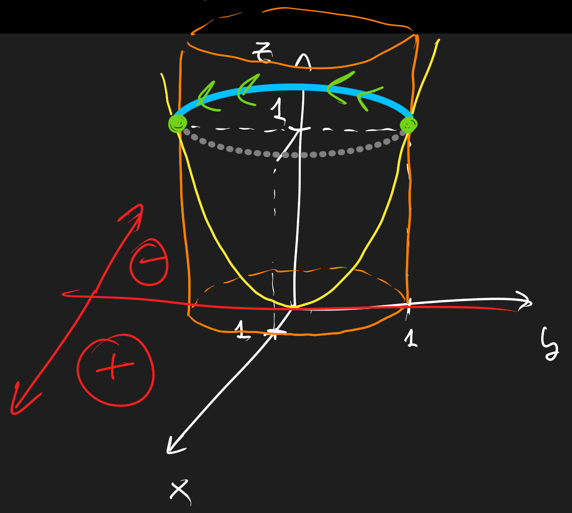
Análisis II / Matemática 3 / Análisis Matemático II
Segundo cuatrimestre 2021 - Primer parcial (13/10/2021)
TEMA 3

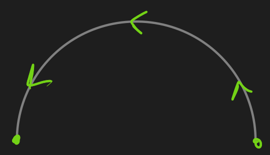
Ejercicio 1

Sea $\mathcal{C} := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1, z=x^2+y^2, x \leq 0\}$.

(a) *Dar una parametrización regular de* $\mathcal{C}$ *que empiece en el* $(0,1,1)$ *y termine en el* $(0,-1,1)$.

(b) *Calcular* $\int_{\mathcal{C}} (0,y,xy) \cdot \mathbf{ds}$, *donde* C *está orientada como en el ítem anterior.*





a) $\sigma(t) = (\cos t, \sin t, 1)$

$t \in [0, \pi]$

b) $F = (0, y, xy)$

$$\int_C F \cdot ds = \int_{t=0}^{\pi} \langle \underbrace{F(\sigma(t)), \sigma'(t)}_{} \rangle \, dt$$

$$\langle (0, \sin t, \cos t \cdot \sin t), (-\sin t, \cos t, 0) \rangle$$

$$= \sin t \cdot \cos t$$

$$= \int_{t=0}^{\pi} \sin t \cdot \cos t \, dt$$

$u = \sin t$

$du = \cos t \, dt$

$$= \int_0^0 u \cdot du = 0$$

Ejercicio 2

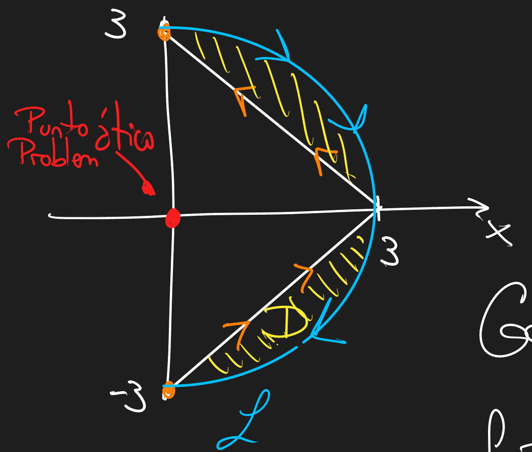
Sea $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$ *el campo dado por*

$$\mathbf{F}(x,y)=\left(x\,\mathrm{sen}(\sqrt{x^2+y^2})-\frac{y}{x^2+y^2},\,y\,\mathrm{sen}(\sqrt{x^2+y^2})+\frac{x}{x^2+y^2}\right).$$

Calcular $\int_{\mathcal{C}}\mathbf{F}\cdot\mathbf{ds}$, *donde* $\mathcal{C}$ *es la curva dada por la unión de los dos segmentos de recta*

$$\begin{cases} y=3-x, & 0\le y\le 3\\ y=x-3, & -3\le y\le 0\end{cases}$$

recorrida desde el $(0,-3)$ *al* $(0,3)$.



$$\int_C F\cdot ds = ?$$

Green

sentido opuesto

$$\int_{\partial D} F\cdot ds = -\iint_D Q_x - P_y \, dx\,dy$$

$$\int_C F\cdot ds + \int_{\mathcal{L}} F\cdot ds = -\iint_D Q_x - P_y\, dx\,dy$$

$$\mathbf{F}(x,y)=\left(\underbrace{x\,\mathrm{sen}(\sqrt{x^2+y^2})-\frac{y}{x^2+y^2}}_{P},\,\underbrace{y\,\mathrm{sen}(\sqrt{x^2+y^2})+\frac{x}{x^2+y^2}}_{Q}\right).$$

$$Q_x = y\cdot\cos\sqrt{x^2+y^2}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x + \frac{1}{x^2+y^2} - \frac{2x^2}{x^2+y^2}$$

$$\underline{CA}:\ \frac{d}{dx}\, x(x^2+y^2)^{-1} = \frac{1}{x^2+y^2} - x(x^2+y^2)^{-2}\cdot 2x$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$\frac{d}{dy}\, -y\left(x^2+y^2\right)^{-1} = -\frac{1}{x^2+y^2} + y\left(x^2+y^2\right)^{-2}\cdot 2y$$

$$= -\frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2}$$

$$P_y = y \cdot \cos\sqrt{x^2+y^2} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x - \frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2}$$

$$Q_x = y \cdot \cos\sqrt{x^2+y^2} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2}$$

$$Q_x - P_y = \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{\left(x^2+y^2\right)^2}$$

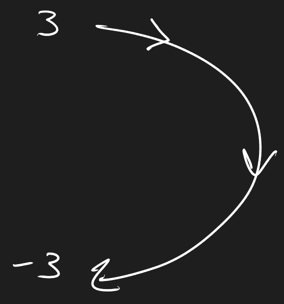
$$= \frac{2}{x^2+y^2} - \underbrace{\frac{2x^2+2y^2}{\left(x^2+y^2\right)^2}}_{= \frac{2\cancel{\left(x^2+y^2\right)}}{\left(x^2+y^2\right)^{\cancel{2}}}} \qquad x^2+y^2 \neq 0$$

$$= 0 \quad //$$

$\therefore$

$$\int_C F \cdot ds + \int_{\mathcal{L}} F \cdot ds = 0$$

$$\int_C F \cdot ds = -\int_{\mathcal{L}} F \cdot ds$$



Parametrizo $\mathcal{L}$

$$\sigma(t) = \left(3 \cdot \cos\left(\frac{\pi}{2} - t\right),\ 3 \cdot \sin\left(\frac{\pi}{2} - t\right)\right)$$

$$t \in [0, \pi]$$

$$\mathbf{F}(x,y) = \left(x\,\mathrm{sen}(\sqrt{x^2+y^2}) - \frac{y}{x^2+y^2},\ y\,\mathrm{sen}(\sqrt{x^2+y^2}) + \frac{x}{x^2+y^2}\right).$$

$$-\int_{\mathcal{L}} F \cdot ds = \int_{t=0}^{\pi} \left\langle \underbrace{F(\sigma(t))}, \left(3 \sin\left(\frac{\pi}{2} - t\right), -3 \sin\left(\frac{\pi}{2} - t\right)\right) \right\rangle$$

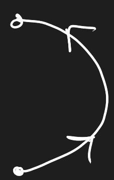
$$\left(3\cos\left(\frac{\pi}{2} - t\right) \cdot \sin(3) - \frac{3 \cdot \sin\left(\frac{\pi}{2} - t\right)}{9},\right.$$

$$\left. 3 \cdot \sin\left(\frac{\pi}{2} - t\right) \cdot \sin(3) + \frac{3 \cdot \cos\left(\frac{\pi}{2} - t\right)}{9}\right)$$

Horrible : sigo con otra param.

Parametrizo $\mathcal{L}^-$ ← sentido opuesto!

$$\sigma(t) = \left(3\cdot\cos t \,,\, 3\cdot\sin t\right)$$



$t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\mathbf{F}(x,y) = \left(x\,sen(\sqrt{x^2+y^2}) - \frac{y}{x^2+y^2}, y\,sen(\sqrt{x^2+y^2}) + \frac{x}{x^2+y^2}\right).$$

$$-\int_{\mathcal{L}} F\cdot ds = \int_{\mathcal{L}^-} F\cdot ds = \int_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle F(\sigma(t)), \sigma'(t)\rangle$$

$$F(\sigma(t)) = \left(3.\cos t \cdot \sin(3) - \frac{\cancel{3}\sin t}{\cancel{9}_3} \,,\, 3.\sin t \cdot \sin(3) + \frac{\cancel{3}\cdot\cos t}{\cancel{9}_3}\right)$$

$$\sigma'(t) = \left(-3.\sin t \,,\, 3.\cos t\right)$$

$$\langle F(\sigma(t)), \sigma(t)\rangle = -9.\cos t\cdot\sin t\cdot\sin 3 + \sin^2 t + 9\cos t\,\sin t\cdot\sin 3 + \cos^2 t$$

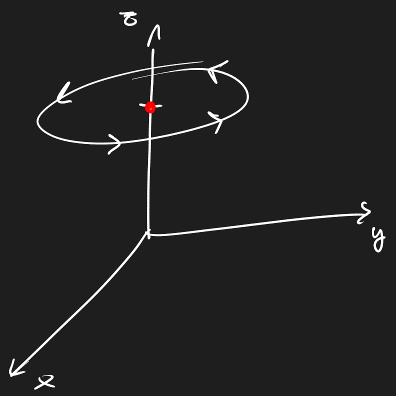
$$= \sin^2 t + \cos^2 t = 1$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, dt = \pi$$

$$\int_C F\cdot ds = \int_{\mathcal{L}^-} F\cdot ds = \pi \quad //$$

Ejercicio 3 *Sea* $\mathcal{C} := \{(x,y,z) \in \mathbb{R}^3 : z = 3, x^2+y^2 = 1\}$ *orientada de manera tal que al proyectarla en el plano* xy *se recorra en sentido positivo. Calcular* $\int_{\mathcal{C}} F \cdot \mathbf{ds}$*, donde*

$$F(x,y,z) = \left(\frac{x}{x^2+y^2+(z-3)^2} + \frac{(z-3)^3}{3}, \frac{y}{x^2+y^2+(z-3)^2} + \frac{x^3}{3}, \frac{z-3}{x^2+y^2+(z-3)^2} + \frac{y^3}{3}\right).$$



$$F = G + H$$

$$G = \left(\frac{x}{x^2+y^2+(z-3)^2}, \frac{y}{x^2+y^2+(z-3)^2}, \frac{z}{x^2+y^2+(z-3)^2}\right)$$

$$H = \left(\frac{(z-3)^3}{3}, \frac{x^3}{3}, \frac{y^3}{3}\right)$$

$$\Rightarrow \int_C F \cdot ds = \int_C G \cdot ds + \int_C H \cdot ds$$

Stokes sobre H pues $H \in \mathcal{C}^1$

$$\int_{C=\partial S} H \cdot ds \overset{\downarrow}{=} \iint_S \nabla \times H \cdot ds$$

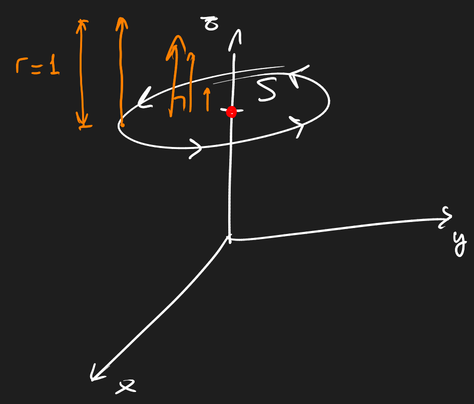
$$\nabla \times H = \begin{vmatrix} i & \overset{(-)}{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{(z-3)^3}{3} & \frac{x^3}{3} & \frac{y^3}{3} \end{vmatrix}$$

$$= \left(y^2, -\left(0 - (z-3)^2\right), x^2\right)$$

$$= \left(y^2, (z-3)^2, x^2\right)$$

$$T(r,\theta) = (r\cdot\cos\theta,\ r\cdot\sin\theta,\ 3)$$

$$r \in [0, 1]$$
$$\theta \in [0, 2\pi)$$



$$T_r = (\cos\theta,\ \sin\theta,\ 0)$$
$$T_\theta = (-r\cdot\sin\theta,\ r\cdot\cos\theta,\ 0)$$

$$T_r \times T_\theta = (0,\ 0,\ r)$$

$$\iint_S \nabla\times H\cdot dS = \int_r\int_\theta \left\langle \underbrace{\nabla\times H\,(T(r,\theta))}_{=},\ T_r\times T_\theta \right\rangle d\theta\, dr$$

$$\left.\begin{aligned}\nabla\times H &= (y^2,\ (z-3)^2,\ x^2)\\ T(r,\theta) &= (r\cdot\cos\theta,\ r\cdot\sin\theta,\ 3)\end{aligned}\right\} \to (r^2\cdot\sin^2\theta,\ 0,\ r^2\cdot\cos^2\theta)$$

$$= \int_r\int_\theta r^3\cdot\cos^2\theta\ d\theta\, dr$$

$$= \int_{r=0}^{1} r^3 \int_{\theta=0}^{2\pi} \cos^2\theta\ d\theta\ dr$$

$$\int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 \, d\theta + \frac{1}{2}\int_0^{2\pi} \cos 2\theta \, d\theta$$

CA

$$\frac{\partial}{\partial\theta}\frac{1}{2}\sin 2\theta = \frac{2}{2}.\cos 2\theta$$

$$= \pi + \frac{1}{4}.\sin 2\theta \Big|_0^{2\pi}$$

$$= \pi$$

$$\int_{r=0}^{1} r^3 \int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta \, dr = \pi . \int_{r=0}^{1} r^3 . dr$$

$$= \pi . \frac{r^4}{4}\Big|_0^1$$

$$= \frac{\pi}{4}$$

$$= \int_C H \cdot ds$$

$$G = \left(\frac{x}{x^2+y^2+(z-3)^2}, \frac{y}{x^2+y^2+(z-3)^2}, \frac{z}{x^2+y^2+(z-3)^2} \right)$$

$$\int_C G \cdot ds = \int_{t=0}^{2\pi} \langle \underbrace{G(\sigma(t))}_{= (\cos t,\, \sin t,\, 3)}, \underbrace{\sigma'(t)}_{= (-\sin t,\, \cos t,\, 0)} \rangle \, dt$$

$$\sigma(t) = (\cos t, \sin t, 3)$$

$$= -\cos t \cdot \sin t + \cos t \cdot \sin t = 0$$

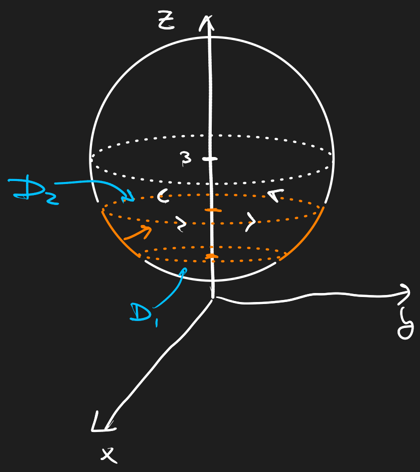
$$\int_C G \cdot ds = 0$$

$$\therefore$$

$$\int_C F \cdot ds = \frac{\pi}{4}$$

//

Ejercicio 4 *Sea* $\mathbf{F}$ *el campo vectorial dado por* $\mathbf{F}(x,y,z) = \left(\text{sen}(z^2) + 3xy, e^{x^3} - y^2, x^2 - yz\right)$. *Calcular el flujo de* $\mathbf{F}$ *a través de la superficie* S *dada por la sección de la esfera de ecuación* $x^2 + y^2 + (z-3)^2 = 5$ *acotada entre los planos* $z = 1$ *y* $z = 2$, *orientada con la normal interior.*



$$\text{div}\, F = 3y - 2y - y = 0$$

$$F \in C^1$$

Uso Gauss

$$\int_{S^-} F \cdot dS + \int_{D_1} F \cdot dS + \int_{D_2} F \cdot dS = 0$$

$$\shortparallel$$

$$-\int_S F \cdot dS$$

$$\Rightarrow \int_S F \cdot dS = \int_{D_1} F \cdot dS + \int_{D_2} F \cdot dS$$

$$\int_{D_1} F \cdot dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left\langle F(T_1(r,\theta)), T_{1r} \times T_{1\theta} \right\rangle dr\, d\theta$$

CA

en $z = 1$: el radio de D_1 es:

$$x^2 + y^2 = 5 - (z-3)^2$$

$$r_1 = 1$$

$$r_2 = 2$$

$$T_1^-(r,\theta) = (r\cdot\cos\theta,\ r\cdot\sin\theta,\ 1)$$

Opuesto a la orientacion de T_1

$r \in [0,1]$

$\theta \in [0,2\pi)$

n apunta hacia abajo.

$$= -\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left\langle\left(\ldots,\ \ldots,\ r^2\cdot\cos^2\theta - r\cdot\sin\theta\right),\ (0,0,r)\right\rangle dr\,d\theta$$

$$= -\int_{\theta=0}^{2\pi}\underbrace{\int_{r=0}^{1} r^3\cdot\cos^2\theta - r^2\cdot\sin\theta\ dr}_{\frac{r^4}{4}\cdot\cos^2\theta\Big|_0^1 - \frac{r^3}{3}\cdot\sin\theta\Big|_0^1}\,d\theta$$

$$= -\int_0^{2\pi}\frac{1}{4}\cos^2\theta - \underbrace{\frac{1}{3}\sin\theta\ d\theta}_{=0}$$

$$= -\frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi}\frac{1}{2}\cdot(1+\cos 2\theta)\,d\theta$$

$$= -\frac{1}{8}\int_0^{2\pi}1+\cos 2\theta\ d\theta$$

$$= -\frac{1}{4}\pi - \frac{1}{8}\underbrace{\int_0^{2\pi}\cos 2\theta\,d\theta}_{\frac{1}{2}\sin 2\theta\Big|_0^{2\pi}}$$

$$= -\frac{1}{4}\pi$$

$$T_2(r,\theta) = (r\cdot\cos\theta,\ r\cdot\sin\theta,\ 2)$$

$r \in [0,1]$
$\theta \in [0,2\pi)$

no invierte la orientación

$$= +\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\langle(\dots,\ \dots,\ r^2\cdot\cos^2\theta - 2\cdot r\cdot\sin\theta),\ (0,0,r)\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\underbrace{\int_0^2 r^3\cdot\cos^2\theta - 2r^2\cdot\sin\theta\, dr}_{\underbrace{\frac{r^4}{4}\cdot\cos^2\theta\Big|_0^2}_{4\cdot\cos^2\theta} - 2\frac{r^3}{3}\cdot\sin\theta\Big|_0^2}\, d\theta$$

$$4\cdot\cos^2\theta - \frac{16}{3}\cdot\sin\theta$$

$$= \int_0^{2\pi} 4\cdot\cos^2\theta - \underbrace{\frac{16}{3}\sin\theta\, d\theta}_{=0}$$

$$= 4\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= 4\pi$$

Finalmente

$$\int_S F\cdot ds = \int_{D_1} F\cdot ds + \int_{D_2} F\cdot ds$$

$$= -\frac{1}{4}\pi + 4\pi = \frac{15}{4}\pi \quad //$$