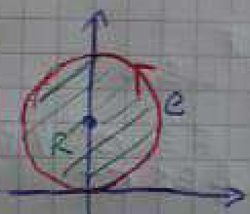
TEOREMA DE GREEN

TEOREMA DE GREEN: SEA $D \subseteq \mathbb{R}^2$ UNA REGIÓN ELEMENTAL Y ∂D SU BORDE ORIENTADO POSITIVAMENTE. SI $P, Q: D \to \mathbb{R}$ SON DE CLASE C^1, ENTONCES

$$\int_{\partial D^+} (P\,dx + Q\,dy) = \iint_D (Q_x - P_y)\,dx\,dy$$

EJEMPLOS: 1) SEA C EL CÍRCULO CERRADO DE RADIO 2 CENTRADO EN $(0,2)$, RECORRIDO EN SENTIDO ANTIHORARIO.

HALLAR $\int_C F\,ds$ DONDE $F(x,y) = (y\,\mathrm{senh}(x), \frac{1}{2}x^2 y + \cosh(x))$



C SE PARAMETRIZA COMO:

$\sigma(t) = (2\cos t, 2+2\,\mathrm{sen}\,t)$, $t \in [0, 2\pi]$

$\sigma'(t) = (-2\,\mathrm{sen}\,t, 2\cos t)$

$$\int_C F\,ds = \int_0^{2\pi} \langle F(\sigma(t)), \sigma'(t) \rangle\,dt =$$

$$= \int_0^{2\pi} \left\langle \left((2+2\,\mathrm{sen}\,t)\,\mathrm{senh}(2\cos t), 2\cos^2 t(2+2\,\mathrm{sen}\,t) + \cosh(2\cos t)\right), (-2\,\mathrm{sen}\,t, 2\cos t) \right\rangle dt$$

$$= \int_0^{2\pi} -2(2+2\,\mathrm{sen}\,t)\,\mathrm{sen}\,t\,\mathrm{senh}(2\cos t) + 4\cos^3 t(2+2\,\mathrm{sen}\,t) + 2\cos t\,\cosh(2\cos t)\,dt$$

MUY DIFÍCIL Y LARGO!!! USEMOS EL TEOREMA DE GREEN

F ES C^1, R ES DE TIPO III Y C ESTÁ ORIENTADA POSITIVAMENTE

$P = y\,\mathrm{senh}(x)$ $\qquad$ $P_y = \mathrm{senh}(x)$

$Q = \frac{1}{2}x^2 y + \cosh(x)$ $\qquad$ $Q_x = xy + \mathrm{senh}(x)$

$$Q_x - P_y = xy + \mathrm{senh}(x) - \mathrm{senh}(x) = xy$$

LUEGO, $\left(T: \begin{cases} x = r\cos\theta & 0 \le r \le 2 \\ y = 2 + r\sin\theta & 0 \le \theta \le 2\pi \end{cases} \qquad |JT| = r \right)$

$$\int_C F ds = \iint_R xy \, dx dy \overset{\uparrow}{=} \int_0^{2\pi}\int_0^{2} r\cos\theta(2 + r\sin\theta)\, r \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (2r^2\cos\theta + r^3\cos\theta\sin\theta)\, dr\, d\theta = \int_0^{2\pi} \frac{2r^3}{3}\cos\theta + \frac{r^4}{4}\cos\theta\sin\theta \Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{16}{3}\cos\theta + 4\cos\theta\sin\theta \, d\theta = \frac{16}{3}\sin\theta + 2\sin^2\theta \Big|_0^{2\pi} = 0$$

2) PROBAR QUE SI F ES DE CLASE C^1, CON $F = \nabla f$ PARA ALGUNA $f: \mathbb{R}^2 \to \mathbb{R}$, Y SI D ES UNA REGIÓN DONDE PUEDO APLICAR EL TEOREMA DE GREEN, ENTONCES

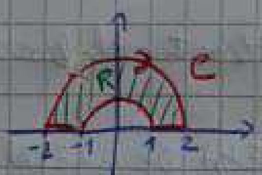
$$\int_{\partial D} F ds = 0$$

NOTAR QUE SI $F = \nabla f$, $P = f_x$ Y $Q = f_y$, LUEGO

$$Q_x - P_y = f_{yx} - f_{xy} = 0$$, PUES F ES C^1 Y POR LO TANTO f ES C^2

LUEGO, $\int_{\partial D} F ds = \int_D 0 \, dx dy = 0$

3) SEA $F(x,y) = (y + \sin(x^2), 2x + e^{\cos(y^2)})$ Y SEA C LA CURVA INDICADA EN EL GRAFICO:



CALCULAR $\int_C F ds$ $\quad (R = \{1 \le x^2 + y^2 \le 4\})$

USEMOS EL TEOREMA DE GREEN:

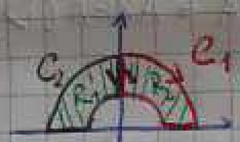
- PROBLEMA: C NO ESTA ORIENTADA POSITIVAMENTE!

  SOLUCIÓN: $\int_{C^-} F ds = -\int_{C^+} F ds$

- F ES C^1 ✓

• PROBLEMA: R NO ES DE TIPO III

SOLUCIÓN:



$$\int_C F\,ds = \int_{C_1} F\,ds + \int_{C_2} F\,ds$$

R_1 Y R_2 DE TIPO III

$$\int_R (Q_x - P_y)\,dA = \int_{R_1} (Q_x - P_y)\,dA + \int_{R_2} (Q_x - P_y)\,dA$$

POR LO TANTO, VALE GREEN EN ESTA REGIÓN

$$\int_{C^-} F\,ds = -\int_{C^+} F\,ds = -\iint_R Q_x - P_y\,dA = -\iint_R (2-1)\,dA = -\iint_R dA = -\text{AREA}(R)$$

$$\text{AREA}(R) = \frac{1}{2}\left(\pi 2^2 - \pi 1^2\right) = \frac{3}{2}\pi$$

LUEGO, $\int_{C^-} F\,ds = \frac{-3}{2}\pi$

4) ES IMPORTANTE QUE EL CAMPO F SEA C^1 EN TODA LA REGIÓN:

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

$$P_y = \frac{-(x^2+y^2) - (-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \; ; \; Q_x = \frac{x^2+y^2-x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

SEA $D = B_1(0,0)$, LUEGO $\iint_D (Q_x - P_y)\,dA = \iint_D 0\,dA = 0$

SI $C = \partial D^+$, UNA PARAMETRIZACIÓN ES $C(t) = (\cos t, \sin t)$, $C'(t) = (-\sin t, \cos t)$

$$\int_C F\,ds = \int_0^{2\pi} \langle F(\cos t, \sin t), (-\sin t, \cos t)\rangle\,dt = \int_0^{2\pi} \left\langle \left(\frac{-\sin t}{\underbrace{\cos^2 t + \sin^2 t}_{1}}, \frac{\cos t}{\underbrace{\cos^2 t + \sin^2 t}_{1}}\right), (-\sin t, \cos t)\right\rangle dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} dt = 2\pi \neq 0$$

ESTO NO CONTRADICE EL TEOREMA DE GREEN PUES F NO ES C^1 EN TODO D.

APLICACIÓN DEL TEOREMA DE GREEN PARA CALCULAR AREAS

SI TOMAMOS $F(x,y)=(0,x) \rightsquigarrow Q_x - P_y =$

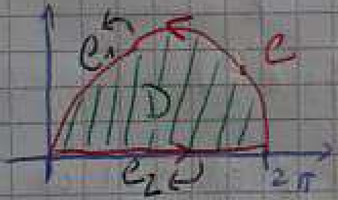
POR LO TANTO, $\int_{\partial D^+} F\,ds = \iint_D dA = AREA(D)$

SI TOMAMOS $G(x,y)=(-y,0) \rightsquigarrow Q_x - P_y = 1$

POR LO TANTO, $\int_{\partial D^+} G\,ds = \iint_D dA = AREA(D)$

EN RESUMEN, $AREA(D) = \int_{\partial D^+} x\,dy = \int_{\partial D^+} -y\,dx = \frac{1}{2}\int_{\partial D^+} -y\,dx + x\,dy$

EJEMPLOS: 1) HALLAR EL AREA LIMITADA POR EL EJE X Y UN ARCO DE CICLOIDE DE PARAMETRIZACIÓN $C(t)=(t-\operatorname{sen} t, 1-\cos t)$, $0 \leq t \leq 2\pi$



POR LO ANTERIOR,

$$AREA(D) = \int_{C^+} x\,dy = \int_{C^+} -y\,dx$$

NOS CONVIENE USAR $\int_{C^+} x\,dy$ (YA VEREMOS POR QUE), PERO AMBAS FUNCIONAN.

$$\int_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy$$

$\underline{C_1}$: $\Gamma(t) = (t - \operatorname{sen} t, 1-\cos t)$ $\quad 0 \leq t \leq 2\pi \longrightarrow$ SENTIDO CONTRARIO *

$\Gamma'(t) = (1-\cos t, \operatorname{sen} t)$

$$\int_{C_1} x\,dy \overset{*}{=} -\int_0^{2\pi} \langle (0, t-\operatorname{sen} t), (1-\cos t, \operatorname{sen} t)\rangle dt = -\int_0^{2\pi} (t - \operatorname{sen} t)\operatorname{sen} t\,dt$$

$$= -\int_0^{2\pi} t\operatorname{sen} t - \operatorname{sen}^2 t\,dt = -\int t\operatorname{sen} t - \frac{(1-\cos 2t)}{2}\,dt = -\int_0^{2\pi} t\operatorname{sen} t + \frac{\cos 2t}{2} - \frac{1}{2}\,dt$$

$$= -\left(-t\cos t + \operatorname{sen} t + \frac{\operatorname{sen} 2t}{4} - \frac{1}{2}t \Big|_0^{2\pi}\right) = 3\pi.$$

$\underline{C_2}$: $\sigma(t) = (t, 0)$, $0 \le t \le 2\pi$

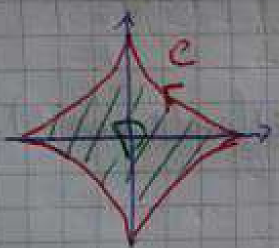
$\sigma'(t) = (1,0)$

$$\int_{C_2^+} x\,dy = \int_0^{2\pi} t \cdot 0\, dt = 0$$

LUEGO, AREA(D) $= \int_{C^+} x\,dy = 3\pi + 0 = 3\pi$

2) HALLAR EL AREA DEL HIPOCICLOIDE, LA REGIÓN CONTENIDA DENTRO DE LA CURVA DE PARAMETRIZACIÓN $\sigma(t) = (\cos^3(t), \operatorname{sen}^3(t))$

$0 \le t \le 2\pi$ → BIEN ORIENTADA



$\sigma(t) = (\cos^3(t), \operatorname{sen}^3(t))$

$\sigma'(t) = (-3\cos^2(t)\operatorname{sen} t, 3\operatorname{sen}^2(t)\cos t)$

AREA(D) $= \int_{C^+} x\,dy = \int_{C^+} -y\,dx$

$$\int_{C^+} -y\,dx = \int_0^{2\pi} -\operatorname{sen}^3(t)(-3)\cos^2 t \operatorname{sen} t\, dt = \int_0^{2\pi} 3\operatorname{sen}^4(t)\cos^2(t)\,dt =$$

$$= \int_0^{2\pi} 3\left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right) dt \qquad \left(\begin{array}{l} \operatorname{sen}^2 t = \frac{1-\cos(2t)}{2} \\ \cos^2(t) = \frac{1+\cos(2t)}{2} \end{array}\right)$$

$$= \int_0^{2\pi} \frac{3}{8}(1-2\cos(2t)+\cos^2(2t))(1+\cos(2t))\,dt$$

$$= \frac{3}{8}\int_0^{2\pi} 1 - 2\cos(2t) + \cos^2(2t) + \cos(2t) - 2\cos^2(2t) + \cos^3(2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi} 1 - \cos(2t) - \cos^2(2t) + \cos^3(2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi} 1 - \cos(2t) - \left(\frac{1+\cos(2t)}{2}\right) + (1-\operatorname{sen}^2(2t))\cos(2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi} \frac{1}{2} - \frac{\cos(2t)}{2} - \operatorname{sen}^2(2t)\cos(2t)\, dt = \frac{3}{8}\left(\frac{1}{2}t - \frac{\operatorname{sen}(2t)}{4} - \frac{\operatorname{sen}^3(2t)}{6}\Big|_0^{2\pi}\right)$$

$$= \frac{3}{8}\pi$$

3) SUPONGAMOS QUE LA CURVA $\mathcal{C}$ ESTÁ PARAMETRIZADA EN POLARES, ES DECIR,

$$\Gamma : [\sigma_1, \sigma_2] \to \mathbb{R}^2 \quad , \quad \Gamma(t) = \left(r(t)\cos t, r(t)\sin t \right) \quad , \quad r > 0$$

RECORRIDA EN SENTIDO POSITIVO

PROBAR QUE $\quad A(D) = \frac{1}{2}\int_{\sigma_1}^{\sigma_2} r^2(t)\, dt$

DEM: $\quad A(D) = \int_{\mathcal{C}^+} x\, dy = \int_{\mathcal{C}^+} -y\, dx \qquad \Gamma(t) = r(t)(\cos t, \sin t)$

$\Gamma'(t) = r'(t)(\cos t, \sin t) + r(t)(-\sin t, \cos t)$

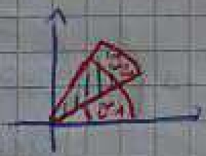
LUEGO, $A(D) = \frac{1}{2}\left(\int_{\mathcal{C}^+} -y\, dx + x\, dy \right)$

$$= \frac{1}{2}\int_{\sigma_1}^{\sigma_2} r(t)(-\sin t, \cos t) \cdot \left(r'(t)(\cos t, \sin t) + r(t)(-\sin t, \cos t) \right) dt$$

$$= \frac{1}{2}\int_{\sigma_1}^{\sigma_2} r(t)r'(t)(-\sin t\cos t + \sin t\cos t) + r^2(t)\overbrace{(\sin^2 t + \cos^2 t)}^{1}\, dt$$

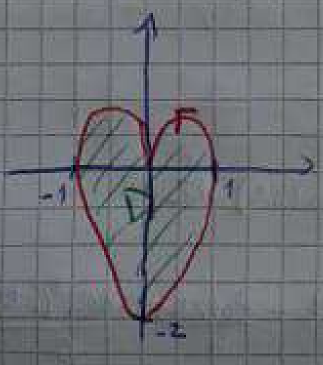
$$= \frac{1}{2}\int_{\sigma_1}^{\sigma_2} r^2(t)\, dt$$

4) $\underline{r = cte} \longrightarrow$ TROZO DE CIRCULO $\qquad \sigma_1 \leq t \leq \sigma_2$



$$A(D) = \frac{1}{2}\int_{\sigma_1}^{\sigma_2} r^2\, dt = r^2 \frac{(\sigma_2 - \sigma_1)}{2}$$

5) $r(t) = 1 - \sin t \quad , \quad 0 \leq t \leq 2\pi \qquad$ "CARDIOIDE"



$$A(D) = \frac{1}{2}\int_0^{2\pi} (1 - \sin t)^2\, dt = \frac{1}{2}\int_0^{2\pi} 1 - 2\sin t + \sin^2 t\, dt$$

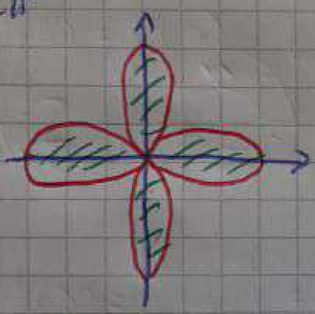
$$= \frac{1}{2}\int_0^{2\pi} 1 - 2\sin t + \frac{1 - \cos(2t)}{2}\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} - 2\sin t - \frac{\cos(2t)}{2}\, dt$$

$$= \frac{1}{2}\left(\frac{3}{2}t + 2\cos t - \frac{\sin(2t)}{4} \Big|_0^{2\pi} \right) = \frac{3}{2}\pi$$

6) $r(t) = \cos(2t)$ "ROSA POLAR"

$0 \le t \le 2\pi$



$$A(D) = \frac{1}{2}\int_0^{2\pi} \cos^2(2t)\,dt = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos(4t)}{2}\,dt = \frac{1}{2}\left(\frac{1}{2}t + \frac{\sin(4t)}{8}\Big|_0^{2\pi}\right)$$

$$= \frac{\pi}{2}$$