

Elliptic Curves

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1 Fermat's Method of Infinite Descent

Suppose we have a right-angled triangle Δ with side lengths a, b, c , so that by Pythagoras we have $a^2 + b^2 = c^2$, and $\text{area}(\Delta) = \frac{1}{2}ab$.

Definition 1.1. Δ is **rational** if $a, b, c \in \mathbb{Q}$, and **primitive** if $a, b, c \in \mathbb{Z}$ coprime.

Lemma 1.2. Every primitive triangle is of the form $a = u^2 - v^2, b = 2uv, c = u^2 + v^2$ for coprime integers $u > v > 0$.

Proof. If a, b were both odd, then $a^2 + b^2 \equiv 2 \pmod{4}$, and we have no solutions for c . If a, b both even, then they are not coprime. So we may assume a is odd, b is even, c is odd.

Then $(\frac{b}{2})^2 = \frac{c+a}{2} \cdot \frac{c-a}{2}$, and the right hand side is a product of coprime positive integers. So by unique prime factorisation in the integers, $\frac{c+a}{2} = u^2, \frac{c-a}{2} = v^2$ for some coprime integers u, v . Rearranging, we have the lemma. \square

Definition 1.3. $D \in \mathbb{Q}_{>0}$ is a **congruent number** if it is the area of a rational triangle.

Note that, by scaling the triangle, it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

For example, $D = 5, 6$ are congruent numbers. $6 = \frac{1}{2} \cdot 3 \cdot 4$, and $3^2 + 4^2 = 5^2$, and 5 is left as an exercise.

Lemma 1.4. $D \in \mathbb{Q}_{>0}$ is congruent if and only if $Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}, y \neq 0$.

Proof. Lemma 1.2 shows that D is congruent if and only if $Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}, w \neq 0$.

Setting $x = \frac{u}{v}, y = \frac{w}{v^2}$ finishes the proof. \square

Fermat showed that 1 is not a congruent number.

Theorem 1.5. There is no solution to

$$w^2 = uv(u+v)(u-v) \quad (*)$$

in integers u, v, w with $w \neq 0$.

Proof. Without loss of generality, u, v are coprime with $u > 0, w > 0$. If $v < 0$ then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2})$. So we may assume that all of $u, v, u+v, u-v$ are coprime positive integers whose product is a square, and hence are all squares, say a^2, b^2, c^2, d^2 respectively, where $a, b, c, d \in \mathbb{Z}_{>0}$.

Since $u \not\equiv v \pmod{2}$, both c, d are odd. Consider the right angled triangle with side lengths, $\frac{c+d}{2}, \frac{c-d}{2}, a$. This is a primitive triangle, and it has area $\frac{c^2-d^2}{8} = \frac{v}{4} = (\frac{b}{2})^2$.

Let $w_1 = \frac{b}{2}$. Then lemma 1.2 gives $w_1^2 = u_1v_1(u_1^2 - v_1^2)$ for some $u_1, v_1 \in \mathbb{Z}$, giving a new solution to (*). But $4w_1^2 = b^2 = v|w^2$, and so $w_1 \leq \frac{1}{2}w$.

So by Fermat's method of infinite descent, if there were a solution we would have a strictly decreasing infinite sequence of positive integers \nexists . Hence there is no solution to (*). \square

1.1 A Variant for Polynomials

Here, K is a field with $\text{char } K \neq 2$. The algebraic closure of K will be \overline{K} .

Lemma 1.6. *Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for four distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.*

Proof. Without loss of generality we may assume $K = \overline{K}$, as that doesn't change the degree of polynomials, and every square is still a square.

Changing coordinates on \mathbb{P}^1 , we may assume the ratios $\alpha : \beta$ are $(1 : 0), (0 : 1), (1 : -1), (1 : -\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$, with $\mu = \sqrt{\lambda}$.

Then $u = a^2, v = b^2, u - v = (a + b)(a - b), u - \lambda v = (a + \mu b)(a - \mu b)$ are all squares. They are also coprime, and so by unique factorisation in $K[t]$, $(a + b), (a - b), (a + \mu b), (a - \mu b)$ are all squares.

But $\max\{\deg a, \deg b\} \leq \frac{1}{2} \max\{\deg u, \deg v\}$. So by Fermat's method of infinite descent, we get that the original $u, v \in K$. \square

Now we have some important definitions:

Definition 1.7.

1. An **elliptic curve** E over a field K is the projective closure of the affine curve $y^2 = f(x)$ where $f \in K[x]$ is a monic cubic polynomial with distinct roots.
2. For L/K any field extension, $E(L) = \{(x, y) \in L^2 : y^2 = f(x)\} \cup \{0\}$. 0 is called the **point at infinity**.

We call the point at infinity 0 because we will see that $E(L)$ is naturally an abelian group under an operation we will denote by $+$, and 0 will be the identity for that group. In this course we will study $E(L)$ for L a finite field, a local field, and a number field.

Lemma 1.4 and theorem 1.5 together imply that, if E is given by $y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$, which we will see is the group $C_2 \times C_2$.

Corollary 1.8. *Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.*

Proof. Without loss of generality, $K = \overline{K}$. By a change of coordinates we may assume $E : y^2 = x(x - 1)(x - \lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ with $u, v \in K[t]$ coprime. Then $w^2 = uv(u - v)(u - \lambda v)$ for some $w \in K[t]$.

Unique factorisation in $K[t]$ gives $u, v, u - v, u - \lambda v$ are all squares, and so by lemma 1.6, $u, v \in K$, and so $x, y \in K$. \square

2 Some Remarks on Algebraic Curves

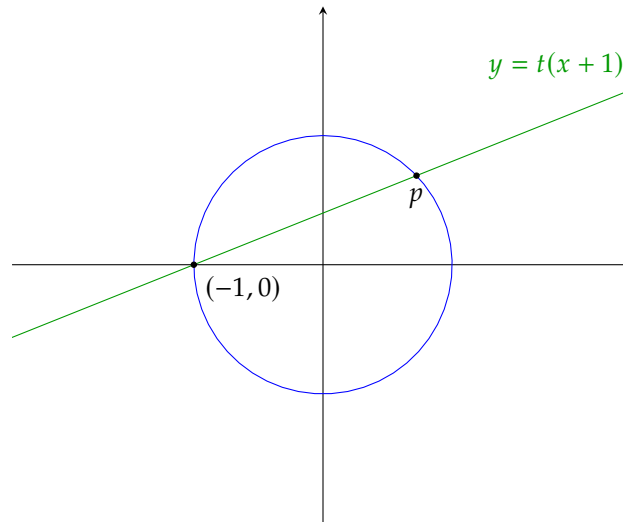
We will be working over an algebraically closed field K .

Definition 2.1. *An (irreducible) plane algebraic curve $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$ is **rational** if it has a rational parametrization, i.e. there are $\phi, \psi \in K(t)$ such that:*

1. $\mathbb{A}^1 \rightarrow \mathbb{A}^2; t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{\text{finite set}\}$.
2. $f(\phi(t), \psi(t)) = 0$.

Examples 2.2.

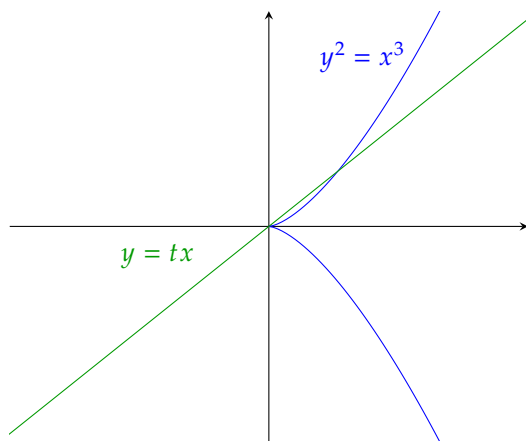
1. Any nonsingular plane conic is rational. For example, take a circle $x^2 + y^2 = 1$. Pick a point on it, $(-1, 0)$. Now draw a line through it with slope t , and solve for the points of intersection between the curve and the line.



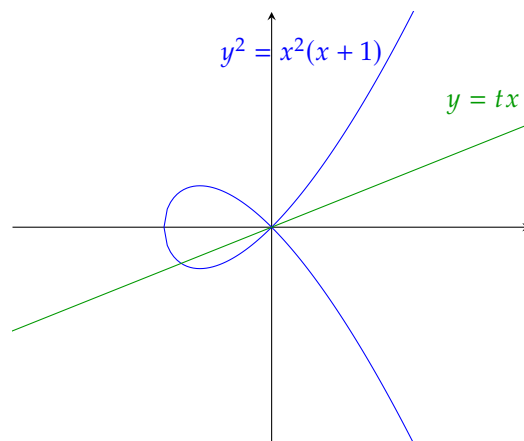
Solving for the coordinates of p , we get the quadratic $x^2 + t^2(x + 1)^2 = 1$, i.e. $x = -1$ or $\frac{1-t^2}{1+t^2}$.

So we have the rational parametrization $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

2. Any singular plane cubic is rational.



(a) Rational Parametrization $(x, y) = (t^2, t^3)$



(b) Left as an example on the first sheet

3. Corollary 1.8 shows that elliptic curves are *not* rational.

Definition 2.3. The **genus** $g(C) \in \mathbb{Z}_{\geq 0}$ is an invariant of a smooth projective curve.

- If $K = \mathbb{C}$, then $g(C)$ = genus of the Riemann surface C .

- A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

Proposition 2.4. Let C be a smooth projective curve over K , an algebraically closed field. Then:

1. C is rational $\iff g(C) = 0$.
2. C is an elliptic curve $\iff g(C) = 1$.

Proof. A proof of 1 is omitted from this course. For 2, we check (on the first example sheet) that elliptic curves are smooth plane curves. Then they have degree 3, so genus $\frac{2 \cdot 1}{2} = 1$. For the other direction, see later on in the course. \square

2.1 Order of Vanishing

C will be an algebraic curve, and $K(C)$ its function field, with $P \in C$ a smooth point. Write $\text{ord}_P(f)$ to mean the order of vanishing of $f \in K(C)$ at P (negative if f has a pole).

Fact: $\text{ord}_P : K(C)^\times \rightarrow \mathbb{Z}$ is a discrete valuation, i.e. $\text{ord}_P(f_1 f_2) = \text{ord}_P(f_1) + \text{ord}_P(f_2)$ and $\text{ord}_P(f_1 + f_2) \geq \min\{\text{ord}_P(f_1), \text{ord}_P(f_2)\}$.

We say $t \in K(C)^\times$ is a **uniformizer** at the point P if $\text{ord}_P(t) = 1$.

Example 2.5. Let $C = \{g(x, y) = 0\} \subseteq \mathbb{A}^2$, where $g \in K[x, y]$ is irreducible. Then $K(C) = \text{Frac} \frac{K[x, y]}{(g)}$, with $g = g_0 + g_1(x, y) + g_2(x, y) + \dots$, g_i homogeneous of degree i .

Suppose $P = (0, 0) \in C$ is a smooth point, i.e. $g_0 = 0, g_1(x, y) = \alpha x + \beta y$ with α, β not both zero.

Let $\gamma, \delta \in K$. It is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\frac{\gamma}{\delta} \neq \frac{\alpha}{\beta}$, i.e. $\alpha\delta - \beta\gamma \neq 0$.

Example 2.6. $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2, \lambda \neq 0, 1$. We take the projective closure, i.e. homogenize the equation as $\{Y^2 Z = X(X-Z)(X-\lambda Z)\} \subset \mathbb{P}^2$ by setting $x = X/Z, y = Y/Z$.

Have we got new points by taking projective closure? We only get these when $Z = 0$, i.e. $0 = X^3 \implies X = 0, Y \neq 0$. Since we're in projective space, this is just one point: $P = (0 : 1 : 0)$. We compute $\text{ord}_P(x)$ and $\text{ord}_P(y)$. Put $t = X/Y, w = Z/Y$ (since we can't return to the original affine piece, as it doesn't contain $Z = 0$). Then we get $w = t(t-w)(t-\lambda w)$. Now P is the point $(t, w) = (0, 0)$. This is a smooth point, as there are linear terms at that point (namely w). So $\text{ord}_P(t) = \text{ord}_P(t-2) = \text{ord}_P(t-\lambda w) = 1$, and $\text{ord}_P(w) = 1 + 1 + 1 = 3$.

Then:

$$\begin{aligned}\text{ord}_P(x) &= \text{ord}_P(X/Z) = \text{ord}_P(t/w) = 1 - 3 = -2 \\ \text{ord}_P(y) &= \text{ord}_P(Y/Z) = \text{ord}_P(1/w) = -3\end{aligned}$$

2.2 Riemann Roch Spaces

Let C be a smooth projective curve. Then a **divisor** is a formal sum of points on C , say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$, and only finitely many n_P are nonzero, and let $\deg D = \sum_{P \in C} n_P$. These divisors form a group under addition, denoted $\text{Div}(C)$.

D is said to be **effective**, written $D \geq 0$ if $n_P \geq 0$ for all $P \in C$.

If $f \in K(C)^\times$, we write $\text{div}(f) = \sum_{P \in C} \text{ord}_P(f) P$.

The Riemann Roch space of $D \in \text{Div}(C)$ is:

$$\mathcal{L}(D) = \{f \in K(C) : \text{div}(f) + D \geq 0\} \cup \{0\}$$

i.e. the K -vector space of rational functions on C with “poles no worse than specified by D .”

Theorem 2.7 (Riemann Roch for genus 1).

$$\dim \mathcal{L}(D) = \begin{cases} 0 & \deg D < 0 \\ 0 \text{ or } 1 & \deg D = 0 \\ \deg D & \deg D > 0 \end{cases}$$

Example 2.6 (revisited). Our curve is $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$, together with $P = (0 : 1 : 0)$, the point at infinity. Recall $\text{ord}_P(x) = -2, \text{ord}_P(y) = -3$.

We thus deduce that $\mathcal{L}(2P) = \langle 1, x \rangle, \mathcal{L}(3P) = \langle 1, x, y \rangle$.

Proposition 2.8. Let K be an algebraically closed field not of characteristic 2. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic, and that $P \in C$ is a point of inflection. Then we may change coordinates such that:

$$C : Y^2Z = X(X-Z)(X-\lambda Z), \quad \lambda \neq 0, 1 \\ P = (0 : 1 : 0)$$

Proof. We make a change of coordinates such that $P = (0 : 1 : 0)$ and the tangent line to C at P , $T_P(C) = \{Z = 0\}$. Now let $C = \{F(X, Y, Z) = 0\}$.

Since $P \in C$ is a point of inflection, $F(t, 1, 0)$ has a triple root at $t = 0$. But F is degree 3, so we have $F(t, 1, 0) = kt^3$ for k some constant. I.e., there are no terms in F of the form X^2Y, XY^2, Y^3 .

So $F \in \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3 \rangle$. The coefficient of Y^2Z is nonzero, as otherwise P would be singular. The coefficient of X^3 is also nonzero, as C is irreducible and otherwise $\{Z = 0\} \subset C$.

We are free to rescale X, Y, Z, F , and so wlog C is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

We call this Weierstrass form.

Since our field doesn't have characteristic 2, we may complete the square by substituting $Y = Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$.

Now $C : Y^2Z = Z^3f(X/Z)$, where f is a monic cubic polynomial. Since C is smooth, f has distinct roots, which are wlog $0, 1, \lambda$. So

$$C : Y^2Z = X(X-Z)(X-\lambda Z)$$

which we call the Legendre form. □

It may be shown that the points of inflection on $C = \{F = 0\} \subset \mathbb{P}^2$ are given by $F = \det \left(\frac{\partial^2 f}{\partial X_i \partial X_j} \right) = 0$

2.3 The Degree of a Morphism

Let $\phi : C_1 \rightarrow C_2$ be a nonconstant morphism of smooth projective curves. Let $\phi^* : K(C_2) \rightarrow K(C_1), f \mapsto f \circ \phi$.

Definition.

1. $\deg \phi = [K(C_1) : \phi^*K(C_2)]$
2. ϕ is separable if $K(C_1)/\phi^*K(C_2)$ is a separable field extension (which by Galois theory is automatic if $\text{char } K = 0$)

Suppose $P \in C_1, Q \in C_2, \phi : P \rightarrow Q$. Let $t \in K(C_2)$ be a uniformizer at Q . We then define $e_\phi(P) = \text{ord}_P(\phi^*t)$, which is always ≥ 1 , and independent of t . $e_\phi(P)$ is called the **ramification index** of ϕ at P .

Theorem 2.9. Let $\phi : C_1 \rightarrow C_2$ be a nonconstant morphism of smooth projective curves. Then

$$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi$$

for any point $Q \in C_2$. Moreover, if ϕ is separable then $e_\phi(P) = 1$ with at most finitely many exceptions.

In particular:

1. ϕ is surjective
2. If ϕ is separable, $\#\phi^{-1}(Q) \leq \deg \phi$, with equality for all but finitely many choices of Q .

Remark 2.10. Let C be an algebraic curve. A rational map is given by $\phi : C \dashrightarrow \mathbb{P}^n, P \mapsto (f_0(P) : \dots : f_n(P))$, where $f_0, \dots, f_n \in K(C)$ are not all zero. If C is smooth then ϕ is a morphism.

3 Weierstrass Equations

In this section, K is a perfect field (so that all finite extensions of K are separable), with algebraic closure \bar{K} .

Definition. An elliptic curve E over K is a smooth projective curve of genus 1 defined over K with a specified K -rational point O_E .

Example: Take $\{X^3 + pY^3 + p^2Z^3 = 0\} \subset \mathbb{P}^2$ for p prime. This is not an elliptic curve over \mathbb{Q} since there is no \mathbb{Q} -points.

Theorem 3.1. Every elliptic curve E is isomorphic over K to a curve in Weierstrass form via an isomorphism taking O_E to $(0 : 1 : 0)$.

Proposition 2.8 treated the special case where E is a smooth plane cubic and O_E is a point of inflection.

If $D \in \text{Div}(E)$ is defined over K (i.e. fixed by the natural action of $\text{Gal}(\bar{K}/K)$), then $\mathcal{L}(D)$ has a basis in $K(E)$, not just in $\bar{K}(E)$.

Proof. Note that

$$\mathcal{L}(2O_E) \subset \mathcal{L}(3O_E)$$

Pick bases of these spaces, say $\{1, x\}$ and $\{1, x, y\}$.

Note that $\text{ord}_{O_E}(x) = -2, \text{ord}_{O_E}(y) = -3$. The 7 elements $\{1, x, y, x^2, xy, x^3, y^2\}$ are rational functions with no pole except at O_E , where they have poles of degree at most 6, so they all lie in $\mathcal{L}(6O_E)$. Riemann-Roch tells us this space has dimension 6, so there is a dependence relation between these elements.

Leaving out x^3 or y^2 gives a basis for $\mathcal{L}(6O_E)$ since each term has a different order pole at O_E , so they are independent.

Therefore this dependence relation *must* involve both x^3 and y^2 . Rescaling x, y we get

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

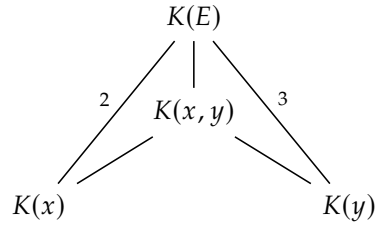
Let E' be the curve defined by this equation (or rather its projective closure).

There is a morphism

$$\begin{aligned}\phi : E &\rightarrow E' \\ P &\mapsto (x(P) : y(P) : 1) = \left(\frac{x}{y}(P) : 1 : \frac{1}{y}(P) \right) \\ O_E &\mapsto (0 : 1 : 0)\end{aligned}$$

$$\begin{aligned}[K(E) : K(x)] &= \deg(E \xrightarrow{x} \mathbb{P}^1) = \text{ord}_{O_E}\left(\frac{1}{x}\right) = 2 \\ [K(E) : K(y)] &= \deg(E \xrightarrow{y} \mathbb{P}^1) = \text{ord}_{O_E}\left(\frac{1}{y}\right) = 3\end{aligned}$$

This gives us a diagram of field extensions



So $[K(E) : K(x, y)]$ divides both 2 and 3 by the tower law, and hence $K(E) = K(x, y)$, and hence $\deg(E \xrightarrow{\phi} E') = 1$, and ϕ is birational. If E' is singular, then it is rational, and so E is also rational $\frac{1}{2}$. So E' is not singular and hence smooth, and we may use remark 2.10 to ϕ^{-1} to see that ϕ^{-1} is a morphism, and hence ϕ is an isomorphism. \square

Proposition 3.2. *Let E, E' be elliptic curves over K in Weierstrass form. Then $E \cong E'$ over K if and only if the Weierstrass equations are related by a change of variables of the form*

$$\begin{aligned}x &= u^2x' + r \\ y &= u^3y' + u^2sx' + t\end{aligned}$$

for $u, r, s, t \in K, u \neq 0$.

Proof. Using the notation of the previous proof,

$$\begin{aligned}\langle 1, x \rangle &= \mathcal{L}(2O_E) = \langle 1, x' \rangle \\ \langle 1, x, y \rangle &= \mathcal{L}(3O_E) = \langle 1, x', y' \rangle \\ \implies \begin{cases} x = \lambda x' + r & \lambda_1 r \in K, \lambda \neq 0 \\ y = \mu y' + \sigma x' + t & \mu, \sigma, t \in K, \mu \neq 0 \end{cases}\end{aligned}$$

Looking at the coefficients of x^3 and y^2 , $\lambda^3 = \mu^2 \implies (\lambda, \mu) = (u^2, u^3)$ for $u \in K^\times$.

Put $s = \sigma/u^2$ □

The effect of this transformation on the coefficients a_i is on the formula sheet for this course. A Weierstrass equation defines an elliptic curve if and only if it defines a smooth curve, if and only if $\Delta(a_1, \dots, a_6) \neq 0$ where Δ is as follows:

$$\begin{aligned}b_2 &:= a_1^2 + 4a_2 \\ b_4 &:= 2a_4 + a_1a_3 \\ b_6 &:= a_3^2 + 4a_6 \\ b_8 &:= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\ \Delta &:= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6\end{aligned}$$

If $\text{char } K \neq 2, 3$, then we can reduce to the case

$$\begin{aligned}E : y^2 &= x^3 + ax + b \\ \Delta &= -16(4a^3 + 27b^2)\end{aligned}$$

Corollary 3.3. Assume $\text{char } K \neq 2, 3$. If we have two elliptic curves

$$\begin{aligned}E : y^2 &= x^3 + ax + b \\ E' : y^2 &= x^3 + a'x + b'\end{aligned}$$

then they are isomorphic over K if and only if

$$\begin{aligned}a' &= u^4a \\ b' &= u^6b\end{aligned}$$

for some $u \in K^\times$.

Proof. E and E' are related as in 3.2 with $r = s = t = 0$. □

Definition. The *j -invariant* is $j(E) = \frac{1728(4a^3)}{4a^3 + 27b^2}$. Note that the denominator is nonzero since the discriminant is nonzero.

Corollary 3.4. $E \cong E' \implies j(E) = j(E')$, and the converse holds if $K = \bar{K}$.

Proof.

$$\begin{aligned}
E \cong E' &\iff a' = u^4 a; b' = u^6 b \text{ for some } u \in K^\times \\
&\implies (a^3 : b^2) = ((a')^3 : (b')^2) \\
&\iff j(E) = j(E')
\end{aligned}$$

and the reverse implication holds in the second line if $K = \bar{K}$. □

4 Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic, and $O_E \in E(K)$. Since E is of degree 3, it meets each line in 3 points counted with multiplicity. Hence, given two points P, Q on E , the line \overline{PQ} meets E at a third point S . Then the line $\overline{O_E S}$ meets E at a third point R . We then define $P \oplus Q = R$.

If $P = Q$, then we take the tangent line at P , likewise if $S = O_E$. We can view this diagrammatically as follows:

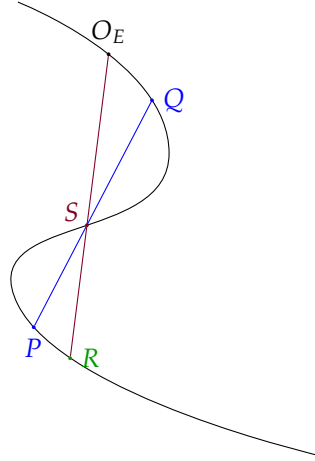


Figure 2: Illustration of the group operation on an elliptic curve

We call this the “chord and tangent process”.

Theorem 4.1. (E, \oplus) is an abelian group.

Proof.

- (i) $P \oplus Q = Q \oplus P$ by construction.
- (ii) O_E is the identity.
- (iii) For inverses, let S be the third point of intersection of T_{O_E} and E , and Q be the third point of intersection of \overline{PS} and E . Then $P \oplus Q = O_E$.
- (iv) Associativity is much harder.

□

Definition. $D_1, D_2 \in \text{Div}(E)$ are **linearly equivalent** (written $D_1 \sim D_2$) if there is $f \in \tilde{K}(E)^\times$ such that $\text{div}(f) = D_1 - D_2$. Then we will let $[D] = \{D' : D' \sim D\}$.

Definition. The **Picard group of E** , $\text{Pic}(E) = \text{Div}(E)/\sim$. We write $\text{Div}^0(E) := \ker \left(\text{Div}(E) \xrightarrow{\deg} \mathbb{Z} \right)$ for the group of degree 0 divisors on E , and then $\text{Pic}^0(E) = \text{Div}^0(E)/\sim$. Sometimes Pic^0 is called the Jacobian.

Proposition 4.2. Let $\psi : E \rightarrow \text{Pic}^0(E); P \mapsto [(P) - (O_E)]$. Then:

1. $\psi(P \oplus Q) = \psi(P) + \psi(Q)$
2. ψ is a bijection

Proof.

1. Referring back to Fig. 2, let $\{\ell = 0\}$ be the line \overline{PQ} , and $\{m = 0\}$ be the line $\overline{O_ER}$. Then:

$$\begin{aligned} \text{div}(\ell/m) &= (P) + (S) + (Q) - (R) - (S) - (O_E) \\ &= (P) + (Q) - (O_E) - (P \oplus Q) \\ \implies (P \oplus Q) + (O_E) &\sim (P) + (Q) \\ \implies (P \oplus Q) - (O_E) &\sim (P) - (O_E) + (Q) - (O_E) \\ \implies \psi(P \oplus Q) &= \psi(P) + \psi(Q) \end{aligned}$$

2. For injectivity, suppose $\psi(P) = \psi(Q)$. Then there is $f \in \tilde{K}(E)^\times$ such that $\text{div}(f) = P - Q$. Then $\deg \left(E \xrightarrow{f} \mathbb{P}^1 \right) = \text{ord}_P(f) = 1$. But then f is a birational morphism, so an isomorphism, and $E \cong \mathbb{P}^1$.

For surjectivity, let $[D] \in \text{Pic}^0(E)$. Then $D + (O_E)$ has degree 1 (as D had degree 0). Then Riemann-Roch tells us $\dim \mathcal{L}(D + (O_E)) = 1$, and so there exists some $f \in \tilde{K}(E)^\times$ such that $\text{div}(f) + D + (O_E) \geq 0$. Since f is rational, $\deg \text{div}(f) = 0$, and $\deg D = 0$. So the coefficients of $\text{div}(f) + D + (O_E)$ are non-negative and sum to 1, hence one of them is 1 and the rest are 0. So $\text{div}(f) + D + (O_E) = (P)$ for some $P \in E$. But then $(P) - (O_E) \sim D$, i.e. $\psi(P) = [D]$.

□

So ψ is a bijection respecting the group law, and so we deduce that \oplus is associative, and then $(E, \oplus) \cong (\text{Pic}^0 E, +)$.

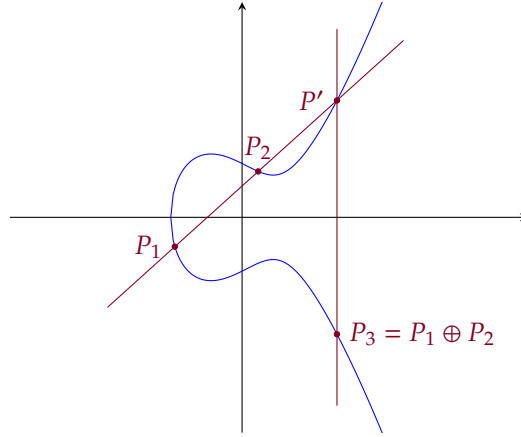
4.1 Explicit Formulae for the Group Law

We consider E in Weierstrass form, with O_E the point at infinity:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (*)$$

Note that O_E is a point of inflection. Now $P_1 \oplus P_2 \oplus P_3 = O_E \iff P_1, P_2, P_3$ are collinear.

We will use the following notation:



and put $P_i = (x_i, y_i)$, $P' = (x', y')$.

Now $\ominus P_1 = (x_1, -(a_1x_1 + a_3) - y_1)$, just by setting $y = -y_1$ in (*).

The line through P_1, P_2 has equation say $y = \lambda x + \nu$. Substituting into (*) and looking at the coefficient of x^2 , we get:

$$\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x'$$

Since $x_3 = x'$, we have:

$$\begin{aligned} x_3 &= \lambda^2 + a_1\lambda - a_2 - x_1 - x_2 \\ y_3 &= -(a_1x' + a_3) - y' \\ &= -(\lambda + a_1)x_3 - \nu - a_3 \end{aligned}$$

It remains to find λ and ν . There are 3 cases:

1. $x_1 = x_2, P_1 \neq P_2$.

Then $P_1 \oplus P_2 = O_E$.

2. $x_1 \neq x_2$.

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \quad \nu = y_1 - \lambda x_1 = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

3. $P_1 = P_2$.

Here we have to compute the equation of the tangent line etc. The solutions are:

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}, \quad \nu = \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}$$

Corollary 4.3. $E(K)$ is an abelian group.

Proof. It is a subgroup of $E (= E(\bar{K}))$.

Identity: $O_E \in E(K)$ by definition.

Closure: See formulae above.

Inverses: See formulae above.

Associativity: Inherited from $E(\bar{K})$.

Commutativity: Inherited from $E(\bar{K})$.

□

If there is no ambiguity (i.e. we are not also adding numbers at the same time), the circles will be dropped from the group operation.

Theorem 4.4. *Elliptic curves are group varieties.*

i.e., $[-1] : E \rightarrow E; P \mapsto -P$ and $+: E \times E \rightarrow E; (P, Q) \mapsto P + Q$ are morphisms of algebraic varieties.

Proof. The above formulae show that $[-1]$ and $+$ are rational maps. We know immediately that $[-1]$ is a morphism, as it is a rational map from a smooth curve to a projective variety.

The formulae also show that $+$ is regular on the set

$$U = \{(P, Q) \in E \times E \mid P, Q, P + Q, P - Q \neq O_E\}$$

For $P \in E$, let $\tau_P : E \rightarrow E; X \mapsto P + X$ be the “translation by P ” map.

Then τ_P is a rational map from a smooth curve to a projective variety, so is a morphism.

We factor $+$ as:

$$E \times E \xrightarrow{\tau_{-A} \times \tau_{-B}} E \times E \xrightarrow{\tau_{A+B}} E \xrightarrow{\tau_{A+B}} E$$

Now $+$ is regular on $(\tau_A \times \tau_B)(U)$ for all $A, B \in E$, and so $+$ is regular on $E \times E$.

□

Definition. For any $n \in \mathbb{Z}_{>0}$, let $[n] : E \rightarrow E; P \mapsto P + \dots + P$, n times, and $[-n] = [-1] \circ [n]$, $[0] : P \mapsto O_E$ (i.e., the standard way of turning an abelian group into \mathbb{Z} module).

Definition. The n -torsion subgroup of E is $E[n] = \ker([n] : E \rightarrow E)$.

Lemma 4.5. *If $\text{char}(K) \neq 2$, and $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$.*

Then $E[2] = (0, (e_1, 0), (e_2, 0), (e_3, 0)) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. Let $P = (x, y) \in E$. Then $[2]P = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. □

4.2 Elliptic Curves over \mathbb{C}

Let $\Lambda = \{a\omega_1 + b\omega_2 : a, b \in \mathbb{Z}\}$, where ω_1, ω_2 form a basis for \mathbb{C} over \mathbb{R} .

Then the meromorphic functions on the Riemann surface (or lattice) \mathbb{C}/Λ are the same as the Λ -invariant meromorphic functions on \mathbb{C} (i.e. $f(z) = f(z + \lambda)$ for $\lambda \in \Lambda$).

This set of functions is a field, and is generated by $\wp(z)$ and $\wp'(z)$, where:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

They satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, for some $g_1, g_3 \in \mathbb{C}$ depending on λ . We call \wp the *Weierstrass p -function*.

One can show that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where E is the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. This is an isomorphism, not only of Riemann surfaces, but moreover of groups

Theorem 4.6 (Uniformisation Theorem). *Every elliptic curve over \mathbb{C} arises in this way.*

Thus, for elliptic curves E/\mathbb{C} , we have:

$$\textcircled{1} \quad E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$$

$$\textcircled{2} \quad \deg[n] = n^2$$

We will show that $\textcircled{2}$ holds over any field K , and $\textcircled{1}$ holds if $\text{char } K \nmid n$.

Summary of Results (N.B. the isomorphisms in 1, 2, 4 respect the relevant topologies)

- | | |
|----------------------------------|--|
| 1. $K = \mathbb{C}$ | $E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ |
| 2. $K = \mathbb{R}$ | $E(\mathbb{R}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \Delta < 0 \end{cases}$ |
| 3. $K = \mathbb{F}_q$ | $ \#E(\mathbb{F}_q) - (q + 1) \leq 2\sqrt{q}$ |
| 4. $[K : \mathbb{Q}_p] < \infty$ | $E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$ |
| 5. $[K : \mathbb{Q}] < \infty$ | $E(K)$ is a finitely generated abelian group. |

5 Isogenies

Let E_1, E_2 be elliptic curves.

Definition. An *isogeny* $\phi : E_1 \rightarrow E_2$ is a non-constant morphism taking O_{E_1} to O_{E_2} , and we say E_1 and E_2 are *isogenous* if there is an isogeny $E_1 \rightarrow E_2$.

Definition. $\text{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \rightarrow E_2\} \cup \{0\}$. This is a group under $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$ are isogenies, then $\psi\phi$ is an isogeny. The tower law tells us that $\deg(\psi\phi) = \deg(\phi)\deg(\psi)$.

Lemma 5.1. *If $0 \neq n \in \mathbb{Z}$, then $[n] : E \rightarrow E$ is an isogeny.*

Proof. Theorem 4.4 tells us that $[n]$ is a morphism. We must show that $[n] \neq 0$.

Assume $\text{char } K \neq 2$, then we can use Lemma 4.5. If $n = 2$, then $\#E[2] = 4$, and so $[2] \neq 0$.

If n is odd, then there is $0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n]$ is not the zero map.

Now $[m][n] = [m] \circ [n]$, and any $n = 2^k m$ for m odd, so $[n]$ is not the zero map for any $n \neq 0$.

If $\text{char } K = 2$, then replace 4.5 with a lemma computing $E[3]$. □

Corollary. $\text{Hom}(E_1, E_2)$ is torsion-free as a \mathbb{Z} -module.

Lemma 5.2. *Let $\phi : E_1 \rightarrow E_2$ be an isogeny. Then $\phi(P + Q) = \phi(P) + \phi(Q)$ for all $P, Q \in E_1$.*

Sketch proof. ϕ induces a map $\phi_* : \text{Div}^0(E_1) \rightarrow \text{Div}^0(E_2)$ given by $\sum_{P \in E_1} n_P P \mapsto \sum_{P \in E_2} n_P \phi(P)$.

Recall that, via a pullback, $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

If $f \in K(E_1)^*$, then $\phi_*(\text{div } f) = \text{div}(N_{K(E_1)/K(E_2)} f)$ - this is a fact that we'll take for granted.

So ϕ_* takes principal divisors to principal divisors. Since $\phi(O_{E_1}) = O_{E_2}$, the following diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ \text{Pic}^0(E_1) & \xrightarrow{\phi_*} & \text{Pic}^0(E_2) \end{array}, \text{ where } \psi_1 : P \mapsto [(P) - (O_{E_1})], \psi_2 : Q \mapsto [(Q) - (O_{E_2})].$$

Since ϕ_* is a group homomorphism, ϕ is also a group homomorphism. \square

Lemma 5.3. *Let $\phi : E_1 \rightarrow E_2$ be an isogeny. Then there is a morphism ξ making the following diagram commute:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow x_1 & & \downarrow x_2 \\ \mathbb{P}^1 & \xrightarrow{\xi} & \mathbb{P}^1 \end{array}$$

where x_i is the x -coordinate in a Weierstrass equation for E_i .

Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ for $r, s \in K[t]$ coprime, then $\deg \phi = \deg \xi = \max(\deg r, \deg s)$.

Proof. For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 extension, since the extension is given by adjoining y_i , which satisfies a quadratic (see the Weierstrass equation). Moreover, it is Galois, as $[-1]^*$ is a non-trivial automorphism of $K(E_i)$ fixing $K(x_i)$.

Since ϕ is a group homomorphism, we have that $\phi(-P) = -\phi(P)$, i.e. $\phi \circ [-1] = [-1] \circ \phi$.

If $f \in K(x_2)$, then $[-1]^* f = f$, and $[-1]^*(\phi^* f) = \phi^*([-1]^* f) = \phi^* f$. Hence $\phi^* f$ is fixed by $[-1]$, so is in $K(x_1)$, and $K(x_2) \leq K(x_1)$.

Taking $f = x_2$, then $\phi^* x_2 \in K(x_1)$, say $\xi(x_1)$ for some rational function ξ . Then ξ is as required.

Since $[K(E_1) : K(x_1)] = [K(E_2) : K(x_2)] = 2$, we have the following diagram of field extensions:

$$\begin{array}{ccccc} & & K(E_1) & & \\ & \swarrow 2 & & \searrow \deg \phi & \\ K(x_1) & & & & K(x_2) \\ & \searrow \deg \xi & & \swarrow 2 & \\ & & K(x_2) & & \end{array}$$

Using the tower law, $\deg \phi = \deg \xi$. Now, $K(x_2) \hookrightarrow K(x_1)$ via $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_1)}$ for $r, s \in K[t]$ coprime.

The minimal polynomial of x_1 over $K(x_2)$ is $f(t) = r(t) - s(t)x_2 \in K(x_2)[t]$ - this is clearly a polynomial for x_1 , but we need to check it's irreducible.

f is irreducible in $K[t][x_2] = K[x_2][t]$ as it is of degree 1 in x_2 , so one of the factors must be constant in x_2 , so divide both r and s which are coprime. Then we can use Gauss's lemma, and it is irreducible in $K(x_2)[t]$.

Hence $\deg \phi = \deg \xi = [K(x_1) : K(x_2)] = \deg(r(t) - s(t)x_2) = \max(\deg r, \deg s)$. \square

Lemma 5.4. $\deg[2] = 4$

Proof. Assume $\text{char } K \neq 2, 3$. Then $E : y^2 = x^3 + ax + b = f(x)$.

If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2 - 8xf(x)}{4f(x)} = \frac{x^4 + \dots}{4f(x)}$.

The numerator and denominator are coprime - suppose there was a common factor. Then $\exists \theta \in \bar{K}$ with $f(\theta) = (3\theta^2 + a)^2 = f'(\theta) = 0$, and so f has a multiple root. But E is an elliptic curve so f doesn't have multiple roots.

Hence $\deg[2] = \max(\deg x^4 + \dots, \deg 4f(x)) = \max(4, 3) = 4$. \square

Definition. Let A be an abelian group. We say that $q : A \rightarrow \mathbb{Z}$ is a *quadratic form* if it satisfies

1. $q(nx) = n^2 q(x) \forall n \in \mathbb{Z}, x \in A$.
2. $(x, y) \rightarrow q(x + y) - q(x) - q(y)$ is \mathbb{Z} -bilinear.

Lemma 5.5. $q : A \rightarrow \mathbb{Z}$ is a quadratic form if and only if it satisfies the parallelogram law:

$$q(x + y) + q(x - y) = 2q(x) + 2q(y) \forall x, y \in A$$

Proof. For the forwards direction, let $\langle x, y \rangle = q(x + y) - q(x) - q(y)$.

Then $\langle x, x \rangle = q(2x) - 2q(x) = 2q(x)$.

Then $\frac{1}{2}\langle x + y, x + y \rangle + \frac{1}{2}\langle x - y, x - y \rangle = \langle x, x \rangle + \langle y, y \rangle$ by bilinearity, and hence $q(x + y) + q(x - y) = 2q(x) + 2q(y)$.

The reverse direction is left as an exercise on example sheet 2. \square

Theorem 5.6.

$$\deg : \text{Hom}(E_1, E_2) \rightarrow \mathbb{Z}$$

is a quadratic form.

Proof. For the proof, we will assume $\text{char } K \neq 2, 3$ for simplicity - the result still holds in those characteristics.

We write $E_2 : y^2 = x^3 + ax + b$.

Let $P, Q \in E_2$ with $P, Q, P + Q, P - Q \neq 0$, and let x_1, \dots, x_4 be the x -coordinates of these 4 points. Then we have:

Lemma 5.7. There exists $w_0, w_1, w_2 \in \mathbb{Z}[a, b][x_1, x_2]$ of degree ≤ 2 in x_1 and in x_2 such that $(1 : x_3 + x_4 : x_3 x_4) = (w_0 : w_1 : w_2)$.

Proof. We could prove this by direct calculation, leading to the formulae:

$$\begin{aligned} w_0 &= (x_1 - x_2)^2 \\ w_1 &= 2(x_1x_2 + a)(x_1 + x_2) + 4b \\ w_2 &= x_1^2x_2^2 - 2ax_1x_2 - 4b(x_1 + x_2) + a^2 \end{aligned}$$

As an alternative proof, let $y = \lambda x + \nu$ be the line through P and Q . Then

$$x^3 + ax + b - (\lambda x + \nu)^2 = (x - x_1)(x - x_2)(x - x_3) = x^3 - s_1x^2 + s_2x - s_3$$

where s_i is the i^{th} symmetric polynomial in (x_1, x_2, x_3) .

Comparing coefficients:

$$\begin{aligned} \lambda^2 &= s_1 \\ -2\lambda\nu + a &= s_2 \\ \nu^2 - b &= s_3 \end{aligned}$$

Eliminating λ, ν , we have $F(x_1, x_2, x_3) := (s_2 - a)^2 - 4s_1(s_3 + b) = 0$. Then F has degree at most 2 in each x_i .

x_3 is a root of the quadratic polynomial $W(t) = F(x_1, x_2, t)$, and repeating this for the line through P and $-Q$ shows that x_4 is the other root. Hence

$$w_0(t - x_3)(t - x_4) = W(t) = w_0t^2 - w_1t + w_2$$

And so $(1 : x_3 + x_4 : x_3x_4) = (w_0 : w_1 : w_2)$. □

We then show that, if $\phi, \psi \in \text{Hom}(E_1, E_2)$, then

$$\deg(\phi + \psi) + \deg(\phi - \psi) \leq 2\deg(\phi) + 2\deg(\psi)$$

We may assume $\phi, \psi, \phi + \psi, \phi - \psi \neq 0$, as otherwise the result is trivial.

$$\begin{aligned} \phi &: (x, y) \mapsto (\xi_1(x), \dots) \\ \psi &: (x, y) \mapsto (\xi_2(x), \dots) \\ \phi + \psi &: (x, y) \mapsto (\xi_3(x), \dots) \\ \phi - \psi &: (x, y) \mapsto (\xi_4(x), \dots) \end{aligned}$$

Then 5.7 gives $(1 : \xi_3 + \xi_4 : \xi_3\xi_4) = ((\xi_1 - \xi_2)^2 : \dots : \dots)$.

Put $\xi_i = \frac{r_i}{s_i}$ where $r_i, s_i \in K[x]$ are coprime:

$$(s_3s_4 : r_3s_4 + r_4s_3 : r_3r_4) = ((r_1s_2 - r_2s_2)^2 : \dots : \dots) \quad (*)$$

So we have:

$$\begin{aligned} \deg(\phi + \psi) + \deg(\phi - \psi) &= \max(\deg r_3, \deg s_3) + \max(\deg r_4, \deg s_4) \\ &= \max(\deg(s_3s_4), \deg(r_3s_4 + r_4s_3), \deg(r_3r_4)) \end{aligned}$$

Suppose $(s_3s_4, r_3s_4 + r_4s_3, r_3r_4)$ are not coprime, so that p irreducible divides all 3. Then p divides one of r_3, r_4 , and one of s_3, s_4 . p can't divide both s_i and r_i as they are coprime, so wlog p divides r_3 and s_4 and not r_4 nor s_3 . Then p doesn't divide $r_3s_4 + r_4s_3$. Hence these polynomials are coprime.

Hence the polynomials on RHS of (*) must be multiples of polynomials on the LHS by some irreducible polynomial, and hence each have degree \geq their corresponding polynomial on LHS, and thus, as w_i are of degree ≤ 2 in r_1, s_1, r_2, s_2 ,

$$\begin{aligned} \deg(\phi + \psi) + \deg(\phi - \psi) &\leq \max(\deg(w_0), \deg(w_1), \deg(w_2)) \\ &\leq 2 \max(\deg r_1, \deg s_1) + 2 \max(\deg r_2, \deg s_2) \\ &= 2 \deg \phi + 2 \deg \psi \end{aligned} \tag{1}$$

Now replace ϕ and ψ by $\phi + \psi$ and $\phi - \psi$ to get

$$\deg(2\phi) + \deg(2\psi) \leq 2 \deg(\phi + \psi) + 2 \deg(\phi - \psi)$$

Since $\deg[2] = 4$,

$$2 \deg(\phi) + 2 \deg(\psi) \leq \deg(\phi + \psi) + \deg(\phi - \psi) \tag{2}$$

(1) and (2) together give

$$2 \deg(\phi) + 2 \deg(\psi) = \deg(\phi + \psi) + \deg(\phi - \psi)$$

so \deg satisfies the parallelogram law, and hence is a quadratic form. \square

Corollary 5.8.

$$\deg(n\phi) = n^2 \deg(\phi) \quad \forall n \in \mathbb{Z}, \phi \in \text{Hom}(E_1, E_2)$$

In particular, $\deg[n] = n^2$.

Example 5.9. Let E/K be an elliptic curve, suppose $\text{char } K \neq 2$, and let $O_E \neq T \in E(K)[2]$.

Then we may take $E : y^2 = x(x^2 + ax + b)$, $a, b \in K$, $b(a^2 - 4b) \neq 0$, $T = (0, 0)$

Then if $P = (x, y)$ and $P' = P + T = (x', y')$, then:

$$\begin{aligned} x' &= (y/x)^2 - a - x = \frac{x^2 + ax + b}{x} - x - a = \frac{b}{x} \\ y' &= -(y/x)x' = \frac{-by}{x^2} \end{aligned}$$

Then let $\xi = x + x' + a = \frac{x^2 + ax + b}{x} = \left(\frac{y}{x}\right)^2$, and $\eta = y + y' = \frac{y}{x}(x - \frac{b}{x})$

$$\text{Then } \eta^2 = \left(\frac{y}{x}\right)^2 \left[\left(x + \frac{b}{x}\right)^2 - 4b\right] = \xi((\xi - a)^2 - 4b) = \xi(\xi^2 - 2a\xi + a^2 - 4b)$$

Let $E' : y^2 = x(x^2 + a'x + b')$ where $a' = -2a$, $b' = a^2 - 4b$. Then there is an isogeny $\phi : E \rightarrow E'$ given by $(x, y) \mapsto \left(\left(\frac{y}{x}\right)^2 : \frac{y(x^2 - b)}{x^2} : 1\right); O_E \mapsto (0 : 1 : 0)$

5.3 tells us, as $x' = \left(\frac{y}{x}\right)^2 = \frac{x^2 + ax + b}{x}$, that $\deg(\phi) = \max(2, 1) = 2$, and we say ϕ is a 2-isogeny.

6 The Invariant Differential

Let C be an algebraic curve over an algebraically closed field. Then the *space of differentials* Ω_C is a vector space over the function field of the curve $K(C)$, generated by df for $f \in K(C)$ subject to the relations

1. $d(f + g) = df + dg$
2. $d(fg) = f dg + g df$
3. $da = 0$ for $a \in K$

It turns out that $\dim \Omega_C = \dim C$, and since C is a curve, Ω_C is a 1-dimensional $K(C)$ -vector space.

Let $0 \neq \omega \in \Omega_C$, and let $P \in C$ be a smooth point, with $t \in K(C)$ a uniformizer at P (has order of vanishing 1 at P). Then $\omega = f dt$ for some $f \in K(C)$.

We define $\text{ord}_P(\omega) = \text{ord}_P(f)$. This does not depend on the choice of uniformizer.

Suppose we have $f \in K(C)^*$, and $\text{ord}_P(f) = n \neq 0$. Then, if $\text{char } K \nmid n$, $\text{ord}_P(df) = n - 1$.

If C is now a smooth projective curve, we define the divisor of $\omega \in \Omega_C$ to be

$$\text{div}(\omega) = \sum_{P \in C} \text{ord}_P(\omega) P \in \text{Div}(C)$$

using the fact that $\text{ord}_P(\omega)$ is zero at all but finitely many points $P \in C$.

The *space of regular differentials* is the finite dimensional vector space over K of all $\omega \in \Omega_C$ for which $\text{div}(\omega)$ is effective, i.e. there are no poles. The dimension of this space is called the *genus* of C , $g(C)$.

As a consequence of Riemann-Roch, we have, for $0 \neq \omega \in \Omega_C$, $\deg(\text{div}(\omega)) = 2g(C) - 2$.

Lemma 6.1. Assume $\text{char } K \neq 2$. Take an elliptic curve $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$, where e_1, e_2, e_3 distinct.

Then $\omega = \frac{dx}{y}$ is a differential on E , and has no zeros and no poles, and so $g(E) = 1$.

Moreover, the space of regular differentials is just $\langle \omega \rangle$.

Proof. Let $T_i = (e_i, 0)$, so that $E[2] = \{O, T_1, T_2, T_3\}$.

Then $\text{div}(y) = (T_1) + (T_2) + (T_3) - 3(O)$ - we know the zeros at T_i are simple as y is rational, so $\deg \text{div}(y) = 0$.

Then for $P \in E$, $\text{div}(x - x_P) = (P) + (-P) - 2(O)$, in the same way as above.

If $P \in E \setminus E[2]$, then $\text{ord}_P(x - x_P) = 1$, so $\text{ord}_P(d(x - x_P)) = \text{ord}_P(dx) = 1 - 1 = 0$.

If $P = T_i$, then $P = -P$, and $\text{ord}_P(x - x_P) = 2$, so $\text{ord}_P(dx) = 2 - 1 = 1$

If $P = O$, then $\text{ord}_P(x) = -2$, so $\text{ord}_P(dx) = -3$.

Hence $\text{div}(dx) = (T_1) + (T_2) + (T_3) - 3(O) = \text{div}(y)$.

So $\text{div}(dx/y) = \text{div}(dx) - \text{div}(y) = 0$. Then Riemann-Roch gives $g(E) = 1$, and so the space of regular differentials is 1-dimensional, so generated by ω . \square

Definition. If $\phi : C_1 \rightarrow C_2$ is a non-constant morphism, then we can pull back to

$$\phi^* : \Omega_{C_1} \rightarrow \Omega_{C_2}; f dg \mapsto \phi^* f d(\phi^* g)$$

Lemma 6.2. Let $P \in E$, $\tau_P : E \rightarrow E; X \mapsto P + X$, and $\omega = dx/y$ be as above.

Then $\tau_P^* \omega = \omega$, and so ω is called the **invariant differential**.

Proof. Since ω had no poles, $\tau_P^* \omega$ is again a regular differential, and hence equal to $\lambda_P \omega$ for some $\lambda_P \in K$, as the regular differentials are a 1-dimensional vector space over K .

The map $E \rightarrow \mathbb{P}^1; P \mapsto \lambda_P$ is a morphism of smooth projective curves, but is not surjective as it misses 0 and ∞ , and so this morphism is constant, by 2.8.

So λ is independent of P . Take $P = O_E$, then τ_P is the identity map, and so λ is 1. \square

If $K = \mathbb{C}$, then $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, via $z \mapsto (\wp(z), \wp'(z))$. Then $\frac{dx}{y} = \frac{\wp'(z)dz}{\wp'(z)} = dz$, which is invariant under $z \mapsto z + \text{const.}$

Lemma 6.3. Let $\phi, \psi \in \text{Hom}(E_1, E_2)$, ω the invariant differential on E_2 . Then

$$(\phi + \psi)^*(\omega) = \phi^* \omega + \psi^* \omega$$

Proof. Write $E = E_2$, and consider the maps:

$$\begin{aligned} E \times E &\rightarrow E \\ \mu : (P, Q) &\mapsto P + Q \\ \text{pr}_1 : (P, Q) &\mapsto P \\ \text{pr}_2 : (P, Q) &\mapsto Q \end{aligned}$$

$\Omega_{E \times E}$ is a 2-dimensional $K(E \times E)$ vector space with basis $\text{pr}_1^* \omega$ and $\text{pr}_2^* \omega$.

Then $\mu^* \omega = f \text{pr}_1^* \omega + g \text{pr}_2^* \omega$ for some $f, g \in K(E \times E)$.

For $Q \in E$, let $\iota_Q : E \rightarrow E \times E; P \mapsto (P, Q)$. Then

$$\begin{aligned} \iota_Q^*(\mu^* \omega) &= (\mu \circ \iota_Q)^* \omega = \iota_Q^* f (\text{pr}_1 \circ \iota_Q)^* \omega + \iota_Q^* g (\text{pr}_2 \circ \iota_Q)^* \omega \\ \tau_Q^* \omega &= \iota_Q^* f \omega + 0 \\ \omega &= \iota_Q^* f \omega \end{aligned}$$

So $\iota_Q^* f = 1$ for all $Q \in E$, so $f(P, Q) = 1$ for all $P, Q \in E$.

Similarly, $g(P, Q) = 1$.

So $\mu^* \omega = \text{pr}_1^* \omega + \text{pr}_2^* \omega$. Now pull back by $E \rightarrow E \times E; P \mapsto (\phi(P), \psi(P))$ to get $(\phi + \psi)^* \omega = \phi^* \omega + \psi^* \omega$. \square

Lemma 6.4. If $\phi : C_1 \rightarrow C_2$ is a non-constant morphism, then ϕ is separable if and only if $\phi^* : \Omega_{C_2} \rightarrow \Omega_{C_1}$ is nonzero

Proof. Omitted. \square

Example: Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$, with group law $\mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m; (x, y) \mapsto xy$.

Let $n \geq 2$ be an integer, $\alpha : \mathbb{G}_m \rightarrow \mathbb{G}_m; x \mapsto x^n$.

Then $\alpha^*(dx) = d(\alpha x) = d(x^n) = nx^{n-1}dx$. So if $\text{char } K \nmid n$, then α is separable. So $\#\alpha^{-1}(Q) = \deg \alpha$ for all but finitely many $Q \in \mathbb{G}_m$.

But α is group homomorphism, so all fibres have the same size, and $\#\alpha^{-1}(Q) = \#\ker \alpha$, hence $\#\ker \alpha = \deg \alpha = n$. So $K(= \bar{K})$ contains exactly n n^{th} roots of unity.

Theorem 6.5. *If $\text{char } K \nmid n$, then $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$.*

Proof. By 6.3 and induction, $[n]^*\omega = n\omega$. So if $\text{char } K \nmid n$, $[n]$ is separable. So all but finitely many fibres of $[n]$ have size $\deg[n]$, and since $[n]$ is a group homomorphism, all fibres have the same size, and hence $\#[n]^{-1}(O_E) = \#E[n] = \deg[n] = n^2$.

By the structure theorem for finite abelian groups, $E[n] \cong \mathbb{Z}/d_1\mathbb{Z} \times \dots \times \mathbb{Z}/d_t\mathbb{Z}$ with $d_i | d_{i+1}$. Since this group is killed by multiplication by n , all $d_i | n$ as well, and $\prod_{i=1}^t d_i = n^2$ by the previous paragraph.

If p is a prime with $p | d_1$, then $E[p] \cong (\mathbb{Z}/p\mathbb{Z})^t$, and by the first paragraph, $t = 2$. Then $d_1 | d_2 | n$, and $d_1 d_2 = n^2$, hence $d_1 = d_2 = n$. \square

Remark (not to be used on example sheet 2). If $\text{char } K = p$, then $[p]$ is not separable. It can be shown that $E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z}$ for all $r \geq 1$ or $E[p] = 0$. The first case is described as “ordinary”, and the second case is “supersingular”.

7 Elliptic Curves over Finite Fields

Lemma 7.1. *Let A be an abelian group and $q : A \rightarrow \mathbb{Z}$ a positive definite quadratic form. If $x, y \in A$ then $\langle x, y \rangle := |q(x+y) - q(x) - q(y)| \leq 2\sqrt{q(x)q(y)}$.*

Proof. We may assume $x \neq 0$ otherwise the result is clear. Let $m, n \in \mathbb{Z}$.

$$\begin{aligned} 0 &\leq q(mx + ny) \\ &= \frac{1}{2} \langle mx + ny, mx + ny \rangle \\ &= m^2 q(x) + mn \langle x, y \rangle + n^2 q(y) \\ &= q(x) \left(m + \frac{\langle x, y \rangle}{2q(x)} n \right)^2 + n^2 \left(q(y) - \frac{\langle x, y \rangle^2}{4q(x)} \right) \end{aligned}$$

Take $m = \langle x, y \rangle$, $n = -2q(x)$, we deduce $\langle x, y \rangle^2 \leq 4q(x)q(y)$, so $|\text{angle } x, y| \leq 2\sqrt{q(x)q(y)}$. \square

Recall that $\text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ is cyclic of order r generated by the Frobenius map $x \mapsto x^q$.

Theorem 7.2 (Hasse). *Let E/\mathbb{F}_q be an elliptic curve. Then $|\#E(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}$*

Proof. Let E have Weierstrass equation with coefficients $a_1, \dots, a_6 \in \mathbb{F}_q$. Define the Frobenius endomorphism $\phi : E \rightarrow E; (x, y) \mapsto (x^q, y^q)$, which is an isogeny of degree q .

Then $E(\mathbb{F}_q) = \{P \in E : \phi(P) = P\} = \ker(1 - \phi)$.

$$\phi^* \omega = \phi^* \left(\frac{dx}{y} \right) = \frac{dx^q}{y^q} = \frac{qx^{q-1}dx}{y^q} = 0, \text{ since } q \equiv 0 \pmod{p}.$$

So $(1 - \phi)^* \omega = 1^* \omega - \phi^* \omega = \omega - 0 = \omega \neq 0$, so $1 - \phi$ is separable.

Hence the size of all but finitely many fibres is $\deg 1 - \phi$, and $1 - \phi$ is a group homomorphism, so $\#E[\mathbb{F}_q] = \# \ker(1 - \phi) = \deg(1 - \phi)$.

By 5.6, $\deg : \text{End}(E) := \text{Hom}(E, E) \rightarrow \mathbb{Z}$ is a positive definite quadratic form.

By 7.1, $|\deg(1 - \phi) - 1 - \deg \phi| \leq 2\sqrt{\deg \phi}$, and hence $|\#E(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q}$. \square

7.1 Zeta Functions

For K a number field:

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{(N_{\mathfrak{a}})^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K \text{ prime}} \left(1 - \frac{1}{(N_{\mathfrak{p}})^s} \right)^{-1}$$

For K a function field, e.g. $K = \mathbb{F}_q(C)$ for C/\mathbb{F}_q a smooth projective curve:

$$\zeta_K(s) = \prod_{x \in |C|} \left(1 - \frac{1}{(Nx)^s} \right)^{-1}$$

where $|C|$ is the set of closed points (i.e. orbit of action of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$) on $C(\bar{\mathbb{F}}_q)$, and $Nx = q^{\deg x}$, where $\deg x$ is the size of the orbit.

We have that $\zeta_K(s) = F(q^{-s})$ for $F \in \mathbb{Q}[[T]]$, where

$$\begin{aligned} F(T) &= \prod_{x \in |C|} (1 - T^{\deg x})^{-1} \\ \log F(T) &= \sum_{x \in |C|} \sum_{m=1}^{\infty} \frac{1}{m} T^{m \deg x} \\ \frac{d}{dT} \log F(T) &= \sum_{x \in |C|} \sum_{m=1}^{\infty} \deg x T^{m \deg x} \\ &= \sum_{n=1}^{\infty} \left(\sum_{\substack{x \in |C| \\ \deg x | n}} \deg x \right) T^n \\ &= \sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n}) T^n \\ \implies F(T) &= \exp \left(\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} T^n \right) =: Z_C(T) \end{aligned}$$

For $\phi, \psi \in \text{Hom}(E_1, E_2)$, we put:

$$\langle \phi, \psi \rangle = \deg(\phi + \psi) - \deg(\phi) - \deg(\psi)$$

We define the *trace map* $\text{tr} : \text{End}(E) \rightarrow \mathbb{Z}; \psi \mapsto \langle \psi, 1 \rangle$.

Lemma 7.3. If $\psi \in \text{End}(E)$ then $\psi^2 - [\text{tr } \psi]\psi + [\deg \psi] = 0$, where $[n]$ means the multiplication by n endomorphism.

Proof. Example sheet 2. □

Definition. The *zeta function of a variety* V/\mathbb{F}_q is

$$Z_v(T) = \exp \left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{q^n})}{n} T^n \right)$$

Lemma 7.4. Let E/\mathbb{F}_q be an elliptic curve, with $\#E(\mathbb{F}_q) = q + 1 - a$. Then

$$Z_E(T) = \frac{1 + aT + qT^2}{(1 - T)(1 - qT)}$$

Proof. Let $\phi : E \rightarrow E$ be the q -power Frobenius map. By the proof of Hasse's theorem,

$$\#E(\mathbb{F}_q) = \deg(1 - \phi) = q + 1 - \text{tr}(\phi)$$

Then $\text{tr}(\phi) = a$, $\deg(\phi) = q$.

Then lemma 7.3 gives $\phi^2 - a\phi + q = 0$. Composing with ϕ^n for $n \geq 0$ gives

$$\begin{aligned} \phi^{n+2} - a\phi^{n+1} + q\phi^n &= 0 \\ \text{tr}(\phi^{n+2}) - a \text{tr}(\phi^{n+1}) + q \text{tr}(\phi^n) &= 0 \end{aligned}$$

This second-order difference equation with initial conditions $\text{tr}(\phi^0) = \text{tr}(1) = 2$, $\text{tr}(\phi^1) = a$ has solutions

$$\text{tr}(\phi^n) = \alpha^n + \beta^n$$

where α, β are the roots of $x^2 - ax + q = 0$.

Hence $\#E(\mathbb{F}_{q^n}) = \deg(1 - \phi^n) = 1 + \deg(\phi^n) - \text{tr}(\phi^n) = 1 + q^n - \alpha^n - \beta^n$.

Substituting, we have:

$$Z_E(T) = \exp \left(\sum_{n=1}^{\infty} \frac{T^n}{n} + \frac{(qT)^n}{n} - \frac{(\alpha T)^n}{n} - \frac{(\beta T)^n}{n} \right)$$

Since $-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, this can be simplified to:

$$\begin{aligned} Z_E(T) &= \frac{(1 - \alpha T)(1 - \beta T)}{(1 - T)(1 - qT)} \\ &= \frac{1 - aT + qT^2}{(1 - T)(1 - qT)} \end{aligned}$$

□

Note that Hasse's theorem gives us $|a| \leq 2\sqrt{q}$, and so the discriminant of $x^2 - aT + q$ is negative, and so $\alpha = \bar{\beta}$, $|\alpha| = |\beta| = \sqrt{q}$.

Let $K = \mathbb{F}_q(E)$. Then $\zeta_K(s) = 0 \implies Z_E(q^{-s}) = 0 \implies q^2 = \alpha$ or β , and hence $\Re(s) = \frac{1}{2}$.

8 Formal Groups

Here, R will be a ring with $I \subset R$ an ideal. The *I -adic topology* on R is the topology with basis $\{r + I^n : r \in R, n \geq 1\}$.

A sequence (x_n) in R is *Cauchy* if, for all k there is some N with $x_m - x_n \in I^k$ for all $m, n \geq k$.

R is *complete* if

1. $\bigcap_{n \geq 0} I^n = \{0\}$ and
2. every Cauchy sequence converges.

Note that, if $x \in I$ then $\frac{1}{1-x} = 1 + x + x^2 + \dots$, and the sequence of partial sums is Cauchy, and hence converges. So $1 - x \in R^\times$.

For example, we could have:

- $R = \mathbb{Z}_p, I = p\mathbb{Z}_p$
- $R = \mathbb{Z}[[t]], I = (t)$.

Lemma 8.1 (Hensel's Lemma). *Let R be an integral domain, complete with respect to I . Let $F \in R[x], s \geq 1$. Suppose $a \in R$ satisfies $F(a) \equiv 0 \pmod{I^s}$, and $F'(a) \in R^\times$.*

Then there is a unique $b \in R$ with $F(b) = 0$ and $b \equiv a \pmod{I^s}$.

Proof. Let $u \in R^\times$ with $F'(a) \equiv u \pmod{I}$, e.g. $u = f'(a)$.

Replacing $F(x)$ by $\frac{F(x+a)}{u}$, we may assume $a = 0$ and $F'(0) \equiv 1 \pmod{I}$.

We put $x_0 = 0, x_{n+1} = x_n - F(x_n)$.

By induction, $x_n \in I_s$ for all n .

$F(x) - F(y) = (x - y)(F'(0) + xG(x, y) + yH(x, y))$ for some polynomials $G, H \in R[x, y]$.

Now we claim $x_{n+1} \equiv x_n \pmod{I^{n+s}}$ for all $n \geq 0$.

This can be proven by induction on n : in the case where $n = 0$, and $x_1 \in I^s$.

Suppose $x_n \equiv x_{n-1} \pmod{I^{n+s-1}}$. Then

$$F(x_n) - F(x_{n-1}) = (x_n - x_{n-1})(1 + c)$$

for some $c \in I$, and hence

$$F(x_n) - F(x_{n-1}) \equiv x_n - x_{n-1} \pmod{I^{n+s}}$$

Rearranging, we have $x_{n+1} \equiv x_n \pmod{I^{n+s}}$, which proves the claim.

Hence (x_n) is Cauchy, and by completeness converges to some $b \in R$. Taking the limit as $n \rightarrow \infty$, we have $b = b - F(b)$, and so $F(b) = 0$, with $b \in I^s$.

For uniqueness, we can use the expression for $F(x) - F(y)$ and the assumption that R is an integral domain. \square

For example, take $E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$

We pass to the affine piece $Y \neq 0, t = X/Y, w = -Z/Y$: Then

$$E : w = t^3 + a_1tw + a_2t^2w + a_3w^2 + a_4tw^2 + a_6w^3 = f(t, w)$$

We can apply Hensel's lemma with $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, $I = (t)$, and $F(x) = x - f(t, x) \in R[x]$. Taking $s = 3, a = 0$, we have:

$$F(0) = -f(t, 0) = -t^3 \equiv 0 \pmod{I^3} \quad F'(0) = 1 - a_t - a_2t^2 \in R^\times$$

So there is a unique root of F , $w(t) \in \mathbb{Z}[a_1, \dots, a_6][[t]]$ such that $w(t) = f(t, w(t))$ and $w(t) \equiv 0 \pmod{t^3}$.

Following the proof of Hensel's lemma with $u = 1$ gives $w(t) = \lim_{n \rightarrow \infty} w_n(t)$ where $w_0(t) = 0, w_{n+1}(t) = f(t, w_n(t))$.

In fact, we may write $w(t) = \sum_{n=2}^{\infty} A_{n-2}t^{n-1}$ with $A_1 = a_1, A_2 = a_1^2 + a_2, A_3 = a_1^3 + 2a_1a_2 + a_3, \dots$

Lemma 8.2. *Let R be an integral domain, complete with respect to $I \trianglelefteq R$, and let $a_1, \dots, a_6 \in R, K = \text{Frac}(R)$.*

Then $\widehat{E}(I) = \{(t, w) \in E(K) : t, w \in I\} = \{(t, w(t)) \in E(K) : t \in I\}$ is a subgroup of $E(K)$.

Proof. The two descriptions of $\widehat{E}(I)$ agree, since given $t \in I$ we can solve for a unique $w \in I$ such that the pair $(t, w) \in E(K)$.

Taking $(t, w) = (0, 0)$ shows that $O_E \in \widehat{E}(I)$. So it suffices to show that, if $P_1, P_2 \in \widehat{E}(I)$, then $-P_1 - P_2 \in \widehat{E}(I)$.

If $P_1 = (t_1, w_1), P_2 = (t_2, w_2)$ lie on the straight line $\lambda t + \nu$, then $-P_1 - P_2$ is the third point of intersection of this line with E .

Then $\lambda = \frac{w(t_2) - w(t_1)}{t_2 - t_1}$ if $t_1 \neq t_2$, and $w'(t_1)$ if $t_1 = t_2$.

$P_1, P_2 \in \widehat{E}(I) \implies t_1, t_2 \in I$.

Thus $\lambda = \sum_{n=2}^{\infty} A_{n-2}(t_1^n + t_1^{n-1}t_2 + \dots + t_2^n) \in I$, and $\nu = w_1 - \lambda t_1 \in I$.

Substituting $w = \lambda t + \nu$ into $w = f(t, w)$ gives $\lambda t + \nu = t^3 + a_1t(\lambda t + \nu) + a_2t^2(\lambda t + \nu) + a_3(\lambda t + \nu)^2 + a_4t(\lambda t + \nu)^3 + a_6(\lambda t + \nu)^3$.

Let A be the coefficient of t^3 , so $A = 1 + a_2\lambda + a_4\lambda^2 + a_6\lambda^3$.

Let B be the coefficient of t^2 , so $B = a_1\lambda + a_2\nu + a_3\lambda^2 + 2a_4\lambda\nu + 3a_6\lambda^2\nu$.

Then $A \in R^\times, B \in I$, and $t_3 = -B/A - t_2 - t_2 \in I$, and $w_3 = \lambda t_3 + \nu \in I$.

Hence $-P_1 - P_2 \in \widehat{E}(I)$, and so $\widehat{E}(I)$ is a subgroup. \square

Taking $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, and $I = (t)$, then the previous lemma tells us there is some power series $\iota \in \mathbb{Z}[a_1, \dots, a_6][[t]]$ with $\iota(0) = 0$ such that $[-1](t, w(t)) = (\iota(t), w(\iota(t)))$

Taking $R = \mathbb{Z}[a_1, \dots, a_6][[t]]$, and $I = (t_1, t_2)$, then we get that there is some power series $F \in I$ such that $(t_1, w(t_1)) + (t_2, w(t_2)) = (F(t_1, t_2), w(F(t_1, t_2)))$.

In fact, we can compute

$$\begin{aligned}\iota(x) &= -x - a_1x^2 - a_2x^3 - (a_1^3 + a_3)x^4 + \dots \\ F(x, y) &= x + y - a_1xy - a_2(x^2y + xy^2) + \dots\end{aligned}$$

By properties of the group law, we can deduce:

1. $F(x, y) = F(y, x)$
2. $F(x, 0) = x, F(0, y) = y$
3. $F(x, F(y, z)) = F(F(x, y), z)$
4. $F(x, \iota(x)) = 0$

This then motivates the following definition:

Definition. Let R be a ring. A **formal group** over R is a power series $F(x, y) \in R[[x, y]]$ satisfying the properties 1, 2, and 3 above.

Exercise. Show that, for any formal group, there is a unique $\iota(x) \in R[[x]]$ such that $F(x, \iota(x)) = 0$.

Examples:

1. $F(x, y) = x + y$
2. $F(x, y) = x + y + xy = (1 + x)(1 + y) - 1$
3. F as above.

We label these formal groups by $\widehat{\mathbb{G}}_a$, $\widehat{\mathbb{G}}_m$, and \widehat{E} respectively.

Definition. Let \mathcal{F}, \mathcal{G} be formal groups over R given by power series F, G respectively. Then:

1. A **morphism** $f : \mathcal{F} \rightarrow \mathcal{G}$ is a power series $f \in R[[t]]$ such that $f(0) = 0$ satisfying $f(F(x, y)) = G(f(x), f(y))$.
2. $\mathcal{F} \cong \mathcal{G}$ if there is some morphism $f : \mathcal{F} \rightarrow \mathcal{G}$, and $g : \mathcal{G} \rightarrow \mathcal{F}$ with $f(g(x)) = g(f(x)) = x$.

Theorem 8.3. If $\text{char}(R) = 0$, then any formal group \mathcal{F} over R is isomorphic to $\widehat{\mathbb{G}}_a$ over $R \otimes \mathbb{Q}$.

More precisely:

1. There is a unique power series $\log : T \mapsto T + \frac{a_2}{2}T^2 + \frac{a_3}{3}T^3 + \dots$ with $a_i \in R$, such that

$$\log(F(x, y)) = \log(x) + \log(y) \quad (*)$$

2. There is a unique power series $\exp : T \mapsto T + \frac{b_2}{2!}T^2 + \frac{b_3}{3!}T^3 + \dots$ with $b_i \in R$ such that

$$\exp(\log(T)) = \log(\exp(T)) = T$$

Proof.

1. Notation: $F_1(x, y) = \frac{\partial F}{\partial x}(x, y)$ (via formal differentiation).

For uniqueness, let $p(T) = \frac{d}{dT} \log(T) = 1 + a_2T + a_3T^2 + \dots$

Differentiating (*) with respect to x , we get: $p(F(x, y))F_1(x, y) = p(x) + 0$. Setting $x = 0$, we get $p(y)F_1(0, y) = 1$, and hence $p(y) = F_1(0, y)^{-1}$, and hence p is uniquely determined, so a_2, a_3, \dots are uniquely determined. But then \log is uniquely determined.

For existence, let $p(T) = F_1(0, T)^{-1} = 1 + a_2T + a_3T^2 + \dots$, where $a_i \in R$.

Integrating up, we let $\log(T) = T + \frac{a_2}{2}T^2 + \frac{a_3}{3}T^3 + \dots$. We now check it satisfied (*).

$$\begin{aligned} F(F(x, y), z) &= F(x, F(y, z)) \\ \frac{\partial}{\partial x} F(F(x, y), z) &= \frac{\partial}{\partial x} F(x, F(y, z)) \\ F_1(F(x, y), z)F_1(x, y) &= F_1(x, F(y, z)) \\ F_1(F(0, y), z)F_1(0, y) &= F_1(0, F(y, z)) \\ F_1(y, z)F_1(0, y) &= F_1(0, F(y, z)) \\ F_1(y, z)p(y)^{-1} &= p(F(y, z))^{-1} \\ F_1(y, z)p(F(y, z)) &= p(y) \\ \log(F(y, z)) &= \log(y) + h(z) \end{aligned}$$

By symmetry between y, z we see that the constant of integration $h(z)$ must be $\log(z)$.

For the second part, we will need the following lemma, which is a generalisation of the statement:

Lemma 8.4. *Let $f(T) = aT + \dots \in R[[T]]$ with $a \in R^\times$. Then there is a unique $g(T) = a^{-1}T + \dots \in R[[t]]$ such that $f(g(T)) = g(f(T)) = T$.*

Proof. We construct polynomials $g_n(T) \in R[T]$ such that $f(g_n(T)) \equiv T \pmod{T^{n+1}}$ and $g_{n+1}(T) \equiv g_n(T) \pmod{T^{n+1}}$. Then we will set $g(T) = \lim_{n \rightarrow \infty} g_n(T)$, satisfying $f(g(T)) = T$.

This is done inductively. To start with, $g_1(T) = a^{-1}T$. Then $f(g_1(T)) = T + T^2(\dots) \equiv T \pmod{T^2}$.

Now suppose $n \geq 1$ and $g_{n-1}(T)$ exists.

Then $f(g_{n-1}(T)) \equiv T + bT^n \pmod{T^{n+1}}$. Let $g_n(T) = g_{n-1}(T) + \lambda T^n$, where $\lambda \in R$ to be chosen later.

Then $f(g_n(T)) = f(g_{n-1}(T) + \lambda T^n) \equiv f(g_{n-1}(T)) + \lambda aT^n \pmod{T^{n+1}} \equiv T + (b + \lambda a)T^n \pmod{T^{n+1}}$.

So pick $\lambda = -ba^{-1}$.

This gives $g(T)$ with $f(g(T)) = T$.

Applying the same argument, we get $h(T)$ such that $g(h(T)) = T$.

Then $f(T) = f(g(h(T))) = h(T)$, and so g is as required. \square

2. We now only have to show that the $b_n \in R$ (not just in $R \otimes \mathbb{Q}$). See example sheet 2 for this. \square

Let \mathcal{F} be a formal group (e.g. $\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_m, \widehat{E}$), given by a power series $F \in R[x, y]$, and suppose that R is I -adically complete. Then for $x, y \in I$, put $x \oplus_{\mathcal{F}} y = F(x, y) \in I$. Then $\mathcal{F} = (I, \oplus_{\mathcal{F}})$ is an abelian group.

For example, $\widehat{\mathbb{G}}_a(I) = (I, +)$, $\widehat{\mathbb{G}}_m(I) = (1 + I, \times)$, and in 8.2, we saw $\widehat{E}(I) \leq E(K)$.

Corollary 8.5. Let \mathcal{F} be a formal group over R , and $n \in \mathbb{Z}$. Suppose $n \in R^\times$. Then:

1. $[n] : \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism.
2. If R is complete with respect to I , then $\mathcal{F}(I) \xrightarrow{\times n} \mathcal{F}(I)$ is an isomorphism.

In particular, $\mathcal{F}(I)$ has no n -torsion.

Proof. We have $[1](T) = T$, $[n](T) = F([n-1]T, T)$ for $n \geq 2$. For $n < 0$, use $[-1](T) = \iota(T)$.

Induction gives us $[n](T) = nT + \dots$, and so by 8.4, $[n]$ is an isomorphism. \square

9 Elliptic Curves over Local Fields

Let K be a field, complete with respect to the discrete valuation $v : K^\times \rightarrow \mathbb{Z}$. Then we define the valuation ring, or ring of integers, the set:

$$\mathcal{O}_K = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}$$

Then $\mathcal{O}_K^\times = \{x \in K^\times : v(x) = 0\}$. There is a unique maximal ideal $\pi\mathcal{O}_K$, where $v(\pi) = 1$, and we define the residue field to be $k = \mathcal{O}_K/\pi\mathcal{O}_K$.

We assume $\text{char } K = 0$, $\text{char } k = p$.

For example, if $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$, $\pi = p$, $k = \mathbb{F}_p$.

Let E/K be an elliptic curve. Then a Weierstrass equation for E with coefficients $a_1, \dots, a_6 \in K$ is *integral* if $a_i \in \mathcal{O}_K$, and minimal if $v(\Delta)$ is minimal among all integral Weierstrass equations for E .

Putting $x = u^2x'$, $y = u^3y'$ give $a_i = u^i a'_i$. So we can clear denominators, and hence every elliptic curve has an integral Weierstrass equation. Moreover, since $a_i \in \mathcal{O}_K$, $\Delta \in \mathcal{O}_K$, and so $v(\Delta) \geq 0$, and hence we can pick a minimal Weierstrass equation.

If $\text{char } k \neq 2, 3$ then there is a minimal Weierstrass equation of the form $y^2 = x^3 + ax + b$.

Lemma 9.1. Let E/K have integral Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Let $0 \neq P = (x, y) \in E(K)$. Then either $x, y \in \mathcal{O}_K$ or $v(x) = -2s$, $v(y) = -3s$ for some $s \geq 1$.

Compare this to example sheet 1, question 5.

Proof. If $v(x) \geq 0$, then consider y .

If $v(y) < 0$, then $v(LHS) < 0$, but $v(RHS) \geq 0$, and hence $x, y \in \mathcal{O}_K$.

Now if $v(x) < 0$, then $v(LHS) \geq \min(2v(y), v(x) + v(y), v(y))$
 $v(RHS) = v(x^3) = 3v(x)$.

Hence $v(y) < v(x)$. But then $v(LHS) = 2v(y)$, and hence $3v(x) = 2v(y)$. \square

If K is complete, then \mathcal{O}_K is complete with respect to the ideal $\pi^r\mathcal{O}_K$ for any $r \geq 1$.

Fix a minimal Weierstrass equation for E/K , and hence a formal group \widehat{E} over \mathcal{O}_K .

Take $I = \pi^r \mathcal{O}_K$ in 8.2, we have

$$\begin{aligned}\widehat{E}(\pi^r \mathcal{O}_K) &= \left\{ (x, y) \in E(K) : -\frac{x}{y}, -\frac{1}{y} \in \pi^r \mathcal{O}_K \right\} \cup \{0\} \\ &= \left\{ (x, y) \in E(K) : v\left(\frac{x}{y}\right) \geq r \text{ \& } v\left(\frac{1}{y}\right) \geq r \right\} \cup \{0\} \\ &= \left\{ (x, y) \in E(K) : v(x) = -2s, v(y) = -3s, s \geq r \right\} \cup \{0\} \\ &= \left\{ (x, y) \in E(K) : v(x) \leq -2r, v(y) \leq -3r \right\} \cup \{0\}\end{aligned}$$

By 8.2, this is a subgroup of $E(K)$, say $E_r(K)$. We have a chain

$$\dots \subset E_3(K) \subset E_2(K) \subset E_1(K)$$

More generally, for \mathcal{F} a formal group over \mathcal{O}_K , we get

$$\dots \subset \mathcal{F}(\pi^3 \mathcal{O}_K) \subset \mathcal{F}(\pi^2 \mathcal{O}_K) \subset \mathcal{F}(\pi \mathcal{O}_K)$$

We will show that $\mathcal{F}(\pi^r \mathcal{O}_K) \cong (\mathcal{O}_K, +)$ for r sufficiently large, and $\mathcal{F}(\pi^r \mathcal{O}_K)/\mathcal{F}(\pi^{r+1} \mathcal{O}_K) \cong (k, +)$.

Theorem 9.2. *Let \mathcal{F} be a formal group over \mathcal{O}_K , and let $e = v(p)$. If $r > \frac{e}{p-1}$, then:*

$$\mathcal{F}(\pi^r \mathcal{O}_K) \cong \widehat{\mathbb{G}}_a(\pi^r \mathcal{O}_K)$$

via the log map, with inverse given by exp.

Note that $\widehat{\mathbb{G}}_a(\pi^r \mathcal{O}_K) = (\pi^r \mathcal{O}_K, +) \cong (\mathcal{O}_K, +)$.

Proof. For $x \in \pi^r \mathcal{O}_K$, we must check that the power series exp, log converge.

Recall $\exp(T) = T + \frac{b_2}{2!}T^2 + \frac{b_3}{3!}T^3 + \dots$, where $b_i \in \mathcal{O}_K$.

Claim: $v_p(n!) \leq \frac{n-1}{p-1}$.

To see this: $v_p(n!) = \sum_{r=1}^{\infty} \lfloor \frac{n}{p^r} \rfloor < \sum_{r=1}^{\infty} \frac{n}{p^r} = \frac{n}{p-1}$.

So $(p-1)v_p(n!) < n$, and as both are integers, $(p-1)v_p(n!) \leq n-1$.

Now $v\left(\frac{b_n x^n}{n!}\right) \geq nr - e \frac{n-1}{p-1} = (n-1)\left(r - \frac{e}{p-1}\right) + r$

This is always $\geq r$ as $r > \frac{e}{p-1}$, and goes to infinity as $n \rightarrow \infty$.

Hence $\exp(x)$ converges, and belongs to $\pi^r \mathcal{O}_K$. A similar argument applies for log. \square

Lemma 9.3. *We have $\frac{\mathcal{F}(\pi^r \mathcal{O}_K)}{\mathcal{F}(\pi^{r+1} \mathcal{O}_K)} \cong (k, +)$ for all $r \geq 1$.*

Proof. By definition of a formal group, $F(x, y) = x + y + xy(\dots)$. So if $x, y \in \mathcal{O}_K$, then:

$$F(\pi^r x, \pi^r y) = \pi^r(x + y) + \pi^{2r}(xy)(\dots) \equiv \pi^r(x + y) \pmod{\pi^{r+1}}$$

So $\mathcal{F}(\pi^r \mathcal{O}_K) \rightarrow (k, +); (\pi^r x) \mapsto (x \pmod{\pi})$ is a surjective group homomorphism, with kernel $\mathcal{F}(\pi^{r+1} \mathcal{O}_K)$, and so apply the first isomorphism theorem. \square

So we have a filtration:

$$(O_K, +) \cong \mathcal{F}(\pi^r O_K) \supseteq \dots \supseteq \mathcal{F}(\pi^2 O_K) \supseteq \mathcal{F}(\pi O_K)$$

where we have equality on the left is $r > \frac{e}{p-1}$, and each quotient is $(k, +)$.

Corollary. If $|k| < \infty$, then $\mathcal{F}(\pi O_K)$ has a subgroup of finite index isomorphic to O_K under addition.

As a point of notation, when we have the map $O_K \rightarrow O_K/\pi O_K$, we write $x \mapsto \tilde{x}$, and call this reduction mod π .

Proposition 9.4. Let E/K be an elliptic curve. The reduction mod π of any two minimal Weierstrass equations for E define isomorphic curves over k .

Proof. Say the Weierstrass equations are related by $[u; r, s, t]; u \in K^\times; r, s, t \in K$.

Then $\Delta_1 = u^{12} \Delta_2$. Both equations are minimal, so $v(\Delta_1) = v(\Delta_2)$, and hence $v(u) = 0, u \in O_K^\times$.

Transformation formulae for a_i and b_i , together with the fact that the valuation ring is integrally closed, give that $r, s, t \in O_K$. The Weierstrass equations for the reduction mod π are related by $[\tilde{u}; \tilde{r}, \tilde{s}, \tilde{t}]$. \square

Definition. The reduction \tilde{E}/k of E/K is defined by the reduction of a minimal Weierstrass equation, and hence is well-defined up to isomorphism by the previous proposition.

We say E has *good reduction* if \tilde{E} is non-singular, i.e. is an elliptic curve. Otherwise, it is *bad*.

For an integral Weierstrass equation, $v(\Delta) = 0 \implies$ good reduction.

If $0 < v(\Delta) < 12$, then we must have a minimal Weierstrass equation, and we get bad reduction.

If $v(\Delta) \geq 12$, beware that the equation might not be minimal.

There is a well defined map from $\mathbb{P}^2(K) \rightarrow \mathbb{P}^2(k); (x : y : z) \mapsto (\tilde{x} : \tilde{y} : \tilde{z})$, when we choose representatives of $(x : y : z)$ with $\min(v(x), v(y), v(z)) = 0$.

We restrict this map to give a map $E(K) \rightarrow \tilde{E}(k); P \rightarrow \tilde{P}$. If $P = (x, y) \in E(K)$, then by 9.1, either $x, y \in O_K$ or $v(x) = -2s, v(y) = -3s$. In the first case $\tilde{P} = (\tilde{x}, \tilde{y})$. In the second, we write $P = (\pi^{3s}x : \pi^{3s}y : \pi^{3s})$, so $\tilde{P} = (0 : 1 : 0)$.

Therefore $E_1(K) = \hat{E}(\pi O_K) = \{P \in E(K) : \tilde{P} = 0\}$, and we call it the *kernel of reduction*.

$$\text{Let } \tilde{E}_{ns} = \begin{cases} \tilde{E} & \text{if } E \text{ has good reduction} \\ \tilde{E} \setminus \{p\} & \text{if } \tilde{E} \text{ has a singular point } p \end{cases}$$

The chord and tangent process still defines a group law on \tilde{E}_{ns} . In cases of bad reduction, we get $\tilde{E}_{ns} \cong \mathbb{G}_a$ or \mathbb{G}_m over k , or possibly only over a quadratic extension of k . We call these cases additive and multiplicative reduction.

For simplicity, suppose $\text{char}(k) \neq 2$. Then $\tilde{E} : y^2 = f(x)$ for f monic cubic. Then \tilde{E} singular $\iff f$ has a repeated root. The cases of double root, triple root correspond to multiplicative, additive reduction respectively.

For multiplicative case, see example sheet 3. Here, we'll illustrate the additive case. We have a triple root, so take $y^2 = x^3$. Then we have an isomorphism

$$\begin{aligned}\tilde{E}_{ns} &\rightarrow \mathbb{G}_a \\ (x, y) &\mapsto \frac{x}{y} \\ (t^{-2}, t^{-3}) &\mapsto t \\ \infty &\mapsto 0\end{aligned}$$

Let P_1, P_2, P_3 lie on the line $ax + by = 1$. Write $P_i = (x_i, y_i)$, $t_i = \frac{x_i}{y_i}$. Then $x_i^3 = y_i^2 = y_i^2(ax_i + by_i)$, and so t_1, t_2, t_3 are the roots of $X^3 - aX - b = 0$. Looking at the coefficient of X^2 , we have $t_1 + t_2 + t_3 = 0$.

Definition. $E_0(K) := \{P \in E(K) : \tilde{P} \in \tilde{E}_{ns}(k)\}$.

Proposition 9.5. $E_0(K)$ is a subgroup of $E(K)$, and reduction mod π is a surjective group homomorphism from $E_0(K) \rightarrow \tilde{E}_{ns}(k)$.

Proof. For the group homomorphism part, a line ℓ in \mathbb{P}^2 defined over K has equation

$$\ell : aX + bY + cZ = 0 \quad a, b, c \in K$$

We may assume $\min(v(a), v(b), v(c)) = 0$. Reduction mod π gives the line $\tilde{\ell}$ with equation

$$\tilde{\ell} : \tilde{a}X + \tilde{b}Y + \tilde{c}Z = 0$$

If $P_1, P_2, P_3 \in E(K)$ with $P_1 + P_2 + P_3 = 0$, then these points lie on a line ℓ , and then $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \in \tilde{E}(k)$ lie on the line $\tilde{\ell}$.

If $\tilde{P}_1, \tilde{P}_2 \in \tilde{E}_{ns}(k)$, then $\tilde{P}_3 \in \tilde{E}_{ns}(k)$, and if $P_1, P_2 \in E_0(K)$, then $P_3 \in E_0(K)$, and $\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 = 0$.

As an exercise, check this still works if the points are not all distinct.

For surjectivity, let $f(x, y) = y^2 + a_1xy + a_3y - (x^3 + \dots)$. Let $\tilde{P} \in \tilde{E}_{ns}(k) \setminus \{0\}$, say $(\tilde{x}_0, \tilde{y}_0)$ for some x_0, y_0 in \mathcal{O}_K .

Since \tilde{P} is non-singular, either

- (i) $\frac{\partial f}{\partial x}(x_0, y_0) \not\equiv 0 \pmod{\pi}$
- (ii) $\frac{\partial f}{\partial y}(x_0, y_0) \not\equiv 0 \pmod{\pi}$

If (i), we put $g(t) = f(t, y_0) \in \mathcal{O}_K[t]$. Then $g(x_0) \equiv 0 \pmod{\pi}$, $g'(x_0) \in \mathcal{O}_K^\times$. Then Hensel's lemma tells us there is some $b \in \mathcal{O}_K$ with $g(b) = 0, b \equiv x_0 \pmod{\pi}$.

Then $P = (b, y_0) \in E(K)$ has reduction \tilde{P} .

Case (ii) is similar. □

Recall for $r \geq 1$, we have $E_r(K) = \{(x, y) \in E(K) : v(x) \leq -2r, v(y) \leq -3r\} \cup \{0\}$. Then:

$$\mathcal{O}_K \cong E_{\lceil e/(p-1) \rceil}(K) \supset \dots \supset E_2(K) \supset E_1(K) \cong \hat{E}(\pi\mathcal{O}_K) \subset E_0(K) \subset E(K)$$

We know the quotients $E_i(K)/E_{i+1}(K) \cong (k, +)$ for $i \geq 1$. The above gives $E_0(K)/E_1(K) \cong \widetilde{E}_{ns}(k)$. The only quotient left to understand is $E(K)/E_0(K)$.

Lemma 9.6. *If $|k| < \infty$, then $E_0(K) \subset E(K)$ has finite index.*

Proof. A compactness argument - see below. \square

Theorem 9.7. *If $[K : \mathbb{Q}_p] < \infty$, then $E(K)$ contains a subgroup of finite index, isomorphic as a group to $(\mathcal{O}_K, +)$.*

Proof. $|k| < \infty$, so this follows from the above. \square

Lemma 9.8. *If $|k| < \infty$, then $\mathbb{P}^n(K)$ is compact with respect to the π -adic topology.*

Proof. $|k| < \infty$, so $\mathcal{O}_K/\pi^r \mathcal{O}_K$ is also finite for $r \geq 1$. Hence

$$\mathcal{O}_K \cong \varprojlim_r \mathcal{O}_K/\pi^r \mathcal{O}_K$$

is compact.

$\mathbb{P}^n(K)$ is the union of compact sets of the form

$$\{(a_0 : a_1 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n) : a_j \in \mathcal{O}_K\}$$

and hence is compact. \square

Proof of 9.6. $E(K) \subset \mathbb{P}^2(K)$ is a closed subset, so $(E(K), +)$ is a compact topological group.

If \widetilde{E} has a singular point $(\widetilde{x}_0, \widetilde{y}_0)$ then $E(K) \setminus E_0(K) = \{(x, y) \in E(K) : v(x - x_0) \geq 1, v(y - y_0) \geq 1\}$, is a closed subset of $E(K)$, and so $E_0(K)$ is an open subgroup of $E(K)$, so any coset is also open.

The cosets of $E_0(K)$ form an open cover of $E(K)$, hence have a finite subcover, and so there are only finitely many cosets.

Hence $[E(K) : E_0(K)] < \infty$. \square

We call this index $c_K(E)$, the *Tamagawa number*.

Remarks.

1. Good reduction $\implies c_K(E) = 1$, but the converse is false.
2. It can be shown that either $c_K(E) = v(\Delta)$ or $c_K(E) \leq 4$, as long as we work with a minimal Weierstrass equation.

Let $[K : \mathbb{Q}_p]$ be finite, and L/K finite, with residue fields k', k (corresponding to L, K respectively), with $f = [k' : k]$ and ramification index e . From local fields, we know $[L : K] = ef$.

If L/K is Galois then there is a natural group homomorphism $\text{Gal}(L/K) \rightarrow \text{Gal}(k'/k)$, and this map is surjective, with kernel of order e . We say the extension is *unramified* if $e = 1$, so if these Galois groups are isomorphic.

For each $m \geq 1$, k has a unique extension of degree m , called k_m (not standard notation). K has a unique unramified extension of degree m , called K_m . Note that then the residue field of K_m is k_m . These extensions are Galois with cyclic Galois group.

We then define $K^{nr} = \bigcup_{m \geq 1} K_m$ inside \bar{K} , the maximal unramified extension.

Theorem 9.9. Suppose $[K : \mathbb{Q}_p] < \infty$, and E/K has good reduction and $p \nmid n$. If $P \in E(K)$ then $K([n]^{-1}P)/K$ is unramified.

Notation: $[n]^{-1}P = \{Q \in E(\bar{K}) : nQ = P\}$, and $K(P_1, \dots, P_r) = K(x_1, \dots, x_r, y_1, \dots, y_r)$, $P_i = (x_i, y_i)$.

Proof. For each $m \geq 1$, there is a short exact sequence $0 \rightarrow K_1(K_m) \rightarrow E(K_m) \rightarrow \tilde{E}(k_m) \rightarrow 0$.

Taking union over all m gives a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1(K^{nr}) & \longrightarrow & E(K^{nr}) & \longrightarrow & \tilde{E}(\bar{k}) \longrightarrow 0 \\ & & \downarrow \times n & & \downarrow \times n & & \downarrow \times n \\ 0 & \longrightarrow & E_1(K^{nr}) & \longrightarrow & E(K^{nr}) & \longrightarrow & \tilde{E}(\bar{k}) \longrightarrow 0 \end{array}$$

The first vertical arrow is an isomorphism by 8.5, as $n \in \mathcal{O}_K^\times$.

The last vertical arrow is surjective by 2.8, with kernel $(\mathbb{Z}/n\mathbb{Z})^2$ by 6.5, as $p \nmid n$.

The snake lemma tells us $E(K^{nr})[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$, $E(K^{nr})/nE(K^{nr}) = 0$.

So if $P \in E(K)$, then multiplication by n is surjective, and there is Q in $E(K^{nr})$ with $nQ = P$, and $[n]^{-1}P = \{Q + T : T \in E[n]\} \subset E(K^{nr})$.

So $K([n]^{-1}P) \subset K^{nr}$, and $K([n]^{-1}P)/K$ is unramified. \square

Corollary 9.10. Let E/K be an elliptic curve with $[K : \mathbb{Q}_p] < \infty$ Then $E(K)_{\text{tors}}$ is finite.

Proof. In 9.7 we saw that $E(K)$ has a subgroup $E_r(K)$ of finite index isomorphic to $(\mathcal{O}_K, +)$. Since $E_r(K)$ is torsion free, $E(K)_{\text{tors}} \hookrightarrow E(K)/E_r(K)$, an injection into a finite group. \square

10 Elliptic Curves over Number Fields

10.1 The Torsion Subgroup

Let $[K : \mathbb{Q}] < \infty$ and E/K an elliptic curve.

Let \mathfrak{p} be a prime of K (i.e. a prime ideal in \mathcal{O}_K). We write $K_{\mathfrak{p}}$ for the \mathfrak{p} -adic completion of K , and $k_{\mathfrak{p}}$ for $\mathcal{O}_K/\mathfrak{p}$. Note that, upon taking completions, the residue field doesn't change.

Definition. \mathfrak{p} is a prime of good reduction for E/K if $E/K_{\mathfrak{p}}$ has good reduction.

Lemma 10.1. E/K has only finitely many primes of bad reduction.

Proof. Take any Weierstrass equation for E , with coefficients in \mathcal{O}_K . E is non-singular, so $0 \neq \Delta \in \mathcal{O}_K$. We can thus write $\Delta = \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_r^{\alpha_r}$ as a unique factorisation into prime ideals, and let $S = \{\mathfrak{p}_i\}$ in this factorisation.

If $\mathfrak{p} \notin S$, then $v_{\mathfrak{p}}(\Delta) = 0$, so $E/K_{\mathfrak{p}}$ has good reduction.

Hence the set of bad primes for E is a subset of S , which is finite. \square

Note that we'd like to say that S is the set of bad primes. If K has class number 1, e.g. $K = \mathbb{Q}$, then we can always find Weierstrass equation for E with the coefficients in \mathcal{O}_K minimal at all primes p , and then S will be all the bad primes.

Lemma 10.2. $E(K)_{\text{tors}}$ is finite.

Proof. Take any prime p and complete at p . Then $K \subseteq K_p$, so $E(K)_{\text{tors}} \subseteq E(K_p)_{\text{tors}}$ is finite by 9.10. \square

Lemma 10.3. Let p be a prime of good reduction, with $p \nmid n$. Then reduction mod p gives an injective group homomorphism

$$E(K)[n] \hookrightarrow \tilde{E}(k_p)[n]$$

Proof. 9.5 tells us that $E(K_p) \rightarrow \tilde{E}(k_p)$ is a group homomorphism. Hence it takes n -torsion points to n -torsion points, as needed. It has kernel $E_1(K_p)$. Since $p \nmid n$, 8.5 tells us $E_1(K_p)$ has no n -torsion, and so the map is injective. \square

Examples.

1. $E/\mathbb{Q} : y^2 + y = x^3 - x^2, \Delta = -11$. E has good reduction at all primes $p \neq 11$.

p	2	3	5	7	11	13
$\# \tilde{E}(\mathbb{F}_p)$	5	5	5	10	-	10

By 10.3 looking at $p = 2$, $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 2^a$ for some $a \geq 0$.

Looking at $p = 3$, $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 3^b$ for some $b \geq 0$.

Hence $\#E(\mathbb{Q})_{\text{tors}} | 5$, so is 1 or 5.

Let $T = (0, 0) \in E(\mathbb{Q})$. Calculation gives that $5T = O_E$, and so $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}$.

2. $E/\mathbb{Q} : y^2 + y = x^3 + x^2, \Delta = -43$. E has good reduction at all primes $p \neq 43$.

p	2	3	5	7	11	13
$\# \tilde{E}(\mathbb{F}_p)$	5	6	10	8	9	19

So $\#E(\mathbb{Q})_{\text{tors}} | 5 \cdot 2^a$, some $a \geq 0$, and $\#E(\mathbb{Q})_{\text{tors}} | 9 \cdot 11^b$, some $b \geq 0$.

So $\#E(\mathbb{Q})_{\text{tors}} = 1$, and $E(\mathbb{Q})_{\text{tors}} = \{O_E\}$.

Now, since $P = (0, 0) \in E(\mathbb{Q})$, it has infinite order, and hence infinitely many rational points on $E(\mathbb{Q})$. This is an example where $\text{rank } E(\mathbb{Q}) \geq 1$.

3. $E_D : y^2 = x^3 - D^2x$ for $D \in \mathbb{Z}$ a squarefree integer. Then $\Delta = 2^6 D^6$.

$$E_D(\mathbb{Q})_{\text{tors}} \supset \{0, (0, 0), (\pm D, 0)\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Let $f(x) = x^3 - D^2x$. Then if p is prime not dividing $2D$, then it is a prime of good reduction.

$$\# \tilde{E}(\mathbb{F}_p) = 1 + \sum_{x \in \mathbb{F}_p} \left(\left(\frac{f(x)}{p} \right) + 1 \right), \text{ where } \left(\frac{f(x)}{p} \right) \text{ is the Legendre symbol.}$$

If $p \equiv 3 \pmod{4}$, then since $f(x)$ is an odd function:

$$\left(\frac{f(-x)}{p} \right) = \left(\frac{-f(x)}{p} \right) = \left(\frac{-1}{p} \right) \left(\frac{f(x)}{p} \right) = - \left(\frac{f(x)}{p} \right)$$

and so $\#E_D(\mathbb{F}_p) = p + 1$.

Let $m = \#E(\mathbb{Q})_{\text{tors}}$. We have $4|m|p + 1$ for all sufficiently large primes p congruent to 3 mod 4, and hence $m = 4$, since otherwise this contradicts Dirichlet's theorem on primes in arithmetic progression.

Hence $E_D(\mathbb{Q})_{\text{tors}} \cong (\mathbb{Z}/2\mathbb{Z})^2$. So $\text{rank } E_D(\mathbb{Q}) \geq 1$ if and only if there are $x, y \in \mathbb{Q}$ with $y \neq 0$ such that $y^2 = x^3 - Dx$, which by the first lecture is equivalent to D being a congruent number.

Lemma 10.4. *Let E/\mathbb{Q} be given by a Weierstrass equation with coefficients in \mathbb{Z} . Then:*

1. $4x, 8y \in \mathbb{Z}$.
2. If $2|a_1$ or $2T \neq O_E$, then $x, y \in \mathbb{Z}$.

Proof. The Weierstrass equation defines a formal group \widehat{E} over \mathbb{Z} . For $r \geq 1$, we have $\widehat{E}(p^r\mathbb{Z}_p) = \{(x, y) \in E(\mathbb{Q}_p) : v_p(x) \leq -2r, v_p(y) \leq -3r\} \cup \{0\}$.

9.2 gives $\widehat{E}(p^r\mathbb{Z}_p) \cong (\mathbb{Z}_p, +)$ if $r > \frac{1}{p-1}$, and hence $\widehat{E}(4\mathbb{Z}_2)$ and $\widehat{E}(p\mathbb{Z}_p)$ are torsion free.

Since T is a nonzero torsion point, it follows that $v_p(x), v_p(y) \geq 0$ for all odd primes p , and $v_2(x) \geq -2, v_2(y) \geq -3$. This proves part 1.

For the second part, suppose that $T \in \widehat{E}(2\mathbb{Z}_2)$, i.e. $v_2(x) = -2, v_2(y) = -3$.

Since $\frac{\widehat{E}(2\mathbb{Z}_2)}{\widehat{E}(4\mathbb{Z}_2)} \cong (\mathbb{F}_2, +)$ and $\widehat{E}(4\mathbb{Z}_2)$ is torsion free, we get $2T = 0$. Also, $(x, y) = T = -T = (x, -y - a_1x - a_3)$, and hence $2y + a_1x + a_3 = 0, 8y + 4xa_1 + 4a_3 = 0$.

$8y$ is odd, $4x$ is odd, $4a_3$ is even, and hence a_1 is odd.

So if $2T \neq 0$ or a_1 is even, then $T \notin \widehat{E}(2\mathbb{Z}_2)$, so $x, y \in \mathbb{Z}$. □

For example, if $y^2 + xy = x^3 + 4x + 1$, then $(-\frac{1}{4}, \frac{1}{8}) \in E(\mathbb{Q})[2]$.

Theorem 10.5 (Lutz-Nagell). *Let E/\mathbb{Q} be given by $y^2 = x^3 + ax + b$, for $a, b \in \mathbb{Z}$.*

Suppose $0 \neq T = (x, y) \in E(\mathbb{Q})_{\text{tors}}$. Then $x, y \in \mathbb{Z}$, and either $y = 0$ or $y^2 | 4a^3 + 27b^2$.

Note that this is not an if and only if - we still have to check the answers we get.

Proof. 10.4 gave us $x, y \in \mathbb{Z}$. If $2T = 0$, then $y = 0$.

Otherwise, $0 \neq 2T = (x_2, y_2) \in E(\mathbb{Q})_{\text{tors}}$, and so 10.4 gives $x_2, y_2 \in \mathbb{Z}$.

But $x_2 = \left(\frac{f'(x)}{2y}\right)^2 - 2x$, and so $y|f'(x)$.

E non-singular, so $f(x)$ and $f'(x)$ are coprime, and so $f(x)$ and $(f'(x))^2$ are coprime, hence $1 = g(x)f(x) + h(x)(f'(x))^2$ for some $g, h \in \mathbb{Q}[x]$.

Doing this calculation and clearing denominators, we get

$$(3x^2 + 4a)f'(x)^2 - 27(x^3 + ax - b)f(x) = 4a^3 + 27b^2$$

Since $y|f'(x)$, $y^2 = f(x)$, so y^2 divides LHS, hence $y^2 | 4a^3 + 27b^2$. □

Mazur showed that, if E/\mathbb{Q} is an elliptic curve, then $E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & 1 \leq n \leq 12, n \neq 11 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} & 1 \leq n \leq 4 \end{cases}$.

Moreover, all 15 possibilities occur.

11 Kummer Theory

K is a field, $\text{char } K \nmid n$, and $\mu_n \subset K$, where μ_n is the set of n^{th} roots of unity.

Lemma 11.1. *Let $\Delta \subset K^\times/(K^\times)^n$ be a finite subgroup, and let $L = K(\sqrt[n]{\Delta})$. Then L/K is Galois, and $\text{Gal}(L/K) \cong \text{Hom}(\Delta, \mu_n)$.*

Proof. L/K is Galois since $\mu_n \subset K$, and $\text{char } K \nmid n$.

Define the Kummer pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \text{Gal}(L/K) \times \Delta &\rightarrow \mu_n \\ (\sigma, x) &\mapsto \frac{\sigma(\sqrt[n]{x})}{\sqrt[n]{x}} \end{aligned}$$

It is well defined: suppose $\alpha, \beta \in L$ are two different choices of $\sqrt[n]{x}$. Then $(\alpha/\beta)^n = 1$, so $\alpha/\beta \in \mu_n \subset K$, so $\sigma(\alpha/\beta) = \alpha/\beta$. Hence $\sigma(\alpha)/\alpha = \sigma(\beta)/\beta$.

It is bilinear: $\langle \sigma\tau, x \rangle = \frac{\sigma\tau(\sqrt[n]{x})}{\tau(\sqrt[n]{x})} = \langle \sigma, x \rangle \langle \tau, x \rangle$, as $\tau(\sqrt[n]{x})$ is another choice of $\sqrt[n]{x}$, and

$$\langle \sigma, xy \rangle = \frac{\sigma(\sqrt[n]{xy})}{\sqrt[n]{xy}} = \frac{\sigma(\sqrt[n]{x})}{\sqrt[n]{x}} \frac{\sigma(\sqrt[n]{y})}{\sqrt[n]{y}} = \langle \sigma, x \rangle \langle \sigma, y \rangle.$$

It is non-degenerate: Let $\sigma \in \text{Gal}(L/K)$. If $\langle \sigma, x \rangle = 1$ for all $x \in \Delta$, then $\sigma(\sqrt[n]{x}) = \sqrt[n]{x}$ for all $x \in \Delta$, and so σ fixes L pointwise. Hence $\sigma = \text{id}$. Now fix $x \in \Delta$, and suppose $\langle \sigma, x \rangle = 1$ for all $\sigma \in \text{Gal}(L/K)$. Then $\sigma(\sqrt[n]{x}) = \sqrt[n]{x}$ for all $\sigma \in \text{Gal}(L/K)$, and hence $\sqrt[n]{x} \in K$, and so $x \in (K^\times)^n$, i.e. $x(K^\times)^n$ is trivial in Δ .

We thus get injective group homomorphisms $\text{Gal}(L/K) \hookrightarrow \text{Hom}(\Delta, \mu_n)$, $\Delta \hookrightarrow \text{Hom}(\text{Gal}(L/K), \mu_n)$.

Hence $\text{Gal}(L/K)$ is abelian of exponent dividing n .

If G is a finite abelian group of exponent dividing n , then $\text{Hom}(G, \mu_n) = G$ (non-canonically).

So $|\text{Gal}(L/K)| \leq |\Delta| \leq |\text{Gal}(L/K)|$, and so $|\Delta| = |\text{Gal}(L/K)|$, and hence the injective homomorphisms are surjective as well, so isomorphisms. \square

For example $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Theorem 11.2. *There is a bijection*

$$\begin{aligned} \{\text{finite subgroups } \Delta \subseteq K^\times/(K^\times)^n\} &\leftrightarrow \{\text{finite abelian extensions } L/K \text{ or exponent dividing } n\} \\ \Delta &\mapsto K(\sqrt[n]{\Delta}) \\ \frac{(L^\times)^n \cap K^\times}{(K^\times)^n} &\hookrightarrow L \end{aligned}$$

Proof. Let L/K be a finite abelian extension of exponent dividing n . Let $\Delta = \frac{(L^\times)^n \cap K^\times}{(K^\times)^n}$. Then $K(\sqrt[n]{\Delta}) \subset L$ and we aim to show equality.

Let $G = \text{Gal}(L/K)$.

The Kummer pairing gives an injection $\Delta \hookrightarrow \text{Hom}(G, \mu_n)$. We claim this is a surjection.

Given the claim, we will then have $\Delta \cong \text{Hom}(G, \mu_n)$, so $[K(\sqrt[n]{\Delta}) : K] = |\Delta|$ by **11.1** $= |G| = [L : K]$, and hence we have the equality.

To prove the claim, let $\chi : G \rightarrow \mu_n$ be a member of $\text{Hom}(G, \mu_n)$. Distinct automorphisms are linearly independent. Then

$$\exists a \in L \text{ s.t. } \underbrace{\sum_{\tau \in G} \chi(\tau)^{-1} \tau(a)}_y \neq 0$$

Let $\sigma \in G$. Then

$$\begin{aligned} \sigma(y) &= \sum_{\tau \in G} \chi(\tau)^{-1} \sigma \tau(a) \\ &= \sum_{\tau \in G} \chi(\sigma^{-1} \tau)^{-1} \tau(a) \\ &= \chi(\sigma) y \end{aligned}$$

So $\sigma(y^n) = y^n$ for all $\sigma \in G$. Then if $x := y^n$, we have $x \in K^\times$, and $x \in (L^\times)^n$.

So $x \in \Delta$, and $\chi(\sigma) = \frac{\sigma(\sqrt[n]{x})}{\sqrt[n]{x}}$, and so χ is the image of x under the injection, and hence it is a surjection.

For the other direction, we start with $\Delta \subset K^\times / (K^\times)^n$ a finite subgroup. Let $L = K(\sqrt[n]{\Delta})$, and $\Delta' = \frac{(L^\times)^n \cap K^\times}{(K^\times)^n}$, and we must show that $\Delta' = \Delta$.

Clearly $\Delta \subseteq \Delta'$. We then compute sizes.

$L = K(\sqrt[n]{\Delta}) \subset K(\sqrt[n]{\Delta'}) \subset L$, and we have equality throughout. So $K(\sqrt[n]{\Delta}) = K(\sqrt[n]{\Delta'})$.

11.1 gives $|\Delta| = |\Delta'|$, and so $\Delta = \Delta'$. □

Proposition 11.3. *Let K be a number field containing μ_n . Let S be a finite set of primes of K . Then there are only finitely many extensions L/K such that:*

1. L/K is abelian of exponent dividing n .
2. L/K is unramified at all primes outside S .

Proof. **11.2** gives us $L = K(\sqrt[n]{\Delta})$ for some $\Delta \in K^\times / (K^\times)^n$ a finite subgroup. Let \mathfrak{p} be a prime of K . Then $\mathfrak{p}\mathcal{O}_L = \mathcal{P}_1^{e_1} \dots \mathcal{P}_r^{e_r}$, where \mathcal{P}_i are primes in \mathcal{O}_L .

If $x \in K^\times$ represents an element of Δ . If $x \in K^\times$ represents an element of Δ , then

$$nv_{\mathcal{P}_i}(\sqrt[n]{x}) = v_{\mathcal{P}_i}(x) = e_i v_{\mathfrak{p}}(x)$$

If $\mathfrak{p} \notin S$, then all $e_i = 1$, so $v_{\mathfrak{p}}(x) \equiv 0 \pmod n$, and so $\Delta \subset K(S, n)$, where

$$K(S, n) := \{x \in K^\times / (K^\times)^n : v_{\mathfrak{p}}(x) \equiv 0 \pmod n \ \forall \mathfrak{p} \in S\}$$

The proof is completed by the following lemma. □

Lemma 11.4. $K(S, n)$ is finite.

Proof. The map $K(S, n) \rightarrow (\mathbb{Z}/n\mathbb{Z})^{|S|}$, $x \mapsto (v_p(x) \bmod n)$ for $p \in S$ is group homomorphism with kernel $K(\emptyset, n)$.

Since $|S| < \infty$, it suffices to prove the lemma with $S = \emptyset$.

If $x \in K^\times$ represents an element of $K(\emptyset, n)$, then $(x) = \mathfrak{a}^n$ for some ideal \mathfrak{a} .

There is then an exact sequence:

$$0 \longrightarrow \frac{\mathcal{O}_K^\times}{(\mathcal{O}_K^\times)^n} \longrightarrow K(\emptyset, n) \longrightarrow Cl_K[n] \longrightarrow 0$$

$$x(K^\times)^n \longmapsto [\mathfrak{a}]$$

Now $|Cl_K| < \infty$ and \mathcal{O}_K^\times is finitely generated by Dirichlet's unit theorem, so $K(\emptyset, n)$ is finite. \square

12 Elliptic Curves over Number Fields II

12.1 The Mordell-Weil Theorem

Lemma 12.1. Let E/K be an elliptic curve. Let L/K be a finite Galois extension. Then the map

$$E(K)/nE(K) \rightarrow E(L)/nE(L)$$

has finite kernel.

Proof. For each element in the kernel, we pick a coset representative $P \in E(K)$, and then $Q \in E(L)$ with $nQ = P$.

Note that, for any $\sigma \in \text{Gal}(L/K)$, $n(\sigma(Q) - Q) = \sigma(P) - P = 0$.

$\text{Gal}(L/K)$ is finite and $E[n]$ is finite, so there are only finitely many possibilities for the map $\text{Gal}(L/K) \rightarrow E[n]$, $\sigma \mapsto \sigma Q - Q$.

But if $P_1, P_2 \in E(K)$ then $P_i = nQ_i$ for $Q_1, Q_2 \in E(L)$, and $\sigma Q_1 - Q_2 = \sigma Q_2 - Q_2$, and so $\sigma(Q_1 - Q_2) = Q_1 - Q_2$ for all $\sigma \in \text{Gal}(L/K)$.

But then $Q_1 - Q_2 \in E(K)$, and so $P_1 - P_2 \in nE(K)$. \square

Theorem 12.2 (Weak Mordell-Weil). If K is a number field and E/K is an elliptic curve, with $n \geq 2$ an integer, then $E(K)/nE(K)$ is finite.

Proof. 12.1 tells us we may replace K by a finite Galois extension. So without loss of generality, $\mu_n \subset K$ and $E[n] \subset E(K)$.

Let $S = \{p|n\} \cup \{\text{primes of bad reduction for } E/K\}$.

For each $P \in E(K)$, the extension $K([n]^{-1}P)/K$ is unramified outside S , by 9.9.

Let $Q \in [n]^{-1}P$. Since $E[n] \subseteq E(K)$, $K(Q) = K([n]^{-1}P)$, and this is a Galois extension of K .

Then we claim that we have an injective group homomorphism:

$$\begin{aligned} \text{Gal}(K(Q)/K) &\hookrightarrow E[n] \\ \sigma &\mapsto \sigma Q - Q \end{aligned}$$

This is a group homomorphism as $\sigma\tau Q - Q = \sigma(\tau Q - Q) + (\sigma Q - Q)$. But $\tau Q - Q$ is an n -torsion point, so lies in K , so this is $\tau Q - Q + \sigma Q - Q$.

It is injective, as if $\sigma Q = Q$ then σ fixes $K(Q)$, and so σ is the identity.

So $K(Q)/K$ is an abelian extension of exponent dividing n , unramified outside of S , so 11.3 tells us there are only finitely many possibilities for $K(Q)/K$ as we vary P .

Let L be the composite of all such extensions of K . Then L/K is a finite Galois extension, and

$$E(K)/nE(K) \rightarrow E(L)/nE(L)$$

is the zero map.

12.1 implies that this has finite kernel, and so $|E(K)/nE(K)| < \infty$. \square

If $K = \mathbb{R}$ or \mathbb{C} or $[K : \mathbb{Q}_p] < \infty$, then $|E(K)/nE(K)| < \infty$, yet $E(K)$ is uncountable. Hence $E(K)$ is not finitely generated.

It is a fact that there is a quadratic form called the *canonical height* $\widehat{h} : E(K) \rightarrow \mathbb{R}_{\geq 0}$ with the property that, for any $B \geq 0$, the set $\{P \in E(K) : \widehat{h}(P) \leq B\}$ is finite.

Theorem 12.3 (Mordell-Weil). *Let K be a number field with E/K an elliptic curve. Then $E(K)$ is a finitely generated abelian group.*

Proof. Fix any integer $n \geq 2$. Weak Mordell-Weil gives $|E(K)/nE(K)| < \infty$. Pick coset representatives for $E(K)/nE(K)$, say P_1, \dots, P_m .

Then let $\Sigma = \{P \in E(K) : \widehat{h}(P) \leq \max_{1 \leq i \leq m} \widehat{h}(P_i)\}$.

We claim Σ generates $E(K)$.

If not, then there is $P \in E(K) \setminus \{\text{subgroup generated by } \Sigma\}$ of minimal point (this exists because there are only finitely many “small points that are too big”).

Then $P = P_i + nQ$ for some $1 \leq i \leq m$, and $Q \in E(K)$.

Note that $Q \in E(K) \setminus \langle \Sigma \rangle$. The minimal choice of P tells us that $4\widehat{h}(P) \leq 4\widehat{h}(Q) \leq n^2\widehat{h}(Q) = \widehat{h}(nQ) = \widehat{h}(P - P_i) \leq \widehat{h}(P - P_i) + \widehat{h}(P + P_i) = 2\widehat{h}(P) + 2\widehat{h}(P_i)$. Hence $\widehat{h}(P) \leq \widehat{h}(P_i)$, and so $P \in \Sigma'_i$.

Hence Σ generates $E(K)$, and is finite, so $E(K)$ is finitely generated. \square