Modular Forms

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0 Introduction

Notation. We will write $\mathbb{H} := \{\tau \in \mathbb{C} : \mathfrak{Im}(\tau) > 0\}$ for the complex upper half plane. This is acted on by two groups:

$$GL_2(\mathbb{R})^+ = \{g \in GL_2(\mathbb{R}) : \det(g) > 0\} \geq SL_2(\mathbb{Z}) = \{g \in GL_2(\mathbb{Z}) : \det(g) = 1\}$$

Lemma 0.1. *$GL_2(\mathbb{R})^+$ acts on $\mathbb{H}$ by Möbius transformations. This action is transitive.*

Proof. Let $\tau \in \mathbb{H}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$. We then write $g\tau = \frac{a\tau+b}{c\tau+d}$. This is an action on $\mathbb{C}$ by theory about Möbius transformations. To see that $g\tau \in \mathbb{H}$, we check:

$$\mathfrak{Im}(g\tau) = \frac{1}{2}(g\tau - \overline{g\tau}) = \det(g)\frac{\mathfrak{Im}(\tau)}{|c\tau + d|^2}$$

Now for transitivity, let $\tau = x + \mathrm{i}y \in \mathbb{H}$. Then $\tau = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mathrm{i}$. $\square$

Definition 0.2. *Let $k \in \mathbb{Z}$, and $f : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$, and let $g \in GL_2(\mathbb{R})^+$. Then we define $f|_k[g] : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$ by the formula*

$$f|_k[g](\tau) = f(g\tau)\det(g)^{k-1}j(g,\tau)^{-k}$$

where $j(g,\tau) = c\tau + d$.

Lemma 0.3. *This defines a right actions of $GL_2(\mathbb{R})^+$ on the set of functions $f : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$.*

Proof. Suppose $g, h \in GL_2(\mathbb{R})^+$. We need to show that $f|_k[gh] = (f|_k[g])|_k[h]$.

$$\begin{aligned} RHS(\tau) &= f|_k[g](h\tau)\det(h)^{k-1}j(h,\tau)^{-k} \\ &= f(gh\tau)\det(g)^{k-1}j(g,h\tau)^{-k}j(h,\tau)^{-k}\det(h)^{k-1} \\ LHS(\tau) &= f(gh\tau)\det(gh)^{k-1}j(gh,\tau) \end{aligned}$$

So we need to check that $j(g,h\tau)j(h,\tau) = j(gh,\tau)$.

Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix}$.

So $gh\begin{pmatrix} \tau \\ 1 \end{pmatrix} = j(gh,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gj(h,\tau)\begin{pmatrix} h\tau \\ 1 \end{pmatrix} = j(h,\tau)j(g,h\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix}$. $\square$

Definition 0.4. *Let $k \in \mathbb{Z}$, and let $\Gamma \leq SL_2(\mathbb{Z})$ be a finite index subgroup. Then a meromorphic function $f : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$ is called a weakly modular function of weight k and level Γ if it satisfies $\forall \gamma \in \Gamma, f|_k[\gamma] = f$.*

Motivating Examples

1. Modular forms were first studied in the context of elliptic functions. Suppose that E is an elliptic curve over $\mathbb{C}$, and let ω be a non-vanishing holomorphic differential on E. Then there's a unique holomorphic isomorphism of Riemann surfaces

$$\mathbb{C}/\Lambda \underset{\psi}{\xrightarrow{\sim}} E(\mathbb{C})$$

such that $\psi^*(\omega) = dz$. Here $\Lambda \subset \mathbb{C}$ is a lattice.

E can be defined by the equation $y^2 = x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where $G_k(\Lambda) = \sum\limits_{\omega\in\Lambda\setminus\{0\}} \omega^{-k}$.

This is absolutely convergent provided $k \geq 4$.

If $\tau \in \mathbb{H}$, then we can write $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$. This is a lattice, and the functions $G_k(\tau) = G_k(\Lambda\tau)$ are examples of modular forms.

2. If $f : \mathbb{H} \to \mathbb{C}$ is a modular form, then f has a Fourier expansion $f(\tau) = \sum\limits_{n\geq 0} a_n e^{2\pi i n\tau/h}$ for some natural number h, and complex numbers a_n. These Fourier coefficients often carry useful arithmetic information.

For example, consider $\theta(\tau) = \sum\limits_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$. If $k \geq 2$ is an even integer, then θ^{2k} is a modular form of weight k. Its Fourier expansion is $\theta^{2k}(\tau) = \sum\limits_{n\geq 1} r_{2k}(n)e^{\pi i n\tau}$ where $r_{2k}(n)$ is the number of ways of writing $n = x_1^2 + \ldots x_{2k}^2$, where $x_i \in \mathbb{Z}$.

By relating θ^{2k} to other modular forms with known Fourier series, we can then get information about the numbers $r_{2k}(n)$. For example, $r_4(n) = 8 \sum\limits_{d|n, 4\nmid d} d$.

3. Recall the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$. This function has some important properties:

   a) It has a meromorphic continuation to all of $\mathbb{C}$.
   b) It has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.
   c) It has a representation as an Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$.

Any series $L(s) = \sum_{n\geq 1} a_n n^{-s}$ with $a_n \in \mathbb{C}$ which has properties analogous to these is called an L-function.

For example, if $N \in \mathbb{N}$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character, we can define the Dirichlet L-function $L(\chi, s) = \sum_{(n,N)=1} \chi(n \mod N)n^{-s}$. These functions can be used to prove Dirichlet's theorem on primes in arithmetic progression.

Modular forms can be used to construct L-functions with these properties. To find the right modular forms, we need to introduce Hecke operators.

4. The Langlands programme predicts relations between objects ocurring in number theory and modular forms. This includes as a special case the Shimura-Taniyama-Weil conjecture, otherwise known as the modularity theorem. This asserts a bijection between elliptic curves over $\mathbb{Q}$ up to isogeny and certain modular forms, given by (L-function of elliptic curve) = (L-function of modular form).

1 Modular forms on $SL_2(\mathbb{Z})$

Recall the definition, for $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, $g \in GL_2(\mathbb{R})^+$, we have

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g,\tau)^{-k}$$

We said f is ***weakly modular of weight k and level*** $SL_2(\mathbb{Z})$ if f is meromorphic on $\mathbb{H}$ and, for all $\gamma \in SL_2(\mathbb{Z})$, $f|_k[\gamma] = f$.

Note that $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ satisfies $f|_k[T](\tau) = f(\tau+1)$. So if f is a weakly modular function, then we can define a new function

$$\widetilde{f} : \{q \in \mathbb{C} : 0 < |q| < 1\} \to \mathbb{C}; e^{2\pi i\tau} \mapsto f(\tau)$$

This function $\widetilde{f}$ is meromorphic, since f is.

Definition 1.1. *We say that the weakly modular function f is:*

- *meromorphic at ∞ if $\widetilde{f}$ is meromorphic at 0.*
- *holomorphic at ∞ if $\widetilde{f}$ is holomorphic at 0.*
- *vanishes at ∞ if $\widetilde{f}$ is holomorphic and vanishes at 0.*

If f is meromorphic at ∞ then $\widetilde{f}$ has a Laurent expansion $\widetilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in some region $\{0 < |q| < \varepsilon\}$, where $a_n \in \mathbb{C}$ and $a_n = 0$ if $n < 0$ and $|n|$ is sufficiently large.

We get a formula $f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n$ where $q = e^{2\pi i\tau}$. This is valid in some region $\{\tau \in \mathbb{H} : \Im\mathfrak{m}\, \tau > R\}$, and is called the q-expansion of f. Then f is holomorphic at ∞ if and only if $a_n = 0$ when $n < 0$, and $f(\infty) = a_0$.

Definition 1.2. *Let f be a weakly modular function of weight k and level $SL_2(\mathbb{Z})$. We say that f is*

- *a* **modular function** *if f is meromorphic at ∞.*
- *a* **modular form** *if f is holomorphic in $\mathbb{H}$ and holomorphic at ∞.*
- *a* **cuspidal modular form** *if f is a modular form vanishing at ∞.*

all with weight k and level $SL_2(\mathbb{Z})$.

We write $M_k(SL_2(\mathbb{Z}))$ for the $\mathbb{C}$-vector space of modular forms of weight k and level $SL_2(\mathbb{Z})$. We write $S_k(SL_2(\mathbb{Z}))$ for the subspace of cuspidal modular forms.

Examples. If $\tau \in \mathbb{H}$, then $\Lambda_\tau = \mathbb{Z}_\tau \oplus \mathbb{Z}$. if $k \in \mathbb{Z}$, then we can define $G_k(\tau) = \sum\limits_{\omega \in \Lambda_\tau \setminus \{0\}} \omega^{-k}$.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\Lambda_{\gamma\tau} = \mathbb{Z}\left(\frac{a\tau+b}{c\tau+d}\right) \oplus \mathbb{Z} = j(\gamma,\tau)^{-1}\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d) = j(\gamma,\tau)^{-1}\Lambda_\tau$.

Finally, we find $G_k|_k[\gamma](\tau) = G_k(\gamma\tau) j(\gamma,\tau)^{-k} = \sum\limits_{\omega \in \Lambda_{\gamma\tau}\setminus\{0\}} (\omega j(\gamma,\tau))^{-k} = \sum\limits_{\omega\in\Lambda_\tau\setminus\{0\}} \omega^{-k} = G_k(\tau)$.

Proposition 1.3. *Suppose $k \geq 4$ and k is even. Then $G_k(\tau)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$. Moreover, $G_k(\tau)$ is holomorphic at ∞ and $G_k(\infty) = 2\zeta(k)$. In particular, $G_k \in M_k(SL_2(\mathbb{Z}))$.*

Remark. We have $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$ and $f|_k[-I] = f \cdot (-1)^k$, so if k were odd then $f \equiv 0$, and hence $M_k(SL_2(\mathbb{Z})) = 0$ when k is odd.

Proof. Fix $A \geq 1$. Define $\Omega_A = \{\tau \in \mathbb{H} : |\mathfrak{Re}(|\tau)| \leq A, \mathfrak{Im}(\tau) \geq \frac{1}{A}\}$. We'll show uniform convergence of G_k in Ω_A. Note that if $\tau \in \Omega_A$, then for any $x \in \mathbb{R}$, $|\tau + x| \geq \frac{1}{A}$, and $|\tau + x| \geq \frac{1}{2}|x|$ if $|x| \geq 2A$. Hence $|\tau + x| \geq \sup(1/A, 1/2A^2|x|) \geq \frac{1}{2A^2}\sup(1, |x|)$ for any $x \in \mathbb{R}$.

If $\tau \in \Omega_A$, then:

$$\begin{aligned}
\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} |m\tau + n|^{-k} &= \sum_{(m,n)} |m|^{-k}|\tau + n/m|^{-k} \\
&\leq \sum_{(m,n)} \frac{|m|^{-k}}{(2A)^{-k}} \sum (1, |n/m|)^{-k} \\
&= \sum_{(m,n)} (2A)^k \sup(|m|^{-k}, |n|^{-k}) \\
&= \sum_{r\in\mathbb{N}} (2A)^k r^k 8r = (2A)^k 8\zeta(k-1)
\end{aligned}$$

This shows absolute and uniform convergence.

To show that G_k is holomorphic at ∞ and $G_k(\infty) = 2\zeta(k)$, it's enough to show that

$$\lim_{\tau\in\Omega_1, \mathfrak{Im}\,\tau\to\infty} G_k(\tau) = 2\zeta(k)$$

This limit equals $\sum_{(m,n)} \lim_{\mathfrak{Im}\,\tau\to\infty} (m\tau + n)^{-k} = \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-k} = 2\zeta(k)$, as all terms with $m \neq 0$ vanish. □

G_k is an example of an ***Eisenstein series***.

Definition 1.4. *We define the* **normalised Eisenstein series** $E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) = 1 + \sum_{n\geq 1} a_n q^n$. *We'll see that the a_n are rational numbers of bounded denominators.*

Remark. If $f \in M_k(SL_2(\mathbb{Z}))$ and $g \in M_\ell(SL_2(\mathbb{Z}))$, then $fg \in M_{k+\ell}(SL_2(\mathbb{Z}))$. So $E_4^3, E_6^2 \in M_{12}(SL_2(\mathbb{Z}))$, and $E_4^3(\infty) = E_6^2(\infty)$, so $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(SL_2(\mathbb{Z}))$. We'll see shortly that $\Delta = \sum_{n\geq 1} b_n q^n$ where $b_1 = 1, b_n \in \mathbb{Z}$ for all $n \geq 1$.

We now study a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. We will write $\Gamma(1) = SL_2(\mathbb{Z})$, and $\overline{\Gamma(1)} = SL_2(\mathbb{Z})/\langle -I \rangle$. This will make sense later.

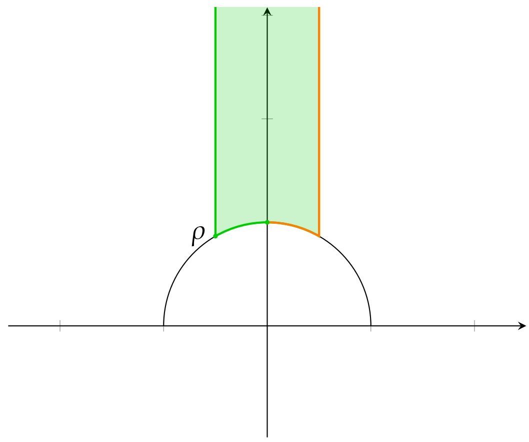
We write

$$\mathcal{F} = \{\tau \in \mathbb{H} : -\frac{1}{2} \leq \mathfrak{Re}\,\tau \leq \frac{1}{2}, |\tau| \geq 1\}$$

and

$$\mathcal{F}' = \{\tau \in \mathcal{F} : \mathfrak{Re}\,\tau < 1/2, |\tau| = 1 \implies \mathfrak{Re}\,\tau \leq 0\}$$

In the following diagram, $\mathcal{F}$ is all of the green + orange regions, whilst $\mathcal{F}'$ is just the green area. We also define $\rho := \exp(2\pi i/3)$



We have elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma(1)$.

Proposition 1.5. *$\mathscr{F}$ is a fundamental domain for the action of $\overline{\Gamma(1)}$ on $\mathbb{H}$. More precisely, if $\tau \in \mathbb{H}$ there is $\gamma \in \overline{\Gamma(1)}$ such that $\gamma\tau \in \mathscr{F}$. If $\gamma\tau \in \mathscr{F}^\circ$, then γ is unique. Moreover, each $\tau \in \mathbb{H}$ is $\overline{\Gamma(1)}$-conjugate to exactly one element of $\mathscr{F}'$.*

Proof. We first prove that any $\tau \in \mathbb{H}$ is $\overline{\Gamma(1)}$-conjugate to an element of $\mathscr{F}$. We proved earlier that if $\tau \in \mathbb{H}$ and $\gamma \in \Gamma(1)$, then $\Im\mathfrak{m}\, \gamma(\tau) = \Im\mathfrak{m}(\tau)/|c\tau + d|^2$.

If $\tau \in \mathbb{H}$, then $\Lambda_\tau = \mathbb{Z}_\tau \oplus \mathbb{Z}$ is a lattice. So as $(c, d) \in \mathbb{Z}^2 \setminus \{0\}$, the numbers $|c\tau + d|$ achieve a minimum. Consequently, the numbers $\Im\mathfrak{m}\, \gamma(\tau)$ for $\gamma \in \Gamma(1)$ achieve a maximum. So WLOG we may assume $\Im\mathfrak{m}(\tau) \geq \Im\mathfrak{m}(\gamma\tau)$ for all $\gamma \in \Gamma(1)$. Also WLOG we may take $-\frac{1}{2} \leq \Re\mathfrak{e}(\tau) \leq \frac{1}{2}$.

We then claim that these properties are sufficient for $\tau \in \mathscr{F}$. It is sufficient to show that $|\tau| \geq 1$. We have $\Im\mathfrak{m}(S\tau) = \Im\mathfrak{m}(\tau)/|\tau|^2 \leq \Im\mathfrak{m}(\tau)$, and hence $|\tau|^2 \geq 1$, so we are done. □

We slightly strengthen this with the following proposition:

Proposition 1.6.

1. *For all $\tau \in \mathbb{H}$, there is a unique $\gamma \in \overline{\Gamma(1)}$ such that $\gamma\tau \in \mathscr{F}'$.*
2. *If $\tau \in \mathscr{F}'$, then $\mathrm{Stab}_{\overline{\Gamma(1)}}(\tau) = \{I\}$, except $\mathrm{Stab}_{\overline{\Gamma(1)}}(\mathrm{i}) = \{I, S\}$ and $\mathrm{Stab}_{\overline{\Gamma(1)}}(\rho) = \{1, ST, (ST)^2\}$.*
3. *$\overline{\Gamma(1)}$ is generated by S and T.*

Proof. To prove the first two parts, it's enough to show that:

a) For all $\tau \in \mathbb{H}$, there is $\gamma \in \overline{\Gamma(1)}$ such that $\gamma\tau \in \mathscr{F}'$.

b) For all $\tau, \tau' \in \mathscr{F}'$ and $\gamma \in \overline{\Gamma(1)}$, $\gamma\tau' = \tau \implies \tau' = \tau$ and either $\begin{cases} \gamma = 1 \\ \tau = \mathrm{i}, \gamma = S \\ \tau = \rho, \gamma = ST, (ST)^2 \end{cases}$.

a) was done above. For b), take $\tau, \tau' \in \mathscr{F}'$ such that $\tau' = \gamma\tau$. We have $\Im\mathfrak{m}(\gamma\tau) = \Im\mathfrak{m}(\tau)/|c\tau + d|^2$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Without loss of generality, we have $\Im(\tau') = \Im(\gamma\tau) \geq \Im(\tau) \implies |c\tau + d| \leq 1$.

So $\Im(\tau) \geq \sqrt{3}/2 \implies |c\tau + d| \geq c\sqrt{3}/2$, and so $|c| \leq 1$, so we can assume $c = 1$ or 0 (if -1, just multiply by $-I$, since we are in $\overline{\Gamma(1)} = \Gamma(1)/\langle -I \rangle$). We then split into cases:

1. $c = 0, \gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. This forces $\gamma = I, \tau = \tau'$.

2. $c = 1, \gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$. Now $|\tau + d| \leq 1$. Then $\tau \in \mathscr{F}' \implies$ either $d = 0, |\tau| = 1$, or $d = 1, \tau = \rho$.

   In the first case, $\gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$, and so $\gamma\tau = a - \frac{1}{\tau}$. We have $\Re(\tau), \Re(\Gamma(\tau)) = a - \Re(\tau)$ both in $[-1/2, 0]$. The only possibilities are $\Re(\tau) = -\frac{1}{2}, a = -1, \tau = \rho, \gamma = (ST)^2$ and $\Re(\tau) = 0, a = 0, \tau = \mathrm{i}, \gamma = S$.

   In the second case, $d = 1, \tau = \rho, \gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$. Then $\gamma\rho = \frac{a\rho+b}{\rho+1}$. We have $\rho^2 + \rho + 1 = 0, \rho^2 = \rho - 1$, so $\gamma\rho = -\rho(a\rho + b) = -a\rho^{-1} - b\rho$.

   We know that $|\rho + 1| = |\tau + d| = 1$, so $\Im(\gamma\rho) = \Im(\rho)/|\rho + 1| = \Im(\rho)$. So $\gamma\rho = \rho$, as ρ is the unique element of $\mathscr{F}'$ of smallest imaginary part, and hence $\rho = -a\rho^{-1} - b\rho \implies a = 0, b = -1$, and so $\gamma = ST$.

For part 3., lets take $G = \langle S, T \rangle$. For all $\tau \in \mathbb{H}$, there is $\gamma \in G$ with $\gamma\tau \in \mathscr{F}$. Why? Without loss of generality, we can assume that, for all $\gamma \in G, \Im(\gamma\tau) \leq \Im(\tau)$, and moreover that $-\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$.

This implies that $\tau \in \mathscr{F}$, as $\Im(S\tau) = \Im(\tau)/|\tau|^2 \leq \Im(\tau) \implies |\tau| \geq 1$.

Choose $\tau \in \mathscr{F}^\circ$. Choose $\gamma \in \overline{\Gamma(1)}$. We'll show that $\gamma \in G$. Note that $\gamma\tau \in \mathbb{H}$, so there is $\delta \in G$ such that $\delta\gamma\tau \in \mathscr{F}$, so $\delta\gamma\tau \in \mathscr{F}^\circ$ and $\delta\gamma = I$, so $\delta = \gamma^{-1} \in G$. □

If $P \in \overline{\Gamma(1)} \backslash \mathbb{H}$ (since $\overline{\Gamma(1)}$ acts on the left on $\mathbb{H}$, this is a left quotient. P is a $\overline{\Gamma(1)}$-orbit, i.e. can be represented as $\overline{\Gamma(1)} \cdot \tau$ for some $\tau \in \mathbb{H}$), then we define $e_P = |\operatorname{Stab}_{\overline{\Gamma(1)}}(\tau)|$.

We've just shown that $e_P = 1$ except for $\begin{cases} e_{\overline{\Gamma(1)}\cdot\rho} = e_\rho = 3 \\ e_{\overline{\Gamma(1)}\cdot\mathrm{i}} = e_\mathrm{i} = 2 \end{cases}$.

Suppose that f is a modular function of weight k and level $SL_2(\mathbb{Z})$. Then we define $v_P(f)$ to be the order of f at τ (where τ is a representative for P).

Note that this independent of the specific choice of representative τ as, for any $\gamma \in \Gamma(1)$ we have $f(\gamma\tau)j(\gamma, \tau)^{-k} = f(\tau)$, and $j(\gamma, \tau)$ is holomorphic and non-vanishing.

We define $v_\infty(f) = \inf\{n \in \mathbb{Z} : a_n \neq 0\}$, where $f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n$ is the q-expansion of f. Equivalently, this is the order of $\widetilde{f}$ at $q = 0$.

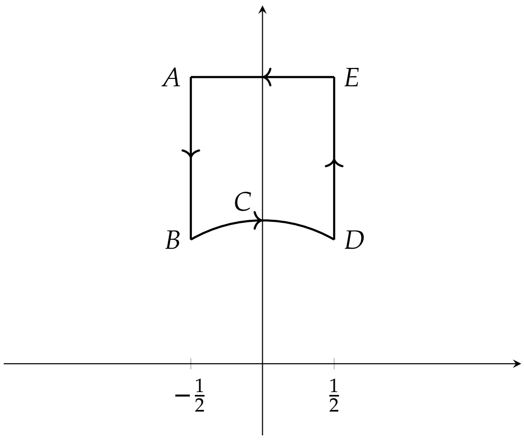
Theorem 1.7. *Let f be a modular function of weight k and level $SL_2(\mathbb{Z})$. Assume that $f \neq 0$. Then:*

$$v_\infty(f) + \sum_{P \in \overline{\Gamma(1)} \backslash \mathbb{H}} \frac{1}{e_P} v_P(f) = \frac{k}{12}$$

Proof. Let $U \subseteq \mathbb{C}$ be an open subset, and $\gamma \subseteq U$ a positively oriented simple closed contour, and $f : U \to \mathbb{C}$ a meromorphic function with no zeros or poles on γ. Then $\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{df}{f} = \sum_{\tau \in \mathrm{Int}(\gamma)} v_\tau(f)$ - this is the argument principle.

Let's first prove the theorem assuming that f has no zeros or poles on the boundary of $\mathscr{F}$. Since f is meromorphic at infinity, there exists a $R > 0$ such that f has no zeros or poles on in $\{\tau \in \mathbb{H} : \Im\mathfrak{m}(\tau) \geq R\}$.

We consider the contour $\gamma = ABCDEA$:



where $A = -\frac{1}{2} + \mathrm{i}R, B = \rho, C = \mathrm{i}, D = \rho + 1, E = \frac{1}{2} + \mathrm{i}R$.

The argument principle gives $\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{df}{f} = \sum_{\tau \in \mathrm{Int}(\mathscr{F})} v_\tau(f)$.

We can break up the integral into the different segments AB, BC, CD, DE, and EA, and make some observations:

- $f(\tau) = f(\tau + 1)$, so $\int_A^B \frac{df}{f} = \int_E^D \frac{df}{f} = -\int_D^E \frac{df}{f}$, so these paths cancel.
- The image of the path EA under the map $\tau \mapsto e^{2\pi \mathrm{i} \tau}$ is a negatively oriented circle c going around $q = 0$, so $\frac{1}{2\pi \mathrm{i}} \int_E^A \frac{df}{f} = \frac{1}{2\pi \mathrm{i}} \oint_c \frac{d\widetilde{f}}{\widetilde{f}} = -v_0(\widetilde{f}) = -v_\infty(f)$.
- The path from CD is the image of the path CB under S. So $\frac{1}{2\pi \mathrm{i}} \int_D^C \frac{df}{f} = \frac{1}{2\pi \mathrm{i}} \int_B^C \frac{d(f \circ S)}{(f \circ S)}$. We have $f(S\tau) = f(\tau)\tau^k$, so $\frac{d(f \circ S)}{f \circ S} = \frac{k d\tau}{\tau} + \frac{df}{f}$.

  Hence this integral is $\frac{1}{2\pi \mathrm{i}} \int_B^C \frac{k}{\tau} d\tau + \int_B^C \frac{df}{f}$, and so we have:

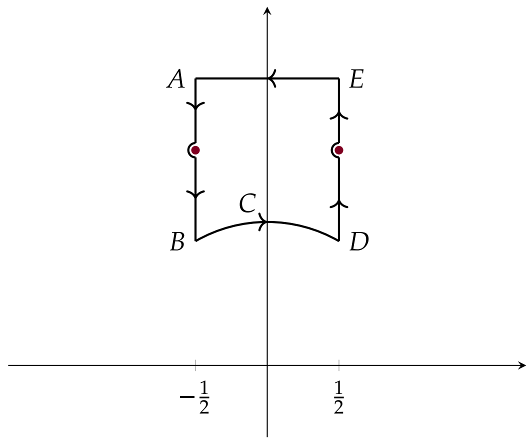
$$\frac{1}{2\pi \mathrm{i}} \int_B^C \frac{df}{f} + \frac{1}{2\pi \mathrm{i}} \int_C^D \frac{df}{f} = \frac{1}{2\pi \mathrm{i}} \int_C^B \frac{k}{\tau} d\tau = \frac{k}{12}$$

Putting this all together, we have:

$$\frac{k}{12} - v_\infty(f) = \sum_{\tau \in \mathrm{Int}(\mathscr{F})} v_\tau(f)$$

Since we're assuming all zeros and poles are in the interior and so have $e_P = 1$, adding in the e_Ps for the result in the theorem doesn't change anything.

If there are zeros or poles on the boundary of $\mathscr{F}$, then we need a modified contour. First suppose that f has a zero or pole on the lines AB and DE, but nowhere else. Then we use the contour γ':



where the small semicircles are chosen so that they avoid all zeros or poles of f, noting that the zeros and poles of a meromorphic function are isolated, and so that AB is mapped to ED by T, in order to still have $\int_B^A \frac{df}{f} + \int_D^E \frac{df}{f} = 0$. The rest of the proof goes through as before. We can make a similar modification if f has a zero/pole on BC.

The remaining case is when f has a zero or pole at ρ or i. In this case, we use the following observation: let $g : U \to \mathbb{C}$ be a meromorphic function defined in an open neighbourhood of $z = 0$.

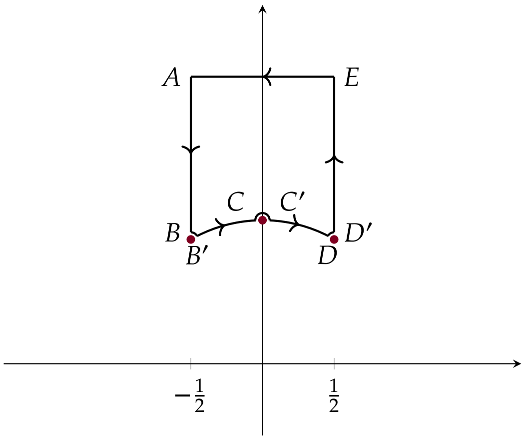
We consider the paths $\gamma_\varepsilon : [0,1] \to U$ given by $\gamma_\varepsilon(t) = \varepsilon e^{2\pi\mathrm{i}(\theta_0 + t\theta)}$. Then:

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi\mathrm{i}} \int_{\gamma_\varepsilon} \frac{dg}{g} = \frac{\theta}{2\pi} v_0(g)$$

To show this, write $g(z) = z^n h(z)$ where $n = v_0(g)$ and $h(z)$ is holomorphic and non-vanishing at 0. Then

$$\frac{1}{2\pi\mathrm{i}} \int_{\gamma_\varepsilon} \frac{dg}{g} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_\varepsilon} \frac{n\,dz}{z} + \frac{1}{2\pi\mathrm{i}} \int_{\gamma_\varepsilon} \frac{dh}{h} \to \frac{\theta}{2\pi} + 0$$

Now suppose that f has zeros or poles at ρ or i, and at no other points on the boundary of $\mathscr{F}$. We consider a family of contours γ_ε given by replacing γ at B, C, and D by small arcs of radius ε.



Then the argument principle gives:

$$\frac{1}{2\pi\mathrm{i}}\left[\int_A^B + \int_B^{B'} + \ldots + \int_E^A \frac{df}{f}\right] = \sum_{\tau\in\mathrm{Int}(\mathscr{F})} v_\tau(f)$$

It's still the case that $\frac{1}{2\pi\mathrm{i}}\int_E^A \frac{df}{f} = -v_\infty(f)$, and that the paths AB and $D'E$ cancel. It's also still the case that $\frac{1}{2\pi\mathrm{i}}\left[\int_{B'}^C + \int_{C'}^D \frac{df}{f}\right] = \frac{\alpha k}{2\pi}$, where α is the angle swept out by CB', which tends to $k/12$ as $\varepsilon \to 0$.

We need to understand the remaining terms given by the paths BB', CC', DD'. Using our previous observation, we see that $\lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_B^{B'}\frac{df}{f} = -\frac{1}{6}v_\rho(f)$. Similarly, we have $\lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_C^{C'}\frac{df}{f} = -\frac{1}{2}v_{\mathrm{i}}(f)$, $\lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_D^{D'}\frac{df}{f} = -\frac{1}{6}v_\rho(f)$.

We finally obtain an identity:

$$v_\infty(f) + \frac{1}{3}v_\rho(f) + \frac{1}{2}v_{\mathrm{i}}(f) + \sum_{\tau\in\mathrm{Int}(\mathscr{F})} v_\tau(f) = \frac{k}{12}$$

giving the result. □

Let's now apply this to some examples. Take $k = 4, f = E_4 \in M_4(SL_2(\mathbb{Z}))$. We get:

$$v_\infty(E_4) + \sum_{P\in\overline{\Gamma(1)}\setminus\mathbb{H}} \frac{1}{e_P}v_P(E_4) = \frac{1}{3}$$

and so $v_\rho(E_4) = 1$ and $v_P(E_4) \neq 0$ for $P \neq \overline{\Gamma(1)}\cdot\rho$. i.e. E_4 has a simple zero at ρ and no other zeros in $\mathscr{F}'$.

Now take $k = 6, f = E_6$. We get $LHS = \frac{1}{2}$, and so $v_{\mathrm{i}}(E_6) = 1, v_P(E_6) = 0$ for all $P \neq \overline{\Gamma(1)}\cdot\mathrm{i}$, i.e. E_6 has a simple zero at i and no other zeros in $\mathscr{F}'$.

We defined $\Delta = (E_4^3 - E_6^2)/1728 \in S_{12}(SL_2(\mathbb{Z}))$. Then $\Delta(\mathrm{i}) = E_4(\mathrm{i})^3/1728 \neq 0$, and so Δ is actually a non-zero cuspidal modular form. We apply our formula to Δ, using that it is non-zero, and get

$$v_\infty(\Delta) + \sum_{P \in \overline{\Gamma(1)}} \frac{1}{e_P} v_P(\Delta) = 1$$

We know Δ is cuspidal so $v_\infty(\Delta) \geq 1$, hence $v_\infty(\Delta) = 1$ and Δ is non-vanishing in $\mathbb{H}$.

Theorem 1.8. *Let $k \in \mathbb{Z}$ be an even integer. Then:*

1. *If $k < 0$ or $k = 2$, then $M_k(SL_2(\mathbb{Z})) = 0$ Moreover, $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$ (identified with the constant functions).*
2. *If $4 \leq k \leq 10$ or $k = 14$, then $M_k(SL_2(\mathbb{Z})) = \mathbb{C} \cdot E_k$*
3. *If $k \geq 0$, then multiplication by Δ induces an isomorphism $M_k(SL_2(\mathbb{Z})) \xrightarrow{\sim} S_{k+12}(SL_2(\mathbb{Z}))$.*

Proof. We use the formula $v_\infty(f) + \sum_P \frac{1}{e_P} v_P(f) = \frac{k}{12}$, valid for any non-zero $f \in M_k(SL_2(\mathbb{Z}))$. If $k < 0, LHS \geq 0, RHS < 0$ and so there are no such f.

If $k = 2, RHS = 1/6, LHS = a + b/2 + c/3$ where $a, b, c \in \mathbb{Z}_{\geq 0} \lightning$.

Suppose $f \in M_0(SL_2(\mathbb{Z}))$ and f is not a scalar. Then there is $\lambda \in \mathbb{C}$ such that $f - \lambda$ is cuspidal and non-zero, so $v_\infty(f - \lambda) \geq 1$. But then $LHS > 0, RHS = 0$, and we have a contradiction. Hence $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$.

Now suppose $f \in M_k(SL_2(\mathbb{Z}))$ and either $4 \leq k \leq 10$ or $k = 14$. Then there is $\lambda \in \mathbb{C}$ such that $f - \lambda E_k \in S_k(SL_2(\mathbb{Z}))$. If $f - \lambda E_k \neq 0$, we get $v_\infty(f - \lambda E_k) + \sum_P \frac{1}{e_P} v_P(f - \lambda E_k) = \frac{k}{12}$. If $k < 12$, then $RHS < 1$ and $LHS \geq 1 \lightning$. If $k = 14$, then we will use part 3 and 1 to show $S_{14}(SL_2(\mathbb{Z})) = 0$, so $M_{14}(SL_2(\mathbb{Z})) = \mathbb{C}E_{14}$.

To prove the final part of the theorem, consider the described map $\times\Delta : M_k(SL_2(\mathbb{Z})) \to S_{k+12}(SL_2(\mathbb{Z}))$. It's injective as Δ is non-vanishing in $\mathbb{H}$, so $f\Delta = g\Delta \implies f = g$. It's surjective as Δ is non-vanishing and $v_\infty(\Delta) = 1$. This means that, if $f \in S_{k+12}(SL_2(\mathbb{Z}))$ then $v_\infty(f/\Delta) = v_\infty(f) - 1 \geq 0$, and so $f/\Delta \in M_k(SL_2(\mathbb{Z}))$.

□

Corollary 1.9. *For any $k \in \mathbb{Z}, k \geq 0$ even, we have*

$$\dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \mod 12 \end{cases}$$

Proof. The theorem shows this is true for $0 \leq k \leq 14$. We have $M_k(SL_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(SL_2(\mathbb{Z}))$, just by subtracting a scalar multiple of E_k to get a cusp form, and so $\dim_{\mathbb{C}} M_{k+12}(SL_2(\mathbb{Z})) = 1 + \dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z}))$, and the result follows by induction. □

Corollary 1.10. *Let $k \geq 0$ be even. Then $M_k(SL_2(\mathbb{Z}))$ is spanned as a $\mathbb{C}$-vector space by the elements $E_4^a E_6^b$ where $a, b \in \mathbb{Z}_{\geq 0}$ and $4a + 6b = k$.*

Proof. This holds when $k \leq 10$. We'll now show that if the corollary holds for k, then it holds for $k + 12$. This will give the general case by induction.

Choose $a, b \in \mathbb{Z}_{\geq 0}$ such that $4a + 6b = k + 12$. Then $E_4^a E_6^b \in M_{k+12}$ with leading term 1 in its q-expansion, so we have $M_{k+12}(SL_2(\mathbb{Z})) = S_{k+12}(SL_2(\mathbb{Z})) \oplus \mathbb{C}E_4^a E_6^b = \Delta M_k(SL_2(\mathbb{Z})) \oplus \mathbb{C}E_4^a E_6^b$.

Note that $\Delta = (E_4^3 - E_6^2)/1728$, so the result follows. $\square$

Definition 1.11. *We define $j : \mathbb{H} \to \mathbb{C}$ by the formula $j(\tau) = E_4^3(\tau)/\Delta(\tau)$. This is a modular function of weight 0 and level $SL_2(\mathbb{Z})$.*

If $\tau \in \mathbb{H}$, then $j(\tau)$ is the j-invariant of the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$.

Theorem 1.12.

1. *j is holomorphic in $\mathbb{H}$ and $v_\infty(j) = -1$.*
2. *j gives a bijection $\overline{\Gamma(1)}\diagdown\mathbb{H} \xrightarrow{\sim} \mathbb{C}$.*
3. *Every modular function of weight 0 and level $SL_2(\mathbb{Z})$ is a rational function of j.*

Later, we'll give $\overline{\Gamma(1)}\diagdown\mathbb{H} \sqcup \{\infty\}$ the structure of a compact Riemann surface. Part 3 of the above theorem says that j gives an isomorphism from this surface to the Riemann sphere.

Proof.

1. $j = E_4^3/\Delta$ is holomorphic in $\mathbb{H}$ as Δ is non-vanishing in $\mathbb{H}$. We have $v_\infty(E_k^3) = 0$ and $v_\infty(\Delta) = 1$, so $v_\infty(E_4^3/\Delta) = 0 - 1 = -1$.
2. We need to show that for all $z \in \mathbb{C}$, there is a unique $\tau \in \mathbb{H}$ modulo $\overline{\Gamma(1)}$ such that $j(\tau) = z$, or equivalently, $E_4^3(\tau) - z\Delta(\tau) = 0$.

   We have $v_\infty(E_4^3 - z\Delta) + \sum_P \frac{1}{e_P} v_P(E_4^3 - z\Delta) = 1$. The first term is 0, since the leading term of $E_4^3 - z\Delta$ is 1. Then we have $a + b/2 + c/3 = 1$ for $a, b, c \in \mathbb{Z}_{\geq 0}$. We see that there's exactly one P such that $v_P(E_4^3 - z\Delta) > 0$, and so have the desired statement.
3. Let f be a non-zero modular function of weight 0 and level $SL_2(\mathbb{Z})$. Then we can multiply f by terms of the form $j - j(\tau_0)$ to conclude that f is holomorphic in $\mathbb{H}$. Then there exists $n \geq 0$ such that $\Delta^n f$ is holomorphic at ∞, and so $\Delta^n f \in M_{12n}(SL_2(\mathbb{Z}))$.

   To prove the theorem, it's enough to show that, if $4a + 6b = 12$, then $E_4^a E_6^b/\Delta^n$ is a rational function of j.

   Note that, if $4a + 6b = 12n$, then $2a + 3b = 6n$, so we can write $a = 3p, b = 2q$ for some $p, q \in \mathbb{Z}_{\geq 0}$, and then $p + q = n$. Hence $E_4^a E_6^b/\Delta^n = (E_4^3/\Delta)^p (E_6^2/\Delta)^q = j^p (E_6^2/\Delta)^q$. So it remains to show that E_6^2/Δ is a rational function of j.

   By definition, $\Delta = (E_4^3 - E_6^2)/1728$, and so $E_6^2/\Delta = E_4^3/\Delta - 1728 = j - 1728$.

$\square$

Proposition 1.13. *Let $k \geq 4$ be an even integer. Then the q-expansion of G_k is*

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Proof. We use the identity $\pi\cot(\pi\tau) = \frac{1}{\tau} + \sum_{n=1}^{\infty}\left(\frac{1}{\tau-n} + \frac{1}{\tau+n}\right)$ as holomorphic functions in $\mathbb{H}$. The RHS is absolutely convergent and uniformly convergent in compact subsets.

Note that $\pi\cot(\pi\tau) = \pi\mathrm{i}\frac{e^{\mathrm{i}\pi\tau}+e^{-\mathrm{i}\pi\tau}}{e^{\mathrm{i}\pi\tau}-e^{-\mathrm{i}\pi\tau}} = \pi\mathrm{i}\frac{q+1}{q-1} = -\pi\mathrm{i}(1+q)(1+q+q^2+\ldots)$.

So $-\pi\mathrm{i} - 2\pi\mathrm{i}\sum_{n\geq 1} q^n = \frac{1}{\tau} + \sum_{n=1}^{\infty}\left(\frac{1}{\tau-n} + \frac{1}{\tau+n}\right)$. Differentiating $k-1$ times gives:

$$-2\pi\mathrm{i}\sum_{n\geq 1}(2\pi\mathrm{i})^{k-1}q^n = (-1)^{k-1}(k-1)!\left[\frac{1}{\tau^k} + \sum_{n=1}^{\infty}\left(\frac{1}{(\tau-n)^k} + \frac{1}{(\tau+n)^k}\right)\right] = (-1)^{k-1}(k-1)!\sum_{n\in\mathbb{Z}}(\tau+n)^{-k}$$

Hence, after rearranging, we have:

$$\sum_{n\in\mathbb{Z}}(\tau+n)^{-k} = \frac{(2\pi\mathrm{i})^k}{(k-1)!}\sum_{n\geq 1} n^{k-1}q^n$$

Applying this to $G_k(\tau)$, we have:

$$\begin{aligned} G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}}(m\tau+n)^{-k} &= 2\zeta(k) + 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}}(m\tau+n)^{-k} \\ &= 2\zeta(k) + 2\frac{(2\pi\mathrm{i})^k}{(k-1)!}\sum_{m,n\geq 1} n^{k-1}q^{nm} \\ &= 2\zeta(k) + 2\frac{(2\pi\mathrm{i})^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n \end{aligned}$$

□

Corollary 1.14. *$E_k(\tau)$ has q-expansion:*

$$E_k(\tau) = 1 + \frac{(2\pi\mathrm{i})^k}{(k-1)!\zeta(k)}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$$

In particular, the coefficients are rationals, and integers when $k = 4$ or 6, in which case:

$$E_4(\tau) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n; E_6(\tau) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$$

Proof. Fact: when $k \in 2\mathbb{Z}_{>0}$, $\pi^k/\zeta(k)$ is a rational number. In particular, $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$. This gives the result. □

Proposition 1.15. *The q-expansion of Δ is $q + \sum_{n\geq 1} a_n q^n$ where $a_n \in \mathbb{Z}$ for all $n \geq 2$. The q-expansion of j is $q^{-1} + \sum_{n\geq 0} b_n q^n$ where $b_n \in \mathbb{Z}$ for all $n \geq 0$.*

Proof. Since $j = E_4^3/\Delta$, it's enough to show that Δ has the claimed properties.

By definition, $\Delta = (E_4^3 - E_6^2)/1728$. Write $E_4 = 1 + 240U, E_6 = 1 - 504V$, where U, V are the sum parts of **1.14**.

Then, expanding, $\Delta = \frac{5}{12}U + \frac{7}{12}V + P(U,V) = \frac{5}{12}(U-V) + (V + P(U,V))$ where $P \in \mathbb{Z}[U,V]$

So we need to show that $\sigma_3(n) \equiv \sigma_5(n) \mod 12$ for all $n \in \mathbb{N}$. It would be enough to show that $n^3 \equiv n^5 \mod 3$ and $\mod 4$ for all $n \in \mathbb{N}$, which is true. This shows that $b_n \in \mathbb{Z}\ \forall n \geq 2$. It remains to show that the leading term of Δ is q. It's $\frac{3\cdot 240 + 2\cdot 504}{1728} = 1$. □