$I=[0,1]$ $\quad a\in\mathbb{R}$

a, $\quad f(x)=a\chi_{[0,1]}(x) \qquad \int f\,d\mu = a\,\mu([0,1]) = a$

$$\int h_n\,d\mu = \int \underbrace{(g_n+\delta_n)}_{g_n+\delta_n\chi_{[0,1]}}\,d\mu = \int g_n\,d\mu + \underbrace{\int \delta_n\chi_{[0,1]}\,d\mu}_{=\delta_n}$$

$$= \int g_n\,d\mu + \delta_n.$$

$$\cdot \quad \sum_{n=1}^{N} \int_{E_n} f\,d\mu = \sum_{n=1}^{N} \int \chi_{E_n} f\,d\mu$$

$$\int_{E_n} f\,d\mu = \int \chi_{E_n} f\,d\mu$$

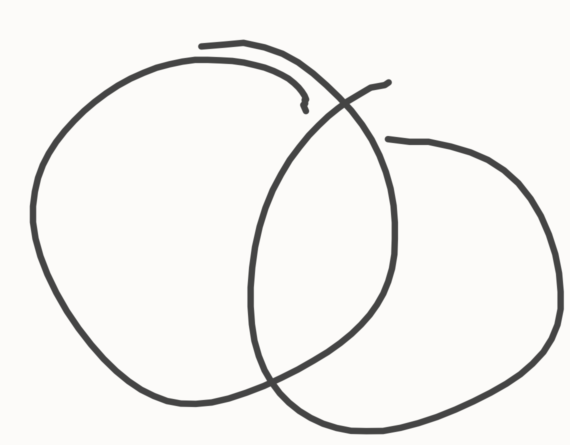
$\int$ es lineal

$$= \int \sum_{n=1}^{N} \chi_{E_n} f\,d\mu$$

$$= \int f\cdot \underbrace{\left(\sum_{n=1}^{N} \chi_{E_n}\right)}_{=\chi_E}\,d\mu \qquad (\text{xq son disj.})$$

$$= \int_E f\,d\mu.$$

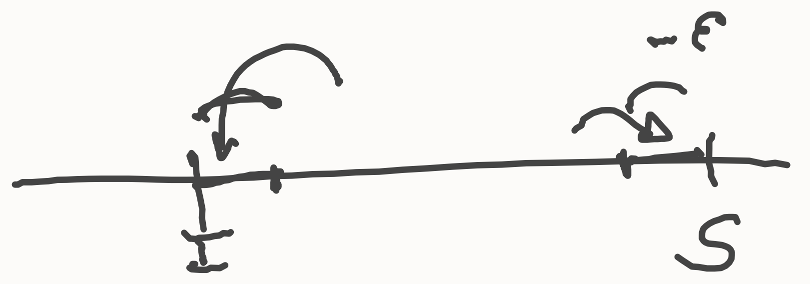
$$\boxed{\chi_A+\chi_B = \chi_{A\cup B} \quad A\cap B=\emptyset}$$



$$\chi_A+\chi_B-\chi_{A\cap B} = \chi_{A\cup B}$$

Ej (10) A acotado en $\mathbb{R}$.

A es medible $\iff$ [$\forall \varepsilon>0$ $\exists$ F, G ; $\underline{F \subseteq A \subseteq G}$; F cerrado y G es abierto $\wedge$ $\mu(G \setminus F) < \varepsilon$.]



$\Rightarrow$) Sea $\varepsilon>0$ $\Rightarrow$ $\exists$ $G \supseteq A$ abierto / $\mu(G) \leq \mu(A) + \varepsilon/4$

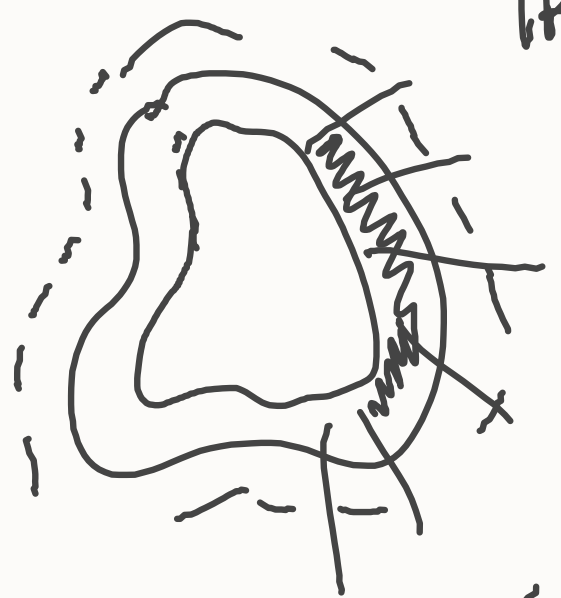
también $\exists$ $F \subseteq A$ cerrado / $\mu(A) - \varepsilon/4 \leq \mu(F)$

$\mu(G) - \mu(A) < \varepsilon$

$\mu(A) - \mu(F) < \varepsilon$

$$G \setminus F = G \cap F^c = \underbrace{G \cap F^c \cap A}_{\substack{(G \supseteq A) \\ = F^c \cap A}} \overset{d}{\cup} \underbrace{G \cap F^c \cap A^c}_{\substack{(F^c \supseteq A^c) \\ = G \cap A^c}}$$

$\mathbb{R} = A \overset{d}{\cup} A^c$



$$= A \setminus F \overset{d}{\cup} G \setminus A$$

$$\Rightarrow \mu(G \setminus F) = \mu(A \setminus F) + \mu(G \setminus A)$$

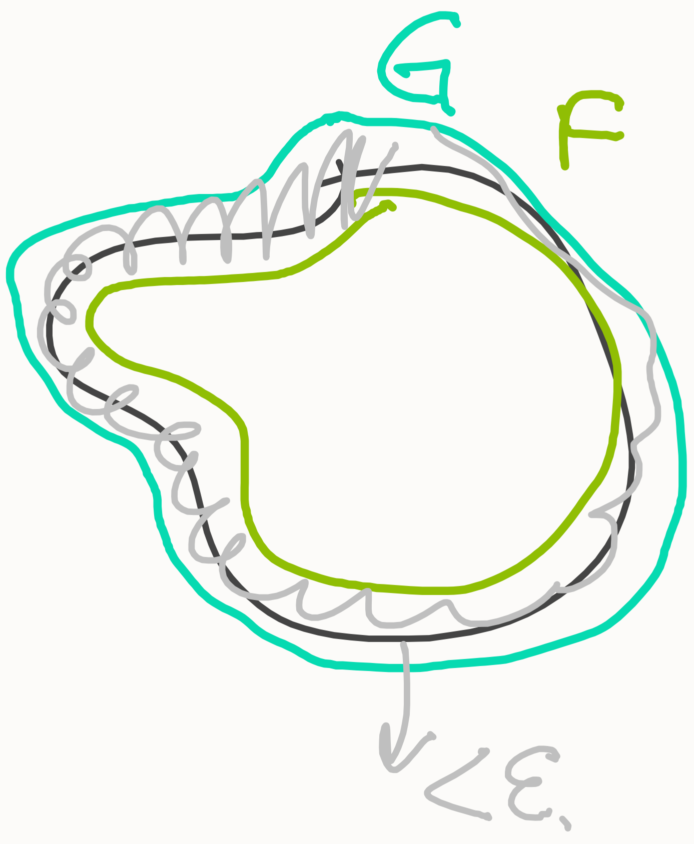
$$= \mu(A) - \mu(F) + \mu(G) - \mu(A)$$

$$\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2 < \varepsilon.$$

A acotado
$\Downarrow$
$\mu(A) < \infty$
$\Downarrow$
$\mu(F) < \infty$

$\Leftarrow$) $\forall \varepsilon > 0$ $\exists$ F, G cerrado y abto / $F \subseteq A \subseteq G \wedge \mu(G \setminus F) < \varepsilon$.

Para cada $n \in \mathbb{N}$, $\varepsilon_n = 1/n$ $\exists$ F_n cerrado y G_n abto / $F_n \subseteq A \subseteq G_n \wedge \mu(G_n \setminus F_n) < 1/n$.



$B = \bigcup_n F_n \in \mathcal{M}$ x ser unión numerable de medibles.

$B \subseteq A$

Si vemos q' $A \setminus B$ es nulo $\Rightarrow$

$A = \underset{\in \mathcal{M}}{B} \cup \underset{\text{nulo} \in \mathcal{M}}{A \setminus B} \in \mathcal{M}$.

Veamos que $A \setminus B$ es nulo.

$$A \setminus B = A \cap B^c \subseteq \underbrace{G_n \cap B^c}_{\in \mathcal{M}_C} \quad \forall n \in \mathbb{N}$$

$$G_n \cap B^c = G_n \cap \left(\bigcup_m F_m \right)^c = G_n \cap \bigcap_m F_m^c$$

$$= \bigcap_m (G_n \cap F_m^c) \subseteq G_n \cap F_n^c \quad \forall n$$

$$\Rightarrow A \setminus B \subseteq \underbrace{G_n \setminus F_n}_{\text{mide } < 1/n} \quad \forall n \in \mathbb{N}$$

. Si supiéramos q' $A \setminus B \in \mathcal{M} \Rightarrow \mu(A \setminus B) < 1/n \ \forall n$ $\Rightarrow \mu(A \setminus B) = 0$.

. Sea $D = \bigcap_n G_n \in \mathcal{M}$

- $D \supseteq A$ ✓
- $A \setminus B \subseteq \underbrace{\overset{= D \cap B^c}{D \setminus B}}_{\text{medible}} \subseteq G_n \setminus F_n \ \forall n$ (mismo cuenta q' antes)

Ahora sí: $D \setminus B \in \mathcal{M} \wedge D \setminus B \subseteq G_n \setminus F_n \ \forall n$

$$\Rightarrow \mu(D \setminus B) \leq \mu(G_n \setminus F_n) < 1/n \ \forall n \Rightarrow \mu(D \setminus B) = 0.$$

$\Rightarrow D \setminus B$ es nulo $\xRightarrow[A \setminus B \subseteq D \setminus B]{}$ $A \setminus B$ es nulo

$\Rightarrow A \setminus B \in \mathcal{M} \wedge \mu(A \setminus B) \Rightarrow A \in \mathcal{M}$.

- $B = \bigcup_n F_n \to$ conjuntos F sigma $\in \mathcal{F}_\sigma$.
- $D = \bigcap_n G_n \to$ conjuntos G delta G_δ.

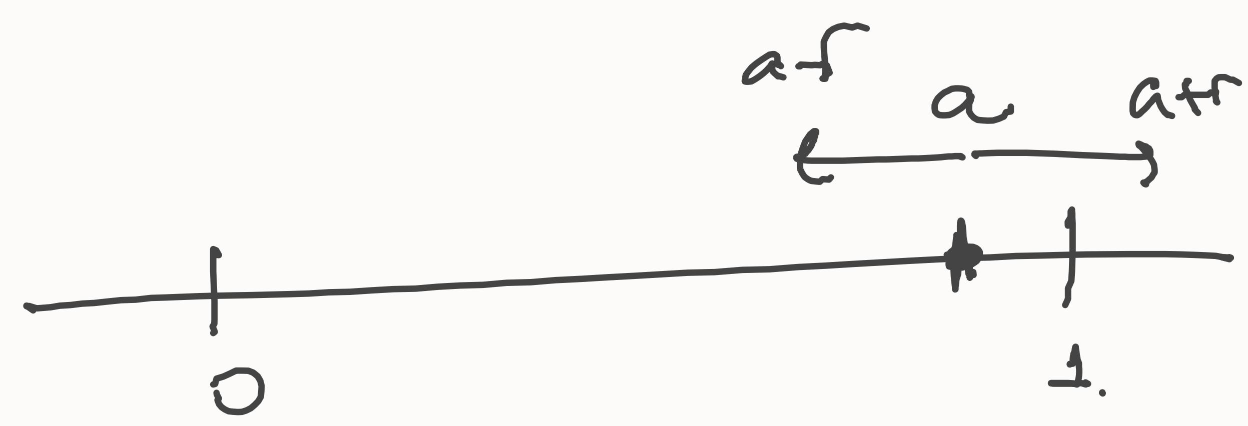
12) $A \subseteq [0,1]$ medible / $\mu(A)=1 \Rightarrow$ A es denso en $[0,1]$.

Qvq A es denso $(\bar{A}=[0,1])$

$\forall a \in A$ y $\forall r>0$, $A \cap (a-r, a+r) \neq \emptyset$.

Si $\exists a \in A$ y $r>0$ / $A \cap (a-r, a+r) = \emptyset$

$$\Rightarrow \underbrace{(a-r, a+r) \cap [0,1]}_{\text{es un intervalo y} \therefore \text{ tiene medida positiva}} \subseteq [0,1] \setminus A$$



$$\Rightarrow \mu([0,1]\setminus A) > 0.$$

$$\mu(A) = \underset{\mu([0,1])}{\overset{}{1}} - \underbrace{\mu([0,1]\setminus A)}_{>0} < 1 \quad \text{ABS!}$$

$\Rightarrow$ A es denso.

$f(x) = \lambda x$. Ej 2 $\mathcal{B} = \{B : f^{-1}(B) \in \mathcal{M}\}$

es σ-algebra

Hay que ver que $\mathcal{B}_\lambda = \mathcal{M}$.

Si vemos que los nulos y los abiertos están en $\mathcal{B} \Rightarrow \mathcal{M} \subseteq \mathcal{B}_\lambda$

↳ "más chico q contiene a los nulos y a los abiertos".

$\frac{1}{\lambda} \rightsquigarrow \beta_{1/\lambda} \Rightarrow \mathcal{C} \subseteq \beta_{1/\lambda}$ y ahí saleq'

$Sm = .$