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Solving a set of global optimization problems by the parallel technique with uniform convergence

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Abstract In this paper, we consider solving a set of global optimization problems in parallel. The proposed novel algorithm provides uniform convergence to the set of solutions for all problems treated simultaneously. The current accuracy for each particular solution is estimated by the difference in each coordinate from the point of global decision. The main statement is given in the corresponding theorem. For the sake of illustration some computational results with hundreds of multidimensional global problems are provided.

Keywords global optimization · Peano-type space filling curves · uniform convergence · parallel computing

1 Introduction

This paper considers global optimization problems of the following form:

$$\varphi(y^*) = \min\{\varphi(y) : y \in D\}, \qquad (1)$$
$$D = \left\{y \in R^N : a_i \le y_i \le b_i, 1 \le i \le N\right\}.$$

The objective function is assumed to satisfy the Lipschitz condition

$$\left|\varphi(y') - \varphi(y'')\right| \le L\left\|y' - y''\right\|,\ y', y'' \in D,\ 0 < L < \infty,$$

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with the constant L unknown a priori. The analytical form of the function may be unknown, i.e. the objective function may be defined by an algorithm for computing the function values within the search domain (a "black-box" function). At the same time, it is assumed that even a single computation of the objective function value may be a time-consuming operation since it involves numerical modeling when solving applied problems.

The assumption regarding the fulfillment of the Lipschitz condition for the objective function is typical of many approaches (see, for example [1]–[10]). In recent years, the theory and the methods for parallel Lipschitz global optimizations have been extensively developed. Parallel implementations have been proposed for many well-known algorithms [11]–[14].

Usually, the methods of global optimization (both sequential and parallel) are intended to solve a single optimization problem. If one needs to solve a series of q problems, the problems in the series are solved sequentially, one after another. Therefore, the optimum estimation in the i-th problem in the series remains undefined until all preceding problems of the series (with the indices $1, 2, ..., i-1$) have been completely solved. In the case of limited computational resources, the optima estimates in the problems $i+1, ..., q$ will not be obtained if the computation resources are exhausted while solving the i-th problem.

Situations when a series of q problems must be solved are not an extraordinary. For example, a series of scalar problems arises when one needs to find a Pareto set in solving multi-objective optimization problems. In this case, the solution of a single scalar problem corresponds to one of the Pareto optimal points of a multi-objective problem (for reference, a review of the scalarization techniques is presented in [15]). A series of optimization problems also arises when using dimension reduction methods to solve multidimensional optimization problems. It is known that a multidimensional problem can be reduced to a set of problems of lower dimension. A detailed description of various dimension reduction schemes is presented in [1,2].

When solving a set of problems, it is useful to get the estimates of the optima for all the problems at once as early as possible. This may be necessary to evaluate the expedience of continuing the search process at any point in time. It is also desirable to have the estimates of optima for all problems with the same precision. Running several independent processes, each solving one of the problems in the series on a parallel computer system, has a number of disadvantages. First, a workload imbalance between the processors will occur. If solving the i-th problem requires considerably fewer iterations of the method than solving the j-th problem, the processor tasked with handling the i-th problem would remain idle after completing the task. Second, the estimates of the optima will be obtained with different precision in different problems. Simpler problems will be solved with higher precision, whereas precision will be lower for more complex problems.

The goal of this study was to develop a method for solving a series of global optimization problems, which would be free of the drawbacks listed above. Namely, the method should ensure:

- a uniform load for all processors/cores employed;
- a uniform convergence to the solutions of all problems in the series.

Here and below uniform convergence is taken to mean proportional decreasing of the distance between the best iteration point and the global optimizer in all the problems in the series.

The proposed method is a further development of the Global Search Algorithm (GSA) and of its parallel generalization described in detail in [16]–[22] as well as in monographs [1,2]. GSA was developed in the framework of a stochastic model as a one-step optimal Bayesian method that distinguishes it from the methods optimal in the worst case (see, for example, [23]). The idea of average optimality of global optimization algorithms rests on the assumption (which is typical of applied problems) that the worst case objective function $\varphi = const$ is not very realistic.

The main part of this article is organized as follows. In Section 2, a description of the GSA is presented, and theoretical statements regarding its convergence are given. The modification of the algorithm that makes it applicable for solving a series of problems simultaneously is described in Section 3. A uniform convergence of the modified algorithm to solving a series of problems is substantiated in Section 4. Section 5 presents the results of the numerical experiments; a comparison is made of various parallelization methods between the processors when solving a series of multidimensional multiextremal test problems. Section 6 concludes the paper.

2 Global search algorithm

Let us consider a global optimization problem

$$\varphi(y^*) = \min\{\varphi(y) : y \in D\}, \tag{2}$$
$$D = \left\{y \in R^N : -2^{-1} \le y_i \le 2^{-1}, 1 \le i \le N\right\},$$

with a Lipschitzian objective function. The issues related to reducing more general problem statements to standard statement (2) were considered in [24].

Using a continuous unambiguous mapping $y(x)$ of the interval $[0,1]$ of the real axis onto hypercube D from (2) (*Peano space-filling curve*) one can reduce the multidimensional problem (2) to a one-dimensional problem

$$\varphi(y^*) = \varphi(y(x^*)) = \min\{\varphi(y(x)) : x \in [0,1]\}. \tag{3}$$

As follows from the properties of the space-filling curve [1], the reduced one-dimensional function $\varphi(y(x))$ satisfies a uniform Hölder condition

$$\left|\varphi(y(x')) - \varphi(y(x''))\right| \le H\left|x' - x''\right|^{1/N},$$

where the Hölder constant H is related to the Lipschitz constant L by the relationship

$$H = 2L\sqrt{N+3}. \tag{4}$$

Problems of numerical construction of Peano-type space filling curves and the corresponding theory are considered in detail in [1,2]. Here we will note that a numerically constructed curve (*evolvent*) is 2^{-m} accurate approximation of the theoretical Peano curve in L_∞ metric, where m is an evolvent construction parameter.

The algorithm considered here generates a series of preimages $x^k \in [0,1], k = 0,1,2,...$, in which first the images $y^k = y(x^k)$ and then the values of the objective function $z^k = \varphi(y^k), k = 0,1,2,...$, are computed. Let us call the process of computing a function value (including the construction of the image $y^k = y(x^k)$) a *trial*, and the pair (x^k, z^k) the result of the trial. The set of pairs

$$\omega_k = \left\{(x^i, z^i) : 0 \le i \le k\right\},$$

corresponding to the first k terms of the sequence of preimages x^k is the *search information* accumulated by the algorithm within k steps.

According to the global search algorithm, the first two trials are executed at the points

$$y^0 = y(0), y^1 = y(1).$$

The choice of the point $y^{k+1}, k \ge 1$, for the next $(k+1)$-th trial is defined by the following rules.

1. Renumber the inverse images of all the points from the trials already performed

$$y^0 = y(x^0), y^1 = y(x^1), ..., y^k = y(x^k) \tag{5}$$

by subscripts in the increasing order of their coordinates, i.e.

$$0 = x_0 < x_1 < \ldots < x_k = 1, \tag{6}$$

and associate these with the values $z_i = \varphi(y(x_i)), 0 \le i \le k$, computed at these points.

2. Compute the maximum absolute value of the first divided differences

$$\mu = \max_{1 \le i \le k} \frac{|z_i - z_{i-1}|}{\Delta_i}, \tag{7}$$

where $\Delta_i = (x_i - x_{i-1})^{1/N}$. If $\mu = 0$, set $\mu = 1$.

3. For each interval $(x_{i-1}, x_i),\ 1 \le i \le k$, calculate the value

$$R(i) = r\mu\Delta_i + \frac{(z_i - z_{i-1})^2}{r\mu\Delta_i} - 2(z_i + z_{i-1}) \tag{8}$$

called the *characteristic* of the interval; the real number $r > 1$ being the input parameter of the algorithm.

4. Select the interval (x_{t-1}, x_t) corresponding to the maximum characteristic

$$R(t) = \max_{1 \le i \le k} R(i). \tag{9}$$

If there are several such intervals, choose the minimum number satisfying condition (9) as t.

5. Assume

$$x^{k+1} = \frac{x_t + x_{t-1}}{2} - \mathrm{sign}(z_t - z_{t-1})\frac{1}{2r}\left[\frac{|z_t - z_{t-1}|}{\mu}\right]^N \tag{10}$$

as the inverse image for the next trial point, i.e. the next trial is to be carried out at the point $y^{k+1} = y(x^{k+1})$ from D.

Rules (5)–(10) describe the sequence of decision functions

$$x^{k+1} = G_k^r(x^0, ..., x^k, z^0, ..., z^k),\ k = 1, 2, ...,$$

generating the sequence of inverse images $x^k \in [0,1]$, $x^{k+1} \notin \left\{x^0, ..., x^k\right\}$, and also the sequence of trial points $y^k \in D$, $y^{k+1} \notin \left\{y^0, ..., y^k\right\}$. This infinite sequence of trials may be terminated by meeting the stopping condition

$$\Delta_t < \epsilon,$$

where t is from (9), and $\epsilon > 0$ is the predefined accuracy. We will use the value

$$\varphi_k^* = \min_{0 \le i \le k} \varphi(y^i)$$

and the point

$$y_k^* = \arg \min_{0 \le i \le k} \varphi(y^i)$$

as the estimate of the optimum corresponding to the k-th step of the algorithm.

The conditions for convergence of the algorithm described above are formulated in the form of the following theorem from [1].

Theorem 1 (sufficient convergence conditions)

Let the point $\overline{y}$ be a limit point of the sequence $\left\{y^k\right\}$ generated by the rules of GSA while minimizing a Lipschitzian function $\varphi(y)$, $y \in D$, with the constant L. Then:

1. If side by side with $\overline{y}$ there exists another limit point y' of the sequence $\left\{y^k\right\}$, then $\varphi(\overline{y}) = \varphi(y')$.
2. For any $z^k = \varphi(y^k) \ge \varphi(\overline{y})$.
3. If at the some step of the search process the value μ from (7) satisfies the condition
$$r\mu > 2^{3-1/N} L\sqrt{N+3}, \tag{11}$$
then $\overline{y}$ is a global minimizer of the function $\varphi(y)$ over D and any global minimizer y^* from (2) is also a limit point of the sequence $\left\{y^k\right\}$.

Remark 1. It follows from relationship (4) between the Lipschitz and Hölder constants, as well as from condition (11) of the Theorem, that for the convergence of the algorithm, parameter r should satisfy the condition

$$r > 2^{2-1/N}. \tag{12}$$

Remark 2. Substantiation of the theorem is obtained by reducing to the contradiction with condition (9), i.e. by deriving the consequence that the subsequent trial will be carried out within an interval without the maximum characteristic.

The algorithm considered above is very flexible and allows an efficient parallelization. Let us assume a multiprocessor system with p cores/processors to be at our disposal. The computational complexity of the objective function is one of the basic assumptions (see Introduction). The computations related to applying the algorithm's decision rules will play a role with negligible overhead costs. In this regard, the execution of p search trials in parallel on different cores/processors within one iteration of the method would be the most efficient parallelization scheme. Thus the obtained parallel algorithm will consist of the following.

The first p trials are executed at the points $x^0 = 0$, $x^1 = 1$ and at the arbitrary internal points $x^2, ..., x^{p-1}$ of the interval $(0, 1)$. Let us assume $n \geq 1$ iterations of the method to be completed, in the course of which the trials in $k = k(n)$ points $x^i, 0 \leq i \leq k$, have been executed. Then, the points $x^{k+1}, ..., x^{k+p}$ of the search trials for the next $(n+1)$-th iteration are determined according to the following rules.

Steps 1–3 of the parallel algorithm repeat steps 1–3 of GSA completely.

4. Arrange characteristics $R(i), 1 \leq i \leq k$, in decreasing order

$$R(t_1) \geq R(t_2) \geq \ldots \geq R(t_k) \tag{13}$$

and select p largest characteristics with interval numbers $t_j, 1 \leq j \leq p$.

5. Carry out new trials at points $x^{k+j} \in (x_{t_j-1}, x_{t_j}), 1 \leq j \leq p$, computed according to the formula

$$x^{k+j} = \frac{x_{t_j} + x_{t_j-1}}{2} - \mathrm{sign}(z_{t_j} - z_{t_j-1}) \frac{1}{2r} \left[\frac{|z_{t_j} - z_{t_j-1}|}{\mu} \right]^N .$$

The algorithm stops if condition $\Delta_{t_j} < \epsilon$ is satisfied for at least one index $t_j, 1 \leq j \leq p$; here $\epsilon > 0$ is the predefined accuracy.

The decision rules of the Parallel Global Search Algorithm (PGSA) allow the following interpretation (see [1]), according to which series (13) arranges the intervals $(x_{t-1}, x_t), 1 \leq t \leq k$, with respect to the decreasing probability of the localization of the global minimizer of the function $\varphi(y(x))$ in these intervals. Therefore, (as part of this interpretation) one can expect the next trial points to be chosen within the intervals with the largest probabilities at the current search step.

The theorem on the convergence of PGSA is a generalization of theorem 1, of which is also presented in [1]. In particular, it has been proven that the introduction of the parallelism described above does not produce limit points different from those in a purely sequential scheme. The results of the investigation of sequential and parallel algorithms (as well as their various modification) are presented in [16, 17].

As an example of the work of GSA and PGSA, let us consider the minimization of a two-dimensional function obtained using a GKLS generator [25]. In Fig. 1 (a) and (b), the contour plots of the objective function are presented. Fig. 1 (a) also shows the points of 261 trials (261 iterations) performed by the sequential version of the method until the required accuracy $\epsilon = 10^{-2}$ was achieved. In Fig. 1 (b) the points of 304 trials (76 iterations) performed by the parallel algorithm using 4 cores, i.e. at $p = 4$, are shown. The points of the trials performed by the parallel algorithm which differ from the trial points of the sequential algorithm are marked by the '+' sign.

3 An algorithm for solving a series of optimization problems

Now, let us assume that we need to solve a series of q problems

$$\min\{\varphi_1(y), y \in D\}, \min\{\varphi_2(y), y \in D\}, ..., \min\{\varphi_q(y), y \in D\}.$$

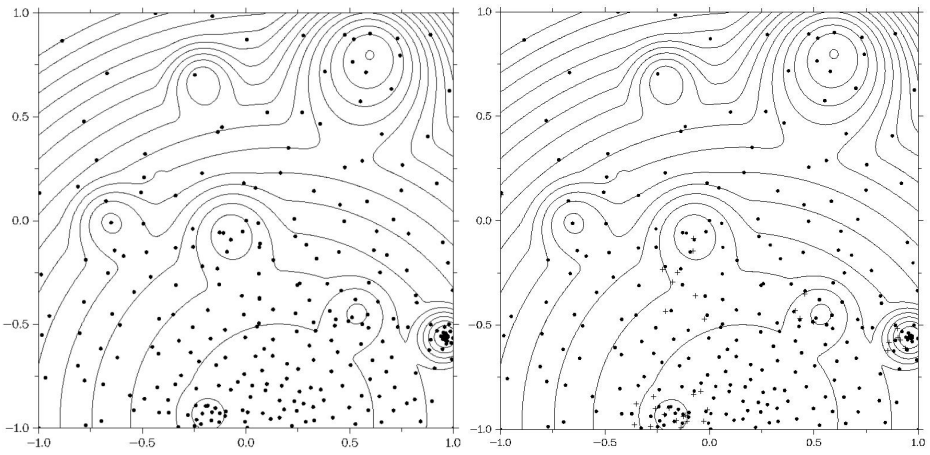


Fig. 1 Solving a two-dimensional problem using (a) GSA and (b) PGSA

It is necessary to introduce some modifications to the algorithm's computational scheme which would allow obtaining the estimates of optima in all the problems simultaneously. It is also necessary to provide the properties of the trial sequences generated during the minimization of different functions that do not depend on a particular form of the function.

The fulfillment of the inequality

$$R(j(l)) > -4\varphi(y(x^*)), \tag{14}$$

where $R(j(l))$ is the characteristic of the interval (x_{j-1}, x_j), $j = j(l)$, containing the global minimizer x^* of the function $\varphi(y(x))$ during the l-th iteration, is one of key properties used in [1] in the proof for the algorithm's convergence.

As seen from the inequality (14), the value of the characteristic of the interval (x_{j-1}, x_j), $j = j(l)$, depends on the value of the function. If the algorithm is applied to minimize different functions, the values of the characteristics of intervals containing the global minimizer will be different as well. Hence in order to use the algorithm to minimize different functions, the algorithm's decision rules should be modified so that the value of the characteristic for the best interval does not depend on a particular form of the function. Let us consider such modifications.

1. If all characteristics $R(i)$ from (8) at a given k are divided by a quantity $r\mu$, where μ is from (7), i.e. if the characteristics are computed according to the formula

$$R(i) = \Delta_i + \frac{(z_i - z_{i-1})^2}{(r\mu)^2 \Delta_i} - \frac{2(z_i + z_{i-1})}{r\mu},$$

the relationship of inequality between the characteristics would not change.

2. If a quantity $4\varphi_*^k / r\mu$ is added to all characteristics at a given k, the relationship of inequality between these would not change either. Thus, the algorithms with characteristics (8) and with characteristics

$$R(i) = \Delta_i + \frac{(z_i - z_{i-1})^2}{(r\mu)^2 \Delta_i} - \frac{2(z_i + z_{i-1} - 2\varphi_*^k)}{r\mu} \tag{15}$$

will generate the same sequences $\{x^k\}$. Consequently, the theorem of convergence is true for an algorithm with the characteristics (15) since the normalization of the characteristics at each step does not change the relationship of their equality and superiority. Then, the inequality (14) transforms into the inequality

$$R(j(l)) > 0, \tag{16}$$

i.e. the lower bound for the best interval characteristic does not depend on the objective function now.

4 Conditions of uniform convergence for the algorithm

Let us consider the problems in the form (2) for a set of q lipschitzian functions $\varphi_1(y), \varphi_2(y), ..., \varphi_q(y)$ defined in D. Let us select (arbitrarily) two of these problems:

$$\begin{aligned} &\min\{\varphi(y), y \in D\}, \\ &\min\{\psi(y), y \in D\}. \end{aligned} \tag{17}$$

Let us denote the characteristics (15) for the first problem as $R_\varphi(i)$ and for the second problem as $R_\psi(j)$. Given that,

$$\begin{aligned} R_\varphi(t_\varphi) &= \max_{1 \le i \le k} R_\varphi(i), \\ R_\psi(t_\psi) &= \max_{1 \le j \le s} R_\psi(j), \end{aligned}$$

where k corresponds to the number of trials in the first problem and s corresponds to the number of trials in the second one. The sequence of trials $\{v^k\}$ corresponds to the first problem and the sequence of trials $\{u^s\}$ corresponds to the second problem. The values $z^k = \varphi(y(v^k))$ correspond to the trial points $v^k \in [0,1]$ and the values $w^s = \psi(y(u^s))$ correspond to the trial points $u^s \in [0,1]$.

Let us consider the case where these two problems are solved simultaneously, and the selection of the interval $(v_{t_\varphi - 1}, v_{t_\varphi})$ or $(u_{t_\psi - 1}, u_{t_\psi})$ to execute the trials is determined by the condition

$$R(t) = \max\{R_\varphi(t_\varphi), R_\psi(t_\psi)\}. \tag{18}$$

Theorem 2 (on the convergence conditions)

Let the conditions for theorem 1 be satisfied for each problem (17) separately. Then solving both problems by the algorithm defining the interval for each next trial according to rule (18) for characteristics (15) will generate two infinite sequences $\{v^k\}$ and $\{u^s\}$, for the limit points of which theorem 1 is also true.

Proof. Let $l = k + s, l = 0, 1, 2, ...$, and the condition $v^0 = u^0 = 0$, $v^1 = u^1 = 1$ be satisfied. Then, the sequence of trials (i.e. $\{y(v^k)\}$) will be infinite at least for one of the functions (let it be $\varphi(y)$), and theorem 1 will be true for this function. We show that the algorithm generates an infinite sequence $\{y(u^s)\}$ for the second function $\psi(y)$ as well, for which, consequently, theorem 1 is also true.

Let the limit point $\overline{v} \in [v_{i-1}, v_i]$ where $i = i(k)$ and the parameter of algorithm r satisfies the condition $r > 2^{2-1/N}$, according to (12). From (7)

$$\mu_\varphi^2 \geq \left(\frac{|z_i - z_{i-1}|}{\Delta_i} \right)^2$$

and, consequently,

$$\frac{(z_i - z_{i-1})^2}{r^2 \mu_\varphi^2 \Delta_i} = \frac{(z_i - z_{i-1})^2 \Delta_i}{(r\mu_\varphi \Delta_i)^2} \leq \frac{\Delta_i}{r^2}.$$

Hence, according to (15)

$$R_\varphi(i) \leq \Delta_i \left(1 + \frac{1}{r^2}\right) - \frac{2(z_i + z_{i-1} - 2\varphi_*^k)}{r\mu_\varphi}.$$

Because $\overline{v}$ is the limit point of the sequence of preimages $\left\{v^k\right\}$ and $\varphi_*^k \geq \varphi(y(\overline{v}))$ at $k \to \infty$

$$\begin{gathered} \Delta_i \to 0, \\ z_i + z_{i-1} - 2\varphi_*^k \to 0. \end{gathered}$$

Therefore, for any small $\delta > 0$ there exists a large value of k such that

$$R_\varphi(i) < \delta. \tag{19}$$

Let $\psi_*^s = w_{j-1}$, i.e. the current minimum value of function $\psi(y)$ is achieved at the left point of the interval (w_{j-1}, w_j). Then

$$\frac{w_j + w_{j-1} - 2\psi_*^s}{r\mu_\psi} = \frac{w_j - w_{j-1}}{r\mu_\psi} = \frac{\Delta_j (w_j - w_{j-1})}{r\mu_\psi \Delta_j} \leq \frac{\Delta_j}{r}$$

and, taking into account that $r > 2^{2-1/N} > 2$,

$$R_\psi(j) = \Delta_j + \frac{(w_j - w_{j-1})^2}{r^2 \mu_\psi^2 \Delta_j} - \frac{2(w_j + w_{j-1} - 2\psi_*^s)}{r\mu_\psi} \geq \Delta_j \left(1 - \frac{2}{r}\right) > 0.$$

Hence, according to (19), the inequality

$$R_\psi(j) > R_\varphi(i).$$

will be true at large enough values of k, and the scheduled iteration falls into the interval (u_{j-1}, u_j), i.e. the sequence $\{u^s\}$ will be infinite as well.

Theorem 3 (on the uniform convergence)

Let the conditions of theorem 2 be satisfied. Then the convergence to the limit points of the sequences $\left\{v^k\right\}$ and $\{u^s\}$ is uniform, i.e. for any step $l = k + s$ the minimum distance between the points of the sequence $\left\{v^k\right\}$ and its limit point does not exceed a similar value for the sequence $\{u^s\}$.

Proof. Let v^* be the preimage of the global optimizer of the function $\varphi(y)$, and u^* be the preimage of the global optimizer of the function $\psi(y)$, given that

$$v^* \in [v_{i-1}, v_i], \; u^* \in [u_{j-1}, u_j],$$

and let

$$R_\psi(j) > R_\varphi(i) \tag{20}$$

at some step $l = k + s$. Consequently, the next trial will fall into the interval (u_{j-1}, u_j) before the interval (v_{i-1}, v_i). At large enough values of s, it follows from the convergence of $\{u^s\}$ to u^* that $\psi_*^s = w_{j-1}$, whence

$$w_j + w_{j-1} - 2\psi_*^s = w_j + w_{j-1} - 2w_{j-1} = w_j - w_{j-1}$$

and

$$R_\psi(j) = \Delta_j + \Delta_j \left(\frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j}\right)^2 - 2\Delta_j \frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j} = \Delta_j \left(1 - \frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j}\right)^2, \tag{21}$$

where $\Delta_j = (u_j - u_{j-1})^{1/N}$. Analogously, if $\varphi_*^k = z_{i-1}$, then

$$R_\varphi(i) = \Delta_i \left(1 - \frac{z_i - z_{i-1}}{r\mu_\varphi \Delta_i}\right)^2, \tag{22}$$

where $\Delta_i = (v_i - v_{i-1})^{1/N}$. Now, it follows from (20)–(22) that

$$\Delta_j \left(1 - \frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j}\right)^2 > \Delta_i \left(1 - \frac{z_i - z_{i-1}}{r\mu_\varphi \Delta_i}\right)^2$$

and

$$\frac{\Delta_i}{\Delta_j} < \frac{\left(1 - \frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j}\right)^2}{\left(1 - \frac{z_i - z_{i-1}}{r\mu_\varphi \Delta_i}\right)^2}.$$

Because

$$0 \le \left|\frac{w_j - w_{j-1}}{r\mu_\psi \Delta_j}\right| \le \frac{1}{r},\ 0 \le \left|\frac{z_i - z_{i-1}}{r\mu_\varphi \Delta_i}\right| \le \frac{1}{r},$$

then

$$\frac{\Delta_i}{\Delta_j} < \frac{\left(1 + \frac{1}{r}\right)^2}{\left(1 - \frac{1}{r}\right)^2} = \left(\frac{r+1}{r-1}\right)^2.$$

Thus, at the moment when the next trial falls into the interval (u_{j-1}, u_j), the length of the interval (v_{i-1}, v_i) cannot exceed the length of the interval (u_{j-1}, u_j) by more than $\left(\frac{r+1}{r-1}\right)^2$ times. The uniform convergence has been proven.

Remark 3. It follows from the properties of the evolvent (see [2]) that any two close points $x', x'' \in (0, 1)$ correspond to close images $y' = y(x')$, $y'' = y(x'')$ in the hypercube D, and if

$$\left|x' - x''\right| \le 2^{-N(m+1)},$$

where $m \ge 1$, then

$$\max\left\{\left|y_i' - y_i''\right| : 1 \le i \le N\right\} \le 2^{-m}.$$

Thus, the uniform convergence of the images in the multidimensional space also follows from the uniform convergence of the preimages on the one-dimensional interval as well.

Remark 4. Since two problems (17) were selected from a set of problems being solved arbitrarily, theorem 2 and theorem 3 will also be true for any pair of problems from the set being solved. At the same time, theorem 2 and theorem 3 will also be true for $q > 2$ problems

$$\min\{\varphi_1(y), y \in D\},\ \min\{\varphi_2(y), y \in D\}, ...,\ \min\{\varphi_q(y), y \in D\}$$

being solved simultaneously by induction.

Remark 5. The theorems on uniform convergence for the parallel algorithm are proved similarly, which allows applying the multiprocessor systems to solve a set of problems simultaneously.

Remark 6. The parallel algorithm will provide a uniform workload distribution for the cores/processors used. We assume that computing the objective function value will be a time-consuming operation, and it accounts for the majority of the computation costs as compared to the application of the parallel algorithm decision rules (the computing and arrangement of the characteristics, the information interchange between parallel processes, etc.). However, just this most computationally costly part is parallelized (p trials are executed at each iteration where p is the number of cores/processors employed). Therefore, the load on the computer system will be close to its full capacity.

5 Results of experiments

One of the well-known approaches to investigating and comparing multiextremal optimization algorithms is based on the application of these methods for solving a set of test problems selected randomly from some constructed class. In this case, each test problem can be considered as a particular realization of a random function defined by means of a special generator. The application of multiextremal optimization algorithms to sets of these functions allows the efficiency of each particular algorithm to be evaluated.

The numerical comparison of the algorithms has been carried out using the GKLS test problem generator [25]. This generator allows multiextremal optimization problems to be generated with known properties (the number of local minima, the size of their domains of attraction, the global minimizer, etc.). Eight GKLS classes of differentiable test functions of the dimensions $N = 2, 3, 4$, and 5 have been used. For each dimension, both *Hard* and *Simple* classes have been considered. The difficulty of a class was increased either by decreasing the radius of the attraction region of the global minimizer, or by decreasing the distance from the global minimizer y^* to the domain boundaries. The global minimizer y^* was considered to be found, if the algorithm generated a trial point y^k in the vicinity of the global minimum, i.e. $\left|y^k - y^*\right| < \delta \|b - a\|$, where $\delta = 0.01$, a and b are the boundaries of the search domain D. When using the Global Search Algorithm, the parameter r was set to 5, the evolvent construction parameter was fixed as $m = 10$. The maximum allowable number of iterations per problem was $K_{max} = 10^6$.

Each series of problems was solved using a scheme to distribute the problems between the parallel threads ($S1$) and using the proposed scheme to simultaneously solve all problems ($S2$). The number of employed parallel threads p was varied from 1 to 24.

Table 1 Average number of iterations for the *Simple* class

p	$N=2$		$N=3$		$N=4$		$N=5$	
	$S1$	$S2$	$S1$	$S2$	$S1$	$S2$	$S1$	$S2$
1	408	408	2502	2502	28254	28254	87261	87261
2	408	415	2502	2496	28254	27206	87261	85994
4	408	405	2502	2430	28254	30121	87261	78064
8	408	402	2502	2505	28254	28980	87261	84337
16	408	415	2502	2501	28254	28206	87261	82564
24	408	400	2502	2416	28254	27868	87261	86252

Table 2 Average number of iterations for the *Hard* class

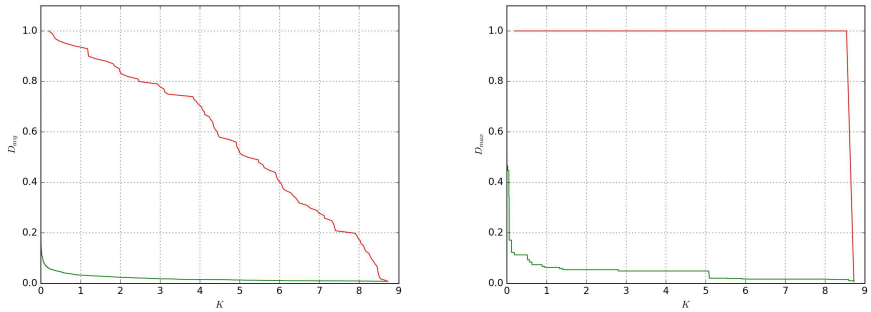
p	$N=2$		$N=3$		$N=4$		$N=5$	
	$S1$	$S2$	$S1$	$S2$	$S1$	$S2$	$S1$	$S2$
1	806	806	3873	3873	60972	60972	162475	162475
2	806	807	3873	4104	60972	58382	162475	167728
4	806	823	3873	4081	60972	57285	162475	172522
8	806	881	3873	4153	60972	55353	162475	173725
16	806	854	3873	4165	60972	60158	162475	178130
24	806	887	3873	4270	60972	55939	162475	182821

The average numbers of trials executed by the algorithm depending on the number of threads when solving *Simple* and *Hard* classes of problems are presented in Tables 1 and 2, respectively. The results demonstrate that both schemes of problem distribution between threads require an almost equal number of trials. The workload of each thread is also the same. In particular, there is an exact match between the number of trials for the sequential runs (at $p = 1$).

In order to demonstrate the presence of the uniform convergence of the proposed algorithm for solving a series of problems, let us plot the functions $D_{av}(K)$ and $D_{max}(K)$ featuring the average and maximum deviation of the current approximation from the exact solution over all problems, depending on the total number of trials. Figs. 2, 3, 4 and 5 show these functions plotted while solving five-dimensional *Simple* class problems using a sequential algorithm and a parallel algorithm. The number of trials is plotted on the abscissa (in millions) against the deviation of the current approximation from the exact solution of the problem on the ordinate. The red curve shows the first scheme of problem distributions while the green curve represents the second scheme.

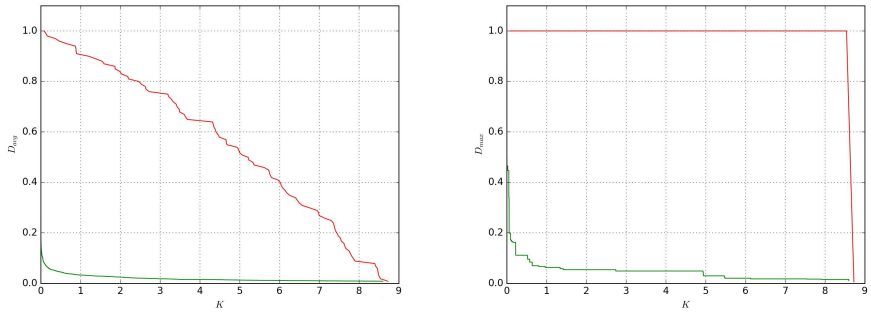
The plots clearly demonstrate the uniform convergence to the solutions of the problems of the series when using Scheme 2. The arrangement of curves in Figs. 2, 3, 4 and 5 demonstrates that when simultaneously solving the problems, one can obtain a good approximation to the solution in all problems within 100000 trials (on average as well as in the worst case). In both cases the scheme of problem distribution between the independent threads is inferior to the scheme for simultaneous solving of all problems.

Previous experiments were carried out with the objective functions of the same structure. To demonstrate the uniform convergence for objective functions of a different structure let us consider two mixed series of two-dimensional problems:



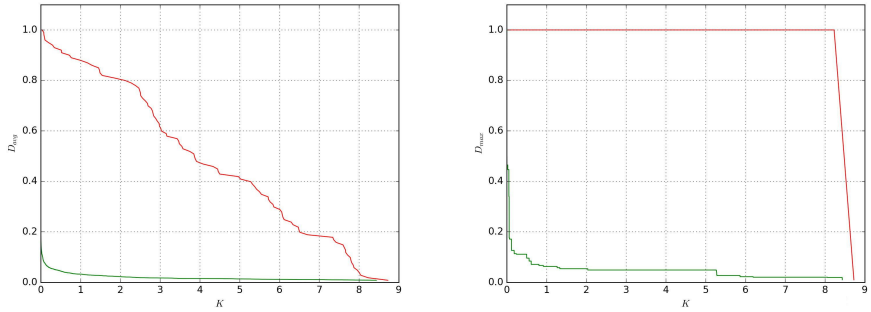
(a) (b)

Fig. 2 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the sequential algorithm.



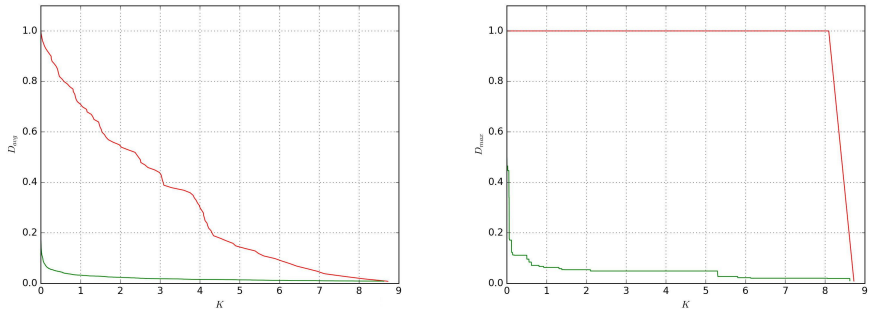
(a) (b)

Fig. 3 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the parallel algorithm; the number of threads $p = 2$.



(a) (b)

Fig. 4 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the parallel algorithm; the number of threads $p = 8$.



(a) (b)

Fig. 5 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the parallel algorithm; the number of threads $p = 24$.

Table 3 Average number of iterations for the mixed series of problems

p	GKLS *Simple* and (23)		GKLS *Hard* and (23)	
	$S1$	$S2$	$S1$	$S2$
1	234	234	472	472
2	234	240	472	476
4	234	248	472	469
8	234	249	472	482
16	234	251	472	479

50 functions from GKLS *Simple* or *Hard* classes and 50 functions of the form

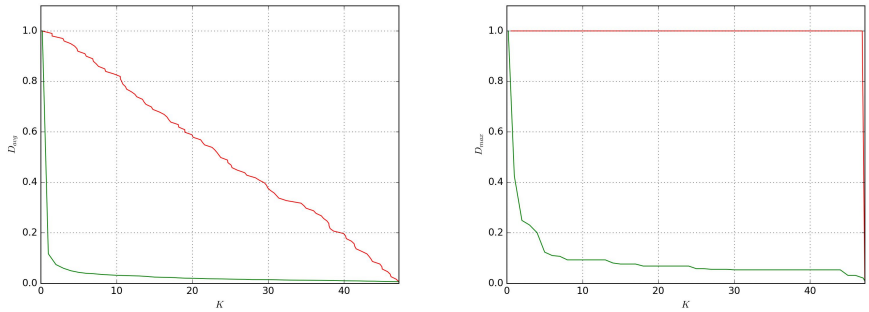
$$\varphi(y) = - \left\{ \left(\sum_{i=1}^{7} \sum_{j=1}^{7} A_{ij} g_{ij}(y) + B_{ij} h_{ij}(y) \right)^2 + \left(\sum_{i=1}^{7} \sum_{j=1}^{7} C_{ij} g_{ij}(y) + D_{ij} h_{ij}(y) \right)^2 \right\}^{1/2}, \quad (23)$$

where

$$y = (y_1, y_2) \in R^2, 0 \le y_1, y_2 \le 1,$$
$$g_{ij}(y) = \sin(i\pi y_1) \sin(j\pi y_2),$$
$$h_{ij}(y) = \cos(i\pi y_1) \cos(j\pi y_2),$$

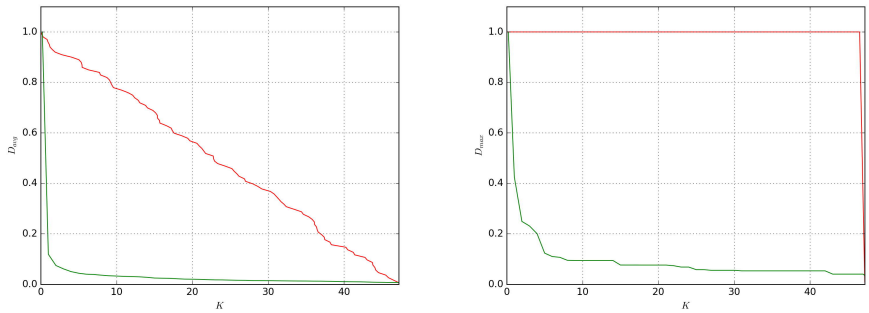
and coefficients $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ are taken uniformly in the interval $[-1, 1]$. The problems with odd numbers are from GKLS classes and the problems with even numbers are based on functions (23).

Each series of problems was solved using a scheme for distributing the problems between the parallel threads ($S1$) and using the scheme for solving all problems simultaneously ($S2$). The number of employed parallel threads p was varied from 1 to 16. The average numbers of trials executed by the algorithm depending on the number of threads when solving mixed classes of problems are presented in Table 3.



(a) (b)

Fig. 6 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the sequential algorithm when solving mixed series of problems.



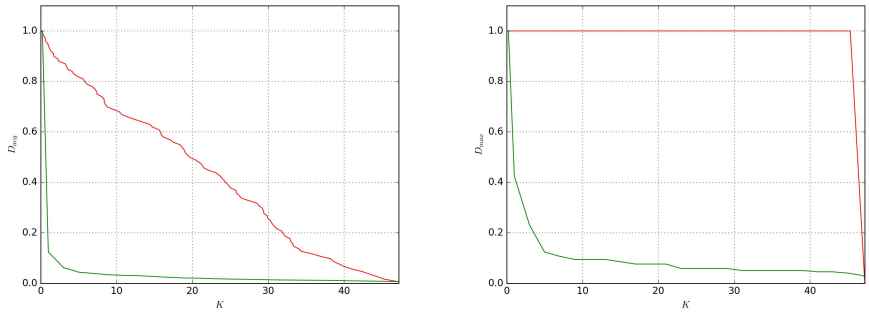
(a) (b)

Fig. 7 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the parallel algorithm when solving mixed series of problems; the number of threads $p = 8$.

Figs. 6, 7, 8 show the functions $D_{av}(K)$ and $D_{max}(K)$ plotted when solving mixed problem series using the sequential algorithm and the parallel algorithm. The number of trials is plotted on the abscissa (in thousands) against the deviation of the current approximation from the exact solution of the problem on the ordinate. The red curve shows the first scheme of problem distributions while the green curve represents the second scheme.

As in the case of objective functions of the same structure, the results show the uniform convergence when using the scheme for solving all problems simultaneously.

The computational experiments were carried out on one of the nodes of a high-performance cluster at Lobachevsky State University of Nizhny Novgorod. The cluster node included 2 Intel Sandy Bridge E5-2660 2.2 GHz CPUs and 64 Gb RAM. The CPU had 8 cores, i.e. a total of 16 physical cores and 16 virtual cores in the hyper-threading mode were available in the node. MS Visual Studio 15 was used to implement the algorithm.



(a) (b)

Fig. 8 Average (a) and maximum (b) deviation of the current approximation from the exact solution for the parallel algorithm when solving mixed series of problems; the number of threads $p = 16$.

6 Conclusions

In this paper, the results obtained when investigating the issues in solving a series of optimization problems are presented. A series of problems may arise when solving a more complex type of problem, for example, a multiple-criteria optimization problem or a larger dimensional one. A modification of the Global Search Algorithm has been proposed to allow several problems to be solved at once and, correspondingly, simultaneously obtain estimates for the solutions to these problems. The theorem regarding the uniform convergence of the novel algorithm has been proven. The computational experiments on a series of several hundred test problems of various dimensions have been carried out. The experiments have clearly demonstrated the uniform convergence.

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