Parallel Computations for Various Scalarization Schemes in Multicriteria Optimization Problems*

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Abstract. In the present paper, a novel approach to parallel computations for solving the time-consuming multicriteria global optimization problems is presented. This approach includes various methods of the scalarization of the vector criteria, the dimensionality reduction with the use of the Peano space-filling curves, and the efficient global search algorithms. The applied criteria scalarization methods can be altered in the course of computations in order to achieve more complete matching to the stated optimality requirements. The overcoming of the computational complexity of the developed approach is provided by means of the reuse of the whole search information obtained in the course of computations and by the parallel computations for the high-performance computational systems. The numerical experiments confirmed the efficiency of the developed approach.

Keywords: Multicriteria optimization · criteria scalarization · global optimization · parallel computations.

1 Introduction

The multicriteria optimization (MCO) problems arise in various applications of science and technology. Such a wide demand of the MCO problems determines a high intensity of research in the field [1–5].

Usually, the criteria of the MCO problems are contradictory, and obtaining the decisions, which provide the best values with respect to all criteria simultaneously, is impossible. In such situations a set of the efficient (non-dominated) decisions is considered for the MCO problems, for which the improvement of the values of some criteria cannot be achieved without the worsening of the efficiency values with respect to other criteria. When solving the MCO problems, it may appear to be necessary to obtain the whole set of the efficient decisions (the Pareto set) that may require performing a large amount of computations. Besides, the analysis of a large set of the efficient decisions may appear to be

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difficult for a decision-making person (decision maker, DM). And, therefore, in practice the finding of relatively small set of the efficient decisions only may be justified. Constructing such a limited set of the efficient decisions is performed usually according to the optimality requirements defined by DM.

The restricted set of the computed efficient decisions leads to a notable reduction in the amount of required computations. However, the criteria of efficiency may have a complex multiextremal form, and the computation of the criteria values may appear to be time-consuming. In such cases, solving the MCO problems is featured by a considerable computation complexity, the overcoming of which may be provided with the use of the high-performance supercomputing systems.

In the present paper, the research results on the development of the highly efficient parallel methods of the multicriteria optimization using the search information obtained in the course of computations [6–9, 16] are presented. A novel contribution consists in the development of an approach, in which it is possible to use different MCO problem criteria scalarization schemes allowing accounting for different optimality requirements to desirable decisions. For the considered scalarization schemes, a general method of the parallel computations was proposed and the computational evaluation of the efficiency of the developed approach was performed.

Further structure of the paper is as follows. In Section 2, the statement of the multicriteria optimization problems is given. In Section 3, a general scalarization scheme for the MCO problem criteria involving various kinds of criteria convolution is proposed. In Section 4, the search information obtained in the course of computations, which can be reused after the altering of the applied scalarization schemes, is considered. Also, this section presents a general scheme of the parallel execution of the global search algorithms, in the framework of which an efficient parallel method for solving the time-consuming global optimization problems is considered. Section 5 contains the results of the numerical experiments confirming the developed approach to be promising. In Conclusion, the obtained results are discussed and possible main directions of further investigations are outlined.

2 Multicriteria optimization problem statement

In the most general form, the multicriteria optimization problem is defined in the following way

$$f(y) \to min, y \in D : g(y) \leq 0, \tag{1}$$

where $y = (y_1, y_2, \ldots, y_N)$ is a vector of varied parameters, $f(y) = (f_1(y), f_2(y), \ldots, f_s(y))$ is the vector of the efficiency criteria, $D \subset R^N$ is the search domain

$$D = \{y \in R^N : a_i \leq y_i \leq b_i, 1 \leq i \leq N\}, \tag{2}$$

for given vectors a and b, and $g(y) = (g_1(y), g_2(y), \ldots, g_m(y))$ are the constraints of the MCO problem (the conditions of feasibility of the chosen decisions $y \in D$).

In the most complex cases, the criteria $f_i(y)$, $1 \le i \le s$, can be multiextremal and computing the values of these ones can appear to be time-consuming. Usually the criteria $f_i(y)$, $1 \le i \le s$, satisfy the Lipschitz condition

$$|f_i(y_1) - f_i(y_2)| \le L_i \|y_1 - y_2\|, 1 \le i \le s, \tag{3}$$

where L_i are the Lipschitz constants for the criteria $w_j(y)$, $1 \le j \le s$ and $\| * \|$ denotes the Euclidean norm in R^N.

Without any loss in generality, in further consideration we will assume the criteria $f_i(y)$, $1 \le i \le s$ to be the non-negative ones and their decreasing corresponds to the increase of efficiency of the chosen decisions.

3 Reducing the multicriteria optimization to the multiple global search problems

As mentioned above, for solving MCO problems it is required to find a set of the efficient (Pareto optimal) decisions, for which the values cannot be improved with respect to all criteria simultaneously. However, the finding of a complete set of the efficient decisions may require performing a large amount of computations. An approach to reducing the computational costs used widely consists in the finding of only a relatively small set of the efficient decisions, defined according to the optimality requirements of the decision-making person (decision maker). In many cases, the finding of the required decisions is performed by means of reducing the vector criteria to some general scalar efficiency function that allows applying a variety of already existing global optimization methods for solving the MCO problems. Among the possible scalarization methods, there are, for example, the weighted sum method, the compromise programming method, the weighted min-max method and many other methods – see, for example, [2, 4].

Within the developed approach, it was proposed to employ the following methods of scalarization of the vector criterion of the MCO problems.

1. One of the widely applied scalarization methods consists in the use of the min-max convolution of the criteria [4, 6]:

$$\begin{gathered} F_1(\lambda, y) = \max\left(\lambda_i f_i(y), 1 \le i \le s\right), \\ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \in \Lambda \subset R^s : \sum_{i=1}^{s} \lambda_i = 1, \lambda_i \ge 0, 1 \le i \le s. \end{gathered} \tag{4}$$

2. In the case when the criteria can be arranged in importance, the method of successive concessions (MSC) is applied usually [5, 6]. In this method, the multistage computations can be reduced to the solving of the following scalar optimization problem

$$\min F_2(\delta, y) = f_s(y), f_i(y) \le f_i^{min} + \delta_i(f_i^{max} - f_i^{min}), 1 \le i < s, g(y) \le 0, y \in D, \tag{5}$$

where f_i^{min}, f_i^{max}, $1 \le i < s$, are the minimum and the maximum values of the criteria in the feasible domain D from (2), and $0 \le \delta_i \le 1$, $1 \le i < s$, are the concessions with respect to each criterion.

3. If an estimate of the criteria values for the required decision exists *a priori* (for example, based on some ideal decision), the solving of a MCO problem can consist in the finding of an efficient decision matching the predefined optimality values [2, 4]:

$$\min F_3(\theta, y) = 1/s \sum_{i=1}^{s} \theta_i (f_i(y) - f_i^*)^2, g(y) \le 0, y \in D, \tag{6}$$

where $F_3(\theta, y)$ is the mean square deviation of the decision $y \in D$ from the sought ideal decision y^*, and the values $0 \le \theta_i \le 1$, $1 \le i < s$, are the importance parameters of the approximation accuracy with respect to each variable y_i, $1 \le i \le N$.

In the general case, the statements of the global optimization problems generated by the criteria scalarization schemes (4)-(6) can be represented in the form:

$$\min \varphi(y) = F(\alpha, y), g(y) \le 0, y \in D, \tag{7}$$

where F is the objective function generated as a result of the criteria scalarization, α is the vector of parameters of the criteria convolution, $g(y)$ are the constraints of the MCO problem, and D is the search domain from (1). It should be noted that due to the possibility of altering the optimality requirements in the course of computations, the form of the function $\varphi(y)$ from (7) can vary. Thus, it may appear to be necessary to alter the scalarization method employed (4)-(6) and/or to change the convolution parameters λ, δ, and θ. Such variations form a set of the scalar global optimization problems (7)

$$\mathbb{F}_T = \{F_k(\alpha_i, y) : 1 \le i \le T, k = 1, 2, 3\}. \tag{8}$$

This set of problems can be formed sequentially in the course of computations. The problems in this set can be solved either sequentially or simultaneously – in the time sharing mode or in parallel with the use of high performance computational systems. The opportunity of forming the set $\mathbb{F}_T$ allows formulating *a new class of the multicriteria global optimization problems.*

Within the framework of the developed approach, one more step of transformation of the problems $F(\alpha, y)$ from the set $\mathbb{F}_T$ is applied – namely, the dimensionality reduction with the use of the Peano space-filling curves (evolvents) $y(x)$ providing an unambiguous mapping of the interval $[0, 1]$ onto an N-dimensional hypercube D [10, 11]. As a result of such reduction, a multidimensional problem (7) is reduced to a one-dimensional global optimization problem:

$$F(\alpha, y(x^*)) = \min \{F(\alpha, y(x)) : g(y(x)) \le 0, x \in [0, 1]\}. \tag{9}$$

The dimensionality reduction allows applying many highly efficient one-dimensional global search algorithms – see, for example, [10–12, 14] – for solving the problems

of the set $\mathbb{F}_T$ from (8). It should be noted also that according to (3) the function $F(\alpha, y(x))$ satisfy the uniform Hölder condition with a constant H [10, 11] i.e.

$$|F(\alpha, y(x') - F(\alpha, y(x'')| \leq H|x' - x''|^{1/N}. \tag{10}$$

4 Parallel computations for solving the multiple global optimization problems

The numerical solving of the MCO problems consists usually in a sequential computing of the values of the criteria $f(y)$ and the constraints $g(y)$ from (1) at the points y^i, $0 \leq i \leq k$, of the search domain D [10, 12–14]. After the scalarization of the vector criterion (7) and the application of the dimensionality reduction (9), the obtained search information takes the form of the set:

$$A_k = \{(x_i, z_i, f_i, g_i)^T : 1 \leq i \leq k\}, \tag{11}$$

where x_i, $1 \leq i \leq k$ are the reduced points of the executed global search iterations arranged in the order of increasing of the coordinates z_i, and f_i, g_i, $1 \leq i \leq k$ are the values of the objective function criteria and constraints correspondingly at these points for current optimization problem $F(\alpha, y(x))$.

The availability of the set A_k from (11) allows to transform the values of the criteria and constraints performed earlier to the values of the current optimization problem $F(\alpha, y(x))$ from (7) without repeating the time-consuming computations of the values of criteria and constraints i. e.

$$(x_i, f_i, g_i) \to F(\alpha, y(x_i)), 1 \leq i \leq k. \tag{12}$$

In this way, the whole search information A_k from (11) recalculated according to (12) can be reused to continue the solving of the MCO problem. Such an opportunity can provide a considerable reduction of the amount of computations performed for solving every next MCO problem of the set $\mathbb{F}_T$ from (8) up to a limited number of global search iterations.

Further increasing of the computational efficiency in solving the MCO problems can be provided by means of parallel computations. Within the framework of the developed approach, a general parallel computation scheme for solving the global optimization problems is applied – namely, the parallel computations is provided by the simultaneous computing of the values of the minimized function $F(\alpha, y(x))$ from (7) at several different points of the search domain D [6, 10]. Such an approach provides the parallelization of the most time-consuming part of the global search computations and is a general one – it can be applied to any global optimization method for a variety of global optimization problems.

The state-of-the-art in the field of global optimization is reviewed, for example, in [10, 12, 13]. In the present paper, the proposed approach is based on the information-statistical theory of global search [10]. This theory has been applied as a basis for the development of a large number of efficient methods of multiextremal optimization – see, for example, [6–11, 15–18]. The main key ideas of

the developed algorithms consist in the dimensionality reduction with the use of the Peano space-filling curves and in parallel computations by simultaneous computing the values of the minimized functions at several points of the search domain. From a general point of view, the developed multidimensional parallel global search algorithms (MPGSA) can be presented by a unified computation scheme as follows [6, 10, 15].

Let $p > 1$ be the number of the applied processors (cores) with shared memory. At the initial iteration of the MGGSA, the computing of the minimized function values at p arbitrary points of the interval $(0, 1)$ is performed (the obtaining of the function value will be called hereafter *a trial*). Then, let us assume τ, $\tau > 1$ global search iterations to be completed. The choice of the trial points for the next $(\tau + 1)^{th}$ iteration is determined by the following rules.

Rule 1. Renumber the points of the trial points by the lower indices in the order of increasing values of coordinate[1]

$$0 = x_0 \leq x_1 \leq \cdots \leq x_i \leq \cdots \leq x_k \leq x_{k+1} = 1 \tag{13}$$

The points x_0, x_{k+1} were introduced additionally for the convenience of further presentation, and $k = \tau p$ is the total number of trials performed earlier.

Rule 2. For each interval (x_{i-1}, x_i), $1 \leq i \leq k+1$, compute the quantity $R(i)$ called hereafter *characteristic* of the interval.

Rule 3. Arrange the characteristics of the intervals in the decreasing order

$$R(t_1) \geq R(t_2) \geq \cdots \geq R(t_{k-1}) \geq R(t_k) \tag{14}$$

Select p intervals with the indices t_j, $1 \leq j \leq p$, having the highest characteristics.

Rule 4. Perform new trials at the points of the interval $(0, 1)$

$$x^{k+1}, x^{k+2}, \ldots, x^{k+p}, \tag{15}$$

placed in the intervals with the maximal characteristics from (14).

The stopping condition, according to which the trials are terminated, is defined as

$$(x_t - x_{t-1})^{1/N} \leq \varepsilon, \tag{16}$$

fulfilled for at least one of the indices t_j, $1 \leq j \leq p$, from (14), N is the dimensionality, and $\varepsilon > 0$ is the predefined accuracy of solving the optimization problem. If the stopping condition is not met, the iteration index τ is incremented by 1, and the new global search iteration is performed.

According to this general scheme, particular global search algorithms are defined by defining the expressions for computing the characteristics $R(i)$, $1 \leq i \leq k + 1$, from (14) and the points of the next trials x^{k+j}, $1 \leq j \leq p$, from (15) in the intervals with the maximal characteristics. Thus, for example, for

[1] It should be noted that the condition (13) determines the necessity of the ordering of the search information stored in the set A_k from (11).

the multidimensional generalized algorithm of global search (MGGSA) [10], the interval characteristic takes the form

$$R(i) = m\varrho_i + \frac{(z_i - z_{i-1})^2}{m\varrho_i} - 2(z_i + z_{i-1}), 1 \le i \le k+1, \tag{17}$$

where m is the numerical estimate of the Hölder constant from (10), z_i, $1 \le i \le k+1$, are the computed values of the minimized function $F(\alpha, y(x))$ from (9) at the points of the performed global search iterations, and $\varrho_i = (x_i - x_{i-1})^{1/N}$, $1 \le i \le k+1$.

The MGGSA rule for computing the next trial point $x^k \in (0,1)$ is defined as

$$x^{k+1} = \frac{x_t + x_{t-1}}{2} - sign(z_t - z_{t-1})\frac{1}{2r}\Big[\frac{|z_t - z_{t-1}|}{m}\Big]^N, \tag{18}$$

where r, $r > 1$ is a parameter of the MGGSA algorithm.

For the clarification of the considered computational scheme, one can note the following. The computed characteristics $R(i)$, $1 \le i \le k+1$, from (14) can be considered as some measures of importance of the intervals from the viewpoint of containing the global minimum point in these ones. Thus, the scheme of selecting the interval for executing the new trial becomes clear – the point of every next trial x^{k+1} from (18) is selected in the interval with the highest value of the interval characteristic (i. e. in the interval, in which the finding of the global minimum is most probable).

The conditions of convergence for the algorithms developed in the framework of the information-statistical theory of global search and of the non-redundancy of the parallel computations were considered in [10].

5 Results of numerical experiments

The numerical experiments were carried out on the Lobachevsky supercomputer of State University of Nizhny Novgorod (the operating system – CentOS 6.4, the management system – SLURM). Each supercomputer node had 2 Intel Sandy Bridge E5-2660 2.2 GHz, 64 Gb RAM processors. Each processor unit had 8 cores (i. e. total 16 CPU cores were available on each node).

In the experiments, the bi-criteria two-dimensional MCO problems (i. e. $N = 2$, $s = 2$) were solved. Multiextremal functions defined by the relations [10]:

$$\begin{aligned} f(y_1, y_2) &= -\{AB + CD\}^{1/2} \\ AB &= \Big(\sum_{i=1}^{7}\sum_{j=1}^{7}[A_{ij}a_{ij}(y_1, y_2) + B_{ij}b_{ij}(y_1, y_2)]\Big)^2 \\ CD &= \Big(\sum_{i=1}^{7}\sum_{j=1}^{7}[C_{ij}a_{ij}(y_1, y_2) - D_{ij}b_{ij}(y_1, y_2)]\Big)^2 \end{aligned} \tag{19}$$

were used as the problem criteria; the expressions

$$a_{ij}(y_1, y_2) = \sin(\pi i y_1)\sin(\pi j y_2), b_{ij}(y_1, y_2) = \cos(\pi i y_1)\cos(\pi j y_2) \tag{20}$$

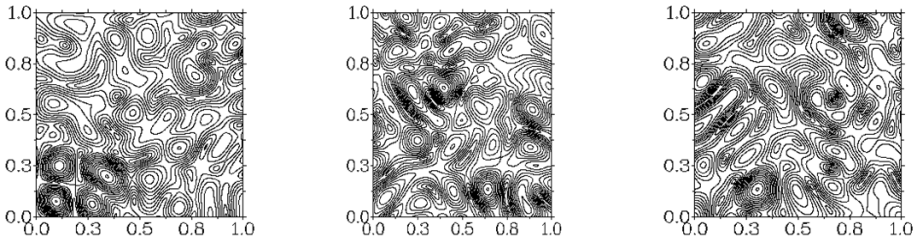

Fig. 1. Contour plots of three multiextremal functions from the test optimization problem family

Table 1. Averaged number of iterations and the speedup of the parallel computations in solving a single MCO problem without the reuse of the search information

Convolution	Number of cores									
	1		2		4		8		16	
	K	S	K	S	K	S	K	S	K	S
$F_1(\lambda, y)$ from (4)	13 221	1	7 187	1,8	3 456	3,8	1 897	7,0	1 058	12,5
$F_2(\delta, y)$ from (5)	16 002	1	9 458	1,7	4 113	3,9	2 356	6,8	1 204	13,3
$F_3(\theta, y)$ from (6)	14 041	1	7 866	1,8	3 558	3,9	1 982	7,1	1 068	13,2

were defined in the range $0 \leq y_1, y_2 \leq 1$ the parameters $1 \leq A_{ij}, B_{ij}, C_{ij}, D_{ij} \leq 1$ were the independent random numbers. The minimization of such functions arises, for example, in the problem of evaluation of the maximum strain (the determining of the strength) in a thin plate under the normal load.

In Fig. 1, the contour plots of three functions of this family are shown. As one can see, the functions of such kind are multiextremal essentially.

In order to draw more justified conclusions on the efficiency of the developed approach, the solving of 100 multicriteria problems formed using the multiextremal functions of the family (19) has been performed. To solve each problem, total 50 different coefficient α of the convolutions (4)-(6) were used. All results presented below were averaged over the family of the solved problems.

First, the averaged numbers of iterations executed for solving the multicriteria problems in the case when the global optimization algorithms with various convolution types (4)-(6) were employed without the use of the accumulated search information A_k from (11) when changing the convolution coefficients are presented.

In Table 1, the columns marked as "K" contain the averaged numbers of iterations; in the columns labeled as "S", the achieved speedup of the parallel computations is shown. As follows from the presented results of experiments, the parallel algorithm from the MPAGS family based on the MGGSA method [10, 16] has a high efficiency. When employing 16 cores, the speedup was more than 12.5 times for any type of convolutions (4)-(6).

The results presented in Table 2 show the reuse of the search information to allow reducing the amount of performed computations not less than 8 times.

Table 2. Averaged numbers of iterations and speedup of the parallel computations in solving a single MCO problem with the reuse of the search information

Convolution	Number of cores									
	1		2		4		8		16	
	K	S	K	S	K	S	K	S	K	S
$F_1(\lambda, y)$ from (4)	1 193	1	595	2,0	300	4,0	173	6,9	103	11,5
$F_2(\delta, y)$ from (5)	2 021	1	934	2,2	475	4,3	272	7,4	147	13,7
$F_3(\theta, y)$ from (6)	990	1	491	2,0	249	4,0	145	6,8	90	10,9

Table 3. Overall speedup of parallel computations based on the developed approach for solving a MCO problem

Convolution	Number of cores				
	1	2	4	8	16
$F_1(\lambda, y)$ from (4)	11,1	22,2	44,1	76,2	127,5
$F_2(\delta, y)$ from (5)	7,9	17,1	33,7	58,7	108,2
$F_3(\theta, y)$ from (6)	14,2	28,6	56,3	96,3	154,9

The efficiency of the developed approach becomes clearer when the obtained speedup of parallel calculations is shown relatively to the initial sequential algorithm (Table 1), which does not use the search information (Table 3). As follows from the results of performed experiments, the overall speedup of parallel computations when using 16 computation cores only was not less 100 times.

6 Conclusion

In the present paper, a novel approach to parallel computations for solving the time-consuming multicriteria global optimization problems is presented. This approach is based on various methods of the scalarization of the vector criteria, the dimensionality reduction with the use of the Peano space-filling curves, and the efficient global search algorithms.

In the present paper, the research results on the development of the highly efficient parallel methods of the multicriteria optimization using the search information obtained in the course of computations are presented. A novel contribution consists in the development of an approach, in which it is possible to use different criteria scalarization schemes allowing accounting for different optimality requirements to desirable decisions. For the considered scalarization schemes, a general method of the parallel computations was proposed and the computational evaluation of the efficiency of the developed approach was performed. The numerical experiments confirmed the efficiency of the developed approach – for example, the speedup of parallel computations when using 16 computational cores was not less 100 times.

In further investigations it is intended to execute the numerical experiments on parallel solving the multicriteria optimization problems for a larger num-

ber of efficiency criteria and for larger dimensionality. Parallel computations on computational nodes with distributed memory can be considered as well.

References

1. Marler, R.T., Arora, J.S.: Multi-Objective Optimization: Concepts and Methods for Engineering. In: VDM Verlag (2009).
2. Ehrgott, M.: Multicriteria Optimization. In: Springer (2nd ed., 2010).
3. Collette, Y., Siarry, P.: Multiobjective Optimization: Principles and Case Studies (Decision Engineering). In: Springer (2011).
4. Pardalos, P.M., Žilinskas, A., Žilinskas, J.: Non-Convex Multi-Objective Optimization. In: Springer (2017).
5. Modorskii, V.Y., Gaynutdinova, D.F., Gergel, V.P., Barkalov, K.A.: Optimization in design of scientific products for purposes of cavitation problems. In: AIP Conference Proceedings, **1738**, pp. 400013 (2016). https://doi.org/10.1063/1.4952201
6. Strongin, R.G., Gergel, V.P.: Parallel computing for globally optimal decision making. In: Lecture Notes in Computer Science, **2763**, pp. 76–88 (2003).
7. Gergel, V., Kozinov, E.: Efficient methods of multicriterial optimization based on the intensive use of search information. In: Springer Proceedings in Mathematics and Statistics, **197**, pp. 27–45 (2017). https://doi.org/10.1007/978-3-319-56829-4_3
8. Barkalov, K., Gergel, V., Lebedev, I.: Solving global optimization problems on GPU cluster. In: AIP Conference Proceedings, **1738**, pp. 400006 (2016). https://doi.org/10.1063/1.4952194
9. Gergel, V.P., Kozinov, E.A.: Efficient multicriterial optimization based on intensive reuse of search information. In: J Glob Optim., **71**(1), pp. 73–90, (2018). https://doi.org/10.1007/s10898-018-0624-3
10. Strongin, R., Sergeyev, Ya.: Global optimization with non-convex constraints. Sequential and parallel algorithms. In: Kluwer Academic Publishers, Dordrecht (2nd ed. 2013, 3rd ed. 2014).
11. Sergeyev Y.D., Strongin R.G., Lera D.: Introduction to global optimization exploiting space-filling curves. In: Springer (2013).
12. Zhigljavsky, A., Žilinskas, A.: Stochastic Global Optimization. In: Springer (2008).
13. Locatelli, M., Schoen, F.: Global Optimization: Theory, Algorithms, and Applications. In: SIAM (2013).
14. Floudas, C.A., Pardalos, M.P.: Recent Advances in Global Optimization. In: Princeton University Press (2016).
15. Gergel, V.P., Sidorov, S.V.: A Two-Level Parallel Global Search Algorithm for Solution of Computationally Intensive Multiextremal Optimization Problems. In: Parallel Computing Technologies / Lecture Notes in Computer Science, **9251**, pp. 505–515 (2015).
16. Gergel, V., Kozinov, E.: Parallel computing for time-consuming multicriterial optimization problems. In: Lecture Notes in Computer Science, **10421**, pp. 446–458 (2017).
17. Sergeyev Y.D., Kvasov D.E.: A deterministic global optimization using smooth diagonal auxiliary functions. In: Communications in Nonlinear Science and Numerical Simulation, **21**(1-3), pp. 99–111 (2015).
18. Gergel, V., Grishagin, V., Gergel, A.: Adaptive nested optimization scheme for multidimensional global search. In: Journal of Global Optimization, **66**(1), pp. 35–51 (2016).