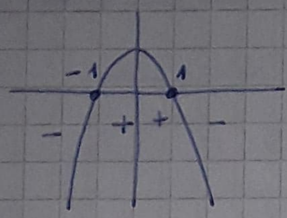
Práctica 7

• Dinámica unidimensional

1. $\dot{x} = f(x)$

a. $\dot{x} = 1 - x^2$

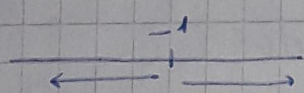


$-1, 1$ son puntos de equilibrio porque $f(x)$ se anula.

$x' = f(x)$ $\qquad x \equiv 1$, $x \equiv -1$ son sol de la ec. dif.

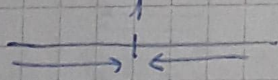
• Como la derivada es negativa en $(-\infty, -1)$
→ la **solución** x decrece.

• Como $f(x) > 0$ en $(-1, 1) \Rightarrow x$ crece.



-1 es un punto de eq. inestable porque $x(t)$ tiende a alejarse.

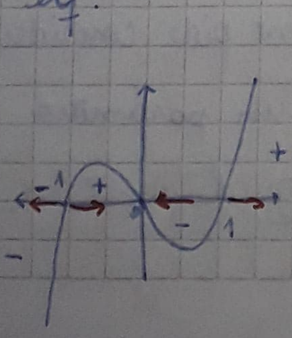
• Como $f(x) < 0$ en $(1, +\infty) \rightarrow x$ decrece.



1 es un punto de eq. estable porque la solución tiende a acercarse.

b. $\dot{x} = x^3 - x = x(x^2 - 1)$

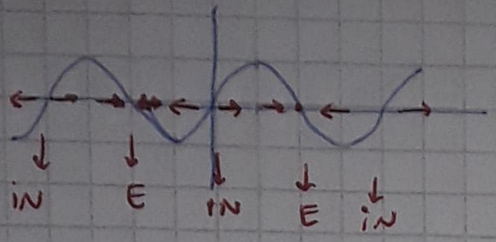
$x \equiv 0$, $x \equiv 1$, $x \equiv -1$ son soluciones y son puntos de eq.



$-1, 1$ son eq inestables

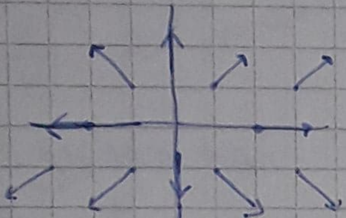
0 es eq estable

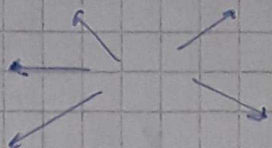
c. $\dot{x} = \text{sen}(x)$



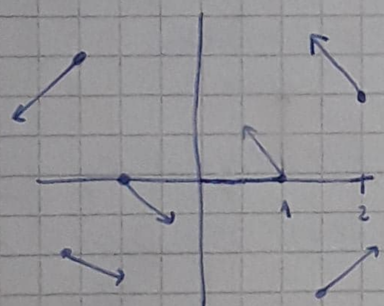
Todos los múltiplos de π son eq. Se alterna 1 estable con uno inestable.

2. a. $F(x,y) = (x,y)$ Para cada punto me da una flecha que apunta en la dirección (x,y).





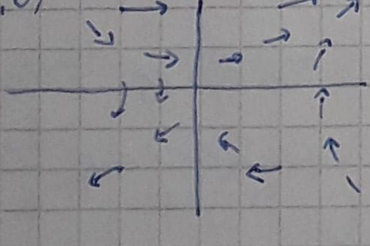
b. $F(x,y) = (-y, x)$



Todas las flechas giran alrededor de 0.

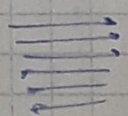
Circ. concéntricas.

c. $(y, 0)$

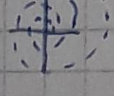


Las flechas se alejan del 0.

~~direc constante~~



d.



Espiral $F(x,y) = (-x+2y, -2x-y)$

$F(1,0) = (-1, 2)$ $F(0,1) = (2, -1)$

3. $\begin{cases} \dot{x} = 2x \\ \dot{y} = \lambda y \end{cases}$ $\lambda \in \mathbb{R}$

Me pide considerar a un parámetro

$$\begin{cases} \dot{x} = 2x \\ \dot{y} = \lambda y \quad \lambda \in \mathbb{R} \end{cases}$$

3. Sol $x(t) = C_1 e^{2t}$ $\qquad x'(t) = C_1 \cdot e^{2t} \cdot 2 = 2x(t)$ ✓

$y(t) = C_2 e^{\lambda t}$ $\qquad y'(t) = C_2 e^{\lambda t} \cdot \lambda = \lambda y(t)$ ✓

Despejo e^t de $x(t)$ para escribir a y en función de $x(t)$

$$x = C_1 (e^t)^2 \underset{C_1 \neq 0}{\Rightarrow} \frac{x}{|C_1|} = (e^t)^2 \Rightarrow \left(\frac{x}{|C_1|}\right)^{\frac{1}{2}} = e^t$$

NUNCA ENTENDÍ ESTA NOTACIÓN:
ASÍ LO HIZO MI PROFE

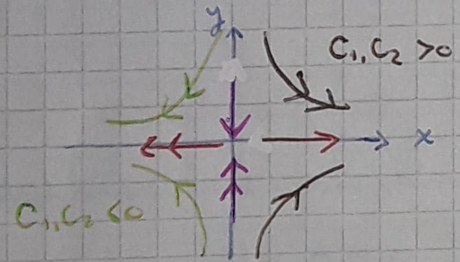
$$y = C_2 (e^t)^\lambda = C_2 \cdot \left(\frac{x}{|C_1|}\right)^{\frac{\lambda}{2}} = \frac{C_2}{|C_1|^{\lambda/2}} |x(t)|^{\frac{\lambda}{2}}$$

$\boxed{\lambda = -1}$ $\quad y = \frac{C_2}{C_1^{-1/2}} x(t)^{-\frac{1}{2}}$

! AUTOVALOR POSITIVO ⇒ SE ALEJA DEL $(0,0)$
AUTOVALOR NEGATIVO ⇒ SE ACERCA AL $(0,0)$ CUANDO $t \to +\infty$

Cuando $t \to +\infty$: $x \to +\infty$ $\quad y \to 0$

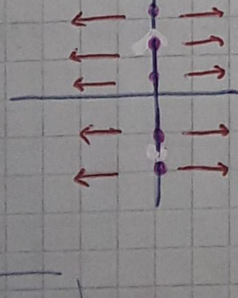
$e^t \to +\infty$
$e^{-t} \to 0$



- $\underline{C_1 = 0}$ ⇒ sol $= y(t)$
- $\underline{C_2 = 0}$ ⇒ sol $= x(t)$

0 eq inestable

$\boxed{\lambda = 0}$ $\quad y = C_2 = \text{cte}$



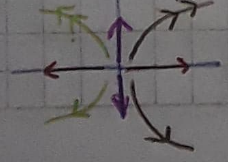
$\underline{C_1 = 0}$ $\quad$ sol $= C_2$ cte!

$C_2 = 0$ $\quad$ sol $= x(t)$

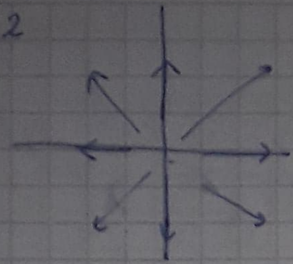
C_2 eq inestable $\to \infty$ $\quad C_2 \in \mathbb{R}$.

$\boxed{\lambda = 1}$ $\quad y = \frac{C_2}{C_1} x(t)^{\frac{1}{2}}$ $\quad$ • $C_1 = 0$ $\quad$ sol $y(t)$

$t \to +\infty$: $x(t) \to +\infty$ $\quad y(t) \to +\infty$



$\lambda = 2$ $\quad$ $t \to +\infty \quad y(t) \to \infty \quad x(t) \to \infty$



4. $\dot{x} = Ax \qquad A \in \mathbb{R}^{2\times 2}$

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ La matriz es diagonal $\begin{cases} \dot{x} = \lambda_1 x \\ \dot{y} = \lambda_2 y \end{cases}$

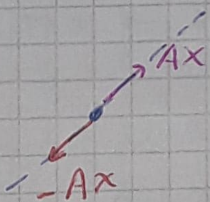
Las soluciones son $X(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$

para que $\lim\limits_{t\to+\infty} x(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \lambda_1, \lambda_2 < 0 \quad \forall c_1, c_2 \in \mathbb{R}$

5. a.b $x \mapsto -Ax$ es el opuesto a $x \mapsto Ax$

$\dot{x} = Ax$

$\dot{x} = -Ax$



Invierten el recorrido

Si $\dot{x} = Ax$ defino $y(t) = x(-t)$
↳ recorre para el otro lado

$\Rightarrow \underbrace{\dot{y}(t) = \dot{x}(-t)\cdot(-1)}_{\text{regla de la cadena}} \underbrace{= -A\cdot x(-t)}_{\dot{x}=Ax} = -A\,y(t)$

$\dot{y} = -Ay \Rightarrow$ Las líneas de flujo son las mismas recorridas en sentidos opuestos.

6. $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

$\det\begin{pmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{pmatrix} = (5-\lambda)(-4-\lambda) + 18 = 0$

$= -20 + 4\lambda - 5\lambda + \lambda^2 + 18 = 0$

$= \lambda^2 - \lambda - 2 = 0$

$\to \boxed{\lambda_1 = 2} \quad \boxed{\lambda_2 = -1}$

• $\lambda = 2 \rightsquigarrow \begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x + y = 0 \Rightarrow x = -y$

$v = \langle (1, -1) \rangle$

• $\lambda = -1 \rightsquigarrow \begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \to \left(\begin{array}{cc|c} 6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ $6x + 3y = 0$

$6x = -3y$

$-2x = y$

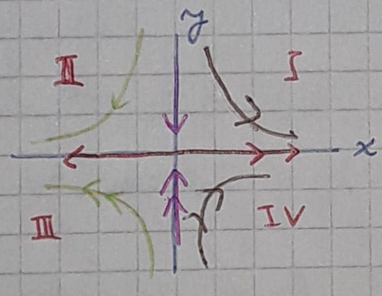
$v = \langle (1, -2) \rangle$

Sol $x(t) = c_1 e^{2t} (1, -1)$

$y(t) = c_2 e^{-t} (1, -2)$

$X(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix} = C \cdot y$ → IMPORTANTE ORDENAR LOS AUTOVALORES $(v_1; v_2)$ $\lambda_1 > \lambda_2$

$y(t)$



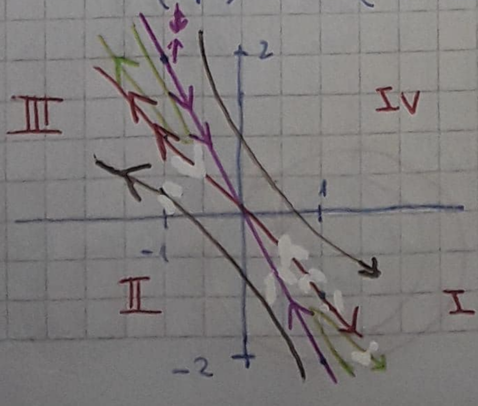
$\det C = -2 + 1 = -1 < 0$

$\Rightarrow$ invierte orientación

$x(t) = C \cdot y(t)$

Ejes: • $C \cdot \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

• $C \cdot \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



C invierte los cuadrantes.

7. (b). $F(x,y) = (-y, x) \Rightarrow \begin{cases} x' = -y \\ y' = x \end{cases}$

$$x' = F(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>Aval.</u> $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = +\lambda^2 + 1 = 0$

$\hookrightarrow \lambda_1 = 0 + i$

$\hookrightarrow \lambda_2 = 0 - i$

<u>Autovectores</u> $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[\cdot\, -i]{} \begin{pmatrix} -i & -1 \\ -i & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$

$$-ix - y = 0$$

$$-ix = y$$

$$v = \langle (1, -i) \rangle$$

~~sol~~ $\cancel{x(t) = e^{it}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$

$\cancel{X(t) = C_1 e^{0t} \cos(t) + C_2 e^{0} \operatorname{sen}(t)}$ $\rightarrow$ Esto era antes

$\cancel{= C_1 \cos(t) + C_2 \operatorname{sen}(t)}$

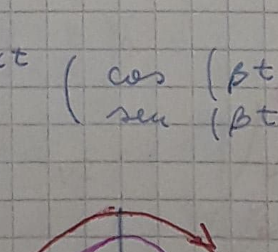
~~$\{\cos(t), \operatorname{sen}(t)\}$ base de soluciones~~

$$x(t) = \begin{pmatrix} \underbrace{1}_{Re} & \underbrace{0}_{Im} \\ 0 & -1 \end{pmatrix} r e^{0} \begin{pmatrix} \cos(t+\theta) \\ \operatorname{sen}(t+\theta) \end{pmatrix} = C \cdot y(t)$$

En general $\quad y(t) = r e^{\alpha t} \begin{pmatrix} \cos(\beta t + \theta) \\ \operatorname{sen}(\beta t + \theta) \end{pmatrix}$

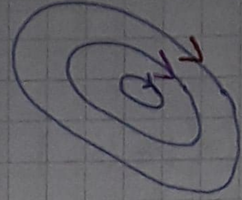
$\lambda = \alpha + i\beta$

Caso $\alpha = 0 \quad \beta < 0$



$y(t)$

$\Rightarrow x(t)$



(d) $F(x,y) = (-x+2y, -2x-y)$

$$\begin{cases} x' = -x + 2y \\ y' = -2x - y \end{cases}$$

$$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

Autovalores $\begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} \Rightarrow \lambda_1 = -1-2i$

$\lambda_2 = -1+2i$

Autovectores $\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \xrightarrow{x -\frac{1}{2}} \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \begin{pmatrix} i \\ 1+i \end{pmatrix}$

$(1+i)x - y = 0 \Rightarrow (1+i)x = y$

$1 \cdot x + (1+i) \cdot y = x + (1+i)^2 x = 0$

$x(1+(1+i)^2) = 0$

$x(1+1+2i-1) = 0$

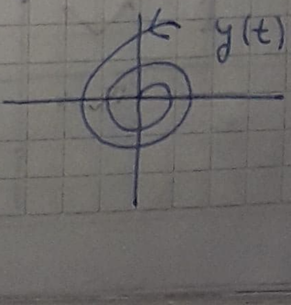
$$\begin{pmatrix} -1-(-1+2i) & 2 \\ -2 & -1-(-1+2i) \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \xrightarrow{\times i} \begin{pmatrix} -2i & 2 \\ -2i & 2 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} -2i & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow -2ix + 2y = 0$

$\cancel{2}y = \cancel{2}ix$

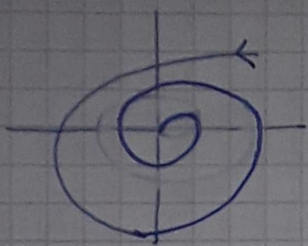
$v = \langle (1, i) \rangle$

$y(t) = r e^{-1t} \begin{pmatrix} \cos(2t+\theta) \\ \operatorname{sen}(2t+\theta) \end{pmatrix}$



Caso $\alpha < 0$ $\beta > 0$

$$x(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y(t)$$



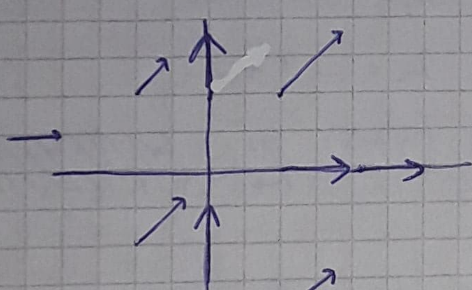
$x(t)$

8. a. $F(x,y) = (x^2, y^2)$

$F(1,0) = (1,0)$ $\quad$ $F(2,0) = (4,0)$

$F(0,2) = (0,4)$ $\quad$ $F(1,1) = (1,1)$

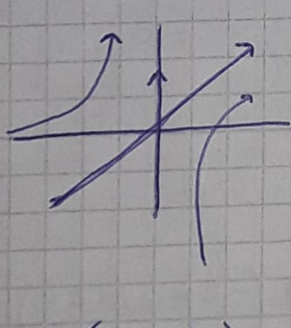
$F(-1,-1) = (1,1)$ $\quad$ $F(0,-2) = (0,4)$



$F(1,-3) = (1,9)$

$F(-1,1) = (1,1)$

$F(-5,1) = (25,1)$

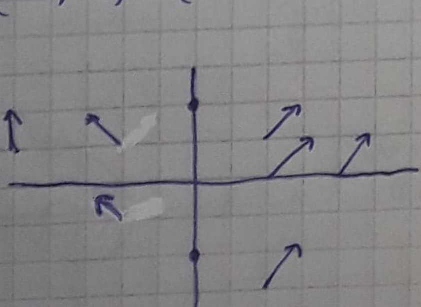


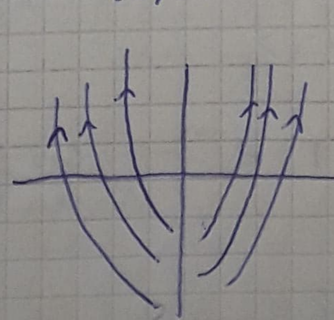
b. $F(x,y) = (x, x^2)$

$F(1,0) = (1,1)$ $\quad$ $F(2,0) = (2,4)$ $\quad$ $F(1,-3) = (1,1)$

$F(0,2) = (0,0)$ $\quad$ $F(1,1) = (1,1)$ $\quad$ $F(-1,1) = (-1,1)$

$F(-1,-1) = (1,1)$ $\quad$ $F(0,-2) = (0,0)$ $\quad$ $F(-5,1) = (-5,25)$





"parábolas"

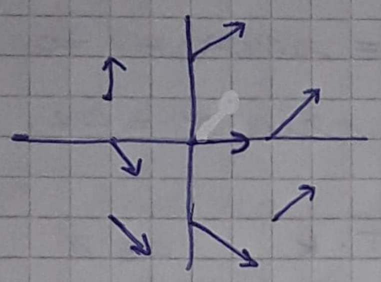
c. $F(x,y) = (1, x+y)$

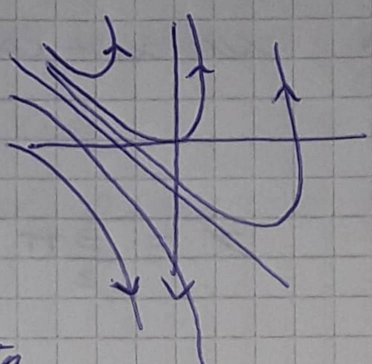
$F(0,0) = (1,0)$ $\quad$ $F(-1,1) = (1,0)$

$F(1,0) = (1,1)$ $\quad$ $F(0,1) = (1,1)$

$F(-1,-1) = (1,-2)$ $\quad$ $F(0,-2) = (1,-2)$

$F(1,-1) = (1,0)$ $\quad$ $F(-1,0) = (1,-1)$





$x+y \rightarrow$ una recta

9. Equilibrios y estabilidad

a. $\begin{cases} \dot{x} = \operatorname{sen} x + \cos y \\ \dot{y} = xy \end{cases}$

$F(x,y) = (\operatorname{sen} x + \cos y, \ xy)$ clase C^1 ✓

$$DF(x,y) = \begin{pmatrix} \cos x & -\operatorname{sen} y \\ y & x \end{pmatrix}$$

Busco puntos de equilibrio

- $\operatorname{sen} x + \cos y = 0 \Rightarrow -\operatorname{sen} x = \cos y$
- $xy = 0$
- $x = 0, y = 0$ $\quad$ $0 + 1 \neq 0 \rightarrow (0,0)$ no es punto de eq.

$-\operatorname{sen} x = \cos y$

$\arccos(-\operatorname{sen} x) = y$

- $x = 0,\ y \neq 0 \Rightarrow \operatorname{sen}(0) + \cos y = 0$

$$y = \frac{\pi}{2} + K\pi \qquad K \in \mathbb{Z}$$

$$\boxed{P_1 = \left(0, \frac{\pi}{2} + K\pi\right)}$$

- $x \neq 0,\ y = 0 \Rightarrow \operatorname{sen} x + \cos(0) = \operatorname{sen} x + 1 = 0$

$$\operatorname{sen} x = -1$$

$$x = \frac{3\pi}{2} + 2\pi \cdot k \qquad k \in \mathbb{Z}$$

si pongo $K\pi$ me puede dar $+1$ y yo solo quiero -1

$$P_2 = \left(\frac{3}{2}\pi + 2K\pi, 0\right)$$

- $x \neq 0,\ y \neq 0 \quad (x, \arccos(-\operatorname{sen}(x))) \rightarrow$ no cumple $xy = 0$

$DF\left(0, \frac{\pi}{2} + K\pi\right)$ Tomo $K = 0$ para simplificar las cuentas.

$$*DF\left(0, \frac{\pi}{2}\right) = \begin{pmatrix} 1 & -1 \\ \frac{\pi}{2} & 0 \end{pmatrix}$$

Autovalores

$$\det\begin{pmatrix} 1-\lambda & -1 \\ \pi/2 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) + \frac{\pi}{2} = 0$$

$$= -\lambda + \lambda^2 + \frac{\pi}{2} = 0$$

$\hookrightarrow$

$$\frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot \pi/2}}{2} = \frac{1 \pm \sqrt{1 - 2\pi}}{2} = \frac{1 \pm \sqrt{(-1)(2\pi - 1)}}{2}$$

$$= \frac{1 \pm \sqrt{2\pi - 1}\, i}{2}$$

$$\rightarrow \lambda_1 = \frac{1}{2} + \frac{\sqrt{2\pi - 1}}{2} i$$

$$\lambda_2 = \frac{1}{2} - \frac{\sqrt{2\pi - 1}}{2} i$$

Para analizar si el punto de eq es estable o inestable alcanza con ver cómo se comporta el sistema lineal asociado $y' = \underbrace{\begin{pmatrix} 1 & -1 \\ \pi/2 & 0 \end{pmatrix}}_{A} y(t)$ en el punto de eq.

Como $\lambda_1, \lambda_2 = \alpha + \beta i$ con parte real positiva

$Re(\lambda) = \frac{1}{2} > 0 \Rightarrow$ El diagrama de fases de

$y' = Ay$ da espirales que se alejan del $(0,0)$ cuando $t \to +\infty \Rightarrow \left(0, \frac{\pi}{2}\right)$ es un $\boxed{\text{eq inestable.}}$

$*$ $P_2 = \left(\frac{3\pi}{2}, 0\right)$ tomo $k=0$

$$DF\left(\frac{3\pi}{2}, 0\right) = \begin{pmatrix} 0 & 1 \\ 0 & \frac{3\pi}{2} \end{pmatrix}$$

<u>Autovalores</u>

$$\det \begin{pmatrix} -\lambda & 1 \\ 0 & \frac{3\pi}{2} - \lambda \end{pmatrix} = -\lambda \left(\frac{3\pi}{2} - \lambda\right) = 0$$

$\hookrightarrow \boxed{\lambda_1 = 0}$

$\hookrightarrow \lambda_2 = \frac{3}{2}\pi$

El teorema de estabilidad no sirve para $\lambda=0$
($DF(x_0)$ no tiene autovalor con parte real 0)

b. $\begin{cases} \dot{x} = (x+1)e^y - 1 \\ \dot{y} = x+y \end{cases}$

$F(x,y) = ((x+1)e^y - 1, \; x+y)$ clase C^1 ✓

$DF(x,y) = \begin{pmatrix} e^y & x+1 \\ 1 & 1 \end{pmatrix}$

Busco ptos de eq.

- $(x+1)e^y - 1 = 0 \Rightarrow (x+1)e^y = 1$
- $x+y = 0 \Rightarrow x = -y$

$x=0, y=0$ $\quad \boxed{P_1 = (0,0)}$ $\quad (0+1)e^0 = 1$ ✓
$0 = 0$ ✓

$x=0, y\neq 0$ $\quad (0+1)e^y = 1 \iff y=0$

$y=0, x\neq 0$ $\quad (x+1)\,1 = 1 \iff x=0$

$x\neq 0, y\neq 0$ $\quad (-y+1)e^y = 1$

$$e^y = \frac{1}{1-y}$$

$$y = \ln\left(\frac{1}{1-y}\right) = \ln(1) - \ln(1-y)$$

$$y = -\ln(\underbrace{1-y}_{=x})$$

$$y = -\ln(1+x)$$

$\Rightarrow (0,0)$ es el único punto de eq.

$$DF(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Autovalores $\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = 0$

$= \lambda^2 - 2\lambda + \cancel{1} - \cancel{1} = 0$

$= \lambda(\lambda - 2) = 0$

$\hookrightarrow \lambda = 0$ → El teorema no se aplica!

$\hookrightarrow \lambda = 2$

C. $\begin{cases} \dot{x} = x^2 - y - 1 \\ \dot{y} = xy \end{cases}$ $F(x,y) = (\dot{x}, \dot{y})$

$$DF(x,y) = \begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}$$

Puntos de eq

- $xy = 0 \rightarrow x = 0$
  $\rightarrow y = 0$

- $x^2 - y - 1 = 0 \Rightarrow x^2 - 1 = y$

si $x = 0 \Rightarrow y \neq 0$ pues $0 - 1 \neq 0$ $(0,0)$ no es

$x = 0, y \neq 0$ $\boxed{-1 = y}$ $P_1 = (0, -1)$

$x \neq 0, y = 0$ $x^2 - 1 = 0 \Rightarrow x = -1$

$x = 1$

$P_2 = (-1, 0)$ $P_3 = (1, 0)$

$x \neq 0, y \neq 0$ $x \cdot (x^2 - 1) = 0 \Rightarrow x^2 - 1 = 0 \begin{cases} x = 1 \\ x = -1 \end{cases}$

no hay otro punto de eq.

Autovalores $\cdot\ DF(0,-1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\det\begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

$\hookrightarrow \lambda = 1$

$\hookrightarrow \lambda = -1$

$\boxed{\lambda_1 > 0 > \lambda_2}$ $y(t) = C_1 e^{t} v_1 + C_2 e^{-t} v_2$

En el diagrama de fases del sist. lineal asociado $y' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} y(t)$ dos trayectorias se acercan a cero cuando $t \to +\infty$ (correspondientes al autovalor negativo) y las otras se alejan.

$\Rightarrow (0,-1)$ es $\boxed{\text{eq inestable}}$

$\cdot\ DF(1,0) = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 0 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) = 0$$

$\hookrightarrow \lambda = 1$

$\hookrightarrow \lambda = 2$

$\boxed{\lambda_1 > \lambda_2 > 0}$ Como los dos tienen parte real positiva las trayectorias se alejan cuando $t \to +\infty \Rightarrow \boxed{\text{eq inestable}}$

$\cdot\ DF(-1,0) = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix}$

$$\det\begin{pmatrix} -2-\lambda & -1 \\ 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-2-\lambda) = 0$$

$\hookrightarrow \lambda = -1$

$\hookrightarrow \lambda = -2$

$\boxed{0 > \lambda_1 > \lambda_2}$ Las trayectorias $\to x_0$ cuando $t \to +\infty$

$\Rightarrow (1,0)$ $\boxed{\text{eq estable}}$

10. d. $x'' + cx' - x^3 = 1$

$x'' = 1 - cx' + x^3$

Armo mi sistema $\begin{cases} x_0 = x \\ x_1 = x' \end{cases}$

$\Rightarrow \quad x_1' = x'' = 1 - cx' + x^3$

$x_0' = x' = x_1$

El sistema queda

$$\begin{cases} x_0' = x_1 \\ x_1' = 1 - cx_1 + x_0^3 \end{cases}$$

~~Busco~~ Quiero hallar puntos de eq.

$F(x_0, x_1) = (x_1, \; 1 - cx_1 + x_0^3)$ clase C^1 ✓

$$DF(x_0, x_1) = \begin{pmatrix} 0 & 1 \\ 3x_0^2 & -c \end{pmatrix}$$

$x_1 = 0$

$1 - cx_1 + x_0^3 = 0 \Rightarrow 1 + x_0^3 = 0$

$\Rightarrow \boxed{x_0 = -1}$

$(-1, 0)$ es el único punto de eq.

Autovalores $\quad DF(-1,0) = \begin{pmatrix} 0 & 1 \\ 3 & -c \end{pmatrix}$

$$\det \begin{pmatrix} -\lambda & 1 \\ 3 & -c-\lambda \end{pmatrix} = (-\lambda)(-c-\lambda) - 3 = 0$$

$$= +\lambda c + \lambda^2 - 3 = 0$$

$$\frac{-c \pm \sqrt{c^2-4\cdot 1\cdot(-3)}}{2\cdot 1} = \frac{-c \pm \sqrt{c^2+12}}{2}$$

$$\lambda_1 = -\frac{c}{2} + \frac{\sqrt{c^2+12}}{2}$$

$$\lambda_2 = -\frac{c}{2} - \frac{\sqrt{c^2+12}}{2}$$

$c^2+12>0$ por lo tant

complejos

si $\boxed{c<0}$ $\Rightarrow$ $-\frac{c}{2}>0$ y

$$\lambda_1 = -\frac{c}{2} + \frac{\sqrt{c^2+12}}{2} > 0$$

$$-\frac{c}{2} > -\frac{\sqrt{c^2+12}}{2}$$

$\Rightarrow$ $\lambda_2 > 0$

$\sqrt{c^2+12} > c$

$\Rightarrow$ los autovalores van a ser siempre positivos

$\Rightarrow$ eq. inestable

(no importa signo de c)

- si $\underline{\lambda_1, \lambda_2 > 0}$ $\Rightarrow$ por el teorema de estabilidad lineal el punto de eq es $\boxed{\text{inestable}}$

- si $\underline{\lambda_1 > 0 > \lambda_2}$ $\Rightarrow$ el eq es inestable:

si $\boxed{c>0}$ $\Rightarrow$ $-\frac{c}{2}<0$

y entonces $-\frac{c}{2} - \frac{\sqrt{c^2+12}}{2} < 0 \Rightarrow \lambda_2 < 0$

$$-\frac{c}{2} < \frac{\sqrt{c^2+12}}{2} \Rightarrow \lambda_1 > 0$$

$\boxed{c=0}$ $\lambda_1 = \frac{\sqrt{12}}{2}$ $\lambda_2 = -\frac{\sqrt{12}}{2}$

$\Rightarrow$ $\boxed{\text{eq inestable}}$

e. $\ddot{x} + x^3 - x = 0$

$\ddot{x} = x - x^3$

$$\left[\begin{array}{l} x_0 = x \\ x_1 = \dot{x} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_0 - x_0^3 \end{array}\right.$$

$F(x_0, x_1) = (x_1, x_0 - x_0^3)$

$$DF(x_0, x_1) = \begin{pmatrix} 0 & 1 \\ 1-3x_0^2 & 0 \end{pmatrix}$$

Puntos de equilibrio

$x_1 = 0$

$x_0 - x_0^3 = 0 \Rightarrow x_0(1 - x_0^2) = 0$

$\hookrightarrow x_0 = 0$

$\hookrightarrow x_0 = 1$

$\hookrightarrow x_0 = -1$

$P_1 = (0,0)$

$P_2 = (1,0)$

$P_3 = (-1,0)$

Autovalores $DF(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$

$\hookrightarrow \lambda = 1, \lambda = -1$

• $\underline{\lambda_1 > 0 > \lambda_2} \Rightarrow \boxed{\text{eq inestable}}$

$DF(1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

$\det \begin{pmatrix} -\lambda & 1 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 + 2 = 0$

$\hookrightarrow \sqrt{2}i = \lambda$

$\hookrightarrow \lambda = -\sqrt{2}\, i$

Como los autovalores tienen parte real $Re(\lambda)=0$ no se puede decidir nada sobre la estabilidad con el teo de estabilidad lineal.

• $DF(-1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

c. $\ddot{x} - x + \cos x = 0 \Rightarrow \ddot{x} = x - \cos x$

$x_0 = x$

$x_1 = \dot{x}$ $\Rightarrow \begin{cases} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_0 - \cos x_0 \end{cases}$

$F(x_0, x_1) = (x_1, x_0 - \cos x_0)$

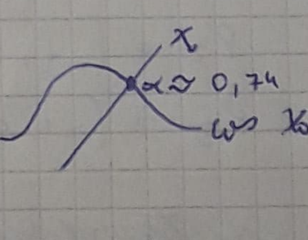
$DF(x_0, x_1) = \begin{pmatrix} 0 & 1 \\ 1 + \operatorname{sen} x_0 & 0 \end{pmatrix}$

Puntos de eq

$x_1 = 0$

$x_0 - \cos x_0 = 0 \Rightarrow x_0 = \cos x_0$

Llamo α $x_0 / \cancel{x_0} = \cos\alpha$, $\alpha > 0$



$DF(\alpha, 0) = \begin{pmatrix} 0 & 1 \\ 1 + \operatorname{sen}\alpha & 0 \end{pmatrix}$

Autovalores $\lambda^2 \underbrace{- 1 - \operatorname{sen}\alpha}_{\simeq -1,67} = 0 \Rightarrow \left.\begin{matrix} \lambda_1 = 1,3 \\ \lambda_2 = -1,3 \end{matrix}\right\} \Rightarrow$ eq inestable

11. $\begin{cases} \dot{x} = x e^y \\ \dot{y} = -1 + y + \operatorname{sen}(x) \end{cases}$

$F(x,y) = (x e^y, -1 + y + \operatorname{sen}(x))$

$DF(x,y) = \begin{pmatrix} e^y & x e^y \\ \cos(x) & 1 \end{pmatrix}$

Puntos de equilibrio

$x e^y = 0 \iff x = 0$ pues $e^y \neq 0$

$-1 + y + \operatorname{sen} x = 0$

$-1 + y + 0 = 0 \Rightarrow \boxed{y = 1}$

El punto de equilibrio es $(0,1)$

Autovalores $DF(0,1) = \begin{pmatrix} e & 0 \\ 1 & 1 \end{pmatrix}$

$\det \begin{pmatrix} e-\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)(e-\lambda) = 0$

$\hookrightarrow \lambda = 1$

$\hookrightarrow \lambda = e$

$\lambda_1 > \lambda_2 > 0 \Rightarrow$ el punto es un eq inestable

Autovectores

$\lambda = 1 \quad \operatorname{Ker} \begin{pmatrix} e-1 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{matrix} (e-1)x = 0 \\ x = 0 \end{matrix} \quad y \in \mathbb{R}$

$\Rightarrow v_1 = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$\lambda = e \quad \operatorname{Ker} \begin{pmatrix} 0 & 0 \\ 1 & 1-e \end{pmatrix} \Rightarrow 1 \cdot x + (1-e) y = 0$

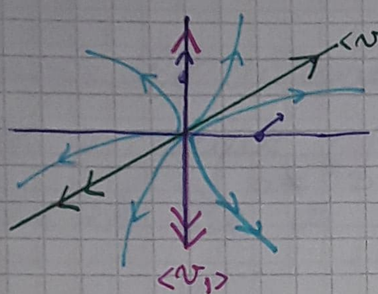
$x = (e-1) y$

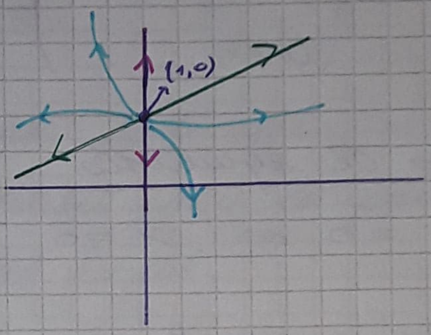
$v_2 = \langle (e-1, 1) \rangle$

Grafico el diagrama de fases para las sol de la forma

$$y(t) = c_1 e^{t}\begin{pmatrix}0\\1\end{pmatrix} + c_2 e^{et}\begin{pmatrix}e-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0 & e-1\\1 & 1\end{pmatrix}\begin{pmatrix}c_1 e^{t}\\c_2 e^{et}\end{pmatrix} \qquad y' = DF(0,1)\,y$$





Miro el vector velocidad en algunos puntos

$$DF(0,1)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e\\1\end{pmatrix} \quad ✓$$

$$\begin{pmatrix}e & 0\\1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \quad ✓$$

b. $\begin{cases}\dot{x} = e^{x-y} - 1\\ \dot{y} = xy - 1\end{cases}$

$$F(x,y) = (e^{x-y} - 1,\ xy - 1)$$

$$DF(x,y) = \begin{pmatrix}e^{x-y} & -e^{x-y}\\ y & x\end{pmatrix}$$

Puntos de equilibrio

$$e^{x-y} - 1 = 0 \iff x = y$$

$$xy - 1 = 0 \Rightarrow x^2 - 1 = 0 \to x = 1, \quad x = -1$$

$P_1 = (1,1)$

$P_2 = (-1,-1)$

$$DF(1,1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Autovalores $(1-\lambda)^2 + 1 = 0$

$= \lambda^2 - 2\lambda + 1 + 1 = 0$

$\hookrightarrow \lambda = 1+i$

$\hookrightarrow \lambda = 1-i$

$Re(\lambda) = 1 \neq 0$ Vale el Teo de linealización

Autovectores $\lambda = 1-i$

$$Ker \begin{pmatrix} 1-1+i & -1 \\ 1 & 1-1+i \end{pmatrix} = Ker \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \xrightarrow[F_2 \times i]{} Ker \begin{pmatrix} i & -1 \\ \cancel{i} & \cancel{-1} \end{pmatrix}$$

$ix - y = 0 \Rightarrow ix = y$

$v_1 = \langle (1, i) \rangle$

$$\lambda = 1+i \quad Ker \begin{pmatrix} 1-1-i & -1 \\ 1 & 1-1-i \end{pmatrix} \to Ker \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[x \cdot i]{} \begin{pmatrix} -i & -1 \\ \cancel{-i} & \cancel{-1} \end{pmatrix}$$

$-ix - y = 0 \Rightarrow ix = y$

$v_2 = \langle (1, -i) \rangle$

$$sol = C_1 e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(1-i)t} \begin{pmatrix} 1 \\ +i \end{pmatrix} \qquad C_1, C_2 \in \mathbb{R}$$

$\Rightarrow \cancel{y(t) = C_1 e^t \cos t + C_2 e^t \operatorname{sen}(t)}$

$$= C_1 e^t \cdot e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^t e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

forma polar $e^{it} = \cos(t) + i \operatorname{sen}(t)$

$$DF(1,1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Autovalores $(1-\lambda)^2 + 1 = 0$

$$= \lambda^2 - 2\lambda + 1 + 1 = 0$$

$\hookrightarrow \lambda = 1 + i$

$\hookrightarrow \lambda = 1 - i$

$Re(\lambda) = 1 \neq 0$ Vale el Teo de linealización

Autovectores $\lambda = 1 - i$

$$Ker\begin{pmatrix} 1-1+i & -1 \\ 1 & 1-1+i \end{pmatrix} = Ker\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \xrightarrow[F_2 \times i]{} Ker\begin{pmatrix} i & -1 \\ \cancel{i} & \cancel{-1} \end{pmatrix}$$

$$ix - y = 0 \Rightarrow ix = y$$

$$v_1 = \langle (1, i) \rangle$$

$$\lambda = 1+i \quad Ker\begin{pmatrix} 1-1-i & -1 \\ 1 & 1-1-i \end{pmatrix} \rightarrow Ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[x \cdot i]{} \begin{pmatrix} -i & -1 \\ \cancel{-i} & \cancel{-1} \end{pmatrix}$$

$$-ix - y = 0 \Rightarrow ix = y$$

$$v_2 = \langle (1, -i) \rangle$$

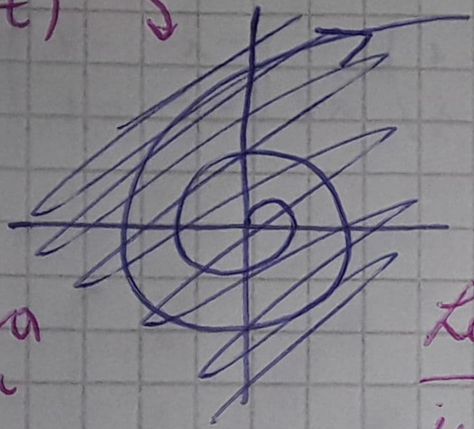
$$sol = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ +i \end{pmatrix} \qquad c_1, c_2 \in \mathbb{R}$$

$\Rightarrow$ ~~$y(t) = c_1 e^t \cos t + c_2 e^t \operatorname{sen}(t)$~~

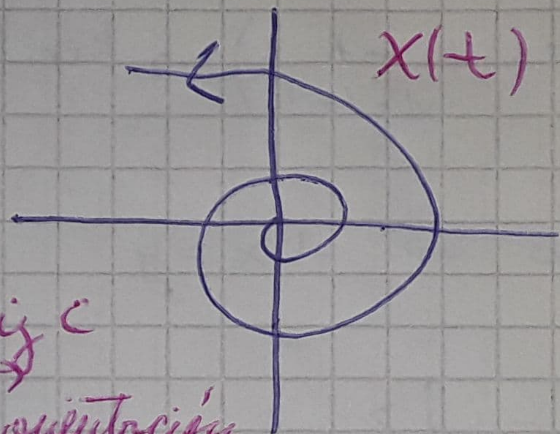
$$= c_1 e^t \cdot e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^t e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

forma polar $e^{it} = \cos(t) + i \operatorname{sen}(t)$

$$y(t) = r\,e^{t} \begin{pmatrix} \cos(t+\theta) \\ \operatorname{sen}(t+\theta) \end{pmatrix} \qquad \lambda = 1 + i$$



La matriz C $\longrightarrow$ invierte orientación



$\det C < 0$

$\Rightarrow$ invierte orientación

Caso $\lambda = \alpha + \beta i$ $\qquad \alpha > 0 \Rightarrow y \to \infty$

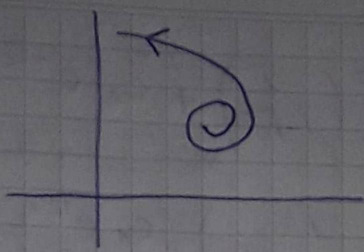
~~[illegible]~~

$\beta > 0 \Rightarrow$ Los espirales se mueven desde $\mathrm{Im}(v)$ a $\mathrm{Re}(v)$

$v = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \mathrm{Im}(v) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \mathrm{Re}(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \quad x(t) =$



$$DF(-1,-1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Autovalores $(1-\lambda)(-1-\lambda) - 1 = 0$

$$= -1 - \cancel{\lambda} + \cancel{\lambda} + \lambda^2 - 1 = 0$$

$$\lambda^2 - 2 = 0$$

$\hookrightarrow \lambda = \sqrt{2}$

$\hookrightarrow \lambda = -\sqrt{2}$

Vale el teo, el punto de eq será inestable porque hay al menos un autovalor positivo

Autovectores

$$\lambda = \sqrt{2} \quad Ker \begin{pmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{pmatrix} \to Ker \begin{pmatrix} 1-\sqrt{2} & -1 \\ 1 & 1+\sqrt{2} \end{pmatrix}$$

$$(1-\sqrt{2})x - y = 0$$

$$(1-\sqrt{2})\, x = y$$

$$x + (1+\sqrt{2})(1-\sqrt{2})\, x = 0$$

$$x + (1-2)\, x = 0 \quad ✓$$

$$v = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

$$\lambda = -\sqrt{2}\ Ker \begin{pmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{pmatrix} \to \quad (1+\sqrt{2})x = y$$

$$-x + (-1+\sqrt{2})(1+\sqrt{2})\, x = 0$$

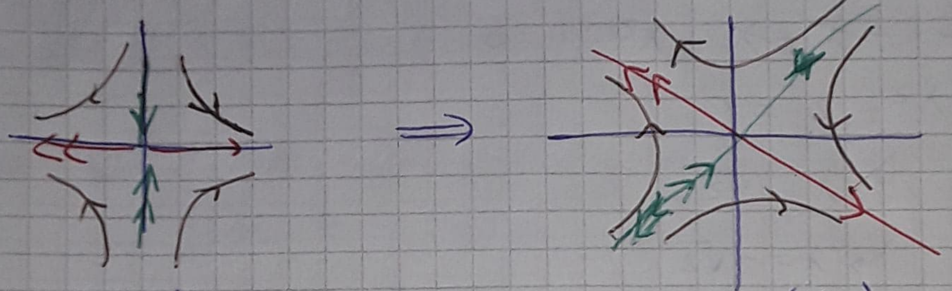
$$-x + (1-2)\, x = 0 \, ✓$$

$$v = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$$\text{Sol } y(t) = c_1 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} + c_2 e^{-\sqrt{2}t}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 e^{\sqrt{2}t} \\ c_2 e^{\sqrt{2}t} \end{pmatrix}$$

$\det C = \cancel{1} + \sqrt{2} - \cancel{1} + \sqrt{2} > 0 \Rightarrow$ preserva orientación



$y(t)$ $\qquad\qquad$ $x(t)$

$$\lambda \begin{pmatrix} 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$$\lambda \begin{pmatrix} & " & \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

c. $\begin{cases} \dot{x} = x(y-1) - 4 \\ \dot{y} = x^2 - (y-1)^2 \end{cases}$ $\qquad F(x,y) = (\dot{x}, \dot{y})$

$$x(y-1) - 4 = 0$$
$$x(y-1) = 4$$
$$(y-1) = \frac{4}{x} \qquad x \neq 0$$

$$x^2 - (y-1)^2 = x^2 - \frac{16}{x^2} = 0$$

$$x^4 - 16 = 0$$

$\hookrightarrow x = 2$

$\hookrightarrow x = -2$

$P_1 = (2, 2) \qquad P_2 = (-2, -2)$

$$DF(x,y) = \begin{pmatrix} (y-1) & x \\ 2x & -2(y-1) \end{pmatrix}$$

$$DF(2,2) = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix}$$

Autovalores $(1-\lambda)(-2-\lambda) - 8 = 0$

$$-2 + 2\lambda - \lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 + \lambda - 10 = 0$$

↳

PAJA

↳

Fin temas para el parcial