07/09/2020

Clase Práctica 2 - José Luna

Longitud de Arco

Recordar

Sea $\mathscr{C}$ una curva simple, $\underbrace{\text{suave a trozos}}_{}$, y sea $\nabla : I \to \mathbb{R}^n$ una parametrización $\underbrace{\text{regular a trozos}}_{}$ de $\mathscr{C}$ — relajación

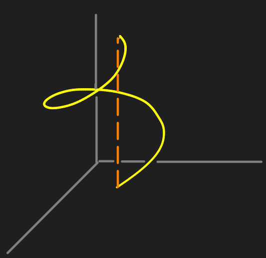
La longitud de $\mathscr{C}$ está dada por

$$\ell(\mathscr{C}) = \int_I \|\nabla'\|$$

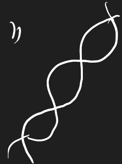
Obs:
La longitud de una curva No depende de la parametrización ∇

• Ejemplos

1) $\nabla(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$



"Hélice"



Tenemos que

$$\ell(\mathscr{C}) = \int_0^{2\pi} \|(-\sin t, \cos t, 1)\| \, dt$$

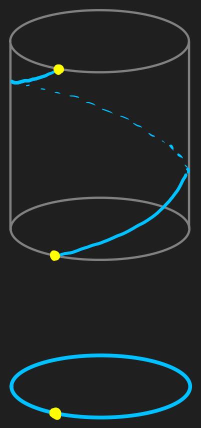
↳ CA

$$(\sin^2 t + \cos^2 t + 1)^{1/2} = \sqrt{2}$$

$$\ell(\mathscr{C}) = 2\sqrt{2}.\pi$$

↳ Notar que es mayor a 2π

$$2\pi < \underbrace{2\sqrt{2}}_{\sim 2.8}\pi$$



Donde 2π es el perímetro de una circunferencia, parametrizado por

$$(\cos t, \sin t, a)$$

para algún a fijo

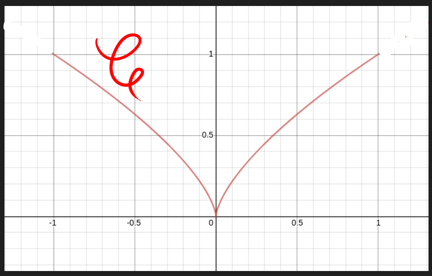
Esta diferencia nos dice cuánto "se estira" el perímetro de un círculo enrollando un elástico sobre un cilindro.

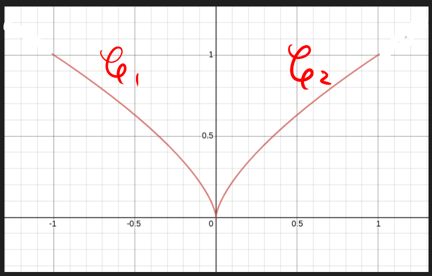
Ej 2. Calcular $\text{long}(\mathcal{C})$

con $\mathcal{C} = \{ x, y \in \mathbb{R} \,/\, x^2 = y^3 \wedge -1 \leq x \leq 1 \}$

Gráfico de $\mathcal{C}$, suave a trozos

↳ ver (0,0)





- $\text{long}(\mathcal{C}) = \text{long}(\mathcal{C}_1) + \text{long}(\mathcal{C}_2)$

$= 2 \cdot \text{long}(\mathcal{C}_2)$

↑ simétrica en y

$$\text{long}(\mathcal{C}_1) = \int_{x=-1}^{x=1} \quad \int_{y=0}^{y=1} \quad ?$$

CA:

$-1 \leq x \leq 1$

$\Big(x^2 = y^3 \Rightarrow y = x^{2/3}$

$0 \leq y \leq 1$

Parametrizo $\mathcal{C}$

Sé que $x^2 = y^3 \Rightarrow (x^2)^{1/3} = y$

conozco y en función de x

$$\Rightarrow \sigma(t) = \left(t, (t^2)^{1/3}\right) \quad \text{con } -1 \leq t \leq 1$$

$$= \left(t, t^{2/3}\right)$$

Mejor parametrizo como:

$$\nabla(t) = \left(t^3, t^2\right) \quad \text{con } -1 \leq t \leq 1$$

Pruebo que efectivamente, $\nabla(t)$ parametriza a $\mathcal{C}$

↳ Pruebo doble inclusión:

- $\mathrm{Im}(\nabla) \subset \mathcal{C}$? sí, pues $(\underbrace{t^3}_{x}, \underbrace{t^2}_{y}) \Rightarrow (t^3)^2 = (t^2)^3$ ✓

$\mathcal{C}$ eran los x, y: $x^2 = y^3 \wedge -1 \leq x \leq 1$

- $\mathcal{C} \subset \mathrm{Im}(\nabla)$?

↳ es decir que todos los puntos de $\mathcal{C}$ se escriben como $\nabla(t)$ para algún t

Sea $(x,y) \in \mathcal{C} \Rightarrow x^2 = y^3$

$$(x^2)^{1/3} = y$$

$\therefore\ y \geq 0$ y existe $\sqrt{y}$

$$x^2 = y^3 = (\sqrt{y})^6 \Rightarrow |x| = (\sqrt{y})^3$$

- Supongamos que $x \geq 0 \Rightarrow x = (\sqrt{y})^3$

Luego

$$(x,y) = \left(\overbrace{(\sqrt{y})^3}^{=x}, \overbrace{(\sqrt{y})^2}^{=y}\right) = \nabla(\sqrt{y})$$

- Si $x \leq 0 \Rightarrow x = -(\sqrt{y})^3$

$$(x,y) = \Big(\overbrace{(-\sqrt{y})^3}^{=x<0}, \overbrace{(-\sqrt{y})^2}^{=y}\Big) = \nabla(-\sqrt{y})$$

Teniendo las parametrizaciones de $\mathcal{C}_1$ y $\mathcal{C}_2$

Calculo:

$0 \leq \sqrt{y} \leq 1 \Rightarrow 0 \leq t \leq 1$

$$\ell(\overset{x\geq 0}{\mathcal{C}_2}) = \int_{t=0}^{t=1} \|\nabla(t)\| dt = \int_0^1 \|(t^3, t^2)'\| dt$$

$$= \int_0^1 \|(3t^2, 2t)\| dt$$

$$= \int_0^1 (9t^4 + 4t^2)^{1/2} dt = \int_0^1 \left(t^2(9t^2+4)\right)^{1/2}$$

$t \geq 0$: saco $|\cdot|$

$$= \int_0^1 |t| \cdot (9t^2+4)^{1/2} dt = \int_{s=4}^{s=13} \frac{(s)^{1/2}}{18} \cdot ds = \frac{1}{18}\int_4^{13} \sqrt{s}\, ds$$

$s = 9t^2+4$

$ds = 18t$

$$= \frac{1}{18} \cdot \int_{u=2}^{u=\sqrt{13}} u \cdot 2u\, du = \frac{1}{9}\int_2^{\sqrt{13}} u^2 du = \frac{1}{9}\left[u^3/3\right]_2^{\sqrt{13}}$$

$u = \sqrt{s}$

$du = \frac{1}{2\underbrace{\sqrt{s}}_{=u}} ds$

$ds = 2u\, du$

$$= \frac{1}{9} \cdot \left(\frac{13^{3/2}}{3} - \frac{2^3}{3}\right) = \frac{1}{27}\left(13^{3/2} - 8\right)$$

$$\therefore \text{long}(\mathcal{C}) = \frac{2}{27}\left(13^{3/2} - 8\right) \approx 2{,}879$$

Ej 3: $\sigma(t) = \left(\cos t, \sin t, \frac{t^2}{2}\right) \qquad 0 \leq t \leq 1$

Longitud

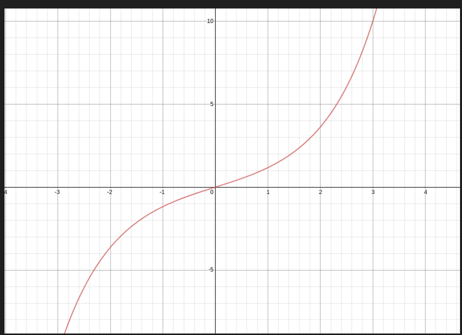
$$\ell(\sigma) = \int_0^1 \|\sigma_t'\|\, dt = \int_0^1 \left\|\left(-\sin t, \cos t, \frac{2t}{2}\right)\right\| dt$$

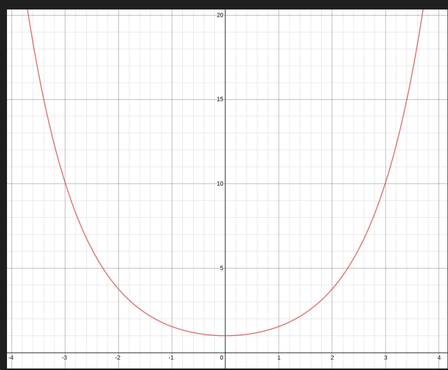
$$= \int_0^1 (\underbrace{\sin^2 t + \cos^2 t}_{=1} + t^2)^{1/2} dt = \int_0^1 \sqrt{1+t^2}\, dt$$

Para resolver esa integral ↗

Introducimos Funciones Hiperbólicas

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$





Propiedades

- Derivadas

$$(\sinh(x))' = \cosh(x) \quad y \quad (\cosh(x))' = \sinh x$$

usaremos →

- $\cosh^2(x) - \sinh^2(x) = 1$

- $\cosh(2x) = 2\cosh^2(x) - 1$

$\sinh(2x) = 2\sinh(x) \cdot \cosh(x) - 1$

- $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

  $\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$

  ↳ rama positiva

Volviendo a la integral

$$\int_0^1 \sqrt{1+t^2}\, dt$$

Propongo cambio de variable

$t = \sinh(x)$

$dt = \cosh(x)\, dx$

para $t=0 \Rightarrow 0 = \sinh x$

$\operatorname{arcsinh}(0) = x$

$x = 0$

para $t = 1$

$\Rightarrow 1 = \sinh x$

$\operatorname{arcsin}(1) = \ln\left(1+\sqrt{2}\right)$

Entonces

$$\int_0^1 \sqrt{1+t^2}\, dt = \int_0^{\ln(1+\sqrt{2})} \underbrace{\sqrt{1+\sinh^2 x}}_{\sqrt{\cosh^2 x}} . \cosh(x) . dx =$$

$$= \int \cosh(x) . \cosh(x)\, dx = \int_0^{\ln(1+\sqrt{2})} \cosh^2 x\, dx$$

$$= \int_0^{\ln} \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = \int_0^{\ln(1+\sqrt{2})} \frac{e^{2x}}{4} + \overbrace{\frac{e^{x-x}}{4}}^{1/4} + \frac{e^{-2x}}{4}\, dx$$

$$= \frac{1}{4}\int_0^{\ln(1+\sqrt{2})} e^{2x} + \frac{1}{4} . \ln\left(1+\sqrt{2}\right) + \frac{1}{4}\int e^{-2x}$$

CA:

$$\frac{1}{4}\cdot\frac{e^{2x}}{2}\Big|_0^{\ln(1+\sqrt{2})} = \frac{e^{\ln(1+\sqrt{2})}\cdot e^{\ln(1+\sqrt{2})}}{8}$$

$$= \frac{(1+\sqrt{2})\cdot(1+\sqrt{2})}{8} = \frac{1}{8}\cdot\left(1+2\sqrt{2}+2\right)$$

$$= \frac{\sqrt{2}}{4}+\frac{3}{8}$$

$$= \frac{(1+\sqrt{2})^2}{8} + \frac{\ln(1+\sqrt{2})}{4} - \frac{(1+\sqrt{2})^{-2}}{8} \quad //$$

Cambio de variable hiperbólico

en general

$$\int \sqrt{a + m\cdot t^2}\, dt$$

ej

$$\sqrt{1+t^2}$$

$$\sqrt{5+3t^2}$$