Análisis II - Análisis Matemático II - Matemática 3

Primer cuatrimestre de 2020

Práctica 3: Teorema de Green

Ejercicio 1. Verificar el teorema de Green para el disco D con centro $(0,0)$ y radio R y las siguientes funciones:

(a) $P(x,y) = xy^2, \quad Q(x,y) = -yx^2$.
(b) $P(x,y) = 2y, \quad Q(x,y) = x$.

Ejercicio 2. Verificar el teorema de Green y calcular $\int_{\mathcal{C}} y^2\,dx + x\,dy$, siendo $\mathcal{C}$ la curva recorrida en sentido positivo:

(a) Cuadrado con vértices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.
(b) Elipse dada por $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(c) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, donde $\mathcal{C}_1 : y = x$, $x \in [0,1]$, y $\mathcal{C}_2 : y = x^2$, $x \in [0,1]$.

Ejercicio 3. Usando el teorema de Green hallar el área de:

(a) El disco D con centro $(0,0)$ y radio R.
(b) La región dentro de la elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ejercicio 4. Sea D la región encerrada por el eje x y el arco de cicloide:

$$x = \theta - \sin\theta,\ y = 1 - \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

Usando el teorema de Green calcular el área de D.

Ejercicio 5. Hallar el área entre las curvas dadas en coordenadas polares por

$$r = 1 + \cos\theta, \quad -\pi \le \theta \le \pi,$$
$$r = \sqrt{\cos^2\theta - \operatorname{sen}^2\theta}, \quad -\pi/4 \le \theta \le \pi/4.$$

Ejercicio 6. Probar la fórmula de *integración por partes*: si $D \subset \mathbb{R}^2$ es un dominio elemental, ∂D su frontera orientada en sentido antihorario y $\mathbf{n} = (n_1, n_2)$ la normal exterior a D, entonces

$$\int_D u\,v_x\,dx\,dy = -\int_D u_x\,v\,dx\,dy + \int_{\partial D} u\,v\,n_1\,ds,$$

para todo par de funciones u, $v \in C(\bar{D}) \cap C^1(D)$.

Ejercicio 7. Sean P y Q funciones continuamente diferenciables en $\mathbb{R}^2$. Verificar que el teorema de Green para estas funciones es válido cuando la región D es el anillo

$$D = \left\{(x,y) : 1 \le x^2 + y^2 \le 4\right\}.$$

Sugerencia: aplicar el teorema de Green en los discos de radios 1 y 2.

Ejercicio 8. Sea $\mathcal{C}$ la curva

$$\begin{aligned}
&x = 0, && 0 \le y \le 4,\\
&y = 4, && 0 \le x \le 4,\\
&y = x, && 0 \le x \le 1,\\
&y = 2 - x, && 1 \le x \le 2,\\
&y = x - 2, && 2 \le x \le 3,\\
&y = 4 - x, && 2 \le x \le 3,\\
&y = x, && 2 \le x \le 4,
\end{aligned}$$

orientada positivamente. Calcular

$$\int_{\mathcal{C}} \frac{y}{(x-1)^2+y^2}\,dx + \frac{1-x}{(x-1)^2+y^2}\,dy.$$

Ejercicio 9. Sea $D = \{(x,y)\,:\,1 \leq x^2+y^2 \leq 4,\ x \geq 0\}$. Calcular

$$\int_{\partial D} x^2y\,dx - xy^2\,dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).

Ejercicio 10. Calcular el trabajo efectuado por el campo de fuerzas $\mathbf{F}(x,y) = (y+3x, 2y-x)$ al mover una partícula rodeando una vez la elipse $4x^2+y^2=4$ en el sentido de las agujas del reloj.

Ejercicio 11. Sea $\mathbf{F}(x,y) = \big(P(x,y), Q(x,y)\big) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$. Calcular $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ donde $\mathcal{C}$ es la circunferencia unitaria centrada en el origen orientada positivamente. Calcular $Q_x - P_y$. ¿Se satisface en este caso el teorema de Green?

Ejercicio 12. Calcular $\int_{\mathcal{C}} f_1dx + f_2dy$ siendo

$$f_1(x,y) = \frac{x\,\operatorname{sen}\left(\dfrac{\pi}{2(x^2+y^2)}\right) - y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}, \quad f_2(x,y) = \frac{y\,\operatorname{sen}\left(\dfrac{\pi}{2\,(x^2+y^2)}\right) + x(x^2+y^2)}{\left(x^2+y^2\right)^2},$$

y $\mathcal{C}$ la curva

$$\mathcal{C} = \begin{cases} y = x+1 & \text{si } -1 \leq x \leq 0, \\ y = 1-x & \text{si } \quad 0 \leq x \leq 1, \end{cases}$$

recorrida del $(-1,0)$ al $(1,0)$.

Ejercicio 13. Determinar todas las circunferencias $\mathcal{C}$ en el plano $\mathbb{R}^2$ sobre las cuales vale la igualdad

$$\int_{\mathcal{C}} -y^2\,dx + 3x\,dy = 6\pi.$$

Ejercicio 14. Calcular la integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$ donde

$$\mathbf{F}(x,y) = \left(y^2e^x + \cos x + (x-y)^2\,,\ 2\,y\,e^x + \operatorname{sen} y\right),$$

y $\mathcal{C}$ es la curva

$$x^2+y^2=1, \quad y \geq 0,$$

orientada de manera tal que comience en $(1,0)$ y termine en $(-1,0)$.

Ejercicio 15. Sean $u,\, v \in C^1(D)$, donde $D = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$. Consideremos los campos definidos por $\mathbf{F}(x,y) = (u(x,y), v(x,y))$, $\mathbf{G}(x,y) = (v_x - v_y, u_x - u_y)$. Calcular

$$\iint_D (\mathbf{F}\cdot\mathbf{G})(x,y)\,dx\,dy$$

sabiendo que sobre el borde de D se tiene $u(x,y) = x, \quad v(x,y) = 1$.

1) a) $P(x,y) = xy^2$, $Q(x,y) = -yx^2$, $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq R^2\} \Rightarrow \sigma(t) = (R\cos(t), R\,\text{sen}(t))$ con $t \in [0, 2\pi]$

$$\Rightarrow \int_{C^+} P\,dx + Q\,dy = \int_0^{2\pi} \langle (P(\sigma(t)), Q(\sigma(t))), \sigma'(t) \rangle\, dt = \int_0^{2\pi} \langle (R^3 \cos t\, \text{sen}^2 t, -R^3 \text{sen}\, t \cos^2 t)(-R\,\text{sen}\, t, R\cos t) \rangle\, dt$$

$$= -\int_0^{2\pi} R^4 \cos t\, \text{sen}^3 t + R^4 \text{sen}\, t \cos^3 t\; dt = -\int_0^{2\pi} R^4 \cos t\, \underline{\text{sen}\, t}\; dt = -R^4 \int_0^0 u\, du = -R^4 \left.\frac{u^2}{2}\right|_0^0 = -R^4 \left.\frac{\text{sen}^2 t}{2}\right|_0^{2\pi} = 0$$

$u = \text{sen}\, t$
$du = \cos t\, dt$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right) dx\,dy = \int_{-R}^{R} \int_{-\sqrt{R^2-R^2}}^{\sqrt{R^2-x^2}} (-2yx - 2yx)\, dx\,dy = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} -4xy\, dx\,dy = \int_{-R}^{R} -4x \left.\frac{y^2}{2}\right|_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dx$$

$$= \int_{-R}^{R} -4x \left(\cancel{\frac{R^2-x^2}{2} - \frac{R^2-x^2}{2}}^{\,0} \right) dx = 0$$

b) $P(x,y) = 2y$, $Q(x,y) = x$

$$\Rightarrow \int_{C^+} P\,dx + Q\,dy = \int_0^{2\pi} \langle (P(\sigma(t)), Q(\sigma(t))), \sigma'(t) \rangle\, dt = \int_0^{2\pi} \langle (2R\,\text{sen}\, t, R\cos t), (-R\,\text{sen}\, t, R\cos t) \rangle\, dt = \int_0^{2\pi} (-2R^2 \text{sen}^2 t + R^2 \cos^2 t)\, dt$$

$$= \int_0^{2\pi} R^2 (\cos^2 t - 2\,\text{sen}^2 t)\, dt = \int_0^{2\pi} R^2 (\cos^2 t - 2(1-\cos^2 t))\, dt = R^2 \int_0^{2\pi} (3\cos^2 t - 2)\, dt$$

$$= R^2 \left(\int_0^{2\pi} \frac{3}{2}\cos(2t) + \frac{3}{2}\, dt - \int_0^{2\pi} 2\, dt \right) = R^2 \left(\cancel{\left.\frac{3}{4}\text{sen}(2t)\right|_0^{2\pi}}^{\,0} + \left.\frac{3}{2} t\right|_0^{2\pi} - \left. 2t \right|_0^{2\pi} \right) = R^2 (3\pi - 4\pi) = -\pi R^2$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right) dx\,dy = \iint (1-2)\, dx\,dy = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} -1\, dx\,dy = -\int_0^R \int_0^{2\pi} -r\,.\, d\theta\, dr = 2\pi \left.\left(-\frac{r^2}{2}\right)\right|_0^R = -\pi R^2$$

uso coordenadas polares

(2) $\int_C y^2dx + x\,dy \Rightarrow P(x,y)=y^2 \,, Q(x,y)=x$

a) $D=\{(x,y)/\ 0\le x\le 2 \,, 0\le y\le 2\}$ $\quad \sigma_1(t)=(t,0)$ con $t\in[0,2]$, $\sigma_2(t)=(2,t)$ con $t\in[0,2]$, $\sigma_3(t)=(t,2)$ con $t\in[0,2]$, $\sigma_4(t)=(0,t)$ con $t\in[0,2]$

$$\Rightarrow \int_C Pdx+Qdy = \int_0^2 \langle(0,t),(1,0)\rangle dt + \int_0^2 \langle(t^2,2),(0,1)\rangle dt - \int_0^2 \langle(4,t),(1,2)\rangle dt - \int_0^2 \langle(t,0),(0,1)\rangle dt = \int_0^2 0\,dt + \int_0^2 2\,dt - \int_0^2 4\,dt + \int_0^2 0\,dt = 2t\Big|_0^2 - 4t\Big|_0^2 = 4-8 = -4$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right)dxdy = \int_0^2\int_0^2 (1-2y)\,dxdy = \int_0^2\int_0^2 1\,dxdy - 2\int_0^2\int_0^2 y\,dxdy = \int_0^2 x\Big|_0^2 dy - 2\int_0^2 yx\Big|_0^2 dy = 2y\Big|_0^2 - 4\frac{y^2}{2}\Big|_0^2$$

$$= 4-4.2 = -4$$

b) $D=\{(x,y)/\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\} \Rightarrow C=\partial D:\ \sigma_1(t)=(a\cos t, b\,sen\,t)$ con $t\in[0,2\pi]$

$$\Rightarrow \int_C Pdx+Qdy = \int_0^{2\pi}(b^2sen^2t, a\cos t)(-a\,sen\,t, b\cos t)\,dt = \int_0^{2\pi}(-ab^2.sen^3t + ab\cos^2 t)\,dt = \int_0^{2\pi} -ab^2 \underbrace{sen^3 t}_{sen^2t\,.\,sen\,t}\,dt + \int_0^{2\pi} ab\cos^2 t\,dt$$

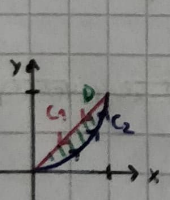
$$= ab\int_0^{2\pi}\frac{1}{2}(\cos(2t)+1)\,dt - ab^2\int_0^{2\pi} sen\,t.\left(\frac{1-\cos(2t)}{2}\right) = \frac{ab}{2}.\left(\cancel{\frac{sen(2t)}{2}}^{0}\Big|_0^{2\pi} + t\Big|_0^{2\pi}\right) - \frac{ab^2}{2}\int_0^{2\pi}\frac{sen\,t}{2} - \underbrace{\frac{sen\,t\cos(2t)}{2}}_{\frac{sen(3t)-sen\,t}{4}}\,dt$$

$$= ab\pi - \frac{ab^2}{2}\left(\int_0^{2\pi}\frac{3\,sen\,t}{4}\,dt - \int_0^{2\pi}\frac{sen(3t)}{4}\right) = ab\pi - \frac{ab^2}{2}\left(\cancel{-\frac{3}{4}\cos t}^{0}\Big|_0^{2\pi} + \cancel{\frac{\cos(3t)}{12}}^{0}\Big|_0^{2\pi}\right) = ab\pi$$

$$\Rightarrow \iint_D (Q_x-P_y)\,dxdy = \iint (1-2y)\,dxdy \overset{\text{uso coordenadas elípticas}}{=} \int_0^1\int_0^{2\pi} abr.(1-2b\,sen\theta)\,d\theta dr = \int_0^1\int_0^{2\pi} abr\,d\theta dr - \int_0^1\int_0^{2\pi} 2ab^2 sen\theta\,drd\theta = 2\pi ab\frac{r^2}{2}\Big|_0^1 - 2ab^2\cancel{(-\cos\theta)}^{0}\Big|_0^{2\pi}$$

$$= 2\pi ab.\frac{1}{2} = ab\pi$$

c)



$D=\{(x,y)/\ 0\le x\le 1 \,, x^2\le y\le x\} \Rightarrow C_1: \sigma_1(t)=(t,t)$ con $t\in[0,1]$, $C_2: \sigma_2(t)=(t,t^2)$ con $t\in[0,1]$

$$\Rightarrow \int_C Pdx+Qdy = \int_{C_1^- + C_2^+} Pdx+Qdy = -\int_{C_1^-}(t^2,t)(1,1)\,dt + \int_{C_2^+}(t^3,t)(1,2t)\,dt = -\int_0^1(t^2+t)\,dt + \int_0^1(t^3+2t^2)\,dt$$

$$= \frac{t^4}{4}\Big|_0^1 + 2\frac{t^3}{3}\Big|_0^1 - \frac{t^2}{2}\Big|_0^1 - \frac{t^3}{3}\Big|_0^1 = \frac{1}{4}+\frac{2}{3}-\frac{1}{2}-\frac{1}{3} = 1/12$$

$$\Rightarrow \iint_D (Q_x-P_x)\,dxdy = \int_0^1\int_{x^2}^{x} (1-2y)\,dxdy = \int_0^1 (y-y^2)\Big|_{x^2}^{x} dx = \int_0^1(x-x^2-(x^2-x^4))\,dx = \int_0^1(x^4-2x^2+x)\,dx = \frac{x^5}{5}\Big|_0^1 - \frac{2}{3}x^3\Big|_0^1 + \frac{x^2}{2}\Big|_0^1$$

$$= \frac{1}{5}-\frac{2}{3}+\frac{1}{2} = 1/30$$

$\Rightarrow 1/12 \ne 1/30 \Rightarrow D$ es una región tipo 2, no una región tipo 3

③ a) Defino $P(x,y)=-y$, $Q(x,y)=x$ $\Rightarrow \left(\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right)=2 \Rightarrow$ de forma que $\iint_D 2\,dx\,dy=\int_C -y\,dx+x\,dy = 2\cdot\text{Area}(D)$

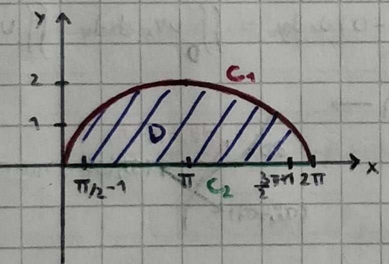
$D=\{(x,y)/\ x^2+y^2\le R^2\} \Rightarrow \sigma(t)=(R\cos t, R\,\text{sen}\,t)$ con $t\in[0,2\pi]$

$$\Rightarrow \frac{1}{2}\int_C -y\,dx+x\,dy = \frac{1}{2}\int_0^{2\pi} <(-R\,\text{sen}\,t, R\cos t),(-R\,\text{sen}\,t, R\cos t)>dt = \frac{1}{2}\int_0^{2\pi}(R^2\text{sen}^2t+R^2\cos^2t)\,dt = \frac{1}{2}\int_0^{2\pi}R^2\,dt = \frac{1}{2}R^2t\Big|_0^{2\pi} = \pi R^2$$

b) $D=\{(x,y)/\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\} \Rightarrow \sigma(t)=(a\cos t, b\,\text{sen}\,t)$ con $t\in[0,2\pi]$

$$\Rightarrow \frac{1}{2}\int_C -y\,dx+x\,dy = \frac{1}{2}\int_0^{2\pi} <(-b\,\text{sen}\,t, a\cos t),(-a\cos t, b\cos t)>dt = \frac{1}{2}\int_0^{2\pi}(ab\,\text{sen}^2t+ab\cos^2t)\,dt = \frac{1}{2}\int_0^{2\pi}ab\,dt = ab\pi$$

④



$\Rightarrow C_1: \sigma_1(t)=(t-\text{sen}\,t,\ 1-\cos t)$ con $t\in[0,2\pi]$

$C_2: \sigma_2(t)=(t,0)$ con $t\in[0,2\pi]$

$\Rightarrow$ Defino $P(x,y)=-y$, $Q(x,y)=x$

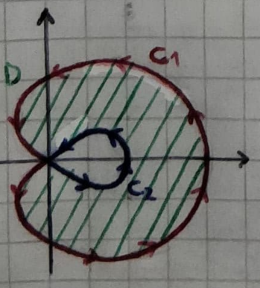
$$\Rightarrow \text{Area}(D)=\frac{1}{2}\int_{C_1\cup C_2}-y\,dx+x\,dy = -\frac{1}{2}\int_{C_1^-}-y\,dx+x\,dx+\frac{1}{2}\int_{C_2^+}-y\,dx+x\,dx = \frac{-1}{2}\int_0^{2\pi}<(\cos t-1, t-\text{sen}\,t),(1-\cos t, \text{sen}\,t)>dt+\frac{1}{2}\int_0^{2\pi}<(0,t),(1,0)>dt$$

$$= -\frac{1}{2}\int_0^{2\pi}(\cos t-1)(1-\cos t)+(t\,\text{sen}\,t-\text{sen}^2t)\,dt = \frac{-1}{2}\int_0^{2\pi}(\cos t-\cos^2t-1+\cos t+t\,\text{sen}\,t-\text{sen}^2t)\,dt$$

$$= -\frac{1}{2}\int_0^{2\pi}(2\cos t-1-1+t\,\text{sen}\,t)\,dt = \frac{-1}{2}\left(2\,\text{sen}\,t\Big|_0^{2\pi}-2t\Big|_0^{2\pi}+(\text{sen}\,t-t\cos t)\Big|_0^{2\pi}\right) = -\frac{1}{2}(-4\pi-2\pi) = 3\pi$$

⑤ $C_1: \sigma_1(t)=((1+\cos t)\cos t, (1+\cos t)\,\text{sen}\,t)$ con $t\in[-\pi,\pi]$

$C_2: \sigma_2(t)=(\sqrt{\cos^2t-\text{sen}^2t}\cos t, \sqrt{\cos^2t-\text{sen}^2t}\,\text{sen}\,t)$ con $t\in[-\pi/4,\pi/4]$



$$\text{Area}(D)=\frac{1}{2}\int_{C_1^+\cup C_2^-}-y\,dx+x\,dy = \frac{1}{2}\int_{C_1^+}-y\,dx+x\,dx-\int_{C_2^-}-y\,dx+x\,dy$$

$\Rightarrow$ Viendo que tanto σ_1 como σ_2 son de la forma $(r(t)\cos t, r(t)\,\text{sen}\,t)$:

$$\int_{C_1/C_2}-y\,dx+x\,dx = \int_{-\pi}^{\pi}(-r(t)\,\text{sen}\,t, r(t)\cos t)(r'(t)\cos t-r(t)\,\text{sen}\,t, r'(t)\,\text{sen}\,t+r(t)\cos t)\,dt$$

$$= \int_{-\pi}^{\pi}(-r(t)\cancel{r'(t)\cos t}\,\text{sen}\,t+r^2(t)\text{sen}^2t+r(t)\cancel{r'(t)\cos t}\,\text{sen}\,t+r^2(t)\cos^2t)\,dt$$

$$= \int_{-\pi}^{\pi}r^2(t)(\cos^2t+\text{sen}^2t)\,dt = \int_{-\pi}^{\pi}r^2(t)\,dt$$

$$\Rightarrow \int_{C_1} -y\,dx + x\,dy = \int_{-\pi}^{\pi} (1+\cos t)^2\,dt = \int_{-\pi}^{\pi} (1 + 2\cos t + \underbrace{\cos^2 t}_{\frac{1}{2}(\cos(2x)+1)})\,dt = t\Big|_{-\pi}^{\pi} + \cancel{2\,\operatorname{sen} t\Big|_{-\pi}^{\pi}}^{\,0} + \cancel{\frac{\operatorname{sen}(2t)}{4}\Big|_{-\pi}^{\pi}}^{\,0} + \frac{t}{2}\Big|_{-\pi}^{\pi} = 2\pi + \pi = 3\pi$$

$$\Rightarrow \int_{C_2} -y\,dx + x\,dx = \int_{-\pi/4}^{\pi/4} (\cancel{\sqrt{}}\cos^2 t - \operatorname{sen}^2 t)^{\cancel{2}}\,dt = \int_{-\pi/4}^{\pi/4} \underbrace{\cos^2 t}_{\frac{1}{2}(\cos(2t)+1)}\,dt - \int_{-\pi/4}^{\pi/4} \underbrace{\operatorname{sen}^2 t\,dt}_{\frac{1}{2}(1-\cos(2t))} = \frac{1}{2}\left(\frac{\operatorname{sen}(2t)}{2}\Big|_{-\pi/4}^{\pi/4} + \cancel{t\Big|_{-\pi/4}^{\pi/4}} + \frac{\operatorname{sen}(2t)}{2}\Big|_{-\pi/4}^{\pi/4} - \cancel{t\Big|_{-\pi/4}^{\pi/4}} \right)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \left(-\frac{1}{2}\right) + \frac{1}{2} - \left(-\frac{1}{2}\right)\right) = 1$$

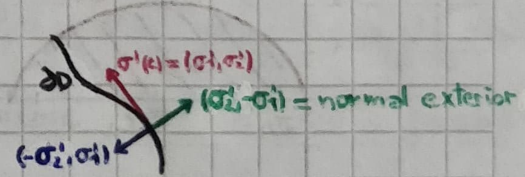
$$\Rightarrow \text{Area}(D) = \frac{1}{2}\int_{C_1} -y\,dx + x\,dy - \int_{C_2} -y\,dx + x\,dy = \frac{3\pi}{2} - \frac{1}{2}$$

(6)

$$\iint_D u v_x\,dxdy = -\iint_D u_x v\,dxdy + \int_{\partial D} u v n_1\,ds \Rightarrow \iint_D u v_x\,dx\,dy + \iint_D u_x v\,dxdy = \iint_D \underbrace{(u v_x + u_x v)}_{\frac{\partial(uv)}{\partial x}}\,dxdx$$

$$\Rightarrow \text{Considero } F(x,y) = (\underbrace{0}_{P}, \underbrace{u.v}_{Q}) \Rightarrow \int_{C=\partial D} F\,ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D (u v_x + u_x v - 0)\,dxdy = \iint_D u v_x\,dxdy + \iint u_x v\,dxdy$$

$\Rightarrow$ Defino parametrización de ∂D: $\sigma(t) = (\sigma_1, \sigma_2)$ con $t \in [a,b]$ $\Rightarrow \sigma'(t) = (\sigma_1', \sigma_2') \longrightarrow$



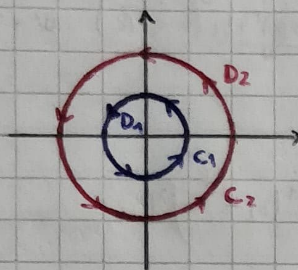
$$\Rightarrow \int_{\partial D} F\,ds = \int_a^b (0, uv(\sigma(t)))(\sigma_1', \sigma_2')\,dt = \int_a^b uv(\sigma(t))\cdot\sigma_2'\,dt$$

$$\Rightarrow n = (n_1, n_2) = \frac{(\sigma_2', -\sigma_1')}{\|\sigma'(t)\|} \Rightarrow n_1 = \frac{\sigma_2'}{\|\sigma'(t)\|}$$

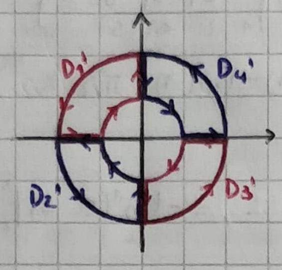
$$\Rightarrow \int_a^b uv(\sigma(t)).\,\sigma_2'\,.\,\frac{\|\sigma'(t)\|}{\|\sigma'(t)\|}\,dt = \int_a^b \underbrace{uv(\sigma(t)).\,n_1}_{F}\,\underbrace{\|\sigma'(t)\|\,dt}_{ds}$$

(7) P, Q funciones C^1 en $\mathbb{R}^2$, $D = \{(x,y) / 1 \leq x^2+y^2 \leq 4\} \Rightarrow$ Divido en D_1 y D_2

$\Rightarrow D_1 = \{(x,y) / x^2+y^2 \leq 1\}$, $D_2 = \{(x,y) / x^2+y^2 \leq 4\}$



$\Rightarrow$



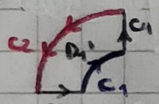
$\Rightarrow D = D_1' \cup D_2' \cup D_3' \cup D_4' \rightarrow$ Divido el anillo en 4 regiones tipo III y las recorro en sentido positivo

$\Rightarrow$ Veo que los bordes rectos los recorro 2 veces, 1 en cada sentido de forma que se cancelan, y resultan una curva que recorre el borde interior en sentido horario y una que recorre el borde exterior en sentido antihorario

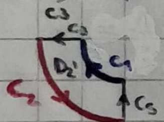
$\Rightarrow C_1: \sigma_1(t) = (\cos t, \operatorname{sen} t)$ con $t \in [0, 2\pi]$

$C_2: \sigma_2(t) = (2\cos t, 2\operatorname{sen} t)$ con $t \in [0, 2\pi]$

$\Rightarrow$



$\Rightarrow \iint_{D_1'} (Q_x - P_y)\,dx\,dy = \int_{C_2^+ \cup C_3^+ \cup C_1^- \cup C_4^+} P\,dx + Q\,dy = \int_{C_2^+} P\,dx + Q\,dy + \int_{C_3^+} P\,dx + Q\,dy - \int_{C_1^-} P\,dx + Q\,dy + \int_{C_4^+} P\,dx + Q\,dy$ (+)



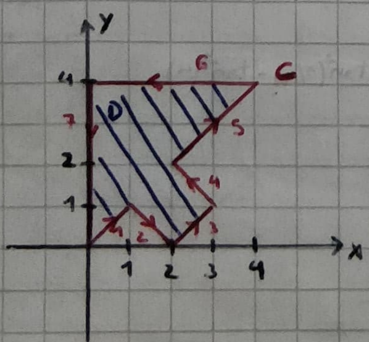
$\Rightarrow \iint_{D_2'} (Q_x - P_y)\,dx\,dy = \int_{C_2^+ \cup C_3^+ \cup C_1^- \cup C_5^-} P\,dx + Q\,dy = \int_{C_2^+} P\,dx + Q\,dy + \int_{C_3^+} P\,dx + Q\,dy - \int_{C_1^-} P\,dx + Q\,dy - \int_{C_5^-} P\,dx + Q\,dy$ (−)

$\Rightarrow$ Como $D = D_1' \cup D_2' \cup D_3' \cup D_4' \Rightarrow \iint_D (Q_x - P_y)\,dx\,dy = \iint_{D_1'} (Q_x - P_y)\,dx\,dy + \iint_{D_2'} (Q_x - P_y)\,dx\,dy + \iint_{D_3'} (Q_x - P_y)\,dx\,dy + \iint_{D_4'} (Q_x - P_x)\,dx\,dy$

$\Rightarrow$ Al sumar las integrales de las 4 divisiones de D, las curvas horizontales y verticales que usamos para cerrar dichas divisiones son recorridas 2 veces cada una, una vez con orientación positiva y otra negativa, por lo que las integrales de curva se cancelan, y solo me quedan los terminos que corresponden a partes de C_1 (orientada positivamente) y C_2 (orientada negativamente)

$\Rightarrow \iint_D (Q_x - P_y)\,dx\,dy = \int_{C_2^+} P\,dx + Q\,dy - \int_{C_1^-} P\,dx + Q\,dx$

(8)



$\Rightarrow \int_C P\,dx + Q\,dy \Rightarrow P = \frac{y}{(x-1)^2+y^2}$, $Q = \frac{1-x}{(x-1)^2+y^2} \Rightarrow F = (P, Q)$

$\Rightarrow$ Primero me fijo que F este definido en $D \subset \mathbb{R}^2$

$\Rightarrow P$ y Q no estan definidas en el punto $(1,0) \rightarrow (1-1)^2 + 0^2 = 0$

Pero el punto $(1,0)$ no esta dentro de D, por lo que F esta definido en D

$\Rightarrow$ Divido C en 7 curvas concatenadas: $\sigma_1(t) = (t,t)$ con $t \in [0,1]$, $\sigma_2(t) = (t, 2-t)$ con $t \in [1,2]$, $\sigma_3(t) = (t, t-2)$ con $t \in [2,3]$

$\sigma_4(t) = (t, 4-t)$ con $t \in [2,3]$, $\sigma_5(t) = (t,t)$ con $t \in [2,4]$, $\sigma_6(t) = (t,4)$ con $t \in [0,4]$, $\sigma_7(t) = (0,t)$ con $t \in [0,4]$

$$\Rightarrow \int_C P\,dx + Q\,dx = \int_{C_1\cup C_2\cup C_3\cup C_4\cup C_5\cup C_6\cup C_7} P\,dx + Q\,dx = \underbrace{\int_{C_1^+} Pdx+Qdy}_{1} + \underbrace{\int_{C_2^+} Pdx+Qdy}_{2} + \underbrace{\int_{C_3^+} Pdx+Qdy}_{3} - \underbrace{\int_{C_4^-} Pdx+Qdx}_{4} + \underbrace{\int_{C_5^+} Pdx+Qdx}_{5} - \underbrace{\int_{C_6^-} Pdx+Qdx}_{6} - \underbrace{\int_{C_7^-} Pdx+Qdx}_{7}$$

$$1 = \int_0^1 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2}\right)(1,1)\,dt = \int_0^1 \frac{t+(1-t)}{(t-1)^2+t^2}\,dt = \int_0^1 \frac{1}{2t^2-2t+1}\,dt = \int_0^1 \frac{1}{(\underbrace{\sqrt{2}t-1/\sqrt{2}}_{u})^2+1/2}\,dt = \int \frac{1}{\sqrt{2}}\cdot\frac{1}{u^2+1/2}\,du = \frac{1}{\sqrt{2}}\int \frac{2}{2u^2+1}\,du$$

$$= \sqrt{2}\int \frac{1}{\underbrace{2u^2+1}_{s=\sqrt{2}u}}\,du = \cancel{\sqrt{2}}\int \frac{1}{\cancel{\sqrt{2}}}\frac{1}{s^2+1}\,ds = \tan^{-1}(s)\Big|_{\sqrt{2}u} = \tan^{-1}(\sqrt{2}u)\Big|_{\sqrt{2}t-1/\sqrt{2}} = \tan^{-1}(2t-1)\Big|_0^1 = \tan^{-1}(1)-\tan^{-1}(-1) = 2\tan^{-1}(1) = 2\cdot\frac{\pi}{4} = \pi/2$$

$$2 = \int_1^2 \left(\frac{2-t}{(t-1)^2+(2-t)^2}, \frac{1-t}{(t-1)^2+(2-t)^2}\right)(1,-1)\,dt = \int_1^2 \frac{1}{(t-1)^2+(2-t)^2}\,dt = \int_1^2 \frac{1}{2t^2-6t+5}\,dt = \int_1^2 \frac{1}{(\underbrace{\sqrt{2}t-3/\sqrt{2}}_{u})^2+1/2}\,dt = \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1/2}\,du \overset{s=\sqrt{2}u}{=} \int \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1}(s)\Big|_{\sqrt{2}u} = \tan^{-1}(\sqrt{2}u)\Big|_{\sqrt{2}t-3/\sqrt{2}} = \tan^{-1}(2t-3)\Big|_1^2 = \tan^{-1}(1)-\tan^{-1}(-1) = \pi/2.$$

$$3 = \int_2^3 \left(\frac{t-2}{(t-1)^2+(t-2)^2}, \frac{1-t}{(t-1)^2+(t-2)^2}\right)(1,1)\,dt = \int_2^3 \frac{-1}{2t^2-6t+5}\,dt = \dots = -\tan^{-1}(2t-3)\Big|_2^3 = -\tan^{-1}(3)+\tan^{-1}(1)$$

$$4 = \int_2^3 \left(\frac{4-t}{(t-1)^2+(4-t)^2}, \frac{1-t}{(t-1)^2+(4-t)^2}\right)(1,-1)\,dt = \int_2^3 \frac{3}{2t^2-10t+17}\,dt = \dots = 2\tan^{-1}(1/3)$$

$$5 = \int_2^4 \left(\frac{t}{(t-1)^2+t^2}, \frac{1-t}{(t-1)^2+t^2}\right)(1,1)\,dt = \int_2^4 \frac{1}{2t^2-2t+1}\,dt = \dots = \tan^{-1}(2t-1)\Big|_2^4 = \tan^{-1}(7)-\tan^{-1}(3)$$

$$6 = \int_0^4 \left(\frac{4}{(t-1)^2+4^2}, \frac{1-t}{(t-1)^2+4^2}\right)(1,0)\,dt = \int_0^4 \frac{4}{t^2-2t+17}\,dt = \int_0^4 \frac{4}{(\underbrace{t-1}_{u})^2+16}\,dt = \int_{-1}^3 \frac{4}{u^2+16}\,du = \int_{-1}^3 \frac{4}{16}\frac{1}{\underbrace{\frac{u^2}{16}}_{s=u/4}+1}\,du = \frac{\cancel{1}}{\cancel{4}}\int_{-1/4}^{3/4} \frac{1}{s^2+1}\,ds$$

$$= \frac{\cancel{1}}{\cancel{4}}\tan^{-1}(s)\Big|_{-1/4}^{3/4} = \frac{\cancel{1}}{\cancel{4}}\left(\tan^{-1}(3/4)-\tan^{-1}(-1/4)\right) = \tan(3/4)+\tan(1/4)$$

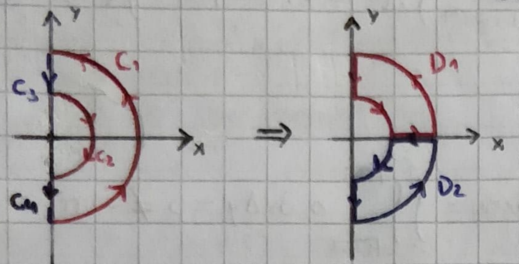
$$7 = \int_0^4 \left(\frac{t}{(0-1)^2+t^2}, \frac{1}{(0-1)^2+t^2}\right)(0,1)\,dt = \int_0^4 \frac{1}{t^2+1}\,dt = \tan^{-1}(t)\Big|_0^4 = \tan^{-1}(4)$$

$$\Rightarrow \int_C Pdx+Qdy = \pi/2+\pi/2-\tan^{-1}(3)+\pi/4-2\tan^{-1}(1/3)+\tan^{-1}(7)-\tan^{-1}(3)-\tan^{-1}(3/4)-\tan^{-1}(1/4)-\tan^{-1}(4) = 0$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \iint_D \left(\frac{\partial\left(\frac{x-1}{(x-1)^2+y^2}\right)}{\partial x} - \frac{\partial\left(\frac{y}{(x-1)^2+y^2}\right)}{\partial y}\right)dxdy = \iint_D \left(\frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2} - \frac{x^2-2x+1-y^2}{((x-1)^2+y^2)^2}\right)dxdy = 0$$

$\Rightarrow$ MUCHO mas fácil

9) $D=\{(x,y)/1\le x^2+y^2\le 4,\ x\ge 0\} \Rightarrow \int_{\partial D} x^2y\,dx - xy^2\,dy \Rightarrow P=x^2y,\ Q=-xy^2$



$\Rightarrow$ Divido D en 2 regiones tipo 3. Veo que el límite horizontal es recorrido por ambas curvas, en sentido contrario. Por lo tanto se cancela.

$\Rightarrow$ Veo que la curva resultante que recorre la región recorre el borde exterior en sentido antihorario y el borde interior en sentido horario

$\Rightarrow$ Divido el recorrido de la región en 4 curvas concatenadas: $\sigma_1(t)=(2\cos t, 2\,\mathrm{sen}\,t)$ con $t\in[-\pi/2,\pi/2]$, $\sigma_2(t)=(\cos t,\mathrm{sen}\,t)$ con $t\in[-\pi/2,\pi/2]$

$\sigma_3(t)=(0,t)$ con $t\in[1,2]$, $\sigma_4(t)=(0,t)$ con $t\in[-2,-1]$

$$\Rightarrow \int_{C=\partial D} P\,dx+Q\,dy = \int_{C_1\cup C_2\cup C_3\cup C_4} P\,dx+Q\,dy = \int_{-\pi/2}^{\pi/2}(4\cos^2 t.2\,\mathrm{sen}\,t, -2\cos t.4\,\mathrm{sen}^2 t)(-2\,\mathrm{sen}\,t, 2\cos t)\,dt - \int_{-\pi/2}^{\pi/2}(\cos^2 t\,\mathrm{sen}\,t, \cos t\,\mathrm{sen}^2 t)(-\mathrm{sen}\,t,\cos t)\,dt + \int_{C_3\cup C_4} P\,dx+Q\,dy$$

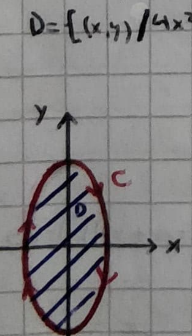
$$= \int_{-\pi/2}^{\pi/2}(-16\cos^2 t.\mathrm{sen}^2 t - 16\cos^2 t.\mathrm{sen}^2 t)\,dt - \int_{-\pi/2}^{\pi/2}(-\cos^2 t\,\mathrm{sen}^2 t + \cos^2 t\,\mathrm{sen}^2 t)\,dt - \int_1^2 0.(0,1)\,dt - \int_{-2}^{-1} 0(0,1)\,dt$$

$$= \int_{-\pi/2}^{\pi/2} -36\underbrace{\cos^2 t.\mathrm{sen}^2 t}_{\frac{1}{4}\mathrm{sen}^2(2x)}\,dt + \int_{-\pi/2}^{\pi/2} 2\cos^2 t\,\mathrm{sen}^2 t\,dt - 0 - 0 = -\frac{36}{4}\int_{-\pi/2}^{\pi/2}\underbrace{\mathrm{sen}^2(2t)}_{\frac{1}{2}-\frac{1}{2}\cos(4t)}\,dt + \int_{-\pi/2}^{\pi/2}\frac{\mathrm{sen}^2(2t)}{2}\,dt$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1}{4}-\frac{1}{4}\cos(4t)\,dt - 8\int_{-\pi/2}^{\pi/2}\frac{1}{2}-\frac{1}{2}\cos(4t)\,dt = \frac{1}{4}t\Big|_{-\pi/2}^{\pi/2} - \cancel{\frac{1}{16}\mathrm{sen}(4t)\Big|_{-\pi/2}^{\pi/2}}^{0} - 4t\Big|_{-\pi/2}^{\pi/2} + \cancel{\mathrm{sen}(4t)\Big|_{-\pi/2}^{\pi/2}}^{0} = \pi/4 - 4\pi = -\frac{15}{4}\pi$$

10) $F(x,y)=(y+3x, 2y-x) \Rightarrow P=(y+3x),\ Q=(2y-x)$

$D=\{(x,y)/4x^2+y^2\le 4\}$



$\Rightarrow C: \sigma(t)=(\cos t, 2\,\mathrm{sen}\,t)$ con $t\in[0,2\pi]$

$$\Rightarrow \int_{C^-} P\,dx+Q\,dy = -\int_0^{2\pi}(2\,\mathrm{sen}\,t+3\cos t, 2\,\mathrm{sen}\,t-\cos t)(-\mathrm{sen}\,t, 2\cos t)\,dt$$

$$= -\int_0^{2\pi}(-2\,\mathrm{sen}^2 t - 3\cos t\,\mathrm{sen}\,t + 2\,\mathrm{sen}\,t\cos t - 2\cos^2 t)\,dt = -\int(-2+\cos t\,\mathrm{sen}\,t)\,dt$$

$$= \int_0^{2\pi}(2+\cos t\,\mathrm{sen}\,t)\,dt = 2t\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\mathrm{sen}(2t)}{2}\,dt = 4\pi + \cancel{\left(\frac{-\cos(2t)}{4}\right)\Big|_0^{2\pi}} = 4\pi$$

$$\Rightarrow \text{Otra forma: } \int_{C^-} P\,dx+Q\,dy = -\iint_D (Q_x-P_y)\,dx\,dy = -\iint_D(-1-1)\,dx\,dy = -\iint_D -2\,dx\,dy = 2\iint_D dx\,dy = 2\int_0^2\int_0^{2\pi} 1\tfrac{1}{2}.r.\,dr\,d\theta = 2\pi\,\frac{r^2}{2}\Big|_0^2$$

$$= 4\pi$$

(11) $F(x,y) = (P(x,y), Q(x,y)) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$

$D = \{(x,y) / x^2+y^2 \leq 1\} \Rightarrow C: \sigma(t) = (\cos t, \operatorname{sen} t)$ con $t \in [0, 2\pi]$

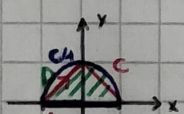
$$\Rightarrow \int_D Q_x - P_y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \overset{\text{Por Green}}{=} \int_C P\,dx + Q\,dy = \int_0^{2\pi} \left(\frac{\operatorname{sen} t}{\cos^2 t + \operatorname{sen}^2 t}, \frac{-\cos t}{\operatorname{sen}^2 t + \cos^2 t} \right) (-\operatorname{sen} t, \cos t)\,dt = \int_0^{2\pi} (-\operatorname{sen}^2 t - \cos^2 t)\,dt$$

$$= \int_0^{2\pi} -1\,dt = -t \Big|_0^{2\pi} = -2\pi$$

$$\Rightarrow \text{Pero por otro lado: } \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} \right) dx\,dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 0\,dx\,dy = 0 \neq -2\pi$$

$\Rightarrow$ Green no se satisface en este caso, ya que F no es C^1 en D (específicamente el punto $(0,0)$)

(12) $f_1(x,y) = \dfrac{x \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) - y(x^2+y^2)}{(x^2+y^2)^2}$, $f_2(x,y) = \dfrac{y \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) + x(x^2+y^2)}{(x^2+y^2)^2}$



$\Rightarrow$ Divido C en 2 curvas: $\sigma_1(t) = (t, t+1)$ con $t \in [-1, 0]$, $\sigma_2(t) = (t, 1-t)$ con $t \in [0,1]$

$C_4: \sigma_4(t): (\cos t, \operatorname{sen} t)$ con $t \in [0, \pi]$

$\Rightarrow$ Cierro la curva con $\cancel{C_3}$: $\sigma_3(t): (t, 0)$ con $t \in [-1, 1]$ X NO, si cierro la curva con C_3, el punto $(0,0)$ queda dentro de D, y f no es continua en este punto

$$\Rightarrow \int_C f_1\,dx + f_2\,dy \overset{\text{Green}}{=} \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy = \iint_D \left(\left(\frac{\frac{-xy\cos\left(\frac{\pi}{2(x^2+y^2)}\right)}{(x^2+y^2)^2} \cdot (x^2+y^2)^2 - y \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) \cdot 2x(x^2+y^2)}{(x^2+y^2)^4} + \frac{y^2-x^2}{(x^2+y^2)^2} \right) - \right.$$

$$\left. \left(\frac{\frac{-xy\cos\left(\frac{\pi}{2(x^2+y^2)}\right)}{(x^2+y^2)^2} \cdot (x^2+y^2)^2 - x \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) \cdot 2y(x^2+y^2)}{(x^2+y^2)^4} - \frac{x^2-y^2}{(x^2+y^2)^2} \right) \right) dx\,dy$$

$$= \iint_D 0\,dx\,dy = 0$$

(13)

$$\int_C -y^2\,dx + 3x\,dy \Rightarrow P = -y^2,\ Q = 3x \Rightarrow Q_x - P_y = 3 + 2y$$

$$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 / (x-x_0)^2 + (y-y_0)^2 \leq R^2\} \Rightarrow \begin{array}{l} x = r\cos\theta + x_0 \\ y = r\operatorname{sen}\theta + y_0 \end{array}$$

$$\Rightarrow \int_C -y^2\,dx + 3x\,dy \overset{\text{Green}}{=} \iint_D (3+2y)\,dx\,dy = \int_{-(R-x_0)}^{R-x_0} \int_{-(\sqrt{R^2-(x-x_0)}-y_0)}^{\sqrt{R^2-(x-x_0)}-y_0} (3+2y)\,dy\,dx \underset{\substack{\uparrow \\ \text{Uso coor.} \\ \text{polares}}}{=} \int_0^R \int_0^{2\pi} r.(3 + 2(r\operatorname{sen}\theta + y_0))\,dr\,d\theta$$

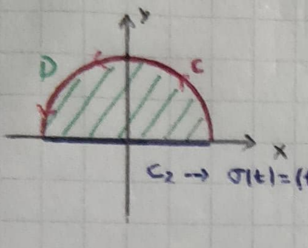
$$= \int_0^R \int_0^{2\pi} 3r\,d\theta\,dr + \int_0^R \int_0^{2\pi} 2r^2 \operatorname{sen}\theta\,dr\,d\theta + \int_0^R \int_0^{2\pi} 2r y_0\,dr\,d\theta = 3.\cancel{2}\pi \frac{r^2}{\cancel{2}} \Big|_0^R + 2.\cancel{2}\pi y_0 \frac{r^2}{\cancel{2}} \Big|_0^R + \int_0^R 2r^2(-\cos\theta) \Big|_0^{2\pi} dr$$

$$= \overset{3}{\cancel{6}}\pi R^2 + \overset{2}{\cancel{4}}\pi y_0 R^2 + 0 = \cancel{2}\pi R^2 (3+2y_0) = 6\pi \Rightarrow R^2 = \frac{6}{3+2y_0}$$

$\Rightarrow C = \left\{ (x,y) \in \mathbb{R}^2 / (x-x_0)^2 + (y-y_0)^2 = \frac{6}{3+2y_0} \right\} \Rightarrow$ Puedo parametrizar C por $\sigma(t) = \left(\sqrt{\frac{6}{3+2y_0}} \cos t + x_0, \sqrt{\frac{6}{3+2y_0}} \operatorname{sen} t + y_0 \right)$ con $t \in [0, 2\pi]$

14) $F(x,y)=(y^2e^x+\cos x+(x-y)^2,\ 2ye^x+\operatorname{sen}y) \Rightarrow P(x,y)=y^2e^x+\cos x+(x-y)^2$, $Q(x,y)=2ye^x+\operatorname{sen}y$

$C=\{(x,y)\ /\ x^2+y^2=1,\ y\geq 0\} \Rightarrow \sigma(t)=(\cos t,\operatorname{sen}t)$ con $t\in[0,\pi]$



$C_2 \to \sigma(t)=(t,0)$ con $t\in[-1,1]$

$$\Rightarrow \int_C F\,ds = ? \Rightarrow \int_{C+C_2} F\,ds = \int_C F\,ds + \int_{C_2} F\,ds = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dy\,dx$$

$$\Rightarrow \int_{C_2} F\,ds = \int_{-1}^{1}(\cos t+t^2,0)(1,0)\,dt = \int_{-1}^{1}(\cos t+t^2)\,dt = \operatorname{sen}t\Big|_{-1}^{1}+\frac{t^3}{3}\Big|_{-1}^{1} = \operatorname{sen}(1)-(\operatorname{sen}(-1))+2/3 = 2/3+2\operatorname{sen}(1)$$

$$\Rightarrow \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\left(2ye^x-(2ye^x+2y-2x)\right)dx\,dy = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}}(2x-2y)\,dy\,dx = \int_0^{\pi}\int_0^{1}2r^2(\cos\theta-\operatorname{sen}\theta)\,d\theta\,dr$$

$$= \int_0^{\cancel{2}\pi}\cos\theta\cdot\frac{2r^3}{3}\Big|_0^1 d\theta - \int_0^{\cancel{2}\pi}\operatorname{sen}\theta\cdot\frac{2r^3}{3}\Big|_0^1 d\theta = \frac{2}{3}\Big(\overset{0}{\cancel{\operatorname{sen}\theta\Big|_0^{2\pi}}}-\underbrace{(-\cos\theta)\Big|_0^{\pi}}_{2}\Big) = \cancel{0}\ -\frac{4}{3}$$

$$\Rightarrow \int_{C+C_2} F\,ds = \overset{-4/3}{\cancel{X}} = \int_C F\,ds + 2/3 + 2\operatorname{sen}(1) \Rightarrow \int_C F\,ds = \overset{-2}{\cancel{-2/3}} - 2\operatorname{sen}(1) = -2(1+\operatorname{sen}(1))$$

15) $D=\left\{(x,y)\in\mathbb{R}^2\ /\ \frac{x^2}{9}+\frac{y^2}{4}\leq 1\right\} \Rightarrow \partial D: \sigma(t)=(3\cos t,\ 2\operatorname{sen}t)$ con $t\in[0,2\pi]$

$F(x,y)=(u(x,y),v(x,y))$, $G(x,y)=(v_x-v_y,\ u_x-u_y)$

$$\Rightarrow F.G=(u,v).(v_x-v_y,\ u_x-u_y) = uv_x-uv_y+vu_x-vu_y = uv_x+vu_x-(uv_y+vu_y) = \frac{\partial(u.v)}{\partial x}-\frac{\partial(u.v)}{\partial y}$$

$$\Rightarrow \iint_D (F.G)(x,y)\,dx\,dy = \iint_D\left(\frac{\partial(uv)}{\partial x}-\frac{\partial(u.v)}{\partial y}\right)dx\,dy = \int_{C=\partial D}(uv)\,dx+(uv)\,dy = \int_0^{2\pi}\langle(uv(\sigma(t)),uv(\sigma(t))),(\sigma'(t))\rangle\,dt$$

$\Rightarrow$ Como $u=x$, $v=1$ sobre $\partial D=C$: $\displaystyle\int_0^{2\pi}\langle(3\cos t,3\cos t),(-3\operatorname{sen}t,2\cos t)\rangle\,dt = \int_0^{2\pi}(-9\cos t\operatorname{sen}t+6\cos^2 t)\,dt$

$$= \int_0^{2\pi}-9\underbrace{\cos t\operatorname{sen}t}_{\frac{1}{2}\operatorname{sen}(2x)}\,dt + \int_0^{2\pi}6\underbrace{\cos^2 t}_{\frac{1}{2}(\cos(2x)+1)}\,dt = -9.\Big(\overset{0}{\cancel{-\frac{\cos(2t)}{4}\Big|_0^{2\pi}}}\Big) + 6\Big(\frac{t}{2}\Big|_0^{2\pi} + \overset{0}{\cancel{\frac{\operatorname{sen}(2t)}{4}\Big|_0^{2\pi}}}\Big) = 6\pi$$

Cont. 12

$$\Rightarrow \int_{C+C^4} f_1dx+f_2dy = \iint_D (f_{2x}-f_{1y})\,dx\,dy = 0 \Rightarrow \int_{C+} f_1dx+f_2dy + \int_{C^4+} f_1dx+f_2dy = 0 \Rightarrow \int_{C^4+} f_1dx+f_2dy = -\int_{C+} f_1dx+f_2dy$$

$$\int_{C^4} f_1dx+f_2dy = \int_0^{\pi}(f_1(\sigma_4(t)),f_2(\sigma_4(t)))(\sigma_4'(t))\,dt = \int_0^{\pi}(\cos t.\overset{1}{\cancel{\operatorname{sen}(\pi/2)}}-\operatorname{sen}t,\ \operatorname{sen}t.\overset{1}{\cancel{\operatorname{sen}(\pi/2)}}+\cos t)(-\operatorname{sen}t,\cos t)\,dt$$

$$= \int_0^{\pi}(\cancel{-\cos t\operatorname{sen}t}+\operatorname{sen}^2 t+\cancel{\operatorname{sen}t\cos t}+\cos^2 t)\,dt = \int_0^{\pi}1\,dt = t\Big|_0^{\pi} = \pi$$

$$\Rightarrow \int_C f_1dx+f_2dy = -\pi$$