Práctica 3: Teorema de Green.

Teo. de Green.

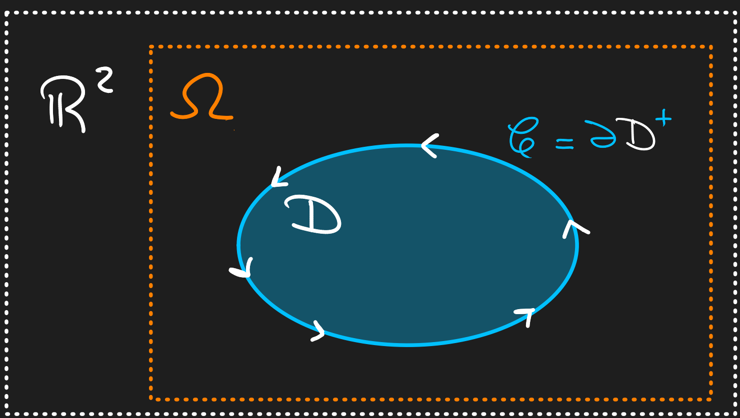
Hipótesis

Volvemos a $\mathbb{R}^2$! $F = (P, Q)$

- $\Omega \in \mathbb{R}^2$ abierto
- $F : \Omega \to \mathbb{R}^2$ es campo C^1
- D es una región de tipo III
- $D \subset \Omega$
- $\partial D = \mathcal{C}$ curva cerrada simple, diferenciable a trozos.
- $\mathcal{C}$ está orientada positivamente $\circlearrowleft$

Teorema

$$\iint_D Q_x - P_y \, dx\,dy = \int_{\mathcal{C}} F \cdot ds$$



Ejercicio 1. Verificar el Teorema de Green para el disco D con centro $(0,0)$ y radio R y las siguientes funciones:

(a) $P(x,y) = xy^2, \quad Q(x,y) = -yx^2.$

(b) $P(x,y) = 2y, \quad Q(x,y) = x.$

$$\iint_D Q_x - P_y \, dy\,dx \overset{?}{=} \int_{\mathcal{C}} F \cdot ds$$

a) $F = (xy^2, -yx^2)$ $\qquad \sigma(t) = (R\cdot\cos t, R\cdot\sin t)$

$t \in [0, 2\pi)$

$$\int_{\mathcal{C}} F \cdot ds = \int_{t=0}^{2\pi} \langle \underbrace{F(\sigma(t))}, \sigma'(t) \rangle \, dt$$

$$= (R^3 \cos t \cdot \sin^2 t, -R^3 \cdot \sin t \cdot \cos^2 t)$$

$$= R^3 \cdot (\cos t \sin^2 t, -\sin t \cos^2 t)$$

$$= \int_{t=0}^{2\pi} R^3 \cdot \underbrace{\langle (\cos t \cdot \sin^2 t, -\sin t \cdot \cos^2 t), (-R \cdot \sin t, R \cdot \cos t) \rangle}\, dt$$

$$= -R \cdot \cos t \cdot \sin^3 t - R \cdot \sin t \cdot \cos^3 t$$

$$= -R \cdot \cos t \cdot \sin t \cdot (\sin^2 t + \cos^2 t)$$

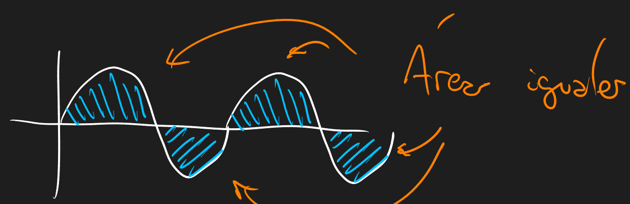
$$= \int_0^{2\pi} -R^4 \cdot \underbrace{\cos t \cdot \sin t}_{\text{es simétrica!}} \, dt$$

$u = \sin t$

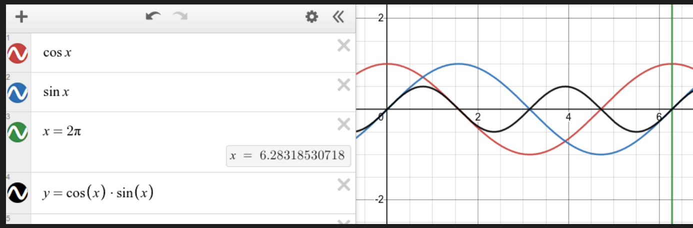
$du = \cos t \, dt$

Separo en intervalos con cortes en cada t tal que cost*sint=0

$$= -R^4 \cdot \int_{u=0} u \cdot du = 0$$



$$\iint_D Q_x - P_y \, dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \; ?$$



$$F = (\underbrace{xy^2}_{P}, \underbrace{-yx^2}_{=Q}) \Rightarrow Q_x = -2xy$$

$$\hookrightarrow P_y = 2xy$$

$$\Bigg\} \; Q_x - P_y = -2xy - 2xy = -4xy$$

Uso coord. polares

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases}$$

Jacobiano

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} -4 \cdot (r^2 \cdot \cos\theta \cdot \sin\theta) \cdot r \; dr\, d\theta$$

$$= -4 \int_{\theta=0}^{2\pi} \cos\theta \cdot \sin\theta \cdot \underbrace{\int_{r=0}^{R} r^3 \cdot dr}_{= \frac{1}{4} r^4 \Big|_0^R = \frac{R^4}{4}} d\theta$$

$$= -\frac{\cancel{4}}{\cancel{4}} R^4 \cdot \underbrace{\int_0^{2\pi} \cos\theta \cdot \sin\theta \, d\theta}_{=0} = 0 \;\checkmark$$

son iguales.

b) $F = (2y, x)$

$$\int_{\mathcal{C}} F \cdot ds = \int_0^{2\pi} \langle (2 \cdot R \cdot \sin\theta, R \cdot \cos\theta), (-R \cdot \sin\theta, R \cdot \cos\theta) \rangle d\theta$$

$$= \int_0^{2\pi} \underbrace{-2R^2 \cdot \sin^2\theta}_{-2R^2 \cdot (1-\cos^2\theta)} + R^2 \cdot \cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} -2R^2 + 2R^2 \cos^2\theta + R^2 \cdot \cos^2\theta \, d\theta$$

$$= \underbrace{\int_0^{2\pi} -2R^2 \, d\theta}_{= -4\pi \cdot R^2} + \underbrace{\int_0^{2\pi} 3R^2 \cdot \cos^2\theta \, d\theta}_{= 3R^2 \cdot \int_0^{2\pi} \cos^2\theta \, d\theta}$$

CA:

Trig ident

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \cos 2\theta + 1 \, d\theta \quad \left[\begin{array}{l} \cos 2\theta = 2\cos^2\theta - 1 \\ \Rightarrow \cos^2\theta = \frac{1}{2} \cos 2\theta + \frac{1}{2} \end{array} \right]$$

$$= \frac{1}{2} \underbrace{\int_0^{2\pi} \cos 2\theta \, d\theta}_{\frac{1}{2} \sin 2\theta \Big|_0^{2\pi} = 0} + \frac{1}{2} \cdot 2\pi$$

$$\frac{\partial}{\partial\theta} \frac{1}{2} \sin 2\theta = \frac{2}{2} \cos 2\theta$$

$$= \pi$$

$$\int_{\mathcal{C}} F \cdot ds = -4\pi \cdot R^2 + 3\pi R^2 = -\pi R^2$$

Veo

$$\iint_D Q_x - P_y \, dy\,dx = ?$$

$$F = (2y, x) \Rightarrow \begin{array}{l} Q_x = 1 \\ P_y = 2 \end{array}$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{R} (1-2) \cdot r \, dr\,d\theta = -\frac{R^2}{2} \cdot 2\pi = -\pi \cdot R^2 \checkmark$$

iguales

Ejercicio 2. Verificar el Teorema de Green y calcular $\int_{\mathcal{C}} y^2\,dx + x\,dy$, siendo $\mathcal{C}$ la curva recorrida en sentido positivo:

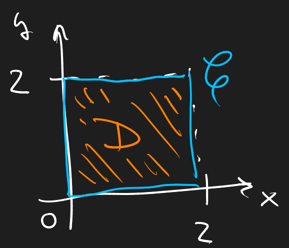
(a) Cuadrado con vértices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.

(b) Elipse dada por $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(c) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, donde $\mathcal{C}_1 : y = x$, $x \in [0,1]$, y $\mathcal{C}_2 : y = x^2$, $x \in [0,1]$.

$$F = (\underbrace{y^2}_{P}, \underbrace{x}_{Q}) \Rightarrow \begin{cases} Q_x = 1 \\ P_y = 2y \end{cases} \Rightarrow \int_{\mathcal{C}} F\,ds = \iint_D 1 - 2y \; dy\,dx$$

a)



$$\sigma_1(t) = (t, 0)$$
$$\sigma_2(t) = (2, t)$$
$$\left.\begin{array}{l} \sigma_3(t) = (t, 2) \\ \sigma_4(t) = (0, t) \end{array}\right\} \text{sentido opuesto!}$$

$$\int_{\mathcal{C}} F\cdot ds = \int_{t=0}^{2} \underbrace{\langle \underbrace{F(t,0)}_{(0,t)}, (1,0)\rangle}_{=0} dt + \int_{t=0}^{2} \underbrace{\langle \underbrace{F(2,t)}_{(t^2,2)}, (0,1)\rangle}_{=2} dt -$$

$$- \int_{t=0}^{2} \underbrace{\langle \underbrace{F(t,2)}_{(4,t)}, (1,0)\rangle}_{=4} dt - \int_{t=0}^{2} \underbrace{\langle \underbrace{F(0,t)}_{(t^2,0)}, (0,1)\rangle}_{=0} dt$$

$$= \int_0^2 2\,dt - \int_0^2 4\,dt$$

$$= 4 - 8 = -4 \,//$$

Con Green

- Región D es de tipo II ✓

$$D = \{ (x,y) \in \mathbb{R}^2 : 0 < x < 2 \ \wedge \ 0 < y < 2 \}$$

$$\iint_D Q_x - P_y \, dy dx = \int_0^2 \int_0^2 1 - 2y \, dy dx$$

$$= \int_0^2 2 - y^2 \Big|_0^2 dx$$

$$= \int_0^2 2 - 4 \, dx$$

$$= -4$$ ✓ son iguales !

b) (b) Elipse dada por $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$F = (\underbrace{y^2}_{P}, \underbrace{x}_{Q}) \Rightarrow \begin{cases} Q_x = 1 \\ P_y = 2y \end{cases} \Rightarrow \int_{\mathcal{C}} F\,ds = \iint_{D} 1 - 2y \; dy\,dx$$

Recordar! Param. de una elipse :

Si elipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \sigma(t) = (a.\cos t, \; b.\sin t)$ $t \in [0, 2\pi)$

sentido ⊕ ✓

$$\int_{\mathcal{C}} F\,ds = \int_{t=0}^{2\pi} \langle \underbrace{F(a.\cos t, b.\sin t)}_{(b^2.\sin^2 t, \; a.\cos t)}, (-a.\sin t, b.\cos t) \rangle \, dt$$

$$= \int_0^{2\pi} \underbrace{-a.b^2.\sin^3 t}_{} + a.b.\cos^2 t \; dt$$

$$= -a.b^2.\sin t.(1-\cos^2 t)$$

$$= -a.b^2.\sin t + a.b^2.\sin t.\cos^2 t$$

Recordar!

Trig ident.

- $2\cos^2 x = 1 + \cos 2x$

$$= a.b^2 \int_0^{2\pi} \underbrace{-\sin t}_{=0} + \sin t \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right).dt$$

$$= a.b^2 \cdot \int_0^{2\pi} \underbrace{\frac{1}{2}\sin t}_{=0} + \underbrace{\frac{1}{2}\cos 2t \, dt}_{=0}$$

$$= 0$$

$$\int_{\mathcal{C}} F\,ds = \int_0^{2\pi} a.b.\cos^2 t \, dt = a.b.\int_0^{2\pi} \cos^2 t \, dt$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$$

$$= a.b \int_0^{2\pi} \underbrace{\frac{1}{2}}_{=\pi} + \underbrace{\frac{1}{2}\cdot \cos 2t \, dt}_{=0}$$

$$= a.b.\pi$$

Recordar!

Coords elípticas

$$x = r.a.\cos\theta$$

$$y = r.b.\cos\theta \qquad r \in [0,1] \quad \theta \in [0,2\pi)$$

Jacob: $a.b.r$

Ahora con Green:

Jacobiano!

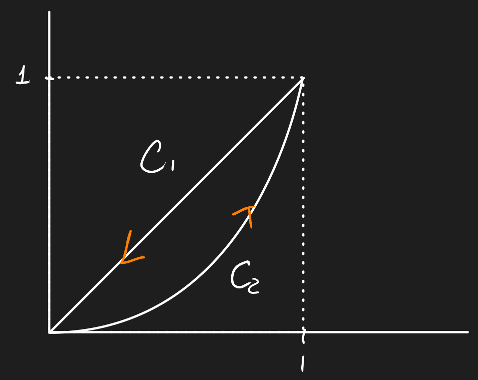
$$\iint_D Q_x - P_y \, dy\,dx = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 - 2.r.b.\cos\theta).\, a.b.r \, dr\, d\theta$$

$$= \int_0^{2\pi} a.b.\frac{r^2}{2}\Big|_0^1 - 2.a.b^2.\frac{r^3}{3}.\cos\theta\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}a.b - \underbrace{\frac{2}{3}a.b^2 \cdot \cos\theta \, d\theta}_{=0}$$

$= a \cdot b \cdot \pi$ // ✓ son iguales!

(c) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, donde $\mathcal{C}_1 : y = x$, $x \in [0,1]$, y $\mathcal{C}_2 : y = x^2$, $x \in [0,1]$.



$$F = (y^2, x)$$

$$Q_x - P_y = 1 - 2y$$

$$\sigma_1(t) = (t, t) \quad t \in [0,1]$$
$$\sigma_2(t) = (t, t^2)$$

$$\int_C F\, ds = -\underbrace{\int_{t=0}^{1} \langle (t^2, t), (1,1) \rangle\, dt}_{\text{I}} + \underbrace{\int_0^1 \langle (t^4, t), (1, 2t) \rangle\, dt}_{\text{II}}$$

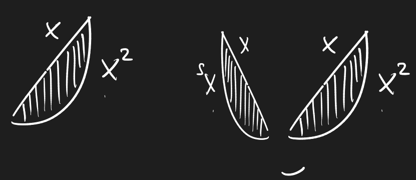
$$\text{I} = -\int_0^1 t^2 + t\; dt = -\left(\overset{2/6}{\frac{1}{3}} + \overset{3/6}{\frac{1}{2}}\right) = -5/6$$

$$\text{II} = \int_0^1 t^4 + 2t^2\; dt = \overset{3/15}{\frac{1}{5}} + \overset{10/15}{\frac{2}{3}} = \frac{13}{15}$$

$$\int_C F\, ds = \frac{1}{30} \;//$$

Con Green:

$$D = \{(x,y) \in \mathbb{R} : 0 \leq x \leq 1 \wedge x^2 \leq y \leq x\}$$



$$\iint_D Q_x - P_y\, dy\,dx = \int_{x=0}^{1} \int_{y=x^2}^{x} 1 - 2y\; dy\,dx$$

$$= \int_0^1 x - x^2 - \left(y^2 \Big|_{x^2}^{x} \right) dx$$

$$= \int_0^1 x - x^2 - x^2 + x^4 \, dx$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5}$$

$$= \frac{1}{30} \checkmark$$ asombrosamente, son iguales!

Ejercicio 3. Usando el teorema de Green, hallar el área de:

(a) El disco D con centro $(0,0)$ y radio R

(b) La región dentro de la elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

a)

Debo definir un campo F tal que $Q_x - P_y = 1$

(o cualquier constante c)

$$\Rightarrow Q_x = 1 + P_y$$

Si $P_y = 1 \Rightarrow Q_x = 2$

$$\Rightarrow \begin{cases} P = y \\ Q = 2x \end{cases} \Rightarrow F = (y, 2x) \quad (\text{hay infinitos } F\,!)$$

$$\iint_D Q_x - P_y \, dy\,dx = \iint_D 2-1 \, dy\,dx = \iint_D 1 \, dy\,dx = \text{Área}(D)$$

Con Polares, resolvería:

$$\begin{cases} x = r \cdot \cos\theta & r \in [0, R] \\ y = r \cdot \sin\theta & \theta \in [0, 2\pi) \end{cases}$$

Jacobiano de Polares

$$\iint_D 1 \, dy\,dx = \int_0^{2\pi} \int_0^R r \, dr\,d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2} = \pi \cdot R^2 \quad //$$

Pero quiero mostrar que

$$\int_{\mathcal{C}} F \cdot ds = \text{Area}(D) \quad \text{pues} \quad \int_{\mathcal{C}} F\, ds = \iint_D Q_x - P_y \, dy\, dx$$

$$= \int_{t=0}^{2\pi} \langle \underbrace{F(R.\cos t, R.\sin t)}_{(R.\sin t,\ 2.R.\cos t)}, (-R.\sin t, R.\cos t) \rangle \, dt$$

$$= \int_0^{2\pi} -R^2 \cdot \sin^2 t + 2.R^2.\cos^2 t \;\; dt$$

Nota: Con $F = (-y, x)$ salía de una!

$$2\cos^2 t = 1 + \cos 2t$$

$$\sin^2 t = 1 - \cos^2 t$$

$$= \int_0^{2\pi} \underbrace{\underbrace{-R^2 \cdot (1 - \cos^2 t)}_{= -R^2 + \underbrace{R^2 \cos^2 t}_{R^2 \cdot \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)}}}_{= -R^2 + \frac{1}{2}R^2} + \underbrace{R^2 \cdot (1 + \cos 2t)}_{= R^2 \quad \left(\text{pues } \int_0^{2\pi} \cos 2t = 0\right)} \, dt$$

$$= \int_0^{2\pi} \frac{1}{2} R^2 \, dt$$

$$= \pi . R^2 \quad // \quad \checkmark$$

(b) La región dentro de la elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Es lo mismo.

Debe dar $\pi . a . b$

quiero que sea 1!

Uso $F = (-y, x) \Rightarrow Q_x - P_y = 1 + 1 = 2$

$$\frac{1}{2}\int F \cdot ds = \frac{1}{2}\int_{t=0}^{2\pi} \left\langle \underbrace{\underbrace{F(a.\cos t, b.\sin t)}_{(-b.\sin t,\ a.\cos t)}, (-a.\sin t, b.\cos t)}_{\underbrace{ab.\sin^2 t + ab.\cos^2 t}_{= a.b}} \right\rangle dt$$

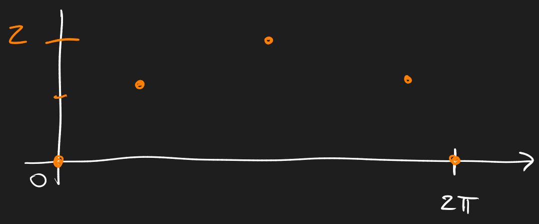
$= \frac{1}{2} a.b . 2\pi$

$= a.b.\pi$ ✓

Ejercicio 4. Sea D la región encerrada por el eje x y el arco de cicloide:

$$x=\theta-\sin\theta,\quad y=\underbrace{1-\cos\theta}_{\geq 0},\quad 0\le\theta\le 2\pi.$$

Usando el teorema de Green, calcular el área de D.



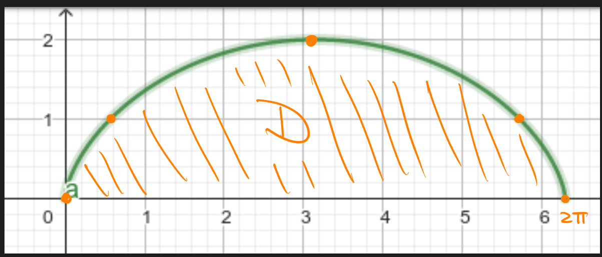
$\theta=0 \Rightarrow (0,0)$

$\theta=\frac{\pi}{2} \Rightarrow \left(\underbrace{\frac{\pi}{2}-1}_{\sim 0{,}57},\ 1\right)$

$\theta=\pi \Rightarrow (\pi,\ 2)$

$\theta=\frac{3}{2}\pi \Rightarrow \left(\frac{3}{2}\pi-1,\ \dots\right.$

Sentido opuesto a Green!



$$\sigma_1(t)=(t,0)\qquad t\in[0,2\pi)$$

$$\sigma_2(t)=(t-\sin t,\ 1-\cos t)\qquad t\in[0,2\pi)$$

sentido opuesto!

$$F=(-y,x)$$

$$\frac{1}{2}\oint_{\mathcal{C}} F\cdot ds=\frac{1}{2}\int_{t=0}^{2\pi}\underbrace{\langle \underbrace{F(t,0)}_{(0,t)},(1,0)\rangle}_{=0}\,dt-\frac{1}{2}\int_{t=0}^{2\pi}\langle \underbrace{F(\sigma_2(t))}_{(\underbrace{\cos t-1}_{-(1-\cos t)},\ t-\sin t)},\ \underbrace{\sigma'(t)}_{(1-\cos t,\ \sin t)}\rangle\,dt$$

$$=-\frac{1}{2}\int_0^{2\pi}\underbrace{-(1-\cos t)^2}_{=-1+2\cos t-\cos^2 t}+t.\sin t-\sin^2 t\,dt$$

$$=-\frac{1}{2}\int_0^{2\pi}-2+2\cos t+t.\sin t\,dt$$

$$=-\frac{1}{2}\left(-4\pi+\underbrace{2\sin t\Big|_0^{2\pi}}_{=0}+\int_0^{2\pi}t.\sin t\,dt\right)$$

$$= 2\pi - \frac{1}{2} \cdot \int_0^{2\pi} t \cdot \sin t \, dt$$

$u = t \qquad du = 1 \cdot dt$

$v = -\cos t \qquad dv = \sin t \, dt$

CA: $\int u \cdot dv = u \cdot v - \int v \cdot du$

$$\int t \cdot \sin t \, dt = -t \cdot \cos t - \int -\cos t \cdot dt$$

$$= -t \cdot \cos t + \sin t$$

Verifico (derivo):

$$-\cancel{\cos t} + t \cdot \sin t + \cancel{\cos t} = t \cdot \sin t \checkmark$$

$$\int_0^{2\pi} t \cdot \sin t \, dt = -t \cdot \cos t + \underbrace{\sin t}_{=0} \Big|_0^{2\pi}$$

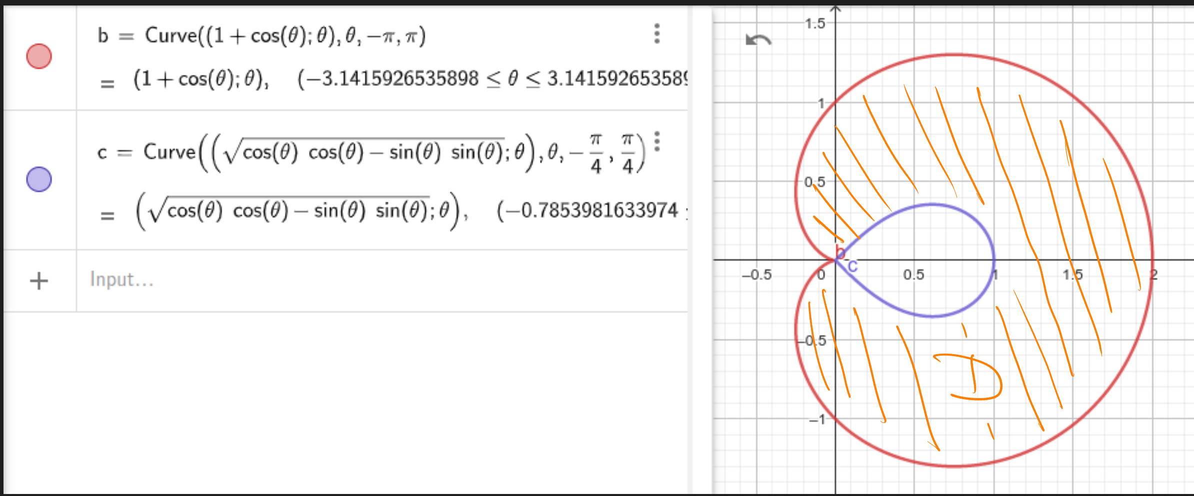
$$= -2\pi$$

$$\frac{1}{2} \int_{\mathcal{C}} F \cdot ds = 2\pi - \frac{1}{2} \cdot (-2\pi)$$

$$\frac{1}{2} \int_{\mathcal{C}} F \cdot ds = 3\pi \; //$$

Ejercicio 5. Hallar el área entre las curvas dadas en polares por

$$r = 1 + \cos\theta \qquad \text{con} \qquad -\pi \leq \theta \leq \pi,$$
$$r = \sqrt{\cos^2\theta - \operatorname{sen}^2\theta} \qquad \text{con} \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}.$$



$F = (-y, x)$

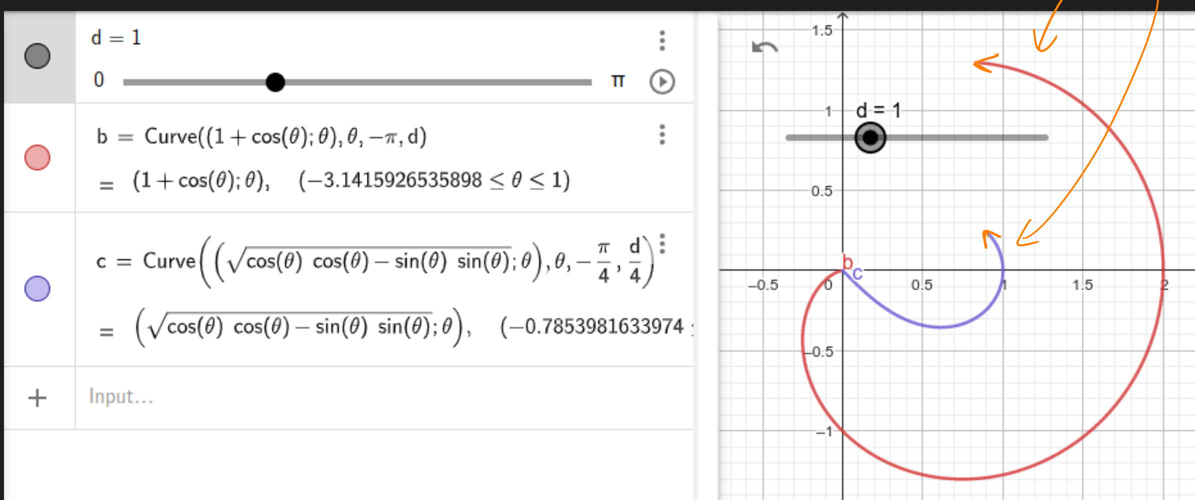
$$\frac{1}{2} \iint_D Q_x - P_y \, dy\,dx = \text{Área}(D)$$

Pero! no es una curva simple! Separo en $\mathcal{C}_{int}$ y $\mathcal{C}_{ext}$

Por Green

$$\frac{1}{2} \iint_D Q_x - P_y \, dy\,dx = \frac{1}{2} \int_{\mathcal{C}_{ext}} F \cdot ds - \frac{1}{2} \int_{\mathcal{C}_{int}} F \cdot ds$$

orientación correcta en ambas!



$$\frac{1}{2}\int_{\mathcal{C}_{ext}} F \cdot ds = \frac{1}{2}\int_{t=-\pi}^{\pi} \langle \underbrace{F(1+\cos t, t)}_{(-t,\ 1+\cos t)}, (-\sin t, 1) \rangle \, dt$$

$$\underbrace{}_{t \cdot \sin t + 1 + \cos t}$$

$$= \underbrace{\frac{1}{2} t \Big|_{-\pi}^{\pi}}_{=\pi} + \underbrace{\frac{1}{2} \sin t \Big|_{-\pi}^{\pi}}_{=0} + \underbrace{\frac{1}{2}\int_{-\pi}^{\pi} t \cdot \sin t \, dt}_{\text{lo resolví arriba}}$$

$$\frac{1}{2}\left(-t \cdot \cos t + \sin t \Big|_{-\pi}^{\pi}\right.$$

$$\frac{1}{2}\left(-\pi - (+\pi)\right) = \pi$$

$$\frac{1}{2}\int_{\mathcal{C}_{ext}} F \cdot ds = 2\pi$$

$$\frac{1}{2}\int_{\mathcal{C}_{int}} F \cdot ds \overset{\circledast}{=} \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \langle F\left(\sqrt{\cos^2 t - \sin^2 t}, t\right), \ldots$$

CA:

$$\frac{d}{dt}\sqrt{\cos^2 t - \sin^2 t} = \frac{1}{2\sqrt{\cos^2 t - \sin^2 t}} \cdot \left(-2\cos t \cdot \sin t - 2\sin t \cdot \cos t\right)$$

$$= -\frac{2\sin t \cdot \cos t}{\sqrt{\cos^2 t - \sin^2 t}}$$

Trig ident.

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$= 1 - 2\sin^2 t$$

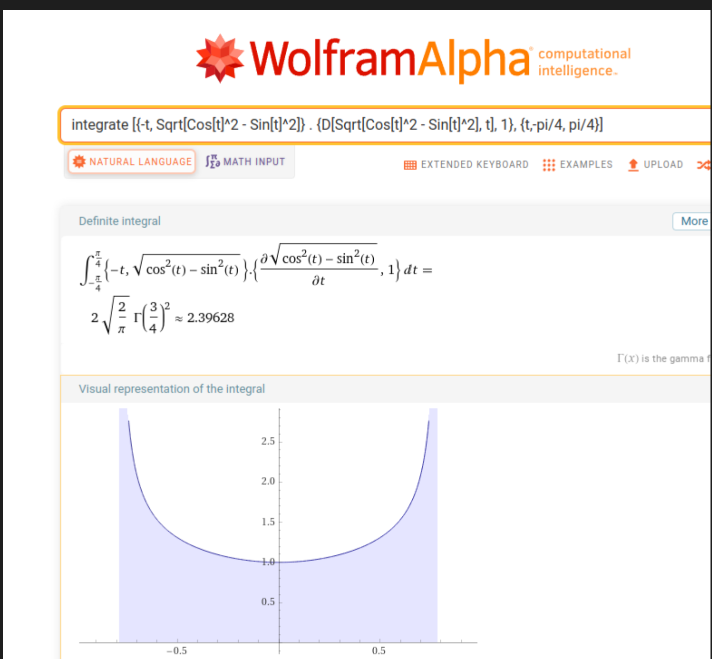
$$= -\frac{2\sin t \cdot \cos t}{\sqrt{\cos 2t}}$$

$$\circledast \quad \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\langle \underbrace{F\left(\sqrt{\cos^2 t - \sin^2 t}, t\right)}_{\left(-t,\ \sqrt{\cos 2t}\right)}, \left(\frac{-2\sin t \cdot \cos t}{\sqrt{\cos 2t}}, 1\right)\right\rangle dt$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2t \cdot \sin t \cdot \cos t}{\sqrt{\cos 2t}} + \sqrt{\cos 2t}\ dt$$

$\neq 0$ ✓

???



No sé cómo seguir.

Da aprox $\frac{1}{2} \cdot 2{,}39628$

$\approx 1{,}19814$

$$\boxed{\text{Total} = 2\pi - 1{,}19814}$$

(2x sin x cos x / sqrt(cos (2x)) + sqrt(cos (2x) **Go!**

CLR + − × ÷ ^ √ ³√ π ( )

This will be calculated:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2x\sin(x)\cdot\frac{\cos(x)}{\sqrt{\cos(2x)}}+\sqrt{\cos(2x)}\right)\mathrm{d}x$$

Not what you mean? *Use parentheses!* Set integration variable and bounds in "*Options*".

Configure the Integral Calculator:

Variable of integration:	x
Upper bound (to):	pi/4 (+∞)
Lower bound (from):	-pi/4 (−∞)
Integrate numerically only?	☐
Simplify expressions?	☐
Simplify all roots? (√x² becomes x, not |x|)	☐
Use complex domain (ℂ)?	☐
Keep decimals?	☐

Result

Done! See the result further below.
In order to not miss anything, please scroll all the way down.

= Check your own answer
= Export the expression (e. g. LaTeX)

YOUR INPUT:
$\mathbf{f(x)} =$

$$\sqrt{\cos(2x)}+\frac{2x\cos(x)\sin(x)}{\sqrt{\cos(2x)}}$$

Note: Your input has been rewritten/simplified.

Simplify/rewrite:

$$\sqrt{\sec(2x)}\,(x\sin(2x)+\cos(2x))$$

Simplify

"MANUALLY" COMPUTED ANTIDERIVATIVE:
$\int \mathbf{f(x)\,dx = F^*(x)} =$

"Manual" integration with steps:
The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might

"Manual" integration with steps:
The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might only be valid for parts of the function.

$$2\,\mathrm{E}(x\,|\,2)-x\sqrt{\cos(2x)}+C$$

This function cannot be graphed.

Hide steps

Problem:

$$\int\left(\sqrt{\cos(2x)}+\frac{2x\cos(x)\sin(x)}{\sqrt{\cos(2x)}}\right)\mathrm{d}x$$

Rewrite/simplify:

$$=\int\left(\frac{x\sin(2x)}{\sqrt{\cos(2x)}}+\sqrt{\cos(2x)}\right)\mathrm{d}x$$

... or choose an alternative:

Skip simplification

Apply linearity:

$$=\int\frac{x\sin(2x)}{\sqrt{\cos(2x)}}\,\mathrm{d}x+\int\sqrt{\cos(2x)}\,\mathrm{d}x$$

Now solving:

$$\int\frac{x\sin(2x)}{\sqrt{\cos(2x)}}\,\mathrm{d}x$$

Rewrite/simplify using trigonometric/hyperbolic identities:

$$=\int\frac{2x\cos(x)\sin(x)}{\sqrt{2\cos^2(x)-1}}\,\mathrm{d}x$$

Apply linearity:

$$=2\int\frac{x\cos(x)\sin(x)}{\sqrt{2\cos^2(x)-1}}\,\mathrm{d}x$$

Now solving:

$$\int\frac{x\cos(x)\sin(x)}{\sqrt{2\cos^2(x)-1}}\,\mathrm{d}x$$

Integrate by parts: $\int \mathrm{f}\mathrm{g}' = \mathrm{f}\mathrm{g}-\int \mathrm{f}'\mathrm{g}$

$$\mathrm{f}=x,\quad \mathrm{g}'=\frac{\cos(x)\sin(x)}{\sqrt{2\cos^2(x)-1}}$$

↓ steps ↓ steps

$$\mathrm{f}'=1,\quad \mathrm{g}=-\frac{\sqrt{2\cos^2(x)-1}}{2}:$$

$$=-\frac{x\sqrt{2\cos^2(x)-1}}{2}-\int-\frac{\sqrt{2\cos^2(x)-1}}{2}\,\mathrm{d}x$$

Now solving:

$$\int-\frac{\sqrt{2\cos^2(x)-1}}{2}\,\mathrm{d}x$$

$$=\int-\frac{\sqrt{1-2\sin^2(x)}}{2}\,\mathrm{d}x$$

Apply linearity:

$$=-\frac{1}{2}\int\sqrt{1-2\sin^2(x)}\,\mathrm{d}x$$

Now solving:

$$\int\sqrt{1-2\sin^2(x)}\,\mathrm{d}x$$

This is a special integral (incomplete elliptic integral of the second kind):

$$=\mathrm{E}(x\,|\,2)$$

Plug in solved integrals:

$$-\frac{1}{2}\int\sqrt{1-2\sin^2(x)}\,\mathrm{d}x$$

$$=-\frac{\mathrm{E}(x\,|\,2)}{2}$$

Plug in solved integrals:

$$-\frac{x\sqrt{2\cos^2(x)-1}}{2}-\int-\frac{\sqrt{2\cos^2(x)-1}}{2}\,\mathrm{d}x$$

$$=\frac{\mathrm{E}(x\,|\,2)}{2}-\frac{x\sqrt{2\cos^2(x)-1}}{2}$$

Plug in solved integrals:

$$2\int\frac{x\cos(x)\sin(x)}{\sqrt{2\cos^2(x)-1}}\,\mathrm{d}x$$

$$=\mathrm{E}(x\,|\,2)-x\sqrt{2\cos^2(x)-1}$$

Now solving:

$$\int\sqrt{\cos(2x)}\,\mathrm{d}x$$

Substitute $u=2x \longrightarrow \mathrm{d}u=2\,\mathrm{d}x$ (steps):

$$=\frac{1}{2}\int\sqrt{\cos(u)}\,\mathrm{d}u$$

Now solving:

$$\int\sqrt{\cos(u)}\,\mathrm{d}u$$

Rewrite/simplify using trigonometric/hyperbolic identities:

$$=\int\sqrt{2\cos^2\left(\frac{u}{2}\right)-1}\,\mathrm{d}u$$

$$=\int\sqrt{1-2\sin^2\left(\frac{u}{2}\right)}\,\mathrm{d}u$$

Substitute $v=\frac{u}{2} \longrightarrow \mathrm{d}v=\frac{1}{2}\,\mathrm{d}u$ (steps):

$$=2\int\sqrt{1-2\sin^2(v)}\,\mathrm{d}v$$

Now solving:

$$\int\sqrt{1-2\sin^2(v)}\,\mathrm{d}v$$

Use previous result:

$$=\mathrm{E}(v\,|\,2)$$

Plug in solved integrals:

$$2\int\sqrt{1-2\sin^2(v)}\,\mathrm{d}v$$

$$=2\,\mathrm{E}(v\,|\,2)$$

Undo substitution $v=\frac{u}{2}$:

$$=2\,\mathrm{E}\left(\frac{u}{2}\,\middle|\,2\right)$$

Plug in solved integrals:

$$\frac{1}{2}\int\sqrt{\cos(u)}\,\mathrm{d}u$$

$$=\mathrm{E}\left(\frac{u}{2}\,\middle|\,2\right)$$

Undo substitution $u=2x$:

$$=\mathrm{E}(x\,|\,2)$$

Plug in solved integrals:

$$\int\frac{x\sin(2x)}{\sqrt{\cos(2x)}}\,\mathrm{d}x+\int\sqrt{\cos(2x)}\,\mathrm{d}x$$

$$=2\,\mathrm{E}(x\,|\,2)-x\sqrt{2\cos^2(x)-1}$$

The problem is solved:

$$\int\left(\frac{x\sin(2x)}{\sqrt{\cos(2x)}}+\sqrt{\cos(2x)}\right)\mathrm{d}x$$

$$=2\,\mathrm{E}(x\,|\,2)-x\sqrt{2\cos^2(x)-1}+C$$

Rewrite/simplify:

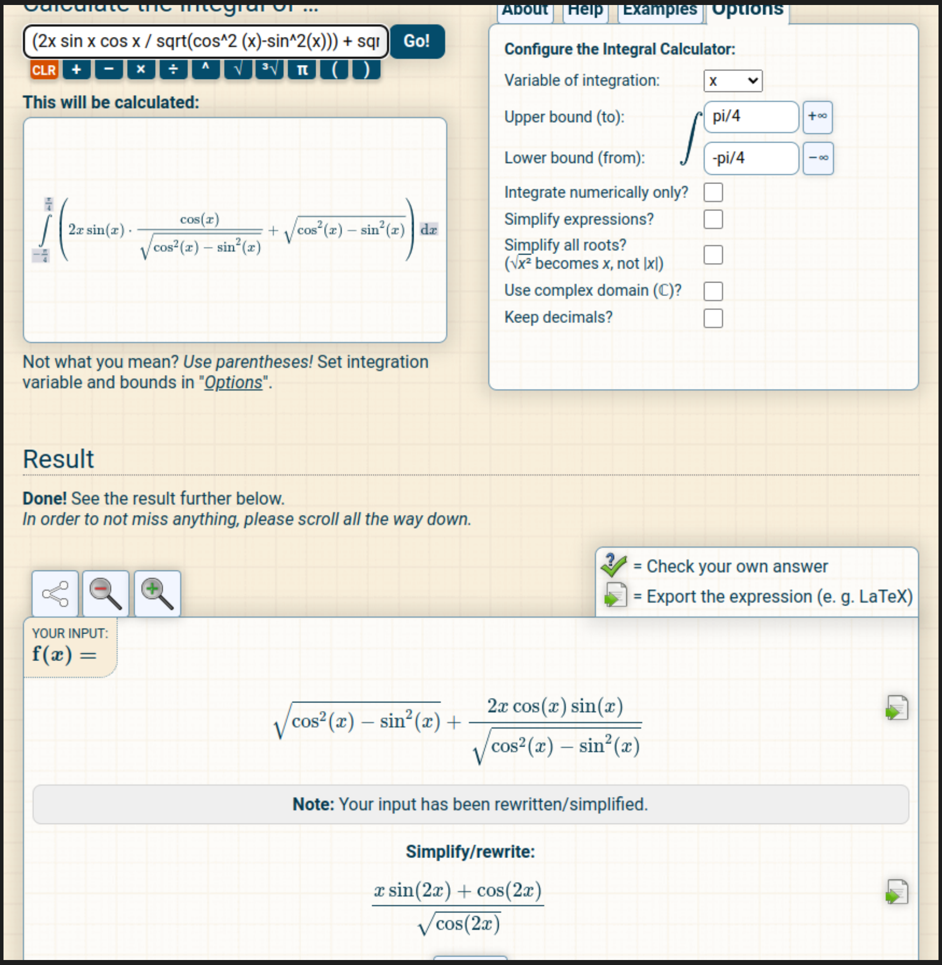
$$=2\,\mathrm{E}(x\,|\,2)-x\sqrt{\cos(2x)}+C$$

Antiderivative or integral could not be found.

Approximation:

2.396280469470934

Lo mismo si :



En los resueltos usa que

$$\sigma_2(t) = (r(t), t)$$

Luego resuelve, y el find reemplaza

Ejercicio 6. Probar la fórmula de integración por partes: Si $D \subset \mathbb{R}^2$ es un dominio elemental, ∂D su frontera orientada en sentido antihorario y $\mathbf{n} = (n_1, n_2)$ la normal exterior a D, entonces

$$\int_D u\, v_x\, dx\, dy = -\int_D u_x\, v\, dx\, dy + \int_{\partial D} u\, v\, n_1\, ds,$$

para todo par de funciones u, $v \in C(\bar{D}) \cap C^1(D)$.

Quiero algo parecido a Green.

$$\underbrace{\iint_D u \cdot v_x \, dx\, dy + \iint_D u_x \cdot v \, dx\, dy}_{\mathrm{I}} = \underbrace{\int_{\partial D} u \cdot v \cdot n_1 \, ds}_{\mathrm{II}}$$

I $\displaystyle\iint_D \underbrace{u \cdot v_x + u_x \cdot v}_{\frac{\partial}{\partial x}(u \cdot v)} \, dx\, dy$

$$\iint_D \frac{\partial}{\partial x}(u \cdot v)\, dx\, dy = \iint_D u \cdot v \quad dy$$

Clave : elijo $F = (0,\ u \cdot v)$

$$\Rightarrow Q_x - P_y = u_x \cdot v - u \cdot v_x$$

Por Green:

$$\int_{\partial D} F \cdot ds = \underbrace{\iint_D Q_x - P_y \, dx\, dy}_{\text{Lo que tengo en I}}$$

Sea σ param de ∂D

$$\Rightarrow \int_{\partial D} F \cdot ds = \int_{t=a}^{b} \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

$$= \int_a^b \left\langle \left(0, Q(\sigma(t))\right), \left(\sigma_1'(t), \sigma_2'(t)\right) \right\rangle dt$$

$$= \int_a^b Q(\sigma(t)) \,.\, \sigma_2'(t)\, dt$$

$$= \int_a^b u.v(\sigma(t)) \cdot \sigma_2'(t) \,.\, dt$$

III

Como n es normal exterior

$\sigma'(t)$ es tangente $\Rightarrow$ $(\sigma_2', -\sigma_1')$ es perpendicular

$$n = \frac{(\sigma_2', -\sigma_1')}{\|\sigma'(t)\|} = \Bigg(\underbrace{\frac{\sigma_2'}{\|\sigma'(t)\|}}_{n_1}, \underbrace{\frac{-\sigma_1'}{\|\sigma'(t)\|}}_{n_2} \Bigg)$$

$$(n_1, n_2)$$

De II:

por ser integral curvilínea

$\downarrow \int_C f\, ds = \int_a^b f(\sigma(t)) . \|\sigma'(t)\|$

$$\int_{\partial D} u.v.\, n_1\, ds = \int_{t=a}^{b} u.v(\sigma(t)) \,.\, \frac{\sigma_2'}{\cancel{\|\sigma'(t)\|}} \,.\, \cancel{\|\sigma'(t)\|} \,.\, dt$$

$$= \int_a^b u.v(\sigma(t)) \,.\, \sigma_2'\, dt$$

Que es igual a III

$\therefore$

$$\iint_D u \cdot v_x \, dx\,dy + \iint_D u_x \cdot v \, dx\,dy = \int_{\partial D} u \cdot v \cdot n_1 \, ds$$

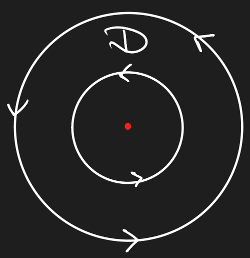
$\blacksquare$

Repaso!

Ejercicio 7. Sean P y Q funciones continuamente diferenciables en $\mathbb{R}^2$. Verificar que el Teorema de Green para estas funcionnes es válido cuando la región D es el anillo

$$D = \{(x, y) \,/\, 1 \leq x^2 + y^2 \leq 4\}.$$

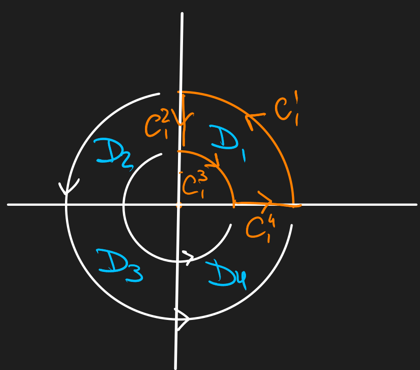
Sugerencia: Aplicar el Teorema de Green en los discos de radios 1 y 2.



$$\iint_D Q_x - P_y \, dx dy = \int_{\mathcal{C}_{ext}} F \cdot ds - \int_{\mathcal{C}_{int}} F \cdot ds$$

$$\int_{\mathcal{C}_{ext}} F \cdot ds = \int_{t=0}^{2\pi} \langle \underbrace{F(2 \cdot \cos t, 2 \cdot \sin t)}_{\text{no puedo hacer nada!}}, (-2 \cdot \sin t, 2 \cdot \cos t) \rangle \, dt$$

Divido en 4 partes



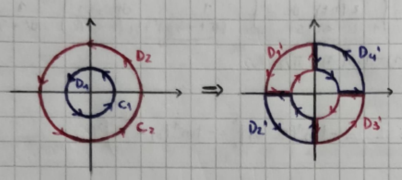
$$\iint_{D_1} Q_x - P_y \, dx dy = \int_{\partial D_1} P \cdot dx + Q \cdot dy$$

$$\partial D_1 = C_1^1 \cup C_1^2 \cup C_1^3 \cup C_1^4$$

$$\int_{\partial D_1} P \cdot dx + Q \cdot dy = \int_{C_1^1}$$

(7) P, Q funciones C^1 en $\mathbb{R}^2$, $D = \{(x,y) / 1 \leq x^2 + y^2 \leq 4\} \Rightarrow$ Divido en D_1 y D_2

$\Rightarrow D_1 = \{(x,y) / \ x^2 + y^2 \leq 1\}$, $D_2 = \{(x,y) / \ x^2 + y^2 \leq 4\}$



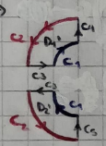
$\Rightarrow D = D_1' \cup D_2' \cup D_3' \cup D_4' \to$ Divido el anillo en 4 regiones tipo III y las recorro en sentido positivo

$\Rightarrow$ Veo que los bordes rectos los recorro 2 veces, 1 en cada sentido de forma que se cancelan, y resultan una curva que recorre el borde interior en sentido horario y una que recorre el borde exterior en sentido antihorario

$\Rightarrow C_1: \sigma_1(t) = (\cos t, \operatorname{sen} t)$ con $t \in [0, 2\pi]$

$C_2: \sigma_2(t) = (2\cos t, 2\operatorname{sen} t)$ con $t \in [0, 2\pi]$

$\Rightarrow$



$$\Rightarrow \iint_{D_1'} (Q_x - P_y) \, dx dy = \int_{C_2^+ \cup C_3^+ \cup C_1^- \cup C_4^+} P dx + Q dy = \int_{C_2^+} P dx + Q dy + \int_{C_3^+} P dx + Q dy - \int_{C_1^-} P dx + Q dy + \int_{C_4^+} P dx + Q dy$$

$\oplus$

$$\Rightarrow \iint_{D_2'} (Q_x - P_y) \, dx dy = \int_{C_2^+ \cup C_5^+ \cup C_1^- \cup C_3^-} P dx + Q dy = \int_{C_2^+} P dx + Q dy + \int_{C_5^+} P dx + Q dy - \int_{C_1^-} P dx + Q dy - \int_{C_3^-} P dx + Q dy$$

$\ominus$

$\Rightarrow$ Como $D = D_1' \cup D_2' \cup D_3' \cup D_4' \Rightarrow \iint_D (Q_x - P_y) \, dx dy = \iint_{D_1'} (Q_x - P_y) \, dx dy + \iint_{D_2'} (Q_x - P_y) \, dx dy + \iint_{D_3'} (Q_x - P_y) \, dx dy + \iint_{D_4'} (Q_x - P_x) \, dx dy$

$\Rightarrow$ Al sumar las integrales de las 4 divisiones de D, las curvas horizontales y verticales que usamos para cerrar dichas divisiones son recorridas 2 veces cada una, una vez con orientación positiva y otra negativa, por lo que las integrales de curva se cancelan, y solo me quedan los terminos que corresponden a partes de C_1 (orientada positivamente) y C_2 (orientada negativamente)

$$\Rightarrow \iint_D (Q_x - P_y) \, dx dy = \int_{C_2^+} P dx + Q dy - \int_{C_1^-} P dx + Q dx$$

Ejercicio 8. Sea $\mathcal{C}$ la curva

$$\begin{aligned}
&x = 0, && 0 \le y \le 4,\\
&y = 4, && 0 \le x \le 4,\\
&y = x, && 0 \le x \le 1,\\
&y = 2 - x, && 1 \le x \le 2,\\
&y = x - 2, && 2 \le x \le 3,\\
&y = 4 - x, && 2 \le x \le 3,\\
&y = x, && 2 \le x \le 4,
\end{aligned}$$

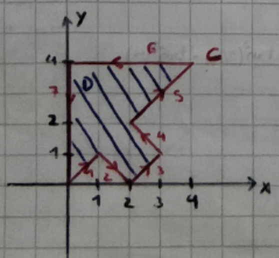
orientada positivamente. Calcular

$$\int_{\mathcal{C}} \frac{y}{(x-1)^2 + y^2}\,dx + \frac{1-x}{(x-1)^2+y^2}\,dy.$$

Ejercicio 9. Sea $D = \{(x,y)\,/\,1 \le x^2 + y^2 \le 4,\ x \ge 0\}$. Calcular

$$\int_{\partial D} x^2 y\,dx - xy^2\,dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).



$\Rightarrow \int_C P\,dx + Q\,dy \Rightarrow P = \dfrac{y}{(x-1)^2+y^2}$, $Q = \dfrac{1-x}{(x-1)^2+y^2} \Rightarrow F = (P, Q)$

$\Rightarrow$ Primero me fijo que F este definido en $D \subset \mathbb{R}^2$

$\Rightarrow$ P y Q no estan definidos en el punto $(1,0) \to (1-1)^2 + 0^2 = 0$

Pero el punto $(1,0)$ no esta dentro de D, por lo que F esta definido en D

$\Rightarrow$ Divido C en 7 curvas concatenadas: $\sigma_1(t) = (t,t)$ con $t \in [0,1]$, $\sigma_2(t) = (t, 2-t)$ con $t \in [1,2]$, $\sigma_3(t) = (t, t-2)$ con $t \in [2,3]$

$\sigma_4(t) = (t, 4-t)$ con $t \in [2,3]$, $\sigma_5(t) = (t,t)$ con $t \in [2,4]$, $\sigma_6(t) = (t,4)$ con $t \in [0,4]$, $\sigma_7(t) = (0,t)$ con $t \in [0,4]$

$$\Rightarrow \int_C P\,dx + Q\,dx = \int_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7} P\,dx + Q\,dx = \underbrace{\int_{C_1^+} P\,dx + Q\,dy}_{1} + \underbrace{\int_{C_2^+} P\,dx + Q\,dy}_{2} + \underbrace{\int_{C_3^+} P\,dx + Q\,dy}_{3} - \underbrace{\int_{C_4^-} P\,dx + Q\,dx}_{4} + \underbrace{\int_{C_5^+} P\,dx + Q\,dx}_{5} - \underbrace{\int_{C_6^-} P\,dx + Q\,dx}_{6} - \underbrace{\int_{C_7^-} P\,dx + Q\,dx}_{7}$$

$$1 = \int_0^1 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2}\right)(1,1)\,dt = \int_0^1 \frac{t + (1-t)}{(t-1)^2+t^2}\,dt = \int_0^1 \frac{1}{2t^2-2t+1}\,dt = \int_0^1 \frac{1}{(\underbrace{\sqrt{2}t - 1/\sqrt{2}}_{u})^2 + 1/2}\,dt = \int \frac{1}{\sqrt{2}} \frac{1}{u^2+1/2}\,du = \frac{1}{\sqrt{2}} \int \frac{2}{2u^2+1}\,du$$

$$= \sqrt{2} \int \frac{1}{\underbrace{2u^2+1}_{s=\sqrt{2}u}}\,du = \cancel{\sqrt{2}} \int \frac{1}{\cancel{\sqrt{2}}} \frac{1}{s^2+1}\,ds = \tan^{-1}(s)\Big|_{\sqrt{2}u} = \tan^{-1}(\sqrt{2}u)\Big|_{\sqrt{2}t - 1/\sqrt{2}} = \tan^{-1}(2t-1)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(-1) = 2\tan^{-1}(1) = 2 \cdot \frac{\pi}{4} = \pi/2$$

$$2 = \int_1^2 \left(\frac{2-t}{(t-1)^2+(2-t)^2}, \frac{1-t}{(t-1)^2+(2-t)^2}\right)(1,-1)\,dt = \int_1^2 \frac{1}{(t-1)^2+(2-t)^2}\,dt = \int_1^2 \frac{1}{2t^2-6t+5}\,dt = \int_1^2 \frac{1}{(\underbrace{\sqrt{2}t - 3/\sqrt{2}}_{u})^2 + 1/2}\,dt = \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1/2}\,du \underset{s=\sqrt{2}u}{=} \int \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1}(s)\Big|_{\sqrt{2}u} = \tan^{-1}(\sqrt{2}u)\Big|_{\sqrt{2}t - 3/\sqrt{2}} = \tan^{-1}(2t-3)\Big|_1^2 = \tan^{-1}(1) - \tan^{-1}(-1) = \pi/2$$

$$3 = \int_2^3 \left(\frac{t-2}{(t-1)^2+(t-2)^2}, \frac{1-t}{(t-1)^2+(t-2)^2}\right)(1,1)\,dt = \int_2^3 \frac{-1}{2t^2-6t+5}\,dt = \dots = -\tan^{-1}(2t-3)\Big|_2^3 = -\tan^{-1}(3) + \tan^{-1}(1)$$

$$4 = \int_2^3 \left(\frac{4-t}{(t-1)^2+(4-t)^2}, \frac{1-t}{(t-1)^2+(4-t)^2}\right)(1,-1)\,dt = \int_2^3 \frac{3}{2t^2-10t+17}\,dt = \dots = 2\tan^{-1}(1/3)$$

$$5 = \int_2^4 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2}\right)(1,1)\,dt = \int_2^4 \frac{1}{2t^2-2t+1}\,dt = \dots = \tan^{-1}(2t-1)\Big|_2^4 = \tan^{-1}(7) - \tan^{-1}(3)$$

$$6 = \int_0^4 \left(\frac{4}{(t-1)^2+4^2}, \frac{1-t}{(t-1)^2+4^2}\right)(1,0)\,dt = \int_0^4 \frac{4}{t^2-2t+17}\,dt = \int_0^4 \frac{4}{(\underbrace{t-1}_{u})^2+16}\,dt = \int_{-1}^3 \frac{4}{u^2+16}\,du = \int_{-1}^3 \frac{4}{16} \frac{1}{\underbrace{\frac{u^2}{16}}_{s=u/4}+1}\,du = \frac{\cancel{4}}{\cancel{4}} \int_{-1/4}^{3/4} \frac{1}{s^2+1}\,ds$$

$$= \frac{\cancel{4}}{\cancel{4}} \tan^{-1}(s)\Big|_{-1/4}^{3/4} = \frac{\cancel{4}}{\cancel{4}}\left(\tan^{-1}(3/4) - \tan^{-1}(-1/4)\right) = \tan(3/4) + \tan(1/4)$$

$$7 = \int_0^4 \left(\frac{t}{(0-1)^2+t^2}, \frac{1}{(0-1)^2+t^2}\right)(0,1)\,dt = \int_0^4 \frac{1}{t^2+1}\,dt = \tan^{-1}(t)\Big|_0^4 = \tan^{-1}(4)$$

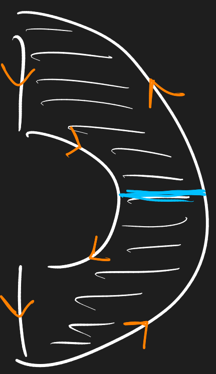
$$\Rightarrow \int_C P\,dx + Q\,dy = \pi/2 + \pi/2 - \tan^{-1}(3) + \pi/4 - 2\tan^{-1}(1/3) + \tan^{-1}(7) - \tan^{-1}(3) - \tan^{-1}(3/4) - \tan^{-1}(1/4) - \tan^{-1}(4) = 0$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \left(\frac{\partial\left(\frac{x-1}{(x-1)^2+y^2}\right)}{\partial x} - \frac{\partial\left(\frac{y}{(x-1)^2+y^2}\right)}{\partial y}\right) dx\,dy = \iint_D \left(\frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2} - \frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2}\right) dx\,dy = 0 \Rightarrow$$ MUCHO mas facil

Ejercicio 9. Sea $D = \{(x,y) \,/\, 1 \leq x^2+y^2 \leq 4,\ x \geq 0\}$. Calcular

$$\int_{\partial D} x^2 y\, dx - xy^2\, dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).



$$\underbrace{\int_{\partial D} x^2 y\, dx - x\cdot y^2\, dy}_{} = \iint_D Q_x - P_y \, dx\, dy$$

$$F = (\underbrace{\underset{P}{x^2 y}, \underset{Q}{-xy^2}}_{})$$

$$\Rightarrow \quad Q_x = -y^2$$
$$P_y = x^2$$

$$\iint_D Q_x - P_y\, dx\, dy = \iint_D -y^2 - x^2 \, dx\, dy =$$

Polares

$$\begin{cases} x = r\cdot \cos t & r \in [1,2] \\ y = r \cdot \sin t & t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

Jacobiano

$$= - \int_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\int_{r=1}^{2} (r^2 \cdot \cos^2 t + r^2 \cdot \sin^2 t)\, r \,.\, dr}_{= \int_1^2 r^3\, dr \,=\, \frac{r^4}{4}\Big|_1^2 \,=\, \frac{1}{4}(16-1) \,=\, \frac{15}{4}}\, dt$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{15}{4}\, dt$$

$$= -\frac{15}{4} \cdot t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{15}{4}\pi$$

// en el resuelto no usó Green

Ejercicio 10. Calcular el trabajo efectuado por el campo de fuerzas $F(x,y) = (y+3x, 2y-x)$ al mover una partÃcula rodeando una vez la elipse $4x^2+y^2=4$ en el sentido de las agujas del reloj.

sentido opuesto a Green!

Me pide

$$\int_{\mathcal{E}} F \cdot ds = -\int_{t=0}^{2\pi} \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

Con $\sigma(t) = (\underbrace{1 \cdot \cos t, \; 2 \cdot \sin t}_{\text{sentido } \oplus \text{, pero quiero } \ominus})$

Puedo usar Green, donde

$$\int_{\mathcal{E}} F \cdot ds = -\iint_D Q_x - P_y$$

Como busco $\int_{\mathcal{E}} F \cdot ds$ con $\mathcal{E}$ recorrida $\circlearrowright$

$\Rightarrow$ Cambio el signo desde ahora (asumiendo que $\mathcal{E}$ se recorre $\circlearrowright$

Calculo:

$$F = (\underbrace{y+3x}_{P}, \underbrace{2y-x}_{Q}) \Rightarrow \begin{cases} Q_x = -1 \\ P_y = 1 \end{cases}$$

$$-\iint_D Q_x - P_y = -\iint_D -2 \, dy\,dx = +2 \text{ Área (Elipse)}$$

$$= 2 \cdot \underbrace{1}_{a} \cdot \underbrace{2}_{b} \cdot \pi$$

$$\text{Trabajo} = 4\pi$$

Ejercicio 11. Sea $\mathbf{F}(x,y) = (P(x,y), Q(x,y)) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$. Calcular $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ donde $\mathcal{C}$ es la circunferencia unitaria centrada en el origen orientada positivamente. Calcular $Q_x - P_y$. ¿Se satisface en este caso el Teorema de Green?

F no está bien definida en el (0,0) !

No se satisface el teorema de Green.

Spoiler : $\iint_D Q_x - P_y \, dy\,dx \neq \int_{\mathcal{C}} F \cdot ds$

$$Q_x = \frac{-1}{x^2+y^2} + \underbrace{-x \cdot (-1) \cdot (x^2+y^2)^{-2} \cdot 2x}_{\frac{+2x^2}{(x^2+y^2)^2}}$$

$$P_y = \frac{1}{x^2+y^2} + \underbrace{y \cdot (-1) \cdot (x^2+y^2)^{-2} \cdot 2y}_{\frac{-2y^2}{(x^2+y^2)^2}}$$

$$Q_x - P_y = \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{-2}{x^2+y^2} + \frac{2\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$= 0$$ //

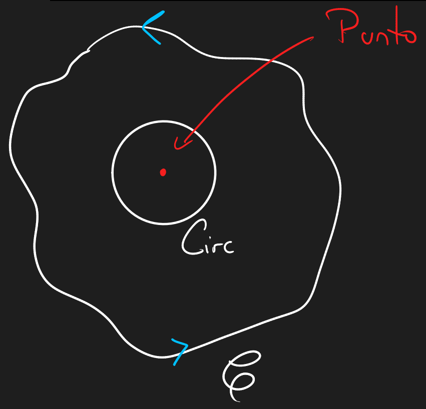
$$\int_C F \cdot ds = \int_{t=0}^{2\pi} \underbrace{\left\langle \left(\frac{\sin t}{1} , \frac{-\cos t}{1} \right) , \left(-\sin t , \cos t \right) \right\rangle}_{-\sin^2 t - \cos^2 t} dt$$

$$= -2\pi //$$

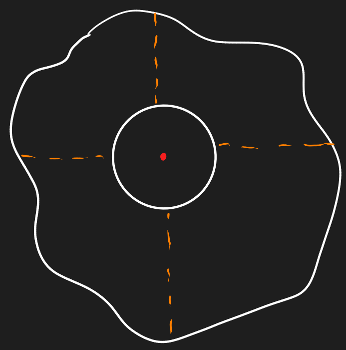
$$-2\pi \neq 0$$

Ejercicio 12. Sea $\mathcal{C}$ una curva cerrada, simple y suave orientada positivamente que encierra la circunferencia unitaria centrada en el origen. Calcular $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ para el campo $\mathbf{F}$ definido en el ejercicio 11.

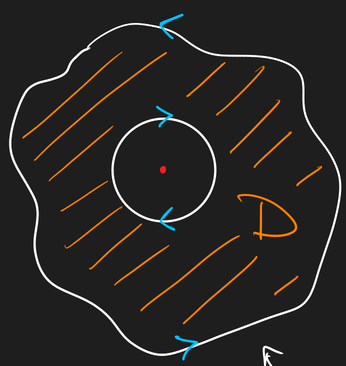
Punto problemático.



Lo pienso así



y como vale Green en cada sección, vale hacer Green con D como



D no contiene al punto problemático

Con Green

$$\int_{\mathcal{C}} F \cdot ds + \underbrace{\int_{Circ^-} F \cdot ds}_{\substack{\text{la del punto 11} \\ \text{(sentido opuesto)}}} = \iint_D \underbrace{Q_x - P_y}_{=0 \text{ por ej 11}} \, dx\,dy$$

$$\int_{\mathcal{C}} F \cdot ds - \int_{Circ^+} F \cdot ds = 0$$

$$\int_{\mathcal{C}} F \cdot ds = \underbrace{\int_{C^{+}_{irc}} F \cdot ds}_{\text{Calculada en ej 11.}}$$

$$\int_{\mathcal{C}} F \cdot ds = -2\pi$$

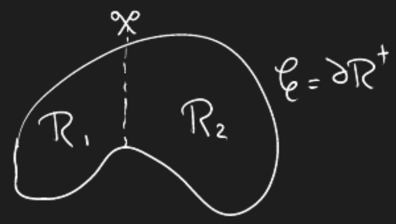
Para cualquier curva $\mathcal{C}$ que encierre a Circ!

Ver notas de Rela más abajo acerca de:

- Regiones con agujeros
- Cerrar curvas

Extensiones del Teorema (Notas de Rela)

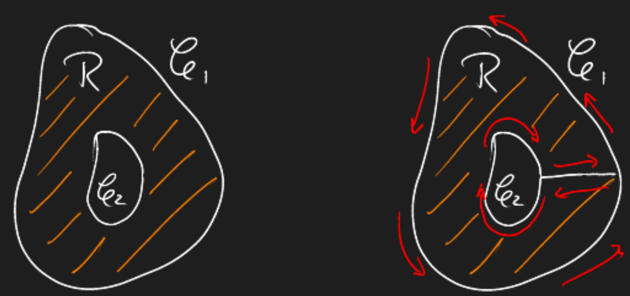
1) a) Regiones que se descomponen



$\mathcal{C}_1 = \partial R_1^+$

$\mathcal{C}_2 = \partial R_2^+$

b) Agujeros R, $\mathcal{C} = \partial R$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$



+ 2 trocitos que se anulen

$$\iint_R Q_x - P_y \, dx\,dy = \int_{\mathcal{C}_1} P\,dx + Q\,dy + \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

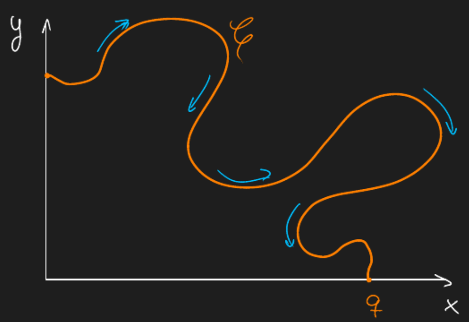
(antihorario) (horario)

Aplicación
(Cerrando Curvas)

Situación:

$\mathcal{C}$ curva (no cerrada) fea

F un campo $F = (P, Q)$ feo



Ejemplo (mismo que la práctica):

$\mathcal{C}$ dada y orientada por

$$\sigma(t) = (\sin t, \cos t)$$

$$t \in [0, \tfrac{3}{2}\pi]$$

$$F(x,y) = \left(2\cos(x^2 y)\cdot xy - 2y \;,\; x^2 \cdot \cos(x^2 \cdot y) + 3x\right)$$

Plan A: Resuelvo a mano $\int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$

- Plan B: Completar la curva y usar Green

Definiendo curvas (Γ_1, Γ_2) que la cierren, de forma que se simplifique el cálculo junto al campo.

Ejercicio 13. Calcular $\int_{\mathcal{C}} f_1 dx + f_2 dy$ siendo

$$f_1(x,y) = \frac{x \text{ sen } \dfrac{\pi}{2(x^2+y^2)} - y(x^2+y^2)}{(x^2+y^2)^2},$$

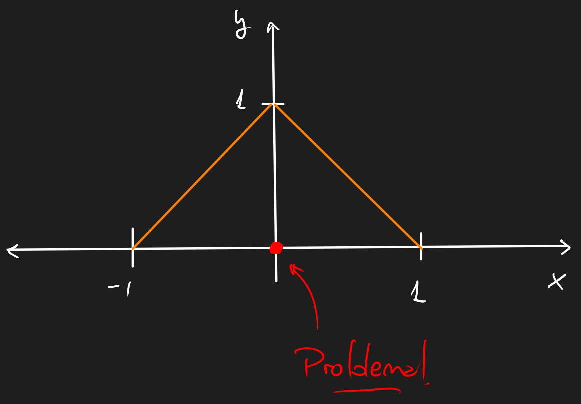
$$f_2(x,y) = \frac{y \text{ sen } \dfrac{\pi}{2(x^2+y^2)} + x(x^2+y^2)}{(x^2+y^2)^2},$$

y

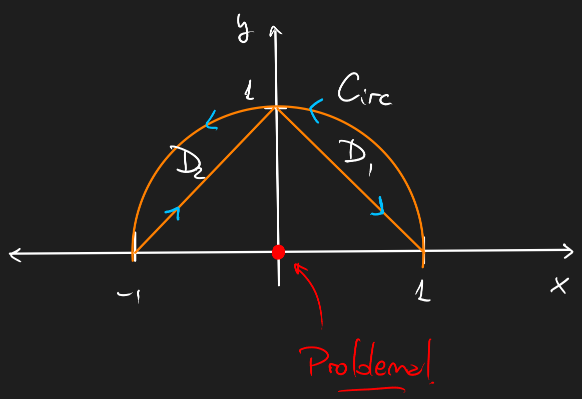
$$\mathcal{C} = \begin{cases} y = x+1 & \text{si } -1 \le x \le 0, \\ y = 1-x & \text{si } \quad 0 \le x \le 1, \end{cases}$$

recorrida del $(-1,0)$ al $(1,0)$.

si $x^2+y^2=1 \Rightarrow$ se simplifica todo!



La cierro así $\Rightarrow$



$$\int_{\mathcal{C}} F \cdot ds + \int_{\mathcal{C}_{irc}} F \cdot ds = \iint_{D_1 \cup D_2} Q_x - P_y \, dx dy$$

$$P_y = \frac{\partial}{\partial y} \cdot \left(\frac{x \cdot \sin \dfrac{\pi}{2(x^2+y^2)} - y(x^2+y^2)}{(x^2+y^2)^2} \right)$$

Resuelvo y veo que $Q_x - P_y = 0$

Entonces calculo:

$$\int_{Circ} F \cdot ds = \int_{t=0}^{\pi} \langle \underbrace{F(\sigma(t))}, \sigma'(t) \rangle \, dt \qquad \sigma(t) = (\cos t, \sin t)$$

$$\left(\cos t \cdot \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} - \sin t \; , \; \sin t \cdot \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + \cos t \right)$$

$$(\cos t - \sin t \; , \; \cos t + \sin t)$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\Rightarrow \int_{Circ} F \cdot ds = \int_0^{\pi} -\cos t \cdot \sin t + \sin^2 t + \cos^2 t + \cos t \cdot \sin t$$

$$= \pi$$

Obtuve

$$\int_{\mathcal{C}} F \cdot ds + \underbrace{\int_{\mathcal{C}irc} F \cdot ds}_{\pi} = \underbrace{\iint_{D_1 \cup D_2} Q_x - P_y \, dx dy}_{=0}$$

$$\therefore \int_{\mathcal{C}} F \cdot ds = -\pi$$

Ejercicio 14. Determinar todas las circunferencias $\mathcal{C}$ en el plano $\mathbb{R}^2$ sobre las cuales vale la siguiente igualdad

$$\int_{\mathcal{C}} -y^2\,dx + 3x\,dy = 6\pi.$$

$F = (-y^2, 3x)$

Bien definido en todo $\mathbb{R}^2$

$$\Rightarrow \int_{\mathcal{C}} F\,ds = \iint_D Q_x - P_y \, dx\,dy = 6\pi$$

Donde

$$Q_x - P_y = 3 + 2y$$

Quiero usar Polares, PERO con circunferencias con cualquier centro (no solo (0,0))

$$\begin{cases} x = r\cdot\cos\theta + x_0 \\ y = r\cdot\sin\theta + y_0 \end{cases} \qquad r \in [0,R] \quad \theta \in [0,2\pi) \qquad x_0, y_0 \in \mathbb{R}$$

$$= \int_{r=0}^{R} \underbrace{\int_{\theta=0}^{2\pi} (3 + 2\cdot r\cdot\sin\theta + 2y_0)\cdot \overset{\text{Jacobiano}}{r}\cdot d\theta}_{\int 3r + 2y_0\cdot r\,d\theta - \left(\underbrace{2\cdot r^2\cdot\cos\theta\Big|_0^{2\pi}}_{=0}\right)} dr$$

$$= 2\pi \int_0^R 3r + 2y_0\cdot r\,dr$$

$$= 2\pi\left(3\frac{r^2}{2}\Big|_0^R + \cancel{2}\cdot y_0\cdot\frac{r^2}{\cancel{2}}\Big|_0^R\right)$$

$$= 2\pi\left(\frac{3}{2}R^2 + y_0\cdot R^2\right)$$

$$= R^2\cdot(3\pi + 2\pi\cdot y_0)$$

Por dato

$$R^2 \cdot \left(3\pi + 2\pi \cdot y_0 \right) = 6\pi$$

$$R^2 = \frac{6}{3 + 2y_0} \quad \text{ó} \quad 3R^2 + 2R^2 y_0 = 6$$

$$2R^2 y_0 = 6 - 3R^2$$

$$y_0 = \frac{6 - 3R^2}{2R^2}$$

$$y_0 = \frac{3}{R^2} - \frac{3}{2}$$

$\therefore$

Son las circunferencias de la forma:

$$C = \left\{ (x,y) \in \mathbb{R}^2 : \; (x - x_0)^2 + \left(y - y_0\right)^2 = \frac{6}{3 + 2y_0} \quad \text{con } x_0, y_0 \in \mathbb{R} \right\}$$

Ejercicio 15. Calcular la integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ donde

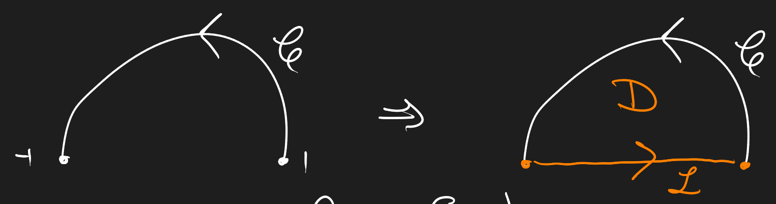
$$\mathbf{F}(x,y) = (y^2e^x + \cos x + (x-y)^2\,,\ 2\,y\,e^x + \operatorname{sen} y),$$

Bien definido

y $\mathcal{C}$ es la curva

$$x^2+y^2=1, \quad y\geq 0,$$

orientada de manera tal que comience en $(1,0)$ y termine en $(-1,0)$.



Primero veo si es Campo Grad, porque es sospechoso:

$$\text{si } f_x = y^2\cdot e^x + \cos x + \underbrace{(x-y)^2}_{x^2-2xy+y^2}$$

$$\Rightarrow f = y^2e^x + \sin x + \frac{x^3}{3} - x^2\cdot y + xy^2 + \varphi(y)$$

Caí! !!

$$\text{si } f_y = 2y\cdot e^x + \sin y$$

$$\Rightarrow f = y^2\cdot e^x - \cos y + \tilde{\varphi}(x)$$

Vuelvo

$$\int_{\mathcal{C}} F\cdot ds + \int_{\mathcal{L}} F\cdot ds = \iint_D Q_x - P_y\,dx\,dy$$

$$F = (\underbrace{y^2\cdot e^x + \cos x + x^2-2xy+y^2}_{P}, \underbrace{2y\cdot e^x + \sin y}_{Q})$$

$$Q_x - P_y = \cancel{2y\cdot e^x} - (\cancel{2y\cdot e^x} - 2x + 2y)$$

$$= 2x - 2y$$

$$\iint_D Q_x - P_y \, dx\,dy = 2 \int_{r=0}^{1} \underbrace{\int_{\theta=0}^{\pi} (\overbrace{r\cdot\cos\theta - r\cdot\sin\theta}^{x-y})\, r \cdot d\theta}_{} \cdot dr$$

$$r^2\cdot\sin\theta\Big|_0^{\pi} + \underbrace{r^2\cdot\cos\theta\Big|_0^{\pi}}_{}$$

$$r^2\cdot(0-0) \qquad r^2\cdot(-1-1)$$

$$-2r^2$$

$$= -4 \int_{r=0}^{1} r^2 \, dr$$

$$= -\frac{4}{3} \; r^3\Big|_0^1 = -\frac{4}{3}$$

Falta

$$\int_{\mathcal{L}} F\cdot ds = \int_{-1}^{1} \langle \underbrace{F(t,0)}_{(\cos t + t^2,\; 0)}, (1,0) \rangle \, dt$$

$$= \sin t\Big|_{-1}^{1} + \frac{t^3}{3}\Big|_{-1}^{1}$$

$$= \sin 1 - \sin(-1) + \frac{1}{3} + \frac{1}{3}$$

CA

$\sin -1 = -\sin 1$

$$= \frac{2}{3} + 2\cdot\sin 1$$

Juntando todo

$$\int_{\mathcal{C}} F\cdot ds + \underbrace{\int_{\mathcal{L}} F\cdot ds}_{\frac{2}{3} + 2\sin 1} = \underbrace{\iint_{D} Q_x - P_y \, dx\, dy}_{-\frac{4}{3}}$$

$$\int_{\mathcal{C}} F\cdot ds = -\frac{4}{3} - \frac{2}{3} - 2\sin 1$$

$$= -2 - 2\sin 1$$

//

Ejercicio 16. Sean $u, v \in C^1(D)$, donde $D = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$. Consideremos los campos definidos por $\mathbf{F}(x,y) = (u(x,y), v(x,y))$, $\mathbf{G}(x,y) = (v_x - v_y, u_x - u_y)$. Calcular

$$\iint_D (\mathbf{F} \cdot \mathbf{G})(x,y)\, dx\, dy$$

sabiendo que sobre el borde de D se tiene $u(x,y) = x$, $v(x,y) = 1$.

$$\langle F, G \rangle = \underbrace{u \cdot (v_x - v_y) + v \cdot (u_x - u_y)}_{\text{se parece a la deriv. de productos}}$$

$$= u \cdot v_x + v \cdot u_x - (u \cdot v_y + v \cdot u_y)$$

$$= \frac{\partial}{\partial x}(u \cdot v) - \frac{\partial}{\partial y}(u \cdot v)$$

$$\iint_D \langle F, G \rangle\, dx dy = \iint_D \underbrace{\frac{\partial}{\partial x}(u \cdot v)}_{Q_x} - \underbrace{\frac{\partial}{\partial y}(u \cdot v)}_{P_y}\, dx dy$$

Por Green: $\displaystyle\iint_D Q_x - P_y\, dx dy = \int_{\partial D} (P, Q) \cdot ds$

$$= \int_{\partial D} (\overbrace{u \cdot v}^{P}, \overbrace{u \cdot v}^{Q}) \cdot ds$$

Parametrizo ∂D (es una elipse)

$$\sigma(t) = (\overbrace{3 \cdot \cos t}^{a \cdot \cos t}, \overbrace{2 \cdot \sin t}^{b \cdot \cos t})$$

$$= \int_{t=0}^{2\pi} \langle \underbrace{F(\sigma(t))}, \sigma'(t)\rangle \, dt$$

$$\Big(\underbrace{u \cdot \gamma(\sigma(t))}, u \cdot v(\sigma(t))\Big)$$

$$\underbrace{\underbrace{u(\sigma(t))}_{} \cdot \underbrace{v(\sigma(t))}_{}}$$

$$\underbrace{\underbrace{u(3 \cdot \cos t, 2 \cdot \sin t)}_{\substack{\text{Por dato del} \\ \text{enunciado} \\ = 3 \cdot \cos t}} \cdot \underbrace{v(3 \cdot \cos t, 2 \cdot \sin t)}_{\substack{\text{Por dato del} \\ \text{enunciado} \\ = 1}}}_{= 3 \cdot \cos t}$$

$$\therefore F(\sigma(t)) = (3 \cdot \cos t, 3 \cos t)$$

$$\sigma'(t) = (-3 \cdot \sin t, 2 \cdot \cos t)$$

$$\int_{t=0}^{2\pi} \langle F(\sigma(t)), \sigma'(t)\rangle \, dt = \int_{t=0}^{2\pi} \underbrace{-9 \cdot \sin t \cdot \cos t}_{\text{I}} + \underbrace{6 \cdot \cos^2 t}_{\text{II}} \, dt$$

$$\text{I} = -9 \underbrace{\int_0^{2\pi} \sin t \cdot \cos t \, dt}_{=0}$$

$$\text{II} = 6 \int_0^{2\pi} \cos^2 t \, dt$$

Ident Trig.

$$\cos^2 t = \frac{1}{2}\cos 2t + \frac{1}{2}$$

$$= \cancel{6}^{3} \int_0^{2\pi} \frac{1}{\cancel{2}} \cos 2t + \frac{1}{\cancel{2}} \, dt$$

$$= 3 \underbrace{\int_0^{2\pi} \cos 2t \, dt}_{=0} + 3 \cdot 2\pi$$

$$= 6\pi$$

$$\int_{t=0}^{2\pi} \langle F(\sigma(t)), \sigma'(t) \rangle \, dt = 6\pi$$

Finalmente

$$\boxed{\iint_D (F \cdot G) \, dx\,dy = 6\pi}$$