

Ramsey Theory

Harry Armitage

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1 Monochromatic Systems

Here, we let $\mathbb{N} = \{1, 2, 3, \dots\}$, and write $[n] = \{1, 2, \dots, n\}$. For a set X and $r \in \mathbb{N}$, we write $X^{(r)} = \{A \subset X : |A| = r\}$, the collection of all r -sets in X .

Suppose we are given a 2-colouring of $\mathbb{N}^{(2)}$, i.e. a function $C : \mathbb{N}^{(2)} \rightarrow \{1, 2\}$. We can think of this being a colouring of the edges of the complete graph on \mathbb{N} . Can we find an infinite monochromatic M , i.e. a set $M \subset \mathbb{N}$ such that C is constant on $M^{(2)}$.

Examples

1. Colour ij (shorthand for the set $\{i, j\}$) red if $i + j$ is even, and blue if $i + j$ is odd. Here, the answer is yes - take $M = 2\mathbb{N} = \{2, 4, 6, \dots\}$.
2. Colour ij red if $\max\{n : 2^n | i + j\}$ is even, and blue if it is odd. The answer is yes - $M = \{4^0, 4^1, 4^2, 4^3, \dots\}$.
3. Colour ij red if $i + j$ has an even number of distinct prime factors, and blue if odd. This is more difficult. To save some time, we shall use the following theorem to answer every question of this form:

Theorem 1.1 (Ramsey's Theorem). *Let $C : \mathbb{N}^{(2)} \rightarrow \{1, 2\}$ be a 2-colouring of $\mathbb{N}^{(2)}$. Then there exists an infinite monochromatic subset of \mathbb{N} .*

Proof. Pick $a_1 \in \mathbb{N}$. Then there are infinitely many edges out of a_1 , so infinitely many have the same colour - say all edges from a_1 to the infinite set B_1 have colour c_1 .

Now pick $a_2 \in B_1$. There must be some infinite set $B_2 \subseteq B_1 \setminus \{a_2\}$ with all edges a_2 to B_2 are the same colour, say c_2 , and repeat inductively.

We then obtain a_1, a_2, a_3, \dots and colours c_1, c_2, c_3, \dots such that $a_i a_j$ for $i < j$ has colour c_i . Now infinitely many of the c_i must be the same colour, say c . Then we may take $M = \{a_i : c_i = c\}$. \square

Remarks

1. This is sometimes called a 2-pass proof - we went through all the numbers to build the sequence a_1, a_2, \dots .
2. In example 3, no explicit example is known.
3. The exact same proof works for n colours. Alternatively, we could deduce this from Ramsey's theorem + induction - view the colours as '1' and '2 or 3 or ...'. If the infinite set is coloured 1, we are done, otherwise repeat with the $n - 1$ colours remaining.
4. An infinite monochromatic set is more than having arbitrarily large finite monochromatic sets. For example, make $\{1\}, \{2, 3\}, \{4, 5, 6\}, \dots$ all monochromatic blue sets, but make all edges between them red. Then there is no infinite monochromatic blue set (there is however an infinite red set - $\{1, 3, 6, \dots\}$).

Example. Any sequence x_1, x_2, \dots in \mathbb{R} (or any totally ordered set) has a monotone subsequence. This was seen in Analysis I, where the proof worked by fiddling around with lim sups and lim infs. Instead, just colour ij 'up' if $x_i \leq x_j$, and 'down' if $x_i > x_j$. Then apply Ramsey's theorem.

What if we coloured $\mathbb{N}^{(r)}$, say for $r = 3, 4, \dots$? Given a 2-colouring $C : \mathbb{N}^{(r)} \rightarrow \{1, 2\}$, must there be an infinite monochromatic set?

For example, take $r = 3$. Colour ijk red if $i|j + k$, and blue if not. Yes - take powers of two. As in the case $r = 2$, we can always do this:

Theorem 1.2 (Ramsey for r -sets). *Let $r \in \mathbb{N}$. Then if $C : \mathbb{N}^{(r)} \rightarrow \{1, 2\}$ is a 2-colouring of $\mathbb{N}^{(r)}$, there exists a monochromatic subset of \mathbb{N} .*

Proof. $r = 1$ is trivial - just infinite pigeonhole. $r = 2$ is **Theorem 1.1**. We then induct on r - suppose our result holds for $r - 1$.

Pick $a_1 \in \mathbb{N}$. We have an induced colouring $D : (\mathbb{N} \setminus \{a_1\})^{(r-1)} \rightarrow \{1, 2\}$ given by $D(F) = C(F \cup \{a_1\})$. By induction, there is an infinite $B_1 \subset \mathbb{N} \setminus \{a_1\}$, monochromatic for the colouring D . That is, $C(F \cup \{a_1\}) = c_1$ for all $F \subset B_1^{(r-1)}$.

Repeating for B_1 , we have $a_2 \in B_1$ and an infinite $B_2 \subset B_1 \setminus \{a_2\}$ with all $F \cup \{a_2\}, F \in B_2^{(r-1)}$ have the same colour, c_2 say.

We thus obtain distinct a_1, a_2, \dots and colours c_1, c_2, \dots such that $C(a_{i_1} a_{i_2} \dots a_{i_r}) = c_{i_1}$. But infinitely many of the c_i agree, say all $i \in I$, and so $\{a_i : i \in I\}$ is our set. \square

Example. We saw from **Theorem 1.1** that, given points $(1, x_1), (2, x_2), \dots$ in \mathbb{R}^2 , there is a subsequence such that the induced function is monotone. In fact, we can ensure that the induced function is either convex or concave. Just 2-colour the triples ijk in $\mathbb{N}^{(r)}$ red if the points $(i, x_i), (j, x_j), (k, x_k)$ are convex, and blue if they are concave (pick either if they are collinear). Then apply **1.2**.

Surprisingly, the infinite version of Ramsey implies the finite version.

Theorem 1.3 (Finite Ramsey). *For all $m, r \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is 2-coloured, there is a monochromatic m -set.*

Proof. Suppose not. We'll show that there is a 2-colouring of $\mathbb{N}^{(r)}$ without a monochromatic m -set, contradicting **Theorem 1.2**. For each $n \geq r$, we have a 2-colouring C_n or $[n]^{(r)}$ with no monochromatic m -set. We would like to take their union to get a bad colouring of $\mathbb{N}^{(r)}$, but can't do this, since they might not agree with each other. There are only finitely many ways to 2-colour $[r]^{(r)}$ (two in fact), so infinitely many of the C_n agree on $[r]^{(r)}$ - say $C_n|_{[r]^{(r)}} = D_r$ for all $n \in A_1$ and some colouring $D_r : [r]^{(r)} \rightarrow \{1, 2\}$, where A_1 is infinite.

There are only finitely many ways to 2-colour $[r+1]^{(r)}$, so infinitely many of the $C_n, n \in A_1$ agree on $[r+1]^{(r)}$ - say they are D_{r+1} on $[r+1]^{(r)}$ for all $n \in A_2 \subset A_1$. Continue inductively.

We obtain colourings $D_n : [n]^{(r)} \rightarrow \{1, 2\}$ for each $n \geq r$, such that:

1. D_n has no monochromatic m -set, as $D_n = C_{n'}|_{[n]^{(r)}}$ for some $n' \geq n$, where $C_{n'}$ has no monochromatic m -set.
2. The D_n are nested in the following way: $D_{n'}|_{[n]^{(r)}} = D_n$ for all $n' > n$.

Now put $C : \mathbb{N}^{(r)} \rightarrow \{1, 2\}; C(F) = D_n(F)$ for any $n \geq \max F$. This is well defined by property 2 and has no monochromatic m -set by 1. ζ \square

Remark. This is called a *compactness argument*: what we are showing is that the space $\{1, 2\}^{\mathbb{N}}$ of 2-colourings, with the product topology, is (sequentially) compact. Note also that this proof gives no information on how large $n = n(m, r)$ has to be. There are direct proofs giving bounds.

What if we colour $\mathbb{N}^{(2)}$ with infinitely many colours, i.e. we have $C : \mathbb{N}^{(2)} \rightarrow X$, some set X . Of course we do not get an infinite monochromatic set ever time - e.g. colour every pair with a different colour. However, this colouring is injective on this set. We might ask, in our quest for order amongst chaos, "Do we always get an infinite set M on which C is either constant or injective?"

It turns out the answer to this question is also no: let $C(ij) = i$ ($i < j$, as usual). However, this is essentially the only way things can go wrong:

Theorem 1.4 (Canonical Ramsey Theorem). *Let C be a colouring of $\mathbb{N}^{(2)}$ with an arbitrary set of colours. Then there exists an infinite set $M \subset \mathbb{N}$ such that one of:*

1. C is constant on $M^{(2)}$.
2. C is injective on $M^{(2)}$.
3. For all $ij, k\ell \in M^{(2)}$, $C(ij) = C(k\ell) \iff i = k$.
4. For all $ij, k\ell \in M^{(2)}$, $C(ij) = C(k\ell) \iff j = \ell$.

Note that this implies Ramsey's theorem, as cases 2,3,4 all require infinitely many colours.

Proof. Define a 2-colouring of $\mathbb{N}^{(4)}$ by giving $ijkl$ the colour *same* if $C(ij) = C(k\ell)$ and *diff* if $C(ij) \neq C(k\ell)$.

By Ramsey for 4-sets, there is an infinite monochromatic M_1 for this colouring. If it has colour *same*, then we have case 1. Indeed, given $ij, k\ell \in M_1^{(2)}$, choose $m < n$ with $m > i, j, k, \ell$. Then $C(ij) = C(mn) = C(k\ell)$.

Otherwise, M_1 has colour *diff*. Now 2-colour $M_1^{(4)}$ by giving $ijkl$ colour *same* if $C(jk) = C(i\ell)$, and *diff* otherwise. Again, by Ramsey for 4-sets we have an infinite $M_2 \subseteq M_1$ monochromatic for this colouring. Note that it cannot be *same*, as then we would have $i < j < k < \ell < m < n$ with $C(jk) = C(in) = C(\ell m)$, giving a contradiction with the previous colouring.

Now 2-colour $M_2^{(4)}$, giving $ijkl$ *same* if $C(ik) = C(j\ell)$, and *diff* if not. Again, get monochromatic infinite M_3 . Now M_3 cannot have colour *same* as above, hence it has colour *diff*.

Now 2-colour $M_3^{(3)}$ by giving ijk colour *same* if $C(ij) = C(jk)$, and *diff* if not. We then have an infinite monochromatic $M_4 \subseteq M_3$. Cannot have M_4 of colour *same*, else have $i < j < k < \ell$ with $C(ij) = C(jk) = C(k\ell)$, so M_4 has colour *diff*.

Now 2-colour $M_4^{(3)}$ by giving ijk colour *left-same* if $C(ij) = C(ik)$, and *left-diff* if not. We obtain infinite monochromatic $M_5 \subseteq M_4$.

Finally, 2-colour $M_5^{(3)}$, giving ijk colour *right-same* if $C(ij) = C(jk)$ and *right-diff* if not. We obtain infinite monochromatic $M_6 \subseteq M_5$.

We then split into the cases:

- M_6 is left-diff and right-diff: This is case 2. All edges are different.
- M_6 is left-same and right-diff: This is case 3. The only edges that agree are ones that share a left vertex.
- M_6 is left-diff and right-same: This is case 4. The only edges that agree are ones that share a right vertex.

- M_6 is *left-same* and *right-same*: This cannot occur. If we have $i < j < k$ in M_6 , then $C(ij) = C(ik) = C(jk)$, giving a contradiction.

□

Remarks.

1. We could use just one colouring, colouring 4-sets $ijkl$ with the partition of $[4]^{(2)}$ induced by C on $\{i, j, k, \ell\}$. This would use as many colours as there are partitions of a set of size $\binom{4}{2}$.
2. Similarly, if $C : \mathbb{N}^{(r)} \rightarrow X$ is an arbitrary colouring, we get an infinite monochromatic $M \subset \mathbb{N}$ and a set $I \subset [r]$ such that, for all $x_1, \dots, x_r, y_1, \dots, y_r \in M^{(r)}$, we have $C(x_1 \dots x_r) = C(y_1 \dots y_r) \iff x_i = y_i \forall i \in I$. These 2^r colourings are called the **canonical colourings** of the r -sets. In the case $r = 2$, we have the correspondence:

Case in 1.4	$I \subset [2]$
1	\emptyset
2	$\{1, 2\}$
3	$\{1\}$
4	$\{2\}$

2 Van der Waerden's Theorem

If we 2-colour \mathbb{N} , can we find 3 consecutive points of the same colour? Answer: of course not - just alternate colours! Ok - what about 3 equally spaced points, i.e. an arithmetic progression of length 3? This is not obviously false, but also not obviously true. What about even hoping for monochromatic arithmetic progressions of length m for any $m \in \mathbb{N}$. (Here, we use "length" to mean the number of terms, not the number of jumps). This is van der Waerden's theorem, which will be the aim of this section.

By our usual compactness argument, this is the same as: for all m , there is n such that whenever $[n]$ is 2-colored, there is a monochromatic AP of length m . Indeed, if this is false, then for every $n \geq m$, there is a colouring $C_n : [n] \rightarrow \{1, 2\}$ with no monochromatic AP of length m . Then infinitely many agree on $[m]$, and of those infinitely many agree on $[m + 1]$ and so on. Put together those nested restrictions to obtain a 2-colouring of \mathbb{N} with no monochromatic AP of length m .

In proving this, one key idea is to generalise: we show that for all m, k there is some n such that whenever $[n]$ is k -coloured there is a monochromatic AP of length m .

Note: proving a stronger result might be easier, e.g. in a proof by induction

Another key idea: given APs A_1, \dots, A_r , each of length m , so $A_i = \{a_i, a_i + d_i, \dots, a_i + (m-1)d_i\}$, we say they are **focused** at f if $a_i + md_i = f$ for all i . For instance, $\{1, 4\}$ and $\{5, 6\}$ are focused at 7.

If in addition each A_i is monochromatic for a given colouring with no two the same colour, we call them **colour-focused**.

So in a k -colouring, if we have APs A_1, \dots, A_k each of length $m-1$ that are colour focused, then we actually have a monochromatic AP of length m , by asking "What colour is the focus?" For

instance, if we are 2-colouring and the sequences $\{1, 4\}$, $\{5, 6\}$ are colour-focused (say $\{1, 4\}$ is red and $\{5, 6\}$ is blue), then if 7 is red we have $\{1, 4, 7\}$ and if 7 is blue then we have $\{5, 6, 7\}$.

We will write $W(m, k)$ for the least n (if it exists) such that whenever $[n]$ is k -coloured, there is a monochromatic AP of length m .

Proposition 2.1. *For all k there is n such that whenever $[n]$ is k -coloured there is a monochromatic AP of length 3.*

Note - this is contained in van der Waerden's theorem - we prove it here for clarity.

Proof. Claim: for all $r \leq k$ there exists n such that whenever $[n]$ is k -coloured, we have either

- a monochromatic AP of length 3.
- or r colour-focused APs of length 2.

Then we will be done - put $r = k$ and look at the focus.

For $r = 1$, this is trivial - take $n = k + 1$. Then we have 2 points of the same colour, so they form a monochromatic AP of length 1.

We now proceed by induction: given n suitable for $r - 1$, we'll show that $(k^{2n} + 1)2n$ is suitable for r .

So now suppose we are given a k -colouring of $[(k^{2n} + 1)2n]$ with no monochromatic AP of length 3. Break up this large set into blocks of length $2n$ - call them $B_1, B_2, \dots, B_{k^{2n}+1}$, where $B_i = [(i - 1) \cdot 2n + 1, i \cdot 2n]$.

Inside any interval of length $2n$ we have $r - 1$ colour-focused APs of length 2 (given by the choice of n) together with their focus (since we took $2n$).

There are k^{2n} ways of k -colouring a block, and therefore some two blocks are identically coloured - say B_s and B_{s+t} .

In B_s we have $r - 1$ colour focused APs of length 2 - say $\{a_1, a_1 + d_1\}, \dots, \{a_{r-1}, a_{r-1} + d_{r-1}\}$ focused at f . But now the APs $\{a_1, a_1 + d_1 + 2nf\}, \dots, \{a_{r-1}, a_{r-1} + d_{r-1} + 2nf\}$ are colour-focused at $f + 4nt$, and $\{f, f + 2nt\}$ is also focused there. Hence we have r colour-focused APs of length 2. \square

Remarks.

1. The idea of looking at the number of ways to colour a block is called a **product argument**.
2. The proof shows $W(3, k) \leq k^{k^{\dots^{k^{4k}}}}$ where the tower has length k . This is called a **tower-type bound**.

Theorem 2.2 (van der Waerden's Theorem). *For all m, k there exists n such that whenever $[n]$ is k -coloured there exists a monochromatic AP of length m .*

Proof. Induction on m . $m = 1$ is trivial - an AP of length 1 is trivial. $m = 2$ is just pigeonhole. $m = 3$ was done above.

We may assume that $W(m - 1, k)$ exists for every k .

Claim: for every $r \leq k$, there is an n such that, whenever $[n]$ is k -coloured, we have either:

- a monochromatic AP of length m .
- or r colour-focused APs of length $m - 1$.

Then, as before, we will be done.

We prove this claim by induction on r . $r = 1$ is done by the induction hypothesis on m - we can take $n = W(m - 1, k)$.

Now given n suitable for $r - 1$, we'll show that $W(m - 1, k^{2n}) \cdot 2n$ is suitable for r .

So, given a k -colouring of $[W(m - 1, k^{2n})2n]$ with no monochromatic AP of length m : break up $[W(m - 1, k^{2n})2n]$ into blocks of length $2n$, say $B_1, B_2, \dots, B_{W(m-1, k^{2n})}$, where $B_i = [(i - 1)2n + 1, i2n]$.

Now, the number of ways to k -colour a block is k^{2n} . So, since we have $W(m - 1, k^{2n})$ blocks, we must have $m - 1$ equally spaced blocks, all coloured identically - say $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$.

Inside B_s we have $r - 1$ colour-focused APs of length $m - 1$ (by the definition of n), together with their focus. Say, A_1, \dots, A_{r-1} focused at f , where A_i has first term a_i and common difference d_i . Then the APs A'_1, \dots, A'_{r-1} , where A'_i has first term a_i and common difference $d_i + 2nt$, are colour focused at $f + (m - 1)2n$. Also, $f, f + 2nt, f + 2 \cdot 2nt, \dots, f + (m - 2)2nt$ is monochromatic of a different colour, focused at $f + (m - 1)2n$.

So we have the r colour focused APs of length $m - 1$. □