group. We say that $q : A \to \mathbb{Z}$ is a ***quadratic form*** if it satisfies

1. $q(nx) = n^2 q(x)\ \forall n \in \mathbb{Z}, x \in A$.
2. $(x, y) \to q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$-bilinear.

Lemma 5.5. *$q : A \to \mathbb{Z}$ is a quadratic form if and only if it satisfies the parallelogram law:*

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)\ \forall x, y \in A$$

Proof. For the forwards direction, let $\langle x, y\rangle = q(x + y) - q(x) - q(y)$.

Then $\langle x, x\rangle = q(2x) - 2q(x) = 2q(x)$.

Then $\frac{1}{2}\langle x+y, x+y\rangle + \frac{1}{2}\langle x-y, x-y\rangle = \langle x, x\rangle + \langle y, y\rangle$ by bilinearity, and hence $q(x+y)+q(x-y) = 2q(x) + 2q(y)$.

The reverse direction is left as an exercise on example sheet 2. □

Theorem 5.6.

$$\deg : \mathrm{Hom}(E_1, E_2) \to \mathbb{Z}$$

is a quadratic form.

Proof. For the proof, we will assume char $K \neq 2, 3$ for simplicity - the result still holds in those characteristics.

We write $E_2 : y^2 = x^3 + ax + b$.

Let $P, Q \in E_2$ with $P, Q, P + Q, P - Q \neq 0$, and let $x_1, \dots, x_4$ be the x-coordinates of these 4 points. Then we have:

Lemma 5.7. *There exists $w_0, w_1, w_2 \in \mathbb{Z}[a, b][x_1, x_2]$ of degree ≤ 2 in x_1 and in x_2 such that* $(1 : x_3 + x_4 : x_3x_4) = (w_0 : w_1 : w_2)$.

Proof. We could prove this by direct calculation, leading to the formulae:

$$\begin{aligned} w_0 &= (x_1 - x_2)^2 \\ w_1 &= 2(x_1x_2 + a)(x_1 + x_2) + 4b \\ w_2 &= x_1^2x_2^2 - 2ax_1x_2 - 4b(x_1 + x_2) + a^2 \end{aligned}$$

As an alternative proof, let $y = \lambda x + \nu$ be the line through P and Q. Then

$$x^3 + ax + b - (\lambda x + \nu)^2 = (x - x_1)(x - x_2)(x - x_3) = x^3 - s_1x^2 + s_2x - s_3$$

where s_i is the i^{th} symmetric polynomial in (x_1, x_2, x_3).

Comparing coefficients:

$$\begin{aligned} \lambda^2 &= s_1 \\ -2\lambda\nu + a &= s_2 \\ \nu^2 - b &= s_3 \end{aligned}$$

Eliminating λ, ν, we have $F(x_1, x_2, x_3) := (s_2 - a)^2 - 4s_1(s_3 + b) = 0$. Then F has degree at most 2 in each x_i.

x_3 is a root of the quadratic polynomial $W(t) = F(x_1, x_2, t)$, and repeating this for the line through P and $-Q$ shows that x_4 is the other root. Hence

$$w_0(t - x_3)(t - x_4) = W(t) = w_0t^2 - w_1t + w_2$$

And so $(1 : x_3 + x_4 : x_3x_4) = (w_0 : w_1 : w_2)$. □

We then show that, if $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$, then

$$\deg(\phi + \psi) + \deg(\phi - \psi) \leq 2\deg(\phi) + 2\deg(\psi)$$

We may assume $\phi, \psi, \phi + \psi, \phi - \psi! = 0$, as otherwise the result is trivial.

$$\begin{aligned} \phi &: (x, y) \mapsto (\xi_1(x), \ldots) \\ \psi &: (x, y) \mapsto (\xi_2(x), \ldots) \\ \phi + \psi &: (x, y) \mapsto (\xi_3(x), \ldots) \\ \phi - \psi &: (x, y) \mapsto (\xi_4(x), \ldots) \end{aligned}$$

Then **5.7** gives $(1 : \xi_3 + \xi_4 : \xi_3\xi_4) = ((\xi_1 - \xi_2)^2 : \ldots : \ldots)$.

Put $\xi_i = \frac{r_i}{s_i}$ where $r_i, s_i \in K[x]$ are coprime:

$$(s_3s_4 : r_3s_4 + r_4s_3 : r_3r_4) = ((r_1s_2 - r_2s_2)^2 : \ldots : \ldots) \tag{$*$}$$

So we have:

$$\begin{aligned} \deg(\phi + \psi) + \deg(\phi - \psi) &= \max(\deg r_3, \deg s_3) + \max(\deg r_4, \deg s_4) \\ &= \max(\deg(s_3s_4), \deg(r_3s_4 + r_4s_3), \deg(r_3r_4)) \end{aligned}$$

Suppose $(s_3s_4, r_3s_4+r_4s_3, r_3r_4)$ are not coprime, so that p irreducible divides all 3. Then p divides one of r_3, r_4, and one of s_3, s_4. p can't divide both s_i and r_i as they are coprime, so WLOG p divides r_3 and s_4 and not r_4 nor s_3. Then p doesn't divide $r_3s_4 + r_4s_3 \lightning$. Hence these polynomials are coprime.

Hence the polynomials on RHS of (∗) must be multiples of polynomials on the LHS by some irreducible polynomial, and hence each have degree ≥ their corresponding polynomial on LHS, and thus, as w_i are of degree ≤ 2 in r_1, s_1, r_2, s_2,

$$\begin{aligned} \deg(\phi+\psi)+\deg(\phi-\psi) &\leq \max(\deg(w_0),\deg(w_1),\deg(w_2)) \\ &\leq 2\max(\deg r_1, \deg s_1) + 2\max(\deg r_2, \deg s_2) \\ &= 2\deg\phi + 2\deg\psi \end{aligned} \tag{1}$$

Now replace ϕ and ψ by $\phi+\psi$ and $\phi-\psi$ to get

$$\deg(2\phi)+\deg(2\psi) \leq 2\deg(\phi+\psi)+2\deg(\phi-\psi)$$

Since $\deg[2] = 4$,

$$2\deg(\phi)+2\deg(\psi) \leq \deg(\phi+\psi)+\deg(\phi-\psi) \tag{2}$$

(1) and (2) together give

$$2\deg(\phi)+2\deg(\psi) = \deg(\phi+\psi)+\deg(\phi-\psi)$$

so deg satisfies the parallelogram law, and hence is a quadratic form. □

Corollary 5.8.

$$deg(n\phi) = n^2\deg(\phi)\ \forall n \in \mathbb{Z}, \phi \in \mathrm{Hom}(E_1, E_2)$$

In particular, $\deg[n] = n^2$.

Example 5.9. Let E/K be an elliptic curve, suppose $\operatorname{char} K \neq 2$, and let $O_E \neq T \in E(K)[2]$.

Then we may take $E : y^2 = x(x^2+ax+b), a, b \in K, b(a^2-4b) \neq 0, T = (0,0)$

Then if $P = (x, y)$ and $P' = P + T = (x', y')$, then:

$$\begin{aligned} x' &= (y/x)^2 - a - x = \frac{x^2+ax+b}{x} - x - a - = \frac{b}{x} \\ y' &= -(y/x)x' = \frac{-by}{x^2} \end{aligned}$$

Then let $\xi = x + x' + a = \frac{x^2+ax+b}{x} = \left(\frac{y}{x}\right)^2$, and $\eta = y + y' = \frac{y}{x}(x - \frac{b}{x})$

Then $\eta^2 = \left(\frac{y}{x}\right)^2\left[\left(x+\frac{b}{x}\right)^2 - 4b\right] = \xi\left((\xi-a)^2-4b\right) = \xi(\xi^2 - 2a\xi + a^2 - 4b)$

Let $E' : y^2 = x(x^2 + a'x + b')$ where $a' = -2a, b' = a^2 - 4b$. Then there is an isogeny $\phi : E \to E'$ given by $(x, y) \mapsto \left(\left(\frac{y}{x}\right)^2 : \frac{y(x^2-b)}{x^2} : 1\right); O_E \mapsto (0:1:0)$

5.3 tells us, as $x' = \left(\frac{y}{x}\right)^2 = \frac{x^2+ax+b}{x}$, that $\deg(\phi) = \max(2,1) = 2$, and we say ϕ is a 2-isogeny.

6 The Invariant Differential

Let C be an algebraic curve over an algebraically closed field. Then the ***space of differentials*** Ω_C is a vector space over the function field of the curve $K(C)$, generated by df for $f \in K(C)$ subject to the relations

1. $d(f+g) = df + dg$
2. $d(fg) = fdg + gdf$
3. $da = 0$ for $a \in K$

It turns out that $\dim \Omega_C = \dim C$, and since C is a curve, Ω_C is a 1-dimensional $K(C)$-vector space.

Let $0 \neq \omega \in \Omega_C$, and let $P \in C$ be a smooth point, with $t \in K(C)$ a uniformizer at P (has order of vanishing 1 at P). Then $\omega = fdt$ for some $f \in K(C)$.

We define $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$. This does not depend on the choice of uniformizer.

Suppose we have $f \in K(C)^*$, and $\mathrm{ord}_P(f) = n \neq 0$. Then, if $\mathrm{char}\, K \nmid n$, $\mathrm{ord}_P(df) = n - 1$.

If C is now a smooth projective curve, we define the divisor of $\omega \in \Omega_C$ to be

$$\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega) P \in \mathrm{Div}(C)$$

using the fact that $\mathrm{ord}_P(\omega)$ is zero at all but finitely many points $P \in C$.

The ***space of regular differentials*** is the finite dimensional vector space over K of all $\omega \in \Omega_C$ for which $\mathrm{div}(\omega)$ is effective, i.e. there are no poles. The dimension of this space is called the ***genus*** of C, $g(C)$.

As a consequence of Riemann-Roch, we have, for $0 \neq \omega \in \Omega_C$, $\deg(\mathrm{div}(\omega)) = 2g(C) - 2$.

Lemma 6.1. *Assume* $\mathrm{char}\, K \neq 2$. *Take an elliptic curve* $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$, *where* e_1, e_2, e_3 *distinct.*

Then $\omega = \frac{dx}{y}$ *is a differential on* E, *and has no zeros and no poles, and so* $g(E) = 1$.

Moreover, the space of regular differentials is just $\langle \omega \rangle$.

Proof. Let $T_i = (e_i, 0)$, so that $E[2] = \{O, T_1, T_2, T_3\}$.

Then $\mathrm{div}(y) = (T_1) + (T_2) + (T_2) - 3(O)$ - we know the zeros at T_i are simple as y is rational, so $\deg \mathrm{div}(y) = 0$.

Then for $P \in E$, $\mathrm{div}(x - x_P) = (P) + (-P) - 2(O)$, in the same way as above.

If $P \in E \setminus E[2]$, then $\mathrm{ord}_P(x - x_P) = 1$, so $\mathrm{ord}_P(d(x - x_P)) = \mathrm{ord}_P(dx) = 1 - 1 = 0$.

If $P = T_i$, then $P = -P$, and $\mathrm{ord}_P(x - x_p) = 2$, so $\mathrm{ord}_P(dx) = 2 - 1 = 1$

If $P = O$, then $\mathrm{ord}_P(x) = -2$, so $\mathrm{ord}_P(dx) = -3$.

Hence $\mathrm{div}(dx) = (T_1) + (T_2) + (T_3) - 3(O) = \mathrm{div}(y)$.

So $\mathrm{div}(dx/y) = \mathrm{div}(dx) - \mathrm{div}(y) = 0$. Then Riemann-Roch gives $g(E) = 1$, and so the space of regular differentials is 1-dimensional, so generated by ω. □

Definition. If $\phi : C_1 \to C_2$ is a non-constant morphism, then we can pull back to

$$\phi^* : \Omega_{C_1} \to \Omega_{C_2}; f dg \mapsto \phi^* f d(\phi^* g)$$

Lemma 6.2. *Let $P \in E$, $\tau_P : E \to E; X \mapsto P + X$, and $\omega = dx/y$ be as above.*

Then $\tau_P^* \omega = \omega$, and so ω is called the **invariant differential**.

Proof. Since ω had no poles, $\tau_P^* \omega$ is again a regular differential, and hence equal to $\lambda_P \omega$ for some $\lambda_P \in K$, as the regular differentials are a 1-dimensional vector space over K.

The map $E \to \mathbb{P}^1; P \mapsto \lambda_P$ is a morphism of smooth projective curves, but is not surjective as it misses 0 and ∞, and so this morphism is constant, by **2.8**.

So λ is independent of P. Take $P = O_E$, then τ_P is the identity map, and so λ is 1. □

If $K = \mathbb{C}$, then $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, via $z \mapsto (\wp(z), \wp'(z))$. Then $\frac{dx}{y} = \frac{\wp'(z)dz}{\wp'(z)} = dz$, which is invariant under $z \mapsto z +$ const..

Lemma 6.3. *Let $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$, ω the invariant differential on E_2. Then*

$$(\phi + \psi)^*(\omega) = \phi^* \omega + \psi^* \omega$$

Proof. Write $E = E_2$, and consider the maps:

$$\begin{aligned} E \times E &\to E \\ \mu : (P, Q) &\mapsto P + Q \\ \mathrm{pr}_1 : (P, Q) &\mapsto P \\ \mathrm{pr}_2 : (P, Q) &\mapsto Q \end{aligned}$$

$\Omega_{E \times E}$ is a 2-dimensional $K(E \times E)$ vector space with basis $\mathrm{pr}_1^* \omega$ and $\mathrm{pr}_2^* \omega$.

Then $\mu^* \omega = f \mathrm{pr}_1^* \omega + g \mathrm{pr}_2^* \omega$ for some $f, g \in K(E \times E)$.

For $Q \in E$, let $\iota_Q : E \to E \times E; P \mapsto (P, Q)$. Then

$$\begin{aligned} \iota_Q^*(\mu^* \omega) = (\mu \circ \iota_Q)^* \omega &= \iota_Q^* f (\mathrm{pr}_1 \circ \iota_Q)^* \omega + \iota_Q^* g (\mathrm{pr}_2 \circ \iota_Q)^* \omega \\ \tau_Q^* \omega &= \iota_Q^* f \omega + 0 \\ \omega &= \iota_Q^* f \omega \end{aligned}$$

So $\iota_Q^* f = 1$ for all $Q \in E$, so $f(P, Q) = 1$ for all $P, Q \in E$.

Similarly, $g(P, Q) = 1$.

So $\mu^* \omega = \mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega$. Now pull back by $E \to E \times E; P \mapsto (\phi(P), \psi(P))$ to get $(\phi + \psi)^* \omega = \phi^* \omega + \psi^* \omega$. □

Lemma 6.4. *If $\phi : C_1 \to C_2$ is a non-constant morphism, then ϕ is separable if and only if $\phi^* : \Omega_{C_2} \to \Omega_{C_1}$ is nonzero*

Proof. Omitted. □

Example: Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$, with group law $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m; (x, y) \mapsto xy$.

Let $n \geq 2$ be an integer, $\alpha : \mathbb{G}_m \to \mathbb{G}_m; x \mapsto x^n$.

Then $\alpha^*(dx) = d(\alpha x) = d(x^n) = nx^{n-1}dx$. So if $\operatorname{char} K \nmid n$, then α is separable. So $\#\alpha^{-1}(Q) = \deg \alpha$ for all but finitely many $Q \in \mathbb{G}_m$.

But α is group homomorphism, so all fibres have the same size, and $\#\alpha^{-1}(Q) = \# \ker \alpha$, hence $\# \ker \alpha = \deg \alpha = n$. So $K(= \bar{K})$ contains exactly n n^{th} roots of unity.

Theorem 6.5. *If* $\operatorname{char} K \nmid n$*, then* $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$.

Proof. By **6.3** and induction, $[n]^*\omega = n\omega$. So if $\operatorname{char} K \nmid n$, $[n]$ is separable. So all but finitely many fibres of $[n]$ have size $\deg[n]$, and since $[n]$ is a group homomorphism, all fibres have the same size, and hence $\#[n]^{-1}(O_E) = \#E[n] = \deg[n] = n^2$.

By the structure theorem for finite abelian groups, $E[n] \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \mathbb{Z}/d_t\mathbb{Z}$ with $d_i | d_{i+1}$. Since this group is killed by multiplication by n, all $d_i | n$ as well, and $\prod_{i=1}^t d_i = n^2$ by the previous paragraph.

If p is a prime with $p | d_1$, then $E[p] \cong (\mathbb{Z}/p\mathbb{Z})^t$, and by the first paragraph, $t = 2$. Then $d_1 | d_2 | n$, and $d_1 d_2 = n^2$, hence $d_1 = d_2 = n$. □

Remark (not to be used on example sheet 2). If $\operatorname{char} K = p$, then $[p]$ is not separable. It can be shown that $E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z}$ for all $r \geq 1$ or $E[p] = 0$. The first case is described as "ordinary", and the second case is "supersingular".

7 Elliptic Curves over Finite Fields

Lemma 7.1. *Let A be an abelian group and $q : A \to \mathbb{Z}$ a positive definite quadratic form. If $x, y \in A$ then $\langle x, y \rangle := |q(x + y) - q(x) - q(y)| \leq 2\sqrt{q(x)q(y)}$.*

Proof. We may assume $x \neq 0$ otherwise the result is clear. Let $m, n \in \mathbb{Z}$.

$$\begin{aligned}
0 &\leq q(mx + ny) \\
&= \frac{1}{2}\langle mx + ny, mx + ny \rangle \\
&= m^2 q(x) + mn\langle x, y \rangle + n^2 q(y) \\
&= q(x)\left(m + \frac{\langle x, y \rangle}{2q(x)} n\right)^2 + n^2\left(q(y) - \frac{\langle x, y \rangle^2}{4q(x)}\right)
\end{aligned}$$

Take $m = \langle x, y \rangle, n = -2q(x)$, we deduce $\langle x, y \rangle^2 \leq 4q(x)q(y)$, so $|anglex, y| \leq 2\sqrt{q(x)q(y)}$. □

Recall that $\operatorname{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ is cyclic of order r generated by the Frobenius map $x \mapsto x^q$.

Theorem 7.2 (Hasse). *Let $E/\mathbb{F}_q$ be an elliptic curve. Then $|\#E(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q}$*

Proof. Let E have Weierstrass equation with coefficients $a_1, \ldots, a_6 \in \mathbb{F}_q$. Define the Frobenius endomorphism $\phi : E \to E; (x, y) \mapsto (x^q, y^q)$, which is an isogeny of degree q.

Then $E(\mathbb{F}_q) = \{P \in E : \phi(P) = P\} = \ker(1 - \phi)$.

$\phi^*\omega = \phi^*\left(\frac{dx}{y}\right) = \frac{dx^q}{y^q} = \frac{qx^{q-1}dx}{y^q} = 0$, since $q \equiv 0 \mod p$.

So $(1-\phi)^*\omega = 1^*\omega - \phi^*\omega = \omega - 0 = \omega \neq 0$, so $1-\phi$ is separable.

Hence the size of all but finitely many fibres is $\deg 1-\phi$, and $1-\phi$ is a group homomorphism, so $\#E[\mathbb{F}_q] = \#\ker(1-\phi) = \deg(1-\phi)$.

By **5.6**, $\deg : \mathrm{End}(E) \coloneqq \mathrm{Hom}(E,E) \to \mathbb{Z}$ is a positive definite quadratic form.

By **7.1**, $|\deg(1-\phi) - 1 - \deg\phi| \leq 2\sqrt{\deg\phi}$, and hence $|\#E(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}$. □

7.1 Zeta Functions

For K a number field:

$$\zeta_K(s) = \sum_{\mathfrak{a}\subset O_K} \frac{1}{(N_\mathfrak{a})^s} = \prod_{\mathfrak{p}\subset O_K \text{ prime}} \left(1 - \frac{1}{(N_\mathfrak{p})^s}\right)^{-1}$$

For K a function field, e.g. $K = \mathbb{F}_q(C)$ for $C/\mathbb{F}_q$ a smooth projective curve:

$$\zeta_K(s) = \prod_{x\in|C|} \left(1 - \frac{1}{(Nx)^s}\right)^{-1}$$

where $|C|$ is the set of closed points (i.e. orbit of action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$) on $C(\bar{\mathbb{F}}_q)$, and $Nx = q^{\deg x}$, where $\deg x$ is the size of the orbit.

We have that $\zeta_K(s) = F(q^{-s})$ for $F \in Q[[T]]$, where

$$\begin{aligned}
F(T) &= \prod_{x\in|C|} (1-T^{\deg x})^{-1} \\
\log F(T) &= \sum_{x\in|C|} \sum_{m=1}^{\infty} \frac{1}{m} T^{m\deg x} \\
\frac{d}{dT}\log F(T) &= \sum_{x\in|C|} \sum_{m=1}^{\infty} \deg x T^{m\deg x} \\
&= \sum_{n=1}^{\infty} \left(\sum_{\substack{x\in|C| \\ \deg x|n}} \deg x \right) T^n \\
&= \sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n}) T^n \\
\implies F(T) &= \exp\left(\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} T^n \right) \eqqcolon Z_C(T)
\end{aligned}$$

For $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$, we put:

$$\langle \phi, \psi \rangle = \deg(\phi+\psi) - \deg(\phi) - \deg(\psi)$$

We define the ***trace map*** $\mathrm{tr} : \mathrm{End}(E) \to \mathbb{Z}; \psi \mapsto \langle \psi, 1 \rangle$.

Lemma 7.3. *If $\psi \in \mathrm{End}(E)$ then $\psi^2 - [\mathrm{tr}\,\psi]\psi + [\deg\psi] = 0$, where $[n]$ means the multiplication by n endomorphism.*

Proof. Example sheet 2. $\square$

Definition. The ***zeta function of a variety*** $V/\mathbb{F}_q$ is

$$Z_v(T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{q^n})}{n} T^n\right)$$

Lemma 7.4. *Let $E/\mathbb{F}_q$ be an elliptic curve, with $\#E(\mathbb{F}_q) = q + 1 - a$. Then*

$$Z_E(T) = \frac{1 + aT + qT^2}{(1-T)(1-qT)}$$

Proof. Let $\phi : E \to E$ be the q-power Frobenius map. By the proof of Hasse's theorem,

$$\#E(\mathbb{F}_q) = \deg(1-\phi) = q + 1 - \mathrm{tr}(\phi)$$

Then $\mathrm{tr}(\phi) = a, \deg(\phi) = q$.

Then lemma **7.3** gives $\phi^2 - a\phi + q = 0$. Composing with ϕ^n for $n \geq 0$ gives

$$\begin{aligned} \phi^{n+2} - a\phi^{n+1} + q\phi^n &= 0 \\ \mathrm{tr}(\phi^{n+2}) - a\,\mathrm{tr}(\phi^{n+1}) + q\,\mathrm{tr}(\phi^n) &= 0 \end{aligned}$$

This second-order difference equation with initial conditions $\mathrm{tr}(\phi^0) = \mathrm{tr}(1) = 2, \mathrm{tr}(\phi^1) = a$ has solutions

$$\mathrm{tr}(\phi^n) = \alpha^n + \beta^n$$

where α, β are the roots of $x^2 - ax + q = 0$.

Hence $\#E(\mathbb{F}_{q^n}) = \deg(1-\phi^n) = 1 + \deg(\phi^n) - \mathrm{tr}(\phi^n) = 1 + q^n - \alpha^n - \beta^n$.

Substituting, we have:

$$Z_E(T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} + \frac{(qT)^n}{n} - \frac{(\alpha T)^n}{n} - \frac{(\beta T)^n}{n}\right)$$

Since $-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, this can be simplified to:

$$\begin{aligned} Z_E(T) &= \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-qT)} \\ &= \frac{1 - aT + qT^2}{(1-T)(1-qT)} \end{aligned}$$

$\square$

Note that Hasse's theorem gives us $|a| \leq 2\sqrt{q}$, and so the discriminant of $x^2 - aT + q$ is negative, and so $\alpha = \bar{\beta}, |\alpha| = |\beta| = \sqrt{q}$.

Let $K = \mathbb{F}_q(E)$. Then $\zeta_K(s) = 0 \implies Z_E(q^{-s}) = 0 \implies q^2 = \alpha$ or β, and hence $\mathfrak{Re}(s) = \frac{1}{2}$.

8 Formal Groups

Here, R will be a ring with $I \subset R$ an ideal. The ***I-adic topology*** on R is the topology with basis $\{r + I^n : r \in R, n \geq 1\}$.

A sequence (x_n) in R is ***Cauchy*** if, for all k there is some N with $x_m - x_n \in I^k$ for all $m, n \geq k$.

R is ***complete*** if

1. $\bigcap_{n \geq 0} I^n = \{0\}$ and
2. every Cauchy sequence converges.

Note that, if $x \in I$ then $\frac{1}{1-x} = 1 + x + x^2 + \ldots$, and the sequence of partial sums is Cauchy, and hence converges. So $1 - x \in R^\times$.

For example, we could have:

- $R = \mathbb{Z}_p, I = p\mathbb{Z}_p$
- $R = \mathbb{Z}[[t]], I = (t)$.

Lemma 8.1 (Hensel's Lemma)**.** *Let R be an integral domain, complete with respect to I. Let $F \in R[x]$, $s \geq 1$. Suppose $a \in R$ satisfies $F(a) \equiv 0 \mod I^s$, and $F'(a) \in R^\times$.*

Then there is a unique $b \in R$ with $F(b) = 0$ and $b \equiv a \mod I^s$.

Proof. Let $u \in R^\times$ with $F'(a) \equiv u \mod I$, e.g. $u = f'(a)$.

Replacing $F(x)$ by $\frac{F(x+a)}{u}$, we may assume $a = 0$ and $F'(0) \equiv 1 \mod I$.

We put $x_0 = 0, x_{n+1} = x_n - F(x_n)$.

By induction, $x_n \in I_s$ for all n.

$F(x) - F(y) = (x - y)(F'(0) + xG(x, y) + yH(x, y))$ for some polynomials $G, H \in R[x, y]$.

Now we claim $x_{n+1} \equiv x_n \mod I^{n+s}$ for all $n \geq 0$.

This can be proven by induction on n: in the case where $n = 0$, and $x_1 \in I^s$.

Suppose $x_n \equiv x_{n-1} \mod I^{n+s-1}$. Then

$$F(x_n) - F(x_{n-1}) = (x_n - x_{n-1})(1 + c)$$

for some $c \in I$, and hence

$$F(x_n) - F(x_{n-1}) \equiv x_n - x_{n-1} \mod I^{n+s}$$

Rearranging, we have $x_{n+1} \equiv x_n \mod I^{n+s}$, which proves the claim.

Hence (x_n) is Cauchy, and by completeness converges to some $b \in R$. Taking the limit as $n \to \infty$, we have $b = b - F(b)$, and so $F(b) = 0$, with $b \in I^s$.

For uniqueness, we can use the expression for $F(x) - F(y)$ and the assumption that R is an integral domain. $\square$

For example, take $E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$

We pass to the affine piece $Y \neq 0, t = X/Y, w = -Z/Y$: Then

$$E : w = t^3 + a_1tw + a_2t^2w + a_3w^2 + a_4tw^2 + a_6w^3 = f(t, w)$$

We can apply Hensel's lemma with $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, $I = (t)$, and $F(x) = x - f(t, x) \in R[x]$. Taking $s = 3, a = 0$, we have:

$$F(0) = -f(t, 0) = -t^3 \equiv 0 \mod I^3 \qquad F'(0) = 1 - a_t - a_2t^2 \in R^\times$$

So there is a unique root of F, $w(t) \in \mathbb{Z}[a_1, \dots, a_6][[t]]$ such that $w(t) = f(t, w(t))$ and $w(t) \equiv 0 \mod t^3$.

Following the proof of Hensel's lemma with $u = 1$ gives $w(t) = \lim_{n\to\infty} w_n(t)$ where $w_0(t) = 0, w_{n+1}(t) = f(t, w_n(t))$.

In fact, we may write $w(t) = \sum_{n=2}^{\infty} A_{n-2}t^{n-1}$ with $A_1 = a_1, A_2 = a_1^2 + a_2, A_3 = a_1^3 + 2a_1a_2 + a_3, \dots$

Lemma 8.2. *Let R be an integral domain, complete with respect to $I \trianglelefteq R$, and let $a_1, \dots, a_6 \in R, K =$* $\mathrm{Frac}(R)$*.*

Then $\widehat{E}(I) = \{(t, w) \in E(K) : t, w \in I\} = \{(t, w(t)) \in E(K) : t \in I\}$ is a subgroup of $E(K)$.

Proof. The two descriptions of $\widehat{E}(I)$ agree, since given $t \in I$ we can solve for a unique $w \in I$ such that the pair $(t, w) \in E(K)$.

Taking $(t, w) = (0, 0)$ shows that $O_E \in \widehat{E}(I)$. So it suffices to show that, if $P_1, P_2 \in \widehat{E}(I)$, then $-P_1 - P_2 \in \widehat{E}(I)$.

If $P_1 = (t_1, w_1), P_2 = (t_2, w_2)$ lie on the straight line $\lambda t + \nu$, then $-P_1 - P_2$ is the third point of intersection of this line with E.

Then $\lambda = \frac{w(t_2)-w(t_1)}{t_2-t_1}$ if $t_1 \neq t_2$, and $w'(t_1)$ if $t_1 = t_2$.

$P_1, P_2 \in \widehat{E}(I) \implies t_1, t_2 \in I$.

Thus $\lambda = \sum_{n=2}^{\infty} A_{n-2}(t_1^n + t_1^{n-1}t_2 + \dots + t_2^n) \in I$, and $\nu = w_1 - \lambda t_1 \in I$.

Substituting $w = \lambda t + \nu$ into $w = f(t, w)$ gives $\lambda t + \nu = t^3 + a_1t(\lambda t + \nu) + a_2t^2(\lambda t + \nu) + a_3(\lambda t + \nu)^2 + a_4t(\lambda t + \nu)^3 + a_6(\lambda t + \nu)^3$.

Let A be the coefficient of t^3, so $A = 1 + a_2\lambda + a_4\lambda^2 + a_6\lambda^3$.

Let B be the coefficient of t^2, so $B = a_1\lambda + a_2\nu + a_3\lambda^2 + 2a_4\lambda\nu + 3a_6\lambda^2\nu$.

Then $A \in R^\times, B \in I$, and $t_3 = -B/A - t_2 - t_2 \in I$, and $w_3 = \lambda t_3 + \nu \in I$.

Hence $-P_1 - P_2 \in \widehat{E}(I)$, and so $\widehat{E}(I)$ is a subgroup. □

Taking $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, and $I = (t)$, then the previous lemma tells us there is some power series $\iota \in \mathbb{Z}[a_1, \dots, a_6][[t]]$ with $\iota(0) = 0$ such that $[-1](t, w(t)) = (\iota(t), w(\iota(t)))$

Taking $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, and $I = (t_1, t_2)$, then we get that there is some power series $F \in I$ such that $(t_1, w(t_1)) + (t_2, w(t_2)) = (F(t_1, t_2), w(F(t_1, t_2))$.

In fact, we can compute

$$\iota(x) = -x - a_1x^2 - a_2x^3 - (a_1^3 + a_3)x^4 + \dots$$
$$F(x, y) = x + y - a_1xy - a_2(x^2y + xy^2) + \dots$$

By properties of the group law, we can deduce:

1. $F(x, y) = F(y, x)$
2. $F(x, 0) = x, F(0, y) = y$
3. $F(x, F(y, z)) = F(F(x, y), z)$
4. $F(x, \iota(x)) = 0$

This then motivates the following definition:

Definition. Let R be a ring. A ***formal group*** over R is a power series $F(x, y) \in R[[x, y]]$ satisfying the properties 1, 2, and 3 above.

Exercise. Show that, for any formal group, there is a unique $\iota(x) \in R[[x]]$ such that $F(x, \iota(x)) = 0$.

Examples:

1. $F(x, y) = x + y$
2. $F(x, y) = x + y + xy = (1 + x)(1 + y) - 1$
3. F as above.

We label these formal groups by $\widehat{\mathbb{G}}_a$, $\widehat{\mathbb{G}}_m$, and $\widehat{E}$ respectively.

Definition. Let $\mathcal{F}$, $\mathcal{G}$ be formal groups over R given by power series F, G respectively. Then:

1. A ***morphism*** $f : \mathcal{F} \to \mathcal{G}$ is a power series $f \in R[[t]]$ such that $f(0) = 0$ satisfying $f(F(x, y)) = G(f(x), f(y))$.
2. $\mathcal{F} \cong \mathcal{G}$ if there is some morphism $f : \mathcal{F} \to \mathcal{G}$, and $g : \mathcal{G} \to \mathcal{F}$ with $f(g(x)) = g(f(x)) = x$.

Theorem 8.3. *If* $\operatorname{char}(R) = 0$*, then any formal group* $\mathcal{F}$ *over* R *is isomorphic to* $\widehat{\mathbb{G}}_a$ *over* $R \otimes \mathbb{Q}$.

More precisely:

1. *There is a unique power series* $\log : T \mapsto T + \frac{a_2}{2}T^2 + \frac{a_3}{3}T^3 + \dots$ *with* $a_i \in R$*, such that*
$$\log(F(x, y)) = \log(x) + \log(y) \tag{$*$}$$
2. *There is a unique power series* $\exp : T \mapsto T + \frac{b_2}{2!}T^2 + \frac{b_3}{3!}T^3 + \dots$ *with* $b_i \in R$ *such that*
$$\exp(\log(T)) = \log(\exp(T)) = T$$

Proof.

1. Notation: $F_1(x, y) = \frac{\partial F}{\partial x}(x, y)$ (via formal differentiation).

   For uniqueness, let $p(T) = \frac{d}{dT}\log(T) = 1 + a_2T + a_3T^2 + \dots$.

Differentiating $(*)$ with respect to x, we get: $p(F(x,y))F_1(x,y) = p(x) + 0$ Setting $x = 0$, we get $p(y)F_1(0,y) = 1$, and hence $p(y) = F_1(0,y)^{-1}$, and hence p is uniquely determined, so $a_2, a_3, \ldots$ are uniquely determined. But then log is uniquely determined.

For existence, let $p(T) = F_1(0,T)^{-1} = 1 + a_2T + a_3T^2 + \ldots$, where $a_i \in R$.

Integrating up, we let $\log(T) = T + \frac{a_2}{2}T^2 + \frac{a_3}{3}T^3 + \ldots$. We now check it satisfied $(*)$.

$$\begin{aligned}
F(F(x,y),z) &= F(x,F(y,z)) \\
\frac{\partial}{\partial x}F(F(x,y),z) &= \frac{\partial}{\partial x}F(x,F(y,z)) \\
F_1(F(x,y),z)F_1(x,y) &= F_1(x,F(y,z)) \\
F_1(F(0,y),z)F_1(0,y) &= F_1(0,F(y,z)) \\
F_1(y,z)F_1(0,y) &= F_1(0,F(y,z)) \\
F_1(y,z)p(y)^{-1} &= p(F(y,z))^{-1} \\
F_1(y,z)p(F(y,z)) &= p(y) \\
\log(F(y,z)) &= \log(y) + h(z)
\end{aligned}$$

By symmetry between y, z we see that the constant of integration $h(z)$ must be $\log(z)$.

For the second part, we will need the following lemma, which is a generalisation of the statement:

Lemma 8.4. *Let $f(T) = aT + \ldots \in R[[T]]$ with $a \in R^\times$. Then there is a unique $g(T) = a^{-1}T + \ldots \in R[[t]]$ such that $f(g(T)) = g(f(T)) = T$.*

Proof. We construct polynomials $g_n(T) \in R[T]$ such that $f(g_n(T)) \equiv T \mod T^{n+1}$ and $g_{n+1}(T) \equiv g_n(T) \mod T^{n+1}$. Then we will set $g(T) = \lim_{n\to\infty} g_n(T)$, satisfying $f(g(T)) = T$.

This is done inductively. To start with, $g_1(T) = a^{-1}T$. Then $f(g_1(T)) = T + T^2(\ldots) \equiv T \mod T^2$.

Now suppose $n \geq 1$ and $g_{n-1}(T)$ exists.

Then $f(g_{n-1}(T)) \equiv T + bT^n \mod T^{n+1}$. Let $g_n(T) = g_{n-1}(T) + \lambda T^n$, where $\lambda \in R$ to be chosen later.

Then $f(g_n(T)) = f(g_{n-1}(T) + \lambda T^n) \equiv f(g_{n-1}(T)) + \lambda a T^n \mod T^{n+1} \equiv T + (b + \lambda a)T^n \mod T^{n+1}$.

So pick $\lambda = -ba^{-1}$.

This gives $g(T)$ with $f(g(T)) = T$.

Applying the same argument, we get $h(T)$ such that $g(h(T)) = T$.

Then $f(T) = f(g(h(T))) = h(T)$, and so g is as required. □

2. We now only have to show that the $b_n \in R$ (not just in $R \otimes \mathbb{Q}$). See example sheet 2 for this.

□

Let $\mathscr{F}$ be a formal group (e.g. $\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_m, \widehat{E}$), given by a power series $F \in R[x,y]$, and suppose that R is I-adically complete. Then for $x, y \in I$, put $x \oplus_{\mathscr{F}} y = F(x,y) \in I$. Then $\mathscr{F} = (I, \oplus_{\mathscr{F}})$ is an abelian group.

For example, $\widehat{\mathbb{G}}_a(I) = (I, +)$, $\widehat{\mathbb{G}}_m(I) = (1 + I, \times)$, and in **8.2**, we saw $\widehat{E}(I) \leq E(K)$.

Corollary 8.5. *Let $\mathcal{F}$ be a formal group over R, and $n \in \mathbb{Z}$. Suppose $n \in R^\times$. Then:*

1. *$[n] : \mathcal{F} \longrightarrow \mathcal{F}$ is an isomorphism.*
2. *If R is complete with respect to I, then $\mathcal{F}(I) \xrightarrow{\times n} \mathcal{F}(I)$ is an isomorphism.*

In particular, $\mathcal{F}(I)$ has no n-torsion.

Proof. We have $[1](T) = T, [n](T) = F([n-1]T, T)$ for $n \geq 2$. For $n < 0$, use $[-1](T) = \iota(T)$.

Induction gives us $[n](T) = nT + \ldots$, and so by **8.4**, $[n]$ is an isomorphism. □

9 Elliptic Curves over Local Fields

Let K be a field, complete with respect to the discrete valuation $v : K^\times \longrightarrow \mathbb{Z}$. Then we define the valuation ring, or ring of integers, the set:

$$O_K = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}$$

Then $O_K^\times = \{x \in K^\times : v(x) = 0\}$. There is a unique maximal ideal πO_K, where $v(\pi) = 1$, and we define the residue field to be $k = O_K / \pi O_K$.

We assume $\operatorname{char} K = 0, \operatorname{char} k = p$.

For example, if $K = \mathbb{Q}_p, O_K = \mathbb{Z}_p, \pi = p, k = \mathbb{F}_p$.

Let E/K be an elliptic curve. Then a Weierstrass equation for E with coefficients $a_1, \ldots, a_6 \in K$ is ***integral*** if $a_i \in O_K$, and minimal if $v(\Delta)$ is minimal among all integral Weierstrass equations for E.

Putting $x = u^2 x', y = u^3 y'$ give $a_i = u^i a_i'$. So we can clear denominators, and hence every elliptic curve has an integral Weierstrass equation. Moreover, since $a_i \in O_K, \Delta \in O_K$, and so $v(\Delta) \geq 0$, and hence we can pick a minimal Weierstrass equation.

If $\operatorname{char} k \neq 2, 3$ then there is a minimal Weierstrass equation of the form $y^2 = x^3 + ax + b$.

Lemma 9.1. *Let E/K have integral Weierstrass equation*

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Let $0 \neq P = (x, y) \in E(K)$. Then either $x, y \in O_K$ or $v(x) = -2s, v(y) = -3s$ for some $s \geq 1$.

Compare this to example sheet 1, question 5.

Proof. If $v(x) \geq 0$, then consider y.

If $v(y) < 0$, then $v(LHS) < 0$, but $v(RHS) \geq 0 \lightning$, and hence $x, y \in O_K$.

Now if $v(x) < 0$, then $v(LHS) \geq \min(2v(y), v(x) + v(y), v(y))$
$v(RHS) = v(x^3) = 3v(x)$.

Hence $v(y) < v(x)$. But then $v(LHS) = 2v(y)$, and hence $3v(x) = 2v(y)$. □

If K is complete, then O_K is complete with respect to the ideal $\pi^r O_K$ for any $r \geq 1$.

Fix a minimal Weierstrass equation for E/K, and hence a formal group $\widehat{E}$ over O_K.

Take $I = \pi^r O_K$ in **8.2**, we have

$$\begin{aligned}
\widehat{E}(\pi^r O_K) &= \left\{(x,y) \in E(K) : -\frac{x}{y}, -\frac{1}{y} \in \pi^r O_K\right\} \cup \{0\} \\
&= \left\{(x,y) \in E(K) : v\left(\frac{x}{y}\right) \geq r \& v\left(\frac{1}{y}\right) \geq r\right\} \cup \{0\} \\
&= \{(x,y) \in E(K) : v(x) = -2s, v(y) = -3s, s \geq r\} \cup \{0\} \\
&= \{(x,y) \in E(K) : v(x) \leq -2r, v(y) \leq -3r\} \cup \{0\}
\end{aligned}$$

By **8.2**, this is a subgroup of $E(K)$, say $E_r(K)$. We have a chain

$$\ldots \subset E_3(K) \subset E_2(K) \subset E_1(K)$$

More generally, for $\mathcal{F}$ a formal group over O_K, we get

$$\ldots \subset \mathcal{F}(\pi^3 O_K) \subset \mathcal{F}(\pi^2 O_K) \subset \mathcal{F}(\pi O_K)$$

We will show that $\mathcal{F}(\pi^r O_K) \cong (O_K, +)$ for r sufficiently large, and $\mathcal{F}(\pi^r O_K)/\mathcal{F}(\pi^{r+1} O_K) \cong (k, +)$.

Theorem 9.2. *Let $\mathcal{F}$ be a formal group over O_K, and let $e = v(p)$. If $r > \frac{e}{p-1}$, then:*

$$\mathcal{F}(\pi^r O_K) \cong \widehat{\mathbb{G}}_a(\pi^r O_K)$$

via the log *map, with inverse given by* exp.

Note that $\widehat{\mathbb{G}}_a(\pi^r O_K) = (\pi^r O_K, +) \cong (O_k, +)$.

Proof. For $x \in \pi^r O_K$, we must check that the power series exp, log converge.

Recall $\exp(T) = T + \frac{b_2}{2!}T^2 + \frac{b_3}{3!}T^3 + \ldots$, where $b_i \in O_K$.

Claim: $v_p(n!) \leq \frac{n-1}{p-1}$.

To see this: $v_p(n!) = \sum_{r=1}^{\infty} \lfloor \frac{n}{p^r} \rfloor < \sum_{r=1}^{\infty} \frac{n}{p^r} = \frac{n}{p-1}$.

So $(p-1)v_p(n!) < n$, and as both are integers, $(p-1)v_p(n!) \leq n-1$.

Now $v(\frac{b_n x^n}{n!}) \geq nr - e\frac{n-1}{p-1} = (n-1)\left(r - \frac{e}{p-1}\right) + r$

This is always $\geq r$ as $r > \frac{e}{p-1}$, and goes to infinity as $n \to \infty$.

Hence $\exp(x)$ converges, and belongs to $\pi^r O_K$. A similar argument applies for log. □

Lemma 9.3. *We have* $\frac{\mathcal{F}(\pi^r O_K)}{\mathcal{F}(\pi^{r+1} O_K)} \cong (k, +)$ *for all* $r \geq 1$.

Proof. By definition of a formal group, $F(x,y) = x + y + xy(\ldots)$. So if $x, y \in O_K$, then:

$$F(\pi^r x, \pi^r y) = \pi^r(x+y) + \pi^{2r}(xy)(\ldots) \equiv \pi^r(x+y) \mod \pi^{r+1}$$

So $\mathcal{F}(\pi^r O_K) \to (k, +); (\pi^r x) \mapsto (x \mod \pi)$ is a surjective group homomorphism, with kernel $\mathcal{F}(\pi^{r+1} O_K)$, and so apply the first isomorphism theorem. □

So we have a filtration:

$$(\mathcal{O}_K,+) \cong \mathscr{F}(\pi^r\mathcal{O}_K) \supseteq \ldots \supseteq \mathscr{F}(\pi^2\mathcal{O}_K) \supseteq \mathscr{F}(\pi\mathcal{O}_K)$$

where we have equality on the left is $r > \frac{e}{p-1}$, and each quotient is $(k,+)$.

Corollary. If $|k| < \infty$, then $\mathscr{F}(\pi\mathcal{O}_K)$ has a subgroup of finite index isomorphic to $\mathcal{O}_K$ under addition.

As a point of notation, when we have the map $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$, we write $x \mapsto \widetilde{x}$, and call this reduction mod π.

Proposition 9.4. *Let E/K be an elliptic curve. The reduction mod π of any two minimal Weierstrass equations for E define isomorphic curves over k.*

Proof. Say the Weierstrass equations are related by $[u;r,s,t]; u \in K^\times; r,s,t \in K$.

Then $\Delta_1 = u^{12}\Delta_2$. Both equations are minimal, so $v(\Delta_1) = v(\Delta_2)$, and hence $v(u) = 0, u \in \mathcal{O}_K^\times$.

Transformation formulae for a_i and b_i, together with the fact that the valuation ring is integrally closed, give that $r,s,t \in \mathcal{O}_K$. The Weierstrass equations for the reduction mod π are related by $[\widetilde{u};\widetilde{r},\widetilde{s},\widetilde{t}]$. □

Definition. The reduction $\widetilde{E}/k$ of E/K is defined by the reduction of a minimal Weierstrass equation, and hence is well-defined up to isomorphism by the previous proposition.

We say E has ***good reduction*** if $\widetilde{E}$ is non-singular, i.e. is an elliptic curve. Otherwise, it is ***bad***.

For an integral Weierstrass equation, $v(\Delta) = 0 \implies$ good reduction.

If $0 < v(\Delta) < 12$, then we must have a minimal Weierstrass equation, and we get bad reduction.

If $v(\Delta) \geq 12$, beware that the equation might not be minimal.

There is a well defined map from $\mathbb{P}^2(K) \to \mathbb{P}^2(k); (x:y:z) \mapsto (\widetilde{x}:\widetilde{y}:\widetilde{z})$, when we choose representatives of $(x:y:z)$ with $\min(v(x),v(y),v(z)) = 0$.

We restrict this map to give a map $E(K) \to \widetilde{E}(k); P \to \widetilde{P}$. If $P = (x,y) \in E(K)$, then by **9.1**, either $x,y \in \mathcal{O}_K$ or $v(x) = -2s, v(y) = -3s$. In the first case $\widetilde{P} = (\widetilde{x},\widetilde{y})$. In the second, we write $P = (\pi^{3s}x : \pi^{3s}y : \pi^{3s})$, so $\widetilde{P} = (0:1:0)$.

Therefore $E_1(K) = \widehat{E}(\pi\mathcal{O}_K) = \{P \in E(K) : \widetilde{P} = 0\}$, and we call it the ***kernel of reduction***.

Let $\widetilde{E}_{ns} = \begin{cases} \widetilde{E} & \text{if } E \text{ has good reduction} \\ \widetilde{E} \setminus \{p\} & \text{if } \widetilde{E} \text{ has a singular point } p \end{cases}$

The chord and tangent process still defines a group law on $\widetilde{E}_{ns}$. In cases of bad reduction, we get $\widetilde{E}_{ns} \cong \mathbb{G}_a$ or $\mathbb{G}_m$ over k, or possibly only over a quadratic extension of k. We call these cases additive and multiplicative reduction.

For simplicity, suppose $\text{char}(k) \neq 2$. Then $\widetilde{E} : y^2 = f(x)$ for f monic cubic. Then $\widetilde{E}$ singular $\iff f$ has a repeated root. The cases of double root, triple root correspond to multiplicative, additive reduction respectively.

For multiplicative case, see example sheet 3. Here, we'll illustrate the additive case. We have a triple root, so take $y^2 = x^3$. Then we have an isomorphism

$$\begin{aligned} \widetilde{E}_{ns} &\to \mathbb{G}_a \\ (x,y) &\mapsto \frac{x}{y} \\ (t^{-2}, t^{-3}) &\mapsfrom t \\ \infty &\mapsfrom 0 \end{aligned}$$

Let P_1, P_2, P_3 lie on the line $ax + by = 1$. Write $P_i = (x_i, y_i), t_i = \frac{x_i}{y_i}$. Then $x_i^3 = y_i^2 = y_i^2(ax_i + by_i)$, and so t_1, t_2, t_3 are the roots of $X^3 - aX - b = 0$. Looking at the coefficient of X^2, we have $t_1 + t_2 + t_3 = 0$.

Definition. $E_0(K) := \{P \in E(K) : \widetilde{P} \in \widetilde{E}_{ns}(k)\}$.

Proposition 9.5. *$E_0(K)$ is a subgroup of $E(K)$, and reduction mod π is a surjective group homomorphism from $E_0(k) \to \widetilde{E}_{ns}(k)$.*

Proof. For the group homomorphism part, a line ℓ in $\mathbb{P}^2$ defined over K has equation

$$\ell : aX + bY + cZ = 0 \qquad\qquad a, b, c \in K$$

We may assume $\min(v(a), v(b), v(c)) = 0$. Reduction mod π gives the line $\widetilde{\ell}$ with equation

$$\widetilde{\ell} : \widetilde{a}X + \widetilde{b}Y + \widetilde{c}Z = 0$$

If $P_1, P_2, P_3 \in E(K)$ with $P_1 + P_2 + P_3 = 0$, then these points lie on a line ℓ, and then $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \widetilde{E}(k)$ lie on the line $\widetilde{\ell}$.

If $\widetilde{P}_1, \widetilde{P}_2 \in \widetilde{E}_{ns}(k)$, then $\widetilde{P}_3 \in \widetilde{E}_{ns}(k)$, and if $P_1, P_2 \in E_0(k)$, then $P_3 \in E_0(k)$, and $\widetilde{P}_1 + \widetilde{P}_2 + \widetilde{P}_3 = 0$.

As an exercise, check this still works if the points are not all distinct.

For surjectivity, let $f(x, y) = y^2 + a_1xy + a_3y - (x^3 + \ldots)$. Let $\widetilde{P} \in \widetilde{E}_{ns}(k) \setminus \{0\}$, say $(\widetilde{x_0}, \widetilde{y_0})$ for some x_0, y_0 in $\mathcal{O}_K$.

Since $\widetilde{P}$ is non-singular, either

$$\begin{aligned} &\text{(i)} \quad \frac{\partial f}{\partial x}(x_0, y_0) \not\equiv 0 \mod \pi \\ &\text{(ii)} \quad \frac{\partial f}{\partial y}(x_0, y_0) \not\equiv 0 \mod \pi \end{aligned}$$

If (i), we put $g(t) = f(t, y_0) \in \mathcal{O}_K[t]$. Then $g(x_0) \equiv 0 \mod \pi$, $g'(x_0) \in \mathcal{O}_K^\times$. Then Hensel's lemma tells us there is some $b \in \mathcal{O}_K$ with $g(b) = 0, b \equiv x_0 \mod \pi$.

Then $P = (b, y_0) \in E(K)$ has reduction $\widetilde{P}$.

Case (ii) is similar. □

Recall for $r \geq 1$, we have $E_r(K) = \{(x, y) \in E(K) : v(x) \leq -2r, v(y) \leq -3r\} \cup \{0\}$. Then:

$$\mathcal{O}_K \cong E_{\lceil e/(p-1) \rceil}(K) \supset \ldots \supset E_2(K) \supset E_1(K) \cong \widehat{E}(\pi\mathcal{O}_K) \subset E_0(K) \subset E(K)$$

We know the quotients $E_i(K)/E_{i+1}(K) \cong (k, +)$ for $i \geq 1$. The above gives $E_0(K)/E_1(K) \cong \widetilde{E}_{ns}(k)$. The only quotient left to understand is $E(K)/E_0(K)$.

Lemma 9.6. *If $|k| < \infty$, then $E_0(K) \subset E(K)$ has finite index.*

Proof. A compactness argument - see below. □

Theorem 9.7. *If $[K : \mathbb{Q}_p] < \infty$, then $E(K)$ contains a subgroup of finite index, isomorphic as a group to $(\mathcal{O}_K, +)$.*

Proof. $|k| < \infty$, so this follows from the above. □

Lemma 9.8. *If $|k| < \infty$, then $\mathbb{P}^n(K)$ is compact with respect to the π-adic topology.*

Proof. $|k| < \infty$, so $\mathcal{O}_K/\pi^r\mathcal{O}_K$ is also finite for $r \geq 1$. Hence

$$\mathcal{O}_K \cong \varprojlim_r \mathcal{O}_K/\pi^r\mathcal{O}_K$$

is compact.

$\mathbb{P}^n(K)$ is the union of compact sets of the form

$$\{(a_0 : a_1 : \ldots : a_{i-1} : 1 : a_{i+1} : \ldots : a_n) : a_j \in \mathcal{O}_K\}$$

and hence is compact. □

*Proof of **9.6**.* $E(K) \subset \mathbb{P}^2(K)$ is a closed subset, so $(E(K), +)$ is a compact topological group.

If $\widetilde{E}$ has a singular point $(\widetilde{x}_0, \widetilde{y}_0)$ then $E(K) \setminus E_0(K) = \{(x, y) \in E(K) : v(x - x_0) \geq 1, v(y - y_0) \geq 1\}$, is a closed subset of $E(K)$, and so $E_0(K)$ is an open subgroup of $E(K)$, so any coset is also open.

The cosets of $E_0(K)$ form an open cover of $E(K)$, hence have a finite subcover, and so there are only finitely many cosets.

Hence $[E(K) : E_0(K)] < \infty$. □

We call this index $c_K(E)$, the ***Tamagawa number***.

Remarks.

1. Good reduction $\implies$ $c_K(E) = 1$, but the converse is false.
2. It can be shown that either $c_K(E) = v(\Delta)$ or $c_K(E) \leq 4$, as long as we work with a minimal Weierstrass equation.

Let $[K : \mathbb{Q}_p]$ be finite, and L/K finite, with residue fields k', k (corresponding to L, K respectively), with $f = [k' : k]$ and ramification index e. From local fields, we know $[L : K] = ef$.

If L/K is Galois then there is a natural group homomorphism $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k'/k)$, and this map is surjective, with kernel of order e. We say the extension is ***unramified*** if $e = 1$, so if these Galois groups are isomorphic.

For each $m \geq 1$, k has a unique extension of degree m, called k_m (not standard notation). K has a unique unramified extension of degree m, called K_m. Note that then the residue field of K_m is k_m. These extensions are Galois with cyclic Galois group.

We then define $K^{nr} = \bigcup_{m \geq 1} K_m$ inside $\bar{K}$, the maximal unramified extension.

Theorem 9.9. *Suppose $[K : \mathbb{Q}_p] < \infty$, and E/K has good reduction and $p \nmid n$. If $P \in E(K)$ then $K([n]^{-1}P)/K$ is unramified.*

Notation: $[n]^{-1}P = \{Q \in E(\bar{K}) : nQ = P\}$, and $K(P_1, \ldots, P_r) = K(x_1, \ldots, x_r, y_1, \ldots, y_r)$, $P_i = (x_i, y_i)$.

Proof. For each $m \geq 1$, there is a short exact sequence $0 \to E_1(K_m) \to E(K_m) \to \widetilde{E}(k_m) \to 0$.

Taking union over all m gives a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E_1(K^{nr}) \arrow[r] \arrow[d, "\times n"] & E(K^{nr}) \arrow[r] \arrow[d, "\times n"] & \widetilde{E}(\bar{k}) \arrow[r] \arrow[d, "\times n"] & 0 \\
0 \arrow[r] & E_1(K^{nr}) \arrow[r] & E(K^{nr}) \arrow[r] & \widetilde{E}(\bar{k}) \arrow[r] & 0
\end{tikzcd}$$

The first vertical arrow is an isomorphism by **8.5**, as $n \in \mathcal{O}_K^\times$.

The last vertical arrow is surjective by **2.8**, with kernel $(\mathbb{Z}/n\mathbb{Z})^2$ by **6.5**, as $p \nmid n$.

The snake lemma tells us $E(K^{nr})[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$, $E(K^{nr})/nE(K^{nr}) = 0$.

So if $P \in E(K)$, then multiplication by n is surjective, and there is Q in $E(K^{nr})$ with $nQ = P$, and $[n]^{-1}P = \{Q + T : T \in E[n]\} \subset E(K^{nr})$.

So $K([n]^{-1}P) \subset K^{nr}$, and $K([n]^{-1}P)/K$ is unramified. □

Corollary 9.10. *Let E/K be an elliptic curve with $[K : \mathbb{Q}_p] < \infty$ Then $E(K)_{tors}$ is finite.*

Proof. In **9.7** we saw that $E(K)$ has a subgroup $E_r(K)$ of finite index isomorphic to $(\mathcal{O}_K, +)$. Since $E_r(K)$ is torsion free, $E(K)_{\text{tors}} \hookrightarrow E(K)/E_r(K)$, an injection into a finite group. □

10 Elliptic Curves over Number Fields

10.1 The Torsion Subgroup

Let $[K : \mathbb{Q}] < \infty$ and E/K an elliptic curve.

Let $\mathfrak{p}$ be a prime of K (i.e. a prime ideal in $\mathcal{O}_K$). We write $K_\mathfrak{p}$ for the $\mathfrak{p}$-adic completion of K, and $k_\mathfrak{p}$ for $\mathcal{O}_K/\mathfrak{p}$. Note that, upon taking completions, the residue field doesn't change.

Definition. $\mathfrak{p}$ is a prime of good reduction for E/K if $E/K_\mathfrak{p}$ is has good reduction.

Lemma 10.1. *E/K has only finitely many primes of bad reduction.*

Proof. Take any Weierstrass equation for E, with coefficients in $\mathcal{O}_K$. E is non-singular, so $0 \neq \Delta \in \mathcal{O}_K$. We can thus write $\Delta = \mathfrak{p}_1^{\alpha_1} \ldots \mathfrak{p}_r^{\alpha_r}$ as a unique factorisation into prime ideals, and let $S = \{\mathfrak{p}_i\}$ in this factorisation.

If $\mathfrak{p} \notin S$, then $v_\mathfrak{p}(\Delta) = 0$, so $E/K_\mathfrak{p}$ has good reduction.

Hence the set of bad primes for E is a subset of S, which is finite. □

Note that we'd like to say that S is the set of bad primes. If K has class number 1, e.g. $K = \mathbb{Q}$, then we can always find Weierstrass equation for E with the coefficients in O_K minimal at all primes $\mathfrak{p}$, and then S will be all the bad primes.

Lemma 10.2. *$E(K)_{tors}$ is finite.*

Proof. Take any prime $\mathfrak{p}$ and complete at $\mathfrak{p}$. Then $K \subseteq K_\mathfrak{p}$, so $E(K)_{\text{tors}} \subseteq E(K_\mathfrak{p})_{\text{tors}}$ is finite by **9.10**. □

Lemma 10.3. *Let $\mathfrak{p}$ be a prime of good reduction, with $\mathfrak{p} \nmid n$. Then reduction mod $\mathfrak{p}$ gives an injective group homomorphism*

$$E(K)[n] \hookrightarrow \widetilde{E}(k_\mathfrak{p})[n]$$

Proof. **9.5** tells us that $E(K_\mathfrak{p}) \longrightarrow \widetilde{E}(k_\mathfrak{p})$ is a group homomorphism. Hence it takes n-torsion points to n-torsion points, as needed. It has kernel $E_1(K_\mathfrak{p})$. Since $\mathfrak{p} \nmid n$, **8.5** tells us $E_1(K_\mathfrak{p})$ has no n-torsion, and so the map is injective. □

Examples.

1. $E/\mathbb{Q} : y^2 + y = x^3 - x^2, \Delta = -11$. E has good reduction at all primes $p \neq 11$.

p	2	3	5	7	11	13
$\#\widetilde{E}(\mathbb{F}_p)$	5	5	5	10	-	10

   By **10.3** looking at $p = 2$, $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 2^a$ for some $a \geq 0$.

   Looking at $p = 3$, $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 3^b$ for some $b \geq 0$.

   Hence $\#E(\mathbb{Q})_{\text{tors}} | 5$, so is 1 or 5.

   Let $T = (0,0) \in E(\mathbb{Q})$. Calculation gives that $5T = O_E$, and so $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}$.

2. $E/\mathbb{Q} : y^2 + y = x^3 + x^2, \Delta = -43$. E has good reduction at all primes $p \neq 43$.

p	2	3	5	7	11	13
$\#\widetilde{E}(\mathbb{F}_p)$	5	6	10	8	9	19

   So $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 2^a$, some $a \geq 0$, and $\#E(\mathbb{Q})_{\text{tors}} | 9 \cdot 11^b$, some $b \geq 0$.

   So $\#E(\mathbb{Q})_{\text{tors}} = 1$, and $E(\mathbb{Q})_{\text{tors}} = \{O_E\}$.

   Now, since $P = (0,0) \in E(\mathbb{Q})$, it has infinite order, and hence infinitely many rational points on $E(\mathbb{Q})$. This is an example where $\operatorname{rank} E(\mathbb{Q}) \geq 1$.

3. $E_D : y^2 = x^3 - D^2x$ for $D \in \mathbb{Z}$ a squarefree integer. Then $\Delta = 2^6 D^6$.

   $E_D(\mathbb{Q})_{\text{tors}} \supset \{0, (0,0), (\pm D, 0)\} \cong (\mathbb{Z}/2\mathbb{Z})^2$.

   Let $f(x) = x^3 - D^2x$. Then if p is prime not dividing $2D$, then it is a prime of good reduction.

   $\#\widetilde{E}(\mathbb{F}_p) = 1 + \sum_{x \in \mathbb{F}_p} \left(\left(\frac{f(x)}{p} \right) + 1 \right)$, where $\left(\frac{f(x)}{p} \right)$ is the Legendre symbol.

   If $p \equiv 3 \mod 4$, then since $f(x)$ is an odd function:

   $$\left(\frac{f(-x)}{p} \right) = \left(\frac{-f(x)}{p} \right) = \left(\frac{-1}{p} \right) \left(\frac{f(x)}{p} \right) = -\left(\frac{f(x)}{p} \right)$$

and so $\#\widetilde{E}_D(\mathbb{F}_p) = p + 1$.

Let $m = \#E(\mathbb{Q})_{\text{tors}}$. We have $4|m|p + 1$ for all sufficiently large primes p congruent to 3 mod 4, and hence $m = 4$, since otherwise this contradicts Dirichlet's theorem on primes in arithmetic progression.

Hence $E_D(\mathbb{Q})_{\text{tors}} \cong (\mathbb{Z}/2\mathbb{Z})^2$. So $\operatorname{rank} E_D(\mathbb{Q}) \geq 1$ if and only if there are $x, y \in \mathbb{Q}$ with $y \neq 0$ such that $y^2 = x^3 - Dx$, which by the first lecture is equivalent to D being a congruent number.

Lemma 10.4. *Let $E/\mathbb{Q}$ be given by a Weierstrass equation with coefficients in $\mathbb{Z}$. Let $0 \neq T \in E(\mathbb{Q})_{tors}$. Then:*

1. $4x, 8y \in \mathbb{Z}$.
2. *If $2|a_1$ or $2T \neq O_E$, then $x, y \in \mathbb{Z}$.*

Proof. The Weierstrass equation defines a formal group $\widehat{E}$ over $\mathbb{Z}$. For $r \geq 1$, we have $\widehat{E}(p^r\mathbb{Z}_p) = \{(x, y) \in E(\mathbb{Q}_p) : v_p(x) \leq -2r, v_p(y) \leq -3r\} \cup \{0\}$.

9.2 gives $\widehat{E}(p^r\mathbb{Z}_p) \cong (\mathbb{Z}_p, +)$ if $r > \frac{1}{p-1}$, and hence $\widehat{E}(4\mathbb{Z}_2)$ and $\widehat{E}(p\mathbb{Z}_p)$ are torsion free.

Since T is a nonzero torsion point, it follows that $v_p(x), v_p(y) \geq 0$ for all odd primes p, and $v_2(x) \geq -2, v_2(y) \geq -3$. This proves part 1.

For the second part, suppose that $T \in \widehat{E}(2\mathbb{Z}_2)$, i.e. $v_2(x) = -2, v_2(y) = -3$.

Since $\frac{\widehat{E}(2\mathbb{Z}_2)}{\widehat{E}(4\mathbb{Z}_2)} \cong (\mathbb{F}_2, +)$ and $\widehat{E}(4\mathbb{Z}_2)$ is torsion free, we get $2T = 0$. Also, $(x, y) = T = -T = (x, -y - a_1x - a_3)$, and hence $2y + a_1x + a_3 = 0, 8y + 4xa_1 + 4a_3 = 0$.

$8y$ is odd, $4x$ is odd, $4a_3$ is even, and hence a_1 is odd.

So if $2T \neq 0$ or a_1 is even, then $T \notin \widehat{E}(2\mathbb{Z}_2)$, so $x, y \in \mathbb{Z}$. □

For example, if $y^2 + xy = x^3 + 4x + 1$, then $(-\frac{1}{4}, \frac{1}{8}) \in E(\mathbb{Q})[2]$.

Theorem 10.5 (Lutz-Nagell)**.** *Let $E/\mathbb{Q}$ be given by $y^2 = x^3 + ax + b$, for $a, b \in \mathbb{Z}$.*

Suppose $0 \neq T = (x, y) \in E(\mathbb{Q})_{tors}$. Then $x, y \in \mathbb{Z}$, and either $y = 0$ or $y^2|4a^3 + 27b^2$.

Note that this is not an if and only if - we still have to check the answers we get.

Proof. **10.4** gave us $x, y \in \mathbb{Z}$. If $2T = 0$, then $y = 0$.

Otherwise, $0 \neq 2T = (x_2, y_2) \in E(\mathbb{Q})_{\text{tors}}$, and so **10.4** gives $x_2, y_2 \in \mathbb{Z}$.

But $x_2 = \left(\frac{f'(x)}{2y}\right)^2 - 2x$, and so $y|f'(x)$.

E non-singular, so $f(x)$ and $f'(x)$ are coprime, and so $f(x)$ and $(f'(x))^2$ are coprime, hence $1 = g(x)f(x) + h(x)(f'(x))^2$ for some $g, h \in \mathbb{Q}[x]$.

Doing this calculation and clearing denominators, we get

$$(3x^2 + 4a)f'(x)^2 - 27(x^3 + ax - b)f(x) = 4a^3 + 27b^2$$

Since $y|f'(x), y^2 = f(x)$, so y^2 divides LHS, hence $y^2|4a^3 + 27b^2$. □

Mazur showed that, if $E/\mathbb{Q}$ is an elliptic curve, then $E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & 1 \le n \le 12, n \ne 11 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} & 1 \le n \le 4 \end{cases}$.
Moreover, all 15 possibilities occur.

11 Kummer Theory

K is a field, $\operatorname{char} K \nmid n$, and $\mu_n \subset K$, where μ_n is the set of n^{th} roots of unity.

Lemma 11.1. *Let $\Delta \subset K^\times/(K^\times)^n$ be a finite subgroup, and let $L = K(\sqrt[n]{\Delta})$. Then L/K is Galois, and $\operatorname{Gal}(L/K) \cong \operatorname{Hom}(\Delta, \mu_n)$.*

Proof. L/K is Galois since $\mu_n \subset K$, and $\operatorname{char} K \nmid n$.

Define the Kummer pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \operatorname{Gal}(L/K) \times \Delta &\to \mu_n \\ (\sigma, x) &\mapsto \frac{\sigma(\sqrt[n]{x})}{\sqrt[n]{x}} \end{aligned}$$

It is well defined: suppose $\alpha, \beta \in L$ are two different choices of $\sqrt[n]{x}$. Then $(\alpha/\beta)^n = 1$, so $\alpha/\beta \in \mu_n \subset K$, so $\sigma(\alpha/\beta) = \alpha/\beta$. Hence $\sigma(\alpha)/\alpha = \sigma(\beta)/\beta$.

It is bilinear: $\langle \sigma\tau, x \rangle = \frac{\sigma\tau\sqrt[n]{x}}{\tau\sqrt[n]{x}} \frac{\tau\sqrt[n]{x}}{\sqrt[n]{x}} = \langle \sigma, x \rangle \langle \tau, x \rangle$, as $\tau\sqrt[n]{x}$ is another choice of $\sqrt[n]{x}$, and $\langle \sigma, xy \rangle = \frac{\sigma\sqrt[n]{xy}}{\sqrt[n]{xy}} = \frac{\sigma\sqrt[n]{x}}{\sqrt[n]{x}} \frac{\sigma\sqrt[n]{y}}{\sqrt[n]{y}} = \langle \sigma, x \rangle \langle \sigma, y \rangle$.

It is non-degenerate: Let $\sigma \in \operatorname{Gal}(L/K)$. If $\langle \sigma, x \rangle = 1$ for all $x \in \Delta$, then $\sigma(\sqrt[n]{x}) = \sqrt[n]{x}$ for all $x \in \Delta$, and so σ fixes L pointwise. Hence $\sigma = \mathrm{id}$. Now fix $x \in \Delta$, and suppose $\langle \sigma, x \rangle = 1$ for all $\sigma \in \operatorname{Gal}(L/K)$. Then $\sigma(\sqrt[n]{x}) = \sqrt[n]{x}$ for all $\sigma \in \operatorname{Gal}(L/K)$, and hence $\sqrt[n]{x} \in K$, and so $x \in (K^\times)^n$, i.e. $x(K^\times)^n$ is trivial in Δ.

We thus get injective group homomorphisms $\operatorname{Gal}(L/K) \hookrightarrow \operatorname{Hom}(\Delta, \mu_n)$, $\Delta \hookrightarrow \operatorname{Hom}(\operatorname{Gal}(L/K), \mu_n)$.

Hence $\operatorname{Gal}(L/K)$ is abelian of exponent dividing n.

If G is a finite abelian group of exponent dividing n, then $\operatorname{Hom}(G, \mu_n) = G$ (non-canonically).

So $|\operatorname{Gal}(L/K)| \le |\Delta| \le |\operatorname{Gal}(L/K)$, and so $|\Delta| = |\operatorname{Gal}(L/K)|$, and hence the injective homomorphisms are surjective as well, so isomorphisms. □

For example $\operatorname{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Theorem 11.2. *There is a bijection*

$$\begin{aligned} \{\textit{finite subgroups } \Delta \subseteq K^\times/(K^\times)^n\} &\leftrightarrow \{\textit{finite abelian extensions } L/K \textit{ or exponent dividing } n\} \\ \Delta &\mapsto K(\sqrt[n]{\Delta}) \\ \frac{(L^\times)^n \cap K^\times}{(K^\times)^n} &\mapsfrom L \end{aligned}$$

Proof. Let L/K be a finite abelian extension of exponent dividing n. Let $\Delta = \frac{(L^\times)^n \cap K^\times}{(K^\times)^n}$. Then $K(\sqrt[n]{\Delta}) \subset L$ and we aim to show equality.

Let $G = \mathrm{Gal}(L/K)$.

The Kummer pairing gives an injection $\Delta \hookrightarrow \mathrm{Hom}(G, \mu_n)$. We claim this is a surjection.

Given the claim, we will then have $\Delta \cong \mathrm{Hom}(G, \mu_n)$, so $[K(\sqrt[n]{\Delta}) : K] = |\Delta|$ by **11.1** $= |G| = [L : K]$, and hence we have the equality.

To prove the claim, let $\chi : G \to \mu_n$ be a member of $\mathrm{Hom}(G, \mu_n)$. Distinct automorphisms are linearly independent. Then

$$\exists\, a \in L \text{ s.t. } \underbrace{\sum_{\tau \in G} \chi(\tau)^{-1} \tau(a)}_{y} \neq 0$$

Let $\sigma \in G$. Then

$$\begin{aligned}
\sigma(y) &= \sum_{\tau \in G} \chi(\tau)^{-1} \sigma\tau(a) \\
&= \sum_{\tau \in G} \chi(\sigma^{-1}\tau)^{-1} \tau(a) \\
&= \chi(\sigma) y
\end{aligned}$$

So $\sigma(y^n) = y^n$ for all $\sigma \in G$. Then if $x := y^n$, we have $x \in K^\times$, and $x \in (L^\times)^n$.

So $x \in \Delta$, and $\chi(\sigma) = \frac{\sigma(\sqrt[n]{x})}{\sqrt[n]{x}}$, and so χ is the image of x under the injection, and hence it is a surjection.

For the other direction, we start with $\Delta \subset K^\times/(K^\times)^n$ a finite subgroup. Let $L = K(\sqrt[n]{\Delta})$, and $\Delta' = \frac{(L^\times)^n \cap K^\times}{(K^\times)^n}$, and we must show that $\Delta' = \Delta$.

Clearly $\Delta \subseteq \Delta'$. We then compute sizes.

$L = K(\sqrt[n]{\Delta}) \subset K(\sqrt[n]{\Delta'}) \subset L$, and we have equality throughout. So $K(\sqrt[n]{\Delta}) = K(\sqrt[n]{\Delta'})$.

11.1 gives $|\Delta| = |\Delta'|$, and so $\Delta = \Delta'$. □

Proposition 11.3. *Let K be a number field containing μ_n. Let S be a finite set of primes of K. Then there are only finitely many extensions L/K such that:*

1. *L/K is abelian of exponent dividing n.*
2. *L/K is unramified at all primes outside S.*

Proof. **11.2** gives us $L = K(\sqrt[n]{\Delta})$ for some $\Delta \in K^\times/(K^\times)^n$ a finite subgroup. Let $\mathfrak{p}$ be a prime of K. Then $\mathfrak{p}\mathcal{O}_L = \mathcal{P}_1^{e_1} \dots \mathcal{P}_r^{e_r}$, where $\mathcal{P}_i$ are primes in $\mathcal{O}_L$.

If $x \in K^\times$ represents an element of Δ. If $x \in K^\times$ represents an element of Δ, then

$$n v_{\mathcal{P}_i}(\sqrt[n]{x}) = v_{\mathcal{P}_i}(x) = e_i v_{\mathfrak{p}}(x)$$

If $\mathfrak{p} \notin S$, then all $e_i = 1$, so $v_{\mathfrak{p}}(x) \equiv 0 \mod n$, and so $\Delta \subset K(S, n)$, where

$$K(S, n) := \{x \in K^\times/(K^\times)^n : v_{\mathfrak{p}}(x) \equiv 0 \mod n\ \forall \mathfrak{p} \in S\}$$

The proof is completed by the following lemma. □

Lemma 11.4. *$K(S,n)$ is finite.*

Proof. The map $K(S,n) \to (\mathbb{Z}/n\mathbb{Z})^{|S|}$, $x \mapsto (v_{\mathfrak{p}}(x) \mod n)$ for $\mathfrak{p} \in S$ is group homomorphism with kernel $K(\emptyset, n)$.

Since $|S| < \infty$, it suffices to prove the lemma with $S = \emptyset$.

If $x \in K^\times$ represents an element of $K(\emptyset, n)$, then $(x) = \mathfrak{a}^n$ for some ideal $\mathfrak{a}$.

There is then an exact sequence:

$$0 \longrightarrow \frac{O_K^\times}{(O_K^\times)^n} \longrightarrow K(\emptyset, n) \longrightarrow Cl_K[n] \longrightarrow 0$$

$$x(K^\times)^n \longmapsto [\mathfrak{a}]$$

Now $|Cl_K| < \infty$ and $O_K^\times$ is finitely generated by Dirichlet's unit theorem, so $K(\emptyset, n)$ is finite. □

12 Elliptic Curves over Number Fields II

12.1 The Mordell-Weil Theorem

Lemma 12.1. *Let E/K be an elliptic curve. Let L/K be a finite Galois extension. Then the map*

$$E(K)/nE(K) \to E(L)/nE(L)$$

has finite kernel.

Proof. For each element in the kernel, we pick a coset representative $P \in E(K)$, and then $Q \in E(L)$ with $nQ = P$.

Note that, for any $\sigma \in \mathrm{Gal}(L/K)$, $n(\sigma(Q) - Q) = \sigma(P) - P = 0$.

$\mathrm{Gal}(L/K)$ is finite and $E[n]$ is finite, so there are only finitely many possibilities for the map $\mathrm{Gal}(L/K) \to E[n], \sigma \mapsto \sigma Q - Q$.

But if $P_1, P_2 \in E(K)$ then $P_i = nQ_i$ for $Q_1, Q_2 \in E(L)$, and $\sigma Q_1 - Q_2 = \sigma Q_2 - Q_2$, and so $\sigma(Q_1 - Q_2) = Q_1 - Q_2$ for all $\sigma \in \mathrm{Gal}(L/K)$.

But then $Q_1 - Q_2 \in E(K)$, and so $P_1 - P_2 \in nE(K)$. □

Theorem 12.2 (Weak Mordell-Weil)**.** *If K is a number field and E/K is an elliptic curve, with $n \geq 2$ an integer, then $E(K)/nE(K)$ is finite.*

Proof. **12.1** tells us we may replace K by a finite Galois extension. So without loss of generality, $\mu_n \subset K$ and $E[n] \subset E(K)$.

Let $S = \{\mathfrak{p}|n\} \cup \{\text{primes of bad reduction for } E/K\}$.

For each $P \in E(K)$, the extension $K([n]^{-1}P)/K$ is unramified outside S, by **9.9**.

Let $Q \in [n]^{-1}P$. Since $E[n] \subseteq E(K)$, $K(Q) = K([n]^{-1}P)$, and this is a Galois extension of K.

Then we claim that we have an injective group homomorphism:

$$\begin{aligned} \mathrm{Gal}(K(Q)/K) &\hookrightarrow E[n] \\ \sigma &\mapsto \sigma Q - Q \end{aligned}$$

This is a group homomorphism as $\sigma\tau Q - Q = \sigma(\tau Q - Q) + (\sigma Q - Q)$. But $\tau Q - Q$ is an n-torsion point, so lies in K, so this is $\tau Q - Q + \sigma Q - Q$.

It is injective, as if $\sigma Q = Q$ then σ fixes $K(Q)$, and so σ is the identity.

So $K(Q)/K$ is an abelian extension of exponent dividing n, unramified outside of S, so **11.3** tells us there are only finitely many possibilities for $K(Q)/K$ as we vary P.

Let L be the composite of all such extensions of K. Then L/K is a finite Galois extension, and

$$E(K)/nE(K) \to E(L)/nE(L)$$

is the zero map.

12.1 implies that this has finite kernel, and so $|E(K)/nE(K)| < \infty$. □

If $K = \mathbb{R}$ or $\mathbb{C}$ or $[K : \mathbb{Q}_p] < \infty$, then $|E(K)/nE(K)| < \infty$, yet $E(K)$ is uncountable. Hence $E(K)$ is not finitely generated.

It is a fact that there is a quadratic form called the ***canonical height*** $\widehat{h} : E(K) \to \mathbb{R}_{\geq 0}$ with the property that, for any $B \geq 0$, the set $\{P \in E(K) : \widehat{h}(P) \leq B\}$ is finite.

Theorem 12.3 (Mordell-Weil)**.** *Let K be a number field with E/K an elliptic curve. Then $E(K)$ is a finitely generated abelian group.*

Proof. Fix any integer $n \geq 2$. Weak Mordell-Weil gives $|E(K)/nE(K)| < \infty$. Pick coset representatives for $E(K)/nE(K)$, say $P_1, \ldots, P_m$.

Then let $\Sigma = \{P \in E(K) : \widehat{h}(P) \leq \max_{1\leq i\leq m} \widehat{h}(P_i)\}$.

We claim Σ generates $E(K)$.

If not, then there is $P \in E(K) \setminus \{\text{subgroup generated by } \Sigma\}$ of minimal point (this exists because there are only finitely many "small points that are too big").

Then $P = P_i + nQ$ for some $1 \leq i \leq m$, and $Q \in E(K)$.

Note that $Q \in E(K) \setminus \langle\Sigma\rangle$. The minimal choice of P tells us that $4\widehat{h}(P) \leq 4\widehat{h}(Q) \leq n^2\widehat{h}(Q) = \widehat{h}(nQ) = \widehat{h}(P - P_i) \leq \widehat{h}(P - P_i) + \widehat{h}(P + P_i) = 2\widehat{h}(P) + 2\widehat{h}(P_i)$. Hence $\widehat{h}(P) \leq \widehat{h}(P_i)$, and so $P \in \Sigma$ ↯.

Hence Σ generates $E(K)$, and is finite, so $E(K)$ is finitely generated. □

Note that the structure theorem for finitely generated abelian groups shows that $E(K)$ is of the form

$$E(K) \cong E(K)_{\text{tors}} \times \mathbb{Z}^r$$

for some non-negative integer r, called the ***rank***. There is no known algorithm for computing the rank of $E(K)$ in all cases.

13 Heights

For simplicity, we will take $K = \mathbb{Q}$. Write $P \in \mathbb{P}^n(\mathbb{Q})$ as $P = (a_0 : \dots : a_n)$ are (not necessarily pairwise) coprime integers.

We define the ***height*** of P, $H(P) = \max_{0 \le i \le n} |a_i|$.

Lemma 13.1. *Let $f_1, f_2 \in \mathbb{Q}[x_1, x_2]$ be coprime homogeneous polynomials of the same degree d. Let $F : \mathbb{P}^1 \to \mathbb{P}^1; (x_1 : x_2) \mapsto (f_1(x_1, x_2) : f_2(x_1, x_2))$.*

Then there are constants $c_1, c_2 > 0$ such that

$$c_1 H(P)^d \le H(F(P)) \le c_2 H(P)^d$$

for all points $P \in \mathbb{P}^1(\mathbb{Q})$.

Proof. Without loss of generality, $f_1, f_2 \in \mathbb{Z}[x_1, x_2]$. For the upper bound, write $P = (a : b)$, coprime. Then

$$\begin{aligned} H(F(P)) &\le \max(|f_1(a, b)|, |f_2(a, b)|) \\ &\le c_2 \max(|a|^d, |b|^d) \end{aligned}$$

where $c_2 = \max($ sum of abs. values of coeffs of $f_i)$. So $H(F(P)) \le c_2 H(P)^d$.

For lower bound, we claim there are $g_{ij} \in \mathbb{Z}[x_1, x_2]$ homogeneous polynomials of degree $d - 1$ and $\kappa \in \mathbb{Z}_{>0}$ such that

$$\sum_{j=1}^{2} g_{ij} f_j = \kappa x_i^{2d-1} \qquad i = 1, 2 \tag{$*$}$$

Indeed, running Euclid's algorithm on $f_1(x, 1), f_2(x, 1)$ give $r, s \in \mathbb{Q}[x]$ of degree $< d$ such that

$$r(x) f_1(x, 1) + s(x) f_2(x, 1) = 1$$

Homogenising and clearing denominators gives $(*)$ for $i = 2$, and likewise for $i = 1$.

Write $P = (a_1 : a_2)$ for integers a_1, a_2 coprime. Then $(*)$ gives that

$$\sum_{j=1}^{2} g_{ij}(a_1, a_2) f_j(a_1, a_2) = \kappa a_i^{2d-1}$$

and so $\gcd(f_1(a_1, a_2), f_2(a_1, a_2))$ divides $\gcd(\kappa a_1^{2d-1}, \kappa a_2^{2d-1}) = \kappa$.

But also $|\kappa a_i^{2d-1}| \le \underbrace{\max_{j=1,2} |f_j(a_1, a_2)|}_{\le \kappa H(F(P))} \underbrace{\sum_{j=1}^{2} |g_{ij}(a_1, a_2)|}_{\le \gamma_i H(P)^{d-1}}$ where $\gamma_i = \sum_{j=1}^{2}($sum of absolute values of coefficients of $g_{ij})$. Hence we have

$$\kappa |a_i|^{2d-1} \le \gamma_i H(F(P)) H(P)^{d-1}$$

and so

$$\frac{1}{\max(\gamma_1, \gamma_2)} H(P)^d \le H(F(P))$$

□

Notation: for $x \in \mathbb{Q}$, we write $H(x) = H((x:1)) = \max(|u|,|v|)$ where $x = \frac{u}{v}$, u, v coprime.

Let $E/\mathbb{Q}$ be an elliptic curve, given by $y^2 = x^3 + ax + b$.

Then we define the height function:

$$\begin{aligned} H : E(\mathbb{Q}) &\to \mathbb{R}_{\geq 1} \\ P &\mapsto \begin{cases} H(x) & P = (x,y) \\ 1 & P = Q \end{cases} \end{aligned}$$

and the ***logarithmic height***

$$\begin{aligned} h : E(\mathbb{Q}) &\to \mathbb{R}_{\geq 0} \\ P &\mapsto \log(H(P)) \end{aligned}$$

Lemma 13.2. *Let E, E' be elliptic curves over $\mathbb{Q}$, and $\phi : E \to E'$ an isogeny defined over $\mathbb{Q}$. Then there is $c > 0$ such that*

$$|h(\phi(P)) - \deg(\phi)h(P)| \leq c \ \forall P \in E(\mathbb{Q})$$

Note that c depends on E, E', ϕ but not on P.

Proof. Recall **5.3** that the x-coordinate of $\phi(P)$ only depends on the x-coordinate of P, say $x(\phi(P)) = \xi(x(P))$. Then **13.1** gives $c_1, c_2 > 0$ such that

$$c_1 H(P)^d \leq H(\phi(P)) \leq c_2 H(P)^d$$

Taking logs gives the result. □

For example, if $\phi = [2] : E \to E$. Then there is $c > 0$ such that

$$|h(2P) - 4h(P)| \leq c \ \forall P \in E(\mathbb{Q})$$

definition The ***canonical height*** is defined as follows:

$$\widehat{h}(P) = \lim_{n\to\infty} \frac{1}{4^n} h(2^n P)$$

We check convergence: let $m \geq n$. Then

$$|\frac{1}{4^m}h(2^m P) - \frac{1}{4^n}h(2^n P)| \leq \sum_{r=n}^{m-1} |\frac{1}{4^{r+1}}h(2^{r+1}P) - \frac{1}{4^r}h(2^r P)| \leq \sum_{r=n}^{\infty} \frac{c}{4^{r+1}} = \frac{c}{3\cdot 4^n} \to 0$$

So the sequence is Cauchy, and $\widehat{h}(P)$ exists.

Lemma 13.3. *$|h(P) - \widehat{h}(P)|$ is bounded.*

Proof. Put $n = 0$ in the above calculation. Then $|\widehat{h}(P) - h(P)| \leq \frac{c}{3}$. □

Corollary 13.4. *The set $\{P \in E(\mathbb{Q}) : \widehat{h}(P) \leq B\}$ is finite.*

Proof. $\widehat{h}(P)$ bounded implies that $h(P)$ is bounded.

But then there are only finitely many possibilities for x, and so finitely many possibilities for P. □

Lemma 13.5. *Let $\phi : E \to E'$ be an isogeny. Then $\widehat{h}(\phi P) = (\deg\phi)\widehat{h}(P)$.*

Proof. **13.2** gives $c > 0$ such that

$$|h(\phi P) - (\deg\phi)h(P)| \leq c$$

Replace P by $2^n P$, divide by 4^n, and take the limit as $n \to \infty$. □

Remarks.

1. H and h depend on a choice of Weierstrass equation. **13.5** applied in the case where ϕ is an isomorphism (so $\deg\phi = 1$) shows that $\widehat{h}$ does not.
2. Taking $\phi = [n] : E \to E$ shows $\widehat{h}(nP) = n^2\widehat{h}(P)$.

Lemma 13.6. *Let $E/\mathbb{Q}$ be an elliptic curve, with Weierstrass equation $y^2 = x^3 + ax + b$. Then there is some $c > 0$ such that*

$$H(P+Q)H(P-Q) \leq cH(P)^2H(Q)^2 \ \forall P, Q \in E(\mathbb{Q}) \text{ with } P, Q, P \pm Q \neq O_E$$

Proof. Let $P, Q, P+Q, P-Q$ have x coordinates $x_1, \ldots, x_4$ respectively. By **5.7** there exist polynomials $W_0, W_1, W_2 \in \mathbb{Z}[x_1, x_2]$ of degree ≤ 2 in x_1 and of degree ≤ 2 in x_2 such that $(1 : x_3 + x_4 : x_3x_4) = (W_0 : W_1 : W_2)$.

Write $x_i = r_i/s_i$ for $r_i, s_i \in \mathbb{Z}$ coprime. Then we get

$$(s_3s_4 : r_3s_4 + r_4s_3 : r_3r_4) = ((r_1s_2 - r_2s_1)^2 : \ldots : \ldots)$$

Note that the three integers on the left are coprime.

Then $H(P+Q)H(P-Q) = \max(|r_3|, |r_3|)\max(|r_4|, |s_4|) \leq 2\max(|s_3s_4|, |r_3s_4 + r_4s_3|, |r_3r_4|)$, and, since the three terms on the right are integers, so must be $\geq$ the corresponding coprime terms on the left, we have:

$$\begin{aligned} H(P+Q)H(P-Q) &\leq 2\max(|r_1s_2 - r_2s_1|^2, \ldots) \\ &\leq cH(P)^2H(Q)^2 \end{aligned}$$

where c depends on E but not on P, Q. □

Theorem 13.7. *$\widehat{h} : E(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ is a quadratic form.*

Proof. Take logs in the last lemma gives

$$h(P+Q) + h(P-Q) \leq 2h(P) + 2h(Q) + c$$

for all P, Q with $P, Q, P+Q, P-Q \neq O_E$. We can remove this last restriction by using the fact that $|h(2P) - 4h(P)|$ is bounded.

Replacing P, Q by $2^n P, 2^n Q$, dividing by 4^n, and taking the limit as $n \to \infty$, we lose the constant, and so

$$\widehat{h}(P+Q) + \widehat{h}(P-Q) \leq 2\widehat{h}(P) + 2\widehat{h}(Q)$$

Replacing P, Q by $P+Q$, $P-Q$ and using $\widehat{h}(2P) = 4\widehat{h}(P)$, we get the inequality the other way round, and so we have equality, so $\widehat{h}$ is a quadratic form. □

The places of a number field K are the finite places (=primes), so that $|x|_{\mathfrak{p}} = c^{v_{\mathfrak{p}}(x)}$ for some fixed constant $c > 1$, and the infinite places (=real & complex embeddings), so that $|x|_\sigma = |\sigma(x)|^d$ for some fixed $d > 0$.

For each place v we may choose a normalisation $|\cdot|_v$ (i.e. make a choice of c and d) so that

$$\prod_v |\lambda|_v = 1 \ \forall \lambda \in K^\times$$

For K a number field, consider $P = (a_0 : a_1 : \ldots : a_n) \in \mathbb{P}^n(K)$. We define $H(P) = \prod_v \max_{0 \leq i \leq n} |a_i|_v$, which is well defined because of the product formula.

Let $\pi_i : E \times E \times E \to E$ be the projection maps onto the i^{th} factor for $i = 1, 2, 3$. Let $\pi_{ij} = \pi_i + \pi_j$, and $\pi_{123} = \pi_1 + \pi_2 + \pi_3$. There is a result called the theorem of the cube which says that, if $D \in \mathrm{Div}(E)$, then

$$\pi_{123}^* D + \pi_1^* D + \pi_2^* D \sim \pi_{12}^* D + \pi_{13}^* D + \pi_{23}^* D$$

This can be used to give alternative proofs of **5.6** and **13.7**.

14 Dual Isogenies & The Weil Pairing

Let K be a perfect field and E/K an elliptic curve.

Proposition 14.1. *Let $\Phi \subseteq E(\bar{K})$ be a finite $\mathrm{Gal}(\bar{K}/K)$-stable subgroup. Then there is an elliptic curve E'/K and a separable isogeny $\phi : E \to E'$ defined over K with kernel Φ such that every isogeny $\psi : E \to E''$ with $\Phi \subseteq \ker \psi$ factors uniquely via ϕ:*

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\phi"'] & E'' \\ E' \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. Omitted, see Silvermann **III.4.12**. □

Proposition 14.2. *Let $\phi : E \to E'$ be an isogeny of degree n. Then there is a unique isogeny $\widehat{\phi} : E' \to E$ such that $\widehat{\phi}\phi = [n]$.*

Proof. In the case that ϕ is separable, $|\ker \phi| = n \implies \ker \phi \subseteq E[n]$. Then apply **14.1** with $\psi = [n]$.

For the case where ϕ is inseparable, see Silvermann **III.6.1**.

For uniqueness, suppose $\psi_1 \phi = \psi_2 \phi = [n]$. Then rearrange to get $(\psi_1 - \psi_2)\phi = 0$, and so $\psi_1 = \psi_2$ as ϕ is nonconstant. □

Remarks.

1. $E_1 \sim E_2 \iff E_1, E_2$ isogenous. Then $\sim$ is an equivalence relation.
2. $\deg[n] = n^2 \implies \deg\widehat{\phi} = \deg\phi = n$, and $\widehat{[n]} = [n]$.
3. $\phi\widehat{\phi}\phi = \phi[n]_E = [n]_{E'}\phi \implies \phi\widehat{\phi} = [n]_{E'}$. In particular, $\widehat{\widehat{\phi}} = \phi$.
4. $\widehat{\phi\psi} = \widehat{\psi}\widehat{\phi}$.
5. If $\phi \in \mathrm{End}(E)$, then $\phi^2 - \mathrm{tr}[\phi] + [\deg\phi] = 0$. Rearranging, we see $\widehat{\phi} = [\mathrm{tr}\,\phi] - \phi$, and so $\mathrm{tr}\,\phi = \phi + \widehat{\phi}$.

Lemma 14.3. *If $\phi, \psi \in \mathrm{Hom}(E, E')$, then $\widehat{\phi + \psi} = \widehat{\phi} + \widehat{\psi}$.*

Proof. If $E = E'$, then this follows from $\mathrm{tr}(\phi + \psi) = \mathrm{tr}\,\phi + \mathrm{tr}\,\psi$.

In general, let $\alpha : E' \to E$ be any isogeny (e.g. $\widehat{\phi}$). Then the first part tells us $\widehat{\alpha\phi + \alpha\psi} = \widehat{\alpha\phi} + \widehat{\alpha\psi}$, so $\widehat{\alpha}(\widehat{\phi + \psi}) = \widehat{\alpha}(\widehat{\phi} + \widehat{\psi})$, and then cancel $\widehat{\alpha}$. □

In Silvermann's book, he proves this lemma before knowing that the degree map is a quadratic form, and uses it to show that deg is a quadratic form.

14.1 The Weil Pairing

Definition. $\mathrm{sum} : \mathrm{Div}(E) \to E; \sum n_P(P) \mapsto \sum n_P P$ is the map taking a formal sum of points to the actual summation of the points using the group law on E.

Recall that $E \cong \mathrm{Pic}^0(E)$, given by $P \mapsto [(P) - (O_E)]$. Then $\sum n_P P \mapsto [\sum n_P(P) - \sum n_P(O_E)]$, and so $\mathrm{sum} D \mapsto [D]$ for all $D \in \mathrm{Div}^0(E)$.

Lemma 14.4. *Let $D \in \mathrm{Div}(E)$. Then $D \sim 0$ if and only if $\deg D = 0$ and $\mathrm{sum} D = O_E$.*

Let $\phi : E \to E'$ be an isogeny of degree n, with dual isogeny $\widehat{\phi} : E' \to E$. Assume that $\mathrm{char}\, K \nmid n$, so that $\phi, \widehat{\phi}$ are separable.

We define the Weil pairing $e_P : E[\phi] \times E'[\widehat{\phi}] \to \mu_n$:

Let $T \in E'[\widehat{\phi}]$. Then $nT = 0$, and so there exists $f \in \bar{K}(E')^*$ such that $\mathrm{div}\, f = n(T) - n(0)$.

Pick $T_0 \in E(\bar{K})$ with $\phi(T_0) = T$. Then $\phi^*(T) - \phi^*(0) = \sum_{P \in E[\phi]}(P + T_0) - \sum_{P \in E[\phi]}(P)$ has sum $nT_0 = \widehat{\phi}\phi T_0 = \widehat{\phi}T = 0$.

So there is some $g \in \bar{K}(E)^*$ such that $\mathrm{div}(g) = \phi^*(T) - \phi^*(0)$.

Now $\mathrm{div}(\phi^* f) = \phi^* \mathrm{div}\, f = n(\phi^*(T) - \phi^*(0)) = n\,\mathrm{div}\, g = \mathrm{div}(g^n)$, so $\phi^* f = g^n$ up to multiplication by a constant $c \in \bar{K}^*$. Rescaling f, we can ignore this constant.

If $S \in E[\phi]$, then $\phi\tau_S = \phi$, i.e. $\tau_S^*\phi^* = \phi^*$, so $\tau_S^*(\mathrm{div}\, g) = \mathrm{div}\, g$, and so $\tau_S^* g = \zeta g$ for some constant ζ.

$\zeta = g(X + S)/g(X)$ for all points X by definition for all $X \in E(\bar{K}) \setminus \{\text{zeros/poles of } g\}$. Then $\zeta^n = g(X + S)^n/g(X)^n = f(\phi(X + S))/f(\phi(X)) = 1$, since $\phi(S) = 0$ and ϕ is an isogeny.

Then we define the Weil pairing $e_\phi(S,T) = \zeta = g(X+S)/g(X)$ for any X not a zero or pole of g.

Proposition 14.5. *e_ϕ is bilinear and non-degenerate.*

Proof. For linearity in the first argument, we have

$$e_\phi(S_1+S_2,T) = \frac{g(X+S_1+S_2)}{g(X+S_2)}\frac{g(X+S_2)}{g(X)} = e_\phi(S_1,T)e_\phi(S_2,T)$$

For linearity in the second argument, let $T_1,T_2 \in E'[\widehat{\phi}]$. Then

$$\mathrm{div}(f_1) = n(T_1) - n(O); \mathrm{div}(f_2) = n(T_2) - n(O)$$

and then

$$\phi^* f_1 = g_1^n; \phi^* f_2 = g_2^n$$

Then there is $h \in \bar{K}(E')^*$ such that $\mathrm{div}(h) = (T_1) + (T_2) - (T_1+T_2) - (O)$.

Then put $f = \frac{f_1 f_2}{h^n}$ and $g = \frac{g_1 g_2}{\phi^* h}$. We then check: $\mathrm{div}(f) = n(T_1+T_2) - n(O)$, and $\phi^* = \frac{\phi^* f_1 \phi^* f_2}{(\phi^* h)^n} = \left(\frac{g_1 g_2}{\phi^* h}\right)^n = g^n$.

So $e_\phi(S, T_1+T_2) = \frac{g(X+S)}{g(X)} = \frac{g_1(X+S)}{g_1(X)}\frac{g_2(X+S)}{g_2(X)}\frac{h(\phi(X))}{h(\phi(X+S))} = e_\phi(S,T_1)e_\phi(S,T_2)$ as $S \in \ker\phi$.

Then to see that e_ϕ is nondegenerate, fix $T \in E'[\widehat{\phi}]$ and suppose that $e_\phi(S,T) = 1$ for all $S \in E[\phi]$. By definition of the Weil pairing, $\tau_S^* g = g$ for all $S \in E[\phi]$.

Then $\bar{K}(E)$ is a Galois field extension of $\phi^*\bar{K}(E')$ with Galois group $E[\phi]$ acting as τ_S^*. So $g = \phi^* h$ for some $h \in \bar{K}(E')$, and hence $\phi^* f = g^n = \phi^*(h^n)$, and so $f = h^n$. Hence $\mathrm{div}(H) = (T) - (O)$, and so $T = O$.

We've shown that the map $E'[\widehat{\phi}] \hookrightarrow \mathrm{Hom}(E[\phi], \mu_n); T \mapsto (S \mapsto e_\phi(S,T))$. This map is thus an isomorphism by counting. □

Remarks.

1. If E, E', ϕ are defined over K, then e_ϕ is Galois equivariant, i.e. $e_\phi(\sigma S, \sigma T) = \sigma(e_\phi(S,T))$.
2. Taking $\phi = [n] : E \to E$, so that $\widehat{\phi} = [n]$, gives $e_n : E[n] \times E[n] \to \mu_{n^2}$. It can be shown, since e_n is bilinear, that we only actually have image in μ_n.

Corollary 14.6. *If $E[n] \subseteq E(K)$, then $\mu_n \subseteq K$.*

Proof. e_n non-degenerate implies that there are $S,T \in E[n]$ such that $e_n(S,T)$ is a primitive n^{th} root of unity, say ζ_n. To see this, pick $T \in E[n]$ of order n. Then the group homomorphism $E[n] \to \mu_n; S \mapsto e_n(S,T)$ has image μ_d for some $d|n$. Then $e_n(S, dT) = 1$ for all $S \in E[n]$, and so by nondegeneracy, $dT = 0$, and so $d = n$.

Then $\sigma(\zeta_n) = e_n(\sigma S, \sigma T) = e_n(S,T)$, and so $\zeta_n \in K$. □

For example, there is no $E/\mathbb{Q}$ with $E(\mathbb{Q})_{\text{tors}} = (\mathbb{Z}/3\mathbb{Z})^2$, since not all of the cube roots of unity are defined over $\mathbb{Q}$.

Remark. In fact, the Weil pairing e_n is alternating, so that $e_n(T, T) = 1$ for all T.

In particular, $e_n(S + T, S + T)$, we have $e_n(S, T) = e_n(T, S)^{-1}$.