

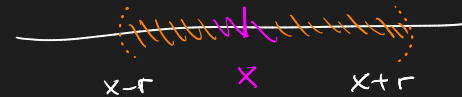
## Práctica 3

1. Probar que los siguientes *son* espacios métricos. Dibujar, en cada caso, una bola abierta.

(a)  $\mathbb{R}$  con  $d(x, y) = |x - y|$ .

$(\mathbb{R}, d)$

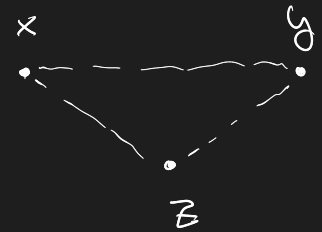
$B(x, r)$



$$1) d(x, x) = 0 \quad \text{y} \quad d \geq 0$$

$$2) d(x, y) = d(y, x)$$

$$3) d(x, y) \leq d(x, z) + d(z, y)$$



$$1) \quad \checkmark$$

$$\begin{aligned} 2) \quad |x - y| &= |(-1)(y - x)| \\ &= |-1| |y - x| \\ &= |y - x| \quad \checkmark \end{aligned}$$

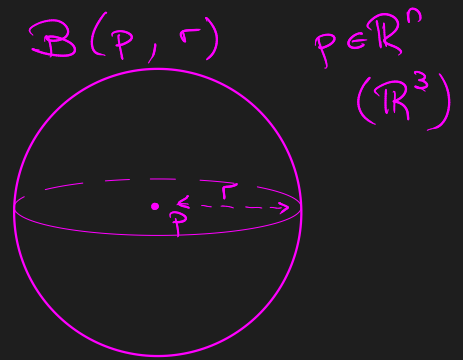
$$\begin{aligned} 3) \quad |x - y| &= |x - z + z - y| \\ &\leq |x - z| + |z - y| \\ &\quad \underbrace{\hspace{1cm}}_{= |y - z|} \quad \checkmark \end{aligned}$$

(b)  $\mathbb{R}^n$  con  $d_2(x, y) = (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$ .

1) ✓

2) ✓

$$3) \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \|x - y\|$$



$$\|x - z + z - y\| \leq \|x - z\| + \|z - y\| \quad z \in \mathbb{R}^n$$

↑  
Cauchy - Schwartz

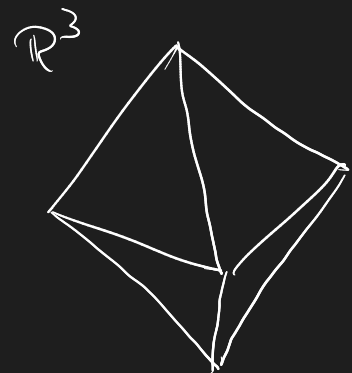
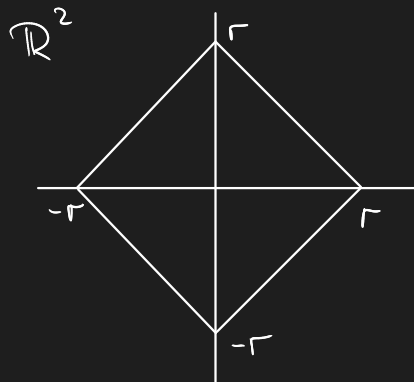
□

(c)  $\mathbb{R}^n$  con  $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$ .

1) ✓

2) ✓

3) ✓

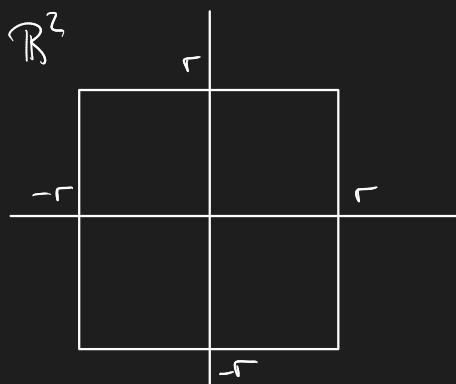


(d)  $\mathbb{R}^n$  con  $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ .

1) ✓

2) ✓

$$3) \max |x_i - y_i| \leq \max \{ |x_i - z_i| + |y_i - z_i| \}$$



$$\leq \max \{ |x_i - z_i| \} + \max \{ |y_i - z_i| \}$$



✓

(e)  $C([0, 1])$  con  $d(f, g) = \max_{0 \leq t \leq 1} |f(t) - g(t)|$ .

1) ✓

$$E = \mathcal{C}([0, 1])$$

2) ✓

$$\mathcal{B}(x, r) = \{g \in E : d(x, g) < r\}$$

3) ✓

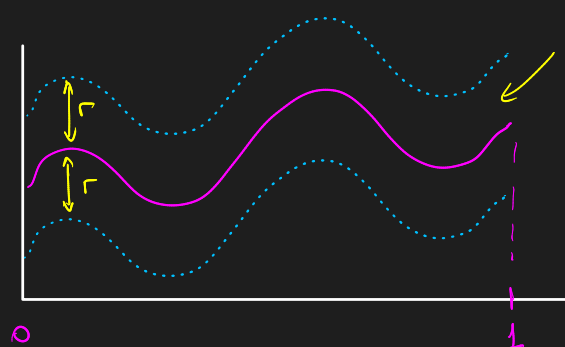
reescribo

$$\mathcal{B}(f, r) = \{g \in E : d(f, g) < r\}$$

• digo  $f(t) \equiv 0$



•  $f(t)$  libre



los  $g \in \mathcal{B}(f, r)$  es tan  
acá adentro

(f)  $E$  cualquier conjunto no vacío, con la métrica

$$d(x, y) = \begin{cases} 0 & \text{si } x = y, \\ 1 & \text{si } x \neq y. \end{cases}$$

1) ✓

2) ✓

$$3) d(x, y) \leq d(x, z) + d(y, z)$$

$$\text{Caso 1: } x = y$$

$$0 \leq \text{Posi} + \text{Positivo} \quad \checkmark$$

$$\text{Caso 2: } x \neq y$$

$$1 \leq 1 \text{ ó } 2 \quad \checkmark$$

$\mathbb{R}^2$

$$B(\vec{0}, 1)$$

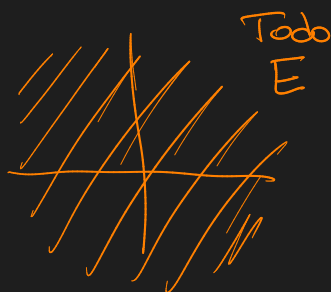


$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

$$\leftarrow d(\vec{0}, y) < r$$

$$B(\vec{0}, 2)$$

$$d(\vec{0}, y) < 2 \quad \leftarrow \text{siempre } \forall y \in \mathbb{R}$$



2. Decidir cuáles de las siguiente funciones definidas en  $\mathbb{R} \times \mathbb{R}$  son métricas en  $\mathbb{R}$ :

$$d_a(x, y) = (x - y)^2, \quad d_b(x, y) = \sqrt{|x - y|}, \quad d_c(x, y) = |x^2 - y^2|.$$

$$d_a(x, y) = (x - y)^2$$

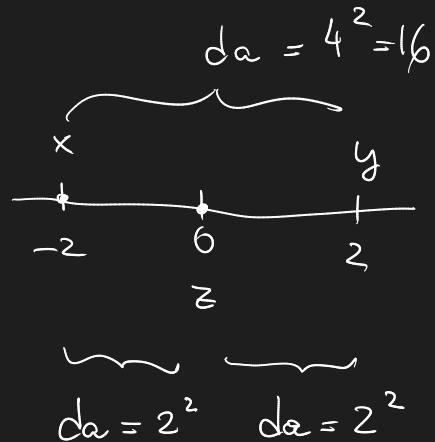
1) ✓

2) ✓

3)  $d_a(x, y) \stackrel{?}{\leq} d_a(x, z) + d_a(z, y)$

$$(x - y)^2 \stackrel{?}{\leq} (x - z)^2 + (y - z)^2$$

quero  
>



$$\text{Si: } z = \frac{x + y}{2}$$

$\Rightarrow$  la desigualdad triangular no vale

$$(x - y)^2 \stackrel{?}{\leq} (x - z)^2 + (y - z)^2$$

$$(x - y)^2 \stackrel{?}{\leq} \left(x - \frac{x + y}{2}\right)^2 + \left(y - \frac{x + y}{2}\right)^2$$

$$(x - y)^2 > \left(\frac{x}{2} - \frac{y}{2}\right)^2 + \left(\frac{x}{2} - \frac{y}{2}\right)^2$$

Pues si

$$E = \mathbb{R}$$

$$x = -2$$

$$y = 2$$

$$z = \frac{-2+2}{2} = 0$$

$$(-2-2)^2 > (-2-0)^2 + (2-0)^2$$

$$16 > 8$$

$\therefore$  da no cumple la desigualdad triangular

$\Rightarrow$  no es distancia.

$$b) d_b(x, y) = \sqrt{|x - y|}$$

$$\boxed{1} \quad \checkmark$$

$$\boxed{2} \quad \checkmark$$

$$\boxed{3} \quad d_b(x, y) \stackrel{?}{\leq} d_b(x, z) + d_b(z, y)$$

$$d_b(x, y)^2 = |x - y| \leq \underbrace{|x - z|}_{d_b(x, z)^2} + \underbrace{|z - y|}_{d_b(z, y)^2}$$

$$d_b(x, y)^2 \leq d_b(x, z)^2 + d_b(z, y)^2$$

$$d_b(x, y) \leq \sqrt{d_b(x, z)^2 + d_b(z, y)^2}$$

$$\sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|}$$

CA:

$$\sqrt{a+b} \stackrel{?}{\leq} \sqrt{a} + \sqrt{b} \quad \text{con } a, b \geq 0$$

Completo cuadrado:

$$\text{Como } 2\sqrt{a \cdot b} > 0$$

$$\Rightarrow a+b \leq \underbrace{a + 2\sqrt{a \cdot b} + b}_{(\sqrt{a} + \sqrt{b})^2}$$

$$\Rightarrow a+b \leq (\sqrt{a} + \sqrt{b})^2$$

$a, b \geq 0$

$$\Rightarrow \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$$

Robé lo que quería

$$\sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|}$$

$$\leq \sqrt{|x-z|} + \sqrt{|z-y|} \quad \checkmark$$

∴  $d_b$  es distancia.

$$c) d_c(x, y) = |x^2 - y^2|$$

Contre ej:

$$d_c(-2, 2) = |(-2)^2 - 2^2|$$

$$= 0$$

$$\text{Pero } -2 \neq 2$$

$\therefore$  no es una métrica.



3. Consideremos en  $\mathbb{R}^n$  las distancias  $d_1$ ,  $d_2$  y  $d_\infty$ . Denotemos por  $B_1(x, r)$ ,  $B_2(x, r)$  y  $B_\infty(x, r)$  a la bola de centro  $x$  y radio  $r$  para cada una de las distancias, respectivamente.

(a) Probar que  $d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n d_\infty(x, y)$ .

(b) Deducir de (a) que  $B_1(x, r) \subseteq B_2(x, r) \subseteq B_\infty(x, r) \subseteq B_1(x, nr)$ .

$$a) \quad d_\infty(x, y) = \sup \{ |x_i - y_i| : i \in [1, n] \}$$

$$d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$$

$$d_\infty \leq d_2 \quad \mathcal{D} := \{ |x_i - y_i| : i \in [1, n] \}$$

$$d_\infty(x, y) \in \mathcal{D}$$

$$d_2(x, y) = \sqrt{\underbrace{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_n - y_n|^2}_{\text{elementos de } \mathcal{D} \text{ al cuadrado}}}$$

$$\Rightarrow (d_\infty(x, y))^2 \text{ es alguno de } \uparrow$$

$$\Rightarrow (d_\infty(x, y))^2 = (x_i - y_i)^2 \quad \text{para algùn } i \in [1, n]$$

Como son todos términos  $(x_i - y_i)^2 \geq 0 \quad \forall i \in [1, n]$

$$\Rightarrow (d_\infty(x, y))^2 \leq \overset{\text{términos } \geq 0}{\leq} |x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_n - y_n|^2$$

Tomo raíz (términos  $\geq 0$ )

$$\Rightarrow d_{\infty}(x, y) \leq \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_n - y_n|^2}$$

$$\therefore \boxed{d_{\infty}(x, y) \leq d_2(x, y)}$$

$$d_2 \leq d_1)$$

$$d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$$

elevo al  $\square$

$$\sum_{i=1}^n (x_i - y_i)^2 \stackrel{?}{\leq} \left( \sum_{i=1}^n |x_i - y_i| \right)^2$$

$$\text{Sea } a_i := |x_i - y_i|$$

$$\sum a_i^2 \stackrel{?}{\leq} \left( \sum_{i=1}^n a_i \right)^2$$

$$\left( \sum_{i=1}^n a_i \right)^2 = \sum_{i=1}^n \sum_{j=1}^n a_i \cdot a_j$$

$$\text{Nota que si } i = j \Rightarrow a_i^2$$

$$= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n a_i \cdot a_j$$

Como  $a_i \geq 0 \quad \forall i \in [1, n]$

Obtuse que

$$\left( \sum_{i=1}^n a_i \right)^2 = \sum_{i=1}^n a_i^2 + s \quad \text{con } s \geq 0$$

∴

$$\left( \sum_{i=1}^n a_i \right)^2 \geq \sum_{i=1}^n a_i^2$$

$$\therefore \sum_{i=1}^n (x_i - y_i)^2 \leq \left( \sum_{i=1}^n |x_i - y_i| \right)^2$$

$$\stackrel{\text{tod } > 0}{\Rightarrow} \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sum_{i=1}^n |x_i - y_i|$$

∴

$$d_2(x, y) \leq d_1(x, y)$$

$$d_1 \leq n \cdot d_\infty$$

$$d_\infty(x, y) = \sup \{ |x_i - y_i| : i \in [1, n] \}$$

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$$

$$\mathcal{D} := \{ |x_i - y_i| : i \in [1, n] \}$$

$$d_\infty(x, y) \in \mathcal{D}$$

además

$$d_{\infty}(x, y) \geq a \in \mathbb{D}$$

$$\text{pues } d_{\infty}(x, y) = \sup \mathbb{D}$$

$$\Rightarrow d_{\infty}(x, y) \geq |x_i - y_i| \quad \forall i \in [1, n]$$

$$\therefore \sum_{i=1}^n |x_i - y_i| \leq \underbrace{\sum_{i=1}^n d_{\infty}(x, y)}_{= n \cdot d_{\infty}(x, y)}$$

$$\Rightarrow \sum_{i=1}^n |x_i - y_i| \leq n \cdot d_{\infty}(x, y)$$

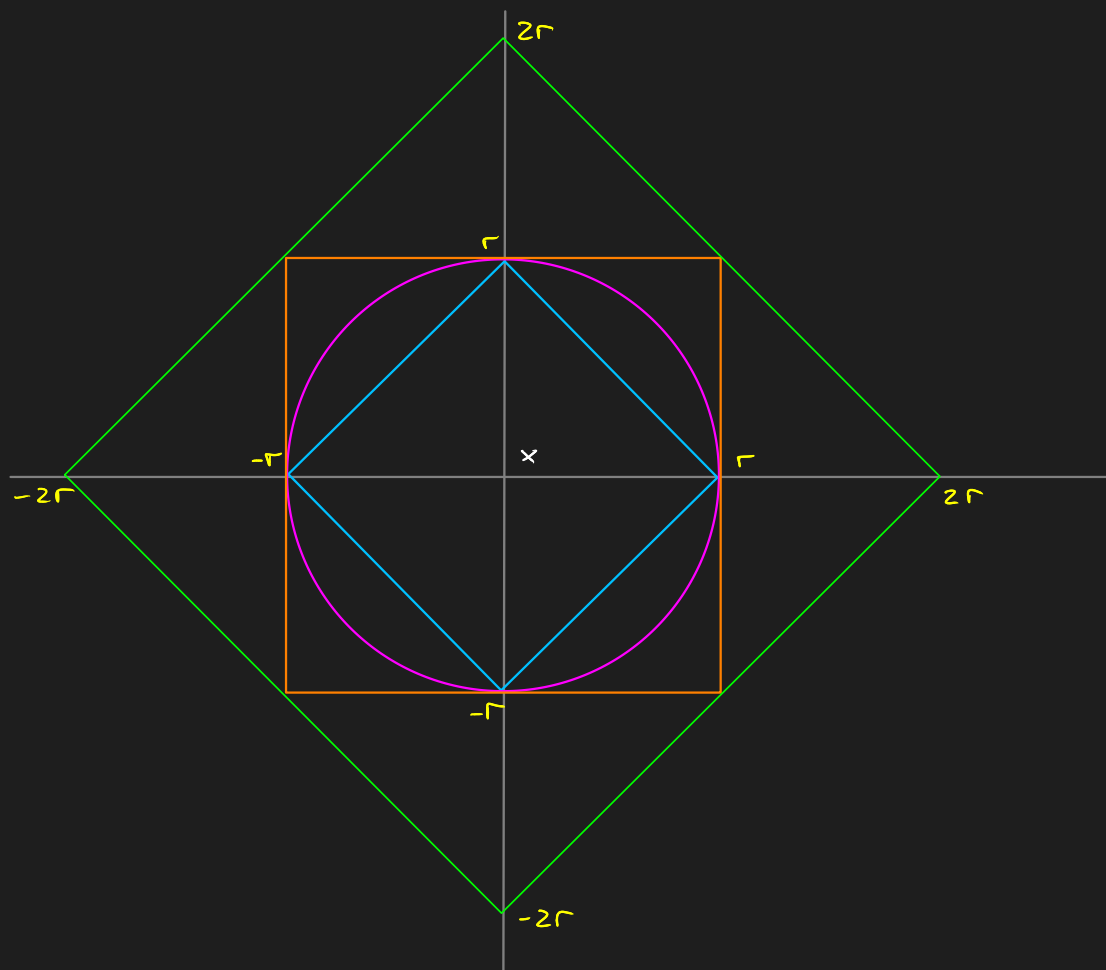
$$d_1(x, y) \leq n \cdot d_{\infty}(x, y)$$

$\therefore$

$$d_{\infty}(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n \cdot d_{\infty}(x, y)$$



(b) Deducir de (a) que  $B_1(x, r) \subseteq B_2(x, r) \subseteq B_\infty(x, r) \subseteq B_1(x, nr)$ .



Como  $d_2 \leq d_1 \quad \forall x, y$

en particular

$\Rightarrow d_2 \leq d_1$  para todo  $x, y \in B_{d_2}(x, r)$

$$B_{d_2}(x, r) = \{y \in E : d_2(x, y) < r\}$$

$$B_{d_1}(x, r) = \{y \in E : d_1(x, y) < r\}$$

$$\underbrace{\quad}_{d_2 \leq d_1}$$

$$\Rightarrow B_{d_1}(x, r) \subseteq B_{d_2}(x, r)$$

Pues para un mismo  $r$ , los  $y \in B_{d_1}$  estarán en  $B_{d_2}$  (pues si miden menos de  $r$  con  $d_1 \Rightarrow$  miden todavía menos con  $d_2$ ), pero no así al revés, pues habrá  $y \in B_{d_2}$  que miden más que  $r$  con  $d_1$ .

$$B_2 \stackrel{?}{\subseteq} B_\infty)$$

$$B_{d_2}(x, r) = \{y \in E : d_2(x, y) < r\}$$

$$B_{d_\infty}(x, r) = \{y \in E : d_\infty(x, y) < r\}$$

de a) obtuve

$$d_\infty \leq d_2$$

$$\Rightarrow B_{d_2}(x, r) \subseteq B_{d_\infty}(x, r) \quad \checkmark$$

$$B_\infty \subseteq B_1(x, n \cdot r)$$

$$\bullet B_{d_\infty}(x, r) = \{y \in E : d_\infty(x, y) < r\}$$

$$\bullet B_{d_1}(x, r) = \{y \in E : d_1(x, y) < n \cdot r\}$$

Se de a) que

$$d_1 \leq n \cdot d_\infty$$

reescribo

$$B_{d_\infty}(x, r) = \{y \in E : n \cdot d_\infty(x, y) < n \cdot r\} \quad n > 0$$

$$B_{d_1}(x, n \cdot r) = \{y \in E : d_1(x, y) < n \cdot r\}$$

$$\Rightarrow \text{como } d_1 \leq n \cdot d_\infty$$

$$B_{d_\infty}(x, r) \subseteq B_{d_1}(x, n \cdot r) \quad \checkmark$$