Ejercicio 4. Verifique que las siguientes ecuaciones en $x = x(t)$ son homogéneas de grado cero y resuelva:

a) $tx' = x + 2t\exp(-x/t)$ b) $txx' = 2x^2 - t^2$ c) $x' = \frac{x+t}{t}$, $x(1) = 0$

Ecuaciones Homogéneas

$f(t, x)$ es Homogénea de Grado n si:

$$f(\lambda . t, \lambda . x) = \lambda^n . f(t, x)$$

a) $$x' = \frac{x}{t} + 2 . e^{-\frac{x}{t}}$$

$$\| $$

$$f(t, x) = \frac{x}{t} + 2 . e^{-\frac{x}{t}}$$

$$f(\lambda t, \lambda x) = \frac{\lambda . x}{\lambda . t} + 2 . e^{-\frac{\lambda x}{\lambda t}} = \frac{x}{t} + 2 . e^{-\frac{x}{t}} = f(t, x)$$

Es homogénea de grado cero!

$$\Rightarrow y(t) = \frac{x(t)}{t}$$

$$x = y . t$$

$$x' = y' . t + y$$

Como $x' = f(t, x) = \frac{x}{t} + 2 . e^{-\frac{x}{t}}$

$\Rightarrow$ Si $\lambda = \dfrac{1}{t}$

$$\underbrace{f(\lambda \cdot t, \lambda \cdot x)}_{= f(t,x)} = f(1, y)$$

$$\underbrace{= x'}$$

$$y' \cdot t + \cancel{y} = \cancel{y} + 2 \cdot e^{-y}$$

$$y' = \frac{2}{t} \cdot e^{-y}$$

$$y' \cdot e^{y} = \frac{2}{t}$$

$$e^{y} = \int \frac{2}{t} \cdot dt$$

$$e^{y} = 2 \ln|t| + C \qquad C \in \mathbb{R}$$

$(\ln \boxed{}$

$$y = \ln\left(2 \cdot \ln|t| + C\right)$$

Como $y = \dfrac{x}{t}$

$$\boxed{x = t \cdot \ln\left(2 \cdot \ln|t| + C\right) \qquad C \in \mathbb{R}}$$

b) $t \cdot x \cdot x' = 2 \cdot x^2 - t^2$

$$x' = \frac{2x^2}{tx} - \frac{t^2}{tx}$$

$x' = f(t, x)$

$$f(\lambda t, \lambda x) = \frac{2\lambda^2 \cdot x^2}{\lambda^2 t x} - \frac{\lambda^2 t^2}{\lambda^2 t x} = f(t, x)$$

$\therefore$ es H. de grado 0,

- Si $\lambda = \frac{1}{t}$, $y(t) = \frac{x(t)}{t}$

$\Rightarrow \quad x = t \cdot y$

$$x' = y + t \cdot y'$$

$$f(\underbrace{\lambda \cdot t}_{=1}, \underbrace{\lambda x}_{\frac{x}{t} = y}) \overset{\text{H.G.O}}{=} f(t, x) = x' = \frac{2x^2}{tx} - \frac{t^2}{tx}$$

$$= f(1, y)$$

$$= \frac{2 \cdot y^2}{y} - \frac{1}{y}$$

$$x' = 2 \cdot y - \frac{1}{y}$$

$$\cancel{y} + t \cdot y' = \cancel{2} \cdot y - \frac{1}{y}$$

$$t \cdot y' = \frac{y^2 - 1}{y}$$

$$\frac{y'}{y^2-1} \cdot y = \frac{1}{t}$$

$t \neq 0$

$y^2 \neq 1$

$$\int \frac{y'}{y^2-1} \cdot y \, dt = \int \frac{1}{t} \, dt$$

CA

$$\frac{\partial}{\partial t} \ln |y^2 - 1| = \frac{1}{y^2-1} \cdot 2y \cdot y'$$

$$\frac{1}{2} \cdot \ln |y^2 - 1| = \ln |t| + c$$

$$|y^2 - 1| = \underbrace{e^{2 \cdot \ln |t|}}_{=\left(e^{\ln |t|}\right)^2} \cdot e^{c}$$

$c \in \mathbb{R}$

$$= |t|^2 \cdot \tilde{C}$$

$\tilde{C} \in \mathbb{R}_{>0}$

$$|y^2 - 1| = \tilde{C} \cdot t^2$$

$$y^2 - 1 = \begin{cases} -\tilde{c} \cdot t^2 & \text{si} \quad y^2 - 1 < 0 \\ \tilde{c} \cdot t^2 & \text{si} \quad y^2 - 1 \geq 0 \end{cases}$$

$$y^2 - 1 = \begin{cases} -\tilde{c} \cdot t^2 & \text{si} \quad y^2 < 1 \\ \tilde{c} \cdot t^2 & \text{si} \quad y^2 \geq 1 \end{cases}$$

- Si $y^2 < 1$

$$\Rightarrow \quad y^2 = 1 - \tilde{c}\, t^2$$

$$y^2 < 1 \Big\downarrow \quad y \in (-1, 1)$$

$$|y| = \underbrace{\sqrt{1 - \tilde{c}\, t^2}}$$

$$\hookrightarrow \quad 1 - \tilde{c} \cdot t^2 \geq 0$$

$$1 \geq \tilde{c}\, t^2 \qquad \tilde{c} > 0$$

$$\frac{1}{\tilde{c}} \geq t^2$$

$$|t| \leq \sqrt{\frac{1}{\tilde{c}}} \Rightarrow t \in \left(-\frac{1}{\sqrt{\tilde{c}}}, \frac{1}{\sqrt{\tilde{c}}}\right)$$

- Si $y^2 \geq 1$

$$\Rightarrow \quad y^2 = 1 + \tilde{c}\, t^2$$

$$y^2 \geq 1 \Big\downarrow$$

$$|y| = \underbrace{\sqrt{1 + \tilde{c}\, t^2}}_{t \in \mathbb{R}}$$

$y \in (-\infty, -1]$

ó

$y \in [-1, +\infty)$

$$\underbrace{t \in \mathbb{R}}_{\text{Intervalo maximal}}$$

Obtuve 2 familias de soluciones

$1°: \quad y = \sqrt{1+\tilde{c}\,t^2} \qquad\qquad \tilde{C} > 0$

$2°: \quad y = -\sqrt{1+\tilde{c}\,t^2}$

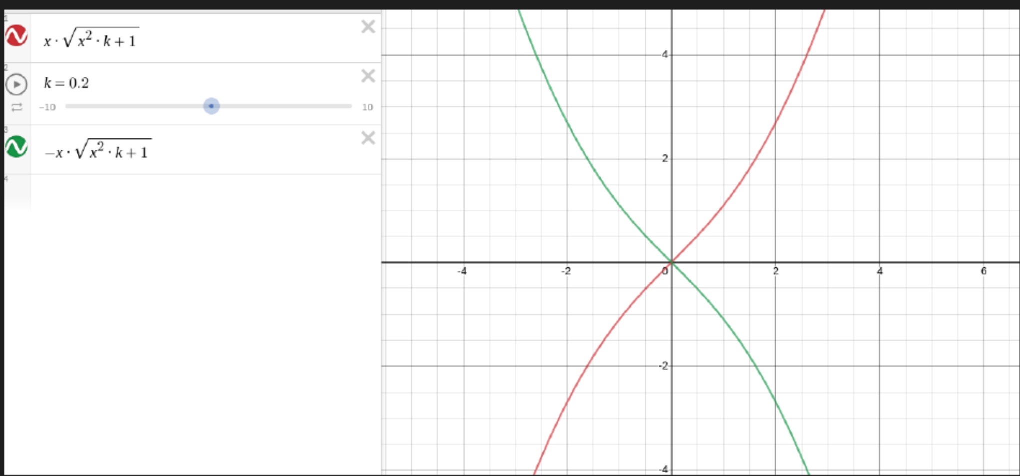
Como $y = \dfrac{x}{t}$

$1°: \quad x = t\sqrt{1+\tilde{c}\,t^2}$

$2°: \quad x = -t\sqrt{1+\tilde{c}\,t^2}$

1°

2°



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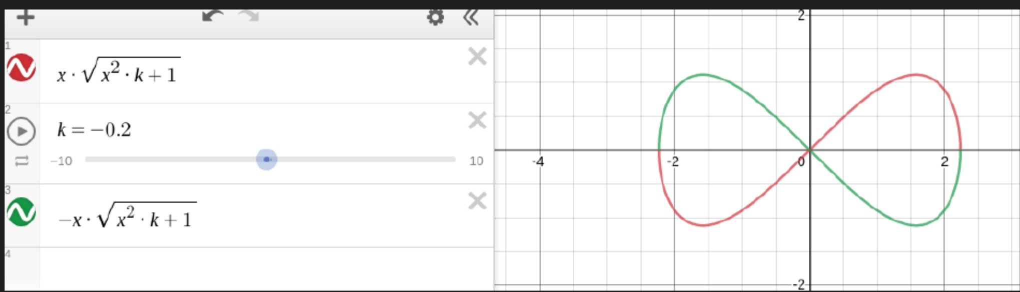

Caso con $k = -\tilde{c}$

c) $x' = \dfrac{x+t}{t}, \quad x(1) = 0$

$$x' = f(t,x) = \frac{x}{t} + \frac{t}{t}$$

$$f(\lambda t, \lambda x) = \frac{\cancel{\lambda} x}{\cancel{\lambda} t} + \frac{\cancel{\lambda} t}{\cancel{\lambda} t} = f(t,x)$$ es homogénea de grado cero

- $\lambda = \frac{1}{t}$

- Si $y = \frac{x}{t} \Rightarrow x = t \cdot y \Rightarrow x' = y + t \cdot y'$ (I)

$$\Rightarrow x' = f(t,x) = f(\lambda t, \lambda x) \overset{\lambda = \frac{1}{t}}{=} f(1, y) = y + 1$$

$$x' = \cancel{y} + 1 \overset{\text{(I)}}{=} \cancel{y} + t \cdot y'$$

$$y' = \frac{1}{t}$$

$$\int y' \, dt = \int \frac{1}{t} \, dt$$

$$y = \ln|t| + c \qquad c \in \mathbb{R}$$

$y = \frac{x}{t}$

$$\boxed{x = t(\ln|t| + c)} \qquad t \in \mathbb{R}$$

Sol. Gral

$$\text{Si } t = 1 \Rightarrow x(1) = 0 = 1\,(\underbrace{\ln|1|}_{=0} + c)$$

$$c = 0$$

Sol. Part.

$$\boxed{x = t \cdot \ln|t|}$$

Verifico: Quiero llegar a $x' = \frac{x+t}{t}$

$$x' = \underbrace{\ln|t|}_{\frac{x}{t}} + \underbrace{\frac{t}{|t|} \cdot \text{signo}(t)}_{=1}$$

✓ Verificado.

Ejercicio 5. Demuestre que la sustitución $y = at + bx + c$ cambia $x' = f(at + bx + c)$ en una ecuación con variables separables y aplique este método para resolver las ecuaciones siguientes:

a) $x' = (x+t)^2$ b) $x' = \operatorname{sen}^2(t - x + 1)$

$$y = at + bx + c$$

Dato: $x' = f(y)$

$$\Rightarrow y' = a + b.x' \qquad \left(y' = \frac{\partial}{\partial t} y \quad y \quad x' = \frac{\partial}{\partial t} x \right)$$

$$y' = a + b.f(y)$$

Una función f aplicada a y de la forma y=(at + bx + c)

Ejs (para algún a,b,c en R)

f(y) = (at + bx + c)^2
f(y) = sin(at + bx + c)
f(y) = e^(at + bx + c)

a) $x' = (x+t)^2$

Como $x' = f(y)$

$$\Rightarrow f(y) = (x+t)^2$$

Supongo $y = x + t$ con $\begin{cases} a = 1 \\ b = 1 \\ c = 0 \end{cases}$

$$\Rightarrow x' = \overbrace{y^2}^{f(y)}$$

Además, sé que

$$y' = a + b.\ f(y)$$

$$y' = 1 + y^2$$

$$\frac{y'}{1+y^2} = 1$$

$$\int \frac{y'}{1+y^2}\,dt = \int 1\ dt$$

$$\arctan y = t + d \qquad d \in \mathbb{R}$$

$$y = \tan(t+d) \qquad d \in \mathbb{R}$$

$$\shortparallel$$

$$x + t = \tan(t+d)$$

$$\boxed{x = \tan(t+d) - t} \qquad \text{con } t+d \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

pues $\tan(-\frac{\pi}{2})$ es indet.

Verifico: Debo llegar a $x' = (x+t)^2$

$$x' = \frac{1}{\cos^2(t+d)} - \underbrace{1}_{\frac{\cos^2(\,)}{\cos^2(\,)}} \overset{?}{=} \underbrace{\left(\tan(t+d) - t + t\right)^2}_{\tan^2(t+d)}$$

$$= \frac{1 - \cos^2(t+d)}{\cos^2(t+d)}$$

$$= \frac{\sin^2(t+d)}{\cos^2(t+d)} = \tan^2(t+d) \quad \checkmark \text{ Verificado}$$

b) $x' = \text{sen}^2(t - x + 1)$

$$y = t - x + 1 \qquad \text{con} \begin{cases} a = 1 \\ b = -1 \\ C = 1 \end{cases}$$

$$x' = f(y)$$

$$y' = 1 - x'$$

$$y' = 1 - f(y)$$

$$= 1 - \sin^2 y$$

$$y' = \cos^2 y$$

$$\frac{y'}{\cos^2 y} = 1 \qquad \text{con } \cos^2 y \neq 0$$

$$\int \frac{y'}{\cos^2 y} dt = \int 1 \, dt$$

CA

$$(\tan(y))' = \frac{1}{\cos^2 y} \cdot y'$$

$$\tan y = t + C \qquad C \in \mathbb{R}$$

$$y = \arctan(t + C)$$

$$t - x + 1 = \arctan(t + C)$$

$$-x = \arctan(t + C) - t - 1$$

$$\boxed{x = t + 1 - \arctan(t + C)}$$

<u>Verifico</u> : Quiero $x' = \sin^2(t - x + 1)$

$$x' = 1 - \frac{1}{1+(t+c)^2} \qquad \underbrace{\sin^2\left(\arctan(t+c)\right)}_{\text{wolfram.}}$$

$$= \frac{1+(t+c)^2 - 1}{1+(t+c)^2} \overset{=}{\longleftrightarrow} \frac{(t+c)^2}{1+(t+c)^2}$$

✓ <u>Verificado</u>

Ejercicio 6.

(a) Si $ae \neq bd$ demuestre que pueden elegirse constantes h, k de modo que las sustituciones $t = s - h$, $x = y - k$ reducen la ecuación:

$$\frac{dx}{dt} = F\left(\frac{at + bx + c}{dt + ex + f}\right)$$

a una ecuación homogénea.

(b) Resuelva las ecuaciones:

a) $x' = \dfrac{2x - t + 4}{x + t - 1}$ b) $x' = \dfrac{x + t + 4}{t - x - 6}$

c) $x' = \dfrac{x + t + 4}{x + t - 6}$, $x(0) = 2$. ¿Se satisface $ae \neq bd$ en este caso?

$$at + bx + c \overset{\substack{t = s - h \\ x = y - k}}{=} a(s-h) + b(y-k) + c$$

$$= as - ah + by - bk + c$$

$$dt + ex + f = ds - dh + ey - ek + f$$

Además

$$x' = (y-k)' = y'$$

$$\Rightarrow y' = F\left(\frac{as - ah + by - bk + c}{ds - dh + ey - ek + f}\right)$$

$$= F\left(\frac{as + by + \underbrace{c - ah - bk}}{ds + ey + \underbrace{f - dh - ek}}\right)$$

Si $\begin{cases} c = ah + bk \\ f = dh + ek \end{cases}$

$$\Rightarrow y' = F\left(\frac{as + by}{ds + ey}\right)$$

Busco h y k / $\begin{cases} c = ah + bk \\ f = dh + ek \end{cases}$

Resuelvo el sistema

$$\underbrace{\begin{pmatrix} a & b \\ d & e \end{pmatrix}}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

$ae - bd \neq 0 \Rightarrow$ determinante no nulo

$\Rightarrow$ tiene solución ($\exists\, h, k$ que resuelven el sistema para cualquier a, b, d, e, c, f con $ae - bd \neq 0$)

$$\Rightarrow y' = F\left(\frac{as + by}{ds + ey}\right) \qquad (y = y(s))$$

$$\| $$

$$f(s, y) = F\left(\frac{as + by}{ds + ey}\right)$$

$$f(\lambda s, \lambda . y) = F\left(\frac{\lambda . as + \lambda . by}{\lambda . ds + \lambda . ey}\right) = f(s, y) \checkmark$$

es homogénea de grado cero.

(b) Resuelva las ecuaciones:

$$\text{a) } x' = \frac{2x - t + 4}{x + t - 1} \qquad \text{b) } x' = \frac{x + t + 4}{t - x - 6}$$

$$\text{c) } x' = \frac{x + t + 4}{x + t - 6}, \quad x(0) = 2. \text{ ¿Se satisface } ae \neq bd \text{ en este caso?}$$

a) $a = -1 \quad b = 2 \quad c = 4$

$d = 1 \quad e = 1 \quad f = -1$

Veo det de $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$: si es $\neq 0 \Rightarrow$ puedo usar sustitución

$\det \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} = -1 - 2 = -3 \neq 0$ ✓ Vale la sustitución de arriba

$$\begin{cases} t = s - h \\ x = y - k \end{cases} \quad (y = y(s))$$

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{cases} -h + 2k = 4 \longrightarrow 1 + k + 2k = 4 \\ h + k = -1 \Rightarrow h = -1 - k \end{cases}$$

$3k = 3$

$k = 1$

$\Rightarrow h = -2$

$$\begin{cases} t = s + 2 \\ x = y - 1 \end{cases}$$

$$\Rightarrow \quad y' = \frac{as + by}{ds + ey}$$

$$f(s,y) = \frac{-s+2y}{s+y} \qquad \text{Si } \lambda = \frac{1}{s}$$

$(f(\lambda s, \lambda y))$

$$f\left(1, \frac{y}{s}\right) = \frac{-1+2\frac{y}{s}}{1+\frac{y}{s}}$$

Llamo $u = \dfrac{y}{s}$

$\Rightarrow u \cdot s = y$

$\Rightarrow y' = u + s \cdot u'$

$$\underbrace{f(1, u)}_{=y'} = \frac{-1+2u}{1+u}$$

$$u + s \cdot u' = \frac{-1+2u}{1+u}$$

$$s \cdot u' = \frac{2u - 1 - u(1+u)}{1+u} = \frac{u - 1 - u^2}{1+u}$$

$$\frac{u' \cdot (1+u)}{u-1-u^2} = \frac{1}{s}$$

$$\int \frac{u' \cdot (1+u)}{u-1-u^2}\, ds = \int \frac{1}{s}\, ds$$

integrate [u'(s) * (1+u) / (u-1-u^2) du]

NATURAL LANGUAGE MATH INPUT EXTENDED KEYB

Indefinite integral

$$\int \frac{u'(s)(1+u(s))}{u(s)-1-u(s)^2}\, du(s) = -\frac{1}{2} u'(s) \left(\log\left(u(s)^2 - u(s) + 1\right) + 2\sqrt{3} \tan^{-1}\left(\frac{2u(s)-1}{\sqrt{3}}\right)\right) + \text{constant}$$

???

Esta ecuación, por lo dicho antes, es homogénea de grado 0.

(b) Resuelva las ecuaciones:

i) $x' = \frac{2x-t+4}{x+t-1}$ ii) $x' = \frac{x+t+4}{t-x-6}$

iii) $x' = \frac{x+t+4}{x+t-6}$, $x(0) = 2$. ¿Se satisface $ae \neq bd$ en este caso?

i) a=2, b=-1, c=4, d=1, e=1, f=-1. $ae - bd = 2.1 + 1.1 = 3 \neq 0$. Podremos llevarla a una homogénea.

Ojo que acá cambié de lugar a x y t (tendría que ser a=-1, b=2, ...)

$c = a.h + b.k \Rightarrow 4 = 2h - k \Rightarrow h = 1,\ k =- 2$
$f = d.h + e.k \Rightarrow - 1 = h + k$

Propongo un reemplazo: $t = s - k,\ x = y - h$ (porque intercambié las letras)

$2x - t + 4 = 2(y - 1) - (s + 2) + 4 = 2y - 2 - s - 2 + 4 = 2y - s$
$x + t - 1 = (y - 1) + (s + 2) - 1 = y - 1 + s + 2 - 1 = y + s$

$$\frac{dx}{dt} = \frac{2x-t+4}{x+t-1} \Rightarrow \frac{dy}{ds} = \frac{2y-s}{y+s}$$

Lo resolvemos con un cambio de variables $z = y/s \Leftrightarrow y = zs$.
$dy/ds = y' = z's + z$

$$z's + z = \frac{2zs-s}{zs+s} = \frac{2z-1}{z+1} \Rightarrow z' = \left(\frac{2z-1}{z+1} - z\right)\frac{1}{s}$$

$$\frac{2z-1}{z+1} - z = \frac{2z-1}{z+1} - \frac{z(z+1)}{z+1} = \frac{-z^2-z+2z-1}{z+1} = \frac{-z^2+z-1}{z+1}$$

falta z'?

$$\Rightarrow \int -\frac{z+1}{z^2-z+1}dz = \int \frac{1}{s}ds$$

Tal vez porque $dz = z'.ds$ pues $z' = \frac{dz}{ds}$

Pero operar con diferenciales es medio rancio

$$-\frac{1}{2}\int \frac{2z+2}{z^2-z+1}dz =- \frac{1}{2}\int \frac{2z-1+3}{z^2-z+1}dz =- \frac{1}{2}\int \frac{2z-1}{z^2-z+1} + \frac{3}{z^2-z+1}dz$$

Sustitución: $u = z^2 - z + 1,\ du = 2z - 1\ dz$

$z^2 - z + 1 = z^2 - z + 1/4 + 3/4 = (z - 1/2)^2 + 3/4$
$w = z - 1/2,\ dw = dz$

$$=- \frac{1}{2}\int \frac{1}{u}du + \int \frac{3}{w^2+3/4}dw$$

$$=- \frac{1}{2}ln|u| + \int \frac{4}{(4/3w^2+1)}dw =- \frac{1}{2}ln|u| + \frac{4}{\sqrt{4/3}}\int \frac{1}{v^2+1}dv$$

$v = \sqrt{4/3}w \Rightarrow dv = \sqrt{4/3}dw$

$$=- \frac{1}{2}ln|u| + \frac{4}{\sqrt{4/3}}arctan(v)$$

Llamo $u = \frac{y}{s} \Rightarrow us = y \Rightarrow y' = u's + u$

$y' = u's + u = \frac{2u-1}{u+1}$

$\frac{du}{ds}s = \frac{2u-1-u\cdot(u+1)}{u+1}$

$\frac{du}{ds}s = \frac{2u-1-u^2-\overset{u}{\cancel{1}}}{u+1}$ → Arrastré error

wat!?

$\frac{du}{ds}s = \frac{-u^2+2u-2}{u+1}$

$\int \frac{u+1}{-u^2+2u-2}du = \int \frac{1}{s}ds$

$= ln|s| + C$

Ejemplo (6)

$$y'(x) = \frac{y + x + 4}{x - y - 6}$$

Resolvemos $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ $\quad h = \cancel{5}1$, $k = 5$

$x = s + 1 \qquad y = u - 5$

$$u'(s) = \frac{u - 5 + s + 1 + 4}{s + 1 - u + 5 - 6} = \frac{u + s}{s - u} \quad \text{homogénea.}$$

Proponemos $u = s.v \to$

$$v + s v' = \frac{sv + s}{s - sv} = \frac{v+1}{1-v}$$

$$s v' = \frac{v+1}{1-v} - v = \frac{\cancel{v} + 1 \cancel{- v} + v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\int \frac{1 - v(s)}{1 + v^2(s)} v'(s)\, ds = \int \frac{1}{s}\, ds$$

$$\arctan(v) - \frac{1}{2}\ln(1+v^2) = \ln(s) + C$$

Volvemos hacia atrás con los cambios de variable.