Ejercicio 7. Resuelva las siguientes ecuaciones en $y = y(x)$:

(a) $(y - x^3)dx + (x + y^3)dy = 0$

(b) $\cos x \cos^2 y \, dx - 2 \operatorname{sen} x \operatorname{sen} y \cos y \, dy = 0$

(c) $(3x^2 - y^2)\, dy - 2xy\, dx = 0$

(d) $x\, dy = (x^5 + x^3y^2 + y)\, dx$

(e) $2(x+y) \operatorname{sen} y\, dx + \big(2(x+y) \operatorname{sen} y + \cos y\big)\, dy = 0$

(f) $3y\, dx + x\, dy = 0$

(g) $\big(1 - y(x+y)\tan(xy)\big)\, dx + \big(1 - x(x+y)\tan(xy)\big)\, dy = 0.$

$$\underbrace{M\,dx + N\,dy = 0}_{\text{es Exacta}} \quad \text{con } M, N \in C^1 \quad \iff \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left(\text{i.e. } \operatorname{Rot}(M, N) = 0\right)$$

Si es Exacta:

$$\Rightarrow \begin{cases} \dfrac{\partial F}{\partial x} = M \\[2ex] \dfrac{\partial F}{\partial y} = N \end{cases} \Rightarrow \text{obtengo } F: \quad F(t, x) = c \qquad c \in \mathbb{R}$$

(a) $(y - x^3)dx + (x + y^3)dy = 0$

$$\underbrace{(y - x^3)}_{M}dx + \underbrace{(x + y^3)}_{N}dy = 0$$

$$\left.\begin{aligned} M_y &= 1 \\ N_x &= 1 \end{aligned}\right\} \quad M_y = N_x \Rightarrow \text{es Exacta}$$

Busco $F(x,y) = c \quad c \in \mathbb{R}$

Si $\dfrac{\partial F}{\partial x} = M = y - x^3 \Rightarrow F = xy - \frac{1}{4}x^4 + \varphi(y)$

Si $\dfrac{\partial F}{\partial y} = N = x + y^3 \Rightarrow F = xy + \frac{1}{4}y^4 + \tilde{\varphi}(x)$

$$F(x,y) = xy + \frac{1}{4}y^4 - \frac{1}{4}x$$

Sol:

$$xy + \frac{1}{4}y^4 - \frac{1}{4}x^4 = C \qquad C \in \mathbb{R}$$

Verifico

Derivo wrt x $\left(y = y(x) \Rightarrow \begin{cases} \frac{\partial}{\partial x} x = 1 \\ \frac{\partial}{\partial x} y = y' \end{cases} \right)$

$$\frac{\partial}{\partial x} F(x,y) = y + x.y' + y^3.y' - x^3 = 0$$

$$y + x.\frac{dy}{dx} + y^3.\frac{dy}{dx} - x^3 = 0$$

$$y.dx + x.dy + y^3.dy - x^3.dx = 0$$

$$(y - x^3)\,dx + (x + y^3)\,dy = 0$$ ✓ Verificado

(b) $\underbrace{\cos x \cos^2 y}_{M}\, dx \underbrace{- 2 \operatorname{sen} x \operatorname{sen} y \cos y}_{N}\, dy = 0$

$$\left.\begin{array}{l} M_y = \cos x \cdot 2 \cdot \cos y \cdot (-\sin y) \\ N_x = -2 \cdot \cos x \cdot \sin y \cdot \cos y \end{array}\right\} M_y = N_x \Rightarrow \text{es Exacta}$$

Busco $F(x,y) = c \quad c \in \mathbb{R}$

Si $\dfrac{\partial F}{\partial x} = M = \cos x \cdot \cos^2 y \quad \Rightarrow F = \sin x \cdot \cos^2 y + \varphi(y)$

Si $\dfrac{\partial F}{\partial y} = N = -2 \sin x \cdot \sin y \cdot \cos y \Rightarrow F = \sin x \cdot \cos^2 y + \varphi(x)$

$$F(x,y) = \sin x \cdot \cos^2 y$$

Sol:

$$\boxed{\sin x \cdot \cos^2 y = C \qquad C \in \mathbb{R}}$$

Verifico:

$$\frac{\partial}{\partial x} F = \cos x \cdot \cos^2 y + \sin x \cdot 2\cos y \cdot (-\sin y) \cdot \underbrace{y'}_{\frac{dy}{dx}} = 0$$

$$\cos x \cdot \cos^2 y \cdot dx + \sin x \cdot 2\cos y \cdot (-\sin y) \cdot dy = 0$$

Verificado

(c) $(3x^2 - y^2)\, dy - 2xy\, dx = 0$

$\underbrace{(3x^2 - y^2)}_{N}\, dy - \underbrace{2xy}_{M}\, dx = 0$ ¡Atentx al orden! $M\,dx + N\,dy$

$$\left.\begin{array}{l} M_y = -2x \\ N_x = 6x \end{array}\right\}$$ No es exacta! Pero!

Puedo multiplicar M dx + N dy = 0 por alguna función con el objetivo de convertirla en una ecuación exacta.

Como multiplico ambos lados por la misma cosa, no estoy cambiando las soluciones originales (solo debo tener en cuenta las indeterminaciones que puedo agregar).

En el ejercicio, -2x y 6x no están muy lejos uno de otro.

Sea μ un factor integrante

$$\mu\left(M\,dx + N\,dy\right) = \mu \cdot 0$$

$$\mu M\,dx + \mu N\,dy = 0$$

Como:

$$\begin{cases} M = -2xy \\ N = 3x^2 - y^2 \end{cases}$$

Quiero $\mu \,/\, (\mu . M)_y = (\mu N)_x$

$$\mu_y . M + \mu . M_y = \mu_x . N + \mu . N_x$$

Si $\mu_x = 0$

$$\mu_y . M + \mu . M_y = \underbrace{\mu_x . N}_{=0} + \mu . N_x$$

$$\mu . (M_y - N_x) = -\mu_y . M$$

$$-\frac{\mu_y}{\mu} = \frac{M_y - N_x}{M}$$

$$-\frac{\mu_y}{\mu} = \frac{-2x - 6x}{-2xy} = \frac{-8\cancel{x}}{-2\cancel{x}y}$$

$$-\frac{\mu_y}{\mu} = \frac{4}{y}$$

$$\int \frac{\mu_y}{\mu}\, dy = -\int \frac{4}{y}\, dy$$

$$\ln|\mu| = -4 \ln|y|$$

$$|\mu| = \left(e^{\ln|y|}\right)^{-4}$$

$$|\mu| = |y|^{-4} \qquad y \neq 0$$

$$|\mu| = y^{-4}$$

Pruebo con $\mu = y^{-4}$

$$\tilde{M} = \mu \cdot M = y^{-4} \cdot (-2xy) = -2x \cdot y^{-3}$$

$$\tilde{N} = \mu \cdot N = y^{-4}(3x^2 - y^2) = 3x^2 \cdot y^{-4} - y^{-2}$$

$$\left.\begin{array}{l} \tilde{M}_y = 6x \cdot y^{-4} \\ \tilde{N}_x = 6x \cdot y^{-4} \end{array}\right\} \checkmark \text{ es } \underline{\text{exacta}}$$

Busco $F(x,y) = c \qquad c \in \mathbb{R}$

$$\frac{\partial}{\partial x} F = \tilde{M} = -2x \cdot y^{-3} \Rightarrow F = -x^2 \cdot y^{-3} + \varphi(y)$$

$$\frac{\partial}{\partial y} F = \tilde{N} = 3x^2 \cdot y^{-4} - y^{-2} \Rightarrow F = -x^2 \cdot y^{-3} + y^{-1} + \tilde{\varphi}(x)$$

$$F(x,y) = y^{-1} - x^2 \cdot y^{-3}$$

$$\boxed{\begin{array}{l} \underline{\text{Sol}}: \\ \quad y^{-1} - x^2 \cdot y^{-3} = c \qquad c \in \mathbb{R} \end{array}}$$

Verifico

$$\frac{\partial}{\partial x} F = -y^{-2} \cdot \underbrace{y'}_{\frac{dy}{dx}} - 2x \cdot y^{-3} + 3x^2 \cdot y^{-4} \cdot \underbrace{y'}_{\frac{dy}{dx}} = 0$$

$$-y^{-2} \cdot dy - 2x \cdot y^{-3} \cdot dx + 3x^2 \cdot y^{-4} \cdot dy = 0$$

$$\underbrace{-2x \cdot y^{-3}}_{= \tilde{M}} \cdot dx + \underbrace{\left(3x^2 \cdot y^{-4} - y^{-2}\right)}_{= \tilde{N}} dy = 0$$

✓ Verificado

(d) $x\,dy = (x^5 + x^3y^2 + y)\,dx$

$$\underbrace{(x^5 + x^3 \cdot y^2 + y)}_{M}\,dx + \underbrace{(-x)}_{N}\,dy = 0$$

$$\left.\begin{aligned} M_y &= 2x^3y + 1 \\ N_x &= -1 \end{aligned}\right\} \text{ \underline{No} es exacta}$$

Busco μ.

Elijo $\mu_y = 0$ ⊛

$$-\frac{\mu_y}{\mu} = \frac{M_y - N_x}{M}$$

⊛ Quiero algo simple en el denominador o que se simplifique con $M_y - N_x$

$$-\frac{\mu_y}{\mu} = \frac{2x^3y + 2}{x^5 + x^3 \cdot y^2 + y} \quad \Big\} \text{ depende de 2 variables } \times$$

Elijo $\mu_x = 0$

$$\frac{\mu_x}{\mu} = \frac{2x^3y + 2}{-x} \quad \Big\} \text{ depende de 2 variables } \times$$

No me sirve esta forma de encontrar μ.

Pruebo buscando $\mu(x,y) = x^a \cdot y^b$ con $a, b \in \mathbb{Q}$

$$(x^5+x^3 \cdot y^2+y)\,dx+(-x)\,dy=0$$

$$x^a \cdot y^b \cdot (x^5+x^3 \cdot y^2+y)\,dx + x^a \cdot y^b\,(-x)\,dy=0$$

$$\underbrace{\left(x^{5+a} \cdot y^b + x^{3+a} \cdot y^{2+b} + x^a \cdot y^{1+b}\right)}_{\tilde{M}} dx + \underbrace{\left(-x^{1+a} \cdot y^b\right)}_{\tilde{N}} dy = 0$$

$$\tilde{M}_y = b \cdot x^{5+a} \cdot y^{b-1} + (2+b) \cdot x^{3+a} \cdot y^{1+b} + (1+b) \cdot x^a \cdot y^b$$

$$\tilde{N}_x = -(1+a) \cdot x^a \cdot y^b$$

Quiero

$$\tilde{M}_y = \tilde{N}_x$$

$$\Rightarrow \quad -(1+a) \cdot x^a \cdot y^b = (1+b) \cdot x^a \cdot y^b$$

$$-1-a = 1+b$$

$$a+b = 2$$

$$b = 2-a$$

Además

$$b \cdot x^{5+a} \cdot y^{b-1} + (2+b) \cdot x^{3+a} \cdot y^{1+b} = 0$$

$$(2-a) \cdot \underbrace{x^{5+a}}_{x^{3+a} \cdot x^2} \cdot y^{\overbrace{2-a-1}^{1-a}} = -(2+2-a) \cdot x^{3+a} \cdot \underbrace{y^{\overbrace{1+2-a}^{3-a}}}_{y^{1-a} \cdot y^2}$$

$$(2-a) \cdot x^{3+a} \cdot x^2 \cdot y^{1-a} = -(2+2-a) \cdot x^{3+a} \cdot y^{1-a} \cdot y^2$$

$$x^{3+a} \neq 0$$

$$y^{1-a} \neq 0$$

$$(2-a) \cdot x^2 = -(2-a) \cdot y^2 - 2y^2$$

No llego a nada ಥ︵ಥ

Reescribo el ejercicio: Método ???

$$(x^5 + x^3 \cdot y^2 + y)\, dx + (-x)\, dy = 0$$

$\times \frac{1}{dx}$

$$(x^5 + x^3 \cdot y^2 + y) \underbrace{\frac{dx}{dx}}_{=1} + (-x) \underbrace{\frac{dy}{dx}}_{y'} = 0$$

$$x^5 + x^3 \cdot y^2 + y - x \cdot y' = 0$$

$$y' = \frac{x^5 + x^3 \cdot y^2 + y}{x}$$

Si $y = \mu \cdot x$ $\quad (\mu = \mu(x))$

$$\Rightarrow y' = \mu'(x) \cdot x + \mu(x) \cdot 1$$

$$\mu + \mu' \cdot x = \frac{x^5 + x^3 \cdot \mu^2 \cdot x^2 + \mu \cdot x}{x}$$

$$\mu + \mu' \cdot x = x^4 + x^4 \cdot \mu^2 + \mu$$

$$\mu' \cdot x = x^4 + x^4 \cdot \mu^2$$

$$\mu' \cdot x = x^4 (1 + \mu^2)$$

$$\frac{\mu'}{1 + \mu^2} = x^3 \quad \leftarrow \text{Se separó!}$$

$$\int \frac{u'}{1+u^2}\,dx = \int x^3\,dx$$

$$\arctan u = \frac{1}{4}x^4 + C$$

$$u = \tan\left(\frac{1}{4}x^4 + C\right)$$

Como $y = u \cdot x$

$\Rightarrow$ $\boxed{\begin{array}{l}\underline{\text{Sol}}: \\ y = x \cdot \tan\left(\frac{1}{4}x^4 + C\right)\end{array}}$ $\quad C \in \mathbb{R}$

(e) $\underbrace{2(x+y)\operatorname{sen} y}_{M}\, dx + \left(\underbrace{2(x+y)\operatorname{sen} y + \cos y}_{N}\right) dy = 0$

$$M = 2x.\sin y + 2y.\sin y$$

$$M_y = 2x.\cos y + 2.\sin y + 2y.\cos y$$

$$N = 2x.\sin y + 2y.\sin y + \cos y$$

$$N_x = 2\sin y$$

No es exacta.

Busco μ.

$$M_y - N_x = 2x.\cos y + 2.\sin y + 2y.\cos y - 2\sin y$$

$$= 2x.\cos y + 2y.\cos y$$

$$= 2\cos y\,(x+y)$$

$$M = 2\sin y\,(x+y)$$

Si $\mu_x = 0$

$$\mu_y.M + \mu.M_y = \overbrace{\mu_x.N}^{=0} + \mu.N_x$$

$$\mu_y = \frac{\mu\left(N_x - M_y\right)}{M}$$

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}$$

$$\frac{\mu_y}{\mu} = \frac{-\cancel{2}\cos y\,\cancel{(x+y)}}{\cancel{2}\sin y\,\cancel{(x+y)}}$$

$$\frac{\mu_y}{\mu} = -\frac{\cos y}{\sin y}$$

$$\int \frac{\mu_y}{\mu}\,dy = \int -\frac{\cos y}{\sin y}\,dy$$

$$v = \ln(\sin y)$$

$$dv = \frac{1}{\sin y}\cdot \cos y\, dy$$

$$\ln|\mu| = -\ln(\sin y) + c$$

$$|\mu| = e^{-\ln(\sin y)} \cdot e^{c} \qquad c \in \mathbb{R}$$

$$|\mu| = \sin^{-1} y \cdot \tilde{C}$$

Elijo $\mu(y) = \sin^{-1} y$

$$\begin{cases} M = 2x.\sin y + 2y.\sin y \\ N = 2x.\sin y + 2y.\sin y + \cos y \end{cases}$$

$$\tilde{M} = \mu.M = 2x + 2y$$

$$\tilde{N} = \mu.N = 2x + 2y + \frac{\cos y}{\sin y}$$

$\Rightarrow \tilde{M}_y - \tilde{N}_x = 2 - 2 = 0$ ✓ es exacta

Busco $F(x,y) = c$ con $c \in \mathbb{R}$

Si $\frac{\partial}{\partial x} F = \tilde{M} = 2x + 2y \Rightarrow F = x^2 + 2xy + \varphi(y)$

Si $\frac{\partial}{\partial y} F = \tilde{N} = 2x + 2y + \frac{\cos y}{\sin y} \Rightarrow F = 2xy + y^2 + \ln(\sin y) + \tilde{\varphi}(x)$

$$F(x,y) = x^2 + 2xy + y^2 + \ln(\sin y)$$

$$\boxed{\text{Sol: } x^2 + 2xy + y^2 + \ln(\sin y) = c} \qquad c \in \mathbb{R}$$

Verifico:

$$\frac{\partial}{\partial x} F = 2x + 2y + 2x\underbrace{y'}_{\frac{dy}{dx}} + 2 \cdot y \cdot \underbrace{y'}_{\frac{dy}{dx}} + \frac{1}{\sin y} \cdot \cos y \cdot \underbrace{y'}_{\frac{dy}{dx}} = 0$$

$$2x \cdot dx + 2y \cdot dx + 2x \cdot dy + 2y \cdot dy + \frac{\cos y}{\sin y} \cdot dy = 0$$

$$\underbrace{(2x + 2y)}_{\tilde{M}} dx + \underbrace{\left(2x + 2y + \frac{\cos y}{\sin y}\right)}_{\tilde{N}} dy = 0$$

✓ Verificado

(f) $\underbrace{3y}_{M}\,dx + \underbrace{x}_{N}\,dy = 0$

$M_y = 3$

$N_x = 1$ No es exacta. Pruebo reordenando antes de usar Factor integrante.

$$3y\underbrace{\frac{dx}{dx}}_{=1} + x.\underbrace{\frac{dy}{dx}}_{=y'} = 0$$

$$3y + x.y' = 0$$

$$x.y' = -3y$$

$$\frac{y'}{y} = \frac{-3}{x}$$

$$\int \frac{y'}{y}dx = \int \frac{-3}{x}dx$$

$$\ln|y| = -3\ln|x| + C \qquad C \in \mathbb{R}$$

$$|y| = |x|^{-3}.\tilde{C} \qquad C \in \mathbb{R}_{>0}$$

Mucho módulo, pruebo con F. Int.

Si uso μ

$$\frac{M_y - N_x}{N} = \frac{3-1}{x} = \frac{2}{x} = \frac{\mu'}{\mu}$$

$\ln|\mu| = 2\ln|x| + c$

$|\mu| = |x|^2 \cdot \tilde{C} \qquad \tilde{C} \in \mathbb{R}_{>0}$

$|\mu| = x^2 \cdot \tilde{C}$

Si $\mu(x) = x^2$

(f) $3y\,dx + x\,dy = 0$

$$\Rightarrow \begin{cases} \tilde{M} = 3x^2 \cdot y & \Rightarrow \tilde{M}_y = 3x^2 \\ \tilde{N} = x^3 & \Rightarrow \tilde{N}_x = 3 \cdot x^2 \end{cases}$$

es exacta ✓

Busco $F(x,y) = c$

$\frac{\partial}{\partial x} F = \tilde{M} = 3x^2 \cdot y \Rightarrow F = x^3 \cdot y + \varphi(y)$

$\tilde{N} = x^3 \Rightarrow F = x^3 \cdot y + \tilde{\varphi}(x)$

$\Rightarrow F(x,y) = x^3 \cdot y$

$$\boxed{\text{Sol:} \quad x^3 \cdot y = c} \qquad c \in \mathbb{R}$$

Verifico

$\frac{\partial}{\partial x} F = 3x^2 \cdot y + x^3 \cdot \underbrace{y'}_{\frac{dy}{dx}} = 0$

$3x^2 \cdot y\,dx + x^3 \cdot dy = 0$ ✓ Verificado.

(g) $\big(1 - y(x+y)\tan(xy)\big)\,dx + \big(1 - x(x+y)\tan(xy)\big)\,dy = 0.$