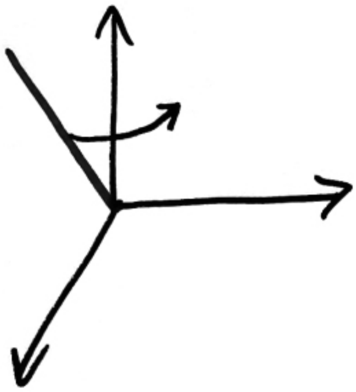
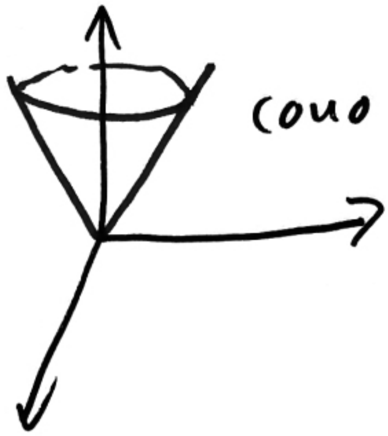
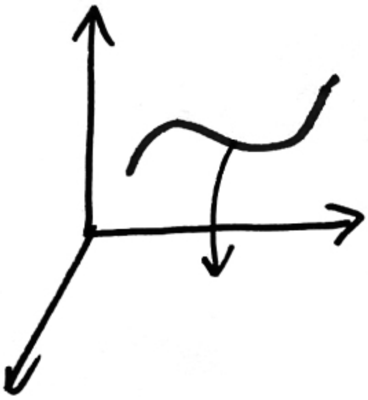
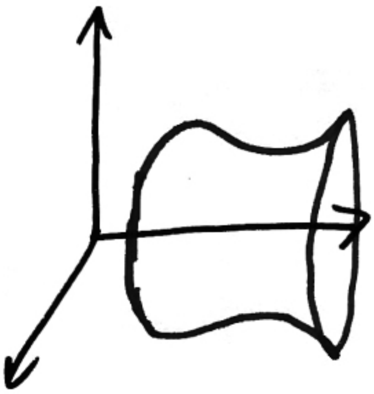
Práctica 2 (superficies de revolución, flujos)

- Idea : Tenemos una curva que gira alrededor de un eje



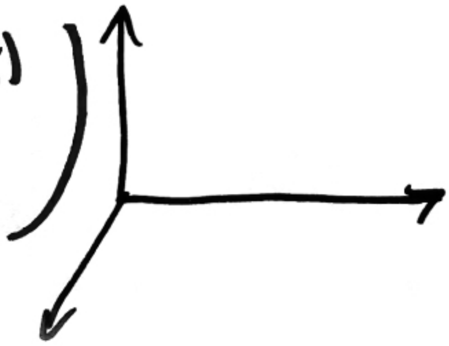


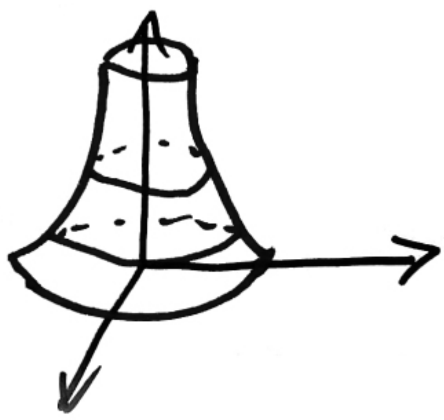




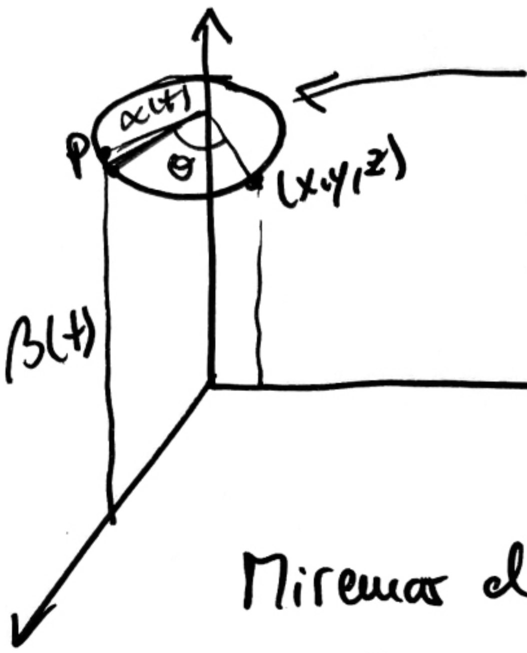
¿Cómo parametrizamos superficies de revolución?

- Consideremos una curva $\sigma(t) = (\alpha(t), \beta(t))$ en el plano xz y hagámosla girar alrededor del eje z. (como en el caso del cono)





Si $P = (\alpha(t), 0, \beta(t))$ (en el plano xz)
y lo rotamos un ángulo θ alrededor del eje z,
obtenemos un punto (x,y,z):

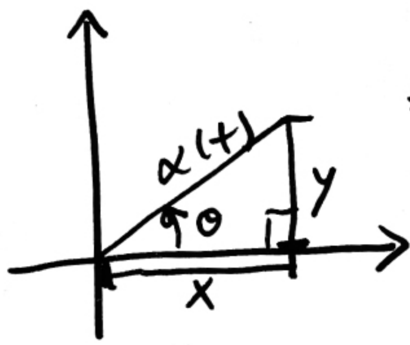


. circunferencia de radio $\alpha(t)$

. $z =$ "altura" se mantiene constante

. x, y ?

Miremos el piso:



$\Rightarrow$. $\cos\theta = \dfrac{x}{\alpha(t)}$

. $\operatorname{sen}\theta = \dfrac{y}{\alpha(t)}$

Luego,

$$T: \begin{cases} x = \alpha(t)\cos(\theta) \\ y = \alpha(t)\operatorname{sen}\theta \\ z = \beta(t) \end{cases}, \quad t \in [a,b] = \operatorname{Dom}(\sigma), \quad \theta \in [0, 2\pi]$$

. Supongamos que σ es regular. Entonces:

1) $T(t,\theta)$ es inyectiva en $[a,b] \times [0,2\pi)$

2) $T \in C^1$

. $T_t \times T_\theta$:

$$T_t = (\alpha'(t)\cos\theta, \alpha'(t)\operatorname{sen}\theta, \beta'(t))$$

$$T_\theta = (-\alpha(t)\operatorname{sen}\theta, \alpha(t)\cos\theta, 0)$$

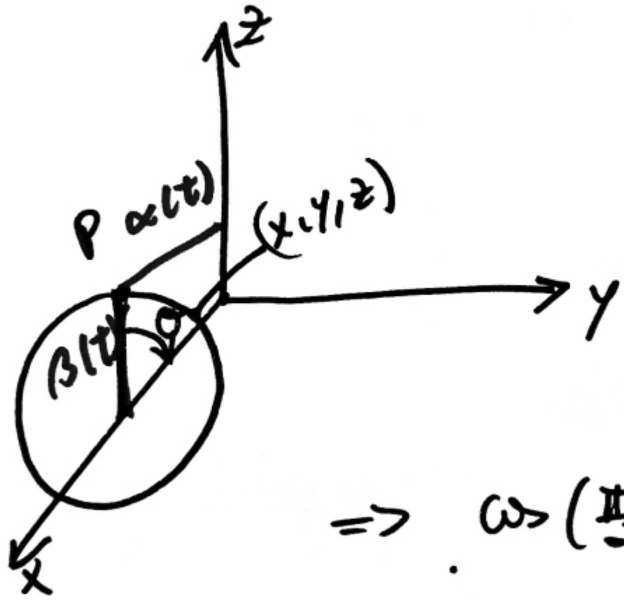
$$\Rightarrow T_t \times T_\theta = (-\alpha(t)\beta'(t)\cos\theta, -\alpha(t)\beta'(t)\operatorname{sen}\theta, \alpha(t)\alpha'(t))$$

$$\Rightarrow \|T_t \times T_\theta\| = \sqrt{(\alpha(t)\beta'(t))^2 + (\alpha(t)\alpha'(t))^2}$$

$$= |\alpha(t)| \cdot \underbrace{\|\gamma'(t)\|}_{\sqrt{(\beta'(t))^2 + (\alpha'(t))^2}}$$

Luego, $\|T_t \times T_\theta\| \neq 0 \Leftrightarrow \alpha(t) \neq 0 \quad \forall t \in [a,b]$.

•) ¿Qué pasa si ahora giramos alrededor del eje x?



. Giramos alrededor del eje x

$\Rightarrow x = \alpha(t)$

. y, z ?

$\Rightarrow$. $\cos(\frac{\pi}{2} - \theta) = \frac{y}{\beta(t)}$

. $\operatorname{sen}(\pi/2 - \theta) = \frac{z}{\beta(t)}$

IV

Como $\cos\left(\frac{\pi}{2}-\theta\right)=\operatorname{sen}\theta$ y $\operatorname{sen}\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$

$$\Rightarrow \quad T: \begin{cases} x=\alpha(t) \\ y=\beta(t)\operatorname{sen}\theta \\ z=\beta(t).\cos\theta \end{cases} \quad , t\in[a,b]=\operatorname{Dom}(\sigma), \quad \theta\in[0,2\pi].$$

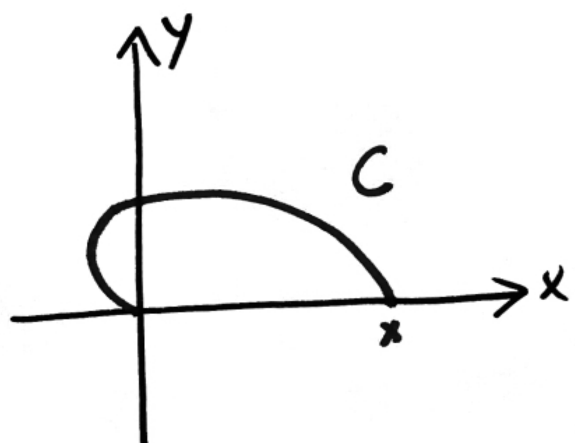
...) Se gira la curva $f(x)=z$, $x\in[a,b]$ alrededor del eje z

$$\Rightarrow \sigma(t)=(t,f(t)), \quad t\in[a,b].$$

Luego (como en ·)), $\underline{T(t,\theta)=(t\cos\theta, t\operatorname{sen}\theta, f(t)).}$

$\underline{\text{Ej}}$: Sea C el cardiode: $r=1+\cos\theta$

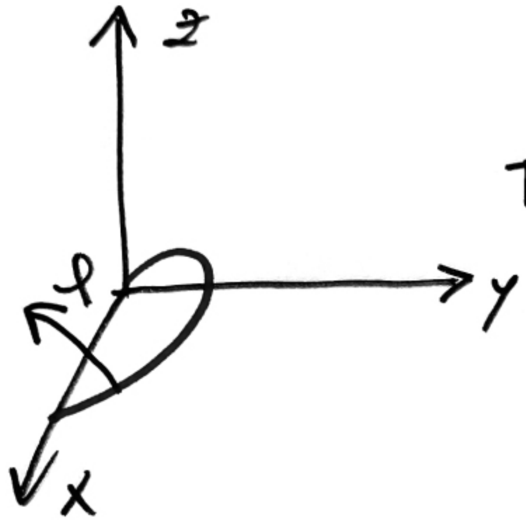
$\theta\in[0,\pi]$.



· Parametricemos en el plano xy:

$$\sigma(\theta): \begin{cases} x=r\cos\theta=(1+\cos\theta)\cos\theta=\alpha(\theta) \\ y=r\operatorname{sen}\theta=(1+\cos\theta)\operatorname{sen}\theta=\beta(\theta) \end{cases}, \quad \theta\in[0,\pi]$$

- Giremos sobre el eje x:



$$T: \begin{cases} x = \alpha(\theta) \\ y = \beta(\theta)\cos\varphi \\ z = \beta(\theta)\operatorname{sen}\varphi \end{cases}, \ \varphi \in [0, 2\pi]$$

("como en ..").

Flujo:

- $S \subseteq \mathbb{R}^3$ superficie
- $T: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ parametrización regular de S "que orienta S".
- $F(x,y,z)$ campo vectorial continuo definido en S.

El flujo de F a través de S es:

$$\iint_S F \cdot dS = \iint_S \langle F, n \rangle \, dS$$

$$\boxed{\iint_S F \cdot dS = \iint_D \langle F(T(u,v)), T_u \times T_v(u,v) \rangle \, du\, dv}$$

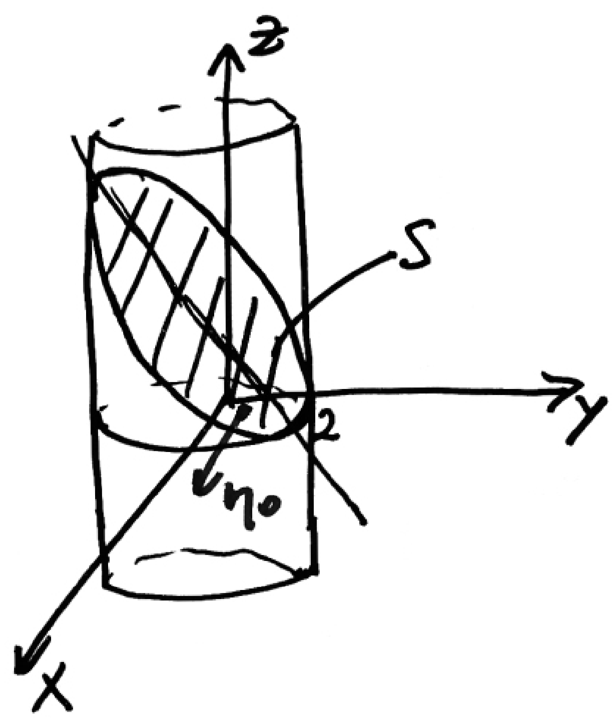
VI

1) Sea $F(x,y,z) = (0,0,4-x^2-y^2)$ y sea S la superficie dada por:

$$S := \begin{cases} x^2+y^2 \leq 4 \\ y+z=1 \end{cases}$$

orientada de manera tal que en $P_0=(1,1,0) \in S$ la normal sea $\eta_0 = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Hallar $\iint_S F \cdot dS$



. $x^2+y^2 \leq 4 \leftarrow$ cilindro

. $y+z=1 \leftarrow$ plano

Parametricemos S:

$$T: \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = 1 - r\,\text{sen}\,\theta \end{cases}, \quad \theta \in [0,2\pi],\ r \in [0,2]$$

. T es inyectiva en $[0,2] \times [0,2\pi)$

. $T \in C^1$

$\cdot T_r = (\cos\theta, \operatorname{sen}\theta, -\operatorname{sen}\theta)$

$T_\theta = (-r\operatorname{sen}\theta, r\cos\theta, -r\cos\theta)$

$T_r \times T_\theta = (0, r, r)$

$\Rightarrow$ T invierte la orientación !

$$p_0 = (1,1,0) = T\left(\underbrace{\tfrac{2}{\sqrt{2}}}_{r}, \overset{\theta}{\overset{\shortparallel}{\pi/4}}\right)$$

$$\Rightarrow \iint_S F \cdot dS = -\int_0^{2\pi}\int_0^2 \left\langle \overset{(0,0,4-r^2)}{\overset{\shortparallel}{F(r\cos\theta, r\operatorname{sen}\theta, 1-r\operatorname{sen}\theta)}}, (0,r,r)\right\rangle dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^2 (4-r^2)\cdot r\,dr\,d\theta$$

$$= -2\pi \cdot \int_0^2 (4r - r^3)\,dr$$

$$= -2\pi \cdot \left(\frac{4r^2}{2} - \frac{r^4}{4}\right)\Big|_0^2$$

$$= -2\pi \cdot (8-4)$$

$$= -8\pi$$

2) Sean $S_1 := \begin{cases} y^2 = z^2 + x^2 \\ 0 \leq y \leq 1 \end{cases}$, $S_2 := \begin{cases} y = 2 - x^2 - z^2 \\ y \geq 1 \end{cases}$ VIII

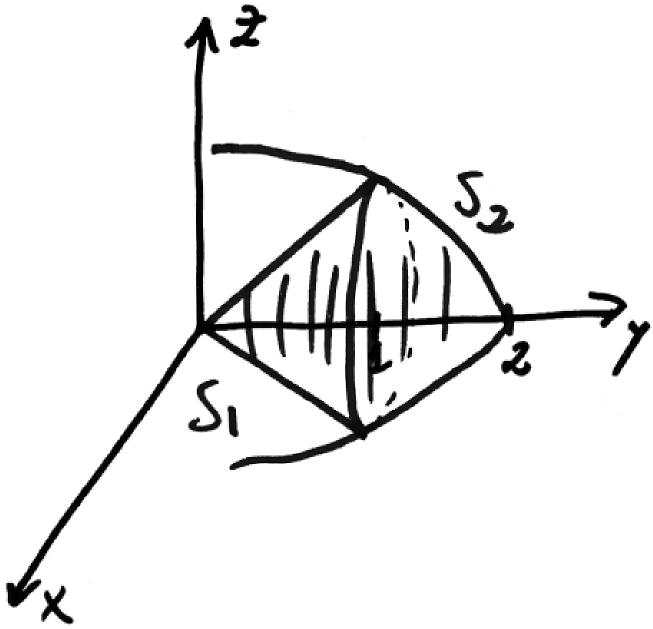
Consideremos $S = S_1 \cup S_2$.

Dado $F(x,y,z) = (x,y,z)$, calcular el flujo saliente a través de S.

(S con normal exterior)

Obs: S_1 : cono en el eje y

S_2: paraboloide en el eje y.



$$\iint_S F \cdot dr = \iint_{S_1} F \cdot dS + \iint_{S_2} F \cdot dS$$ con S_1 y S_2 orientadas con normal exterior.

. Parametrizamos S_1 :

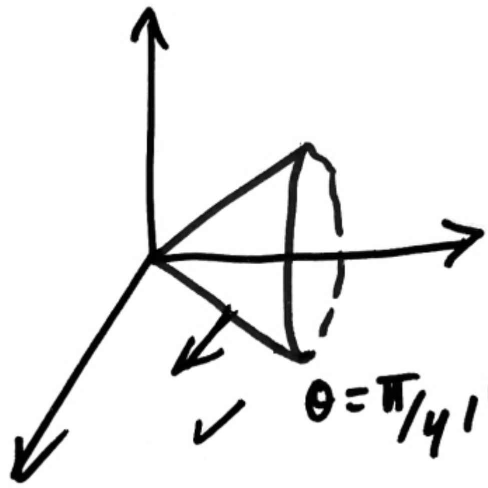
$$T(r,\theta) = (r\cos\theta, r, r\sen\theta)$$

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$

$T_r = (\cos\theta, 1, \operatorname{sen}\theta)$

$T_\theta = (-r\operatorname{sen}\theta, 0, r\cos\theta)$

$\Rightarrow T_r \times T_\theta = (r\cos\theta, -r, r\operatorname{sen}\theta)$



$\Rightarrow$ respeta la orientación ✓

$$\Rightarrow \iint_{S_1} F.dS = \int_0^{2\pi}\int_0^1 \langle F(T(r,\theta)), T_r \times T_\theta \rangle \, dr\, d\theta$$

$T_r \times T_\theta (1, \pi/4) =$

$= \left(\frac{\sqrt{2}}{2}, -1, \frac{\sqrt{2}}{2}\right)$

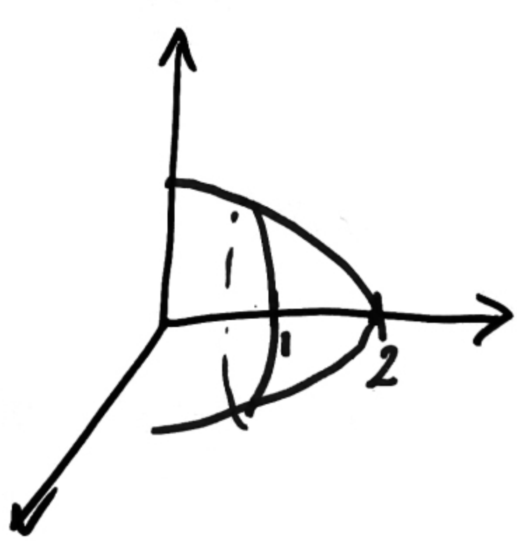
$$= \int_0^{2\pi}\int_0^1 \langle (r\cos\theta, r, r\operatorname{sen}\theta), (r\cos\theta, -r, r\operatorname{sen}\theta) \rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2 - r^2 \, dr\, d\theta = \underline{0}$$

. <u>Parametricemos S_2</u>:

$T(x,z) = (x, 2-x^2-z^2, z)$

$T: D = \{(x,z) / x^2+z^2 \leq 1\}$



X

$T_x = (1, -2x, 0)$

$T_z = (0, -2z, 1)$

$\Rightarrow T_x \times T_z = (-2x, -1, -2z)$

$T(0,0) = (0,2,0)$, $T_x \times T_z (0,0) = (0,-1,0)$

apunta hacia adentro

$$\Rightarrow \iint_{S_2} F . dS = -\iint_D \langle (x, 2-x^2-z^2, z), (-2x,-1,-2z) \rangle \, dx dz$$

$$= -\iint_D (-2x^2 - 2 + x^2 + z^2 - 2z^2) \, dx dz$$

$$= -\iint_D (-2 - x^2 - z^2) \, dx \, dz$$

$$= \iint_D (2 + x^2 + z^2) \, dx dz$$

Polares:

$$\left(\begin{array}{l} x = r\cos\theta \\ z = r\operatorname{sen}\theta \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array} \right)$$

$$= \int_0^{2\pi} \int_0^1 (2 + r^2) \, r \, dr \, d\theta$$

$$= 2\pi \cdot \left(r^2 + \frac{r^4}{4} \right) \Big|_0^1$$

$$= 2\pi \cdot \frac{5}{4} = \frac{5}{2}\pi$$

$$\Rightarrow \iint_S F \cdot dS = 0 + \frac{5}{2}\pi = \frac{5}{2}\pi.$$