

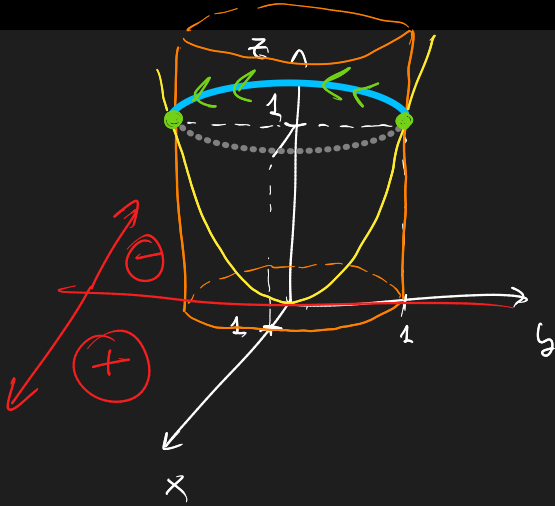
ANÁLISIS II / MATEMÁTICA 3 / ANÁLISIS MATEMÁTICO II
SEGUNDO CUATRIMESTRE 2021 - PRIMER PARCIAL (13/10/2021)
TEMA 3

Ejercicio 1

Sea $C := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = x^2 + y^2, x \leq 0\}$.

(a) Dar una parametrización regular de C que empiece en el $(0, 1, 1)$ y termine en el $(0, -1, 1)$.

(b) Calcular $\int_C (0, y, xy) \cdot ds$, donde C está orientada como en el ítem anterior.



$$a) \sigma(t) = (\cos t, \sin t, 1) \\ t \in [0, \pi]$$

$$b) F = (0, y, xy)$$

$$\int_C F \cdot ds = \int_{t=0}^{\pi} \left\langle F(\sigma(t)), \sigma'(t) \right\rangle dt \\ \left\langle (0, \sin t, \cos t \cdot \sin t), (-\sin t, \cos t, 0) \right\rangle \\ = \sin t \cdot \cos t$$

$$= \int_{t=0}^{\pi} \sin t \cdot \cos t \, dt$$

$$u = \sin t$$

$$du = \cos t \, dt$$

Ejercicio 2

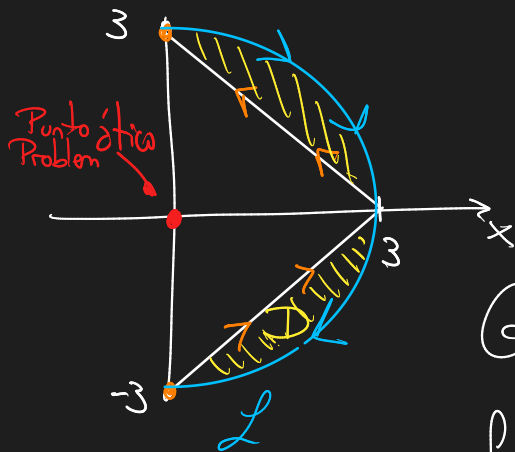
Sea $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ el campo dado por

$$\mathbf{F}(x, y) = \left(x \operatorname{sen}(\sqrt{x^2 + y^2}) - \frac{y}{x^2 + y^2}, y \operatorname{sen}(\sqrt{x^2 + y^2}) + \frac{x}{x^2 + y^2} \right).$$

Calcular $\int_C \mathbf{F} \cdot d\mathbf{s}$, donde C es la curva dada por la unión de los dos segmentos de recta

$$\begin{cases} y = 3 - x, & 0 \leq y \leq 3 \\ y = x - 3, & -3 \leq y \leq 0 \end{cases}$$

recorrida desde el $(0, -3)$ al $(0, 3)$.



$$\int_C \mathbf{F} \cdot d\mathbf{s} = ?$$

Green

sentido opuesto

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = - \iint_D (Q_x - P_y) \, dx \, dy$$

$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_L \mathbf{F} \cdot d\mathbf{s} = - \iint_D (Q_x - P_y) \, dx \, dy$$

$$\mathbf{F}(x, y) = \left(\underbrace{x \operatorname{sen}(\sqrt{x^2 + y^2}) - \frac{y}{x^2 + y^2}}_P, \underbrace{y \operatorname{sen}(\sqrt{x^2 + y^2}) + \frac{x}{x^2 + y^2}}_Q \right).$$

$$Q_x = y \cdot \cos \sqrt{x^2 + y^2} \cdot \frac{1}{2 \sqrt{x^2 + y^2}} \cdot 2x + \frac{1}{x^2 + y^2} - \frac{2x^2}{x^2 + y^2}$$

$$\begin{aligned} \text{CA: } \frac{d}{dx} x (x^2 + y^2)^{-1} &= \frac{1}{x^2 + y^2} - x (x^2 + y^2)^{-2} \cdot 2x \\ &= \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} \end{aligned}$$

$$\begin{aligned}\frac{d}{dy} -y(x^2+y^2)^{-1} &= -\frac{1}{x^2+y^2} + y(x^2+y^2)^{-2} \cdot 2y \\ &= -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\end{aligned}$$

$$P_y = y \cdot \cos \sqrt{x^2+y^2} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$Q_x = y \cdot \cos \sqrt{x^2+y^2} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$Q_x - P_y = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{2}{x^2+y^2} - \frac{2x^2 + 2y^2}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2)}{(x^2+y^2)^2} \quad x^2+y^2 \neq 0$$

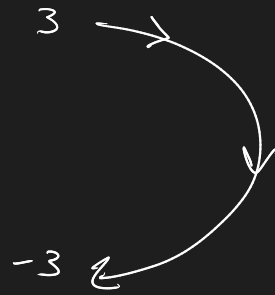
$$= 0$$

//

o.o

$$\oint_C \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 0$$

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = - \int_L \mathbf{F} \cdot d\mathbf{s}$$



Parametrizo L

$$\sigma(t) = \left(3 \cdot \cos\left(\frac{\pi}{2} - t\right), 3 \cdot \sin\left(\frac{\pi}{2} - t\right) \right)$$

$t \in [0, \pi]$

$$\mathbf{F}(x, y) = \left(x \operatorname{sen}(\sqrt{x^2 + y^2}) - \frac{y}{x^2 + y^2}, y \operatorname{sen}(\sqrt{x^2 + y^2}) + \frac{x}{x^2 + y^2} \right).$$

$$- \int_L \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{\pi} \left\langle \mathbf{F}(\sigma(t)), \left(3 \sin\left(\frac{\pi}{2} - t\right), -3 \sin\left(\frac{\pi}{2} - t\right) \right) \right\rangle dt$$

$$\left(3 \cos\left(\frac{\pi}{2} - t\right) \cdot \sin(3) - \frac{3 \cdot \sin\left(\frac{\pi}{2} - t\right)}{9}, \right.$$

$$\left. 3 \cdot \sin\left(\frac{\pi}{2} - t\right) \cdot \sin(3) + \frac{3 \cdot \cos\left(\frac{\pi}{2} - t\right)}{9} \right)$$

Horrible: sigo con otra param.

Parametrizo $\mathcal{L}^- \leftarrow$ sentido opuesto!

$$\sigma(t) = (3 \cdot \cos t, 3 \cdot \sin t)$$



$$t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\mathbf{F}(x, y) = \left(x \operatorname{sen}(\sqrt{x^2 + y^2}) - \frac{y}{x^2 + y^2}, y \operatorname{sen}(\sqrt{x^2 + y^2}) + \frac{x}{x^2 + y^2} \right).$$

$$-\int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{L}^-} \mathbf{F} \cdot d\mathbf{s} = \int_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle$$

$$\mathbf{F}(\sigma(t)) = \left(3 \cdot \cos t \cdot \sin(3) - \frac{3 \sin t}{9}, 3 \cdot \sin t \cdot \sin(3) + \frac{3 \cdot \cos t}{9} \right)$$

$$\sigma'(t) = (-3 \cdot \sin t, 3 \cdot \cos t)$$

$$\langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle = -9 \cdot \cos t \cdot \sin t \cdot \sin 3 + \sin^2 t + 9 \cos t \sin t \cdot \sin 3 + \cos^2 t$$

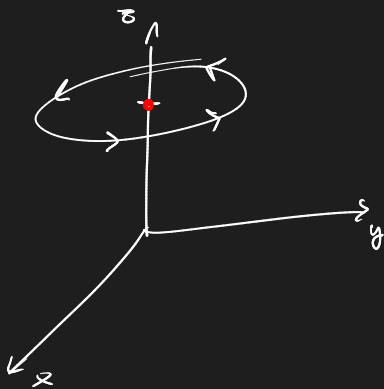
$$= \sin^2 t + \cos^2 t = 1$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dt = \pi$$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{L}^-} \mathbf{F} \cdot d\mathbf{s} = \pi //$$

Ejercicio 3 Sea $C := \{(x, y, z) \in \mathbb{R}^3 : z = 3, x^2 + y^2 = 1\}$ orientada de manera tal que al proyectarla en el plano xy se recorra en sentido positivo. Calcular $\int_C F \cdot ds$, donde

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + (z-3)^2} + \frac{(z-3)^3}{3}, \frac{y}{x^2 + y^2 + (z-3)^2} + \frac{x^3}{3}, \frac{z-3}{x^2 + y^2 + (z-3)^2} + \frac{y^3}{3} \right).$$



$$F = G + H$$

$$G = \left(\frac{x}{x^2 + y^2 + (z-3)^2}, \frac{y}{x^2 + y^2 + (z-3)^2}, \frac{z}{x^2 + y^2 + (z-3)^2} \right)$$

$$H = \left(\frac{(z-3)^3}{3}, \frac{x^3}{3}, \frac{y^3}{3} \right)$$

$$\Rightarrow \int_C F \cdot ds = \int_C G \cdot ds + \int_C H \cdot ds$$

Stokes sobre H por $H \in \mathcal{C}^1$

$$\int_{C=\partial S} H \cdot ds = \int \int_S \nabla \times H \cdot ds$$

$$\nabla \times H = \begin{vmatrix} \begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{(z-3)^3}{3} & \frac{x^3}{3} & \frac{y^3}{3} \end{matrix} \end{vmatrix}$$

$$= \left(y^2, -(0 - (z-3)^2), x^2 \right)$$

$$= \left(y^2, (z-3)^2, x^2 \right)$$

$$T(r, \theta) = (r \cdot \cos \theta, r \cdot \sin \theta, 3)$$

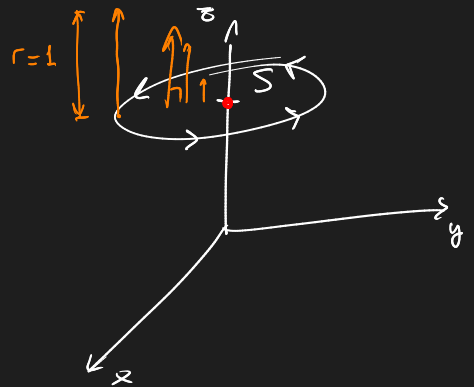
$$r \in [0, 1]$$

$$\theta \in [0, 2\pi)$$

$$T_r = (\cos \theta, \sin \theta, 0)$$

$$T_\theta = (-r \cdot \sin \theta, r \cdot \cos \theta, 0)$$

$$T_r \times T_\theta = (0, 0, r)$$



$$\iint_S \nabla \times H \cdot d\mathbf{s} = \int_0^1 \int_0^{2\pi} \underbrace{\left\langle \nabla \times H(T(r, \theta)), T_r \times T_\theta \right\rangle}_{" } d\theta dr$$

$$\nabla \times H = (y^2, -(z-3)^2, x^2) \quad \left\{ \begin{array}{l} \rightarrow (r^2 \cdot \sin^2 \theta, 0, r^2 \cdot \cos^2 \theta) \\ T(r, \theta) = (r \cdot \cos \theta, r \cdot \sin \theta, 3) \end{array} \right.$$

$$= \int_0^1 \int_0^{2\pi} r^3 \cdot \cos^2 \theta \, d\theta \, dr$$

$$= \int_{r=0}^1 r^3 \int_{\theta=0}^{2\pi} \cos^2 \theta \, d\theta \, dr$$

$$\int_{\theta=0}^{2\pi} \cos^2 \theta \, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 1 \, d\theta + \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, d\theta$$

CA

$$\frac{d}{d\theta} \frac{1}{2} \sin 2\theta = \frac{2}{2} \cdot \cos 2\theta$$

$$= \pi + \frac{1}{4} \cdot \sin 2\theta \Big|_0^{2\pi}$$

$$= \pi$$

$$\int_{r=0}^1 r^3 \int_{\theta=0}^{2\pi} \cos^2 \theta \, d\theta \, dr = \pi \cdot \int_{r=0}^1 r^3 \, dr$$

$$= \pi \cdot \frac{r^4}{4} \Big|_0^1$$

$$= \frac{\pi}{4}$$

$$= \int_C H \cdot ds$$

$$G = \left(\frac{x}{x^2+y^2+(z-3)^2}, \frac{y}{x^2+y^2+(z-3)^2}, \frac{z}{x^2+y^2+(z-3)^2} \right)$$

$$\int_C G \cdot ds = \int_{t=0}^{2\pi} \underbrace{\langle G(\sigma(t)), \sigma'(t) \rangle}_{\text{" "}} dt$$

$$\sigma(t) = (\cos t, \sin t, 3) \rightarrow (\cos t, \sin t, 3), (-\sin t, \cos t, 0)$$

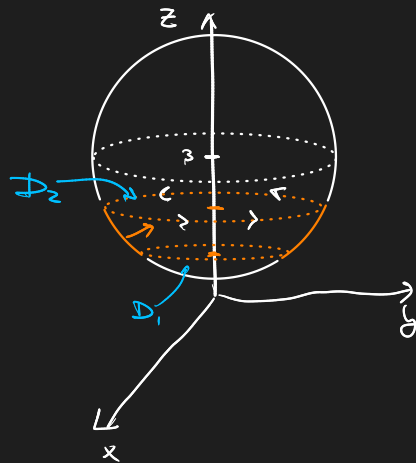
$$\text{" " } -\cos t \cdot \sin t + \cos t \cdot \sin t = 0$$

$$\int_C G \cdot ds = 0$$

0
0 0

$$\int_C F \cdot ds = \frac{\pi}{4} //$$

Ejercicio 4 Sea \mathbf{F} el campo vectorial dado por $\mathbf{F}(x, y, z) = (\sin(z^2) + 3xy, e^{x^3} - y^2, x^2 - yz)$. Calcular el flujo de \mathbf{F} a través de la superficie S dada por la sección de la esfera de ecuación $x^2 + y^2 + (z - 3)^2 = 5$ acotada entre los planos $z = 1$ y $z = 2$, orientada con la normal interior.



$$\operatorname{div} \mathbf{F} = 3y - 2y - y = 0$$

$$\mathbf{F} \in C^1$$

Uso Gauss

$$\int_{S^-} \mathbf{F} \cdot d\mathbf{s} + \int_{D_1} \mathbf{F} \cdot d\mathbf{s} + \int_{D_2} \mathbf{F} \cdot d\mathbf{s} = 0$$

||

$$- \int_S \mathbf{F} \cdot d\mathbf{s}$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{s} = \int_{D_1} \mathbf{F} \cdot d\mathbf{s} + \int_{D_2} \mathbf{F} \cdot d\mathbf{s}$$

$$\int_{D_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \langle \mathbf{F}(T_1(r, \theta)), T_{1r} \times T_{1\theta} \rangle dr d\theta$$

CA

en $z=1$: el radio de D_1 es:

$$x^2 + y^2 = 5 - (z-3)^2$$

$$r_1 = 1$$

$$r_2 = 2$$

$$\vec{T}_1(r, \theta) = (r \cos \theta, r \sin \theta, 1)$$

$$r \in [0, 1]$$

$$\theta \in [0, 2\pi)$$

Opuesto a la orientación de T_1

n apunta hacia abajo.

$$= - \int_{\theta=0}^{2\pi} \int_{r=0}^1 \left\langle (---, ---, r^2 \cos^2 \theta - r \sin \theta), (0, 0, r) \right\rangle dr d\theta$$

$$= - \int_{\theta=0}^{2\pi} \int_{r=0}^1 (r^3 \cos^2 \theta - r^2 \sin \theta) dr d\theta$$

$$\left(\frac{r^4}{4} \cos^2 \theta \Big|_0^1 - \frac{r^3}{3} \sin \theta \Big|_0^1 \right)$$

$$= - \int_0^{2\pi} \left(\frac{1}{4} \cos^2 \theta - \underbrace{\frac{1}{3} \sin \theta}_{=0} \right) d\theta$$

$$= - \frac{1}{4} \int_0^{2\pi} \cos^2 \theta d\theta$$

$$= - \frac{1}{4} \int_0^{2\pi} \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= - \frac{1}{8} \int_0^{2\pi} (1 + \cos 2\theta) d\theta$$

$$= - \frac{1}{4} \pi - \frac{1}{8} \underbrace{\int_0^{2\pi} \cos 2\theta d\theta}_{\frac{1}{2} \sin 2\theta \Big|_0^{2\pi}}$$

$$= - \frac{1}{4} \pi$$

$$T_2(r, \theta) = (r \cos \theta, r \sin \theta, 2)$$

$$r \in [0, 1]$$

$$\theta \in [0, 2\pi)$$

no invierte la orientación

$$= + \int_{\theta=0}^{2\pi} \int_{r=0}^2 \left\langle \left(\dots, \dots, r^2 \cos^2 \theta - 2, r \sin \theta, (0, 0, r) \right) \right\rangle dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 \cos^2 \theta - 2r^2 \sin \theta dr d\theta$$

$$\underbrace{\left(\frac{r^4}{4} \cos^2 \theta \Big|_0^2 - 2 \frac{r^3}{3} \sin \theta \Big|_0^2 \right)}_{4 \cos^2 \theta - \frac{16}{3} \sin \theta}$$

$$= \int_0^{2\pi} 4 \cos^2 \theta - \underbrace{\frac{16}{3} \sin \theta}_{=0} d\theta$$

$$= 4 \int_0^{2\pi} \cos^2 \theta d\theta$$

$$= 4\pi$$

Finalmente

$$\int_S F \cdot ds = \int_{D_1} F \cdot ds + \int_{D_2} F \cdot ds$$

$$= -\frac{1}{4}\pi + 4\pi = \frac{15}{4}\pi //$$

