11/Sep/2020

Parte 2 - Integral de Longitud de arco

$$\operatorname{long}(\mathcal{C}) = \int_I \|\sigma'(t)\|$$

Ej 8 - $\sigma(t) = (t - \sin t \,,\, 1 - \cos t)$

Velocidad: $\sigma'(t) = (1 - \cos t \,,\, \sin t)$ con $t \in [0, 2\pi]$

rapidez: $\|\sigma'(t)\| = \left((1 - \cos t)^2 + \sin^2 t\right)^{1/2} =$

$$= \left(1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1}\right)^{1/2} =$$

$$= (2 - 2\cos t)^{1/2} \quad \text{con } t \in [0, 2\pi]$$

$$\ell(\sigma(t)) = \int_{t=0}^{t=2\pi} \|\sigma'(t)\| \, dt = \int_0^{2\pi} (2 - 2\cos t)^{1/2} \, dt =$$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{1 - \cos t} \, dt$$

uso identidad

$$1 - \cos t = 2 \sin^2 t/2$$

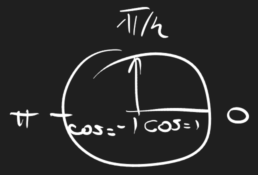
$$\sqrt{1 - \cos t} = \sqrt{2} \cdot \sin t/2 \leftarrow >0 \text{ pues } 0 \le \tfrac{t}{2} \le \tfrac{2\pi}{2}$$

$$= \sqrt{2} \cdot \sqrt{2} \int_0^{2\pi} \sin \frac{t}{2} \, dt$$

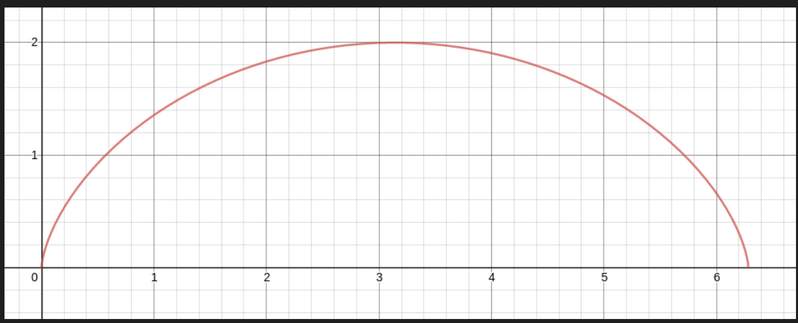
$$= 2 \int_0^{\pi} \sin u \cdot 2 \, du = 4 \int_0^{\pi} \sin u \, du$$

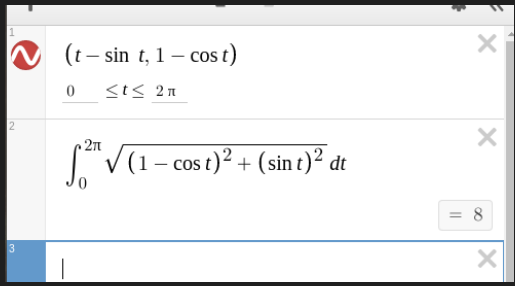
$u = \frac{t}{2}$

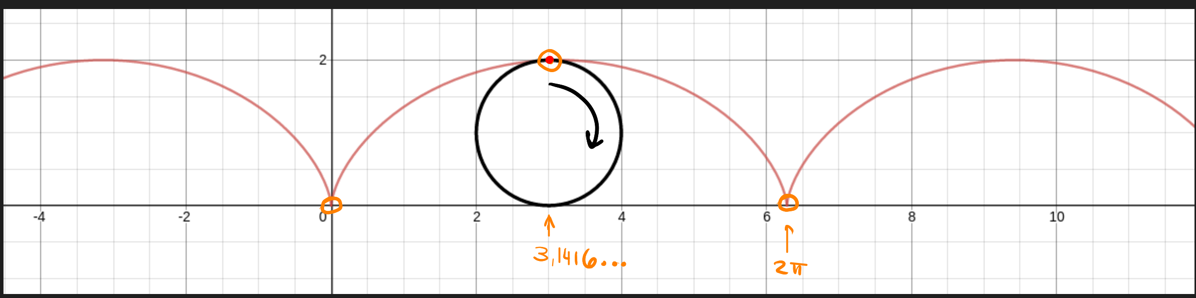
$du = \frac{1}{2} dt$

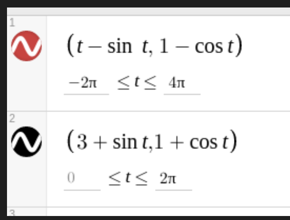


$$= 4 \cdot \left[-\cos u\right]_0^{\pi} = 4 \cdot (+1 + 1) = 8$$





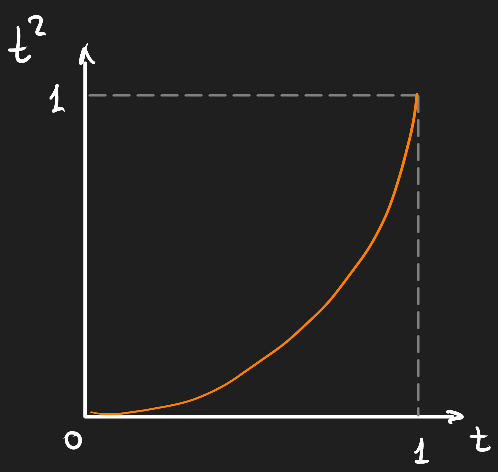




Cicloide: curva generada al girar y desplazar como una rueda, un círculo de radio 1

Ejer 9: a) $\sigma(t) = (t, t^2)$ con $a = 0$, $b = 1$

$\Rightarrow t \in [0, 1]$



$\sigma'(t)$

$$\ell(\mathscr{C}) = \int_0^1 \|(1, 2t)\| \, dt$$

$$= \int_0^1 \sqrt{1^2 + 4t^2} \, dt$$

$$= \int_0^1 \sqrt{1 + (2t)^2} \, dt$$

tiene pinta de función hiperbólica

Rcdo:

contrario a $\sin^2 x + \cos^2 x = 1$

- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh(2x) = 2 \cdot \cosh^2(x) - 1$

CA

tengo

$$\sqrt{1+(2t)^2}\,dt = \sqrt{1+\sinh^2 x}\cdot\frac{\cosh x\,dx}{2} = \sqrt{\cosh^2 x}\cdot\frac{\cosh x\,dx}{2}$$

$2t = \sinh x$ $\qquad$ $= \cosh^2 x$

$2dt = \cosh x\,dx$

$$= |\cosh x|\cdot\frac{\cosh x\,dx}{2} = \frac{\cosh^2 x}{2}\,dx$$

$\cosh x > 0$

Ψ

Volviendo

tenía $\displaystyle\int_{t=0}^{t=1}\sqrt{1+(2t)^2}\,dt = \int_{x=?}^{x=?}$

$2t = \sinh x$

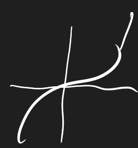
$2dt = \cosh x\,dx$

CA

para $t=0$:

$\sinh x = 0$

$x = 0$



para $t=1$:

$\sinh x = 2\cdot 1$

$\operatorname{arcsinh}(\sinh x) = \operatorname{arcsinh}(2)$

$x = \ln\left(2+\sqrt{2^2+1}\right)$

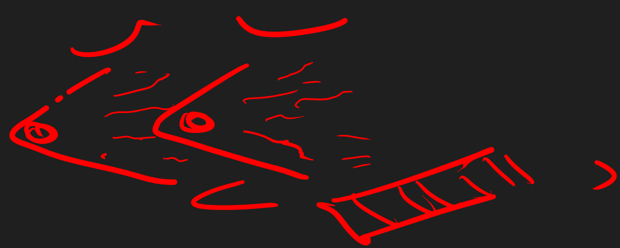
$x = \ln\left(2+\sqrt{5}\right)$

$$= \int_{x=0}^{x=\ln(2+\sqrt{5})}\frac{1}{2}\cdot\cosh^2(x)\,dx =$$

Recordo:

$$\cosh(x) = \frac{e^x+e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x-e^{-x}}{2}$$



$$= \int_{x=0}^{x=\ln(2+\sqrt{5})} \frac{1}{2} \cdot \cosh^2(x)\, dx =$$

$$= \frac{1}{2} \cdot \int_0^{\ln(2+\sqrt{5})} \left(\frac{e^{x}+e^{-x}}{2}\right)^2 dx = \circledast$$

CA

$$\underbrace{\left(\frac{e^{x}+e^{-x}}{2}\right)^2} = \frac{e^{2x}}{4} + \frac{2\,\overbrace{e^{x}\cdot e^{-x}}^{=1}}{4} + \frac{e^{-2x}}{4} =$$

$$= \underbrace{\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)}$$

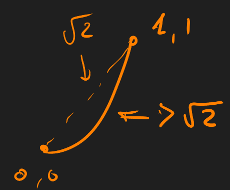
$$\circledast = \frac{1}{2}\cdot\frac{1}{4}\left(\int_0^{\ln(2+\sqrt{5})} e^{2x}\,dx + \int_0^{\ln} 2\,dx + \int_0^{\ln} e^{-2x}\right) =$$

$$= \frac{1}{8}\cdot\left(\left[\frac{e^{2x}}{2}\right]_0^{\ln(2+\sqrt{5})} + 2\ln(2+\sqrt{5}) + \left[-\frac{e^{-2x}}{2}\right]_0^{\ln(2+\sqrt{5})}\right) =$$

$$= \frac{1}{8}\cdot\left(\underbrace{\frac{1}{2}\left((2+\sqrt{5})^2 - \frac{1}{2}\right)} + 2\ln(2+\sqrt{5}) - \underbrace{\frac{1}{2}\left((2+\sqrt{5})^2 - \frac{1}{2}\right)}\right)$$

$$\text{long}(\mathcal{C}) = \frac{2}{8}\cdot\ln(2+\sqrt{5})$$

$$\text{long}(\mathcal{C}) = \frac{1}{4}\cdot\ln(2+\sqrt{5}) \approx 0{,}36\ldots$$ Mal! tiene que ser $> \sqrt{2}$



Revisar!

b) $\sigma(t) = \left(\sqrt{t},\ t+1,\ t\right)$ $\qquad a=10,\ b=20$

$\sigma'(t) = \left(\frac{1}{2\sqrt{t}},\ 1,\ 1\right)$ $\qquad t \in [10,20]$

$$\|\sigma'(t)\| = \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 1^2 + 1^2}$$

$$= \sqrt{\frac{1}{4t} + 2} = \sqrt{2}\cdot\sqrt{\frac{1}{8t}+1}$$

$$\sqrt{2}\cdot\int_{t=10}^{20} \sqrt{\frac{1}{8t}+1}\,dt = \int_{80}^{160}\sqrt{\frac{1}{u}+1}\cdot\frac{1}{8}\,du = \frac{1}{8}\int_{80}^{160}\sqrt{1+\frac{1}{u}}\,du$$

$u = 8t$

$du = 8\cdot dt$

Cambio de variable

$v = \sqrt{1+\frac{1}{u}}$

$dv = \frac{1}{2\sqrt{1+\frac{1}{u}}}\cdot\left(-u^{-2}\right)\cdot du$

$du = -2u^2\underbrace{\sqrt{1+\frac{1}{u}}}_{=v}\,dv$

$v^2 = 1+\frac{1}{u}$

$u = \left(v^2-1\right)^{-1}$

$$= \frac{1}{8}\int_{v=\sqrt{1+\frac{1}{80}}}^{v=\sqrt{1+\frac{1}{160}}} -v\cdot v\cdot\frac{1}{\left(v^2-1\right)^2}\,dv$$

$$= -\frac{1}{8}\int\frac{v^2}{\left(v^2-1\right)^2}\,dv$$

otra u

$\hat{u} = v^2-1 \Rightarrow v \overset{v>0}{=} \sqrt{\hat{u}+1}$

$d\hat{u} = 2v\,dv$

$dv = \frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}$

v^2

$$= -\frac{1}{8}\int\frac{\hat{u}+1}{\left(\hat{u}+\cancel{1}-\cancel{1}\right)}\,\frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}$$

con $\hat{u}_b = \frac{1}{160}$

$\hat{u}_a = \frac{1}{80}$

$$= -\frac{1}{8}\cdot\int_{\frac{1}{80}}^{\frac{1}{160}}\underbrace{\frac{\hat{u}+1}{\hat{u}}}_{=\frac{\hat{u}}{\hat{u}}+\frac{1}{\hat{u}}}\cdot\frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}$$

$$= -\frac{1}{8}\Bigg(\underbrace{\int \frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}}_{A} + \underbrace{\int \frac{1}{\hat{u}}\cdot\frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}}_{B}\Bigg) =$$

$$A: \quad \sqrt{\hat{u}+1}\,\Big|_{\frac{1}{80}}^{\frac{1}{160}} = \left(\left(\frac{1}{160}+1\right)^{1/2} - \left(\frac{1}{80}+1\right)^{1/2} \right) < 0$$

$$B: \quad \frac{1}{2}\int_{\frac{1}{80}}^{\frac{1}{160}} \frac{1}{\hat{u}\sqrt{\hat{u}+1}}\,d\hat{u} = \frac{1}{2}\int \frac{1}{(t^2-1)\,t}\cdot 2t\,dt = \int_{t=\sqrt{\frac{1}{80}+1}}^{t=\sqrt{\frac{1}{160}+1}} \frac{1}{t^2-1}\,dt$$

$(t-1)(t+1)$

$t = \sqrt{\hat{u}+1} \Rightarrow \hat{u} = t^2-1$

$dt = \frac{1}{2\sqrt{\hat{u}+1}}\,d\hat{u}$

$d\hat{u} = 2\sqrt{\hat{u}+1}\,dt = 2\sqrt{t^2} \overset{t>0}{=} 2t\,dt$

CA

$$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{1}{2}\,\frac{1}{(t-1)(t+1)} \quad (t+1)$$

?

Ej 10) $\mathcal{C}$; σ parametriza $\mathcal{C}$

$g:[\bar{a},\bar{b}]\to[a,b]$ una biyección C^1

con $g'(s)\neq 0 \ \forall s\in(\bar{a},\bar{b})$

Sea $\bar{\sigma}:[\bar{a},\bar{b}]\to\mathbb{R}^3$

$\bar{\sigma}(s)=\sigma(g(s))$ una reparametrización de σ

Definición 1.4. Sea $\mathcal{C}$ una curva que admite una con parametrización $\sigma:[a,b]\to\mathbb{R}^3$ y sea $h:[a,b]\to[c,d]$ una biyección continua [a]. Si definimos $\tilde{\sigma}:[c,d]\to\mathbb{R}^3$ dada por $\tilde{\sigma}(\tau)=\sigma(h^{-1}(\tau))$. Entonces, $\tilde{\sigma}$ es una parametrización de $\mathcal{C}$. Decimos que $\tilde{\sigma}$ es una "reparametrización de σ".

[a]Notar que esto implica la existencia y continuidad de $h^{-1}:[c,d]\to[a,b]$.

a) $\bar{\sigma}$ es param regular si

- $\bar{\sigma}$ es continua
- $\mathrm{Im}(\bar{\sigma})=\mathcal{C}$

- Como h es una biyección continua $\Rightarrow h^{-1}$ es continua

y como σ es continua pues es param reg. de $\mathcal{C}$

$\Rightarrow \bar{\sigma}=\sigma(h^{-1}(t))$ es continua por ser compo. de continuas.

- $\mathrm{Im}(\bar{\sigma})=\mathcal{C}$?

$\mathrm{Im}(\bar{\sigma})=\mathrm{Im}(\sigma(g(s)))$ con $s\in(\bar{a},\bar{b})$

como $g:[\bar{a},\bar{b}]\to[a,b]$

$g((\bar{a},\bar{b}))\to(a,b)$

g biyectiva con $g'\neq 0 \therefore$ es creciente ó decreciente

$$\Rightarrow \operatorname{Im}(\bar{\sigma}) = \operatorname{Im}(\sigma) = \mathcal{C}$$

¿Preguntar?

b) $f : \mathbb{R}^3 \to \mathbb{R}$ continua

$$\int_{\mathcal{C}} f\, ds = \int_{t=a}^{t=b} \|\sigma'\|\, dt$$

$$\overset{?}{=} \int_{\bar{a}}^{\bar{b}} \|\bar{\sigma}'\|\, dt = \int_{\bar{a}}^{\bar{b}} \|\sigma'(g(t))\|\, dt$$

$$\sqrt{\left((\sigma_1'(g(t)))^2 + (\sigma_2'(g(t)))^2 + (\sigma_3'(g(t)))^2 \right)}$$

Ej 11) $\mathcal{C}$

$$h(t) \overset{\text{long. de arco}}{=} \int_a^t \|\sigma'(\tau)\|\, d\tau$$

$\bar{\sigma}(s) = \sigma(h^{-1}(s))$ ← reparametrización por long. de arco.

Probar que

$$\int_0^s \|\bar{\sigma}(r)\|\, dr = \int_0^s \|\sigma(h^{-1}(v))\|\, dv$$

¿?

Ej 12) a) $\sigma(t) = (\cos t, \operatorname{sen} t, t)$ $\quad a=0,\ b=1$

$t \in [a,b]$

Parametrización por Longitud de Arco

$$\|\sigma'(t)\| = 1 \quad \forall t$$

Notar que si $\|\sigma'(t)\| = 1$

$$\Rightarrow \int_{t_1}^{t_2} \|\sigma'(t)\|\, dt = t_2 - t_1$$

$$\sigma'(t) = (-\sin t, \cos t, 1)$$

$$\|\sigma'(t)\| = \left(\sin^2 t + \cos^2 t + 1\right)^{1/2} = \sqrt{2}$$

quiero $\|\dot{\sigma}\| = 1$

$$s(t) = \int_0^t \|\sigma'(s)\|\, ds$$

Para un t $\Rightarrow$ me da la longitud hasta

Admite inversa

$$t(s) =$$

Para una longitud devuelve el t correspondiente a $\sigma(t)$

Ademas

$$t'(s) = \frac{1}{\|\sigma'(t(s))\|}$$

Consideramos la reparametrización

$$\tilde{\sigma}(s) := \sigma(t(s)) \qquad \tilde{\sigma} : [0, \mathcal{L}(\mathcal{C})] \to \mathcal{C}$$

- Notar

$$\tilde{\sigma}'(s) = \sigma'(t(s)).t'(s) = \frac{\sigma'(t(s))}{\|\sigma'(t(s))\|}$$

$$\Rightarrow \|\tilde{\sigma}'(s)\| = 1 \quad \forall\ s \in [0, \mathcal{L}(\mathcal{C})]$$

Función de longitud de arco

$$h(s) = \int_0^s \|\sigma'(r)\| dr = \int_0^s \sqrt{2}\, dr = \sqrt{2}\, s$$

$$y(x) = a \cdot x$$

$$x(y) = \frac{1}{a} \cdot y$$

$$h^{-1}(s) = \frac{1}{\sqrt{2}} s$$

$\cos t \ , \text{sen}\, t, \ t$

$\therefore \quad \tilde{\sigma}(t) = \sigma(h^{-1}(t)) = \left(\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \cdot \text{sen}\, t, \frac{t}{\sqrt{2}} \right)$

← notar que coincide con la norma de σ

b) $\sigma(t) = (2e^t, 3e^t + 1, -6e^t)$ $\qquad a = 0$, $b = \ln 3$

$\sigma'(t) = (2e^t, 3e^t, -6e^t)$

$\|\sigma'(t)\| = \left(4e^{2t} + 9e^{2t} + 36e^{2t}\right)^{1/2}$

$= (49e^{2t})^{1/2} = 7 \cdot e^t$

$S(t) = \int_0^t \|\sigma'(r)\| dr = \int_0^t 7e^r \, dr =$

$= 7(e^t - 1)$

$t(s) = \ln\left(\frac{s}{7} + 1\right)$

$\tilde{\sigma}(s) = \sigma(t(s)) =$

$\sigma(t) = (2e^t, 3e^t + 1, -6e^t)$

$= \left(2e^{\ln(\frac{s}{7}+1)}, 3e^{\ln(\frac{s}{7}+1)} + 1, -6e^{\ln(\frac{s}{7}+1)}\right)$

$\tilde{\sigma}(s) = \left(2\left(\frac{s}{7}+1\right), 3\left(\frac{s}{7}+1\right) + 1, -6\left(\frac{s}{7}+1\right)\right)$

Ej. 13 12/Sep/2020

a) $f(x,y,z) = x + y + z$

$\sigma(t) = (\text{sen}\, t, \cos t, t)$

$\sigma'(t) = (\cos t, -\text{sen}\, t, 1)$

$\|\sigma'(t)\| = \sqrt{2}$

Evaluo

$$\int_{\mathcal{C}} f(x,y,z)\, ds = \int_0^{2\pi} f(\sigma(s)) \cdot \overbrace{\|\sigma'(s)\|}^{\sqrt{2}} \cdot ds$$

$$= \int_0^{2\pi} f(\text{sen}\, s, \cos s, s) \cdot \sqrt{2}\, ds$$

$$= \int_0^{2\pi} (\sin s + \cos s + s) \cdot \sqrt{2}\, ds$$

$$= \int_0^{2\pi} (\sin s) \sqrt{2}\, ds + \int_0^{2\pi} (\cos s) \sqrt{2}\, ds + \int_0^{2\pi} s \cdot \sqrt{2}\, ds$$

$$= \sqrt{2} \cdot \left(-\cos s \Big|_0^{2\pi} + \sqrt{2} \cdot \sin s \Big|_0^{2\pi} + \sqrt{2} \cdot \frac{s^2}{2} \Big|_0^{2\pi}\right.$$

$$= \frac{\sqrt{2}}{2} \cdot (4\pi^2 - 0)$$

$$= 2\sqrt{2} \cdot \pi^2 \,//$$

$$\int_{\mathcal{C}} f(x,y,z)\, ds = 2\sqrt{2} \cdot \pi^2$$

Ej 13) b)

mismo que a)

$$f(x,y,z) = \cos z \qquad \sigma(t) = (\cos t, \sin t, t)$$

$$\int_{\mathcal{C}} f\, ds = \int_0^{2\pi} f(\sigma(s)) \cdot \overbrace{\|\sigma'(s)\|}^{\sqrt{2}}\, ds$$

$$= \int_0^{2\pi} \cos\big(\sigma(s)\big) \cdot \sqrt{2}\, ds$$

$$= \int_0^{2\pi} \cos(s) \cdot \sqrt{2}\, ds$$

$$= \sqrt{2} \cdot \sin s \Big|_0^{2\pi} = \sqrt{2}\left(\cancel{\sin 2\pi - \sin 0}\right)$$

$$= 0$$

$$\int_{\mathcal{C}} f\, ds = 0$$

c) $f(x,y,z) = x \cdot \cos z$

$$\sigma(t) = (t, t^2, 0) \qquad t \in [0,1]$$

$$\sigma'(t) = (1, 2t, 0)$$

$$\|\sigma'(t)\| = \sqrt{1+4t^2}$$

tiene pinta de trigonométrica hiper.

$$\int_{\mathcal{C}} f\, ds = \int_0^1 f(\sigma(s)) \cdot \sqrt{1+4s^2}\, ds$$

Rehago abajo!

$$= \int_0^1 (s \cdot \cos 0) \cdot \sqrt{1+4s^2}\, ds$$

$$= \int_0^1 s \cdot \sqrt{1+4s^2} \cdot ds \overset{\circledast}{}$$

$$u = \sqrt{1+4s^2} \Rightarrow |s| = \sqrt{u^2-1} \overset{s \geq 0}{=} s = \sqrt{u^2-1}$$

$$du = \frac{1}{2\sqrt{1+4s^2}} \cdot 8s\, ds$$

$$s = 0 \Rightarrow u = 1$$

$$ds = \frac{2\sqrt{1+4s^2}}{8s}\, du$$

$$s = 1 \Rightarrow u = \sqrt{5}$$

$$\overset{\circledast}{=} \int_1^{\sqrt{5}} \cancel{\sqrt{u^2-1}} \; \frac{2\sqrt{1+4(u^2-1)}}{8 \cdot \cancel{\sqrt{u^2-1}}}\, du$$

$$= \frac{1}{4} \int_1^{\sqrt{5}} \sqrt{1+4u^2-4}\, du$$

$$= \frac{1}{4} \int_1^{\sqrt{5}} \underbrace{\sqrt{4u^2-3}}_{\text{vuelvo a } 1+4(u^2-1)}\, du$$

Recuerdo

$$\cosh^2(x) - \operatorname{senh}^2(x) = 1$$

$$\cosh(2x) = 2\cosh^2(x) - 1$$

$$= \frac{1}{4} \int_1^{\sqrt{5}} \sqrt{1+4(u^2-1)}\, du$$

$u = \cosh x$

$du = \sinh x \, dx$

$$= \frac{1}{3} \int_{\ln(1+\sqrt{1^2+1})}^{\ln(1+\sqrt{5+1})} \sqrt{1 + 4 . \underbrace{(\cosh^2 x - 1)}} \; \sinh x \, dx$$

identidad trig.

$$= \frac{1}{3} \int \sqrt{\underbrace{1 + 4 \sinh^2 x}} \; . \; \sinh x \, dx$$

$$\underbrace{-3 + \underbrace{4 + 4 \sinh^2 x}_{4\cosh^2 x}}_{-3 + 4\cosh^2 x}$$

Acá muero y decido rehacer

De nuevo :

$$\int_0^1 x \sqrt{4x^2+1} \, dx = \int_1^5 \cancel{x} . \sqrt{u} \, \frac{1}{8\cancel{x}} . du$$

$u = 4x^2 + 1$

$du = 8x \, dx$ era una boludez

$dx = \frac{1}{8x} du$

$$= \int_1^5 \frac{1}{8} . \sqrt{u} \, du$$

$$\frac{1}{8} . \left(\frac{2}{3} u^{\frac{3}{2}} \right|_1^5 = \frac{1}{8} . \frac{2}{3} . \left(5^{3/2} - 1^{3/2} \right)$$

$$= \frac{1}{12} . \left(5^{3/2} - 1 \right)$$

$$\int_{\mathcal{C}} f\, ds = \frac{1}{12}\cdot\left(5^{3/2} - 1\right)$$

Ej 14) $r = r(\theta)$ con $\theta_1 \leq \theta \leq \theta_2$

$$\int_{\theta_1}^{\theta_2} f(r\cdot\cos\theta,\ r\cdot\sin\theta)\cdot\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\sigma(\theta) = (r\cdot\cos\theta,\ r\cdot\sin\theta)$$

$$= (\underbrace{r(\theta)\cdot\cos\theta}_{\mathrm{I}},\ \underbrace{r(\theta)\cdot\sin\theta}_{\mathrm{II}})$$

r es función de θ!

Derivo I:

$$\frac{d}{d\theta}\left(r(\theta)\cdot\cos\theta\right) = \frac{dr}{d\theta}\cdot\cos\theta - r(\theta)\cdot\sin\theta$$

Derivo II:

$$\frac{d}{d\theta}\left(r(\theta)\cdot\sin\theta\right) = \frac{dr}{d\theta}\cdot\sin\theta + r(\theta)\cdot\cos\theta$$

$$\sigma'(\theta) = \Big(\underbrace{\frac{dr}{d\theta}\cdot\cos\theta - r(\theta)\cdot\sin\theta}_{(A)},\ \underbrace{\frac{dr}{d\theta}\cdot\sin\theta + r(\theta)\cdot\cos\theta}_{(B)}\Big)$$

$$\|\sigma'(\theta)\| = \left((A)^2 + (B)^2\right)^{1/2}$$

$$(A)^2 = \frac{dr^2}{d\theta^2} \cdot \cos^2\theta - 2 \cdot \frac{dr}{d\theta} \cdot \cos\theta \cdot \sin\theta \cdot r(\theta)$$
$$+ (r(\theta))^2 \cdot \sin^2\theta$$

$$(B)^2 = \frac{dr^2}{d\theta^2} \sin^2\theta + 2 \cdot \frac{dr}{d\theta} \cdot \cos\theta \cdot \sin\theta \cdot r(\theta)$$
$$+ (r(\theta))^2 \cdot \cos^2\theta$$

Junto en la norma

$$\|\sigma'(\theta)\| = \left(\frac{dr^2}{d\theta^2} \cdot \cos^2\theta + r^2(\theta) \cdot \sin^2\theta \right.$$

se cancelan

$$+ 0 +$$

$$\left. + \frac{dr^2}{d\theta^2} \cdot \sin^2\theta + r^2(\theta) \cdot \cos^2\theta \right)^{1/2}$$

$$= \left(\frac{dr^2}{d\theta^2} \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1} + r^2(\theta) \cdot \underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{=1} \right)^{1/2}$$

$$= \left(\left(\frac{dr}{d\theta}\right)^2 + r^2(\theta) \right)^{1/2}$$

// que es lo que quería ¡¡ :)

∴ vale que $\displaystyle\int_{\theta_1}^{\theta_2} f(r\cos, r\sin\theta) \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

es la integral de long. de arco de $f(x,y)$ a lo largo de una curva dada en coord. polar es

Ej 14) b) $r = 1 + \cos\theta$ con $0 \leq \theta \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} f(r\cos, r\sin\theta) \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta =$$

$$\int_0^{2\pi} 1 \cdot \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta =$$

CA

$$(1+\cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{1 + \cos\theta}\, d\theta$$

hay una identidad para ésto

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \cos\theta = 1 - 2 \cdot \sin^2\left(\frac{\theta}{2}\right)$$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{1 + 1 - 2\sin^2\left(\frac{\theta}{2}\right)}$$

$$= \sqrt{2} \cdot \int_0^{2\pi} \sqrt{2 - 2\sin^2\left(\frac{\theta}{2}\right)} = \sqrt{2} \cdot \int_0^{2\pi} \sqrt{2} \cdot \sqrt{1 - \sin^2\frac{\theta}{2}}\, d\theta$$

$$= 2 \cdot \int_0^{2\pi} \sqrt{1 - \sin^2 \frac{\theta}{2}}\, d\theta =$$

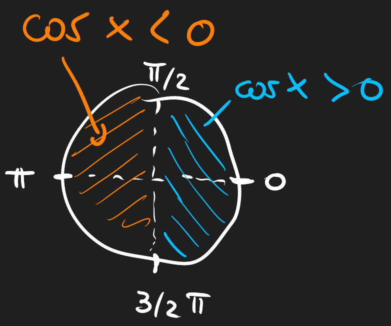
Rcdo

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x = 1 - \sin^2 x$$

$$= 2 \cdot \int_0^{2\pi} \sqrt{\cos^2 \frac{\theta}{2}}\, d\theta$$

$$= 2 \cdot \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta$$



Obs!

$$\cos x < 0 \quad \text{con} \quad x \in \left(\frac{\pi}{2}, \frac{3}{2}\pi \right)$$

Separo la integral en 2:

$$= 2 \cdot \left(\int_0^{\pi} \overset{>0}{\left| \cos \frac{\theta}{2} \right|} d\theta + \int_{\pi}^{2\pi} \overset{<0}{\left| \cos \frac{\theta}{2} \right|} d\theta \right)$$

?

$$= 2 \cdot \left(\int_0^{\pi} \cos \frac{\theta}{2}\, d\theta + \left| \int_{\pi}^{2\pi} \cos \frac{\theta}{2}\, d\theta \right| \right)$$

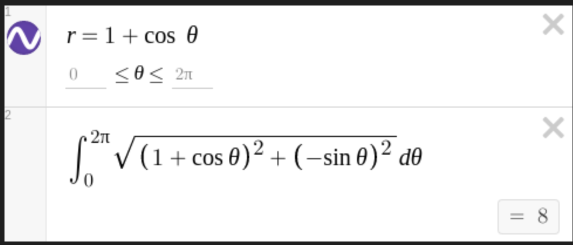
$$= 2 \cdot \left(\left[2 \cdot \sin \frac{\theta}{2} \right]_0^{\pi} + \left| \left[2 \cdot \sin \frac{\theta}{2} \right]_{\pi}^{2\pi} \right| \right)$$

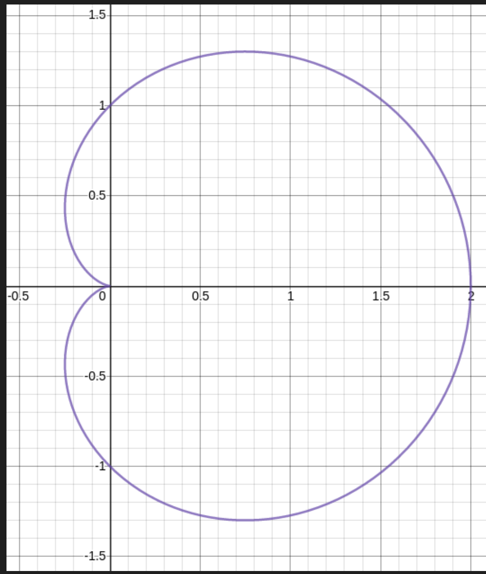
$$= 2 \cdot \left(2 \cdot \left(\underbrace{\sin \frac{\pi}{2}}_{1} - \underbrace{\cancel{\sin \frac{0}{2}}}_{=0} \right) + \left| 2 \cdot \left(\underbrace{\sin \pi}_{=0} - \underbrace{\sin \frac{\pi}{2}}_{=1} \right) \right| \right)$$

$$= 2 \cdot \left(2 \cdot 1 + \left| 2 \cdot (-1) \right| \right)$$

$$= 2 \cdot 4 = 8 \;//$$

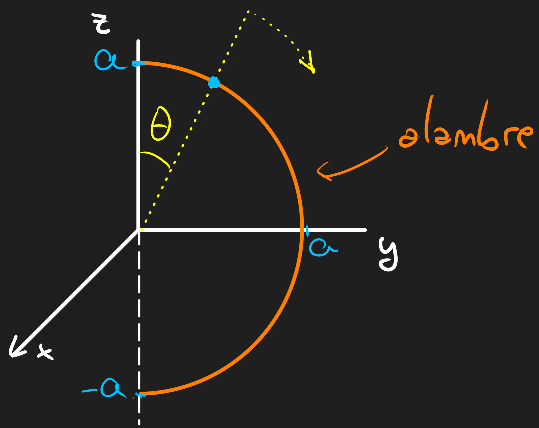
La longitud de la curva es 8





Ej 15) $\sigma(\theta) = (0, a.\sin\theta, a.\cos\theta)$ con $\theta \in [0, \pi]$

$a > 0$



densidad uniforme

$\rho = 2 \frac{\text{gramos}}{\text{long}}$

a) $\int_{\theta=0}^{\theta=\pi} 2 . \|\sigma'(\theta)\| \, d\theta$

$\sigma'(\theta) = (0, a.\cos\theta, -a.\sin\theta)$

$\|\sigma'(\theta)\| = (a^2.\cos^2\theta + a^2.\sin^2\theta)^{1/2}$

$\underset{a>0}{=} a . \sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{=1}}$

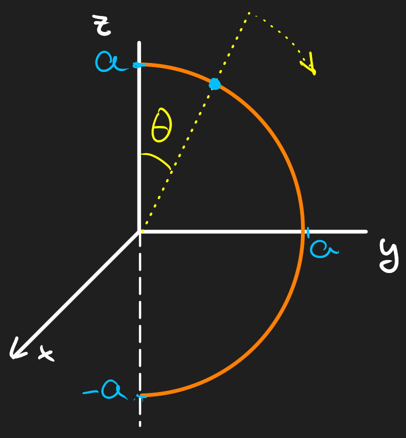
$\|\sigma'(\theta)\| = a$

$\Rightarrow$ masa $= \int_0^{\pi} 2 . a \, d\theta = 2 . a . \pi$ //

↑ un semicírculo de radio a = 1

pesa 2π

Ej 15) b) Centro de masa.



Como la densidad es constante el centro de masa está en

$$\underset{x}{(0,}\ \underset{y}{a,}\ \underset{z}{0)}$$

Formalmente

quiero el θ que separa en 2 la masa del alambre

sé de a)

$$\text{masa} = \int_{\theta=0}^{\theta=\pi} 2 \cdot a \cdot d\theta$$

$$= \underbrace{\int_0^{\pi/2} 2a \cdot d\theta} + \underbrace{\int_{\pi/2}^{\pi} 2a\, d\theta}$$

$$= 2 \cdot a \cdot \frac{\pi}{2} + 2 \cdot a \cdot \frac{\pi}{2}$$

$\therefore$ $\theta = \frac{\pi}{2}$ separa la masa en 2 partes iguales

$\therefore$ el centro de masa está en

$$\sigma\left(\overset{\theta}{\frac{\pi}{2}}\right) = \left(0,\ a \operatorname{sen}\frac{\pi}{2},\ a \cdot \cos\frac{\pi}{2}\right)$$

$$= (0,\ a,\ 0) \quad //$$

15) c) $T(x,y,z) = x + y - z$

Calculo la temperatura promedio del alambre.

$$\int_{\theta=0}^{\theta=\pi} T(\sigma(\theta)) \cdot \|\sigma'(\theta)\| \, d\theta =$$

$$\sigma(\theta) = (0, a.\sin\theta, a.\cos\theta)$$

$$= \int_0^{\pi} (a.\sin\theta - a.\cos\theta) . a \, d\theta$$

$$= \int_0^{\pi} a^2 (\sin\theta - \cos\theta) \, d\theta$$

$$= \int_0^{\pi} a^2 . \sin\theta \, d\theta - \int_0^{\pi} a^2 . \cos\theta \, d\theta$$

$$= a^2 . \underbrace{[\sin\theta]_0^{\pi}}_{=0} - a^2 . \underbrace{[\cos\theta]_0^{\pi}}_{=0} = 0$$

temperatura promedio es cero.

Ej 16) $\sigma(t) = (t, f(t))$ $\quad t \in [a,b]$

a) gráfico de f

$$\operatorname{long}(\mathscr{C}) = \int_a^b \|\sigma'(t)\| \, dt$$

$$\sigma'(t) = (1, f'(t))$$

$$\|\sigma'(t)\| = \left(1 + (f'(t))^2\right)^{1/2}$$

$$\operatorname{long}(\mathcal{C}) = \int_a^b \sqrt{1+(f'(t))^2}\, dt$$

16) b) $y = \ln x \qquad 1 \leq x \leq 2$

$\mathcal{C}_y$ = gráfico de y

que es una curva suave C^1 inyectiva (suave)

una param:

$$\sigma(t) = (t, \ln t)$$

$$\operatorname{long}(\mathcal{C}_y) = \int_{x=1}^{x=2} \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx$$

Trigo:

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

$$\frac{1}{x} = \cosh u$$

$$-\frac{1}{x^2}\, dx = \sinh u\, du$$

$$= -\frac{1}{\cos^2 u}$$

$$\operatorname{long}(\mathcal{C}_2) = \int_{1/\ln(1+\sqrt{1^2+1}) > 0}^{1/\ln(2+\sqrt{4+1}) > 0} \sqrt{1+\cosh^2 u}\,(-x^2) \cdot \sinh(u) \cdot du$$

$$= -\int \sqrt{\sinh^2 u} \cdot \frac{1}{\cosh^2 u} \cdot \sinh(u) \cdot du$$

$\sinh u > 0$

$$= -\int \left(\frac{\sinh^2 u}{\cosh^2 u}\right) du$$

$$= -\int \underbrace{\left(\frac{\sinh u}{\cosh u}\right)^2}_{\tanh^2 u} du$$

$$= -\int \tanh^2 u \, du$$

Prop trig

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

y sé que

$(\tanh x)' = \operatorname{sech}^2 x$

$$= -\int 1 - \operatorname{sech}^2 u \, du$$

$$= -\int 1 \, du - \int \operatorname{sech}^2 u \, du$$

$$= \underbrace{-\left(\frac{1}{\ln(2+\sqrt{5})} - \frac{1}{\ln(1+\sqrt{2})} \right)}_{\approx 0,442} - \underbrace{\left(\tanh u \Big|_{\frac{1}{\ln(1+\sqrt{2})}}^{\frac{1}{\ln(2+\sqrt{5})}} \right.}_{\approx +0,2129}$$

$$= \frac{1}{\ln(1+\sqrt{2})} - \frac{1}{\ln(2+\sqrt{5})} - \tanh\left(\frac{1}{\ln(2+\sqrt{5})}\right) + \tanh\left(\frac{1}{\ln(1+\sqrt{2})}\right)$$

Creo que está mal ☺ ✓