Práctica 4: Stokes.

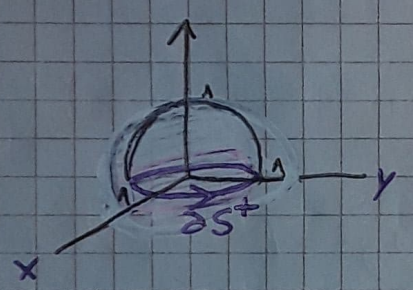
Teorema de Stokes

1. F campo C^1 definido en S sup orientada.

$$\iint_S \underbrace{\nabla \times F}_{\text{rotor}} \, dS = \int_{\partial S^+} F \, ds$$

Las orientaciones deben coincidir.

"Camino con la cabeza apuntando para el mismo lado que la normal y S queda a mi izquierda".



Campo $F(x,y,z) = (x,y,z)$

$S = \{(x,y,z) \in \mathbb{R}^3 / z = \sqrt{1-x^2-y^2},\ z \geq 0\}$

$\int_{\partial S^+} F \, ds$ El borde de S se puede parametrizar como $\sigma(t) = (\cos t, \operatorname{sen} t, 0)$ $0 \leq t \leq 2\pi$

- recorre la curva en sentido antihorario ✓
- F es C^1 ✓
- $\sigma'(t) = (-\operatorname{sen} t, \cos t, 0)$

$$\int_{\partial S^+} F \, dS = \int_{\partial S^+} F(\sigma(t)) \cdot \sigma'(t) \, dt$$

$$= \int_0^{2\pi} (\cos t, \operatorname{sen} t, 0) \cdot (-\operatorname{sen} t, \cos t, 0) \, dt$$

$$= \int_0^{2\pi} \cancel{-\operatorname{sen} t \cos t} + \cancel{\operatorname{sen} t \cos t} + 0 \, dt = \boxed{0}$$

La integral sobre la curva da 0.

$$\iint_S \nabla \times F = \iint_D \nabla \times F(T(x,y)) \cdot T_x \times T_y \quad dx\,dy.$$

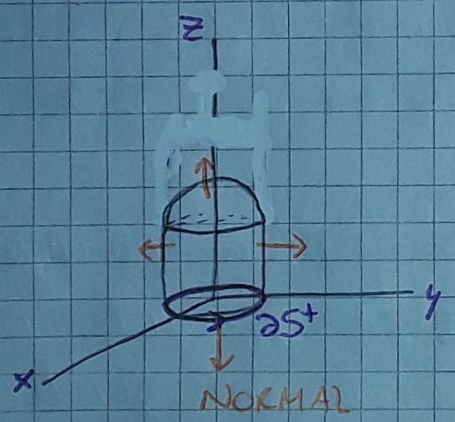
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \boxed{\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, -\frac{\partial z}{\partial x} + \frac{\partial x}{\partial z}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right)}$$

$$\nabla \times F = (0,0,0)$$

$$\rightsquigarrow \int_{\partial S^+} F\,ds = \iint_S \nabla \times F = 0$$ se cumple Stokes.

2.



$S = S_1 \cup S_2$

$S_1 : x^2 + y^2 = 1 \quad 0 \leq z \leq 1$

Cilindro

$S_2 : x^2 + y^2 + (z-1)^2 = 1 \quad z \geq 1$

Esfera unitaria centrada en $(1,0,0)$

- S orientada con normal exterior.

$$F = (zx + z^2 y + x, \quad z^3 y x + y, \quad z^4 x^2)$$

$$\int_S (\nabla \times F)\, dS$$

$$\nabla \times F = \left(\frac{\partial}{\partial y}(z^4 x^2) - \frac{\partial}{\partial z}(zx + z^2 y + x), \frac{\partial}{\partial z}(zx + z^2 y + x) \right.$$

$$\left. - \frac{\partial}{\partial x}(z^4 x^2), \frac{\partial}{\partial x}(z^3 y x + y) - \frac{\partial}{\partial y}(zx + z^2 y + x) \right)$$

$\nabla \times F = (0 - x + 2zy \,,\; x + 2zy - z^4 2x,\; z^3 y - z^2)$

Parametrizo S:

Despejo z de S_2

$$(z-1)^2 = 1 - x^2 - y^2 \geq 0$$

$$\rightsquigarrow \quad z = \sqrt{1 - x^2 - y^2} + 1$$

$$S = \{(x,y,z) \in \mathbb{R}^3 \quad z = 1 + \sqrt{1 - x^2 - y^2} \text{ con } x^2 + y^2 \leq 1\}$$

$$T(x,y) = (x, y, 1 + \sqrt{1 - x^2 - y^2})$$

$$T : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$D = \{(x,y) \in \mathbb{R}^2 / \quad x^2 + y^2 \leq 1\}$$

$$\int_S \nabla \times F \, dS = \int_D \nabla \times F(T(x,y)) \cdot (T_x \times T_y) \, dx\, dy$$

$$T_x = (1, 0, ??) \quad T_y = (0, 1, ??)$$

Veamos si $\int_{\partial S^+} F \, ds$ es más fácil.

Como F es C^1 uso Stokes.

$$\iint_S \nabla \times F \, dS = \int_{\partial S^+} F \, ds$$

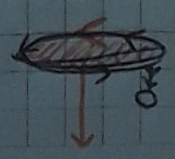
∂S^+: parametrizo $x^2 + y^2 \leq 1$

$$\sigma(t) = (\cos t, \operatorname{sen} t, 0) \qquad 0 \leq t \leq 2\pi$$

$$\sigma'(t) = (-\operatorname{sen} t, \cos t, 0)$$

todo lo que tenga z desaparece

• Respeta sentido antihorario ✓



deja la sup a la izq.

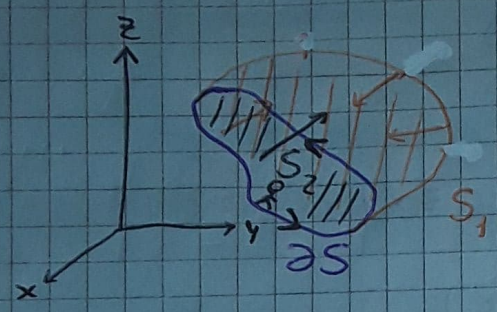
$$\int_0^{2\pi} (0+0+\cos t,\ 0+\operatorname{sen} t,\ 0)(-\operatorname{sen} t, \cos t, 0)\ dt$$

$$= \int_0^{2\pi} -\cancel{\cos t \cdot \operatorname{sen} t} + \cancel{\operatorname{sen} t \cdot \cos t}\ dt = \boxed{0}$$

más fácil

$$\overset{\text{stokes}}{\Longrightarrow} \int_{\partial S^+} F\, dS = \iint_S \nabla \times F\ dS = \boxed{0}$$

3.



∂S está sobre un plano

S_2 es el relleno de ∂S sobre el plano

S_1 es la cosa que tiene como borde a ∂S

Cuando me pone en ∂S quiero dejar la sup a mi izquierda y para eso ambas sup deben tomar orientaciones que permitan que la cabeza de la persona apunte hacia el mismo lado (o sea si S_2 apunta para arriba, S_2 no puede apuntar para abajo).

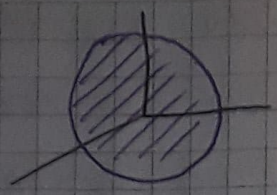
$\Rightarrow \int_{\partial S} F\ dS$ es la misma porque tienen el mismo borde con la misma orientación.

$$\int_{\partial S} F\ dS = \int_{S_1} \nabla \times F\ dS \qquad \text{Stokes}$$

$$\int_{\partial S} F\ dS = \int_{S_2} \nabla \times F\ dS$$

$$\Rightarrow \int_{S_1} \nabla \times F\ dS = \int_{S_2} \nabla \times F\ dS.$$

b.



La esfera no tiene borde porque es una superficie cerrada.

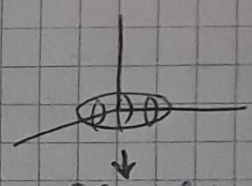
$\Rightarrow \partial S = 0$

$$\int_S \nabla \times F \, dS = \int_{\underbrace{\partial S}_{=0}} F \, dS = 0$$

c. $S = x^2 + y^2 + 2z^2 = 10$ Elipsoide

$$\frac{x^2}{10} + \frac{y^2}{10} + \frac{2z^2}{10} = \frac{10}{10}$$

$$\left(\frac{x}{\sqrt{10}}\right)^2 + \left(\frac{y}{\sqrt{10}}\right)^2 + \left(\frac{z}{\sqrt{5}}\right)^2 = 1$$



Obs: El elipsoide es una superficie cerrada

$\rightarrow \int_{\partial S} F \, dS = 0$

y por Stokes $\int_S \nabla \times F \, dS = 0$ ✓

$F = (\operatorname{sen} xy, e^x, -yz)$

4. $F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ clase C^1 ✓

$\downarrow$

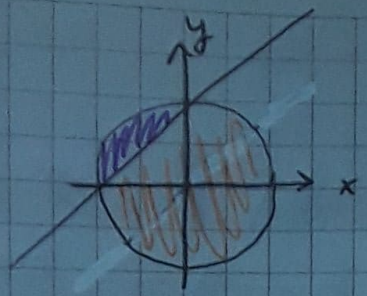
$x^2 + y^2 \neq 0$

a. S círculo de radio $a > 0$ centrado en el <u>origen</u> en el plano $z = 0$

No vale Stokes (ni Green) porque en el origen el campo F no está definido, mucho menos es C^1 en toda la región.

b. $z = 0$ entre $\begin{cases} x^2 + y^2 = 1 \\ x + y = 1 \end{cases}$

$z=0$



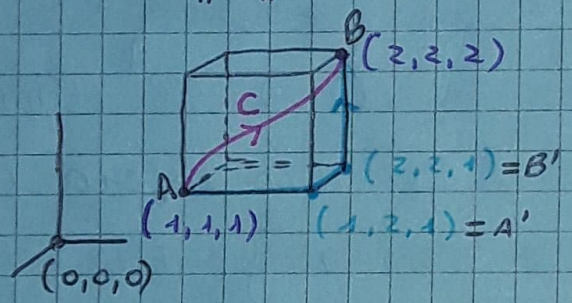
En no vale Stokes porque F no está definido en el origen. D tiene que ser una región donde vale Green para que valga Stokes (entre otras cosas).

En está todo bien y vale Green y Stokes.

5.a. $F = -GmM \dfrac{x}{\|x\|^3}$ Campo gravitacional

$f(x) = \dfrac{1}{\|x\|}$ $(GmM = 1)$

Sabemos que es un campo gradiente



(I) (II) me piden el trabajo de F por dos caminos distintos, pero sabiendo que F es el campo gradiente de la función potencial f chiquita, podemos ver que sólo depende de A y B.

b. $\nabla \times F = 0$ porque $\nabla \times \nabla f$ y las derivadas cruzadas coinciden.

$$F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$rot(F) = \det \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \right)$$

$= (0, 0, 0)$ Que el rotor sea cero me dice que existe $f \,/\, \nabla f = F$

Como F es campo grad vale que (Ej 23 práctica 1)

$$\int_C F\,dS = \int_C \nabla f \cdot dS = f(B) - f(A) = \frac{1}{\|B\|} - \frac{1}{\|A\|}$$

$$= \frac{1}{\sqrt{2^2+2^2+2^2}} - \frac{1}{\sqrt{1^2+1^2+1^2}} = \boxed{\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{3}}}$$

No estamos usando Stokes porque C no es el borde de una superficie

$$\int_C F\,dS \underset{NO}{\neq} \int_S \underset{0}{\underset{\shortparallel}{rot(F)}} =$$

C es una curva abierta que no es el borde de ninguna superficie! !

6. a. $F(x,y)=(x,y)$ Es el gradiente de $f=\frac{x^2}{2}+\frac{y^2}{2}$

Sale con integrales

Busco $f / \frac{\partial f}{\partial x} = x \ , \ \frac{\partial f}{\partial y} = y$

Condición 1) $\int_0^x x\,dx = \frac{x^2}{2}\Big|_0^x = \underbrace{\frac{x^2}{2}}_{\text{Llamo}} + C(y)$

depende de y

$\Rightarrow f(x,y) = h(x) + C(y)$ $h(x) \to$ no depende de y.

Condición 2) $\frac{\partial f}{\partial y} = 0 + \frac{\partial C}{\partial y} = y$

$$\Rightarrow \frac{\partial C}{\partial y} = y$$

$$\Rightarrow \int \frac{\partial C}{\partial y}\,dy = \int y\,dy$$

$$C = \frac{y^2}{2} + C'$$

$C \in \mathbb{R}$

$$\boxed{f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C}$$

b. $F(x,y) = (x^2+y^2, 2xy)$ $\quad f / \frac{\partial f}{\partial y} = 2xy \ , \ \frac{\partial f}{\partial x} = x^2+y^2$

Condición 1) $f = \int_0^y 2xt\,dt = \cancel{2}x \cdot \frac{t^2}{\cancel{2}}\Big|_0^y = \boxed{xy^2 + C(x)}$

Condición 2) $\frac{\partial f}{\partial x} = x^2+y^2 = y^2 + \frac{\partial C}{\partial x}$

$$x^2 + \cancel{y^2} - \cancel{y^2} = \frac{\partial C}{\partial x}$$

$$\int x^2\,dx = \int \frac{\partial C}{\partial x}\,dx$$

$$\frac{x^3}{3} + C' = C$$

$$\frac{x^3}{3} + C' = C$$

$$\Rightarrow f = xy^2 + \frac{x^3}{3} + C$$

$$\nabla f = \left(y^2 + \frac{\cancel{3}x^2}{\cancel{3}} , \quad 2yx \right) \qquad F \text{ es } \nabla f \checkmark$$

c. $F(x,y) = (\cos xy - xy \operatorname{sen}(xy), \quad x^2 \operatorname{sen} xy)$

Condición 1) $\frac{\partial f}{\partial y} = x^2 \operatorname{sen} xy$

$$f = \int_0^y x^2 \operatorname{sen}(xt)\, dt \underset{\substack{u = xt \\ du = x\,dt}}{=} x^2 \int \operatorname{sen}(u) \frac{du}{x}$$

$$= x(-\cos(u)) = x\left(-\cos(xt)\right)\Big|_0^y = \boxed{-x\cos(xy) + C(x)}$$

Obs: si $f = -x\cos(xy) + C \rightsquigarrow \nabla f = \left(-\cos(xy) + yx\operatorname{sen}(xy), x^2\operatorname{sen}(xy)\right)$

si $f = +x\cos(xy) + C \rightsquigarrow \nabla f = \left(\cos(xy) - xy\operatorname{sen}(xy), -x^2\operatorname{sen}(xy)\right)$

Como los signos siempren quedan opuestos, $F \neq \nabla f$

$\rightsquigarrow$ F no es campo gradiente.

7.a. $C: \sigma(t) = (\cos^5 t, \operatorname{sen}^3 t, t^4)$ $\quad 0 \leq t \leq \pi$

$$F = (\overbrace{2xyz + \operatorname{sen} x}^{F_1}, \overbrace{x^2 z}^{F_2}, \overbrace{x^2 y}^{F_3})$$

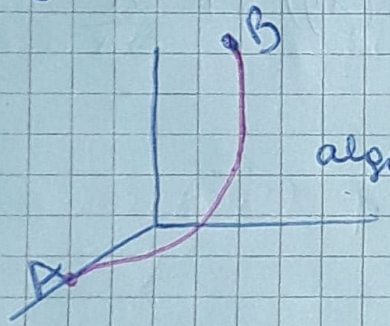
$$\int_C F\, ds = \int_C F(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$= \int_0^\pi (2 \cdot \cos^5 t \cdot \operatorname{sen}^3 t \cdot t^4 + \operatorname{sen}(\cos^5 t), \cos^{10} t\, t^4, \cos^{10} t \operatorname{sen}^3 t)\, \sigma'(t)\, dt$$

$\downarrow$

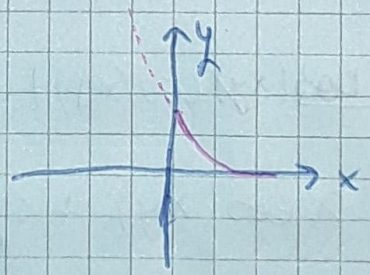
no quiero integrar esto.

¿Quién es C?



algo así, solo toca al eje x en $(1, 0, 0)$

visto desde arriba



$\sigma(0) = (1, 0, 0)$

$\sigma(\pi) = (-1, 0, \pi^4)$

$$\boxed{\nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}$$

$$= (x^2 - x^2, -2xy + 2xy, 2xz - 2xz)$$

$\nabla \times F = (0,0,0) \rightsquigarrow$ Es campo gradiente.

$\rightsquigarrow \int_C F\,ds = f(B) - f(A)$.

¿Quién es f? Sale a ojo.

$$f = x^2 y z - \cos(x)$$

$$\nabla f = (2xyz + \operatorname{sen}(x),\ x^2 z,\ x^2 y) = F \checkmark$$

$$\int_C F\,ds = f(1,0,0) - f(-1,0,\pi)$$

$$= \underbrace{-\cos(1)}_{\approx -0{,}540} - (\underbrace{-\cos(-1)}_{\approx -0{,}540}) = \boxed{0}$$

b. $C: \sigma(t) = (e^t, e^{t+1}, 0) \qquad -1 \leq t \leq 0$

$$\sigma(-1) = (e^{-1}, 1, 0)$$

$$\sigma(0) = (1, e, 0)$$

$$F = (\cos xy^2 - xy^2 \operatorname{sen} xy^2,\ -2x^2 y \operatorname{sen} xy^2,\ 0)$$

$$(\nabla \times F) = (0,\ 0,\ -4xy \operatorname{sen}(xy^2) + (-2x^2 y)\cdot \cos(xy^2)\cdot y^2$$

$$+ \operatorname{sen}(xy^2)\cdot x + (2yx \cdot \operatorname{sen} xy^2 + xy^2 \cdot \cos(xy^2)\cdot x)$$

$$= (0, 0, -4xy \operatorname{sen}(xy^2) - 2x^2 y^3 \cos(xy^2)$$

$$+ 4yx \operatorname{sen}(xy^2) + 2x^2 y^3 \cos(xy^2))$$

CA: $\dfrac{\partial[-2y(x^2 \cdot \operatorname{sen}(xy^2))]}{\partial x} = -2y((2x \cdot \operatorname{sen}(xy^2) + x^2 \cdot \cos(xy^2)\cdot y^2)$

$$= -4xy \operatorname{sen}(xy^2) - 2x^2 y^3 \cos(xy^2)$$

$$\frac{\partial(\cos xy^2 - xy^2 \operatorname{sen} xy^2)}{\partial y} = -\operatorname{sen}(xy^2)\cdot 2xy - (yx2 \operatorname{sen}(xy^2) + (xy^2 \cdot (\cos(xy^2)\cdot 2xy)) =$$

$$-4 \operatorname{sen}(xy^2)\, xy - 2x^2 y^3 \cos(xy^2)$$

Como el rotor es $(0,0,0)$ F es campo gradiente. $\exists f \,/\, \nabla f = F$

$$\frac{\partial f}{\partial x} = \cos(xy^2) - xy^2 \operatorname{sen} xy^2$$

$$\frac{\partial f}{\partial y} = -2x^2 y \operatorname{sen} xy^2$$

$$\frac{\partial f}{\partial z} = 0.$$

Condición 1) $f = \int_0^x \cos(ty^2) - ty^2 \operatorname{sen}(ty^2)\, dt$

$$= \frac{\operatorname{sen}(ty^2)}{y^2}\Big|_0^x - \int s \cdot \operatorname{sen}(s) \frac{ds}{y^2}$$

$s = ty^2$

$ds = y^2 dt$

uso partes

$\int u\,dv = u \cdot v - \int v\,du$

$u = s$

$dv = \operatorname{sen}(s)$

$$= \frac{\operatorname{sen}(xy^2)}{y^2} - \left(-s\cos(s) - \int -\cos(s)\, ds\right)\frac{1}{y^2}$$

$$= \quad '' \quad - \left(-t\cos(ty^2) + \frac{\operatorname{sen}(ty^2)}{y^2}\right)\Big|_0^x$$

$$f = \cancel{\frac{\operatorname{sen}(xy^2)}{y^2}} + x\cos(xy^2) - \cancel{\frac{\operatorname{sen}(xy^2)}{y^2}} + C(y,z)$$

- $\frac{\partial f}{\partial z} = 0 = 0 + \frac{\partial C(y,z)}{\partial z}$

$\rightsquigarrow \frac{\partial C}{\partial z} = 0 \rightarrow$ no depende de z.

- $\frac{\partial f}{\partial y} = -2x^2 y \operatorname{sen} xy^2 = -x \operatorname{sen}(xy^2) \cdot 2xy + \frac{\partial C(y)}{\partial y}$

son iguales

$\rightsquigarrow 0 = \frac{\partial C(y)}{\partial y}$

$$\Rightarrow \boxed{f = x\cos(xy^2)}$$

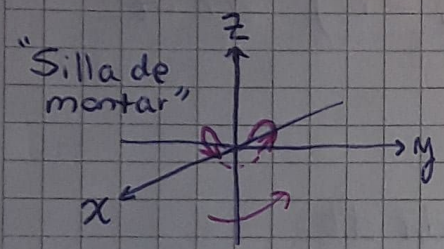
$$\int_C F\,dS = -f(e^{-1},1,0) + f(1,e,0)$$

$$= \cos(e^2) - e^{-1}\cdot\cos(e^{-1})$$

$$= \boxed{\cos(e^2) - \frac{\cos(e^{-1})}{e}}$$

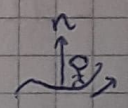
8. $\sigma(t) = (\operatorname{sen} t, \cos t, \operatorname{sen} 2t) \quad 0 \le t \le 2\pi$

$\left.\begin{array}{l}\sigma(0) = (0,1,0)\\ \sigma(2\pi) = (0,1,0)\end{array}\right\}$ Es una curva cerrada



de costado

de arriba



$$F = \left(y + \operatorname{sen} x,\ \frac{3}{2}z^2 + \cos y,\ 2x^3\right)$$

$\nabla\times F = (-3z, -6x^2, -1) \neq \vec{0}$ entonces no es campo gradiente, qué mal

F es $C^1 \to$ vale stokes

necesito una superficie

Uso la sugerencia $z = 2xy \to$ gráfico de una función

$S: \phi(x,y) = (x, y, 2xy) \rightsquigarrow T(r,t) = (r\operatorname{sen} t, r\cos t, \operatorname{sen}(2t))$

$\phi: D \to \mathbb{R}^3$, $D = \{x^2+y^2 \le 1\} \rightsquigarrow$ La puedo describir en polares.

STOKES $\boxed{\iint_S \nabla\times F\,dS = \int_{\partial S^+} F\,dS}$

$$\iint_S (\nabla\times F)\,dS = \iint_D (\nabla\times F)(T(x,y))\cdot(T_r\times T_t)\,dx\,dy$$

$$\int_0^{2\pi}\int_0^1 \left(-3\cdot\operatorname{sen}(2t), -6(\operatorname{sen} t)^2, -1\right)\cdot(T_r\times T_t)\,dx\,dy$$

$\phi(x,y) = (x, y, z)$

$T(r,\theta) = (r\,\text{sen}\,t, r\cos t, \text{sen}(t\cdot 2))$

$T_r = (\text{sen}\,t, \cos t, 0)$

$T_t = (r\cos t, -r\,\text{sen}\,t, 2\cos(2t))$

$T_r \times T_t = (2\cos t\cdot\cos 2t, -2\,\text{sen}\,t\cdot\cos(2t), -r)$ yo quería la normal hacia arriba!

$$-\int_0^{2\pi}\int_0^1 (-3\cdot \text{sen}\,t2, -6\cdot\text{sen}^2 t, -1)\cdot$$

$$(2\cos t\cdot\cos 2t, -2\,\text{sen}\,t\cdot\cos(2t), -r)\ dr\,dt$$

$$= -\int_0^{2\pi}\int_0^1 -6\,\text{sen}(2t)\cos t\cdot\cos 2t$$

$$+12\cdot\text{sen}^3 t\cdot\cos(2t) + r \quad dr\,dt$$

CA $\int_0^{2\pi} \text{sen}(2t)\cdot\cos(t)\cdot\cos(2t)$

Identidad $\text{sen}(2t) = 2\,\text{sen}\,t\cdot\cos t$

$\cos(2t) = \cos^2(t) - \text{sen}^2(t)$ $\quad \text{sen}^2 + \cos^2 = 1$, $\text{sen}^2 = 1 - \cos$

$$= \int_0^{2\pi} 2\cdot\text{sen}\,t\cdot\cos^2 t\cdot(\cos^2 t - \text{sen}^2 t)\ dt$$

$$= \int_0^{2\pi} 2\cdot\text{sen}\,t\cdot\cos^2 t\cdot(\cos^2 t - (1-\cos^2 t))\ dt$$

$$= \int_0^{2\pi} 2\,\text{sen}\,t\cos^2 t\cdot(2\cos^2 t - 1)\ dt$$

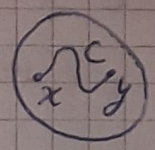
$$= \int_0^{2\pi}(4\underbrace{\text{sen}\,t\cdot\cos^4 t}_{=0 \text{ entre } 0 \text{ y } 2\pi} - 2\underbrace{\text{sen}\,t\cdot\cos^2 t}_{=0})\,dt = 0$$

CA $\int_0^{2\pi} \text{sen}^3 t\cdot\cos(2t)\cdot dt = 0$ porque son funciones trigonométricas impares entre 0 y 2π.

$$\Rightarrow = -\int_0^{2\pi}\int_0^1 r\,dr\,dt = -2\pi\cdot\frac{r^2}{2}\Big|_0^1 = -\pi$$

9 Sea $f \in C^1(B)$ donde B es una bola en $\mathbb{R}^3$.
Si $\nabla f = 0$ en $B \Rightarrow f$ es cte en B.

Una bola es una región arcoconexa
Dados dos puntos x e $y \in B$ y C una curva C^1 que los une.



Considero $F = \nabla f \Rightarrow$ la integral sobre la curva sólo depende de los extremos

$$\int_C F\,dS = \int_x^y \nabla f\,dS = f(y) - f(x) = 0$$

$$\Rightarrow f(y) = f(x)$$

$\Rightarrow f$ es constante para todos los puntos de la bola.

10. $F(x,y,z) = (2xy + z^2,\ x^2 - 2yz,\ 2xz - y^2)$

$$\nabla \times F = (-2y + 2y,\ 2z - 2z,\ 2x - 2x) = (0,0,0)$$

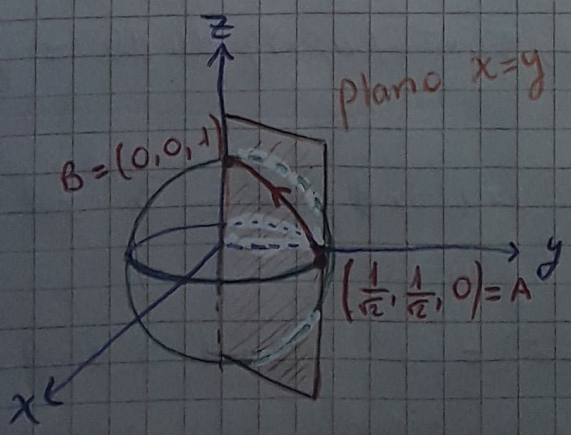
$\Rightarrow F$ es campo gradiente. $\exists f / \nabla f = F$

$$\int_C F\,dS = \int_C \nabla f \cdot dS$$

C es la curva contenida en la esfera $x^2 + y^2 + z^2 = 1$ y el plano de ecuación $y = x$ recorrida desde $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ al polo norte.

Como F es campo gradiente

$$\int_C F\,dS = \int_A^B \nabla f\,dS = f(B) - f(A)$$



3 condiciones 1) $\frac{\partial f}{\partial x} = 2xy + z^2$ 2) $\frac{\partial f}{\partial y} = x^2 - 2yz$ 3) $\frac{\partial f}{\partial z} = 2xz - y^2$

1) $f = \int_0^x 2ty + z^2 \, dt = x \left.\frac{t^2}{2}\right|_0^x y + t\Big|_0^x z^2 = x^2 y + x z^2 + C(y,z)$

2) $f = x^2 y + x z^2 + C(y,z)$

$$\frac{\partial f}{\partial y} = \cancel{x^2} + \frac{\partial C(y,z)}{\partial y} = \cancel{x^2} - 2yz$$

$$\int \frac{\partial C(y,z)}{\partial y} dy = \int -2yz \, dy$$

$$C(y,z) = -\cancel{2}z \cdot \frac{y^2}{\cancel{2}}$$

3) $\boxed{f = x^2 y + x z^2 - z y^2}$

$$\frac{\partial f}{\partial z} = 2xz - y^2 = 2xz - y^2 \checkmark$$

$$\Rightarrow \int_C F \, ds = f(0,0,1) - f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= 0 - \frac{1}{2\cdot\sqrt{2}} + 0 + 0 = \boxed{-\frac{1}{2\sqrt{2}}}$$

11. (17 P2) $F(x,y,z) = (x,y,z)$ flujo saliente del cubo $[0,1] \times [0,1] \times [0,1]$

$$\iint_{S=\partial\Omega} F \eta \, dS \underset{\text{Gauss}}{=} \iiint_{\Omega} \operatorname{div}(F) \, dx\, dy\, dz$$

En la práctica 2 calculamos

$$\iint_S F \cdot \eta \, dS = \iint_D \langle F(T(u,v)), T_u \times T_v(u,v) \rangle \, du\, dv = \boxed{3}$$

$$\iiint_\Omega \operatorname{div}(F) \, dx\, dy\, dz = \iiint_\Omega (1+1+1) \, dx\, dy\, dz = 3 \cdot \operatorname{vol}(\Omega)$$

El vol de un cubo es $\operatorname{vol}(\Omega) = a^3$ donde a representa cada lado (base · altura · profundidad) $a = 1 \Rightarrow 3\operatorname{vol}(\Omega) = \boxed{3}$

12 $\iint_S (x+y+z)\,dS$ donde $S = \partial\Omega$ borde de la bola unitaria

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

$$x+y+z = F \cdot \eta = \underbrace{(1,1,1)}_{F(x,y,z)}\underbrace{(x,y,z)}_{\text{NORMAL UNITARIA EXTERIOR EN } (x,y,z)}$$

$$\iint_S x+y+z\,dA = \iint_{S=\partial\Omega} F\cdot\eta\,dS = \iiint_\Omega \underbrace{\operatorname{div}(F)}_{=0} = 0$$

Otra manera: S es superficie simétrica y $f(x,y,z) = x+y+z$ es impar ya que $f(-x) = -f(x)$

$\Rightarrow$ La integral da cero.

13 $F = -G\cdot m\cdot M\cdot \dfrac{x}{\|x\|^3}$ Ω = Bola unitaria en $\mathbb{R}^3$.

Gauss funciona para campos F C^1 y el campo gravitatorio no está definido en todo $\mathbb{R}^3$ sino en $\mathbb{R}^3 \setminus \{\text{eje } z\}$ si la bola unitaria está centrada en $(0,0,0)$ o contiene al eje z Gauss no se puede aplicar a F.

14 $\iint_S F\,dS$ $F = (x^3, y^3, z^3)$ S esfera radio R con la normal hacia adentro.

- F es C^1 ✓
- S es sup cerrada ✓
- orientación η exterior ✗ Para aplicar Gauss quiero η exterior

$\rightsquigarrow$ S invierte orientación.

$$-\iint_S F\cdot\eta\,dS \underset{\text{Gauss}}{=} \iiint_\Omega \operatorname{div}(F)\,dx\,dy\,dz$$

Puedo parametrizar S y calcular el flujo saliente :(

o puedo integrar $\operatorname{div}(F)$ con esféricas :)

$$\operatorname{div}(F) = 3x^2 + 3y^2 + 3z^2$$

$$S = \{(x,y,z) : x^2+y^2+z^2 = R^2\}$$

En esféricas

$$T(r,\theta,\varphi) = \begin{cases} x = r\cos\theta \operatorname{sen}\varphi \\ y = r \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = r\cos\varphi \end{cases}$$

$$\boxed{J = r^2 \operatorname{sen}\varphi} \; !$$

$$0 \leq r \leq R$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$

$$\iiint_\Omega 3(x^2+y^2+z^2)\,dx\,dy\,dz =$$

$$3\int_0^R \int_0^{2\pi} \int_0^{\pi} (r^2 \cos^2\theta \operatorname{sen}^2\varphi + r^2 \operatorname{sen}^2\theta \operatorname{sen}^2\varphi + r^2\cos^2\varphi) \cdot r^2 \operatorname{sen}\varphi \, d\varphi\, d\theta\, dr$$

$$= 3\int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \left(\operatorname{sen}^2\varphi \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} + \underbrace{\cos^2\varphi}_{=1}\right) \cdot r^2 \operatorname{sen}\varphi \, d\varphi\, d\theta\, dr$$

$$= 3\int_0^R r^4\, dr \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \operatorname{sen}\varphi\, d\varphi$$

$$(-\cos\varphi|_0^{\pi})$$

$$= 3 \cdot \frac{R^5}{5} \cdot 2\pi \cdot 2 = \boxed{\frac{12}{5} R^5 \cdot \pi} \Rightarrow \boxed{\iint_S F\,dS = -\frac{12}{5} R^5 \pi}$$

15 $F(x,y,z) = (x, y, -2z)$

$\operatorname{div}(F) = 1 + 1 - 2 = 0$ F es C^1 sin divergencia

$\sigma(\theta) = (r(\theta) \cdot \operatorname{sen}\theta, 0, r(\theta)\cos(\theta))$ Plano xz (curva C)

donde $r(\theta) = \frac{4\sqrt{3}}{9}(2 - \cos(2\theta))$ $\quad \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

S no es cerrada:

revolución σ en eje z

$$\iint_S F\eta\, dS = \iiint_\Omega \nabla F\, dS$$ Vale cuando S es cerrada

Considero $S' = S \cup S_1$ / S' es cerrada ✓

¿Quién es S_1?

$$\sigma\left(\frac{\pi}{6}\right) = \left(\underbrace{\frac{\cancel{4}\sqrt{3}}{9} \cdot \frac{3}{\cancel{2}}}_{r\left(\frac{\pi}{6}\right)} \cdot \frac{\cancel{1}}{\cancel{2}},\ 0,\ \underbrace{\frac{\cancel{4}\sqrt{3}}{9} \cdot \frac{3}{\cancel{2}}}_{r\left(\frac{\pi}{6}\right)} \frac{\sqrt{3}}{\cancel{2}} \right)$$

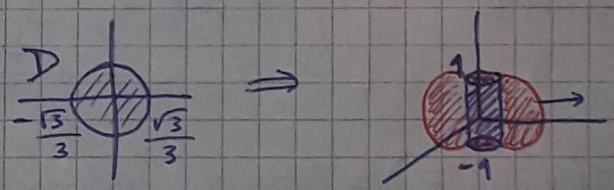
$$= \left(\frac{\sqrt{3}}{3},\ 0,\ 1 \right)$$

$$\sigma\left(\frac{5\pi}{6}\right) = \left(\frac{\cancel{4}\sqrt{3}}{\cancel{9}} \cdot \frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{2}},\ 0,\ \frac{4\sqrt{3}}{9} \cdot \frac{3}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) \right)$$

$$= \left(\frac{\sqrt{3}}{3},\ 0,\ -1 \right)$$

Quiero un cilindro de radio $\frac{\sqrt{3}}{3}$ para que S sea sup. cerrada.

En el plano xy



$$S = \left(r(\theta) \cdot \operatorname{sen}\theta \cdot \cos\varphi,\ r(\theta) \cdot \operatorname{sen}\theta \cdot \operatorname{sen}\varphi,\ r(\theta)\cos(\theta) \right)$$

$$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$$

$$0 \leq \varphi \leq 2\pi$$

$$S_{1_{(\theta,z)}} = \left(\frac{\sqrt{3}}{3}\cos\theta,\ \frac{\sqrt{3}}{3}\operatorname{sen}\theta,\ z \right) \qquad 0 \leq \theta \leq 2\pi$$

$$-1 \leq z \leq 1$$

$S' = S \cup S_1$ vale Gauss

$$\Rightarrow \iint_{S'} F\, dS = \iiint_\Omega \underbrace{\operatorname{div}(F)}_{=0}\, dS = 0 \quad \Bigg\| \quad \iint_{S'} F\, dS = \iint_S F\, dS + \iint_{S_1} F\, dS$$

$$\Rightarrow \iint_S F\, dS = \ominus \iint_{S_1} F\, dS.$$

$$S_1 = \left(\frac{\sqrt{3}}{3}\cos\theta, \frac{\sqrt{3}}{3}\operatorname{sen}\theta, z\right)$$

$$S_{1\theta}(\theta,z) = \left(-\frac{\sqrt{3}}{3}\operatorname{sen}\theta, \frac{\sqrt{3}}{3}\cos\theta, 0\right)$$

$$S_{1z}(\theta,z) = (0, 0, 1)$$

$$S_{1\theta} \times S_{1z}(\theta,z) = \left(\frac{\sqrt{3}}{3}\cos\theta, \frac{\sqrt{3}}{3}\operatorname{sen}\theta, 0\right) = \eta$$ invierte orientación $\ominus$

$$\Rightarrow \oplus \iint_{S_1} F\eta\, dS = \iint_{S_1} \left(\frac{\sqrt{3}}{3}\cos\theta, \frac{\sqrt{3}}{3}\operatorname{sen}\theta, -2z\right) \cdot \left(\frac{\sqrt{3}}{3}\cos\theta, \frac{\sqrt{3}}{3}\operatorname{sen}\theta, 0\right) d\theta\, dz$$

los $-$ se cancelan

$$= \int_{-1}^{1}\int_0^{2\pi} \left(\frac{\sqrt{3}}{3}\right)^2 (\cos^2\theta + \operatorname{sen}^2\theta)\, d\theta\, dz$$

$$= \frac{3}{9}\int_{-1}^{1} dz \int_0^{2\pi} d\theta = \frac{1}{3}\cdot 2 \cdot 2\pi = \boxed{\frac{4\pi}{3}}$$

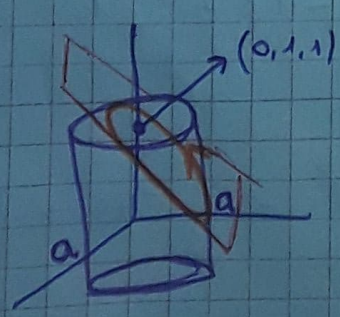
Finalmente $\boxed{\iint_S F\, dS = \frac{4\pi}{3}}$

16. $F(x,y,z) = (0, 0, a^2 - x^2 - y^2)$

$\operatorname{div}(F) = 0$ sin divergencia

Cilindro $x^2 + y^2 \leq a^2$

a. $y + z = 1$



La intersección entre el cilindro y el plano me deja una circunferencia de radio a sobre el plano $y + z = 1$

La normal es exterior.

Si considero la superficie cerrada vale Gauss

$$S: \begin{cases} x^2+y^2 \leq a^2 \\ y+z \leq 1 \end{cases} \quad 0 \leq z \leq 1-y$$

$$\iint_S F\, dS = \iiint_\Omega \underbrace{\operatorname{div}(F)}_{=0}\, dx\,dy\,dz = 0$$

$$\iint_S F\, dS = \underbrace{\iint_{S_1} F\, dS} + \iint_{S_2} F\, dS + \iint_{S_3} F\, dS = 0$$

S_1 → es la que quiero saber

S_2 → tapas laterales (cilindro)

S_3 → tapa de abajo

$$S_2(\theta, z) = (a\cos(\theta),\ a\,\mathrm{sen}(\theta),\ z)$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq z \leq 1 - a\,\mathrm{sen}(\theta)$$

$$S_{2\theta}(\theta, z) = (-a\,\mathrm{sen}(\theta),\ a\cos(\theta),\ 0)$$

$$S_{2z}(\theta, z) = (0,\ 0,\ 1)$$

$$S_{2\theta} \times S_{2z} = (0,\ a\,\mathrm{sen}(\theta),\ 0) \quad \text{respeta orientación}$$

$$\int_0^{2\pi}\int_0^{1-\mathrm{sen}(\theta)} (0, 0, *)\cdot(0, *, 0)\, dz\, d\theta = 0$$

$$S_3(r, \theta) = (r\cos\theta,\ r\,\mathrm{sen}\,\theta,\ 0) \qquad 0 \leq \theta \leq 2\pi$$

$$S_{3r}(r, \theta) = (\cos\theta,\ \mathrm{sen}\,\theta,\ 0) \qquad 0 \leq r \leq a$$

$$S_{3\theta}(r, \theta) = (-r\,\mathrm{sen}\,\theta,\ r\cos\theta,\ 0)$$

$$S_{3r} \times S_{3\theta} = (0,\ 0,\ r) \rightarrow \text{invierte orientación}$$

$$-\int_0^{2\pi}\int_0^a (0, 0, a^2 - r^2\cos^2\theta - r^2\,\mathrm{sen}\,\theta)(0, 0, r)\, dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^a \left[a^2 - r^2(\cos^2\theta + \mathrm{sen}^2\theta)\right]\cdot r\, dr\, d\theta = -2\pi\cdot\left[\int_0^a r\cdot a^2 - r^3\, dr\right]$$

$$= -2\pi \left[\frac{r^2}{2}\Big|_0^a a^2 - \frac{r^4}{4}\Big|_0^a \right] = 2\pi \cdot \left[\frac{a^2}{2} a^2 - \frac{a^4}{4} \right] = \ominus 2\pi \cdot \frac{a^4}{4}$$

$$= \boxed{\frac{-\pi \cdot a^4}{2}}$$

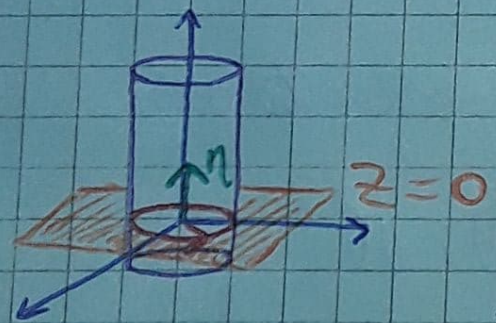
Entonces

$$\iint_{S_1} F\,dS + \iint_{S_2} F\,dS + \iint_{S_3} F\,dS = 0$$

$$\iint_{S_1} F\,dS + 0 - \frac{\pi \cdot a^4}{2} = 0$$

$$\boxed{\iint_{S_1} F\,dS = \frac{\pi \cdot a^4}{2}}$$

b.



$$\iint_S F\,dS = \iint_D F(T(r,\theta)) \cdot T_r \times T_\theta \; dr\,d\theta$$

$$T(r,\theta) = (r\cos\theta,\ r\,\text{sen}\,\theta,\ 0) \;\Big\|\; T_r \times T_\theta = (0,0,r)$$

$$\int_0^{2\pi}\int_0^{a} (0,0,a^2 - r^2\cos^2\theta - r^2\,\text{sen}^2\theta)(0,0,r)\,dr\,d\theta \rightarrow \text{La hice antes}$$

$$= \boxed{\frac{\pi \cdot a^4}{2}}$$

El flujo del área no depende de la sección

17. $F(x,y,z) = \left(x, y, z - \frac{1}{2}\right)$

$\operatorname{div}(F) = (3)$

$S = \left(f(z)\cos\varphi,\ f(z)\operatorname{sen}\varphi,\ z \right)$ $\quad 0 \leq \varphi \leq 2\pi$

$0 \leq z \leq 1$

donde $f(z) = \frac{1}{2} z e^{2-2z}$

$\iint_S F\, dS$

$S \rightarrow$ no es cerrada

Como $S' = S \cup \underbrace{S_1}_{\text{disco "tapita" de arriba en } z=1}$ / S' es cerrada y vale Gauss

$\sigma(z) = \left(\frac{1}{2} z e^{2-2z},\ 0,\ z \right) \qquad 0 \leq z \leq 1$

$\sigma(0) = (0,0,0)$

$\sigma(1) = \left(\frac{1}{2}, 0, 1 \right)$

$S_1(\theta, r) = (r\cos\theta, r\operatorname{sen}\theta, 1) \qquad 0 \leq \theta \leq 2\pi$

$0 \leq r \leq \frac{1}{2}$

$$\iint_{S'} F\, dS = \iiint_{\Omega} 3\, dx\,dy\,dz = 3\, \mathrm{vol}(\Omega)$$

$$\iint_{S'} F\, dS = \boxed{\iint_S F\, dS} + \iint_{S_1} F\, dS$$

Es la que quiero averiguar

$$\iint_{S'} F\, dS - \iint_{S_1} F\, dS = \iint_S F\, dS$$

$$\left.\begin{array}{l} S_{1\theta}(\theta,r) = (-r\operatorname{sen}\theta,\ r\cos\theta,\ 0) \\ S_{1r}(\theta,r) = (\cos\theta,\ \operatorname{sen}\theta,\ 0) \end{array}\right\} S_{1\theta} \times S_{1r} = (0, 0, -r)$$ invierte orientación

$$-\iint_{S_1} F\cdot\eta\, ds = \iint_{S_1} \left(r\cos\theta, r\,\text{sen}\,\theta, \frac{1}{2}\right)(0,0,-r)\, dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^{1/2} -\frac{r}{2}\, dr\, d\theta = +\frac{\cancel{2}\pi}{\cancel{2}} \left.\frac{r^2}{2}\right|_0^{1/2} = -\frac{\pi}{2}\cdot\frac{1}{4} = \boxed{+\frac{\pi}{8}}$$

$$\iint_{S'} F\, dS - \frac{\pi}{8} = \iint_S F\, dS$$

$$3\,\text{vol}(\Omega) - \frac{\pi}{8} = \iint_S F\, dS$$

Tengo que calcular $\text{vol}(\Omega)$ → sólido interior de una superficie de revolución

$$T(r,\theta,z) = (r\, f(z)\cos\theta,\ r\, f(z)\,\text{sen}\,\theta,\ z)$$

↓

relleno cada punto de Ω

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 1$

salvo los bordes e bizes ✓

$$DT = \begin{pmatrix} f(z)\cos\theta & f(z)\,\text{sen}\,\theta & 0 \\ -r f(z)\,\text{sen}\,\theta & r f(z)\cos\theta & 0 \\ r f'(z)\cos\theta & r f'(z)\,\text{sen}\,\theta & 1 \end{pmatrix}$$

lugar 3*3 → $(-1)^{3+3} = 1$

$$\rightsquigarrow \det(DT) = 1\cdot f^2(z)\cdot r + 0 + 0$$

$$\text{Vol}(\Omega) = \int_\Omega 1 = \int_\Omega \text{jacobiano}$$

$$\Rightarrow \int_0^1\int_0^{2\pi}\int_0^1 f^2(z)\cdot r\, dz\, d\theta\, dr = \int_0^1 f^2(z)\, dz \cdot \int_0^{2\pi} d\theta \int_0^1 r\, dr$$

$$= \cancel{2}\pi\cdot\frac{1}{\cancel{2}}\int_0^1 \frac{1}{4}\cdot z^2\cdot e^{(2-2z)\cdot 2}\, dz$$

CA $\int z^2 \cdot e^{4-4z}\, dz = z^2 \cdot \frac{e^{4-4z}}{-4} - \int \underbrace{\frac{e^{4-4z}}{-4} 2z}_{::}$

$u = z^2 \longrightarrow du = z\ 2$

$dv = e^{4-4z} \rightarrow v = \frac{e^{4-4z}}{-4}$

$\int u\, dv = u \cdot v - \int v\, du$

CA $\int z^2 \cdot e^{4-4z}\, dz = \int \left(\frac{u-4}{-4}\right)^2 \cdot e^{u} \cdot \frac{du}{-4}$

$4 - 4z = u \Rightarrow z = \frac{u-4}{-4}$

$-4\, dz = du$

$= \int (u-4)^2 \cdot \frac{1}{16} \cdot e^{u}\, \frac{du}{-4} = -\frac{1}{64} \int (u^2 - 8u + 16) e^{u}\, du$

$= -\frac{1}{64} \left[\underbrace{\int u^2 \cdot e^{u}\, du}_{①} - 8 \underbrace{\int u \cdot e^{u}\, du}_{②} + 16 \int e^{u}\, du \right]$

① $\int u^2 \cdot e^{u}\, du = e^{u} \cdot u^2 - 2 \underbrace{\int e^{u} \cdot u\, du}_{②} = *$

$f = u^2 \rightarrow df = 2u\, du$

$dg = e^{u} du \rightarrow g = e^{u}$

② $\int e^{u} \cdot u\, du = u \cdot e^{u} - \int e^{u}\, du = \boxed{u e^{u} - e^{u} + C}$

$f = u \rightarrow df = 1\, du$

$dg = e^{u} du \rightarrow g = e^{u}$

$\Rightarrow$ ① $* = e^{u} \cdot u^2 - 2(u e^{u} - e^{u}) + C$

$= e^{u} \cdot u^2 - 2u e^{u} + 2e^{u} = \boxed{e^{u}(u^2 - 2u + 2)} + C$

Volviendo a la integral

$= -\frac{1}{64} \left[e^{u}(u^2 - 2u + 2) - 8 \cdot (u e^{u} - e^{u}) + 16 \cdot e^{u} \right]$

$= -\frac{1}{64} \left[e^{u} \cdot u^2 - 2u e^{u} + 2e^{u} - 8u e^{u} + 8e^{u} + 16 e^{u} \right]$

$$= -\frac{1}{64}\left[e^{u}\cdot u^2 - \underline{10\,u\,e^{u}} + \underline{26\,e^{u}}\right]$$

$$= -\frac{1}{64}\left[e^{u}\left(u^2 - 10u + 26\right)\right]$$

$$u = 4 - 4z$$

$$= -\frac{1}{64}\cdot e^{4-4z}\left((4-4z)^2 - 10(4-4z) + 26\right)$$

$$= -\frac{1}{64}\cdot e^{4-4z}\left(16 - \underline{32z} + 16z^2 - 40 + \underline{40z} + 26\right)$$

$$= -\frac{e^{4-4z}}{64}\left(16z^2 + 8z + 2\right)$$

Ahora vuelvo al problema original

$$= \frac{\pi}{4}\left[-\frac{e^{4-4z}}{64}\left(16z^2 + 8z + 2\right)\right]_0^1 = \pi\left[-\frac{e^{4-4z}}{128}\left(8z^2 + 4z + 1\right)\right]_0^1$$

$$= -\frac{\pi}{128}\left[13 - e^{4}\right] = \boxed{\frac{\pi}{128}\left(e^{4} - 13\right) = vol(\Omega)}$$

Finalmente: el flujo térmico saliente a través del mate:

$$\iint_S F\,dS = 3\,vol(\Omega) - \frac{\pi}{8} = \boxed{\frac{3\pi}{128}\left(e^{4} - 13\right) - \frac{\pi}{8}}$$

//

18. Para probar que existe $G / F = \nabla \times G$ si $\operatorname{div}(F) = 0$ usamos Gauss. Como $F = \frac{x}{\|x\|^3}$ no es C^1 en todo $\mathbb{R}^3$ no se estaría cumpliendo una de las hipótesis.

19. $\nabla \times G = (G_{3y} - G_{2z}, G_{1z} - G_{3x}, G_{2x} - G_{1y})$

$G = (G_1, G_2, G_3)$

¿$F = \nabla \times G$? Es equivalente a $\operatorname{div}(F) = 0$

a. $F = (x, y, z) \Rightarrow \operatorname{div}(F) = 3 \neq 0 \rightsquigarrow F$ no es el rotor de otro campo

b. $F = (\overbrace{x^2+1}^{F_1}, \overbrace{x - 2xy}^{F_2}, \overbrace{y}^{F_3})$

$\operatorname{div}(F) = 2x - 2x + 0 = 0 \Rightarrow \exists\, G / F = \nabla \times G$

Tomo $G = (G_1, G_2, G_3)$ del ejercicio 18

$$G_1 = \int_0^z F_2(x, y, t)\, dt - \int_0^y F_3(x, t, 0)\, dt$$

$$G_1 = \int_0^z x - 2xy\, dt - \int_0^y dt$$

$$G_1 = xz - 2xyz - \frac{y^2}{2}$$

$$G_2 = -\int_0^z F_1(x, y, t)\, dt$$

$$G_2 = -\int_0^z x^2 + 1\, dt = -zx^2 - z$$

$$G_3 = 0$$

$$\Rightarrow \boxed{G = (xz - 2xyz, -zx^2 - z, 0)}$$

$$\nabla \times G = (x^2 - 1, x - 2xy, -\cancel{2xz} + \cancel{2xz} + y) = F \checkmark$$

verifico

21. $V = (x, y, xy - z)$ campo v. de un fluido

$\operatorname{div}(V) = 1 + 1 - 1 = 1 > 0$ El fluido se expande

la $\operatorname{div} > 0$ tiene un significado físico.