Elliptic Curves

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1 Fermat's Method of Infinite Descent

Suppose we have a right-angled triangle Δ with side lengths a, b, c, so that by Pythagoras we have $a^2 + b^2 = c^2$, and area$(\Delta) = \frac{1}{2}ab$.

Definition 1.1. *Δ is* **rational** *if $a, b, c \in \mathbb{Q}$, and* **primitive** *if $a, b, c \in \mathbb{Z}$ coprime.*

Lemma 1.2. *Every primitive triangle is of the form $a = u^2 - v^2, b = 2uv, c = u^2 + v^2$ for coprime integers $u > v > 0$.*

Proof. If a, b were both odd, then $a^2 + b^2 \equiv 2 \mod 4$, and we have no solutions for c. If a, b both even, then they are not coprime. So we may assume a is odd, b is even, c is odd.

Then $(\frac{b}{2})^2 = \frac{c+a}{2}\frac{c-a}{2}$, and the right hand side is a product of coprime positive integers. So by unique prime factorisation in the integers, $\frac{c+a}{2} = u^2$, $\frac{c-a}{2} = v^2$ for some coprime integers u, v. Rearranging, we have the lemma. □

Definition 1.3. *$D \in \mathbb{Q}_{>0}$ is a* **congruent number** *if it is the area of a rational triangle.*

Note that, by scaling the triangle, it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

For example, $D = 5, 6$ are congruent numbers. $6 = \frac{1}{2} \cdot 3 \cdot 4$, and $3^2 + 4^2 = 5^2$, and 5 is left as an exercise.

Lemma 1.4. *$D \in \mathbb{Q}_{>0}$ is congruent if and only if $Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}$, $y \neq 0$.*

Proof. Lemma **1.2** shows that D is congruent if and only if $Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}$, $w \neq 0$.

Setting $x = \frac{u}{v}$, $y = \frac{w}{v^2}$ finishes the proof. □

Fermat showed that 1 is not a congruent number.

Theorem 1.5. *There is no solution to*

$$w^2 = uv(u+v)(u-v) \tag{$*$}$$

in integers u, v, w with $w \neq 0$.

Proof. Without loss of generality, u, v are coprime with $u > 0, w > 0$. If $v < 0$ then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2})$. So we may assume that all of $u, v, u+v, u-v$ are coprime positive integers whose product is a square, and hence are all squares, say a^2, b^2, c^2, d^2 respectively, where $a, b, c, d \in \mathbb{Z}_{>0}$.

Since $u \not\equiv v \mod 2$, both c, d are odd. Consider the right angled triangle with side lengths, $\frac{c+d}{2}, \frac{c-d}{2}, a$. This is a primitive triangle, and it has area $\frac{c^2-d^2}{8} = \frac{v}{4} = (\frac{b}{2})^2$.

Let $w_1 = \frac{b}{2}$. Then lemma **1.2** gives $w_1^2 = u_1 v_1 (u_1^2 - v_1^2)$ for some $u_1, v_1 \in \mathbb{Z}$, giving a new solution to $(*)$. But $4w_1^2 = b^2 = v | w^2$, and so $w_1 \leq \frac{1}{2}w$.

So by Fermat's method of infinite descent, if there were a solution we would have a strictly decreasing infinite sequence of positive integers $\lightning$. Hence there is no solution to $(*)$. □

1.1 A Variant for Polynomials

Here, K is a field with char $K \neq 2$. The algebraic closure of K will be $\overline{K}$.

Lemma 1.6. *Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for four distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.*

Proof. Without loss of generality we may assume $K = \overline{K}$, as that doesn't change the degree of polynomials, and every square is still a square.

Changing coordinates on $\mathbb{P}^1$, we may assume the ratios $\alpha : \beta$ are $(1:0), (0:1), (1:-1), (1:-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$, with $\mu = \sqrt{\lambda}$.

Then $u = a^2, v = b^2, u - v = (a+b)(a-b), u - \lambda v = (a + \mu b)(a - \mu b)$ are all squares. They are also coprime, and so by unique factorisation in $K[t]$, $(a+b), (a-b), (a+\mu b), (a - \mu b)$ are all squares.

But $\max\{\deg a, \deg b\} \leq \frac{1}{2} \max\{\deg u, \deg v\}$. So by Fermat's method of infinite descent, we get that the original $u, v \in K$. □

Now we have some important definitions:

Definition 1.7.

1. *An* **elliptic curve** *E over a field K is the projective closure of the affine curve $y^2 = f(x)$ where $f \in K[x]$ is a monic cubic polynomial with distinct roots.*
2. *For L/K any field extension, $E(L) = \{(x, y) \in L^2 : y^2 = f(x)\} \cup \{0\}$. 0 is called the* **point at infinity**.

We call the point at infinity 0 because we will see that $E(L)$ is naturally an abelian group under an operation we will denote by $+$, and 0 will be the identity for that group. In this course we will study $E(L)$ for L a finite field, a local field, and a number field.

Lemma **1.4** and theorem **1.5** together imply that, if E is given by $y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0,0), (\pm 1, 0)\}$, which we will see is the group $C_2 \times C_2$.

Corollary 1.8. *Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.*

Proof. Without loss of generality, $K = \overline{K}$. By a change of coordinates we may assume $E : y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ with $u, v \in K[t]$ coprime. Then $w^2 = uv(u-v)(u - \lambda v)$ for some $w \in K[t]$.

Unique factorisation in $K[t]$ gives $u, v, u - v, u - \lambda v$ are all squares, and so by lemma **1.6**, $u, v \in K$, and so $x, y \in K$. □

2 Some Remarks on Algebraic Curves

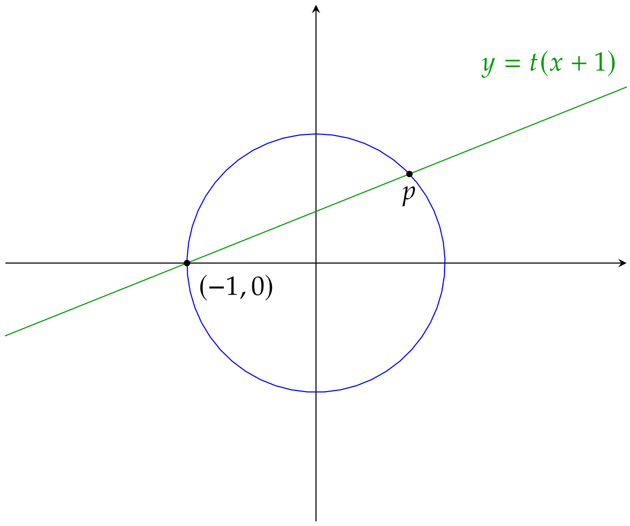
We will be working over an algebraically closed field K.

Definition 2.1. *An (irreducible) plane algebraic curve $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$ is* **rational** *if it has a rational parametrization, i.e. there are $\phi, \psi \in K(t)$ such that:*

1. *$\mathbb{A}^1 \to \mathbb{A}^2; t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{$finite set$\}$.*
2. *$f(\phi(t), \psi(t)) = 0$.*

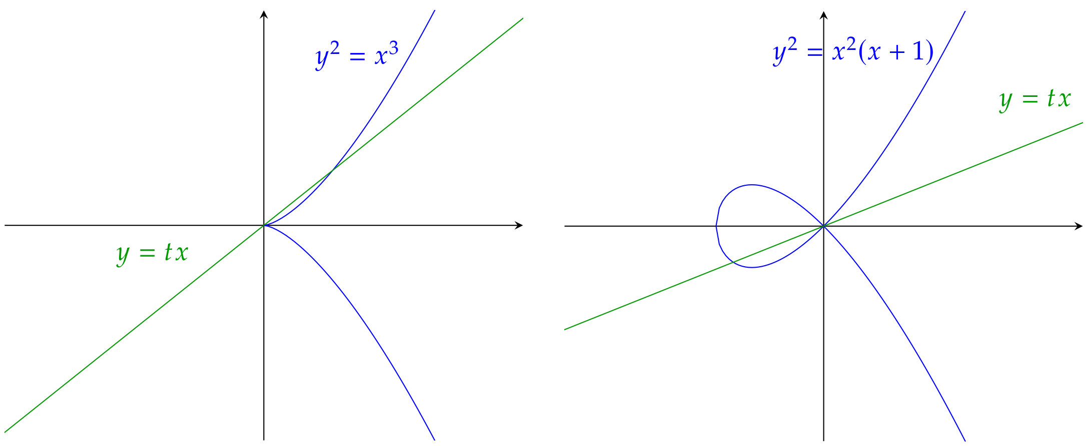
Examples 2.2.

1. Any nonsingular plane conic is rational. For example, take a circle $x^2 + y^2 = 1$. Pick a point on it, $(-1, 0)$. Now draw a line through it with slope t, and solve for the points of intersection between the curve and the line.



Solving for the coordinates of p, we get the quadratic $x^2 + t^2(x+1)^2 = 1$, i.e. $x = -1$ or $\frac{1-t^2}{1+t^2}$. So we have the rational parametrization $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

2. Any singular plane cubic is rational.



(a) Rational Parametrization $(x, y) = (t^2, t^3)$

(b) Left as an example on the first sheet

3. Corollary **1.8** shows that elliptic curves are *not* rational.

Definition 2.3. *The* **genus** $g(C) \in \mathbb{Z}_{\geq 0}$ *is an invariant of a smooth projective curve.*

- *If* $K = \mathbb{C}$, *then* $g(C) =$ *genus of the Riemann surface* C.

- *A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.*

Proposition 2.4. *Let C be a smooth projective curve over K, an algebraically closed field. Then:*

1. *C is rational $\iff g(C) = 0$.*
2. *C is an elliptic curve $\iff g(C) = 1$.*

Proof. A proof of *1* is omitted from this course. For 2, we check (on the first example sheet) that elliptic curves are smooth plane curves. Then they have degree 3, so genus $\frac{2 \cdot 1}{2} = 1$. For the other direction, see later on in the course. □

2.1 Order of Vanishing

C will be an algebraic curve, and $K(C)$ its function field, with $P \in C$ a smooth point. Write $\mathrm{ord}_P(f)$ to mean the order of vanishing of $f \in K(C)$ at P (negative if f has a pole).

Fact: $\mathrm{ord}_P : K(C)^\times \to \mathbb{Z}$ is a discrete valuation, i.e. $\mathrm{ord}_P(f_1 f_2) = \mathrm{ord}_P(f_1) + \mathrm{ord}_P(f_2)$ and $\mathrm{ord}_P(f_1 + f_2) \geq \min\{\mathrm{ord}_P(f_1), \mathrm{ord}_P(f_2)\}$.

We say $t \in K(C)^\times$ is a ***uniformizer*** at the point P if $\mathrm{ord}_P(t) = 1$.

Example 2.5. Let $C = \{g(x, y) = 0\} \subseteq \mathbb{A}^2$, where $g \in K[x, y]$ is irreducible. Then $K(C) = \mathrm{Frac}\,\frac{K[x,y]}{(g)}$, with $g = g_0 + g_1(x, y) + g_2(x, y) + \ldots$, g_i homogeneous of degree i.

Suppose $P = (0, 0) \in C$ is a smooth point, i.e. $g_0 = 0, g_1(x, y) = \alpha x + \beta y$ with α, β not both zero.

Let $\gamma, \delta \in K$. It is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\frac{\gamma}{\delta} \neq \frac{\alpha}{\beta}$, i.e. $\alpha\delta - \beta\gamma \neq 0$.

Example 2.6. $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2, \lambda \neq 0, 1$. We take the projective closure, i.e. homogenize the equation as $\{Y^2 Z = X(X - Z)(X - \lambda Z)\} \subset \mathbb{P}^2$ by setting $x = X/Z, y = Y/Z$.

Have we got new points by taking projective closure? We only get these when $Z = 0$, i.e. $0 = X^3 \implies X = 0, Y \neq 0$. Since we're in projective space, this is just one point: $P = (0 : 1 : 0)$. We compute $\mathrm{ord}_P(x)$ and $\mathrm{ord}_P(y)$. Put $t = X/Y, w = Z/Y$ (since we can't return to the original affine piece, as it doesn't contain $Z = 0$). Then we get $w = t(t - w)(t - \lambda w)$. Now P is the point $(t, w) = (0, 0)$. This is a smooth point, as there are linear terms at that point (namely w). So $\mathrm{ord}_P(t) = \mathrm{ord}_P(t - 2) = \mathrm{ord}_P(t - \lambda w) = 1$, and $\mathrm{ord}_P(w) = 1 + 1 + 1 = 3$.

Then:

$$\mathrm{ord}_P(x) = \mathrm{ord}_P(X/Z) = \mathrm{ord}_P(t/w) = 1 - 3 = -2$$
$$\mathrm{ord}_P(y) = \mathrm{ord}_P(Y/Z) = \mathrm{ord}_P(1/w) = -3$$

2.2 Riemann Roch Spaces

Let C be a smooth projective curve. Then a ***divisor*** is a formal sum of points on C, say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$, and only finitely many n_P are nonzero, and let $\deg D = \sum_{P \in C} n_P$. These divisors form a group under addition, denoted $\mathrm{Div}(C)$.

D is said to be ***effective***, written $D \geq 0$ if $n_p \geq 0$ for all $P \in C$.

If $f \in K(C)^\times$, we write $\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) P$.

The Riemann Roch space of $D \in \text{Div}(C)$ is:

$$\mathscr{L}(D) = \{f \in K(C) : \text{div}(f) + D \geq 0\} \cup \{0\}$$

i.e. the K-vector space of rational functions on C with "poles no worse than specified by D."

Theorem 2.7 (Riemann Roch for genus 1)**.**

$$\dim \mathscr{L}(D) = \begin{cases} 0 & \deg D < 0 \\ 0 \text{ } or \text{ } 1 & \deg D = 0 \\ \deg D & \deg D > 0 \end{cases}$$

Example 2.6 (revisited). Our curve is $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$, together with $P = (0:1:0)$, the point at infinity. Recall $\text{ord}_P(x) = -2, \text{ord}_P(x) = -3$.

We thus deduce that $\mathscr{L}(2P) = \langle 1, x\rangle, \mathscr{L}(3P) = \langle 1, x, y\rangle$.

Proposition 2.8. *Let K be an algebraically closed field not of characteristic 2. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic, and that $P \in C$ is a point of inflection. Then we may change coordinates such that:*

$$\begin{aligned} C : Y^2Z &= X(X-Z)(X-\lambda Z), \quad \lambda \neq 0, 1 \\ P &= (0:1:0) \end{aligned}$$

Proof. We make a change of coordinates such that $P = (0:1:0)$ and the tangent line to C at P, $T_P(C) = \{Z = 0\}$. Now let $C = \{F(X, Y, Z) = 0\}$.

Since $P \in C$ is a point of inflection, $F(t, 1, 0)$ has a triple root at $t = 0$. But F is degree 3, so we have $F(t, 1, 0) = kt^3$ for k some constant. I.e., there are no terms in F of the form X^2Y, XY^2, Y^3.

So $F \in \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3\rangle$. The coefficient of Y^2Z is nonzero, as otherwise P would be singular. The coefficient of X^3 is also nonzero, as C is irreducible and otherwise $\{Z = 0\} \subset C$.

We are free to rescale X, Y, Z, F, and so WLOG C is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

We call this Weierstrass form.

Since our field doesn't have characteristic 2, we may complete the square by substituting $Y = Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$.

Now $C : Y^2Z = Z^3f(X/Z)$, where f is a monic cubic polynomial. Since C is smooth, f has distinct roots, which are WLOG $0, 1, \lambda$. So

$$C : Y^2Z = X(X-Z)(X-\lambda Z)$$

which we call the Legendre form. $\square$

It may be shown that the points of inflection on $C = \{F = 0\} \subset \mathbb{P}^2$ are given by $F = \det\left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right) = 0$

2.3 The Degree of a Morphism

Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Let $\phi^* : K(C_2) \to K(C_1), f \mapsto f \circ \phi$.

Definition.

1. $\deg \phi = [K(C_1) : \phi^* K(C_2)]$
2. *ϕ is separable if $K(C_1)/\phi^* K(C_2)$ is a separable field extension (which by Galois theory is automatic if* $\operatorname{char} K = 0$*)*

Suppose $P \in C_1, Q \in C_2, \phi : P \to Q$. Let $t \in K(C_2)$ be a uniformizer at Q. We then define $e_\phi(p) = \operatorname{ord}_P(\phi^* t)$, which is always ≥ 1, and independent of t. $e_\phi(P)$ is called the ***ramification index*** of ϕ at p.

Theorem 2.9. *Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then*

$$\sum_{p \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi$$

for any point $Q \in C_2$. Moreover, if ϕ is separable then $e_\phi(P) = 1$ with at most finitely many exceptions.

In particular:

1. *ϕ is surjective*
2. *If ϕ is separable, $\#\phi^{-1}(Q) \leq \deg \phi$, with equality for all but finitely many choices of Q.*

Remark 2.10. Let C be an algebraic curve. A rational map is given by $\phi : C \dashrightarrow \mathbb{P}^n, P \mapsto (f_0(P) : \ldots : f_n(P))$, where $f_0, \ldots, f_n \in K(C)$ are not all zero. If C is smooth then ϕ is a morphism.

3 Weierstrass Equations

In this section, K is a perfect field (so that all finite extensions of K are separable), with algebraic closure $\bar{K}$.

Definition. An elliptic curve E over K is a smooth projective curve of genus 1 defined over K with a specified K-rational point O_E.

Example: Take $\{X^3 + pY^3 + p^2 Z^3 = 0\} \subset \mathbb{P}^2$ for p prime. This is not an elliptic curve over $\mathbb{Q}$ since there is no $\mathbb{Q}$-points.

Theorem 3.1. *Every elliptic curve E is isomorphic over K to a curve in Weierstrass form via an isomorphism taking O_E to* $(0 : 1 : 0)$.

Proposition **2.8** treated the special case where E is a smooth plane cubic and O_E is a point of inflection.

If $D \in \operatorname{Div}(E)$ is defined over K (i.e. fixed by the natural action of $\operatorname{Gal}(\bar{K}/K)$, then $\mathscr{L}(D)$ has a basis in $K(E)$, not just in $\bar{K}(E)$).

Proof. Note that

$$\mathscr{L}(2O_E) \subset \mathscr{L}(3O_E)$$

Pick bases of these spaces, say $\{1, x\}$ and $\{1, x, y\}$.

Note that $\mathrm{ord}_{O_E}(x) = -2, \mathrm{ord}_{O_E}(y) = -3$. The 7 elements $\{1, x, y, x^2, xy, x^3, y^2\}$ are rational functions with no pole except at O_E, where they have poles of degree at most 6, so they all lie in $\mathscr{L}(6O_E)$. Riemann-Roch tells us this space has dimension 6, so there is a dependence relation between these elements.

Leaving out x^3 or y^2 gives a basis for $\mathscr{L}(6O_E)$ since each term has a different order pole at O_E, so they are independent.

Therefore this dependence relation *must* involve both x^3 and y^2. Rescaling x, y we get

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Let E' be the curve defined by this equation (or rather its projective closure).

There is a morphism

$$\begin{aligned} \phi : E &\to E' \\ P &\mapsto (x(P) : y(P) : 1) = \left(\frac{x}{y}(P) : 1 : \frac{1}{y}(P)\right) \\ O_E &\mapsto (0 : 1 : 0) \end{aligned}$$

$[K(E) : K(x)] = \deg(E \xrightarrow{x} \mathbb{P}^1) = \mathrm{ord}_{O_E}\left(\frac{1}{x}\right) = 2$
$[K(E) : K(y)] = \deg(E \xrightarrow{y} \mathbb{P}^1) = \mathrm{ord}_{O_E}\left(\frac{1}{y}\right) = 3$

This gives us a diagram of field extensions

$$\begin{tikzcd}
& & K(E) \arrow[dll, dash, "2"'] \arrow[d, dash] \arrow[drr, dash, "3"] & & \\
& & K(x,y) \arrow[dll, dash] \arrow[drr, dash] & & \\
K(x) & & & & K(y)
\end{tikzcd}$$

So $[K(E) : K(x, y)]$ divides both 2 and 3 by the tower law, and hence $K(E) = K(x, y)$, and hence $\deg(E \xrightarrow{\phi} E') = 1$, and ϕ is birational. If E' is singular, then it is rational, and so E is also rational $\lightning$. So E' is not singular and hence smooth, and we may use remark **2.10** to ϕ^{-1} to see that ϕ^{-1} is a morphism, and hence ϕ is an isomorphism. □

Proposition 3.2. *Let E, E' be elliptic curves over K in Weierstrass form. Then $E \cong E'$ over K if and only if the Weierstrass equations are related by a change of variables of the form*

$$\begin{aligned} x &= u^2 x' + r \\ y &= u^3 y' + u^2 s x' + t \end{aligned}$$

for $u, r, s, t \in K, u \neq 0$.

Proof. Using the notation of the previous proof,

$$\begin{aligned}
\langle 1, x\rangle =& \mathscr{L}(2O_E) = \langle 1, x'\rangle \\
\langle 1, x, y\rangle =& \mathscr{L}(3O_E) = \langle 1, x', y'\rangle \\
\implies & \begin{cases} x = \lambda x' + r & \lambda_1 r \in K, \lambda \neq 0 \\ y = \mu y' + \sigma x' + t & \mu, \sigma, t \in K, \mu \neq 0 \end{cases}
\end{aligned}$$

Looking at the coefficients of x^3 and y^2, $\lambda^3 = \mu^2 \implies (\lambda, \mu) = (u^2, u^3)$ for $u \in K^\times$.

Put $s = \sigma/u^2$ □

The effect of this transformation on the coefficients a_i is on the formula sheet for this course. A Weierstrass equation defines an elliptic curve if and only if defines a smooth curve, if and only if $\Delta(a_1, \dots, a_6) \neq 0$ where Δ is as follows:

$$\begin{aligned}
b_2 &:= a_1^2 + 4a_2 \\
b_4 &:= 2a_4 + a_1 a_3 \\
b_6 &:= a_3^2 + 4a_6 \\
b_8 &:= a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2 \\
\Delta &:= -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6
\end{aligned}$$

If $\operatorname{char} K \neq 2, 3$, then we can reduce to the case

$$\begin{aligned}
E : y^2 &= x^3 + ax + b \\
\Delta &= -16(4a^3 + 26b^2)
\end{aligned}$$

Corollary 3.3. *Assume* $\operatorname{char} K \neq 2, 3$. *If we have two elliptic curves*

$$\begin{aligned}
E : y^2 &= x^3 + ax + b \\
E' : y^2 &= x^3 + a'x + b'
\end{aligned}$$

then they are isomorphic over K if and only if

$$\begin{aligned}
a' &= u^4 a \\
b' &= u^6 b
\end{aligned}$$

for some $u \in K^\times$.

Proof. E and E' are related as in **3.2** with $r = s = t = 0$. □

Definition. The ***j-invariant*** is $j(E) = \frac{1728(4a^3)}{4a^3+27b^2}$. Note that the denominator is nonzero since the discriminant is nonzero.

Corollary 3.4. $E \cong E' \implies j(E) = j(E')$, *and the converse holds if $K = \bar{K}$.*

Proof.

$$\begin{aligned} E \cong E' &\iff a' = u^4 a; b' = u^6 b \text{ for some } u \in K^\times \\ &\implies (a^3 : b^2) = ((a')^3 : (b')^2) \\ &\iff j(E) = j(E') \end{aligned}$$

and the reverse implication holds in the second line if $K = \bar{K}$. □

4 Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic, and $O_E \in E(K)$. Since E is of degree 3, it meets each line in 3 points counted with multiplicity. Hence, given two points P, Q on E, the line $\overline{PQ}$ meets E at a third point S. Then the line $\overline{O_E S}$ meets E at a third point R. We then define $P \oplus Q = R$.

If $P = Q$, then we take the tangent line at P, likewise if $S = O_E$. We can view this diagrammatically as follows:

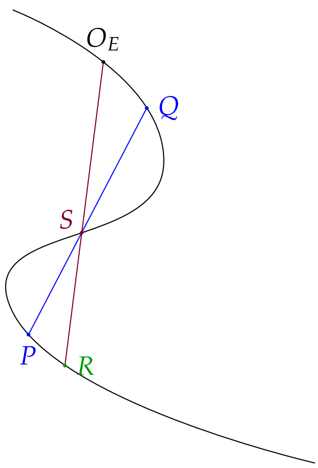


Figure 2: Illustration of the group operation on an elliptic curve

We call this the “chord and tangent process".

Theorem 4.1. $(E, \oplus)$ *is an abelian group.*

Proof.

(i) $P \oplus Q = Q \oplus P$ by construction.

(ii) O_E is the identity.

(iii) For inverses, let S be the third point of intersection of T_{O_E} and E, and Q be the third point of intersection of $\overline{PS}$ and E. Then $P \oplus Q = O_E$.

(iv) Associativity is much harder.

□

Definition. $D_1, D_2 \in Div(E)$ are ***linearly equivalent*** (written $D_1 \sim D_2$) if there is $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) = D_1 - D_2$. Then we will let $[D] = \{D' : D' \sim D\}$.

Definition. The ***Picard group of E***, $\operatorname{Pic}(E) = \operatorname{Div}(E)/\sim$. We write $\operatorname{Div}^0(E) := \ker\left(\operatorname{Div}(E) \xrightarrow{\deg} \mathbb{Z}\right)$ for the group of degree 0 divisors on E, and then $\operatorname{Pic}^0(E) = \operatorname{Div}^0(E)/\sim$. Sometimes Pic^0 is called the Jacobian.

Proposition 4.2. *Let $\psi : E \to \operatorname{Pic}^0(E); P \mapsto [(P) - (O_E)]$. Then:*

1. $\psi(P \oplus Q) = \psi(P) + \psi(Q)$
2. *ψ is a bijection*

Proof.

1. Referring back to Fig. 2, let $\{\ell = 0\}$ be the line $\overline{PQ}$, and $\{m = 0\}$ be the line $\overline{O_E R}$. Then:

$$\begin{aligned}
\operatorname{div}(\ell/m) &= (P) + (S) + (Q) - (R) - (S) - (O_E) \\
&= (P) + (Q) - (O_E) - (P \oplus Q) \\
\implies (P \oplus Q) + (O_E) &\sim (P) + (Q) \\
\implies (P \oplus Q) - (O_E) &\sim (P) - (O_E) + (Q) - (O_E) \\
\implies \psi(P \oplus Q) &= \psi(P) + \psi(Q)
\end{aligned}$$

2. For injectivity, suppose $\psi(P) = \psi(Q)$. Then there is $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) = P - Q$. Then $\deg\left(E \xrightarrow{f} \mathbb{P}^1\right) = \operatorname{ord}_P(f) = 1$. But then f is a birational morphism, so an isomorphism, and $E \cong \mathbb{P}^1 \lightning$.

   For surjectivity, let $[D] \in \operatorname{Pic}^0(E)$. Then $D + (O_E)$ has degree 1 (as D had degree 0). Then Riemann-Roch tells us $\dim \mathscr{L}(D + (O_E)) = 1$, and so there exists some $f \in \bar{K}(E)^\times$ such that $\operatorname{div}(f) + D + (O_E) \geq 0$. Since f is rational, $\deg \operatorname{div}(f) = 0$, and $\deg D = 0$. So the coefficients of $\operatorname{div}(f) + D + (O_E)$ are non-negative and sum to 1, hence one of them is 1 and the rest are 0. So $\operatorname{div}(f) + D + (O_E) = (P)$ for some $P \in E$. But then $(P) - (O_E) \sim D$, i.e. $\psi(P) = [D]$.

□

So ψ is a bijection respecting the group law, and so we deduce that $\oplus$ is associative, and then $(E, \oplus) \overset{\psi}{\cong} (\operatorname{Pic}^0 E, +)$.

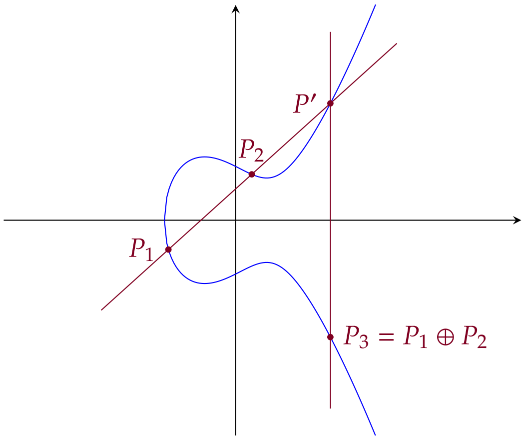
4.1 Explicit Formulae for the Group Law

We consider E in Weierstrass form, with O_E the point at infinity:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \qquad (*)$$

Note that O_E is a point of inflection. Now $P_1 \oplus P_2 \oplus P_3 = O_E \iff P_1, P_2, P_3$ are collinear.

We will use the following notation:



and put $P_i = (x_i, y_i)$, $P' = (x', y')$.

Now $\ominus P_1 = (x_1, -(a_1x_1 + a_3) - y_1)$, just by setting $y = -y_1$ in $(*)$.

The line through P_1, P_2 has equation say $y = \lambda x + \nu$. Substituting into $(*)$ and looking at the coefficient of x^2, we get:

$$\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x'$$

Since $x_3 = x'$, we have:

$$\begin{aligned} x_3 &= \lambda^2 + a_1\lambda - a_2 - x_1 - x_2 \\ y_3 &= -(a_1x' + a_3) - y' \\ &= -(\lambda + a_1)x_3 - \nu - a_3 \end{aligned}$$

It remains to find λ and ν. There are 3 cases:

1. $x_1 = x_2, P_1 \neq P_2$.

   Then $P_1 \oplus P_2 = O_E$.

2. $x_1 \neq x_2$.

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \quad \nu = y_1 - \lambda x_1 = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

3. $P_1 = P_2$.

   Here we have to compute the equation of the tangent line etc. The solutions are:

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}, \quad \nu = \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}$$

Corollary 4.3. *$E(K)$ is an abelian group.*

Proof. It is a subgroup of E $(= E(\bar{K}))$.

Identity: $O_E \in E(K)$ by definition.

$$\begin{aligned} \text{Closure:} &\quad \text{See formulae above.} \\ \text{Inverses:} &\quad \text{See formulae above.} \\ \text{Associativity:} &\quad \text{Inherited from } E(\bar{K}). \\ \text{Commutativity:} &\quad \text{Inherited from } E(\bar{K}). \end{aligned}$$

□

If there is no ambiguity (i.e. we are not also adding numbers at the same time), the circles will be dropped from the group operation.

Theorem 4.4. *Elliptic curves are group varieties.*

i.e., $[-1] : E \longrightarrow E; P \mapsto -P$ *and* $+ : E \times E \longrightarrow E; (P, Q) \mapsto P + Q$ *are morphisms of algebraic varieties.*

Proof. The above formulae show that $[-1]$ and $+$ are rational maps. We know immediately that $[-1]$ is a morphism, as it is a rational map from a smooth curve to a projective variety.

The formulae also show that $+$ is regular on the set

$$U = \{(P, Q) \in E \times E \mid P, Q, P + Q, P - Q \neq O_E\}$$

For $P \in E$, let $\tau_P : E \longrightarrow E; X \mapsto P + X$ be the "translation by P" map.

Then τ_P is a rational map from a smooth curve to a projective variety, so is a morphism.

We factor $+$ as:

$$E \times E \xrightarrow[\tau_{-A} \times \tau_{-B}]{} E \times E \xrightarrow[\tau_{A+B}]{} E \xrightarrow[\tau_{A+B}]{} E$$

Now $+$ is regular on $(\tau_A \times \tau_B)(U)$ for all $A, B \in E$, and so $+$ is regular on $E \times E$. □

Definition. For any $n \in \mathbb{Z}_{>0}$, let $[n] : E \longrightarrow E; P \mapsto P + \ldots + P$, n times, and $[-n] = [-1] \circ [n]$, $[0] : P \mapsto O_E$ (i.e., the standard way of turning an abelian group into $\mathbb{Z}$ module).

Definition. The ***n*-torsion** subgroup of E is $E[n] = \ker([n] : E \longrightarrow E)$.

Lemma 4.5. *If* $\mathrm{char}(K) \neq 2$, *and* $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$.

Then $E[2] = (0, (e_1, 0), (e_2, 0), (e_3 0)) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. Let $P = (x, y) \in E$. Then $[2]P = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. □

4.2 Elliptic Curves over $\mathbb{C}$

Let $\Lambda = \{a\omega_1 + b\omega_2 : a, b \in \mathbb{Z}\}$, where ω_1, ω_2 form a basis for $\mathbb{C}$ over $\mathbb{R}$.

Then the meromorphic functions on the Riemann surface (or lattice) $\mathbb{C}/\Lambda$ are the same as the Λ-invariant meromorphic functions on $\mathbb{C}$ (i.e. $f(z) = f(z + \lambda)$ for $\lambda \in \Lambda$).

This set of functions is a field, and is generated by $\wp(z)$ and $\wp'(z)$, where:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

They satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, for some $g_1, g_3 \in \mathbb{C}$ depending on λ. We call $\wp$ the ***Weierstrass p-function***.

One can show that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where E is the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. This is an isomorphism, not only of Riemann surfaces, but moreover of groups

Theorem 4.6 (Uniformisation Theorem). *Every elliptic curve over $\mathbb{C}$ arises in this way.*

Thus, for elliptic curves $E/\mathbb{C}$, we have:

1 $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$

2 $\deg[n] = n^2$

We will show that 2 holds over any field K, and 1 holds if $\operatorname{char} K \nmid n$.

Summary of Results (N.B. the isomorphisms in 1, 2, 4 respect the relevant topologies)

1. $K = \mathbb{C}$	$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$
2. $K = \mathbb{R}$	$E(\mathbb{R}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \Delta < 0 \end{cases}$
3. $K = \mathbb{F}_q$	$\lvert \#E(\mathbb{F}_q) - (q+1) \rvert \leq 2\sqrt{q}$
4. $[K : \mathbb{Q}_p] < \infty$	$E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$
5. $[K : \mathbb{Q}] < \infty$	$E(K)$ is a finitely generated abelian group.

5 Isogenies

Let E_1, E_2 be elliptic curves.

Definition. An ***isogeny*** $\phi : E_1 \to E_2$ is a non-constant morphism taking O_{E_1} to O_{E_2}, and we say E_1 and E_2 are ***isogenous*** if there is an isogeny $E_1 \to E_2$.

Definition. $\operatorname{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \to E_2\} \cup \{0\}$. This is a group under $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$ are isogenies, then $\psi\phi$ is an isogeny. The tower law tells us that $\deg(\psi\phi) = \deg(\phi)\deg(\psi)$.

Lemma 5.1. *If $0 \neq n \in \mathbb{Z}$, then $[n] : E \to E$ is an isogeny.*

Proof. Theorem **4.4** tells us that $[n]$ is a morphism. We must show that $[n] \neq 0$.

Assume $\operatorname{char} K \neq 2$, then we can use Lemma **4.5**. If $n = 2$, then $\#E[2] = 4$, and so $[2] \neq 0$.

If n is odd, then there is $0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n]$ is not the zero map.

Now $[m][n] = [m] \circ [n]$, and any $n = 2^k m$ for m odd, so $[n]$ is not the zero map for any $n \neq 0$.

If $\operatorname{char} K = 2$, then replace **4.5** with a lemma computing $E[3]$. □

Corollary. $\operatorname{Hom}(E_1, E_2)$ *is torsion-free as a $\mathbb{Z}$-module.*

Lemma 5.2. *Let $\phi : E_1 \to E_2$ be an isogeny. Then $\phi(P+Q) = \phi(P) + \phi(Q)$ for all $P, Q \in E_1$.*

Sketch proof. ϕ induces a map $\phi_* : \mathrm{Div}^0(E_1) \to \mathrm{Div}^0(E_2)$ given by $\sum_{P \in E_1} n_p P \mapsto \sum_{P \in E_1} n_p \phi(P)$.

Recall that, via a pullback, $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

If $f \in K(E_1)^*$, then $\phi_*(\mathrm{div}\, f) = \mathrm{div}(N_{K(E_1)/K(E_2)} f)$ - this is a fact that we'll take for granted.

So ϕ_* takes principal divisors to principal divisors. Since $\phi(O_{E_1}) = O_{E_2}$, the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "\psi_1"] & E_2 \arrow[d, "\psi_2"] \\ \mathrm{Pic}^0(E_1) \arrow[r, "\phi_*"] & \mathrm{Pic}^0(E_2) \end{tikzcd}$$

,where $\psi_1 : P \mapsto [(P) - (O_{E_1})]$, $\psi_2 : Q \mapsto [(Q) - (O_{E_2})]$.

Since ϕ_* is a group homomorphism, ϕ is also a group homomorphism. ◻

Lemma 5.3. *Let $\phi : E_1 \to E_2$ be an isogeny. Then there is a morphism ξ making the following diagram commute:*

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "x_1"] & E_2 \arrow[d, "x_2"] \\ \mathbb{P}^1 \arrow[r, "\xi"] & \mathbb{P}^1 \end{tikzcd}$$

where x_i is the x-coordinate in a Weierstrass equation for E_i.

Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ for $r, s \in K[t]$ coprime, then $\deg \phi = \deg \xi = \max(\deg r, \deg s)$.

Proof. For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 extension, since the extension is given by adjoining y_i, which satisfies a quadratic (see the Weierstrass equation). Moreover, it is Galois, as $[-1]^*$ is a non-trivial automorphism of $K(E_i)$ fixing $K(x_i)$.

Since ϕ is a group homomorphism, we have that $\phi(-P) = -\phi(P)$, i.e. $\phi \circ [-1] = [-1] \circ \phi$.

If $f \in K(x_2)$, then $[-1]^* f = f$, and $[-1]^*(\phi^* f) = \phi^*([-1]^* f) = \phi^* f$. Hence $\phi^* f$ is fixed by $[-1]$, so is in $K(x_1)$, and $K(x_2) \leq K(x_1)$.

Taking $f = x_2$, then $\phi^* x_2 \in K(x_1)$, say $\xi(x_1)$ for some rational function ξ. Then ξ is as required.

Since $[K(E_1) : K(x_1)] = [K(E_2) : K(x_2)] = 2$, we have the following diagram of field extensions:

$$\begin{tikzcd} & K(E_1) \arrow[dl, dash, "2"'] \arrow[dr, dash, "\deg\phi"] & \\ K(x_1) \arrow[dr, dash, "\deg\xi"'] & & K(x_2) \arrow[dl, dash, "2"] \\ & K(x_2) & \end{tikzcd}$$

Using the tower law, $\deg \phi = \deg \xi$. Now, $K(x_2) \hookrightarrow K(x_1)$ via $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_2)}$ for $r, s \in K[t]$ coprime.

The minimal polynomial of x_1 over $K(x_2)$ is $f(t) = r(t) - s(t)x_2 \in K(x_2)[t]$ - this is clearly a polynomial for x_1, but we need to check it's irreducible.

f is irreducible in $K[t][x_2] = K[x_2][t]$ as it is of degree 1 in x_2, so one of the factors must be constant in x_2, so divide both r and s which are coprime. Then we can use Gauss's lemma, and it is irreducible in $K(x_2)[t]$.

Hence $\deg \phi = \deg \xi = [K(x_1) : K(x_2)] = \deg(r(t) - s(t)x_2) = \max(\deg r, \deg s)$. $\square$

Lemma 5.4. $\deg[2] = 4$

Proof. Assume char $K \neq 2, 3$. Then $E : y^2 = x^3 + ax + b = f(x)$.

If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2-8xf(x)}{4f(x)} = \frac{x^4+\ldots}{4f(x)}$.

The numerator and denominator are coprime - suppose there was a common factor. Then $\exists\ \theta \in \bar{K}$ with $f(\theta) = (3\theta^2 + a)^2 = f'(\theta) = 0$, and so f has a multiple root. But E is an elliptic curve so f doesn't have multiple roots.

Hence $\deg[2] = \max(\deg x^4 + \ldots, \deg 4f(x)) = \max(4, 3) = 4$. $\square$