

ANÁLISIS II – ANÁLISIS MATEMÁTICO II – MATEMÁTICA 3

Práctica 3: Teorema de Green.

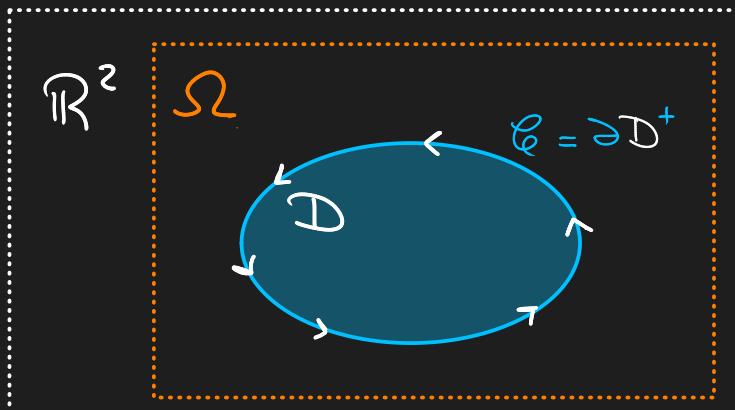
Teo. de Green.

Hipótesis ↙ Volumen $\in \mathbb{R}^2$! $\mathcal{F} = (P, Q)$

- $\Omega \in \mathbb{R}^2$ abierto
- $\mathcal{F}: \Omega \rightarrow \mathbb{R}^2$ es campo C^1
- D es una región de tipo III
- $D \subset \Omega$
- $\partial D = C$ curva cerrada simple, diferenciable \geq veces.
- C está orientada positivamente

Teorema

$$\iint_D Q_x - P_y \, dx dy = \int_C \mathcal{F} \cdot ds$$



Ejercicio 1. Verificar el Teorema de Green para el disco D con centro $(0,0)$ y radio R y las siguientes funciones:

- (a) $P(x,y) = xy^2$, $Q(x,y) = -yx^2$.
 (b) $P(x,y) = 2y$, $Q(x,y) = x$.

$$\iint_D Q_x - P_y \, dy dx \stackrel{?}{=} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

a) $\mathbf{F} = (xy^2, -yx^2)$ $\sigma(t) = (R \cos t, R \sin t)$
 $t \in [0, 2\pi]$

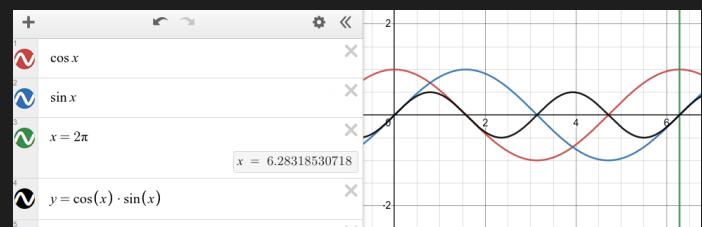
$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_{t=0}^{2\pi} \langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle dt \\ &= \overbrace{\left(R^3 \cos t \sin^2 t, -R^3 \sin t \cos^2 t \right)} \\ &= R^3 \cdot (\cos t \sin^2 t, -\sin t \cos^2 t) \end{aligned}$$

$$\begin{aligned} &= \int_{t=0}^{2\pi} R^3 \cdot \underbrace{\langle (\cos t \sin^2 t, -\sin t \cos^2 t), (-R \sin t, R \cos t) \rangle}_{dt} \\ &= -R \cos t \sin^3 t - R \sin t \cos^3 t \\ &\quad < -R \cos t \sin t \cdot (\sin^2 t + \cos^2 t) \end{aligned}$$

$$\begin{aligned} &= \int_0^{2\pi} -R^4, \underbrace{\cos t \sin t}_{\text{es simétrica!}} dt \quad u = \sin t \\ &\quad du = \cos t dt \end{aligned}$$

$$\begin{aligned} &= -R^4 \cdot \int_{u=0}^1 u \cdot du = 0 \quad \text{Separo en intervalos con cortes en cada } t \text{ tal que } \cos t \sin t = 0 \\ &\quad \text{Área igual!} \end{aligned}$$

$$\iint_D Q_x - P_y \, dx \, dy = \int_{\theta=0}^{2\pi} \int_{r=0}^R ?$$



$$\begin{aligned} F &= \left(xy^2, -yx^2 \right) \\ P &\quad -Q \Rightarrow Q_x = -2xy \\ \hookrightarrow P_y &= 2xy \end{aligned} \quad \left. \begin{array}{l} Q_x - P_y = -2xy - 2xy \\ = -4xy \end{array} \right\}$$

Use coord. polærer

$$\begin{cases} x = r \cdot \cos \theta \\ y = r \cdot \sin \theta \end{cases} \quad \text{Jacobiato}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^R -4 \cdot (r^2 \cdot \cos \theta \cdot \sin \theta) \cdot r \, dr \, d\theta$$

$$= -4 \int_{\theta=0}^{2\pi} \cos \theta \cdot \sin \theta \cdot \underbrace{\int_{r=0}^R r^3 \, dr \, d\theta}_{= 0} = -4 \cdot \frac{1}{4} r^4 \Big|_0^R = \frac{R^4}{4}$$

$$= -\frac{4}{4} R^4 \cdot \int_0^{2\pi} \cos \theta \cdot \sin \theta \, d\theta = 0 \quad \checkmark \text{ son igualer.}$$

$$\underbrace{\quad}_{= 0}$$

$$b) \mathbf{F} = (2y, x)$$

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \left\langle (2 \cdot R \sin \theta, R \cos \theta), (-R \sin \theta, R \cos \theta) \right\rangle d\theta \\
&= \int_0^{2\pi} -2R^2 \sin^2 \theta + R^2 \cos^2 \theta \, d\theta \\
&\quad \underbrace{-2R^2(1-\cos^2 \theta)}_{-2R^2} \\
&= \int_0^{2\pi} -2R^2 + 2R^2 \cos^2 \theta + R^2 \cos^2 \theta \, d\theta \\
&= \underbrace{\int_0^{2\pi} -2R^2 \, d\theta}_{-4\pi R^2} + \underbrace{\int_0^{2\pi} 3R^2 \cos^2 \theta \, d\theta}_{3R^2 \int_0^{2\pi} \cos^2 \theta \, d\theta}
\end{aligned}$$

CA:

$$\begin{aligned}
\int_0^{2\pi} \cos^2 \theta \, d\theta &= \frac{1}{2} \int_0^{2\pi} \cos 2\theta + 1 \, d\theta \quad \text{Tang ident} \\
&\quad \left. \begin{array}{l} \cos 2\theta = 2 \cos^2 \theta - 1 \\ \Rightarrow \cos^2 \theta = \frac{1}{2} \cos 2\theta + \frac{1}{2} \end{array} \right] \\
&= \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, d\theta + \frac{1}{2} \cdot 2\pi \\
&\quad \underbrace{\frac{1}{2} \sin 2\theta \Big|_0^{2\pi}}_{=0} \quad \frac{\partial}{\partial \theta} \frac{1}{2} \sin 2\theta = \frac{1}{2} \cos 2\theta
\end{aligned}$$

$$= \pi$$

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = -4\pi R^2 + 3\pi R^2 = -\pi R^2$$

$\nabla \phi$

$$\iint_D Q_x - P_y \, dy dx = ?$$

$$F = (2y, x) \Rightarrow Q_x = 1 \\ P_y = 2$$

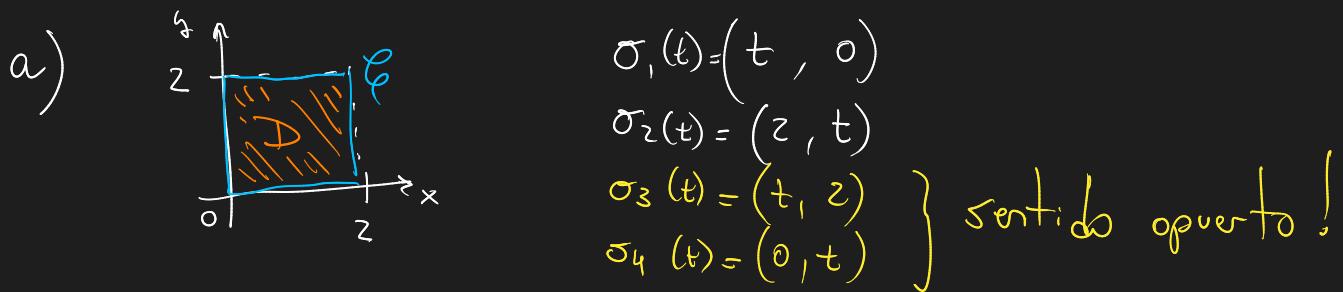
$$\int_{\theta=0}^{2\pi} \int_{r=0}^R (1 - z) \cdot r \, dr d\theta = -\frac{R^2}{2} 2\pi = -\pi R^2 \quad \checkmark$$

ignores

Ejercicio 2. Verificar el Teorema de Green y calcular $\int_{\mathcal{C}} y^2 dx + x dy$, siendo \mathcal{C} la curva recorrida en sentido positivo:

- (a) Cuadrado con vértices $(0,0), (2,0), (2,2), (0,2)$.
- (b) Elipse dada por $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- (c) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, donde $\mathcal{C}_1 : y = x, x \in [0, 1]$, y $\mathcal{C}_2 : y = x^2, x \in [0, 1]$.

$$\mathbf{F} = \begin{pmatrix} y^2 & x \end{pmatrix} \quad \xrightarrow{\text{P} \quad \text{Q}} \quad \Rightarrow \begin{cases} Q_x = 1 \\ P_y = 2y \end{cases} \Rightarrow \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_D 1 - 2y \, dy \, dx$$



$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^1 \underbrace{\left\langle \mathbf{F}(t, 0), (1, 0) \right\rangle}_{(0, t)} dt + \int_{t=0}^1 \underbrace{\left\langle \mathbf{F}(1, t), (0, 1) \right\rangle}_{(t^2, 2)} dt -$$

$$- \int_{t=0}^1 \underbrace{\left\langle \mathbf{F}(1-t, 1), (1, 0) \right\rangle}_{(4, t)} dt - \int_{t=0}^1 \underbrace{\left\langle \mathbf{F}(0, 1-t), (0, 1) \right\rangle}_{(t^2, 0)} dt = 2$$

$$= \int_0^1 2 \, dt - \int_0^1 4 \, dt$$

$$= 4 - 8 = -4$$

$$= 4 - 8 = -4$$

Con Green

- Región D en el tipo II ✓

$$D = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2 \wedge 0 < y < 2\}$$

$$\iint_D Q_x - P_y \, dy dx = \int_0^2 \int_0^2 1 - 2y \, dy dx$$

$$= \int_0^2 \left[z - y^2 \right]_0^2 \, dx$$

$$= \int_0^2 z - 4 \, dx$$

$$= -4 \quad \text{✓ son iguales !}$$

b) (b) Elipse dada por $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\mathcal{F} = \left(\begin{array}{c} y^2 \\ \overline{P} \quad \overline{Q} \end{array}, x \right) \Rightarrow \begin{cases} Q_x = 1 \\ P_y = 2y \end{cases} \Rightarrow \int_{\mathcal{C}} \mathcal{F} ds = \iint_{D} 1 - 2y \, dy \, dx$$

Recordar! Param. de una elipse:

Si elipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \sigma(t) = (a \cdot \cos t, b \cdot \sin t)$
 $t \in [0, 2\pi]$

$$\int_{\mathcal{C}} \mathcal{F} ds = \int_{t=0}^{2\pi} \underbrace{\left\langle \mathcal{F}(a \cdot \cos t, b \cdot \sin t), (-a \cdot \sin t, b \cdot \cos t) \right\rangle}_{(b^2 \cdot \sin^2 t, a \cdot \cos t)} dt$$

sentido \rightarrow ✓

$$= \int_0^{2\pi} -a \cdot b^2 \cdot \sin^3 t + a \cdot b \cdot \cos^2 t \, dt$$

$\underbrace{}$

$$= -a \cdot b^2 \cdot \sin t \cdot (1 - \cos^2 t)$$

$$= -a \cdot b^2 \cdot \sin t + a \cdot b^2 \cdot \sin t \cdot \cos^2 t$$

Recordar!

$$= a \cdot b^2 \int_0^{2\pi} -\sin t + \sin t \left(\frac{1}{2} + \frac{1}{2} \cos 2t \right) \, dt$$

$\underbrace{\phantom{a \cdot b^2 \int_0^{2\pi} -\sin t}}_{=0}$

Trig ident.
• $2 \cos^2 x = 1 + \cos 2x$

$$= ab^2 \cdot \int_0^{2\pi} \underbrace{\frac{1}{2} \sin t}_{=0} + \underbrace{\frac{1}{2} \cos^2 t}_{=0} dt$$

≈ 0

$$\oint_C F ds = \int_0^{2\pi} ab \cdot \cos^2 t dt = ab \cdot \int_0^{2\pi} \cos^2 t dt$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t$$

$$= ab \int_0^{2\pi} \underbrace{\frac{1}{2}}_{=\pi} + \underbrace{\frac{1}{2} \cdot \cos 2t}_{=0} dt$$

$$= ab \cdot \pi$$

Records !

Ahora con Green:

Words ellipticas

$$x = r \cdot a \cdot \cos \theta$$

$$y = r \cdot b \cdot \cos \theta \quad r \in [0,1] \\ \theta \in [0, 2\pi)$$

Jacob: $a \cdot b \cdot r$

$$\iint_D Q_x - P_y dy dx = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \left(1 - z \cdot r \cdot b \cdot \cos \theta \right) \cdot a \cdot b \cdot r dr d\theta$$

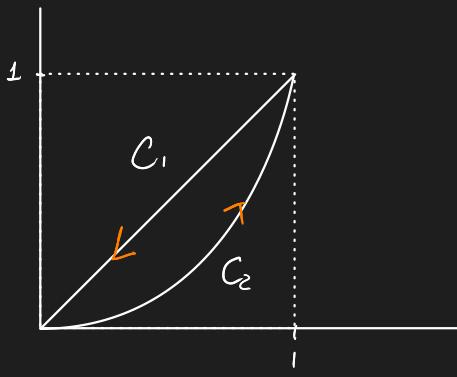
Jacobiano !

$$= \int_0^{2\pi} a \cdot b \cdot \frac{r^2}{2} \Big|_0^1 - z \cdot a \cdot b^2 \cdot \frac{r^3}{3} \cdot \cos \theta \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a \cdot b - \underbrace{\frac{2}{3} a \cdot b^2 \cdot \cos \theta}_{=0} d\theta$$

$$= a \cdot b \cdot \pi // \checkmark \text{ son } \text{ oggi uo[er]}$$

(c) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, donde $\mathcal{C}_1 : y = x$, $x \in [0, 1]$, y $\mathcal{C}_2 : y = x^2$, $x \in [0, 1]$.



$$\mathcal{F} = (y^2, x)$$

$$Q_x - P_y = 1 - 2y$$

$$\sigma_1(t) = (t, t) \quad t \in [0, 1]$$

$$\sigma_2(t) = (t, t^2)$$

$$\int_C \mathcal{F} ds = - \int_{t=0}^1 \langle (t^2, t), (1, 1) \rangle dt + \int_0^1 \langle (t^4, t), (1, 2t) \rangle dt$$

①

②

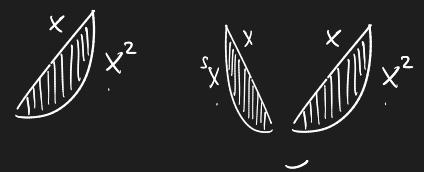
$$\textcircled{I} = - \int_0^1 t^2 + t \, dt = - \left(\frac{1}{3} + \frac{1}{2} \right) = -\frac{5}{6}$$

$$\textcircled{II} = \int_0^1 t^4 + 2t^2 \, dt = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

$$\int_C \mathcal{F} ds = \frac{1}{30}$$

Con Green:

$$\mathcal{D} = \{(x, y) \in \mathbb{R} : 0 \leq x \leq 1 \wedge x^2 \leq y \leq x\}$$



$$\iint_D Q_x - P_y \, dy \, dx = \int_{x=0}^1 \int_{y=x^2}^x 1 - 2y \, dy \, dx$$

$$= \int_0^1 x - x^2 - \left(y^2 \right) \Big|_{x^2}^x dx$$

$$= \int_0^1 x - x^2 - x^2 + x^4 dx$$

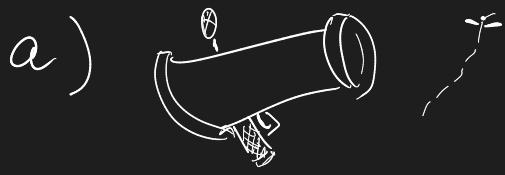
$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5}$$

$$= \frac{1}{30}$$

asombrosamente, son iguales!

Ejercicio 3. Usando el teorema de Green, hallar el área de:

- (a) El disco D con centro $(0,0)$ y radio R
- (b) La región dentro de la elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Debo definir un campo F tal que $Q_x - P_y = 1$
(o cualquier constante c)

$$\Rightarrow Q_x = 1 + P_y$$

$$\text{Si } P_y = 1 \Rightarrow Q_x = 2$$

$$\Rightarrow \begin{cases} P = y \\ Q = 2x \end{cases} \Rightarrow F = (y, 2x) \quad (\text{hay infinitos } F!)$$

$$\iint_D Q_x - P_y \, dy \, dx = \iint_D 2 - 1 \, dy \, dx = \iint_D 1 \, dy \, dx = \text{Área}(D)$$

Con Polares, resolveré:

$$\begin{cases} x = r \cdot \cos \theta & r \in [0, R] \\ y = r \cdot \sin \theta & \theta \in [0, 2\pi) \end{cases}$$

$$\begin{aligned} \iint_D 1 \, dy \, dx &= \int_0^{2\pi} \int_0^R r \, dr \, d\theta && \text{Jacobiano de Polares} \\ &= \int_0^{2\pi} \frac{R^2}{2} = \pi \cdot R^2 // \end{aligned}$$

Pero quiero marcar que

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \text{Area}(\mathcal{D}) \quad \text{pero} \quad \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} Q_x - P_y \, dy \, dx$$

$$= \int_{t=0}^{2\pi} \underbrace{\langle \mathbf{F}(R \cos t, R \sin t), (-R \sin t, R \cos t) \rangle}_{(R \sin t, z \cdot R \cos t)} dt$$

$$= \int_0^{2\pi} -R^2 \cdot \sin^2 t + z \cdot R^2 \cdot \cos^2 t \, dt$$

Nota: Con $\mathbf{F} = (-y, x)$ saldrá de una!

$$z \cos^2 t = 1 + \cos zt$$

$$\sin^2 t = 1 - \cos^2 t$$

$$\begin{aligned} &= \int_0^{2\pi} -R^2 \cdot (1 - \cos^2 t) + R^2 \cdot (1 + \cos zt) \, dt \\ &= \underbrace{-R^2 + R^2 \cos^2 t}_{= -R^2 + \frac{R^2}{2} \cos 2t} + \underbrace{R^2 \cdot (1 + \cos zt)}_{= R^2 \left(1 + \frac{1}{2} \cos zt\right)} \quad (\text{pues } \int_0^{2\pi} \cos zt = 0) \\ &= -R^2 + \frac{1}{2} R^2 \end{aligned}$$

$$= \int_0^{2\pi} \frac{1}{2} R^2 \, dt$$

$$= \pi \cdot R^2$$

(b) La región dentro de la elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Es lo mismo.

Debe dar $\pi \cdot a \cdot b$

$$\text{Us } F = (-y, x) \Rightarrow Q_x - P_y = 1 + 1 = 2 \quad \text{quiero que sea 1!}$$

$$\frac{1}{2} \int_{t=0}^{2\pi} \left\langle F(a \cos t, b \sin t), (-a \sin t, b \cos t) \right\rangle dt$$
$$(-b \sin t, a \cos t)$$
$$\underbrace{ab \sin^2 t + ab \cos^2 t}_{= ab} \checkmark$$

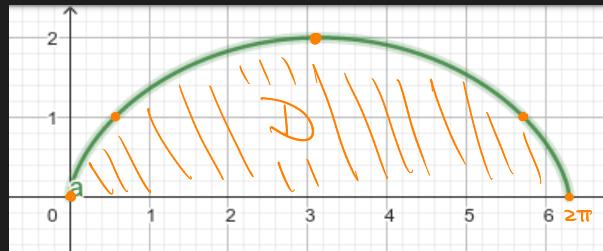
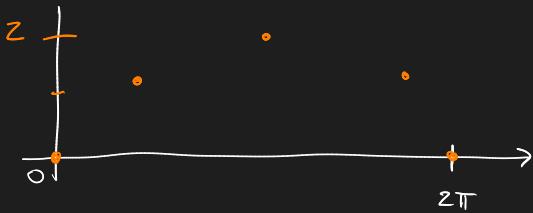
$$= \frac{1}{2} a \cdot b \cdot 2\pi$$

$$= a \cdot b \cdot \pi \checkmark$$

Ejercicio 4. Sea D la región encerrada por el eje x y el arco de cicloide:

$$x = \theta - \sin \theta, \quad y = 1 - \cos \theta, \quad 0 \leq \theta \leq 2\pi.$$

Usando el teorema de Green, calcular el área de D .



$$\mathcal{F} = (-y, x)$$

$$\left. \begin{array}{l} \theta = 0 \Rightarrow (0, 0) \\ \theta = \frac{\pi}{2} \Rightarrow (\frac{\pi}{2} - 1, 1) \\ \theta = \pi \Rightarrow (\pi, 0) \\ \theta = \frac{3\pi}{2} \Rightarrow (\frac{3\pi}{2} - 1, 1) \end{array} \right\} \text{Sentido opuesto a Green!}$$

$$\sigma_1(t) = (t, 0) \quad t \in [0, 2\pi]$$

$$\sigma_2(t) = (t - \sin t, 1 - \cos t) \quad t \in [0, 2\pi] \quad \text{sentido opuesto!}$$

$$\frac{1}{2} \int_C \mathcal{F} \cdot d\sigma = \frac{1}{2} \int_{t=0}^{2\pi} \underbrace{\langle \mathcal{F}(t, 0), (1, 0) \rangle}_{(0, t)} dt - \frac{1}{2} \int_{t=0}^{2\pi} \underbrace{\langle \mathcal{F}(\sigma_2(t)), \sigma'_2(t) \rangle}_{\begin{array}{c} (\cos t - 1, t - \sin t) \\ - (1 - \cos t) \end{array}} dt$$

$$= - \frac{1}{2} \int_0^{2\pi} \underbrace{- (1 - \cos t)^2 + t \cdot \sin t - \sin^2 t}_{-1 + 2\cos t - \cos^2 t} dt$$

$$= - \frac{1}{2} \int_0^{2\pi} -2 + 2\cos t + t \cdot \sin t dt$$

$$= -\frac{1}{2} \left(-4\pi + \underbrace{2\sin t}_{=0} \Big|_0^{2\pi} + \int_0^{2\pi} t \cdot \sin t + dt \right)$$

$$= 2\pi - \frac{1}{2} \cdot \int_0^{2\pi} t \cdot \sin t + dt$$

$$u = t \quad du = 1 \cdot dt$$

$$v = -\cos t \quad dv = \sin t dt$$

$$\text{CA: } \int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int t \cdot \sin t + dt = -t \cdot \cos t - \int -\cos t \cdot dt \\ = -t \cdot \cos t + \sin t$$

Vorhiss (dennus):

$$\cancel{-\cos t} + t \cdot \sin t + \cancel{\cos t} = t \cdot \sin t \checkmark$$

$$\int_0^{2\pi} t \cdot \sin t + dt = -t \cdot \cos t + \underbrace{\sin t}_{=0} \Big|_0^{2\pi} \\ = -2\pi$$

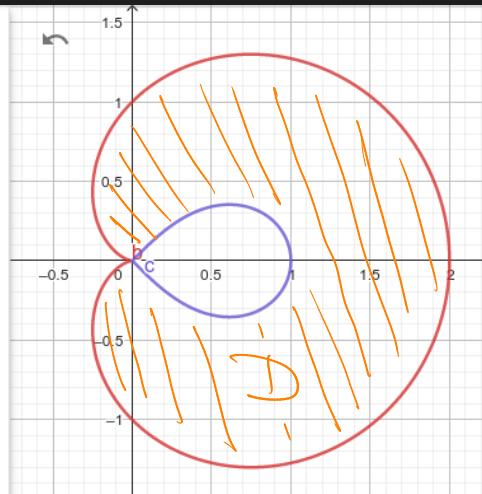
$$\frac{1}{2} \int_C F \cdot ds = 2\pi - \frac{1}{2} \cdot (-2\pi)$$

$$\frac{1}{2} \int_C F \cdot ds = 3\pi //$$

Ejercicio 5. Hallar el área entre las curvas dadas en polares por

$$\begin{aligned} r &= 1 + \cos \theta & \text{con } -\pi \leq \theta \leq \pi, \\ r &= \sqrt{\cos^2 \theta - \sin^2 \theta} & \text{con } -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}. \end{aligned}$$

•	$b = \text{Curve}((1 + \cos(\theta); \theta), \theta, -\pi, \pi)$	⋮
•	$= (1 + \cos(\theta); \theta), \quad (-3.1415926535898 \leq \theta \leq 3.1415926535898)$	
•	$c = \text{Curve}\left(\left(\sqrt{\cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta)}; \theta\right), \theta, -\frac{\pi}{4}, \frac{\pi}{4}\right)$	⋮
•	$= \left(\sqrt{\cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta)}; \theta\right), \quad (-0.7853981633974 \leq \theta \leq 0.7853981633974)$	
+	Input...	



$$\mathcal{F} = (-y, x)$$

$$\frac{1}{2} \iint_D Q_x - P_y \, dy \, dx = \text{Área}(D)$$

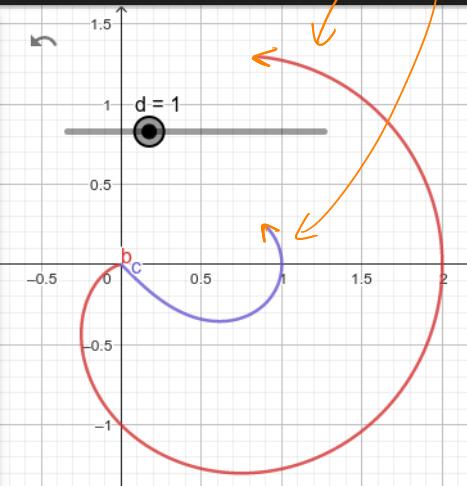
→ P_{ext} no es una curva simple | Separo en \mathcal{C}_{int} y \mathcal{C}_{ext}

Por Green

$$\frac{1}{2} \iint_D Q_x - P_y \, dy \, dx = \frac{1}{2} \int_{\mathcal{C}_{ext}} \mathcal{F} \cdot ds - \frac{1}{2} \int_{\mathcal{C}_{int}} \mathcal{F} \cdot ds$$

orientación correcta
en ambos!

•	$d = 1$	⋮
•	$0 \quad \bullet \quad \pi$	⋮
•	$b = \text{Curve}((1 + \cos(\theta); \theta), \theta, -\pi, d)$	⋮
•	$= (1 + \cos(\theta); \theta), \quad (-3.1415926535898 \leq \theta \leq 1)$	
•	$c = \text{Curve}\left(\left(\sqrt{\cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta)}; \theta\right), \theta, -\frac{\pi}{4}, \frac{d}{4}\right)$	⋮
•	$= \left(\sqrt{\cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta)}; \theta\right), \quad (-0.7853981633974 \leq \theta \leq 0.25)$	
+	Input...	



$$\frac{1}{2} \int_{C_{\text{ext}}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2} \int_{t=-\pi}^{\pi} \underbrace{\left\langle \mathbf{F}(1 + \cos t, t), (-\sin t, 1) \right\rangle}_{\begin{array}{c} \\ \\ \end{array}} dt +$$

$(-\sin t, 1 + \cos t)$

$t \cdot \sin t + 1 + \cos t$

$$= \underbrace{\frac{1}{2} t \Big|_{-\pi}^{\pi}}_{= \pi} + \underbrace{\frac{1}{2} \sin t \Big|_{-\pi}^{\pi}}_{= 0} + \underbrace{\frac{1}{2} \int_{-\pi}^{\pi} t \cdot \sin t \cdot dt}_{\text{to resolve}}$$

$$\frac{1}{2} \left(-t \cdot \cos t + \sin t \Big|_{-\pi}^{\pi} \right)$$

$$\frac{1}{2} \left(-\pi - (+\pi) \right) = -\pi$$

$$\frac{1}{2} \int_{C_{\text{ext}}} \mathbf{F} \cdot d\mathbf{s} = 2\pi$$

$$\frac{1}{2} \int_{C_{\text{int}}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2} \oint_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\langle \mathbf{F} \left(\sqrt{\cos^2 t - \sin^2 t}, t \right), \dots \right\rangle$$

CA:

$$\frac{1}{2} \sqrt{\cos^2 t - \sin^2 t} = \frac{1}{2 \sqrt{\cos^2 t - \sin^2 t}} \cdot (-2 \cos t \cdot \sin t - 2 \sin t \cdot \cos t)$$

$$= -\frac{2 \sin t \cdot \cos t}{\sqrt{\cos^2 t - \sin^2 t}}$$

Trig ident.

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$= 1 - 2 \sin^2 t$$

$$= -\frac{2 \sin t \cdot \cos t}{\sqrt{\cos 2t}}$$

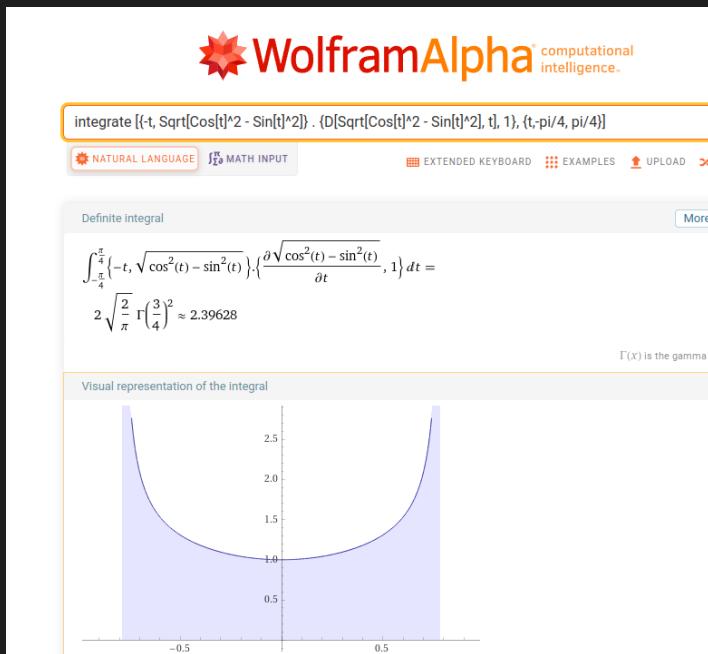
$$\textcircled{1} \quad \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\langle F \left(\sqrt{\cos^2 t - \sin^2 t}, t \right), \left(\frac{-2 \sin t \cdot \cos t}{\sqrt{\cos 2t}}, 1 \right) \right\rangle dt$$

$\underbrace{}_{(-t, \sqrt{\cos 2t})}$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2t \cdot \sin t \cdot \cos t}{\sqrt{\cos 2t}} + \sqrt{\cos 2t} \ dt$$

$\nwarrow \neq 0 \checkmark$

???



No sé cómo seguir.

Da approx $\frac{1}{2} \cdot 2,39628$

$\approx 1,19814$

Total = $\pi - 1,19814$

(2x sin x cos x / sqrt(cos(2x))) + sqrt(cos(2x)) Go!

CLR

+

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x

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v

3v

π

()

This will be calculated:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2x \sin(x) \cdot \frac{\cos(x)}{\sqrt{\cos(2x)}} + \sqrt{\cos(2x)} \right) dx$$

Not what you mean? Use parentheses! Set integration variable and bounds in "Options".

Result

Done! See the result further below.
In order to not miss anything, please scroll all the way down.



YOUR INPUT:
 $f(x) =$

$$\sqrt{\cos(2x)} + \frac{2x \cos(x) \sin(x)}{\sqrt{\cos(2x)}}$$



Note: Your input has been rewritten/simplified.

Simplify/rewrite:

$$\sqrt{\sec(2x)} (x \sin(2x) + \cos(2x))$$



[Simplify](#)

'MANUALLY' COMPUTED ANTIDERIVATIVE:
 $\int f(x) dx = F^*(x) =$

"Manual" integration with steps:

The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might only be valid for parts of the function.

Manual integration with steps:

The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might only be valid for parts of the function.

$$2 E(x | 2) - x \sqrt{\cos(2x)} + C$$



This function cannot be graphed.

[Hide steps](#)

Problem:

$$\int \left(\sqrt{\cos(2x)} + \frac{2x \cos(x) \sin(x)}{\sqrt{\cos(2x)}} \right) dx$$

Rewrite/simplify:

$$= \int \left(\frac{x \sin(2x)}{\sqrt{\cos(2x)}} + \sqrt{\cos(2x)} \right) dx$$

... or choose an alternative:

[Skip simplification](#)

Apply linearity:

$$= \int \frac{x \sin(2x)}{\sqrt{\cos(2x)}} dx + \int \sqrt{\cos(2x)} dx$$

Now solving:

$$\int \frac{x \sin(2x)}{\sqrt{\cos(2x)}} dx$$

Rewrite/simplify using trigonometric/hyperbolic identities:

$$= \int \frac{2x \cos(x) \sin(x)}{\sqrt{2 \cos^2(x) - 1}} dx$$

Apply linearity:

$$= 2 \int \frac{x \cos(x) \sin(x)}{\sqrt{2 \cos^2(x) - 1}} dx$$

Now solving:

$$\int \frac{x \cos(x) \sin(x)}{\sqrt{2 \cos^2(x) - 1}} dx$$

Integrate by parts: $\int fg' = f g - \int f' g$

$$\cos(x) \sin(x)$$

$$f = x, \quad g' = \frac{\cos(x) \sin(x)}{\sqrt{2 \cos^2(x) - 1}}$$

[↓ steps](#)

$$f' = 1, \quad g = -\frac{\sqrt{2 \cos^2(x) - 1}}{2}$$

$$= -\frac{x \sqrt{2 \cos^2(x) - 1}}{2} - \int -\frac{\sqrt{2 \cos^2(x) - 1}}{2} dx$$

Now solving:

$$\int -\frac{\sqrt{2 \cos^2(x) - 1}}{2} dx$$

$$= \int -\frac{\sqrt{1 - 2 \sin^2(x)}}{2} dx$$

Apply linearity:

$$= -\frac{1}{2} \int \sqrt{1 - 2 \sin^2(x)} dx$$

Configure the Integral Calculator:

Variable of integration:

Upper bound (to):

Lower bound (from):

Integrate numerically only?

Simplify expressions?

Simplify all roots?

(\sqrt{x} becomes x, not $|x|$)

Use complex domain (C)?

Keep decimals?

Now solving:

$$\int \sqrt{1 - 2 \sin^2(x)} dx$$

This is a special integral (incomplete elliptic integral of the second kind):

$$= E(x | 2)$$

Plug in solved integrals:

$$-\frac{1}{2} \int \sqrt{1 - 2 \sin^2(x)} dx \\ = -\frac{E(x | 2)}{2}$$

Plug in solved integrals:

$$-\frac{x \sqrt{2 \cos^2(x) - 1}}{2} - \int -\frac{\sqrt{2 \cos^2(x) - 1}}{2} dx \\ = \frac{E(x | 2)}{2} - \frac{x \sqrt{2 \cos^2(x) - 1}}{2}$$

Plug in solved integrals:

$$2 \int \frac{x \cos(x) \sin(x)}{\sqrt{2 \cos^2(x) - 1}} dx \\ = E(x | 2) - x \sqrt{2 \cos^2(x) - 1}$$

Now solving:

$$\int \sqrt{\cos(2x)} dx$$

Substitute $u = 2x \rightarrow du = 2 dx$ (steps):

$$= \frac{1}{2} \int \sqrt{\cos(u)} du$$

Now solving:

$$\int \sqrt{\cos(u)} du$$

Rewrite/simplify using trigonometric/hyperbolic identities:

$$= \int \sqrt{2 \cos^2\left(\frac{u}{2}\right) - 1} du \\ = \int \sqrt{1 - 2 \sin^2\left(\frac{u}{2}\right)} du$$

Substitute $v = \frac{u}{2} \rightarrow dv = \frac{1}{2} du$ (steps):

$$= 2 \int \sqrt{1 - 2 \sin^2(v)} dv$$

Now solving:

$$\int \sqrt{1 - 2 \sin^2(v)} dv$$

Use previous result:

$$= E(v | 2)$$

Plug in solved integrals:

$$2 \int \sqrt{1 - 2 \sin^2(v)} dv \\ = 2 E(v | 2)$$

Undo substitution $v = \frac{u}{2}$:

$$= 2 E\left(\frac{u}{2} | 2\right)$$

Plug in solved integrals:

$$\frac{1}{2} \int \sqrt{\cos(u)} du \\ = E\left(\frac{u}{2} | 2\right)$$

Undo substitution $u = 2x$:

$$= E(x | 2)$$

Plug in solved integrals:

$$\int \frac{x \sin(2x)}{\sqrt{\cos(2x)}} dx + \int \sqrt{\cos(2x)} dx \\ = 2 E(x | 2) - x \sqrt{2 \cos^2(x) - 1}$$

The problem is solved:

$$\int \left(\frac{x \sin(2x)}{\sqrt{\cos(2x)}} + \sqrt{\cos(2x)} \right) dx \\ = 2 E(x | 2) - x \sqrt{2 \cos^2(x) - 1} + C$$

Rewrite/simplify:

$$= 2 E(x | 2) - x \sqrt{\cos(2x)} + C$$

Antiderivative or integral could not be found.

Approximation:

$$2.396280469470934$$

Lo mismo si:

Calculate the integral of ...

($2x \sin x \cos x / \sqrt{\cos^2(x) - \sin^2(x)} + \sqrt{1 - \cos^2(x)}$) + sqrt(pi)

This will be calculated:

$$\int_{-\pi/4}^{\pi/4} \left(2x \sin(x) \cdot \frac{\cos(x)}{\sqrt{\cos^2(x) - \sin^2(x)}} + \sqrt{\cos^2(x) - \sin^2(x)} \right) dx$$

About Help Examples Options

Configure the Integral Calculator:

Variable of integration: x

Upper bound (to): $\pi/4$

Lower bound (from): $-\pi/4$

Integrate numerically only?

Simplify expressions?

Simplify all roots?

($\sqrt{x^2}$ becomes x , not $|x|$)

Use complex domain (\mathbb{C})?

Keep decimals?

Not what you mean? Use parentheses! Set integration variable and bounds in "Options".

Result

Done! See the result further below.

In order to not miss anything, please scroll all the way down.

YOUR INPUT:
 $f(x) =$

= Check your own answer
 = Export the expression (e.g. LaTeX)

$$\sqrt{\cos^2(x) - \sin^2(x)} + \frac{2x \cos(x) \sin(x)}{\sqrt{\cos^2(x) - \sin^2(x)}}$$

Note: Your input has been rewritten/simplified.

Simplify/rewrite:
$$\frac{x \sin(2x) + \cos(2x)}{\sqrt{\cos(2x)}}$$

$$(\tilde{x})$$

En lo resuelto se que

$$\sigma_2(t) = (r(t), t)$$

Luego resuelve, y el final reemplaza

Ejercicio 6. Probar la fórmula de integración por partes: Si $D \subset \mathbb{R}^2$ es un dominio elemental, ∂D su frontera orientada en sentido antihorario y $\mathbf{n} = (n_1, n_2)$ la normal exterior a D , entonces

$$\int_D u v_x \, dx \, dy = - \int_D u_x v \, dx \, dy + \int_{\partial D} u v n_1 \, ds,$$

para todo par de funciones $u, v \in C(\bar{D}) \cap C^1(D)$.

Quiero algo parecido a Green,

$$\underbrace{\iint_D u v_x \, dx \, dy + \iint_D u_x \cdot v \, dx \, dy}_{\text{I}} = \underbrace{\int_{\partial D} u \cdot v \cdot n_1 \, ds}_{\text{II}}$$

$$\text{I} \quad \iint_D u \cdot v_x + u_x \cdot v \, dx \, dy = \frac{\partial}{\partial x} (u \cdot v)$$

$$\iint_D \frac{\partial}{\partial x} (u \cdot v) \, dx \, dy = \iint_D u \cdot v \, dx \, dy$$

Clave: elijo $\mathbf{F} = (0, u \cdot v)$

$$\Rightarrow Q_x - P_y = u_x \cdot v - u \cdot v_x$$

Por Green:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D Q_x - P_y \, dx \, dy$$

$\underbrace{\qquad\qquad\qquad}_{\text{Lo que tengo en I}}$

Sos σ param de ∂D

$$\Rightarrow \int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{t=a}^b \langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle dt$$

$$\begin{aligned}
 &= \int_a^b \left\langle (\sigma, Q(\sigma(t))), (\sigma'_1(t), \sigma'_2(t)) \right\rangle dt \\
 &= \int_a^b Q(\sigma(t)) \cdot \sigma'_2(t) dt \\
 &= \int_a^b u \cdot v(\sigma(t)) \cdot \sigma'_2(t) dt \quad \textcircled{III}
 \end{aligned}$$

Como n é normal exterior

$\sigma'(t)$ é tangente $\Rightarrow (\sigma'_2, -\sigma'_1)$ é perpendicular

$$n = \frac{(\sigma'_2, -\sigma'_1)}{\|\sigma'(t)\|} = \underbrace{\left(\frac{\sigma'_2}{\|\sigma'(t)\|}, \frac{-\sigma'_1}{\|\sigma'(t)\|} \right)}_{\left(\underbrace{n_1}_{n_1}, \underbrace{n_2}_{n_2} \right)}$$

De \textcircled{II} :

$$\int_{\partial D} u \cdot v \cdot n_1 ds = \int_{t=a}^b u \cdot v(\sigma(t)) \cdot \frac{\sigma'_2}{\|\sigma'(t)\|} \cdot \frac{\|\sigma'(t)\|}{\|\sigma'(t)\|} dt$$

por ser integral curvilínea
 $\downarrow \int f d\sigma = \int_a^b f(\sigma(s)) \cdot \|\sigma'(s)\| ds$

$$= \int_a^b u \cdot v(\sigma(t)) \cdot \sigma'_2 dt$$

Que é igual a \textcircled{III}

$$\iint_D u \cdot v_x \, dx dy + \iint_D u_x \cdot v \, dx dy = \oint_{\partial D} u \cdot v \cdot n_1 \, ds$$

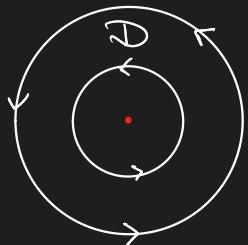
✓

Reporrt!

Ejercicio 7. Sean P y Q funciones continuamente diferenciables en \mathbb{R}^2 . Verificar que el Teorema de Green para estas funciones es válido cuando la región D es el anillo

$$D = \{(x, y) / 1 \leq x^2 + y^2 \leq 4\}.$$

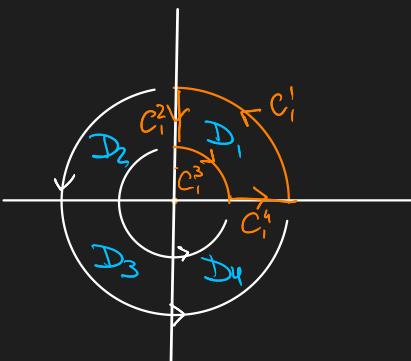
Sugerencia: Aplicar el Teorema de Green en los discos de radios 1 y 2.



$$\iint_D Q_x - P_y \, dx dy = \int_{C_{\text{ext}}} F \cdot ds - \int_{C_{\text{int}}} F \cdot ds$$

$$\int_{C_{\text{ext}}} F \cdot ds = \int_{t=0}^{2\pi} \underbrace{\langle F(z \cdot \cos t, z \cdot \sin t), (-z \cdot \sin t, z \cdot \cos t) \rangle}_{\text{no puedo hacer nada!}} dt$$

Dividido en 4 partes

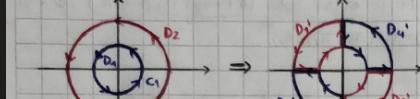


$$\iint_D Q_x - P_y \, dx dy = \int_{\partial D_1} P \cdot dx + Q \cdot dy$$

$$\partial D_1 = C_1^1 \cup C_1^2 \cup C_1^3 \cup C_1^4$$

⑦ P, Q funciones C^1 en \mathbb{R}^2 , $D = \{(x, y) / 1 \leq x^2 + y^2 \leq 4\} \Rightarrow$ Dividido en D_1 y D_2

$$\Rightarrow D_1 = \{(x, y) / x^2 + y^2 \leq 1\}, D_2 = \{(x, y) / x^2 + y^2 \leq 4\}$$



$\Rightarrow D = D_1 \cup D_2 \cup D_3 \cup D_4 \Rightarrow$ Dividido el anillo en 4 regiones tipo III y las recorren en sentido positivo

\Rightarrow Veo que los bordes rectos los recorre 2 veces, 1 en cada sentido de forma que se cancelan, y resultan una curva que recorre el borde interior en sentido horario y una que recorre el borde exterior en sentido antihorario

$$\Rightarrow C_1: C_1(t) = (\cos t, \sin t) \text{ con } t \in [0, 2\pi]$$

$$C_2: C_2(t) = (2\cos t, 2\sin t) \text{ con } t \in [0, 2\pi]$$

$$\begin{aligned} &\Rightarrow \iint_{D_1} (Q_x - P_y) \, dx dy = \int_{C_2^1} P \, dx + Q \, dy + \int_{C_1^1} P \, dx + Q \, dy - \int_{C_1^2} P \, dx + Q \, dy - \int_{C_2^2} P \, dx + Q \, dy \\ &\Rightarrow \iint_{D_1} (Q_x - P_y) \, dx dy = \int_{C_2^1} P \, dx + Q \, dy + \int_{C_2^2} P \, dx + Q \, dy - \int_{C_1^1} P \, dx + Q \, dy - \int_{C_1^2} P \, dx + Q \, dy \end{aligned}$$

$$\Rightarrow \text{Como } D = D_1 \cup D_2 \cup D_3 \cup D_4 \Rightarrow \iint_D (Q_x - P_y) \, dx dy = \iint_{D_1} (Q_x - P_y) \, dx dy + \iint_{D_2} (Q_x - P_y) \, dx dy + \iint_{D_3} (Q_x - P_y) \, dx dy + \iint_{D_4} (Q_x - P_y) \, dx dy$$

\Rightarrow Al sumar las integrales de las 4 divisiones de D , las curvas horizontales y verticales que usamos para cerrar dichas divisiones son recorridas 2 veces cada una, una vez con orientación positiva y otra negativa, por lo que las integrales de curva se cancelan, y solo me quedan los términos que corresponden a partes de C_1 (orientada positivamente) y C_2 (orientada negativamente)

$$\Rightarrow \iint_D (Q_x - P_y) \, dx dy = \int_{C_2^1} P \, dx + Q \, dy - \int_{C_1^2} P \, dx + Q \, dy$$

Ejercicio 8. Sea \mathcal{C} la curva

$$\begin{aligned}x &= 0, & 0 \leq y \leq 4, \\y &= 4, & 0 \leq x \leq 4, \\y &= x, & 0 \leq x \leq 1, \\y &= 2-x, & 1 \leq x \leq 2, \\y &= x-2, & 2 \leq x \leq 3, \\y &= 4-x, & 2 \leq x \leq 3, \\y &= x, & 2 \leq x \leq 4,\end{aligned}$$

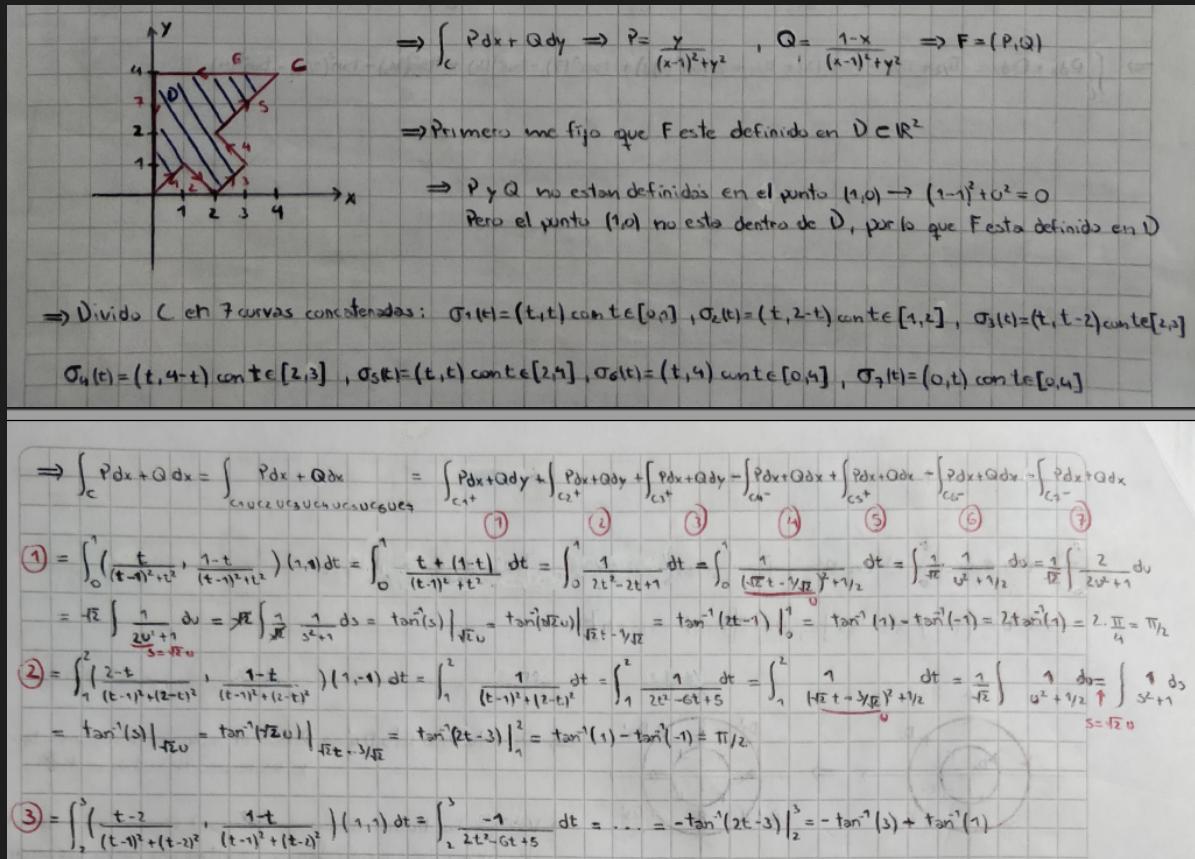
orientada positivamente. Calcular

$$\int_{\mathcal{C}} \frac{y}{(x-1)^2+y^2} dx + \frac{1-x}{(x-1)^2+y^2} dy.$$

Ejercicio 9. Sea $D = \{(x, y) / 1 \leq x^2 + y^2 \leq 4, x \geq 0\}$. Calcular

$$\int_{\partial D} x^2 y dx - xy^2 dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).



$$\begin{aligned}\int_{\mathcal{C}} P dx + Q dy &= \int_{C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7} P dx + Q dy = \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy + \int_{C_3} P dx + Q dy + \int_{C_4} P dx + Q dy + \int_{C_5} P dx + Q dy + \int_{C_6} P dx + Q dy + \int_{C_7} P dx + Q dy \\① &= \int_0^1 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2} \right) (1,1) dt = \int_0^1 \frac{t+1-t}{(t-1)^2+t^2} dt = \int_0^1 \frac{1}{2t^2-2t+1} dt = \int_0^1 \frac{1}{(\sqrt{2}t-\sqrt{2})^2+\sqrt{2}/2} dt = \int_0^{\pi/4} \frac{1}{\sqrt{2}} \frac{1}{u^2+1/2} du = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{2}{2u^2+1} du \\&= \frac{\sqrt{2}}{2} \int_0^{\pi/4} \frac{1}{u^2+1} du = \frac{\sqrt{2}}{2} \left[\frac{1}{\sqrt{2}} \arctan(u) \right]_0^{\pi/4} = \frac{1}{2} \arctan(\sqrt{2}/2) = \frac{1}{2} \arctan(2t-1) \Big|_0^1 = \frac{1}{2} \arctan(1) - \frac{1}{2} \arctan(-1) = \frac{1}{2} (\arctan(1) - \arctan(-1)) = \frac{\pi}{4} \\② &= \int_1^2 \left(\frac{2-t}{(t-1)^2+(2-t)^2}, \frac{1-t}{(t-1)^2+(2-t)^2} \right) (1,-1) dt = \int_1^2 \frac{1}{(t-1)^2+(2-t)^2} dt = \int_1^2 \frac{1}{2t^2-6t+5} dt = \int_1^2 \frac{1}{(2t-3)^2+1/2} dt = \frac{1}{\sqrt{2}} \int_1^2 \frac{1}{u^2+1/2} du = \frac{1}{\sqrt{2}} \int_1^2 \frac{2}{2u^2+1} du \\&= \frac{1}{\sqrt{2}} \arctan(u) \Big|_1^2 = \frac{1}{\sqrt{2}} \arctan(2t-3) \Big|_1^2 = \frac{1}{\sqrt{2}} (\arctan(1) - \arctan(-1)) = \frac{\pi}{4} \\③ &= \int_2^3 \left(\frac{t-2}{(t-1)^2+(t-2)^2}, \frac{1-t}{(t-1)^2+(t-2)^2} \right) (1,-1) dt = \int_2^3 \frac{-1}{2t^2-6t+5} dt = \dots = -\tan^{-1}(2t-3) \Big|_2^3 = -\tan^{-1}(3) + \tan^{-1}(1)\end{aligned}$$

$$④ = \int_2^3 \left(\frac{4-t}{(t-1)^2+(4-t)^2}, \frac{1-t}{(t-1)^2+(4-t)^2} \right) (1,-1) dt = \int_2^3 \frac{3}{2t^2-10t+17} dt = \dots = 2 \tan^{-1}(1/3)$$

$$⑤ = \int_2^4 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2} \right) (1,-1) dt = \int_2^4 \frac{1}{2t^2-2t+1} dt = \dots = \tan^{-1}(2t-1) \Big|_2^4 = \tan^{-1}(7) - \tan^{-1}(3)$$

$$⑥ = \int_0^4 \left(\frac{4}{(t-1)^2+4^2}, \frac{1-t}{(t-1)^2+4^2} \right) (1,0) dt = \int_0^4 \frac{4}{t^2-2t+17} dt = \int_0^4 \frac{4}{(t-1)^2+16} dt = \int_{-1}^3 \frac{4}{u^2+16} du = \int_{-1}^3 \frac{4}{16} \frac{1}{u^2+1} du = \frac{1}{4} \int_{-1}^3 \frac{1}{u^2+1} du = \frac{1}{4} \arctan(u) \Big|_{-1}^3 = \frac{1}{4} (\arctan(3) - \arctan(-1)) = \frac{1}{4} (\arctan(3) + \arctan(1/4))$$

$$⑦ = \int_0^4 \left(\frac{t}{(t-1)^2+t^2}, \frac{1}{(t-1)^2+t^2} \right) (0,1) dt = \int_0^4 \frac{1}{t^2+1} dt = \tan^{-1}(t) \Big|_0^4 = \tan^{-1}(4)$$

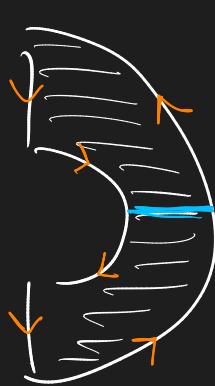
$$\Rightarrow \int_{\mathcal{C}} P dx + Q dy = \frac{\pi}{4} + \frac{\pi}{4} - \tan^{-1}(2) + \tan^{-1}(1/2) + \tan^{-1}(7) - \tan^{-1}(3) - \tan^{-1}(1/4) - \tan^{-1}(1/4) - \tan^{-1}(4) = 0$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_D \left(\frac{\partial}{\partial x} \left(\frac{x-1}{(x-1)^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{1-x}{(x-1)^2+y^2} \right) \right) dx dy = \iint_D \left(\frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2} - \frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2} \right) dx dy = 0 \Rightarrow \text{MUCHO mas facil}$$

Ejercicio 9. Sea $D = \{(x, y) / 1 \leq x^2 + y^2 \leq 4, x \geq 0\}$. Calcular

$$\int_{\partial D} x^2 y \, dx - xy^2 \, dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).



$$\int_{\partial D} x^2 y \, dx - xy^2 \, dy = \iint_D Q_x - P_y \, dx \, dy$$

$$\mathcal{F} = (x^2 y, -xy^2)$$

$$Q_x = -y^2$$

$$P_y = x^2$$

$$\iint_D Q_x - P_y \, dx \, dy = \iint_D -y^2 - x^2 \, dx \, dy =$$

Polarer

$$\begin{cases} x = r \cos t & r \in [1, 2] \\ y = r \sin t & t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

$$= - \int_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^2 (r^2 \cos^2 t + r^2 \sin^2 t) r \, dr \, dt$$

$$= \int_1^2 r^3 \, dr = \frac{r^4}{4} \Big|_1^2 = \frac{1}{4} (16 - 1) = \frac{15}{4}$$

↓ Jacobiano

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{15}{4} \, dt$$

$$= - \frac{15}{4} \cdot t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = - \frac{15}{4} \pi$$

// en el resuelto no usó Green

Ejercicio 10. Calcular el trabajo efectuado por el campo de fuerzas $\mathbf{F}(x, y) = (y + 3x, 2y - x)$ al mover una partícula rodeando una vez la elipse $4x^2 + y^2 = 4$ en el sentido de las agujas del reloj.

sentido opuesto a Green!

Se pide

$$\int_{\mathcal{E}} \mathbf{F} \cdot d\mathbf{s} = \underbrace{- \int_{t=0}^{2\pi} \langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle dt}_{\text{Con } \sigma(t) = (1 \cdot \cos t, 2 \cdot \sin t)}$$

sentido \oplus , pero quiero \ominus

Puedo usar Green, donde

$$\int_{\mathcal{E}} \mathbf{F} \cdot d\mathbf{s} = - \iint_D Q_x - P_y$$

Como busco $\int_{\mathcal{E}} \mathbf{F} \cdot d\mathbf{s}$ con \mathcal{E} recorrida \curvearrowleft

\Rightarrow Cambio el signo desde ahora (escribiendo que \mathcal{E} se recorre \curvearrowleft)

Calculo:

$$\mathbf{F} = \left(\underbrace{y + 3x}_P, \underbrace{2y - x}_Q \right) \Rightarrow \begin{cases} Q_x = -1 \\ P_y = 1 \end{cases}$$

$$-\iint_D Q_x - P_y = - \iint_D -z dy dx = +2 \text{ Área (Elipse)} \\ = \frac{2 \cdot 1 \cdot 2 \cdot \pi}{\tilde{a} \tilde{b}}$$

$$\text{Trabajo} = 4\pi$$

Ejercicio 11. Sea $\mathbf{F}(x, y) = (P(x, y), Q(x, y)) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$. Calcular $\int_C \mathbf{F} \cdot d\mathbf{s}$ donde C es la circunferencia unitaria centrada en el origen orientada positivamente. Calcular $Q_x - P_y$. ¿Se satisface en este caso el Teorema de Green?

\mathbf{F} no está bien definida en el $(0,0)$!

No se satisface el teorema de Green.

$$\text{Sí: } \iint_D Q_x - P_y \, dy \, dx \neq \int_C \mathbf{F} \cdot d\mathbf{s}$$

$$Q_x = \underbrace{\frac{-1}{x^2 + y^2}}_{+} + \underbrace{-x \cdot (-1) \cdot (x^2 + y^2)^{-2} \cdot 2x}_{+ 2x^2} \\ \frac{(x^2 + y^2)^{-2}}{(x^2 + y^2)^2}$$

$$P_y = \underbrace{\frac{1}{x^2 + y^2}}_{+} + \underbrace{y \cdot (-1) \cdot (x^2 + y^2)^{-2} \cdot 2y}_{- 2y^2} \\ \frac{(x^2 + y^2)^{-2}}{(x^2 + y^2)^2}$$

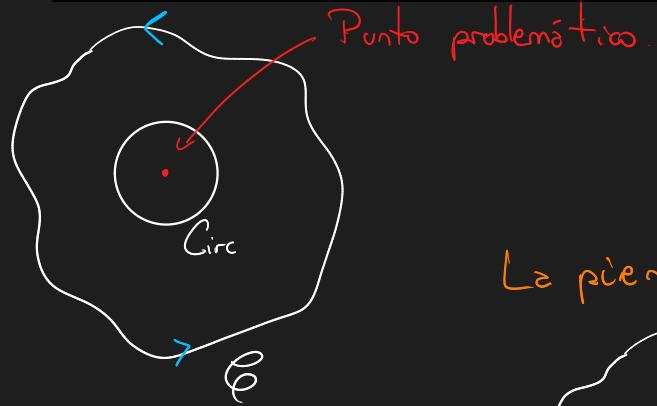
$$Q_x - P_y = \underbrace{\frac{-1}{x^2 + y^2}}_{+} + \underbrace{\frac{2x^2}{(x^2 + y^2)^2}}_{-} - \underbrace{\frac{1}{x^2 + y^2}}_{+} + \underbrace{\frac{2y^2}{(x^2 + y^2)^2}}_{-}$$

$$= \underbrace{\frac{-2}{x^2 + y^2}}_{+} + \underbrace{\frac{2(x^2 + y^2)}{(x^2 + y^2)^2}}_{//}$$

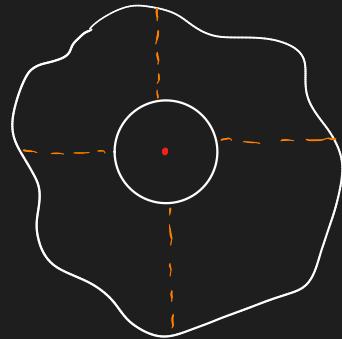
$$= 0 //$$

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{t=0}^{2\pi} \underbrace{\left\langle \left(\frac{\sin t}{1}, -\frac{\cos t}{1} \right), \left(-\sin t, \cos t \right) \right\rangle}_{-\sin^2 t - \cos^2 t} dt \\
 &= -2\pi //
 \end{aligned}$$

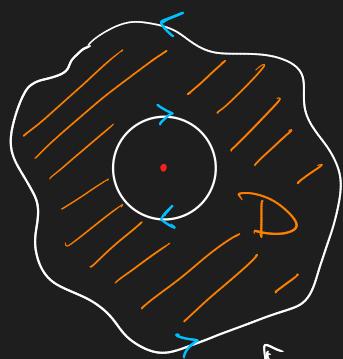
Ejercicio 12. Sea \mathcal{C} una curva cerrada, simple y suave orientada positivamente que encierra la circunferencia unitaria centrada en el origen. Calcular $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ para el campo \mathbf{F} definido en el ejercicio 11.



Punto problemático.



La pienso así



D no contiene el punto problemático

Con Green

$$\underbrace{\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} + \int_{\text{Circ}} \mathbf{F} \cdot d\mathbf{s}}_{\text{lado del punto II}} = \iint_D Q_x - P_y \, dx \, dy$$

= 0 por ej 11

(sentido opuesto)

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} - \int_{\text{Circ}^+} \mathbf{F} \cdot d\mathbf{s} = 0$$

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \oint_{C_{\text{circ}}^+} \mathbf{F} \cdot d\mathbf{s}$$

Calculada en ej 11.

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -2\pi$$

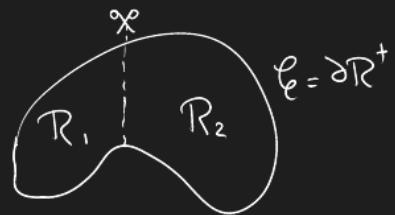
Para cualquier curva \mathcal{C} que encierra a C_{circ} !

Ver notas de Rela más abajo acerca de:

- Regiones con agujeros
- Cerrar curvas

Extensiones del Teorema (Notas de Relo)

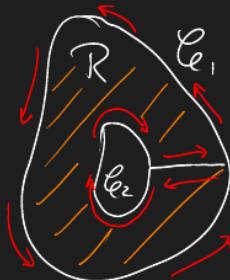
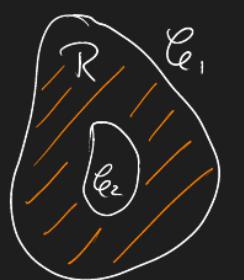
1) a) Regiones que se descomponen



$$\ell_1 = \partial R_1^+$$

$$\mathcal{C}_2 = \partial R_2^+$$

b) Agujeros R , $\mathcal{C} = \partial R$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$



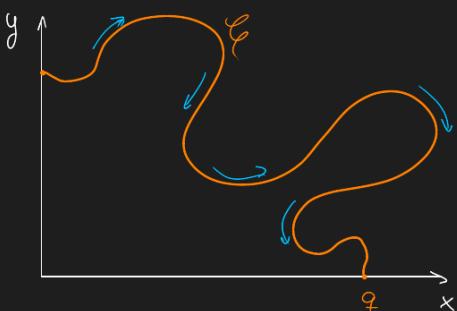
✓ + 2 trocitos que se
enrolan

Aplicación (Cerrando Curvas)

Situación:

\mathcal{C} curva (no cerrada) f_{eo}

F un campo $F = (P, Q)$ f_{eo}



Ejemplo (mismo que la práctica):

C dada y orientada por

$$\sigma(t) = (\sin t, \cos t)$$

$$t \in [0, \frac{3}{2}\pi]$$

$$F(x,y) = \left(2 \cos(x^2y) \cdot xy - 2y, x^2 \cdot \cos(x^2y) + 3x \right)$$

Plan A: Resuelvo a mano $\int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle dt$

• Plan B: Completar la curva y usar Green

Definiendo curvas que la cierran, de forma (Γ_1, Γ_2)

que se simplifique el cálculo junto al campo.

Ejercicio 13. Calcular $\int_C f_1 dx + f_2 dy$ siendo

$$f_1(x, y) = \frac{x \operatorname{sen} \frac{\pi}{2(x^2 + y^2)} - y(x^2 + y^2)}{(x^2 + y^2)^2},$$

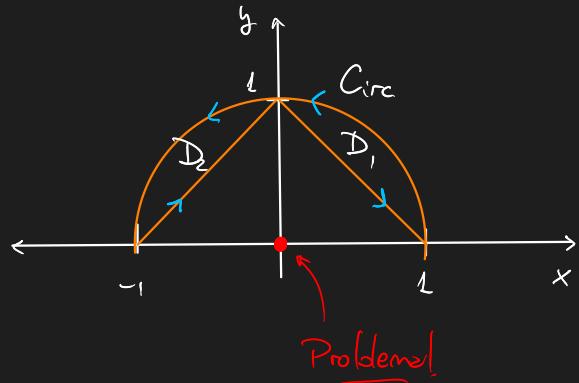
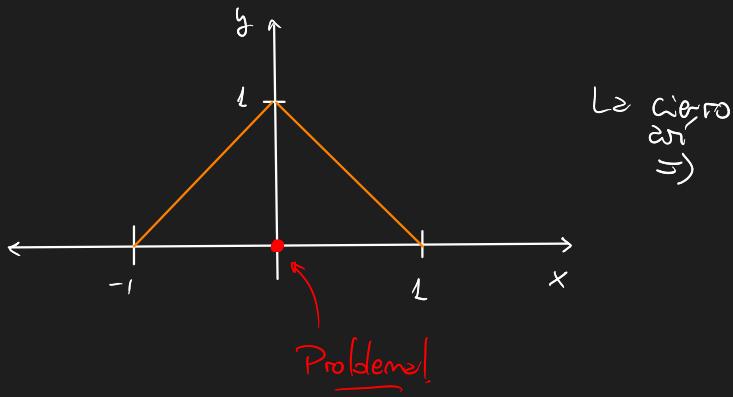
$$f_2(x, y) = \frac{y \operatorname{sen} \frac{\pi}{2(x^2 + y^2)} + x(x^2 + y^2)}{(x^2 + y^2)^2},$$

y

$$C = \begin{cases} y = x + 1 & \text{si } -1 \leq x \leq 0, \\ y = 1 - x & \text{si } 0 \leq x \leq 1, \end{cases}$$

recorrida del $(-1, 0)$ al $(1, 0)$.

Si $x^2 + y^2 = 1 \Rightarrow$ se simplifica todo!



$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{C_{\text{arc}}} \mathbf{F} \cdot d\mathbf{s} = \iint_{D_1 \cup D_2} Q_x - P_y \, dx \, dy$$

$$P_y = \frac{\partial}{\partial y} \cdot \left(\frac{x \operatorname{sen} \frac{\pi}{2(x^2 + y^2)} - y(x^2 + y^2)}{(x^2 + y^2)^2} \right)$$

Revisémos y veo que $Q_x - P_y = 0$

Entonces calculo:

$$\int_{C_{\text{arc}}} \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{\pi} \underbrace{\langle \mathbf{F}(\sigma(t)), \sigma'(t) \rangle}_{\sigma(t) = (\cos t, \sin t)} dt$$

$$\left(\cos t \cdot \sin \left(\frac{\pi}{2} \right) - \sin t, \sin t \cdot \sin \left(\frac{\pi}{2} \right) + \cos t \right)$$

$$(\cos t - \sin t, \cos t + \sin t)$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\Rightarrow \int_{Circ} F \cdot ds = \int_0^{\pi} -\cos t \cdot \sin t + \sin^2 t + \cos^2 t + \cos t \cdot \sin t$$

$$= \pi$$

Observe

$$\int_{\mathcal{C}} F \cdot ds + \int_{Circ} F \cdot ds = \iint_{D_1 \cup D_2} Q_x - P_y \, dx dy$$

$\underbrace{\qquad}_{\pi}$ $\underbrace{\qquad}_{=0}$

$$\therefore \int_{\mathcal{C}} F \cdot ds = -\pi$$

Ejercicio 14. Determinar todas las circunferencias C en el plano \mathbb{R}^2 sobre las cuales vale la siguiente igualdad

$$\int_C -y^2 dx + 3x dy = 6\pi.$$

$F = (-y^2, 3x)$
Buenas líneas en todo \mathbb{R}^2

$$\Rightarrow \int_C F ds = \iint_D Q_x - P_y dx dy = 6\pi$$

Donde

$$Q_x - P_y = 3 + 2y$$

Quiero usar Polares, PERO con circunferencias con cualquier centro (no solo $(0,0)$)

$$\begin{cases} x = r \cos \theta + x_0 & r \in [0, R] \quad x_0, y_0 \in \mathbb{R} \\ y = r \sin \theta + y_0 & \theta \in [0, 2\pi) \end{cases}$$

$$= \int_{r=0}^R \underbrace{\int_{\theta=0}^{2\pi} (3 + 2 \cdot r \sin \theta + 2y_0) \cdot r}_{\text{Jacobiano}} \cdot d\theta dr - \left(\underbrace{2 \cdot r^2 \cos \theta}_{=0} \right) \Big|_0^{2\pi}$$

$$= 2\pi \int_0^R 3r + 2y_0 \cdot r dr$$

$$= 2\pi \left(\frac{3}{2} R^2 + 2y_0 \cdot \frac{R^2}{2} \right)$$

$$= R^2 \left(\frac{3}{2} \pi + 2\pi \cdot y_0 \right)$$

Por dato

$$R^2 \cdot (3\pi + 2\pi \cdot y_0) = 6\pi$$

$$R^2 = \frac{6}{3 + 2y_0}$$

$$3R^2 + 2R^2 y_0 = 6$$

$$2R^2 y_0 = 6 - 3R^2$$

$$y_0 = \frac{6 - 3R^2}{2R^2}$$

$$y_0 = \frac{3}{R^2} - \frac{3}{2}$$

o^o

Sea la circunferencia de la forma:

$$C = \left\{ (x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = \frac{6}{3 + 2y_0} \quad \text{con } x_0, y_0 \in \mathbb{R} \right\}$$

Ejercicio 15. Calcular la integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ donde

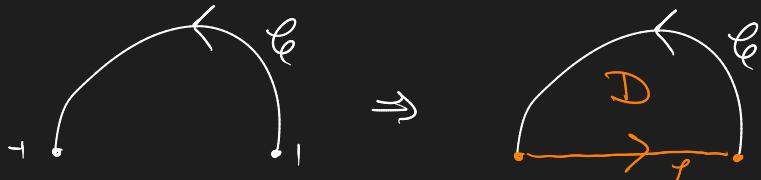
$$\mathbf{F}(x, y) = (y^2 e^x + \cos x + (x-y)^2, 2y e^x + \sin y),$$

y C es la curva

$$x^2 + y^2 = 1, \quad y \geq 0,$$

orientada de manera tal que comience en $(1, 0)$ y termine en $(-1, 0)$.

Bien definido



Primero veo si es Campo Grad, porque es sospechoso:

$$\text{si } f_x = y^2 \cdot e^x + \cos x + (x-y)^2$$

$$\underbrace{x^2 - 2xy + y^2}_{x^2 - 2xy + y^2}$$

$$\Rightarrow f = y^2 e^x + \sin x + \frac{x^3}{3} - x^2 y + xy^2 + \varphi(y)$$

$$\text{si } f_y = 2y \cdot e^x + \sin y \quad \text{Can!} \quad !!$$

$$\Rightarrow f = y^2 e^x - \cos y + \tilde{\varphi}(x)$$

Vuelvo

$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_L \mathbf{F} \cdot d\mathbf{s} = \iint_D Q_x - P_y dx dy$$

$$\mathbf{F} = \underbrace{\left(y^2 \cdot e^x + \cos x + x^2 - 2xy + y^2 \right)}_P, \quad \underbrace{\left(2y \cdot e^x + \sin y \right)}_Q$$

$$Q_x - P_y = \cancel{2y \cdot e^x} - \left(\cancel{2y \cdot e^x} - 2x + 2y \right)$$

$$= 2x - 2y$$

$$\begin{aligned}
 \iint_D Q_x - P_y dx dy &= 2 \int_{r=0}^1 \int_{\theta=0}^{\pi} \underbrace{(r \cos \theta - r \sin \theta) r}_{x-y} dr d\theta \\
 &\quad \left. \underbrace{r^2 \sin \theta}_{r^2(0-0)} \right|_0^\pi + \underbrace{r^2 \cos \theta}_{r^2(-1-1)} \Big|_0^\pi \\
 &= -2r^2 \\
 &= -4 \int_{r=0}^1 r^2 dr \\
 &= -\frac{4}{3} \Big|_0^1 = -\frac{4}{3}
 \end{aligned}$$

Feliz

$$\begin{aligned}
 \int_L \vec{F} \cdot d\vec{s} &= \int_{-1}^1 \underbrace{\langle \vec{F}(t, 0), (1, 0) \rangle}_{(\cos t + t^2, 0)} dt \\
 &= \sin t \Big|_{-1}^1 + \frac{t^3}{3} \Big|_{-1}^1 \\
 &= \sin 1 - \sin(-1) + \frac{1}{3} + \frac{1}{3} \\
 &= \frac{2}{3} + 2 \sin 1
 \end{aligned}$$

CA
 $\sin(-1) = -\sin 1$

Juntando todo

$$\int_C \vec{F} \cdot d\vec{s} + \underbrace{\int_L \vec{F} \cdot d\vec{s}}_{\frac{2}{3} + 2\sin 1} = \iint_D Q_x - P_y dxdy - \underbrace{\frac{4}{3}}$$

$$\int_C \vec{F} \cdot d\vec{s} = -\frac{4}{3} - \frac{2}{3} - 2\sin 1$$

$$= -2 - 2\sin 1$$

\approx

Ejercicio 16. Sean $u, v \in C^1(D)$, donde $D = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$. Consideremos los campos definidos por $\mathbf{F}(x, y) = (u(x, y), v(x, y))$, $\mathbf{G}(x, y) = (v_x - v_y, u_x - u_y)$. Calcular

$$\iint_D (\mathbf{F} \cdot \mathbf{G})(x, y) dx dy$$

sabiendo que sobre el borde de D se tiene $u(x, y) = x$, $v(x, y) = 1$.

$$\langle \mathbf{F}, \mathbf{G} \rangle = \underbrace{u \cdot (v_x - v_y) + v \cdot (u_x - u_y)}_{\text{se parece a la deriv. de productos}}$$

$$= u \cdot v_x + v \cdot u_x - (u \cdot v_y + v \cdot u_y)$$

$$= \underbrace{\frac{\partial}{\partial x} (u \cdot v)}_{Q_x} - \underbrace{\frac{\partial}{\partial y} (u \cdot v)}_{P_y}$$

$$\iint_D \langle \mathbf{F}, \mathbf{G} \rangle dx dy = \iint_D \underbrace{\frac{\partial}{\partial x} (u \cdot v)}_{Q_x} - \underbrace{\frac{\partial}{\partial y} (u \cdot v)}_{P_y} dx dy$$

Por Green : $\iint_D Q_x - P_y dx dy = \oint_{\partial D} (P, Q) \cdot ds$

$$= \oint_{\partial D} (u \cdot v, u \cdot v) \cdot ds$$

Parametrizando ∂D (es una ellipse)

$$\underbrace{a \cdot \cos t}_{\text{a. cost}} \quad \underbrace{b \cdot \cos t}_{\text{b. cost}}$$

$$\sigma(t) = (3 \cdot \cos t, 2 \cdot \sin t)$$

$$\begin{aligned}
 &= \int_{t=0}^{2\pi} \left\langle F(\sigma(t)), \dot{\sigma}(t) \right\rangle dt \\
 &\quad \underbrace{\left(u \cdot \sigma(\sigma(t)), u \cdot v(\sigma(t)) \right)}_{\underbrace{u(\sigma(t))}_{\text{Por dato del enunciado}} \cdot \underbrace{v(\sigma(t))}_{\text{Por dato del enunciado}}} \\
 &\quad = 3 \cdot \cos t \cdot 2 \cdot \sin t \cdot 3 \cdot \cos t + 2 \cdot \sin t \cdot 3 \cdot \cos t \\
 &\quad = 18 \cdot \cos^2 t + 12 \cdot \sin t \cdot \cos t
 \end{aligned}$$

$$\therefore \vec{F}(\sigma(t)) = (3 \cdot \cos t, 3 \sin t)$$

$$\sigma'(t) = (-3 \cdot \sin t, 2 \cdot \cos t)$$

$$\int_{t=0}^{2\pi} \langle F(\sigma(t)), \dot{\sigma}(t) \rangle dt = \int_{t=0}^{2\pi} -9 \cdot \sin t \cdot \cos t + 6 \cdot \cos^2 t dt$$



I II

$$\textcircled{I} = -q \int_0^{2\pi} \sin t \cdot \cos t \, dt$$

$\brace{ }_{=0}$

$$\textcircled{I}\textcircled{I} = 6 \int_0^{2\pi} \cos^2 t \ dt$$

Ident Trig.

$$\cos^2 t = \frac{1}{2} \cos 2t + \frac{1}{2}$$

$$= 6 \int_0^{2\pi} \frac{1}{2} \cos 2t + \frac{1}{2} dt$$

$$= 3 \int_0^{2\pi} \cos 2t dt + 3 \cdot 2\pi$$

$\underbrace{\hspace{10em}}$

$$= 0$$

$$= 6\pi$$

$$\int_{t=0}^{2\pi} \langle F(\sigma(t)), \sigma'(t) \rangle dt = 6\pi$$

Fundamente

$$\iint_D (F \cdot G) dx dy = 6\pi$$