

Guía Práctica 2 - Integrales de Superficies

Ejercicio 1. Dadas las siguientes superficies en coordenadas esféricas, determinar su correspondiente ecuación en coordenadas cartesianas y graficar.

- (a) $r = r_0$, $r_0 > 0$ constante.
- (b) $\varphi = \varphi_0$, $\varphi_0 \in (0, \pi/2]$ constante.

En cada uno de los casos anteriores dé un vedor normal en cada punto.

a) Si r es fijo, se lo verán los ángulos θ y φ



esfera:

$$x^2 + y^2 + z^2 = r^2$$

$$\begin{aligned} \sigma(r_0, \theta, \varphi) = & \left(r_0 \cdot \cos \theta \cdot \sin \varphi, \right. \\ & r_0 \cdot \sin \theta \cdot \sin \varphi, \\ & \left. r_0 \cdot \cos \varphi \right) \end{aligned}$$

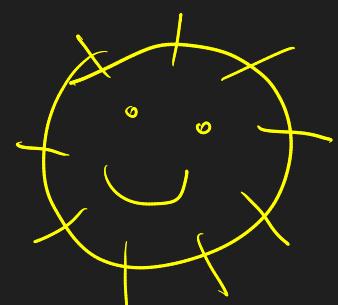
$$x = r_0 \cdot \cos \theta \cdot \sin \varphi$$

$$y = r_0 \cdot \sin \theta \cdot \sin \varphi$$

$$z = r_0 \cdot \cos \varphi$$

o Busco vedor normal a la esfera en cada punto

$$\tau_\theta(r_0, \theta, \varphi) = \begin{bmatrix} -r_0 \cdot \sin \theta \cdot \sin \varphi \\ r_0 \cdot \cos \theta \cdot \sin \varphi \\ 0 \end{bmatrix}$$



$$T_\varphi(r_0, \theta, \varphi) = \begin{bmatrix} r_0 \cdot \cos \theta \cdot \cos \varphi \\ r_0 \cdot \sin \theta \cdot \cos \varphi \\ -r_0 \cdot \sin \varphi \end{bmatrix}$$

$$T_\theta \times T_\varphi = \det \begin{vmatrix} i & j & k \\ r_0 \cdot \sin \theta \cdot \sin \varphi & r_0 \cdot \cos \theta \cdot \sin \varphi & 0 \\ -r_0 \cdot \cos \theta \cdot \cos \varphi & r_0 \cdot \sin \theta \cdot \cos \varphi & -r_0 \cdot \sin \varphi \end{vmatrix}$$

$$\begin{aligned} &= i \left(-r_0^2 \cdot \cos \theta \cdot \sin^2 \varphi \right) + j \cdot r_0^2 \cdot \sin \theta \cdot \sin^2 \varphi + \\ &+ k \cdot \underbrace{\left(r_0^2 \cdot \sin^2 \theta \cdot \sin \varphi \cdot \cos \varphi + r_0^2 \cdot \cos^2 \theta \cdot \sin \varphi \cdot \cos \varphi \right)}_{= r_0^2 \cdot \sin \varphi \cos \varphi \cdot (\cos^2 \varphi + \sin^2 \varphi)} \\ &= r_0^2 \cdot \sin \varphi \cos \varphi \end{aligned}$$

$$T_\theta \times T_\varphi = \left(\underbrace{-r_0^2 \cdot \cos \theta \cdot \sin^2 \varphi}_{\text{red circles}}, \underbrace{r_0^2 \cdot \sin \theta \cdot \sin^2 \varphi}_{\text{green arrow}}, \underbrace{r_0^2 \cdot \sin \varphi \cdot \cos \varphi}_{\text{green arrow}} \right)$$

$$\| T_\theta \times T_\varphi \| = \left(r_0^4 \cdot \cos^2 \theta \cdot \sin^4 \varphi + r_0^4 \cdot \sin^2 \theta \cdot \sin^4 \varphi + r_0^4 \cdot \sin^2 \varphi \cdot \cos^2 \varphi \right)^{1/2}$$

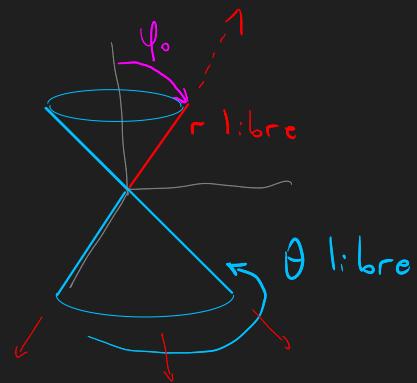
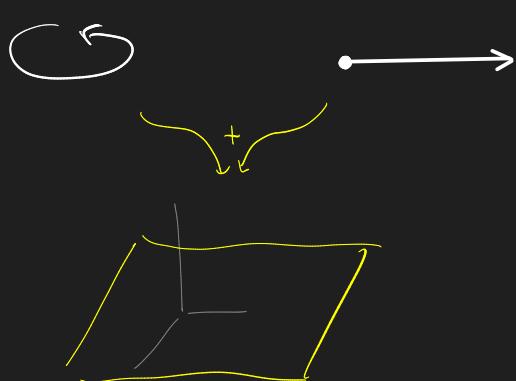
$$\begin{aligned} &= \left(r_0^4 \cdot \left(\sin^4 \varphi + \sin^4 \varphi + \sin^2 \varphi \cdot \cos^2 \varphi \right) \right)^{1/2} \\ &= r_0^4 \cdot \left(2 \sin^4 \varphi + \sin^2 \varphi \cdot \cos^2 \varphi \right)^{1/2} \end{aligned}$$

$$n = \frac{T_\theta \times T_\varphi}{\| T_\theta \times T_\varphi \|}$$

Vector normal en oord2 (r_0, θ, φ)

$$b) \quad \psi = \psi_0 \quad \psi_0 \in (0, \frac{\pi}{2}] \quad \text{constante}$$

$$\theta \text{ libre} \quad r \text{ libre} \quad \psi = \psi_0$$



$$x = r \cdot \cos \theta \cdot \sin \psi_0$$

$$y = r \cdot \sin \theta \cdot \sin \psi_0$$

$$z = r \cdot \cos \psi_0$$

Derivo wrt las 2 variables que quedan libres:

$$T_r(r, \theta, \psi_0) = \begin{bmatrix} \cos \theta \cdot \sin \psi_0 \\ \sin \theta \cdot \sin \psi_0 \\ \cos \psi_0 \end{bmatrix}$$

$$T_\theta(r, \theta, \psi_0) = \begin{bmatrix} -r \cdot \sin \theta \cdot \sin \psi_0 \\ r \cdot \cos \theta \cdot \sin \psi_0 \\ 0 \end{bmatrix}$$

$$T_r \times T_\theta = \det \begin{bmatrix} i & j & k \\ \cos \theta \cdot \sin \psi_0 & \sin \theta \cdot \sin \psi_0 & \cos \psi_0 \\ -r \cdot \sin \theta \cdot \sin \psi_0 & r \cdot \cos \theta \cdot \sin \psi_0 & 0 \end{bmatrix}$$

$$= i \cdot \left(-r \cdot \cos \theta \cdot \sin \varphi_0 \cdot \cos \varphi_0 \right) + \\ + j \cdot \left(- \left(r \cdot \sin \theta \cdot \sin \varphi_0 \cdot \cos \varphi_0 \right) \right) \\ + k \cdot \left(r \cdot \cos^2 \theta \cdot \sin^2 \varphi_0 + r \cdot \sin^2 \theta \cdot \sin^2 \varphi_0 \right)$$

$$T_{r \times T_\theta} = \left(\underbrace{-r \cdot \cos \theta \cdot \sin \varphi_0 \cdot \cos \varphi_0}_b, \underbrace{-r \cdot \sin \theta \cdot \sin \varphi_0 \cdot \cos \varphi_0}_c, \underbrace{r \cdot \sin^2 \varphi_0}_a \right)$$

$$a^2 = r^2 \cdot \cos^2 \theta \cdot \sin^2 \varphi_0 \cdot \cos^2 \varphi_0$$

$$b^2 = r^2 \cdot \sin^2 \theta \cdot \sin^2 \varphi_0 \cdot \cos^2 \varphi_0$$

$$c^2 = r^2 \cdot \sin^4 \varphi_0$$

$$\bullet a^2 + b^2 =$$

$$= r^2 \left(\sin^2 \varphi_0 \cdot \cos^2 \varphi_0 \cdot \cos^2 \theta + \sin^2 \varphi_0 \cdot \cos^2 \varphi_0 \cdot \sin^2 \theta \right)$$

$$= r^2 \cdot \sin^2 \varphi_0 \cdot \cos^2 \varphi_0 \cdot \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1}$$

$$= r^2 \cdot \sin^2 \varphi_0 \cdot \cos^2 \varphi_0$$

$$a^2 + b^2 + c^2 = r^2 \cdot \left(\sin^2 \varphi_0 \cdot \cos^2 \varphi_0 + \sin^4 \varphi_0 \right)$$

$$= r^2 \cdot \left(\sin^2 \varphi_0 \left(\underbrace{\cos^2(\varphi_0) + \sin^2(\varphi_0)}_{=1} \right) \right)$$

$$\| T_r \times T_\theta \| = \left(r^2 \sin^2 \varphi_0 \right)^{1/2}$$

$$= r \cdot \sin \varphi_0$$

//

$$\gamma = \frac{T_r \times T_\theta}{r \cdot \sin \varphi_0} = \frac{(a, b, c)}{r \cdot \sin \varphi_0}$$

a mimir ... $\psi^z z^2$

Sep 16

Ej 2.

Ejercicio 2. Sean $a, b > 0$.

(a) Mostrar que $\Phi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ y $\Phi_2 : \mathbb{R}_{\geq 0} \times [0, 2\pi) \rightarrow \mathbb{R}^3$ dadas por

$$\Phi_1(u, v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2} \right),$$

$$\Phi_2(u, v) = (au \cos(v), bu \sin(v), u^2),$$

son dos parametrizaciones del *paraboloide elíptico* dado cartesianamente por

$$z = \left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2.$$

1. Puedo probar las 2 doble inducciones

2. Puedo probar una, y mostrar que

$\exists h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, biyectiva /

$$\phi_2(u, v) = h(\phi_1(u, v))$$

Muestro que $\text{Im}(\phi_1) = S$

Donde

$$S = \left\{ (x, y) \in \mathbb{R}^2 \mid z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \right\}$$

$\text{Im}(\phi_1) \subseteq S \right)$

Para $(x, y) \in \mathbb{R}^2$, $\text{Im}(\phi_1) = f(x, y)$

donde $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

que se corresponde con el parabolóide elíptico

dado por

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

\Rightarrow Lo mismo.

$\therefore \text{Im}(\phi_1) = S$

Ahora digo que

$$\phi_2(u, v) = h(\phi_1(u, v))$$

quiero h : si la encuentro, ϕ_2 es representación de ϕ_1 y \therefore parametrizan la misma superficie

$$\begin{aligned}\Phi_1(u, v) &= \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right), \\ \Phi_2(u, v) &= (au \cos(v), bu \sin(v), u^2)\end{aligned}$$

Se vé que:

$$\phi_1(u, v) = (u, v, \omega) \quad \text{donde} \quad \begin{cases} u = u \\ v = v \\ \omega = \frac{u^2}{a^2} + \frac{v^2}{b^2} \end{cases}$$

$$h = (a \cdot u \cdot \cos(v), b u \cdot \sin(v), u^2)$$

y como h es C^1 y biyectiva $\begin{cases} \phi_2 = h(\phi_1) \\ h^{-1}(\phi_2) = \phi_1 \end{cases}$

$\Rightarrow \phi_2$ parametriza la superficie S .

(b) Supongamos $b < a$. Mostrar que

$$\Phi(u, v) = ((a + b \cos(u)) \sin(v), (a + b \cos(u)) \cos(v), b \sin(u)),$$

con $u, v \in [0, 2\pi]$, es una parametrización del *toro* dado cartesianamente por

$$z^2 = b^2 - \left(a - \sqrt{x^2 + y^2}\right)^2.$$

Hecho en /p2-ej2.pdf

Ej 3)

Ejercicio 3. Considerar la superficie dada por la parametrización:

$$x = u \cos(v), \quad y = u \sin(v), \quad z = u.$$

¿Es diferenciable esta parametrización? ¿Es suave la superficie?

$$\phi(u, v) = (u \cdot \cos(v), u \cdot \sin(v), u)$$

- Es diferenciable pues todas sus componentes son C^1
- Es suave? No! es un cono 

Para ser curva:

- Tiene plano tangente en todos sus puntos.
- La recta tangente al plano $L(p)$ en el punto p varía con continuidad.

Calculo:

Derivadas parciales:

$$T_u(u, v) = (\cos v, \sin v, 1)$$

$$T_v(u, v) = (-u \cdot \sin v, u \cdot \cos v, 0)$$

Vector Perpendicular al plano (si existe!)

$$T_u \times T_v = \det \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u \cdot \sin v & u \cdot \cos v & 0 \end{vmatrix}$$

$$= i \cdot (-u \cdot \cos v) - j (u \cdot \sin v) + k \underbrace{(u \cdot \cos^2 v + u \cdot \sin^2 v)}_{= u} = u$$

$$T_u \times T_v = (-u \cdot \cos v, -u \cdot \sin v, \underset{\uparrow}{u})$$

T

$$\|T_u \times T_v\| = \left(u^2 \cdot \cos^2 v + u^2 \cdot \sin^2 v + u^2 \right)^{1/2}$$
$$= \left(u^2 \cdot (1 + 1) \right)^{1/2} = \sqrt{2} \cdot u$$

vector normal

$$\begin{aligned}\vec{n}(\mu, \nu) &= \frac{T_u \times T_\nu}{\|T_u \times T_\nu\|} = \frac{(-\mu \cdot \cos \nu, -\mu \cdot \sin \nu, \mu)}{\sqrt{2} \cdot \mu} \\ &= \frac{1}{\sqrt{2}} \cdot (-\cos \nu, -\sin \nu, 1)\end{aligned}$$

Veo que para dos ν fijos ν_0 y ν ,

$$n(\mu, \nu_0) = \text{constante } \forall \mu$$

$$n(\mu, \nu_1) = \text{constante PERO } \neq n(\mu, \nu_0)$$

∴ n no varía con continuidad en $\rho = (\mu, \nu_0) \Rightarrow$ No es suave.

Ej 4)

Ejercicio 4. Sea C la curva en el plano xy dada en coordenadas polares por:

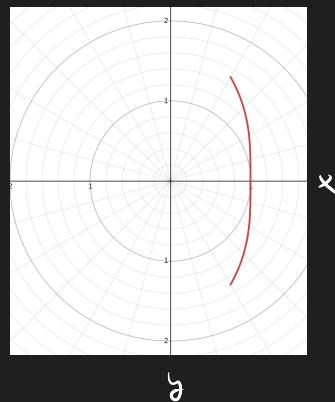
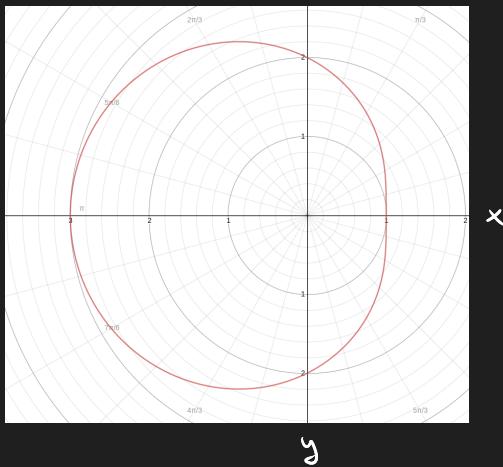
$$r = 2 - \cos \theta, \quad -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}.$$

Sea S la superficie que se obtiene por revolución de esta curva alrededor del eje y .

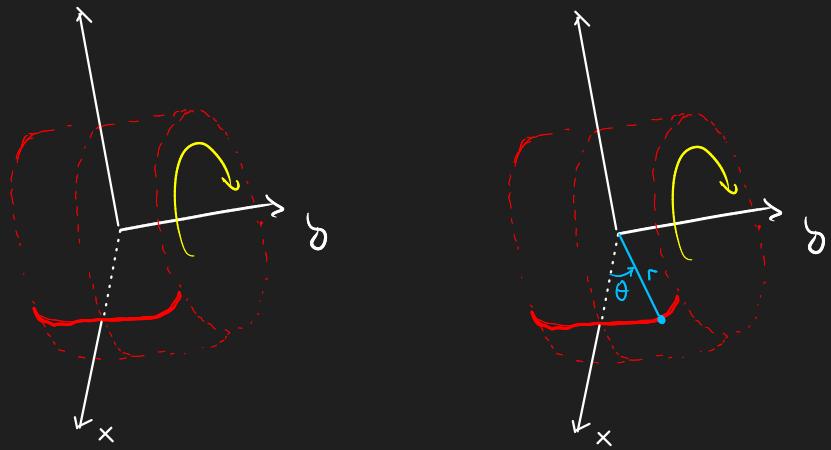
- Dar una parametrización de S .
- ¿Es suave esta superficie?

$$0 \leq \theta \leq 2\pi :$$

$$-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$$

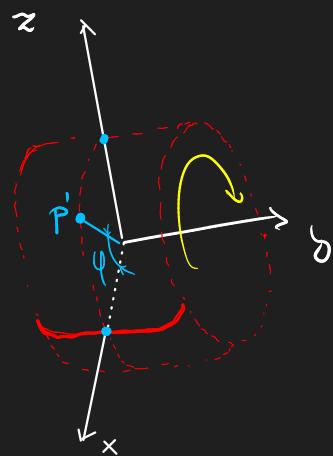


- Quiero: Revolución en eje y ($\in \mathbb{R}^3$!)



$$\text{con } r = 2 - \cos \theta \quad , \quad -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$$

Como mostró Vicky (con ℓ en xz y revol. en z)



Quiero revolucionar alrededor de y

1º. introducir ángulo ϕ

↳ con $\phi = 0$ obtengo la curva original sobre xy :

$$\sigma(\theta) = (r \cdot \cos \theta, r \cdot \sin \theta)$$

con $r = 2 - \cos \theta$

2º. Quiero las coordenadas de cada P' para cada ϕ

$$P' = (x', y', z')$$

• Gire alrededor de y

$\Rightarrow y'$ no cambia con ϕ

y' solo depende de θ (como en la curva original)

$$\Rightarrow y' = r \cdot \sin \theta$$

$$= (2 - \cos \theta) \cdot \sin \theta$$

$$= \beta(\theta)$$

↑ \parallel eje

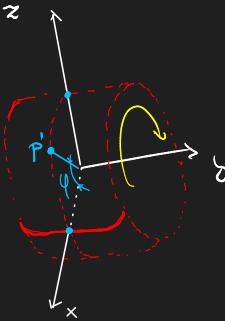
• Para x y z hay que calcular cómo varían

Tip: imaginarse

desde \downarrow

arriba, y

ver qué pasa
en x



- Sabemos que $\alpha(\theta)$ (la distancia al eje de revolución)

eje z $r_z(\theta) = z'$ \Rightarrow nunca cambia (para un θ fijo)

\Rightarrow Calcular los componentes en ejes x y z

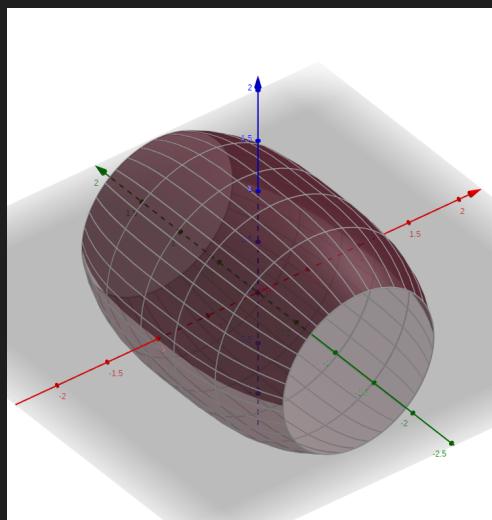
$$\cos \varphi = \frac{x'}{\alpha(\theta)} \Rightarrow x' = \alpha(\theta) \cdot \cos \varphi$$

$$\alpha(\theta) = (z - \cos \theta) \cdot \cos \theta$$

La componente en x de la curva original $\sigma(\theta)$ $\sin \varphi = \frac{z'}{\alpha(\theta)} \Rightarrow z' = \alpha(\theta) \cdot \sin \varphi$

Finalmente

$$\left\{ \begin{array}{l} x' = \alpha(\theta) \cdot \cos \varphi = (z - \cos \theta) \cdot \cos \theta \cdot \cos \varphi \\ y' = \beta(\theta) = (z - \cos \theta) \cdot \sin \theta \\ z' = \alpha(\theta) \cdot \sin \varphi = (z - \cos \theta) \cdot \cos \theta \cdot \sin \varphi \end{array} \right.$$



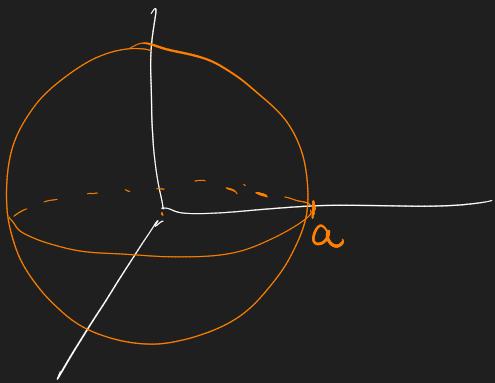
Ej5)

Ejercicio 5. Hallar la ecuación del plano tangente a la esfera de radio a y centro en el origen en un punto (x_0, y_0, z_0) genérico de la esfera.

$$\text{Esfera } x^2 + y^2 + z^2 = a^2$$

Puedo armarme el pleno así

- $\frac{\partial f}{\partial x} = 2x$
- $\frac{\partial f}{\partial y} = 2y \quad \text{TI: } \langle \nabla f(p), (x-p) \rangle = 0$
- $\frac{\partial f}{\partial z} = 2z$



O usando herramientas de parametrización

$$\phi(\theta, \varphi) = \left(a \cdot \cos \theta \cdot \sin \varphi, a \cdot \sin \theta \cdot \sin \varphi, a \cdot \cos \varphi \right)^T$$

$$\phi_\theta(\theta, \varphi) = (-a \cdot \sin \theta \cdot \sin \varphi, a \cdot \cos \theta \cdot \sin \varphi, 0)$$

$$\phi_\varphi(\theta, \varphi) = (a \cdot \cos \theta \cdot \cos \varphi, a \cdot \sin \theta \cdot \cos \varphi, -a \cdot \sin \varphi)$$

Vector tangente al pleno

$$\phi_\theta \times \phi_\varphi = \det \begin{vmatrix} i & j & k \\ -a \cdot \sin \theta \cdot \sin \varphi & a \cdot \cos \theta \cdot \sin \varphi & 0 \\ a \cdot \cos \theta \cdot \cos \varphi & a \cdot \sin \theta \cdot \cos \varphi & -a \cdot \sin \varphi \end{vmatrix}$$

$$= i \cdot (-a^2 \cdot \cos \theta \cdot \sin \varphi \cdot \cos \varphi) - j \cdot a^2 \cdot \sin \theta \cdot \sin \varphi \cdot \cos \varphi$$

$$+ k \cdot (-a^2 \cdot \sin^2 \theta \cdot \sin \varphi \cdot \cos \varphi - a^2 \cdot \cos^2 \theta \cdot \sin \varphi \cdot \cos \varphi)$$

$$\phi_\theta \times \phi_\varphi = \left(-a^2 \cdot \cos \theta \cdot \sin \varphi \cdot \cos \varphi, -a^2 \cdot \sin \theta \cdot \sin \varphi \cdot \cos \varphi, -a^2 \cdot \sin \varphi \cdot \cos \varphi \right)^T$$

$\neq (0,0,0)$ si $\sin \varphi \cdot \cos \varphi \neq 0$
 $\varphi \neq \pi, \frac{\pi}{2}, \frac{3\pi}{2}$

$$\begin{aligned}
\|\phi_\theta \times \phi_\psi\| &= \left(a^4 \cdot \cos^2 \theta \cdot \sin^2 \phi \cdot \cos^2 \psi + \right. \\
&\quad + a^4 \cdot \sin^2 \theta \cdot \sin^2 \phi \cdot \cos^2 \psi + \\
&\quad \left. + a^4 \cdot \sin^2 \theta \cdot \cos^2 \phi \right)^{1/2} \\
&= \left(2 \cdot a^4 \cdot \sin^2 \phi \cdot \cos^2 \psi \right)^{1/2} \\
&= \sqrt{2} \cdot a^2 \cdot \sin \phi \cos \psi
\end{aligned}$$

$$\frac{\phi_\theta \times \phi_\psi}{\|\phi_\theta \times \phi_\psi\|} = -\frac{1}{\sqrt{2}} \cdot \left(\cos \theta, \sin \theta, 1 \right) = h(\theta, \psi)$$

$$\text{Tr} : \left\langle h(\theta, \psi), (x - x_0, y - y_0, z - z_0) \right\rangle = 0$$

Ejercicio 6. Encontrar una ecuación para el plano tangente en el punto $(0,1,1)$ a la superficie dada por la parametrización

Sept 19

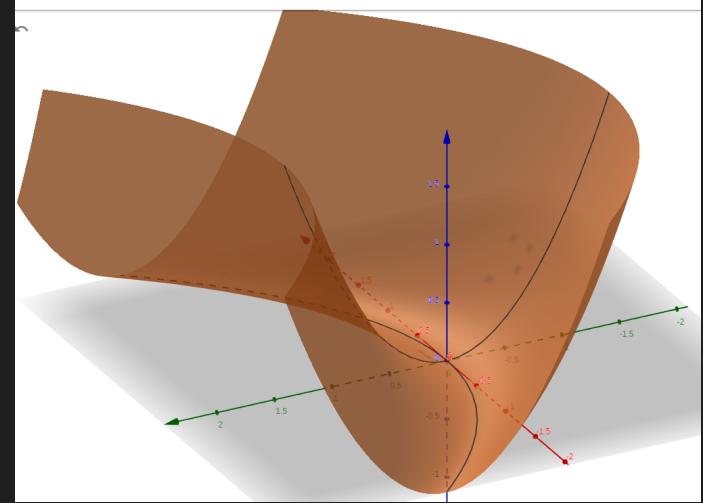
$$x = 2u, \quad y = u^2 + v, \quad z = v^2.$$

$$\phi(u, v) = (2u, u^2 + v, v^2)$$

$$\phi_u(u, v) = (2, 2u, 0)$$

$$\phi_v(u, v) = (0, 1, 2v)$$

$$(\phi_u \times \phi_v)(u, v) = \det \begin{vmatrix} i & j & k \\ 2 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix}$$



$$= i \cdot 4uv - j \cdot 4v + k \cdot 2$$

$$= (4uv, -4v, 2) \neq (0, 0, 0) \quad \forall (u, v) \in \mathbb{R}^2$$

Norma:

$$\|\phi_u \times \phi_v\| = \left(16u^2v^2 + 16v^2 + 4 \right)^{1/2}$$

Vector normal

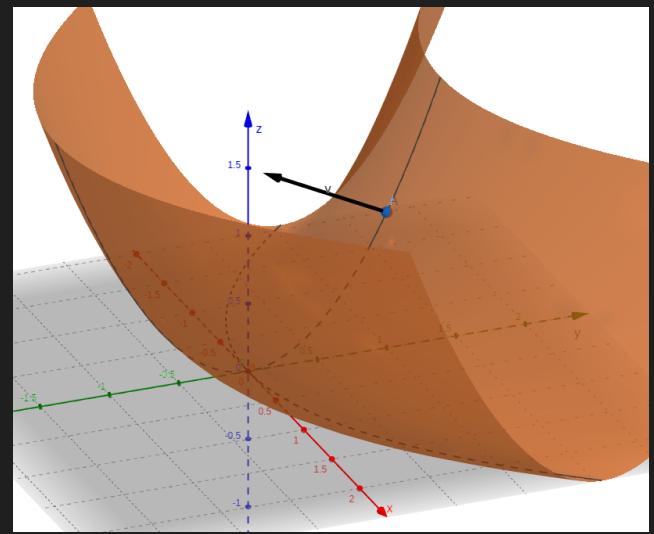
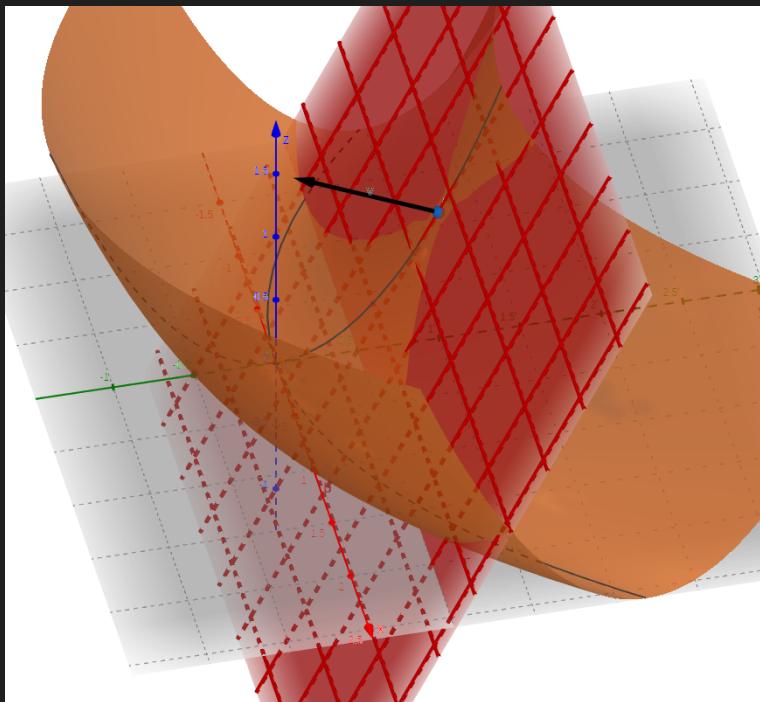
$$\bullet \quad n(u, v) = \frac{(4uv, -4v, 2)}{\|\phi_u \times \phi_v\|}$$

Vector normal en $(0, 1, 1)$

$$\left. \begin{array}{l} 0 = 2u \Rightarrow u = 0 \\ 1 = u^2 + v \xrightarrow{u=0} v = 1 \\ 1 = v^2 \Rightarrow v = 1 \end{array} \right\} \text{vector normal en } \left\{ \begin{array}{l} u = 0 \\ v = 1 \end{array} \right.$$

$$n(0, 1) = \frac{(0, -4, 2)}{\sqrt{20}} = \frac{1}{2\sqrt{5}} (0, -2, 1)$$

$$h(\phi_1) = \frac{1}{\sqrt{S}} \cdot (\phi_1 - z_1)$$



$$\nabla \cdot \langle h(\phi_1), (p - p_0) \rangle = 0$$

$$p = (x, \phi_1, z)$$

$$p_0 = (0, 1, 1)$$

$$\nabla \cdot -\frac{2}{\sqrt{S}}(\phi_1 - 1) + \frac{1}{\sqrt{S}}(z - 1) = 0$$

Ejercicio 7. Sea S la superficie parametrizada por la función $\Phi(r, \theta) : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ dada por

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = \theta,$$

Graficar S , hallar un vector normal en cada punto y hallar su área.

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$$

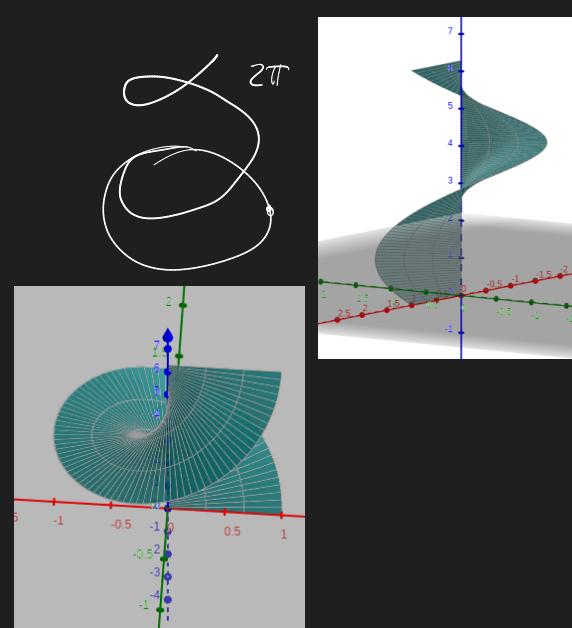
$$\phi_r = (\cos \theta, \sin \theta, 0)$$

$$\phi_\theta = (-r \sin \theta, r \cos \theta, 1)$$

$$\phi_r \times \phi_\theta = \det \begin{vmatrix} i & j & k \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix} =$$

$$= i \cdot \sin \theta - j \cdot \cos \theta + k \cdot (r \cos^2 \theta + r \sin^2 \theta)$$

$$= (\sin \theta, -\cos \theta, r)$$



$$\|\phi_r \times \phi_\theta\| = \left(\sin^2 \theta + \cos^2 \theta + r^2 \right)^{1/2} \\ = \sqrt{r^2 + 1}$$

$$n(r, \theta) = \frac{1}{\sqrt{r^2 + 1}} \cdot \left(\sin \theta, -\cos \theta, r \right) \neq$$

Hallar su área: $\Phi(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$

Otra forma:

Calcular determinantes

$$\textcircled{1} \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

$$\textcircled{2} \quad \frac{\partial(y, z)}{\partial(r, \theta)} = \det \begin{vmatrix} \sin \theta & 0 \\ -r \sin \theta & 1 \end{vmatrix} = \sin \theta$$

$$\textcircled{3} \quad \frac{\partial(x, z)}{\partial(r, \theta)} = \det \begin{vmatrix} \cos \theta & 0 \\ -r \sin \theta & 1 \end{vmatrix} = \cos \theta$$

$$\|\phi_r \times \phi_\theta\| = \left(\underbrace{\sin^2 \theta + \cos^2 \theta}_{1} + r^2 \right)^{1/2} = \sqrt{1 + r^2}$$

$$S = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} dr \quad \sinh^2 x - \cosh^2 x = 1 \\ r = \cosh x$$

$$\text{d}r = \sinh x \text{d}x \\ = \int_0^{2\pi} \int_{\ln(0+\sqrt{r^2+1})}^{\ln(1+\sqrt{r^2+1})} \sqrt{\sinh^2 x} \sinh x \text{d}x = \int_0^{2\pi} \int_{0}^{\sinh^2 x} \sinh^2 x \text{d}x$$

Para practicar

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\iint_0^{2\pi} \sinh^2 x \, dx = \iint \left(\frac{e^x - e^{-x}}{2} \right)^2 \, dx =$$

$$= \iint \frac{1}{4} \cdot (e^{2x} - 2e^0 + e^{-2x}) \, dx = \frac{1}{4} \iint e^{2x} - 2 + e^{-2x} \, dx$$

$$= \frac{1}{4} \cdot \left(\iint e^{2x} \, dx - 2 \iint 1 \, dx + \iint e^{-2x} \, dx \right)$$

$$(1) \quad \iint e^{2x} \, dx = \int_0^{2\pi} \frac{e^{2x}}{2} \Big|_{\ln(1)=0}^{\ln(1+\sqrt{2})} = \int_0^{2\pi} \frac{1}{2} \left(e^{2 \cdot \ln(1+\sqrt{2})} - 1 \right)$$

$$= \int_0^{2\pi} \frac{1}{2} \left(\ln(1+\sqrt{2}) \right)^2 - 1 \, d\theta$$

$$= 2\pi \cdot \frac{1}{2} \left(\dots \right)$$

$$= \pi \left(\ln^2(1+\sqrt{2}) - 1 \right)$$

(2)

$$-2 \iint 1 \, dx = -4 \pi \cdot \ln(1+\sqrt{2})$$

$$(3) \quad \iint e^{-2x} \, dx = \pi \left(\ln^2(1+\sqrt{2}) - 1 \right)$$

$$\begin{aligned}
\text{Superficie} &= \frac{1}{4} \left(\pi \left(\ln^2(1+\sqrt{2}) - 1 \right) - 4 \pi \cdot \ln(1+\sqrt{2}) \right. \\
&\quad \left. + \pi \left(\ln^2(1+\sqrt{2}) - 1 \right) \right) \\
&= -\frac{3}{4}\pi \ln^2(1+\sqrt{2}) - \frac{1}{4}\pi + \frac{1}{4}\pi \cdot \ln^2(1+\sqrt{2}) - \frac{1}{4}\pi \\
&= -\frac{3}{4}\pi \ln^2(1+\sqrt{2}) + \frac{1}{4}\pi \cdot \ln^2(1+\sqrt{2}) - \frac{1}{2}\pi
\end{aligned}$$

Método (es menor a cero)

Ej 8)

Ejercicio 8. Sea D el disco unitario centrado en el origen. Sea S la superficie parametrizada por la función $\Phi(u, v) : D \rightarrow \mathbb{R}^3$ dada por

$$\Phi(u, v) = (u - v, u + v, uv).$$

Calcular el área de S .

$$\Phi_u(u, v) = (1, 1, v)$$

$$\Phi_v(u, v) = (-1, 1, u)$$

$$\Phi_u \times \Phi_v = \det \begin{vmatrix} i & j & k \\ 1 & 1 & v \\ -1 & 1 & u \end{vmatrix}$$

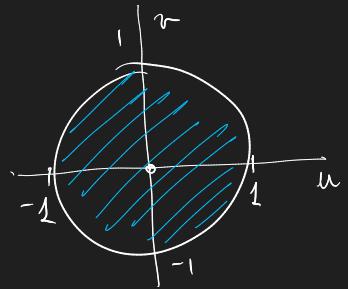
$$= i(u-v) - j(u+v) + k z$$

$$= (u-v, -u-v, z)$$

$$\begin{aligned}
\|\Phi_u \times \Phi_v\| &= \left((u-v)^2 + (-1)^2 \cdot (u+v)^2 + 4 \right)^{1/2} \\
&= \left(u^2 + 2uv + v^2 + u^2 + 2uv + v^2 + 4 \right)^{1/2} \\
&= \left(2u^2 + 2v^2 + 4 \right)^{1/2}
\end{aligned}$$

$$= \sqrt{2} \cdot (\mu^2 + v^2 + z)^{1/2}$$

$$\iint_D \sqrt{2} \cdot (\mu^2 + v^2 + z)^{1/2} dA \quad \text{on } D =$$



Perso = coördinaten der polaars

$$\mu = r \cdot \cos \theta$$

$$v = r \cdot \sin \theta$$

$$0 \leq r \leq 1$$

$$\iint_D \sqrt{2} \cdot (\mu^2 + v^2 + z)^{1/2} dA =$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \sqrt{2} \cdot (\mu^2 + v^2 + z)^{1/2} \cdot r \ dr \ d\theta$$

$$= \iint_D \sqrt{2} \cdot (r^2 + 4)^{1/2} \cdot r \ dr \ d\theta =$$

$$\begin{aligned} s &= r^2 + 4 \\ ds &= 2r \cdot dr \end{aligned} \quad = \sqrt{2} \int_{s=4}^{s=s} \int_{\frac{s}{2}}^{s=s} \sqrt{s} \frac{ds}{2} \ d\theta$$

$$= \sqrt{2} \int_{\theta=0}^{\theta=2\pi} \frac{z}{3} s^{3/2} \Big|_4^s \ d\theta$$

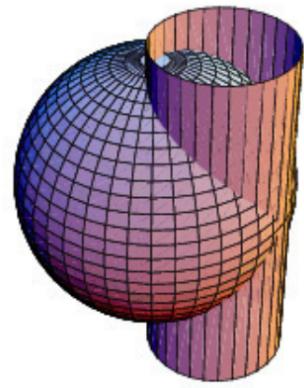
$$= \sqrt{2} \cdot 2\pi \cdot \frac{2}{3} \cdot (s^{3/2} - 4^{3/2})$$

$$= \frac{4}{3} \sqrt{2} \cdot \pi \cdot (s^{3/2} - 8)$$

//Dudo so ...

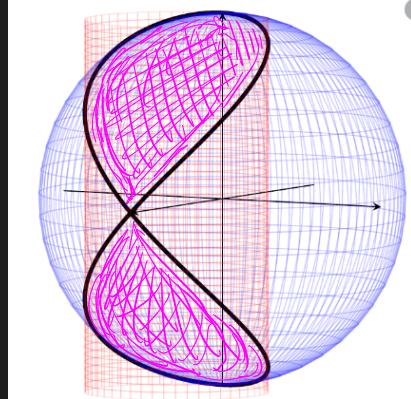
Ej 9)

Ejercicio 9. Sea $R > 0$. Calcular el área de la superficie que se obtiene de interseccar la esfera $x^2 + y^2 + z^2 = R^2$ con el cilindro (relleno) $(x - R/2)^2 + y^2 \leq (R/2)^2$.



Esta superficie se conoce como *bóveda de Viviani*.

En la intersección de la cónica de la esfera y todo el cilindro se obtiene la superficie



$$\text{Esfera } x^2 + y^2 + z^2 = R^2$$

$$\text{cilindro } \left(x - \frac{R}{2}\right)^2 + y^2 \leq \left(\frac{R}{2}\right)^2$$

Solo debo integrar la esfera, pero restringida al int. del cilindro

$$\iint_D \|T_u \times T_v\| ds$$

Despejo el cilindro

$$y^2 \leq \frac{R^2}{4} - \left(x - \frac{R}{2}\right)^2 = \frac{R^2}{4} - \left(x^2 - Rx + \frac{R^2}{4}\right)$$

$$y^2 \leq Rx - x^2$$

$$-\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$$

Polarés / Cilíndricos

$$x = r \cdot \cos \theta$$

$$y = r \cdot \sin \theta$$

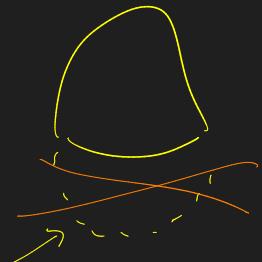
z libre

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \rightarrow \iint_D \|T_x \times T_y\| ds =$$

- param $\hat{\phi}(t) = (r \cdot \cos \theta \cdot \sin \varphi, r \cdot \sin \theta \cdot \sin \varphi, r \cos \varphi)$

- otra param por partes:

$$\left\{ \begin{array}{l} \phi_1(x, y) = (x, y, \sqrt{R^2 - x^2 - y^2}) \\ \phi_2(x, y) = (x, y, -\sqrt{R^2 - x^2 - y^2}) \end{array} \right.$$



$$\|T_x \times T_y\| :$$

$$T_x(x, y) = \left(1, 0, \frac{1}{2\sqrt{R^2 - x^2 - y^2}} \cdot (-2x) \right)$$

$$T_y(x, y) = \left(0, 1, \frac{1}{2\sqrt{R^2 - x^2 - y^2}} \cdot (-2y) \right)$$

Calculo determinantes

$$\frac{\partial(x, y)}{\partial(x, y)}^2 = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\frac{\partial(x, z)}{\partial(x, y)}^2 = \det \begin{vmatrix} 1 & \frac{1}{2\sqrt{R^2 - x^2 - y^2}} \\ 0 & \frac{1}{2\sqrt{R^2 - x^2 - y^2}} \cdot (-2y) \end{vmatrix} = \frac{1}{2\sqrt{R^2 - x^2 - y^2}} \cdot (-2y)$$

$$\frac{\partial(y_1 z)}{\partial(x_1 y)} = \det \begin{vmatrix} 0 & \frac{1}{z\sqrt{R^2-x^2-y^2}} \cdot (-2x) \\ 1 & \frac{1}{z\sqrt{R^2-x^2-y^2}} \cdot (-2y) \end{vmatrix} = -\frac{1}{z\sqrt{R^2-x^2-y^2}} \cdot (-2x)$$

$$\|T_u \times T_v\| = \left(1^2 + 4y^2 \cdot \frac{1}{4(R^2-x^2-y^2)} + 4x^2 \cdot \frac{1}{4(R^2-x^2-y^2)} \right)^{1/2}$$

$$= \left(1 + \frac{y^2+x^2}{R^2-x^2-y^2} \right)^{1/2}$$

$|x| \leq R$
 $|y| \leq R$

$$y = \sqrt{R^2 - x^2}$$

$$\int \int_{x=0}^{x=R} 1 + \frac{x^2+y^2}{R^2-x^2-y^2} dx dy$$

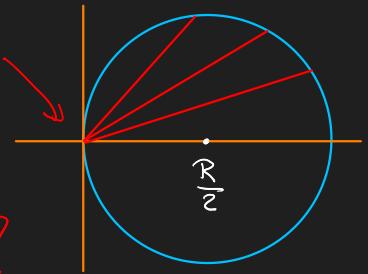
Polares Pero en $(0,0)$!

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Como depende r de θ ?

$$r(\theta) = ?$$



$$(x_1 y) = (r \cos \theta, r \sin \theta)$$

$$(r \cos \theta - \frac{R}{2})^2 + (r \sin \theta)^2 \leq \left(\frac{R}{2}\right)^2$$

$$r^2 \cos^2 \theta - r R \cos \theta + \left(\frac{R}{2}\right)^2 + r^2 \sin^2 \theta \leq \left(\frac{R}{2}\right)^2$$

$$\Leftrightarrow r^2 \leq r \cdot R \cdot \cos \theta$$

Nuevos param

$$\begin{aligned} x^2 &= (r \cos \theta)^2 \\ y^2 &= (r \sin \theta)^2 \end{aligned}$$

$$T(r, \theta) = \left(r \cos \theta, r \sin \theta, \sqrt{R^2 - r^2} \right)$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq R \cos \theta$$

$$\mathbf{T}_r = \left(\cos \theta, \sin \theta, \frac{-2r}{2\sqrt{R^2 - r^2}} \right)$$

$$\mathbf{T}_\theta = \left(-r \cdot \sin \theta, r \cdot \cos \theta, 0 \right)$$

$$\mathbf{T}_r \times \mathbf{T}_\theta = \left(\frac{r^2 \cdot \cos \theta}{\sqrt{R^2 - r^2}}, \frac{r^2 \cdot \sin \theta}{\sqrt{R^2 - r^2}}, r \right)$$

$$\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \left(\frac{r^4}{\sqrt{R^2 - r^2}}, \right)$$

$$\frac{A}{2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{R \cos \theta} \|\mathbf{T}_r \times \mathbf{T}_\theta\| dr d\theta$$

$$A = 2 \cdot R \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{R \cos \theta} \frac{r}{\sqrt{R^2 - r^2}} dr d\theta$$

$$u = R^2 - r^2$$

$$du = -2r dr \quad R^2 \cdot \sin^2 \theta \leftarrow (R^2 - R^2 \cdot \sin^2 \theta)$$

$$A = 2 \cdot R \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{R^2}^{R^2 \cdot \sin^2 \theta} \frac{1}{-2\sqrt{u}} du d\theta$$

$$= 2R \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-R^2}^{R^2 \cdot \sin^2 \theta} \frac{1}{\sqrt{R^2 \cdot \sin^2 \theta}} du d\theta = -2R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sqrt{R^2 \cdot \sin^2 \theta} - \sqrt{R^2}} d\theta$$

$$= -2R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (|\sin \theta| - 1) d\theta =$$

$$\begin{aligned}
&= -2R^2 \left(- \int_{-\pi/2}^0 (\sin \theta + 1) d\theta + \int_0^{\pi/2} (\sin \theta - 1) d\theta \right) \\
&= -2R^2 \cdot \left(1 - \frac{\pi}{2} + 1 - \frac{\pi}{2} \right) \\
&= -2R^2 (2 - \pi)
\end{aligned}$$

Area $\triangle = 2 R^2 (\pi - 2)$

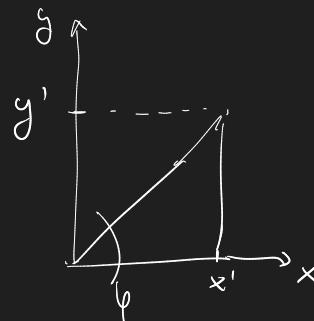
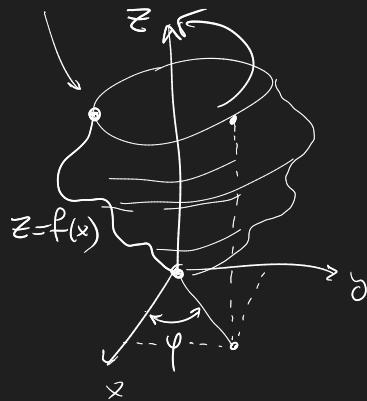
Ej 10

Ejercicio 10. Sea $\alpha > 0$ y sea $f : [\alpha, \beta] \rightarrow \mathbb{R}$ una función positiva. Consideremos la curva $z = f(x)$ girada alrededor del eje z . Mostrar que el área de la superficie barrida es

$$A = 2\pi \int_{\alpha}^{\beta} x \sqrt{1 + (f'(x))^2} dx.$$

Aplicar a la superficie dada en el ejercicio (2) item a) para calcular el área del paraboloide elíptico con $1 \leq z \leq 2$, y $a = b = 1$.

$$P = (x, 0, f(x))$$



$$\sin \phi = \frac{y'}{x'}$$

$$\cos \phi = \frac{x'}{x}$$

$$\begin{cases} x' = x \cdot \cos \phi \\ y' = x \cdot \sin \phi \\ z' = f(x) \end{cases}$$

$$\phi(x, \phi) = (x \cdot \cos \phi, x \cdot \sin \phi, f(x))$$

$$\phi_x(x, \phi) = (\cos \phi, \sin \phi, f'(x))$$

$$\phi_\phi(x, \phi) = (-x \cdot \sin \phi, x \cdot \cos \phi, 0)$$

Cálculo determinantes

$$\frac{\partial(x, y)}{\partial(x, \phi)} = \det \begin{vmatrix} \cos \phi & \sin \phi \\ -x \cdot \sin \phi & x \cdot \cos \phi \end{vmatrix} = x$$

$$\frac{\partial(x, z)}{\partial(x, \phi)} = \det \begin{vmatrix} \cos \phi & f'(x) \\ -x \cdot \sin \phi & 0 \end{vmatrix} = x \cdot f'(x) \cdot \sin \phi$$

$$\frac{\partial(y, z)}{\partial(x, \phi)} = \det \begin{vmatrix} \sin \phi & f'(x) \\ x \cdot \cos \phi & 0 \end{vmatrix} = x \cdot f'(x) \cdot \cos \phi$$

$$\|\phi_x \times \phi_\varphi\| = \left(x^2 \cdot \left(1 + (f'(x))^2 \cdot \sin^2 \varphi + (f'(x))^2 \cdot \cos^2 \varphi \right) \right)^{1/2}$$

$$= |x| \cdot \sqrt{1 + (f'(x))^2}$$

\uparrow //

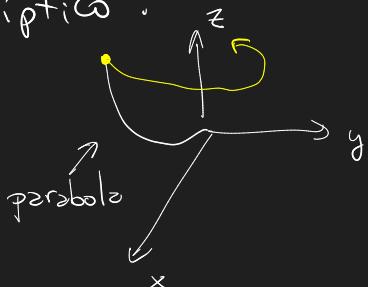
$x > 0$ pues $x \in [\alpha, \beta]$ con $\alpha > 0$

Aplicar a la superficie dada en el ejercicio (2) item a) para calcular el área del parabolóide elíptico con $1 \leq z \leq 2$, y $a = b = 1$.

en el parabolóide elíptico:

$$f(x) = x^2$$

$$\uparrow z$$



$$1 \leq z \leq 2$$

$$1 \leq x \leq \sqrt{2}$$

$$A = 2\pi \int_1^{\sqrt{2}} x \cdot \sqrt{1 + (2x)^2} dx =$$

$$= 2\pi \int_1^{\sqrt{2}} x \cdot \sqrt{1 + 4x^2} dx =$$

$$\begin{aligned} u &= 1 + 4x^2 \\ du &= 8x dx \end{aligned}$$

$$= 2\pi \int_5^9 \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{4}\pi \left(\frac{2}{3} u^{3/2} \right) \Big|_5^9$$

$$= \frac{1}{6}\pi \cdot (9^{3/2} - 5^{3/2}) =$$

//

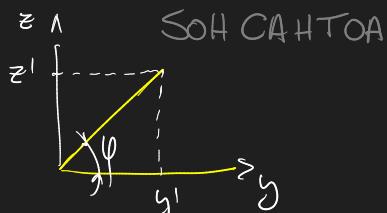
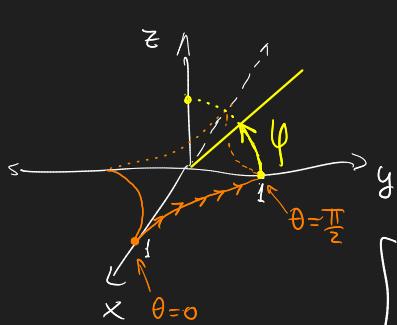
$$= \frac{1}{6}\pi \cdot (27 - 125) =$$

Ejercicio 11. Sea C la curva en el plano xy dada por

$$\begin{cases} x = \cos^3 \theta \\ y = \sin^3 \theta \end{cases}$$

con $0 \leq \theta \leq 2\pi$. Sea S la superficie que se obtiene al girar la curva C alrededor del eje x .

- (a) Hallar una parametrización de S .
- (b) Hallar el área de S .



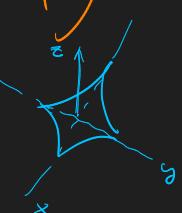
$$\begin{aligned} \sin \varphi &= \frac{z'}{y} = \frac{z'}{\sin^3 \theta} \\ \cos \varphi &= \frac{y'}{y} = \frac{y'}{\sin^3 \theta} \end{aligned}$$

$$\left\{ \begin{array}{l} x' = x = \cos^3 \theta \\ y' = y \cdot \cos \varphi = \sin^3 \theta \cdot \cos \varphi \\ z' = y \cdot \sin \varphi = \sin^3 \theta \cdot \sin \varphi \end{array} \right.$$

$$T(\theta, \varphi) = \left(\cos^3 \theta, \sin^3 \theta \cdot \cos \varphi, \sin^3 \theta \cdot \sin \varphi \right)$$

con $\varphi \in [0, \pi]$ pues

b) Área de S



$$T_\theta(\theta, \varphi) = \left(-3 \cos^2 \theta \cdot \sin \theta, 3 \sin^2 \theta \cdot \cos \theta \cdot \cos \varphi, 3 \sin^2 \theta \cdot \cos \theta \cdot \sin \varphi \right)$$

$$T_\varphi(\theta, \varphi) = \left(0, -\sin^3 \theta \cdot \sin \varphi, \sin^3 \theta \cdot \cos \varphi \right)$$

$$\frac{\partial(x, y)}{\partial(\theta, \varphi)} = \det \begin{vmatrix} -3 \cos^2 \theta \cdot \sin \theta & 3 \sin^2 \theta \cdot \cos \theta \cdot \sin \varphi \\ 0 & -\sin^3 \theta \cdot \cos \varphi \end{vmatrix} =$$

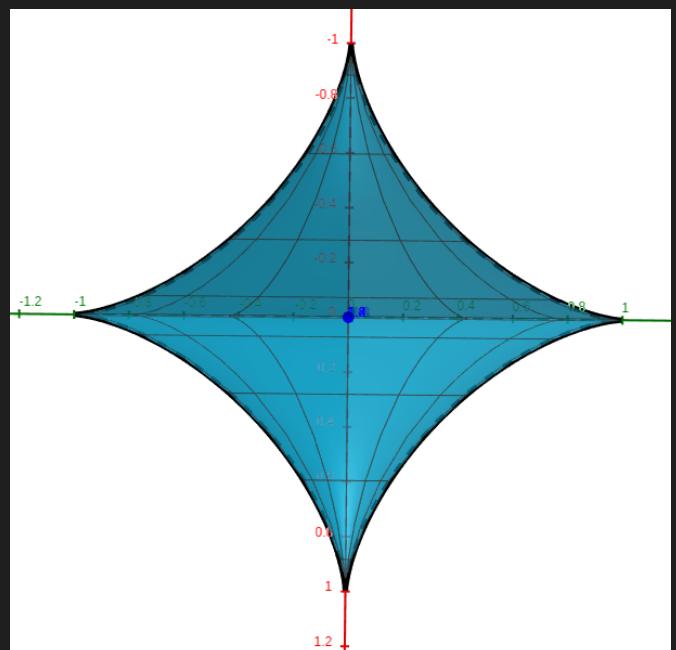
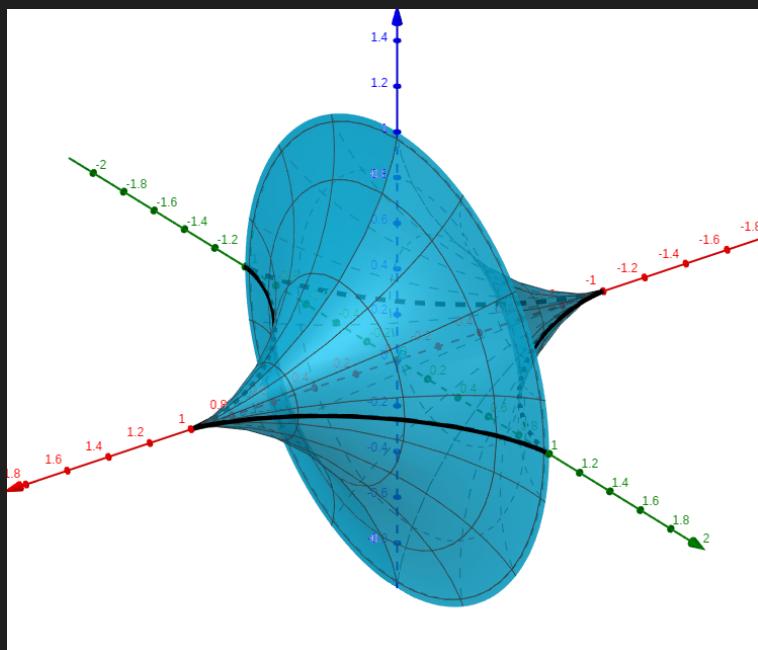
$$= 3 \cos^2 \theta \cdot \sin^4 \theta \cdot \cos \varphi$$

$$\frac{\partial(x, z)}{\partial(\theta, \varphi)} = \det \begin{vmatrix} -3 \cos^2 \theta \cdot \sin \theta & 3 \sin^2 \theta \cdot \cos \theta \cos \varphi \\ 0 & \sin^3 \theta \cdot \sin \varphi \end{vmatrix} =$$

$$= -3 \cos^2 \theta \cdot \sin^4 \theta \cdot \sin \varphi$$

$$\begin{aligned}
 \bullet \quad \frac{\partial(y, z)}{\partial(\theta, \varphi)} &= \text{Det} \left| \begin{array}{cc} 3\sin^2\theta \cdot \cos\theta \cdot \sin\varphi & 3\sin^2\theta \cdot \cos\theta \cdot \cos\varphi \\ -\sin^3\theta \cdot \cos\varphi & \sin^3\theta \cdot \sin\varphi \end{array} \right| = \\
 &= 3\sin^5\theta \cdot \cos\theta \cdot \sin^2\varphi + 3 \cdot \sin^5\theta \cdot \cos\theta \cdot \cos^2\varphi \\
 &= 3\sin^5\theta \cdot \cos\theta
 \end{aligned}$$

$$\begin{aligned}
 \|T_\theta \times T_\varphi\| &= \left[\left(3 \cos^2\theta \cdot \sin^4\theta \cdot \cos\varphi \right)^2 + \right. \\
 &\quad \left(-3 \cos^2\theta \cdot \sin^4\theta \cdot \sin\varphi \right)^2 + \\
 &\quad \left. \left(3\sin^5\theta \cdot \cos\theta \right)^2 \right]^{1/2} \\
 &= \left[9 \cdot \cos^4\theta \cdot \sin^8\theta + 9 \cdot \sin^{10}\theta \cos^2\theta \right]^{1/2} \\
 &= 3 \cdot \left(\cos^4\theta \cdot \sin^8\theta + \cos^2\theta \cdot \sin^{10}\theta \right)^{1/2} =
 \end{aligned}$$

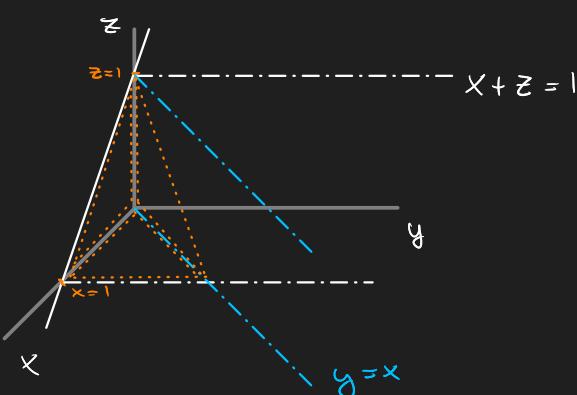


$$\begin{aligned}
 &= 3 \cdot \left(\cos^4 \theta \sin^8 \theta + \cos^2 \theta \sin^{10} \theta \right)^{1/2} \\
 &= 3 \cdot \left(\cos^2 \theta \sin^8 \theta \left(\cos^2 \theta + \sin^2 \theta \right) \right)^{1/2} \\
 &= 3 |\cos \theta| \sin^4 \theta = \|T_\theta \times T_\varphi\|
 \end{aligned}$$

$$\begin{aligned}
 A &= \iint_{\substack{\pi \\ \varphi=0 \\ \theta=0}}^{\pi} 3 |\cos \theta| \sin^4 \theta \, d\theta \, d\varphi & = \text{Revisar fórmula de } \cos^2 \theta \\
 &= \iint_{\substack{\pi \\ \varphi=0 \\ \theta=0}}^{\pi} 3 \cdot u^4 \, d\theta \, d\varphi & = \text{me olvidé el módulo} \\
 u &= \sin \theta & \theta=0: & u=1 \\
 du &= \cos \theta \, d\theta & \theta=\pi: & u=0 \\
 &\quad \left. \begin{array}{l} u=0 \\ \theta=2\pi: \\ u=0 \end{array} \right\} & & \\
 &\quad \theta=\pi: & & \\
 &\quad u=1 & & \\
 && & = 6\pi \cdot \frac{1}{5} u^5 \Big|_0^1 \\
 && & = \frac{6}{5}\pi \\
 && & //
 \end{aligned}$$

Sep 21

12) Ejercicio 12. Calcular $\iint_S xy \, dS$ donde S es el borde del tetraedro con lados $z=0, y=0, x+z=1, x=y$.



Son 4 caras \Rightarrow 4 integrales

1. Para $z = 0$:

como el borde

es $y = x$

$$0 \leq x \leq 1 \Rightarrow 0 \leq y \leq x$$

$$F(T(x,y)) = x, y$$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} x \cdot y \, dS = \int_{x=0}^{x=1} \frac{x \cdot y^2}{2} \Big|_{y=0}^{y=x} = \int_0^1 \frac{x}{2} \cdot x^2 \, dx = \frac{1}{2} \cdot \frac{x^4}{4} \Big|_0^1 = \frac{1}{8}$$

También podría

$$T_1(x,y) = (x, y, 0)$$

$$\mathcal{D}_1 = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$$

$$T_x^{(1)}(x,y) = (1, 0, 0)$$

$$T_y^{(1)}(x,y) = (0, 1, 0)$$

$$T_x^{(1)} \times T_y^{(1)} = \det \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$$

$$\| T_x^{(1)} \times T_y^{(1)} \| = 1$$

$$\int \int x \cdot y \, dS = \int_{x=0}^{x=1} \int_{y=0}^{y=x} x \cdot y \cdot 1 \cdot dy \, dx = \int_{x=0}^{x=1} \frac{x \cdot y^2}{2} \Big|_{y=0}^{y=x} = \frac{1}{8}$$

vector normal al
plano $z=0$ es xy

2. Para $y = 0$:

$$0 \leq x \leq 1 \quad \text{y} \quad 0 \leq z \leq 1 - x$$

$$T_z(x,z) = (x, 0, z)$$

$$\mathcal{D}_2 = \{(x,z) : 0 \leq x \leq 1, 0 \leq z \leq 1-x\}$$

$$\Rightarrow \int_{x=0}^{x=1} \int_{y=0}^{y=x} x \cdot y \cdot \sqrt{1+x^2+y^2} \, dx \, dy = 0$$

3. Parz $x+z=1$

$$0 \leq x \leq 1$$

$$0 \leq y \leq x$$

$$T_3(x,y) = (x, y, 1-x)$$

$$D_3 = \{(x,y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$$

$$T_x^{(3)}(x,y) = (1, 0, -1)$$

$$T_y^{(3)}(x,y) = (0, 1, 0)$$

$$T_x^{(3)} \times T_y^{(3)} = \det \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= i + k = (1, 0, 1)$$



$$\|(1,0,1)\| = \sqrt{2}$$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} x \cdot y \cdot \sqrt{2} \, dx \, dy = \int_0^1 \sqrt{2} \cdot x \cdot \frac{x^3}{2} \Big|_0^x \, dx = \frac{\sqrt{2}}{2} \cdot \frac{x^4}{4} \Big|_0^1 = \frac{\sqrt{2}}{8}$$

4. Falte $y=x$

$$T_4(x,z) = (x, x, z)$$

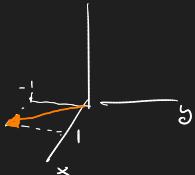
$$D_4 = \{(x,z) : 0 \leq x \leq 1 \text{ and } 0 \leq z \leq x\}$$

$$T_x^{(4)}(x, z) = (1, 1, 0)$$

$$T_z^{(4)}(x, z) = (0, 0, 1)$$

$$T_x^{(4)} \times T_z^{(4)} = \det \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = i - j$$

$$= (1, -1, 0)$$



$$\|(1, -1, 0)\| = \sqrt{2}$$

$$\int_{x=0}^{x=1} \int_{z=0}^{z=1-x} x \cdot y \cdot \sqrt{2} dz dx = \sqrt{2} \int_0^1 x^2 \cdot (1-x) dx$$

$$= \sqrt{2} \cdot \left(\frac{x^3}{3} \Big|_0^1 - \frac{x^4}{4} \Big|_0^1 \right)$$

$$= \sqrt{2} \cdot \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\sqrt{2}}{12}$$

$$\frac{4}{12} - \frac{3}{12}$$

//

Junto todo

$$\int_S xy \cdot dS = \frac{1}{8} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{12}$$

$$= \frac{1+\sqrt{2}}{8} + \frac{\sqrt{2}}{12}$$

//

Ejercicio 13. Calcular $\int_S (x + y + z) dS$ donde S es el borde de la bola unitaria, es decir

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

Lo hago en 2 partes



$$\phi_x(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$$

$$\phi_x^{(1)}(x, y) = (1, 0, \frac{-x}{\sqrt{1-x^2-y^2}})$$

$$\phi_y^{(1)}(x, y) = (0, 1, \frac{-y}{\sqrt{1-x^2-y^2}})$$

$$\phi_x^{(1)} \times \phi_y^{(1)} = \det \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{\sqrt{1-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{1-x^2-y^2}} \end{vmatrix} =$$

$$= i \cdot \frac{-x}{\sqrt{1-x^2-y^2}} + j \cdot \frac{-y}{\sqrt{1-x^2-y^2}} + k \cdot 1.$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1 \right)$$

$$\|\phi_x^{(1)} \times \phi_y^{(1)}\| = \left(\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1 \right)^{1/2}$$

$$\iint (x + y + \sqrt{1-x^2-y^2}) \cdot \sqrt{1 + \frac{x^2+y^2}{1-x^2-y^2}}$$

esto es la malla, rehacer con coordenadas esféricas.

$$x = r \cos \theta \sin \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \varphi$$

con $r = 1$ siempre.

Ejer 14)

Ejercicio 14. Hallar la masa de una superficie esférica de radio r tal que en cada punto $(x, y, z) \in S$ la densidad de masa es igual a la distancia entre (x, y, z) y el punto $(0, 0, r)$.

