Práctica "0"

Hoja 1

Ejercicio 1

Uso coordenadas cilíndricos

$x = r\cos\theta$ $\qquad 0 \leq r \leq r$

$y = r\sin\theta$ $\qquad 0 \leq \theta \leq 2\pi$

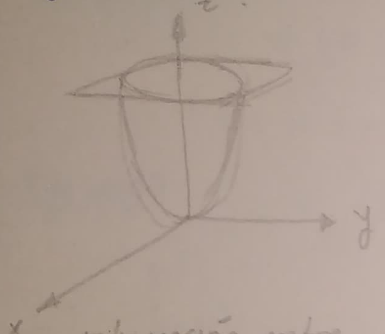
$z = z$ $\qquad 0 \leq z \leq z$

jacobiano!

$$\int_0^z\int_0^{2\pi}\int_0^r r\,dr\,d\theta\,dz = \int_0^z\int_0^{2\pi} \frac{1}{2}r^2\,d\theta\,dz = \frac{1}{2}r^2\theta\Big|_0^{2\pi} dz = \frac{1}{2}r^2\,2\pi\,z$$

Volumen cilindro: $\pi r^2 h$ (donde $h = z$)

Ejercicio 2



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

intersección entre el plano y el paraboloide.

$x^2 + y^2 = r^2 = 2$

$r = \sqrt{2}$

Entonces los límites de integración son:

$0 \leq r \leq \sqrt{2}$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 2$

jacobiano!!

$$\int_0^2\int_0^{2\pi}\int_0^{\sqrt{2}} r\,dr\,d\theta\,dz = \int_0^2\int_0^{2\pi} \frac{1}{2}r^2\Big|_0^{\sqrt{2}} d\theta\,dz = \int_0^2 2\pi\,dz = 4\pi$$

Ejercicio 3

$-1 \leq x \leq 1$

$0 \leq y \leq 1$

a) $$\int_0^1\int_{-1}^1 x^2 y\,dy\,dx = \int_0^1 \frac{1}{3}x^3\Big|_{-1}^1 y\,dy = \left(\frac{1}{3}+\frac{1}{3}\right)\int_0^1 y\,dy$$

$$= \frac{1}{6}\,\frac{1}{2}y^2\Big|_0^1 = \frac{1}{12}$$

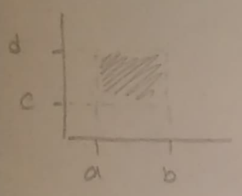
b) $\left.\begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array}\right\}$ $\iint_R x \cos(xy)\, dy\, dx = \int_{-1}^{1}\int_{t=0}^{t=x} \cos t \; dt\; dx$

Sustitución
$t = xy$
$dt = x\, dy$

$$= \int_{-1}^{1} \sin t \Big|_{t=0}^{t=x} dx = \int_{-1}^{1} \sin(xy) \Big|_{0}^{1} dx = \int_{-1}^{1} \sin x \; dx$$

$$= -\cos x \Big|_{-1}^{1} = -\cos 1 - \cos(-1)$$

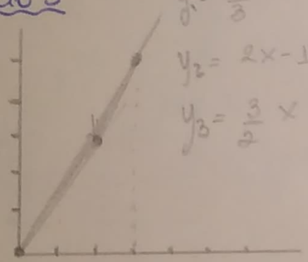
Ejercicio 4



a) $\int_c^d \int_a^b f(x) \cdot g(y)\; dx\, dy = \int_a^b f(x)\, dx \cdot \int_c^d g(y)\; dy$

b) $\int_c^d \int_a^b f(x) + g(y)\; dx\, dy = \int_a^b f(x)\, dx + \int_c^d g(y)\; dy$

Ejercicio 5



$y_1 = \frac{5}{3}x$

$y_2 = 2x - 1$

$y_3 = \frac{3}{2}x$

Región tipo I

$$D = \begin{cases} \frac{3}{2}x \leq y \leq \frac{5}{3}x & \text{si} \quad 0 \leq x \leq 2 \\ 2x-1 \leq y \leq \frac{5}{3}x & \text{si} \quad 2 \leq x \leq 3 \end{cases}$$

Región tipo II.

$$D = \begin{cases} \frac{3}{5}y \leq x \leq \frac{2}{3}y & \text{si} \quad 0 \leq y \leq 3. \\ \frac{3}{5}y \leq x \leq \frac{1}{2}y + \frac{1}{2}. & \text{si} \quad 3 \leq y \leq 5 \end{cases}$$

- Ahora voy a hallar el área, usando la región tipo I

$A_1 : \int_0^2 \frac{5}{3}x - \frac{3}{2}x \; dx = \int_0^2 \frac{10-9}{6}\; dx = \frac{1}{9} \Big|_0^2 = 2/9$

$A_2 : \int_2^3 \frac{5}{3}x - 2x + 1 \; dx = \int_2^3 -\frac{1}{3}x + 1\; dx = -\frac{1}{6}x^2 + x \Big|_2^3 = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$

$A_{total} = A_1 + A_2$

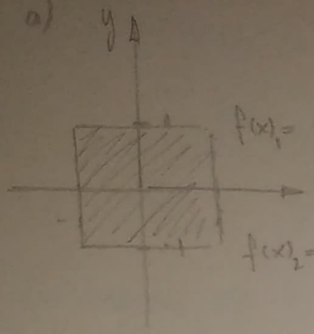
$A_{total} = \frac{7}{18}$

Practica "0"

Hoja 2

Ejercicio 6

a)

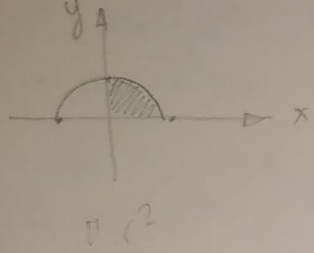


$f(x)_1 = 1$, $f(x)_2 = -1$, $\quad -1 \leq x \leq 1$

$$\text{Area} = \int_{-1}^{1} f_1(x) - f_2(x)\, dx = \int_{-1}^{1} 1-(-1)\, dx = \int_{-1}^{1} 2\, dx$$

$$= 2x \Big|_{-1}^{1} = 2-(-2) = 4$$

b)



πr^2

$0 \leq r \leq 1$

$0 \leq \theta \leq \pi/2$

$x = r\cos\theta$

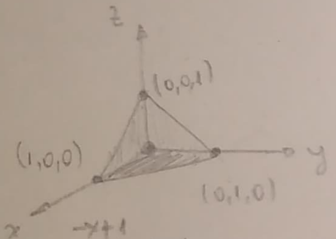
$y = r\sin\theta$

$$\text{Area} = \int_0^{\pi/2}\int_0^1 r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2} r^2 \Big|_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\,\theta \Big|_0^{\pi/2} = \frac{\pi}{4}$$

Ejercicio 7

Tetraedro $= x+y+z = 1$.



$0 \leq x \leq 1$

$0 \leq y \leq -x+1$

$0 \leq z \leq 1-x-y$

$0 \leq x \leq 1$

$0 \leq y \leq 1$

$0 \leq z \leq 1-x-y$.

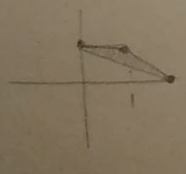
$$\int_0^1\int_0^{-x+1}\int_0^{1-x-y} dz\, dy\, dx = \int_0^1\int_0^{1-x} 1-x-y\, dy\, dx = \int_0^1 y - xy - \frac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2\, dx = \int_0^1 \left(1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2\right) dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2\, dx = \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \Big|_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

Ejercicio 8

El valor medio de f en la region T, se define como $\dfrac{\iint_T f(x,y)\, dx\, dy}{\iint_T dx\, dy}$



- Describo como region I

$$\begin{cases} \text{Si } 0 \leq x \leq 1 & -\frac{1}{2}x + 1 \leq y \leq 1 \\ \text{Si } 1 \leq x \leq 2 & -\frac{1}{2}x + 1 \leq y \leq -x+2 \end{cases}$$

$$\iint_T f(x,y)\,dx\,dy = \iint_{0\ -\frac{1}{2}x+1} x^2 y\,dy\,dx + \iint_{1\ -\frac{1}{2}x+1} x^2 y\,dy\,dx$$

$$= \int_0^1 \frac{x^2 y^2}{2}\Big|_{-\frac{1}{2}x+1}^{1} dx + \int_1^2 \frac{x^2 \cdot y^2}{2}\Big|_{-\frac{1}{2}x+1}^{-x+2} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^2\left(-\frac{1}{2}x+1\right)^2}{2}\right)dx + \int_1^2 \frac{x^2\cdot(-x+2)^2}{2} - \frac{x^2\left(-\frac{1}{2}x+1\right)^2}{2}\,dx$$

$$= \int_0^1 \frac{1}{2}x^2 - \frac{1}{2}x^2\left(\frac{1}{4}x^2 - x + 1\right)dx + \int_1^2 \frac{x^2\left(x^2-4x+4\right)}{2} - \frac{x^2}{2}\left(\frac{1}{4}x^2 - x + 1\right)dx$$

$$= \int_0^1 \cancel{\frac{1}{2}x^2} - \frac{1}{8}x^4 + \frac{1}{2}x^3 - \cancel{\frac{1}{2}x^2}\,dx + \int_1^2 \frac{1}{2}x^4 - 2x^3 + 2x^2 - \frac{1}{8}x^4 + \frac{1}{2}x^3 - \frac{1}{2}x^2\,dx$$

$$= \int_0^1 -\frac{1}{8}x^4 + \frac{1}{2}x^3\,dx + \int_1^2 \frac{3}{8}x^4 - \frac{3}{2}x^3 + \frac{3}{2}x^2\,dx$$

$$= \left(-\frac{1}{40}x^5 + \frac{1}{8}x^4\right)\Big|_0^1 + \left(\frac{3}{40}x^5 - \frac{3}{8}x^4 + \frac{3}{6}x^3\right)\Big|_1^2$$

$$= -\frac{1}{40} + \frac{1}{8} + \left(\frac{3}{40}\cdot 32 - \frac{3}{8}\cdot 16 + \frac{1}{2}\cdot 8\right) - \left(\frac{3}{40} - \frac{3}{8} + \frac{3}{6}\right)$$

$$= -\frac{1}{40} + \frac{1}{8} + \frac{96}{40} - 6 + 4 - \frac{3}{40} + \frac{3}{8} - \frac{1}{2}$$

$$= \frac{92}{40} + \frac{4}{8} - \frac{5}{2} = \frac{3}{10}$$

* Ahora calculo la integral del denominador $\iint_T dx\,dy$.

$$\iint_{0\ -\frac{1}{2}x+1}^{1\ 1} dy\,dx + \iint_{1\ -\frac{1}{2}x+1}^{2\ -x+2} dy\,dx = \int_0^1 y\Big|_{-\frac{1}{2}x+1}^{1} dx + \int_1^2 y\Big|_{-\frac{1}{2}x+1}^{-x+2} dx =$$

$$\int_0^1 \cancel{1} + \frac{1}{2}x - \cancel{1}\,dx + \int_1^2 -x+2+\frac{1}{2}x-1\,dx = \int_0^1 \frac{1}{2}x\,dx + \int_1^2 -\frac{1}{2}x+1\,dx = \frac{1}{4}x^2\Big|_0^1 + \left(-\frac{1}{4}x^2 + x\right)\Big|_1^2$$

$$= \frac{1}{4} + \left(-\frac{1}{4}4 + 2\right) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4} + \cancel{1} + \frac{1}{4} - \cancel{1} = \frac{1}{2}$$

Finalmente el "valor medio" $\dfrac{\iint_T f(x,y)\,dx\,dy}{\iint_T dx\,dy} = \dfrac{3}{10} : \dfrac{1}{2} = \dfrac{6}{10} = \dfrac{3}{5}$ //

Practica "0"

Ejercicio 9

$x^2+y^2+(z-R)^2=R^2$

Voy a usar coordenadas esfericas para hallar el V

$$\left.\begin{aligned} x &= r\cos\theta\,\sin\varphi \\ y &= r\,sen\,\theta\,.\,sin\,\varphi \\ z &= r\cos\varphi + r \end{aligned}\right\} \quad J = r^2 \sin\varphi$$

$d = \lambda . z$

Los limites de integracion son los siguientes

$0 \leq r \leq R$

$0 \leq \theta \leq 2\pi$

$0 \leq \varphi \leq \pi/2$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi/2} \lambda(r\cos\varphi + r)\, r^2 \sin\varphi \, d\varphi . d\theta\, dr$$

$(\lambda r\cos\varphi + \lambda r)\, r^2 \sin\varphi$

$\lambda r^3 \sin\varphi . \cos\varphi + \lambda r^3 \sin\varphi$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi/2} (\lambda r^3 \cos\varphi . \sin\varphi + \lambda r^3 \sin\varphi)\, d\varphi . d\theta\, dr$$

Uso de identidad

- $\sin(2\varphi) = 2 \sin\varphi . \cos\varphi$

$\frac{\sin(2\varphi)}{2} = \sin\varphi . \cos\varphi$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi/2} \left(\frac{\lambda r^3}{2} \sin(2\varphi) + \lambda r^3 \sin\varphi\right) d\varphi\, d\theta\, dr$$

- $\int_0^{\pi/2} \sin(2\varphi)\, d\varphi = \frac{1}{2} \int \sin t\, dt = -\frac{1}{2} \cos 2\varphi \Big|_0^{\pi/2}$

$t = 2\varphi$

$dt = 2d\varphi$

$= -\frac{1}{2}(-1-1) = 1$ ✓

$\int_0^{\pi/2} \sin\varphi\, d\varphi = -\cos\varphi \Big|_0^{\pi/2} = -(0-1) = 1$ ✓

$$= \int_0^R \int_0^{2\pi} \left(\frac{\lambda r^3}{2} + \lambda r^3\right) d\theta\, dr = \int_0^R \int_0^{2\pi} \frac{3}{2} \lambda r^3\, d\theta\, dr = \int_0^r \frac{3}{\cancel{2}} \lambda r^3\, 2\pi\, dr = \int_0^R 3\lambda\pi r^3 dr$$

$$= \boxed{\frac{3}{4}\lambda\pi r^4}$$ masa de la esfera

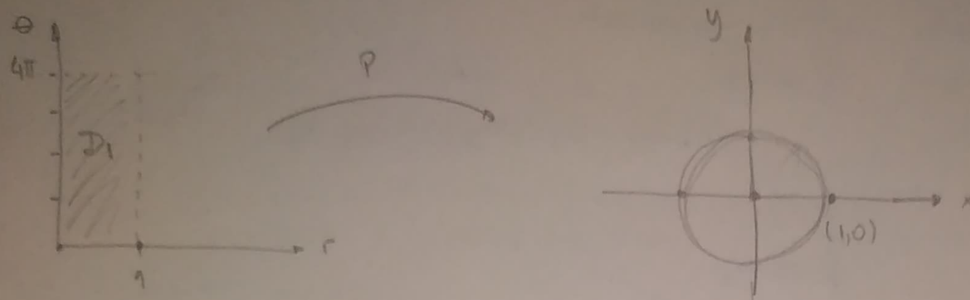
$\frac{4}{3}$

Ejercicio 13

$D_1 = \{(r,\theta) : 0 \le r \le 1 ;\ 0 \le \theta \le 4\pi\}$

$P(r,\theta) = (r\cos\theta, r\,\text{sen}\,\theta)$

a) Hallar $D = P(D_1)$



$x = r\cos\theta$

$y = r\,\text{sen}\,\theta$

$x^2+y^2 \le 1$

$x^2 \le 1-y^2$

$|x| \le \sqrt{1-y^2}$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$-1 \le y \le 1$

b) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2+y^2\,dx\,dy = \int_{-1}^{1} \frac{1}{3}x^3 + y^2x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy = \int_{-1}^{1} \frac{1}{3}\left[\left(\sqrt{1-y^2}\right)^3 - \left(-1\sqrt{1-y^2}\right)^3\right] + y\,2\sqrt{1-y^2}$

$\int_{-1}^{1} \frac{1}{3}\cdot 2\left(\sqrt{1-y^2}\right)^3 + 2y\sqrt{1-y^2}\,dy =$

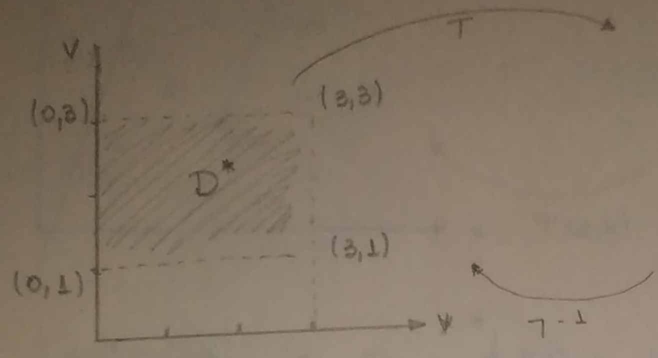
Por otro lado voy a calcular $\int_0^{4\pi}\int_0^{1} r^2 r\,dr\,d\theta = \int_0^{4\pi} \frac{1}{4}r^4\Big|_0^1 d\theta = \int_0^{4\pi} \frac{1}{4}\,d\theta = \pi$

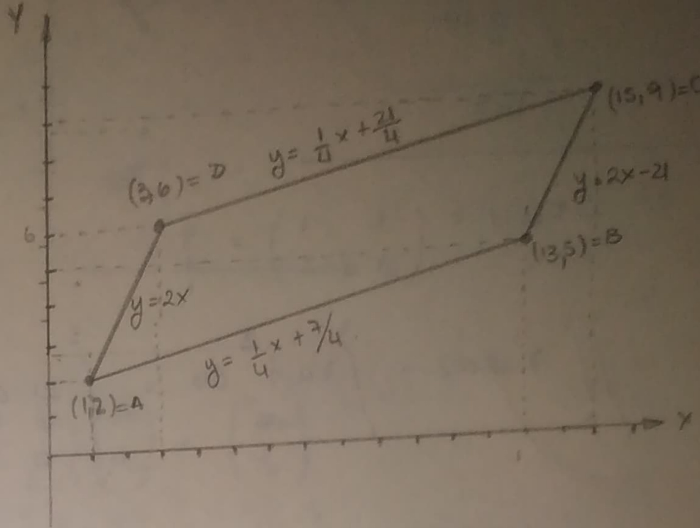
Practica "0"

EJERCICIO 11

$T(u,v) = (4u+v,\ u+2v)$

$D^*: [0,3] \times [1,3]$





$(4u+v,\ u+2v)|_{(0,1)} = (1,2)$

$(4u+v,\ u+2v)_{(0,3)} = (3,6)$

$(4u+v,\ u+2v)_{(3,1)} = (12+1,\ 3+2) = (13,5)$

$(4u+v;\ u+2v)_{(3,3)} = (12+3,\ 3+6) = (15,9)$

Busco las rectas

$\overline{AB}$: $m = \frac{5-2}{13-1} = \frac{3}{12} = \frac{1}{4}$

b: $2 = \frac{1}{4}1 + b \qquad b = 7/4$

$\overline{AD}$: $m = \frac{6-2}{3-1} = \frac{4}{2} = 2$

b: $2 = 2.1 + b \qquad b = 0$

$\overline{DC}$: $m = \frac{9-6}{15-3} = \frac{3}{12} = \frac{1}{4}$

b: $6 = \frac{1}{4}3 + b \qquad b = 21/4$

$\overline{BC}$: $m = \frac{9-5}{15-13} = \frac{4}{2} = 2$

b: $9 = 2\,15 + b \qquad b = -21$

* Para calcular el area, voy a calcular la distancia entre A y B. (BASE)

$d^2 = (13-1)^2 + (5-2)^2$

$d^2 = 144 + 9$

$\{d = \sqrt{153} \to$ BASE $\}$

* Busco la ALTURA, buscando una recta perpendicular a AB que pase por C

$6 = -4\,3 + b$

$6 + 12 = b \qquad y = -4x + 18$

$b = 18$

* Busco la interseccion de AB y $L_{(\perp AB)}$

$\frac{1}{4}x + \frac{7}{4} = -4x + 18$

$\frac{17}{4}x = \frac{65}{4}$

$x = \frac{65}{17} \longrightarrow y = \frac{46}{17}$

* Busco la distancia entre C y el punto de interseccion para saber la altura

$d^2 = \left(3 - \frac{65}{17}\right)^2 + \left(6 - \frac{46}{17}\right)^2$

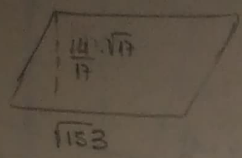
$d^2 = \frac{196}{289} + \frac{3136}{289}$

$d^2 = \frac{196}{17}$

$d = \frac{14}{\sqrt{17}} \frac{\sqrt{17}}{\sqrt{17}} \qquad \left\{d = \frac{14}{17}\sqrt{17}\right\}$ ALTURA

$$\text{Area} \; \square = b \cdot h = \sqrt{153} \cdot \frac{14}{17}\sqrt{17} = \sqrt{2601} \cdot \frac{14}{17}$$

$$= 42$$

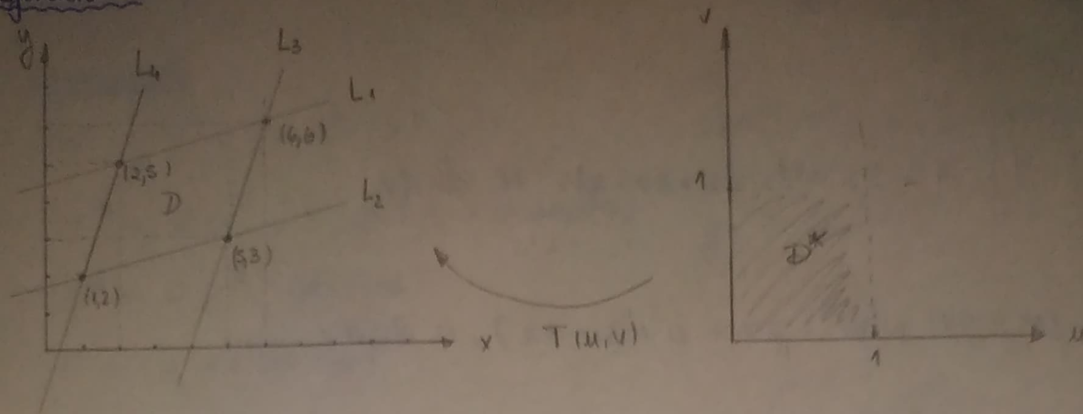


b) $DT(u,v) = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = 7$

$$\int_1^3\int_0^3 7\,du\,dv = \int_1^3 7u\Big|_0^3 dv = \int_1^3 21\,dv = 21v\Big|_1^3 = 63 - 21 = 42$$

Practica "0"

Ejercicio 12



$T(0,0) = (1,2)$

$T(1,0) = (5,3)$

$T(0,1) = (2,5)$

$T(1,1) = (6,6)$

$$T(u,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

$$T(u,v) = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Bigg\} \text{ TRANSFORMACIÓN LINEAL}$$

. Aplico la transformacion lineal para cada punto

$$(1,2) = \begin{pmatrix} a0 + b0 \\ c.0 + d.0 \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \qquad \text{Ecuacion I}$$

$$(5,3) = \begin{pmatrix} a1 + b0 \\ c1 + d0 \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \qquad \text{Ecuación II}$$

$$(2,5) = \begin{pmatrix} a0 + b1 \\ c0 + d1 \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \qquad \text{Ecuacion III}$$

$$(6,6) = \begin{pmatrix} a1 + b1 \\ c1 + d1 \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} a + b \\ c + d \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \qquad \text{Ecuación IV}$$

Ecuacion I $\quad u_0 = 1$

$v_0 = 2$.

Ecuacion II $\quad \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad a = 4, \; c = 1$

Ecuacion III $\quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad b = 1, \; d = 3$.

Compruebo con la ecuación IV $\quad \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 + 1 \\ 1 + 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓

Entonces la transformacion lineal $T(u,v) = (4u + v + 1, \; u + 3v + 2)$ ✓

$$DT(u,v) = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} = 11$$

a) $$\int_D xy\,dx\,dy = \int_0^1\int_0^1 (4u+v+1).(u+3v+2).\,11\,du\,dv$$

$$\int_0^1\int_0^1 (4u^2 + 12uv + 8u + vu + 3v^2 + 2v + u + 3v + 2).\,11\,du\,dv$$

$$\int_0^1\int_0^1 (4u^2 + 13uv + 9u + 5v + 3v^2 + 2).\,11\,du\,dv$$

$$\int_0^1\int_0^1 44u^2 + 143uv + 99u + 55v + 33v^2 + 22\,du\,dv$$

$$\int_0^1 \left(\frac{44}{3}u^3 + \frac{143}{2}u^2v + \frac{99}{2}u^2 + 55uv + 33v^2u + 22u\right)\Big|_0^1 dv$$

$$\int_0^1 \left(\frac{44}{3} + \frac{143}{2}v + \frac{99}{2} + 55v + 33v^2 + 22\right) dv$$

$$\int_0^1 \frac{517}{6} + \frac{253}{2}v + 33v^2\,dv = \left(\frac{517}{6}v + \frac{253}{4}v^2 + \frac{33}{3}v^3\right)\Big|_0^1$$

$$= \frac{1925}{12}$$

b) $$\int_D (x-y)\,dx\,dy = \int_0^1\int_0^1 (4u+v+1-u-3v-2).\,11\,du\,dv = \int_0^1\int_0^1 (3u-2v-1).\,11\,du\,dv$$

$$= \int_0^1 \left(\frac{33}{2}u^2 + 22vu - 11u\right)\Big|_0^1 dv = \int_0^1 \frac{33}{2} + 22v - 11\,dv$$

$$= \left(\frac{33}{2}v + \frac{22}{2}v^2 - 11v\right)\Big|_0^1 = \frac{33}{2} + 11 - 11 = \frac{33}{2}$$

Ejercicio 14

Lemniscata

$(x^2+y^2)^2 = 2a^2(x^2-y^2)$

Uso coordenadas polares

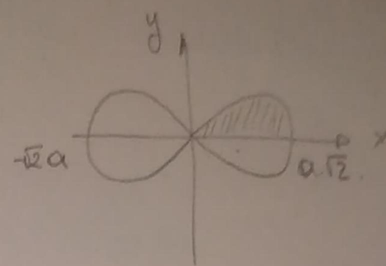
$x = r\cos\theta$

$y = r\sin\theta$

$r^4 = 2a^2(r^2\cos^2\theta - r\sin^2\theta)$

$r^2 = 2a^2\cos(2\theta)$

$r = a\sqrt{2\cos(2\theta)}$



- Si $r = a\sqrt{2}$ — $\theta = 0$
- Si $r = 0$ — $\theta = \frac{\pi}{4}$

$$4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{2\cos(2\theta)}} r\,dr\,d\theta = 4\int_0^{\pi/4}\frac{1}{2}r^2\Big|_0^{a\sqrt{2\cos(2\theta)}} = \frac{4}{2}\int_0^{\pi/4} a^2\, 2\cos(2\theta)\,d\theta$$

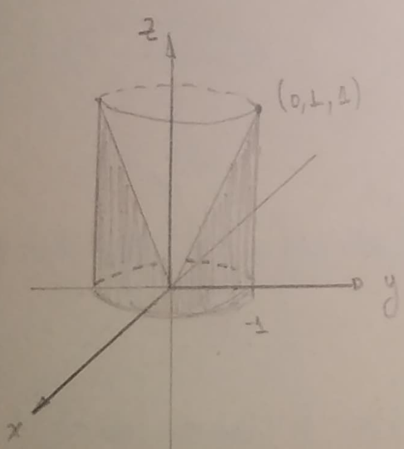
$t = 2\theta$

$dt = 2d\theta$

$\frac{1}{2}\int\cos t\,dt = \frac{1}{2}\sin 2\theta$

$$= 4a^2\frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4} = \boxed{2a^2}$$

Ejercicio 15



Voy a usar coordenadas cilíndricas.

$0 \le r \le 1$

$0 \le \theta < 2\pi$

$0 \le z \le r$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$J = r$

$$\int_0^1\int_0^{2\pi}\int_0^r z\,r\,dz\,d\theta\,dr = r\int_0^1\int_0^{2\pi}\frac{1}{2}z^2\Big|_0^r d\theta\,dr = \int_0^1\int_0^{2\pi}\frac{1}{2}r^3\,d\theta\,dr$$

$z = (x^2+y^2)^{1/2}$

$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

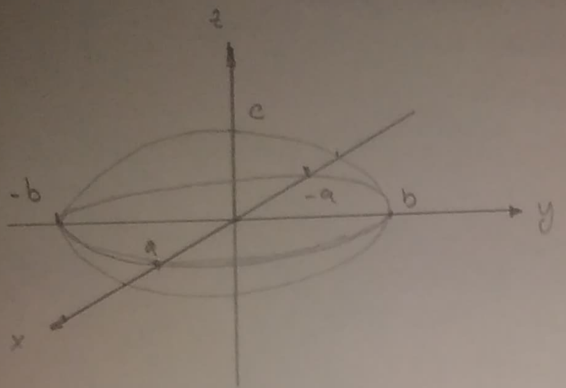
$z = \sqrt{r^2}$

$z = r$

$$\int_0^1\frac{1}{2}r^3 2\pi\,dr = \int_0^1 r^3\pi\,dr = \frac{1}{4}r^4\Big|_0^1\pi = \frac{1}{4}\pi$$

EJERCICIO 16

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$



Uso coordenadas elípticas

$$\begin{cases} x = a\,r \cos\theta \sin\varphi \\ y = b\,r \sin\theta \sin\varphi \\ z = c\,r \cos\varphi \end{cases} \qquad J = abc\,r^2 \sin\varphi$$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \varphi \leq \pi$

a) Volumen del elipsoide

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} abc\, r^2 \sin\varphi \, dr\, d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi} \frac{abc}{3} r^3 \sin\varphi \Big|_0^1 d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi} \frac{abc}{3} \sin\varphi \, d\theta\, d\varphi$$

$$= \int_0^{\pi} \frac{abc}{3} 2\pi \sin\varphi \, d\varphi = \frac{abc}{3} 2\pi \,(-\cos\varphi)\Big|_0^{\pi} = \frac{abc}{3} 2\pi . [-(-1-1)]$$

$$= \frac{abc}{3} 2\pi . 2 = \frac{4}{3} abc\pi$$

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b) $$\frac{(ar\cos\theta \sin\varphi)^2}{a^2} + \frac{(br\sin\theta \sin\varphi)^2}{b^2} + \frac{(cr\cos\varphi)^2}{c^2}$$

$$r^2 \cos^2\theta \sin^2\varphi + r^2 \sin^2\theta \sin^2\varphi + r^2\cos^2\varphi$$

$$r^2 \sin^2\varphi . (\cos^2\theta + \sin^2\theta) + r^2\cos^2\varphi$$

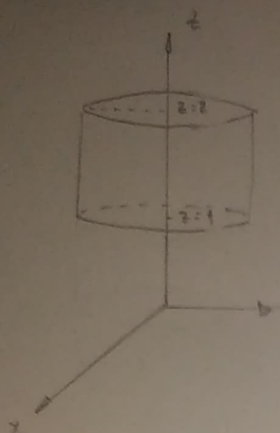
$$r^2 \sin^2\varphi + r^2 \cos^2\varphi = r^2$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2 .(abc\,r^2 \sin\varphi)\, dr\, d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi} \frac{abc}{5} \sin\varphi\, r^5 \Big|_0^1 d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi} \frac{abc}{5} \sin\varphi \, d\theta\, d\varphi$$

$$= \int_0^{\pi} \frac{abc}{5} \sin\varphi \, 2\pi \, d\varphi = 2\pi \frac{abc}{5} (-\cos\varphi) \Big|_0^{\pi} = 2\pi \frac{abc}{5} [-(\cos\pi - \cos 0)]$$

$$= 2\pi \frac{abc}{5} [-(-1-1)] = \frac{4\pi\, abc}{5}$$

Ejercicio 17



$\rho(x,y,z) = (x^2+y^2)\,z^2$

$z_{CM} = \dfrac{\int_V z\,\rho\,d\bar{x}}{\int \rho\,d\bar{x}} = \dfrac{\frac{15}{8}\pi}{\frac{7}{8}\pi} = \dfrac{15}{14}$

Centro de masa: $(0, 0, 15/14)$

Uso coordenadas cilíndricas

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$1 \leq z \leq 2$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right\}$ Jac $= r$

$\rho(x,y,z) = (x^2+y^2)\,z^2$

(I) $\int_1^2\int_0^{2\pi}\int_0^1 z\,(r^2 \cdot z^2)\; r\,dr\,d\theta\,dz = \int_1^2\int_0^{2\pi}\int_0^1 z^3 r^3\,dr\,d\theta\,dz$

$\int_1^2\int_0^{2\pi} \left.\frac{1}{4}z^3 r^4\right|_0^1 d\theta\,dz = \int_1^2 \frac{1}{4}z^3\,2\pi\,dz = \left.\frac{1}{2}\pi\,\frac{1}{4}z^4\right|_1^2$

$= \frac{1}{8}\pi\,(16-1) = \frac{15}{8}\pi$

(II) $\int_1^2\int_0^{2\pi}\int_0^1 (r^2 z^2)\; r\,dr\,d\theta\,dz = \int_1^2\int_0^{2\pi} \frac{1}{4}z^2\,d\theta\,dz = \int_1^2 \frac{2\pi}{4}z^2\,dz = \frac{2\pi}{4}\,\frac{1}{3}z^3 = \frac{1}{6}\pi\,(8-1) = \frac{7}{6}\pi$

Ejercicio 18