Práctica 1

1. a. Probar que

$$\sigma(t) = \begin{cases} x_1(t) = r\cos(2\pi t) \\ y_1(t) = r\,sen(2\pi t) \end{cases} \quad y \quad \tilde{\sigma}(t) = \begin{cases} x_2(t) = r\cos(4\pi t) \\ y_2(t) = r\,sen(4\pi t) \end{cases}$$

son param. $\mathcal{C}^1$ de la circ. de de centro $(0,0)$ y radio r.

$$C = \{ (x,y) \in \mathbb{R}^2 / x^2 + y^2 = r^2 \}$$

$\sigma(t)$ y $\tilde{\sigma}(t)$ son continuas por comp. de continuas y son $\mathcal{C}^1$ porque sus derivadas $\exists$ y son continuas.

Queremos que ver que $Im(\sigma(t)) = Im(\tilde{\sigma}(t)) = C$

Tienen que coincidir como conjuntos de $\mathbb{R}^2$.

$\boxed{Im(\sigma(t)) \subset C}$ y $\boxed{C \subset Im(\sigma(t))}$ y biyectiva!

$\exists\, t \in \mathbb{R} \,/\, \sigma(t)$ se puede escribir como

$(r\cos(2\pi t), r\,\text{sen}(2\pi t))$ y cumple que $x^2+y^2=r^2$

$(r\cos(2\pi t))^2 + (r\,\text{sen}(2\pi t))^2 = r^2\cos^2(2\pi t) + r^2\,\text{sen}^2(2\pi t)$

$r^2\big(\underbrace{\cos^2(2\pi t) + \text{sen}^2(2\pi t)}_{=1}\big) = \boxed{r^2}$

$\Rightarrow \sigma(t)$ [illegible]

Veamos con $t \,/\, \tilde{\sigma}(t) = (r\cos(4\pi t), r\,\text{sen}(4\pi t))$

$r^2\cos^2(4\pi t) + r^2\,\text{sen}^2(4\pi t) = r^2\big(\cos^2(4\pi t) + \text{sen}^2(4\pi t)\big)$

$= \boxed{r^2}$ también cumple.

Obs: $\tilde{\sigma}(t)$ recorre la curva "más rápido" que $\sigma(t)$ ya que $4\pi t$ da el doble de vueltas a la circunferencia que $2\pi t$.

Ya probé que si agarro un $t \in Im(\sigma(t))$ o a la $Im(\tilde{\sigma}(t))$ ambas cumplen $x^2+y^2=r^2$. Ahora hay que probar que si tomo un (x,y) que cumpla la ecuación $x^2+y^2=r^2$ debe $\exists$ t que esté en la imagen de $\sigma(t)$ y $\tilde{\sigma}(t)$

Tomo $(r,0)$ $\quad r^2+0^2=r^2$ ✓ $\quad$ tomo $t=0$ y

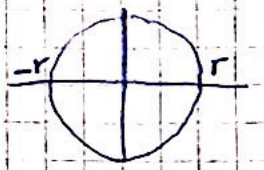
$\sigma(0)=(r,0)$ ✓

$\Rightarrow \sigma$ es parametrización ✓

$\tilde{\sigma}(0)=(r,0) \Rightarrow \tilde{\sigma}$ también ✓

Escaneado con CamScanner

b. Es una curva cerrada y simple: admite una param.



$\sigma: [a,b] \to C \subset \mathbb{R}^2$ que es inyectiva en $[a,b)$ y $\sigma(a) = \sigma(b)$

$\sigma: [0,1] \to C$ $\quad \sigma(0) = (r,0)$ y $\sigma(1) = (r,0)$

$\sigma(a) = \sigma(b)$

Si la parametrización es regular, entonces la curva es suave

- Inyectiva ✓
- $\mathcal{C}^1 = \sigma'(t) = (-r\,\text{sen}(2\pi t)2\pi,\ r\cos(2\pi t)2\pi)$

son continuas ambas coordenadas.

$\sigma'(0) = (0,1) \neq (0,0)$

$\Rightarrow$ es regular y la curva es suave.

c $\tilde{\sigma}(t): [0,1] \to C$

$\tilde{\sigma}(t)$ no es inyectiva porque repite valores:

$\tilde{\sigma}\left(\frac{1}{2}\right) = (r,0) \quad \tilde{\sigma}(0) = (r,0)$

2 $\sigma(t) \begin{cases} (0,(1-t)^2) & 0 \leq t \leq 1 \\ ((t-1)^2, 0) & 1 \leq t \leq 2 \end{cases}$ son dos curvas unidas

$\sigma'(t) = \begin{cases} (0, -2(1-t)) \\ (+2(t-1), 0) \end{cases}$ Las derivadas existen y son continuas $= \mathcal{C}^1$

Como $t = 1$ $\sigma(1) = (0,0) \in \mathcal{C}$

Como $(1,0) \in \mathcal{C}$ y se puede escribir como $t = 2$

$\sigma(2) = (1,0) \Rightarrow$ Es parametrización.

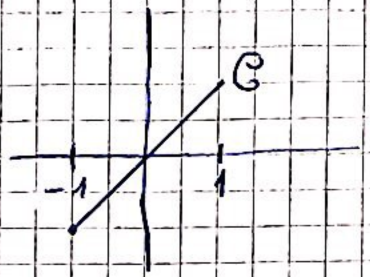
Obs: en $(0,0)$ la derivada no existe porque en el ángulo las derivadas "tendrían diferente dirección" y por ende no admite param regular y no puede tener recta tangente.

3. $\sigma(t) = (t^3, t^3) \quad t \in [-1, 1]$

$\sigma'(t) = (3t^2, 3t^2)$ Las derivadas existen y son continuas $\Rightarrow C^1$

$\sigma'(t) = (0, 0) \rightsquigarrow$ no es una parametrización regular.

$y = x \quad x \in [-1, 1]$



Si tomo $t = 1$

$\sigma(1) = (1, 1)$

si tomo $x = 0$

$(x, y) = (0, 0)$

$\sigma(0) = (0, 0) \rightsquigarrow$ es parametrización porque

$Im(\sigma(t)) \subset C$ y $C \subset Im(\sigma(t))$

C es una curva suave porque admite una parametrización regular. $\sigma(t)$ no es regular.

$\tilde{\sigma}(t) = (t, t)$ con $t \in [-1, 1]$

$t = 1 \quad \tilde{\sigma}(1) = 1, 1)$

con $(x, y) = (-1, 1) \quad \tilde{\sigma}(-1) = (-1, 1) \rightsquigarrow$ es parametrización

$\tilde{\sigma}'(t) = (1, 1) \rightsquigarrow$ la derivada $\neq (0, 0) \Rightarrow$ es regular

$\Rightarrow$ C es suave.

4.

$y = x^2$

$0 \leq x \leq 1$

$f: [0, 1] \to \mathbb{R}$

$f(x) = x^2$

a. curva abierta

$f(0) = (0, 0)$

$f(1) = (1, 1)$ $\neq$

Los extremos no coinciden

$\{(x, y) \in \mathbb{R}^2 / y = f(x), 0 \leq x \leq 1\}$

curva simple: $f(x)$ es inyectiva ✓

Curva suave: tomo $\sigma_1(t) = (t, t^2)$ $\quad \sigma_1'(t) = (1, 2t)$

Admite una param regular $\Rightarrow$ C suave. $\quad \sigma_1'(0) = (1, 0) \neq (0, 0) \; \forall t$

c. $\sigma_1(t)$ es param porque $\sigma(0) = (0,0) \in G$
y $(1,1) \in G = \sigma(1) = (1,1)$ ✓

b. $\bar{\sigma}(s) = (\bar{x}(s), \bar{y}(s))$ $s \in [0, \ln(2)]$

$$\bar{\sigma}(s) = \begin{cases} \bar{x}(s) = e^s - 1 \\ \bar{y}(s) = (e^s - 1)^2 \end{cases}$$ es param. regular de G

$\bar{\sigma}'(s) = (e^s, 2(e^s-1)e^s)$

$e^s \neq 0 \rightsquigarrow$ nunca se hace $(0,0)$

$\bar{\sigma}(0) = (0,0) \in G$ y $(1,1) = \sigma(\ln(2)) = (1,1)$
$\rightsquigarrow$ es param y como su derivada no se anula es regular.

d.
$$\begin{array}{ccc} [0,1] & \xrightarrow{\sigma(t)} & \mathbb{R} \\ & {\scriptstyle g} \searrow \quad [0, \ln(2)] \quad \nearrow {\scriptstyle \bar{\sigma}(s)} & \end{array}$$

$\bar{\sigma}(g(t)) = \sigma(t)$

$\bar{\sigma}(s) = (e^s - 1, (e^s-1)^2) = (t, t^2)$

$$\begin{aligned} e^s - 1 &= t \\ e^s &= t+1 \\ s &= \ln(t+1) \end{aligned}$$

$\underbrace{(e^s-1)^2}_{>0} = t^2$
Como $g: [0,1]$ y $t > 0$ puedo aplicar raíz.
$e^s - 1 = t$
$s = \ln(t+1)$

$g(t): [0,1] \to [0, \ln(2)]$

$g(t) = (\ln(t+1), \ln(t+1))$

$g(0) = (0,0)$; $g(1) = [\ln(2), \ln(2)]$ es biyectiva (además manda extremos de intervalo en extremos de intervalo)

$g'(t) = \left(\frac{1}{t+1}, \frac{1}{t+1}\right) \neq (0,0) \rightsquigarrow$ es C^1 y además sus derivadas nunca se anulan.

5. $L_{P_0} = \lambda\, \sigma'(t_0) + \underset{\sigma(t_0)}{\overset{\shortparallel}{P_0}}$ recta tangente en t_0

a. $\sigma(0) = (0,0,0)$

$\sigma'(t) = (6,\ 6t,\ 3t^2) \qquad \sigma'(0) = (6,0,0)$

$L_{(0,0,0)} = (6,0,0)\,t$

b. $\sigma(0) = (1,\ 0,\ 0)$

$\sigma'(t) = (2\cos t\,(-\operatorname{sen} t),\ 3-3t^2,\ 1)$

$\sigma'(0) = (\ 0,\ 3,\ 1)$

$L_{(1,0,0)} = t\ (0,3,1) + (1,0,0)$

c. $\sigma(1) = (\operatorname{sen} 3,\ \cos 3,\ 2)$

$\sigma'(t) = \left(\cos(3t)\cdot 3,\ -\operatorname{sen}(3t)\cdot 3,\ 2\cdot\frac{3}{2}\sqrt{t}\right)$

$\sigma'(1) = (3\cos(3)\ ,\ -\operatorname{sen}(3)\cdot 3,\ 3\ )$

$L_{(\operatorname{sen}3,\cos3,2)} = (t-1)(3\cos(3),\ -\operatorname{sen}(3)\cdot 3\ ,3) + (\operatorname{sen}3, \cos 3, 2)$

d. $\sigma(1) = (0,0,1)$

$\sigma'(t) = (0,0,1) = \sigma'(1)$

$L_{(0,0,1)} = (t-1)(0,0,1) + (0,0,1)$

6. La fuerza podemos definirla como $\vec{F} = m\cdot\vec{a}$.

$\vec{a} = \sigma''(t) = (2(-\operatorname{sen} t)(-\operatorname{sen} t) + 2\cos(t)(-\cos t),\ -6t, 0)$

$\vec{a} = (2\operatorname{sen}^2(t) - 2\cos^2(t),\ -6t,\ 0)$ $\qquad \operatorname{sen}^2 + \cos^2 = 1$

$\vec{a} = (2\operatorname{sen}^2(t) - 2(1-\operatorname{sen}^2(t)),\ -6t,\ 0)$ $\qquad \cos^2 = 1 - \operatorname{sen}^2$

$\vec{a} = (2\operatorname{sen}^2(t) - 2 + 2\operatorname{sen}^2(t),\ -6t,\ 0)$

$\vec{a} = (4\operatorname{sen}^2(t) - 2,\ 6t,\ 0) \qquad \vec{a}(0=t) = (-2,0,0)$

$\vec{F} = (-2m, 0, 0)$

7. $\sigma(t) = (e^{t}, e^{-t}, \cos t)$ $\quad \sigma(1) = \left(e, \frac{1}{e}, \cos(1)\right)$

$\sigma'(t) = (e^{t}, -e^{-t}, -\operatorname{sen} t)$

$\sigma'(1) = \left(e, -\frac{1}{e}, -\operatorname{sen}(1)\right)$

$$L_{(e,\frac{1}{e},\cos(1))} = (t-1)\left(e, -\frac{1}{e}, -\operatorname{sen}(1)\right) + \left(e, \frac{1}{e}, \cos(1)\right)$$

Obs: cuando $t=1$ la posición es $\sigma(1)$

En $t=1$ la partícula sigue la trayectoria dada por ~~el~~ vector velocidad $\sigma'(1)$

$t=2$ $\quad L = 1\left(e, -\frac{1}{e}, -\operatorname{sen}(1)\right) + \left(e, \frac{1}{e}, \cos(1)\right)$

$$\boxed{L = \left(2e, 0, \cos(1) - \operatorname{sen}(1)\right)}$$ Posición de la partícula en $t=2$

Integral de longitud de arco

$\mathcal{L}(\rho) = \int_a^b \underbrace{\|\sigma'(t)\|}_{\text{"rapidez"}} dt$ Me dice la longitud del vector tangente a la curva en cada punto según la parametrización σ, cuando integro la función entre a y b obtengo la longitud de la curva.

Parámetro de longitud de arco

$S \in [0, \mathcal{L}(c)]$ $\quad S = \int_a^t \|\sigma'(l)\|\, dl$

8. $\sigma(t) = (t - \operatorname{sen}(t), 1 - \cos(t))$

Velocidad $\quad \sigma'(t) = (1 - \cos(t), \operatorname{sen}(t))$

Rapidez $\quad \|\sigma'(t)\| = \sqrt{(1-\cos(t))^2 + \operatorname{sen}^2(t)}$

$$= \sqrt{1 - 2\cos(t) + \underbrace{\cos^2(t) + \operatorname{sen}^2(t)}_{=1}}$$

$$= \sqrt{1 - 2\cos(t) + 1}$$

Longitud de arco entre $\sigma(0)$ y $\sigma(2\pi)$.

$$\mathcal{L}(C) = \int_0^{2\pi} \sqrt{2-2\cos(t)}\, dt = \int_0^{2\pi} \sqrt{2(1-\cos(t))}\, dt$$

Identidad: $\boxed{1-\cos(t) = 2\,\text{sen}^2\left(\frac{t}{2}\right)}$

$$= \int_0^{2\pi} \sqrt{2\,\text{sen}^2\left(\frac{t}{2}\right)}\,\sqrt{2}\, dt = \int_0^{2\pi} \sqrt{2}\cdot\sqrt{2}\,\underbrace{\sqrt{\text{sen}^2\left(\frac{t}{2}\right)}}_{\geq 0}\, dt$$

$$= 2\int_0^{2\pi} \left|\text{sen}\left(\frac{t}{2}\right)\right| dt = 2\int_0^{2\pi} \text{sen}\left(\frac{t}{2}\right) dt =$$

entre 0 y 2π es positivo

$u = \frac{t}{2}$

$du = \frac{1}{2}\, dt$

$$= 2\int_{u(0)}^{u(2\pi)} \text{sen}(u)\, 2\,du = +4\, -\cos(u)\Big|_{u(0)}^{u(2\pi)}$$

$$= 4\left(-\cos\frac{t}{2}\Big|_0^{2\pi}\right) = 4\left(-\cos\pi + \cos 0\right) = \boxed{8}$$

9. a $\sigma(t): [0,1] \to C \quad \sigma(t) = (t, t^2)$

$$\sigma'(t) = (1, 2t)$$

$$\|\sigma'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$$

$$\mathcal{L}(c) = \int_0^1 \sqrt{1+(2t)^2}\, dt = \int_0^2 \sqrt{1+u^2}\,\frac{du}{2} = \int_0^{\text{senh}^{-1}(2)} \sqrt{1+\text{senh}^2(t)}\,\text{Cosh}(t)\frac{dt}{2}$$

$u = 2t$ $\qquad u = \text{senh}(t)$

$du = 2\,dt$ $\qquad du = \text{Cosh}(t)\,dt$

Cuando $t = \text{senh}^{-1}(2)$ $u = 2$

Identidad hiperbólica $\quad \boxed{\begin{array}{l}\text{Cosh}^2(t) - \text{senh}^2(t) = 1 \\ \text{Cosh}^2(t) = 1 + \text{senh}^2(t)\end{array}}$

$$= \int_0^{\text{senh}^{-1}(2)} \sqrt{\text{cosh}^2(t)}\cdot\text{cosh}(t)\,\frac{dt}{2}$$

Wolfram Alpha   Debería dar entre $\sqrt{2}$ y 2

$$= \int_0^{\text{senh}^{-1}(2)} |\underbrace{\cosh(t)}_{>0}| \cdot \cosh(t) \frac{dt}{2}$$

$$= \int_0^{\text{senh}^{-1}(2)} \cosh(t) \cdot \cosh(t) \frac{dt}{2} = \int_0^{\text{senh}^{-1}(2)} \cosh^2(t) \frac{dt}{2}$$

Identidad $\boxed{\cosh^2(t) = \frac{1 + \cosh(2t)}{2}}$

$$= \frac{1}{2} \int_0^{\text{senh}^{-1}(2)} \frac{1 + \cosh(2t)}{2} \, dt$$

$$= \frac{1}{4} \left(\int_0^{\text{senh}^{-1}(2)} 1 \, dt + \int_0^{\text{senh}^{-1}(2)} \cosh(2t) \, dt \right) \qquad \begin{array}{l} u = 2t \\ du = 2dt \end{array}$$

$$= \frac{1}{4} \left(1 \Big|_0^{\text{senh}^{-1}(2)} + \int_0^{\text{senh}^{-1}(2) \cdot 2} \cosh(u) \frac{du}{2} \right)$$

$$= \frac{1}{4} \left(\text{senh}^{-1}(2) + \frac{\text{senh}(u)}{2} \Big|_0^{\text{senh}^{-1}(2) \cdot 2} \right)$$

$$= \frac{1}{4} \left(\text{senh}^{-1}(2) + \frac{\text{senh}(2t)}{2} \Big|_0^{\text{senh}^{-1}(2)} \right)$$

$$= \frac{1}{4} \left(\text{senh}^{-1}(2) + \frac{\text{senh}(2 \cdot \text{senh}^{-1}(2)) - \overset{0}{\text{senh}(0)}}{2} \right)$$

$\mathcal{L}(c) =$

$$= \frac{1}{4} \left(\text{senh}^{-1}(2) + \frac{2 \cdot \cancel{4}\sqrt{5}}{\cancel{2}} \right) = \boxed{\frac{1}{4} \left(\text{senh}^{-1}(2) + 2\sqrt{5} \right)}$$

9.b $\sigma : [10, 20] \to C$

$\sigma(t) = (\sqrt{t}, t+1, t)$ $\qquad \sigma'(t) = \left(\frac{1}{2\sqrt{t}}, 1, 1\right)$

$\|\sigma'(t)\| = \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 2} = \sqrt{\frac{1}{4t} + 2} = \sqrt{2\left(\frac{1}{8t} + 1\right)}$

$\int_{10}^{20} \sqrt{2} \sqrt{\frac{1}{8t} + 1} \underset{\downarrow}{=} \int \sqrt{\frac{1}{u} + 1} \sqrt{2} \frac{du}{8} = \frac{\sqrt{2}}{8} \int \sqrt{\frac{1+u}{u}} du$

$u = 8t$

$\frac{du}{8} = dt$

$= \frac{\sqrt{2}}{8} \int \frac{\sqrt{1+u}}{\sqrt{u}} du \underset{\downarrow}{=} \left(\int 2\cancel{\sqrt{u}} \frac{\sqrt{1+s^2}}{\cancel{\sqrt{u}}} ds\right) \frac{\sqrt{2}}{8}$

$s = \sqrt{u} \rightsquigarrow \sqrt{u+1} = \sqrt{1+s^2}$

$ds = \frac{1}{2\sqrt{u}} du$

$= \frac{\sqrt{2}}{8} \cdot 2 \int \underbrace{\sqrt{1+s^2}}\, ds$

$\hookrightarrow$ Es similar al item a.

Identidades hiperbólicas

$$\cosh^2(x) - \operatorname{senh}^2(x) = 1$$

$$\cosh'(x) = \operatorname{senh}(x) \qquad \operatorname{senh}'(x) = \cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \operatorname{senh}^2(x)$$

$$\operatorname{senh}(2x) = 2\operatorname{senh}(x)\cosh(x)$$

$$1 + \operatorname{tg}^2 = \sec^2$$

10 C suave, $\sigma : [a,b] \to \mathbb{R}^3$ param. regular $\mathcal{C}$
$g : [\bar{a}, \bar{b}] \to [a,b]$ biyección $\mathcal{C}^1$ $g'(s) \neq 0$ $\forall s \in (a,b)$

$\bar{\sigma} : [\bar{a}, \bar{b}] \to \mathbb{R}^3$ dada por $\bar{\sigma}(s) = \sigma(g(s))$

$\bar{\sigma}$ es una reparametrización de σ

a. $\bar{\sigma}'(s) = \sigma'(\underbrace{g(s)}) \cdot \underbrace{g'(s)}_{\neq 0}$
$=$

$g(s) \neq 0$ $\forall s$

$\sigma'(g(s)) \neq 0$ porque es $\mathcal{C}^1$

Como la derivada no se anula, si $\bar{\sigma}(s)$ es param $\Rightarrow$ es regular.
Como $\bar{\sigma}$ es reparametrización de σ sabemos que es param. de $\mathcal{C}$

$\|\bar{\sigma}'(s)\| = \|\sigma'(s)\| \cdot |g'(s)| =$

b.
$$\begin{tikzcd}
{[\bar{a},\bar{b}]} \arrow[r, "g"] \arrow[dr, "\bar{\sigma}"'] & {[a,b]} \arrow[d, "\sigma"] \\
& \mathbb{R}^3 \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

Evaluar la función f sobre la param σ o $\bar{\sigma}$ es lo mismo:

$$\int_C f = \int_a^b f(\sigma(t))\,\|\sigma'(t)\|\,dt$$

$$\int_C f = \int_{\bar a}^{\bar b} f(\bar\sigma(t))\,\|\sigma'(t)\|\,dt$$

$$= \int_{\bar a}^{\bar b} f(\sigma(g(s)))\,\|\sigma'(g(s))\|\,|g'(s)|\,ds.$$

Cambia el extremo de integración y se multiplica por una constante $\Rightarrow$ son iguales.

11. C curva simple, suave

$\sigma : [a,b] \to \mathbb{R}^3$ parametrización regular de C

Para cada $t \in [a,b]$ (dominio de σ) sea $h(t)$ la longitud de arco de la curva C entre los puntos $\sigma(a)$ y $\sigma(t)$

$$h(t) = \int_a^t \|\sigma'(\ell)\|\,d\ell \qquad h:[a,b] \to [0,\mathcal{L}(C)]$$

Continuamente diferenciable y $h'(t) \neq 0 \ \forall t$

Por Teo Fundamental del Cálculo $\therefore h'(t) = \|\sigma'(\ell)\| > 0$ $\mathcal{C}^1$ (pues σ es regular). $h(t)$ admite inversa: $\ell(h) : [0,\mathcal{L}(C)] \to [a,b]$ con derivada $\ell'(h) = \dfrac{1}{\|\sigma'(\ell(h))\|}$

Reparametrización por longitud de arco:

$$\bar\sigma(s) = \sigma(h^{-1}(s)) = \sigma(\ell(s))$$

$$\begin{tikzcd} & {[a,b]} \arrow[dl, "\sigma"'] \arrow[dr, bend left, "h"] & \\ \mathbb{R}^3 & & {[0,\mathcal{L}(C)]} \arrow[ul, "\ell"] \arrow[ll, bend left, "\bar\sigma"] \end{tikzcd}$$

$$\bar\sigma(s) : [0,\mathcal{L}(C)] \to C \subset \mathbb{R}^3$$

$\|\bar\sigma'(s)\| = 1$ pues

$$\bar\sigma'(s) = \sigma'(\ell(s))\,\ell'(s) = \frac{\sigma'(\ell(s))}{\|\sigma'(\ell(s))\|}$$

Escaneado con CamScanner

Entonces $\int_0^s \|\bar\sigma'(s)\|\, ds = \int_0^s 1\, ds = s\big|_0^s = \boxed{s}$

12. a. $\sigma(t) = (\cos t, \operatorname{sen} t, t) \quad a=0 \quad b=1$

$\sigma'(t) = (-\operatorname{sen} t, +\cos t, 1)$

$\|\sigma'(t)\| = \sqrt{(-\operatorname{sen} t)^2 + (\cos t)^2 + 1^2}$

$= \sqrt{\operatorname{sen}^2 t + \cos^2 t + 1} = \sqrt{2}$

$h(t) = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t = s$ si $t=1 \Rightarrow \mathcal{L}(c) = \sqrt{2}$

$\boxed{t = \frac{s}{\sqrt{2}}}$

$\boxed{\bar\sigma(s) = \sigma\left(\frac{s}{\sqrt{2}}\right) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \operatorname{sen}\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)}$ $\bar\sigma(s): [0, \sqrt{2}] \to [0,1]$

$\bar\sigma'(s) = \left(-\operatorname{sen}\left(\frac{s}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}, \cos\left(\frac{s}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

b $\sigma(t) = (2e^t, 3e^t + 1, -6e^t)$

$\sigma'(t) = (2e^t, 3e^t, -6e^t)$

$\|\sigma'(t)\| = \sqrt{4e^{2t} + 9e^{2t} + 36e^{2t}} = \sqrt{49 e^{2t}} = 7\sqrt{e^{2t}}$

$= 7e\sqrt{e^t}$

$h(t) = \int_0^{\ln(t)} 7e\, e^{\frac{l}{2}}\, dl = 7e \frac{e^{\frac{l}{2}+1}}{\frac{l}{2}+1}\Bigg|_0^{\ln(t)} = \frac{e^{\frac{l+2}{2}}}{\frac{l+2}{2}}\Bigg|_0^{\ln(t)} 7e$

$h(t) = 7e\cdot\left(\frac{e^{\frac{\ln(t)+2}{2}}}{\ln(t)+2}\cdot 2 - \frac{e}{1}\right) = s$ *

$\mathcal{L}(c) = 7e\,(4{,}7\cdot 0{,}64 - e) = 6{,}000$ OMG

$\bar\sigma(s) = h^{-1}(t) \qquad \bar\sigma: [0,6] \to [0, \ln(3)]$

$* = 14\, \frac{e^{\frac{\ln(t)+2}{2}}}{\ln(t)+2} - 7e^2 = s$

$14\, e^{\frac{\ln(t)+2}{2}} = (s - 7e^2)(\ln(t)+2)$

12.b. $\sigma(t) = (2e^t, 3e^t + 1, -6e^t)$

$\sigma'(t) = (2e^t, 3e^t, -6e^t)$

$\|\sigma'(t) = \sqrt{4e^{2t} + 9e^{2t} + 36e^{2t}} = \sqrt{49e^{2t}}$

$= 7 e^{\frac{2t}{2}}$

$= \boxed{7e^t}$

$h(t) = \int_0^{\ln(t)} 7e^l \, dl = 7e^l \Big|_0^{\ln(t)}$

$= 7e^{\ln(t)} - 7e^0 = 7t - 7$

$S = \boxed{7(t-1)}$

$\mathcal{L}(c) = 7e^l \Big|_0^{\ln(3)} = 7e^{\ln(3)} - 7e^0 = 21 - 7 = 14$

$\bar{\sigma} : [0, 14] \to [0, \ln(3)]$

$h^{-1}(t) = \bar{\sigma}(s) = \sigma\left(\frac{s}{7} + 1\right) = \left(2e^{\frac{s}{7}+1}, 3e^{\frac{s}{7}+1} + 1, -6e^{\frac{s}{7}+1}\right)$

13. $\sigma(t) = (\operatorname{sen} t, \cos t, t) \quad t \in [0, 2\pi]$

$\sigma'(t) = (\cos t, -\operatorname{sen} t, 1)$

$\|\sigma'(t) = \sqrt{\underbrace{\cos^2 t + \operatorname{sen}^2 t}_{=1} + 1} = \sqrt{2}$

$\int_C f(\sigma(t)) \cdot \|\sigma'(t)\| \, dt =$

a. $\sqrt{2} \int_0^{2\pi} \operatorname{sen} t + \cos t + t \, dt$

$= \sqrt{2}\left(-\cos t + \operatorname{sen} t + \frac{t^2}{2}\right)\Big|_0^{2\pi} = \sqrt{2}\left(-1 + 0 + \frac{4}{2}\pi^2 - (-1 + 0 + 0)\right)$

$= \sqrt{2}\left(\cancel{-1} + 2\pi^2 \cancel{+1}\right) = \boxed{\sqrt{2}\, 2\pi^2}$

b. $\int_0^{2\pi} \cos(t) \cdot \sqrt{2} \, dt = \operatorname{sen} t \Big|_0^{2\pi} \cdot \sqrt{2} = \boxed{0}$

C. $\sigma(t) = (t, t^2, 0) \quad t \in [0,1]$

$\sigma'(t) = (1, 2t, 0)$

$\|\sigma'(t)\| = \sqrt{1^2 + 4t^2}$

$$\int_0^1 t \cos(0) \sqrt{1+4t^2}\, dt$$

$$= \int_0^1 1\, t \sqrt{1+4t^2}\, dt \underset{\downarrow}{=} \int_1^5 \cancel{t} \sqrt{u} \frac{du}{8\cancel{t}} = \left.\frac{u^{3/2}}{\frac{3}{2}}\right|_1^5$$

$u = 1+4t^2$

$du = 8t\,dt$

$$= \frac{\sqrt[2]{5^2\, 5}}{3} \cdot 2 - \frac{1}{3} \cdot 2 = \frac{5}{3}\sqrt{5} \cdot 2 - \frac{2}{3} = \boxed{\frac{1}{3}\left(10\sqrt{5} - 2\right)}$$

14. a Coordenadas polares

$\boxed{r = r(\theta)} \qquad \theta_1 \leq \theta \leq \theta_2$

r es función de θ

$$\sigma(t) = \begin{cases} x = r(\theta)\cos(\theta) \\ y = r(\theta)\operatorname{sen}(\theta) \end{cases}$$

$$\int_C f(x,y)\, dx\,dy = \int_C f(\sigma(t)) \cdot \|\sigma'(t)\|\, dt$$

$$\sigma'(t) = r(\theta)(-\operatorname{sen}(\theta), \cos(\theta)) + r'(\theta)(\cos\theta, \operatorname{sen}\theta)$$

regla de la cadena

$$\sigma'(t) = (-r(\theta)\operatorname{sen}(\theta), r(\theta)\cos(\theta)) + (r'(\theta)\cos(\theta), r'(\theta)\operatorname{sen}(\theta))$$ voy a omitir (θ).

$\sigma'(t) = (-r \operatorname{sen} + r'\cos, r\cos + r'\operatorname{sen})$

$\|\sigma'(t)\|^2 = (-r\operatorname{sen} + r'\cos)^2 + (r\cos + r'\operatorname{sen})^2$

$= r^2\operatorname{sen}^2 - \cancel{2rr'\operatorname{sen}\cos} + r'^2\cos^2$

$+ r^2\cos^2 + \cancel{2rr'\operatorname{sen}\cos} + r'^2\operatorname{sen}^2$

$= r^2(\operatorname{sen}^2 + \cos^2) + r'^2(\cos^2 + \operatorname{sen}^2) = \boxed{r^2 + r'^2}$

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$$\int_C f(\sigma(t)) \cdot \|\sigma'(t)\| \, dt =$$

$$\int_{\theta_1}^{\theta_2} f\big(r(\theta)\cos(\theta),\, r(\theta)\,\mathrm{sen}(\theta)\big) \cdot \sqrt{r^2(\theta) + r'^2(\theta)}\; d\theta$$

//

b. $r = 1 + \cos\theta$ $\quad 0 \leq \theta \leq 2\pi$ $\quad r' = -\,\mathrm{sen}\,\theta$ $\quad f = 1$

$$\int_0^{2\pi} f \ \sqrt{(1+\cos\theta)^2 + (-\,\mathrm{sen}\,\theta)^2}\; d\theta$$

$$\int_0^{2\pi} \sqrt{2\cos\theta + 1 + \underbrace{\cos^2\theta + \mathrm{sen}^2\theta}_{=1}}\; d\theta$$

$$\int_0^{2\pi} \sqrt{2\cos\theta\ \text{[illegible]}}\; d\theta = \boxed{8}$$

//

15c $f(x,y,z) = x + y - z$

$f(\sigma(t)) = 0 + a \operatorname{sen}\theta - a\cos\theta$

$$\frac{\int_\sigma f}{\ell(\sigma)} = \int_0^\pi \frac{\cancel{a}(\operatorname{sen}\theta - \cos\theta)\cdot a}{2\pi\cancel{a}}\, d\theta$$

$$= \frac{a}{2\pi}\int_0^\pi \operatorname{sen}\theta - \cos\theta \, d\theta = \frac{a}{2\pi}\left(-\cos\theta + \operatorname{sen}\theta\right)\Big|_0^\pi$$

$$= \frac{a}{2\pi}\cancel{2} = \boxed{\frac{a}{\pi}}$$

16. $f:[a,b] \to \mathbb{R}$ dif a trozos

$C = \text{graf}(f)$ se parametriza $\sigma(t) = (t, f(t))$ $t \in [a,b]$

a La longitud del gráfico de f en $[a,b]$

$\sigma'(t) = (1, f'(t))$

$\|\sigma'(t)\| = \sqrt{1 + (f'(t))^2}$

Longitud de arco (en este caso: gráfico) $\int_C \|\sigma'(t)\|\, dt$

$$\int_a^b \sqrt{1 + (f(t))^2}\, dt$$

b. $y = \ln(\cancel{x})$ $t = 1$ a $t = 2$ $\sigma(t) = (t, \ln(t))$

$\sigma'(t) = \left(1, \frac{1}{t}\right)$

$$\int_1^2 \sqrt{1 + \frac{1}{t^2}}\, dt$$

CA $1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$

$$\int_1^2 \sqrt{\frac{t^2+1}{t^2}}\, dt =$$

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$$\int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt = \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2-1}\,du$$

1° $\boxed{\begin{aligned}\sqrt{t^2+1} &= u \\ \frac{1}{\cancel{2}\sqrt{t^2+1}}\,\cancel{2}t\,dt &= du\end{aligned}}$

2° CA $\boxed{\frac{t}{\sqrt{t^2+1}} = \frac{t}{\sqrt{t^2+1}} \cdot \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}} = \frac{t\sqrt{t^2+1}}{t^2+1} = \frac{t^2\sqrt{t^2+1}}{t(t^2+1)}}$

$$\frac{t}{\sqrt{t^2+1}}\,dt = du$$

$$\frac{t^2\sqrt{t^2+1}}{t(t^2+1)}\,dt = du$$

$$\boxed{\frac{\sqrt{t^2+1}}{t}\,dt = \frac{t^2+1}{t^2}\,du}\ 3°$$

Esta es exactamente la integral que tengo, ahora quiero escribir $\frac{t^2+1}{t^2}$ en términos de $\underline{u}$.

$$\left.\begin{aligned}\sqrt{t^2+1} &= u \\ \boxed{t^2+1 = u^2} \\ \boxed{t^2 = u^2-1}\end{aligned}\right\} \Rightarrow \boxed{\frac{t^2+1}{t^2} = \frac{u^2}{u^2-1}}\ 4°$$

$$\int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2-1}\,du \qquad \text{CA } \frac{u^2}{u^2-1}+1-1 = \frac{u^2-(u^2-1)}{u^2-1}+1 = \frac{\cancel{u^2}-\cancel{u^2}+1}{u^2-1}+1$$

$$= \boxed{\frac{1}{u^2-1}+1}$$

CA $\dfrac{1}{u^2-1} = \dfrac{A}{u+1} + \dfrac{B}{u-1}$ $\quad A, B \in R$

$$\frac{1}{\cancel{u^2-1}} = \frac{A(u-1)+B(u+1)}{\cancel{u^2-1}}$$

$$1 = A(u-1)+B(u+1) = Au - A + Bu + B = \boxed{u(A+B)+(B-A)}$$

Raíces $\bullet\ u=-1 \Rightarrow 1=-2A \Rightarrow A = -\frac{1}{2}$

$\bullet\ u = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$$\boxed{\frac{1}{u^2-1} = -\frac{1}{2(u+1)} + \frac{1}{2(u-1)}}$$

$$\int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2-1} = \int \frac{1}{u^2-1} + 1 = \int -\frac{1}{2(u+1)} + \frac{1}{2(u-1)} + 1 \ du$$

$$\left[-\frac{1}{2} \log(u+1) + \frac{1}{2} \log(u-1) + u \right]_{\sqrt{2}}^{\sqrt{5}}$$

$$\left(-\frac{1}{2} \log(\sqrt{5}+1) + \frac{1}{2} \log(\sqrt{5}-1) + \sqrt{5} \right) -$$

$$\left(-\frac{1}{2} \log(\sqrt{2}+1) + \frac{1}{2} \log(\sqrt{2}-1) + \sqrt{2} \right)$$

//

Integrales curvilíneas

17.a. $F(x,y,z) = (x,y,z)$

$\sigma(t) = (t,t,t) \qquad 0 \leq t \leq 1$

$\sigma'(t) = (1,1,1)$

$\cancel{\|\sigma'(t)\| = \sqrt{3}}$

$$\boxed{\int_C F(\sigma(t)) \cdot \sigma'(t)\, dt}$$

$$\int_0^1 (t,t,t)(1,1,1)\,dt = \int_0^1 (t+t+t)\,dt = \frac{t^2}{2} + \frac{t^2}{2} + \frac{t^2}{2}\Big|_0^1$$

$= \boxed{\frac{3}{2}}$

b. $\sigma(t) = (\text{sen}\, t, 0, \cos t) \mid \sigma'(t) = (\cos t, 0, -\text{sen}\, t)$

$0 \leq t \leq 2\pi$

$$\int_0^{2\pi} (\text{sen}\, t, 0, \cos t)(\cos t, 0, -\text{sen}\, t)\, dt$$

$$= \int_0^{2\pi} \cancel{\text{sen}\, t \cos t} + 0 - \cancel{\cos t\, \text{sen}\, t}\, dt = 0$$

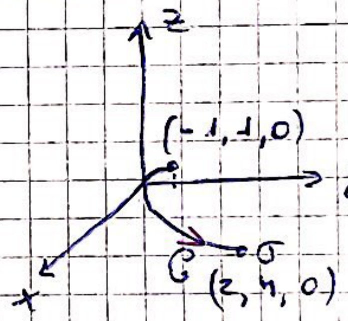
18.a. $F(x,y) = (-y, x) \qquad \sigma(t) = (\cos t, \text{sen}\, t)$

$$\int_\sigma F = \int_0^{2\pi} (-\text{sen}\, t, \cos t)(-\text{sen}\, t, \cos t)\, dt$$

$$\int_0^{2\pi} +\text{sen}^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

19. $F(x,y,z)=(x,y,z)$ $\quad y=x^2$, $z=0$ $\quad -1\leq x\leq 2$

$\sigma(t)=(t,t^2,0)$ $\quad -1\leq t\leq 2$



Trabajo $=\int_C F\,ds=\int_{-1}^{2} F(\sigma(t))\cdot\sigma'(t)\,dt$

$=\int_{-1}^{2}(t,t^2,0)(1,2t,0)\,dt$

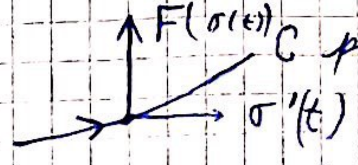
$=\int_{-1}^{2} t+2t\cdot t^2+0\,dt=$

$\int_{-1}^{2} t+2t^3\,dt=\left.\frac{t^2}{2}\right|_{-1}^{2}+2\left.\frac{t^4}{4}\right|_{-1}^{2}=\left(\frac{4}{2}-\frac{1}{2}\right)+$

$\left(2\cdot\frac{16}{4}-2\cdot\frac{1}{4}\right)=\frac{3}{2}+8-\frac{1}{2}=8+1=\boxed{9}$

20. C curva param por σ (orientada por σ) suave

a. F es perpendicular a $\sigma'(t)$ en $\sigma(t)$ $\forall t$.



Prop: $\vec{u}\cdot\vec{v}=|u||v|\underbrace{\cos(90^\circ)}_{=0}=0$

$$\int_C F\,ds = \int_C F(\sigma(t)) \cdot \sigma'(t) = 0$$

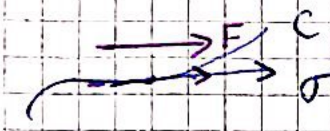
$\downarrow$

F evaluado en el punto $\sigma(t)$

Al ser ortogonales entre sí el producto escalar es cero.

b. F tiene el mismo sentido que $\sigma'(t)$ en $\sigma(t)$ $\forall t$.

Es decir



$$F(\sigma(t)) = \underbrace{\lambda(t)}_{>0} \cdot \sigma'(t)$$

Prop: $\vec{u} \cdot \vec{v} = |u||v| \underbrace{\cos\sigma}_{=1} = |u|\,|v|$

son paralelos y tienen el mismo sentido

$$\int_C F\,ds = \int_C F(\sigma(t)) \cdot \sigma'(t)\,dt = \int_a^b \lambda(t)\ \sigma'(t) \cdot \sigma'(t)\,dt$$

$$\int_a^b \lambda(t) \cdot \|\sigma'(t)\|^2 = \int_a^b \underbrace{\lambda\,\|\sigma'(t)\|}_{\|F(\sigma(t))\|} \cdot \|\sigma'(t)\| = \int_\sigma \|F\|\,dt$$

$$\boxed{\|F(\sigma(t))\|} = \|\lambda(t)\,\sigma'(t)\| = \boxed{\|\sigma'(t)\|}$$

21. Curva cerrada C. Propiedad de campo gradiente: $F = \nabla f$

C es una curva suave simple orientada que comienza en P y termina en q $\Rightarrow$

$$\int_C F\,ds = \int_C \nabla f\,ds = f(q) - f(P) = 0$$

Como C es una curva cerrada los extremos coinciden $f(q) = f(P)$ $\rightarrow$ La integral curvilínea de un campo grad sobre una curva cerrada es 0.

22 $\nabla f(x,y,z) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2}) = F$

$f(0,0,0) = 5$ ¿$f(1,1,2)$?

$\int_C F\,ds = \int_C \nabla f\,ds = f(q) - f(p) = \boxed{2e}$

$f(1,1,2) - f(0,0,0) =$

$f(x,y,z) = yze^{x^2} + c$ → a ojo sacamos F ¿qué ecuación cumple $\nabla f(x,y,z)$? también podemos integrar

$f(0,0,0) = 5$

$f(0,0,0) = 0 \cdot 0 e^{x^0} + 5$

$f(1,1,2) = 1 \cdot 2 \cdot e^{x^2} + 5 = \boxed{2e+5}$ //

$\sigma(t) = (t, t, 2t = tq \quad 0 \le t \le 1$

$\int_0^1 t^3 4e^{t}, 2te^{t^2}, te^{t^2})(1,1,2)\,dt.$

$\int_0^1 4t^3e^t + 2te^{t^2} + 2te^{t^2} = 4\int_0^1 t^3e^t + 4\int_0^1 te^t\,dt$

$u = t^2$, $du = 2t\,dt$ $\qquad$ $u = t^2$, $du = 2t\,dt$

$\ldots = 2t^2 \cdot e^{t^2}\Big|_0^1 = 2e^1 - 0 = \boxed{2e}$

$f(1,1,2) = 2e + f(0,0,0) = \boxed{2e+5}$ //

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24. $f: \mathbb{R}^3 \to \mathbb{R}$ $\mathcal{C}^1$ $G: \mathbb{R}^3 \to \mathbb{R}^3$ Campo $\mathcal{C}^1$

$F = \nabla f + G$ C es cerrada, simple, suave, orientada

$$\int_C F\, ds = \int_C (\nabla f + G)\, ds = \underbrace{\int_C \nabla f\, ds}_{} + \int_C G\, ds$$

como C es cerrada
$f(P) = f(Q)$
$\Rightarrow f(Q) - f(P) = 0.$

$$\int_C F\, ds = \int_C G\, ds.$$

25. C suave $\sigma: [a,b] \to \mathbb{R}^3$ param regular de C. Orientada por σ.

$\tilde{\sigma}$ reparametrización de σ, $F: \mathbb{R}^3 \to \mathbb{R}^3$ continua.

$$\int_C F \cdot ds = \int_a^b F(\sigma(t)) \cdot \sigma'(t)\, dt.$$

$\tilde{\sigma}$ preserva la orientación de C si $\forall P \in C$.

$$\tau(P) = \frac{\sigma'(t)}{\|\sigma'(t)\|} = \frac{\tilde{\sigma}'(t)}{\|\tilde{\sigma}'(t)\|}$$

si $P = \sigma(t) = \tilde{\sigma}(\tilde{t})$ con $t \in [a,b]$, $\tilde{t} \in [\tilde{a}, \tilde{b}]$

Obs si $\tilde{\sigma}$ no preserva la orientación se tiene

$$\frac{\tilde{\sigma}'(t)}{\|\tilde{\sigma}'(t)\|} = -\tau(P) \quad \text{para } P = \tilde{\sigma}(\tilde{t}) \ \forall \tilde{t} \in [\tilde{a}, \tilde{b}]$$

si $\tilde{\sigma}$ preserva la orientación de C se tiene

$$\int_a^b F(\tilde{\sigma}(\tilde{t})) \cdot \tilde{\sigma}'(t)\, d\tilde{t} = \int_C F\, ds.$$

y si no preserva la orientación $\quad$ " $\quad = -\int_C F \cdot ds.$ //

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