

## Espaces Métriques

## Rappeler

$$A \subseteq E, \quad E \text{ métrique}$$

- $\overline{A} = \{ x \in E : B(x, r) \cap A \neq \emptyset \quad \forall r > 0 \}$
- $A \text{ est fermé} \iff A = \overline{A}$

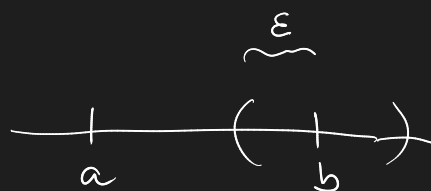
## Exemples :

$$1) \overline{(a, b)} = [a, b]$$

$$\supseteq) \text{ ya sabemos que } (a, b) \subseteq \overline{(a, b)}$$

Veamos que

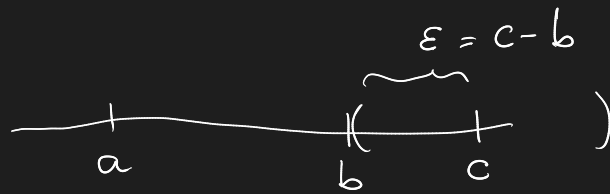
$$b \in \overline{(a, b)}$$



$$\forall \varepsilon > 0, \quad b - \frac{\varepsilon}{2} \in (a, b) \cap B(b, \varepsilon) \quad \checkmark$$

$$\subseteq) \text{ Sea } c > b, \text{ veamos que (por contradicción)}$$

$$c \stackrel{?}{\notin} \overline{(a, b)}$$



Afirmo

$$\underbrace{B(c, c-b)}_{(b, 2c-b)} \cap (a, b) = \emptyset$$

$$\Rightarrow c \notin \overline{(a, b)}$$

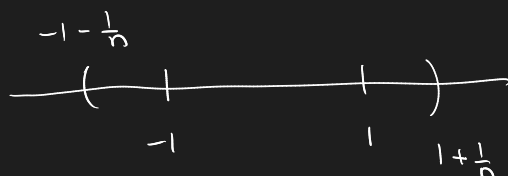
en particular,  $[a, b]$  es cerrado.

También:

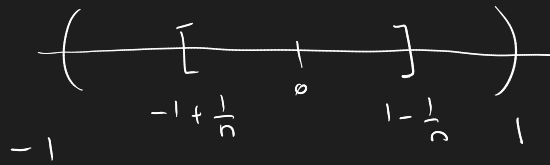
$$\mathbb{R} \setminus [a, b] = \underbrace{(-\infty, a)}_{\text{Abierto}} \cup \underbrace{(b, +\infty)}_{\text{Abierto}}$$

Obs:

$$\underbrace{[-1, 1]}_{\text{no es abierto}} = \bigcap_{n \geq 1} \underbrace{\left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)}_{\text{Abiertos}}$$



$$\bullet \underbrace{(-1, 1)}_{\substack{\text{no es} \\ \text{cerrado}}} = \bigcup_{n \in \mathbb{N}} \left[ -1 + \frac{1}{n}, 1 - \frac{1}{n} \right]$$



2)  $[a, b)$  (con  $b > a$ ) no es cerrado ni abierto

$$\overline{[a, b)} = [a, b]$$

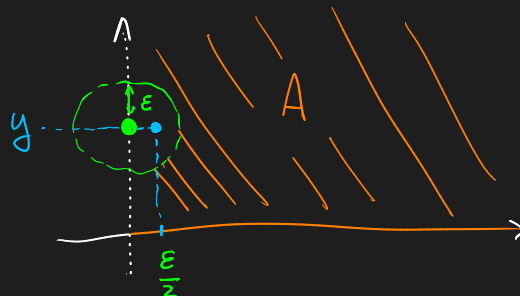
$$3) A = \{ (x, y) \in \mathbb{R}^2 : x > 0, y \geq 0 \}$$



$A$  primo

$$\overline{A} = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \} =: F$$

$\supseteq$ ) Sea  $(0, y) \in F$ ,  $\varepsilon > 0$



$$\Rightarrow \underbrace{\left( \underbrace{\varepsilon/2}_{>0}, \underbrace{y}_{>0} \right)}_{\in A} \in B((0,y), \varepsilon) \cap A$$

$$\Rightarrow d((\varepsilon/2, y), (0, y)) = |\varepsilon/2| < \varepsilon \quad \checkmark$$

$\subseteq$ ) Uso contra recíproco

$$\text{Sea } (x, y) \notin F$$

$$\text{v.q. } (x, y) \notin \overline{A}$$



$$\text{Sup. } x \geq 0, y < 0$$

Afirmo:

$$\underbrace{B((x, y), |y|)}_{\substack{\text{más aún, si } (z, w) \in \\ \Rightarrow w < 0}} \cap A = \emptyset$$

$$\text{más aún, si } (z, w) \in$$

$$\Rightarrow w < 0$$

Sea  $(z, w)$  en  $l_2$  b0  $l_2$

$$\begin{aligned} |y| > d((z, w), (x, y)) &= \sqrt{(z-x)^2 + (w-y)^2} \\ &\geq \sqrt{(w-y)^2} \\ &= |w-y| < |y| \end{aligned}$$

Como

$$\begin{aligned} |w-y| < |y| \\ \Rightarrow w \text{ es negativo.} \end{aligned}$$

Falta ver otros casos

$$\begin{array}{c|c} 2 & A \\ \hline 3 & 1 \end{array}$$

$$\sup. \quad x \quad 0, y \quad 0$$

$$\sup. \quad x \quad 0, y \quad 0$$

EXERCICIO:  $A^\circ = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$

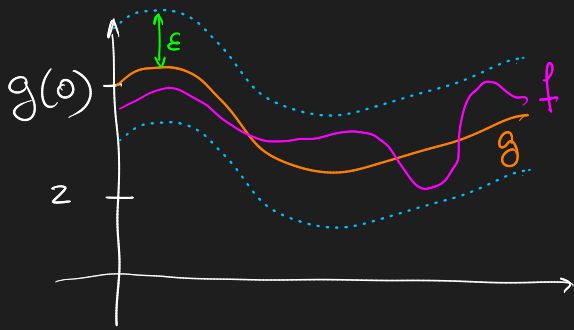
3)  $A = \{f \in C([0, 1]) : f(0) = 2\}$

Probar:

•  $A$  es cerrado para  $d_\infty$

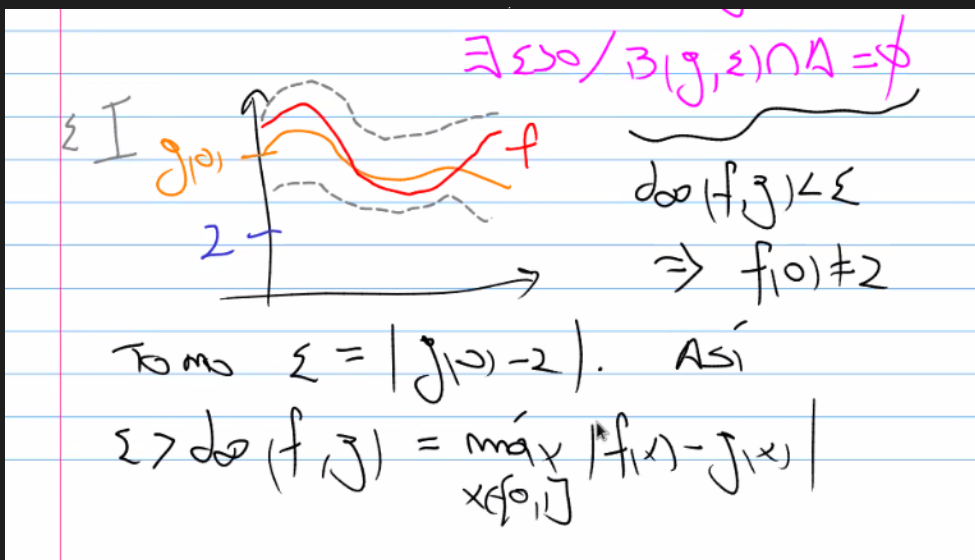
$\underbrace{\hspace{10em}}$

$A^c$  es Abierto



Sea  $g \in A^c$ ,

$$\forall \varepsilon > 0 \quad \exists \varepsilon > 0 \mid B(g, \varepsilon) \cap A = \emptyset$$



$$\geq |f(0) - g(0)|$$

$$\Rightarrow |f(0) - g(0)| < |g(0) - 2| \quad \checkmark$$

$\neq 2$  pues  $|g| < |g|$  absur!

Probar

- $A$  no es cerrado para  $d_1$

$A^c$  no es abierto:  $B \vee g$

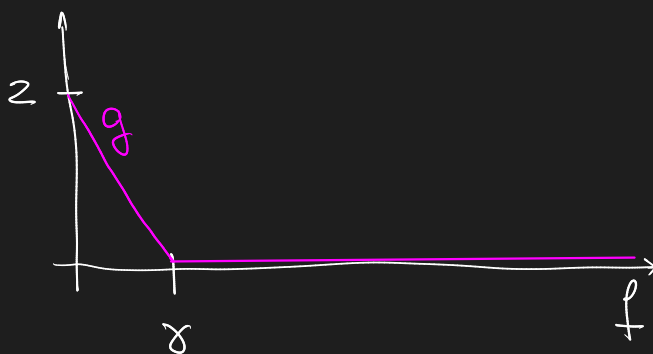
$$\exists f \in A^c \mid \forall \varepsilon > 0, B(f, \varepsilon) \cap A \neq \emptyset$$

es decir

$$\mathcal{B}(f, \varepsilon) \not\subset A^c$$

$$\Rightarrow \exists g \in \mathcal{B}(f, \varepsilon) \mid g(0) = z$$

$$\text{Tomemos } f \equiv 0$$



$$\text{Así, } g \in A$$

$$\text{y } d_1(f, g) = \gamma < \varepsilon \quad \text{si tomamos } \gamma = \frac{\varepsilon}{2}$$

Prop :

Si  $F \subseteq C([0, 1])$  cerrado

para  $d_1$ ,  $b$  es para todo  $d_\infty$

Ejercicio : Probar ↗

Recordar:

$\Sigma (E, d)$  métrico

$$\bar{B}(x, r) = \{y \in E : d(x, y) \leq r\}$$

$$\left( \Rightarrow \bar{B}(x, r) \supseteq \overline{B(x, r)} \right)$$

Afirmar:

en  $C([0, 1])$  con  $d_\infty$ ,

$$\forall f, \forall r, \quad \bar{B}(f, r) = \overline{B(f, r)}$$

Para  $d_1$  también vale

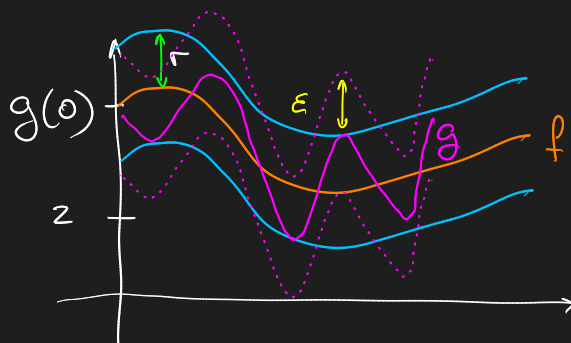
Dem:

$\supseteq$  Vale siempre

$\subseteq$   $\forall g$  si  $d_\infty(f, g) \leq r$

$\Rightarrow \forall \varepsilon > 0, B(g, \varepsilon) \cap B(f, r) \neq \emptyset$

↓ Pto de adherencia





$$\text{Then } h(x) = g(x) + (f(x) - g(x)) \cdot \delta$$

$\delta > 0$   
 $\Rightarrow$  determine

$$\begin{aligned} \bullet \quad d_{\infty}(h, g) &= \max_{x \in [0, 1]} |h(x) - g(x)| \\ &= \delta \cdot \max_{x \in [0, 1]} |f(x) - g(x)| \end{aligned}$$

$$= \delta \cdot \underbrace{d_{\infty}(f, g)}_{\leq r}$$

$$\leq \delta \cdot r < \varepsilon$$

$\uparrow$   
 si then

$$\delta < \frac{\varepsilon}{r}$$

$$\bullet \quad d_{\infty}(h, f) = \max_{x \in [0, 1]} |g(x) - f(x) + (f(x) - g(x))\delta|$$

$$= \max_{x \in [0, 1]} |(g(x) - f(x)) \cdot (1 - \delta)|$$

$\delta < 1$

$$\downarrow = (1 - \delta) \cdot \max_{x \in [0, 1]} |g(x) - f(x)|$$

$$= (1 - \delta) \cdot \underbrace{d_{\infty}(g, f)}_{\leq r}$$

$$\langle \cdot, \cdot \rangle \quad \forall \delta \quad \text{can} \quad \delta < \min \left\{ 1, \frac{\varepsilon}{L} \right\}$$

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