(18) $G \in C^1 \Rightarrow \nabla\cdot(\nabla \times G) = 0$

$F \in C^1 \Rightarrow \nabla \cdot F = 0$ $\Rightarrow G \in C^2 / F = \nabla \times G$

$$\Rightarrow G_1 = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt$$

$$G_2 = -\int_0^z F_1(x,y,t)\,dt \qquad \Rightarrow \nabla \times G = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt & -\int_0^z F_1(x,y,t)\,dt & 0 \end{vmatrix}$$

$$G_3 = 0$$

$$= \left(\frac{\partial}{\partial z}\left(\int_0^z F_1(x,y,t)dt\right), \frac{\partial}{\partial z}\left(\int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)\,dt\right), \frac{\partial}{\partial x}\left(-\int_0^z F_1(x,y,t)dt\right) - \frac{\partial}{\partial y}\left(\int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)\,dt\right)\right)$$

$$= \left(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\right) = F$$

$$\Rightarrow \text{Si } F = -GmM \frac{x}{\|x\|^3} \Rightarrow \nabla \cdot F = \left(\frac{\partial}{\partial x}\left(\frac{-GmM\,x}{\|(x,y,z)\|^3}\right) + \frac{\partial}{\partial y}\left(\frac{-GmM\,y}{\|(x,y,z)\|^3}\right) + \frac{\partial}{\partial z}\left(\frac{-GmM\,z}{\|(x,y,z)\|^3}\right)\right)$$

$$= -GmM\left(\frac{\cancel{-2x^2} + \cancel{y^2} + \cancel{z^2} \quad \cancel{x^2} - \cancel{2y^2} + \cancel{z^2} \quad \cancel{x^2} + \cancel{y^2} - \cancel{2z^2}}{(x^2+y^2+z^2)^{5/2}}\right) = 0$$

$$\Rightarrow G(x,y,z) = \Bigg(\underbrace{\int_0^z -GmM \frac{y}{(x^2+y^2+t^2)^{3/2}}\,dt - \int_0^y -GmM \cdot \frac{z}{(x^2+t^2+0^2)^{3/2}}\,dt}_{G_1}, \underbrace{-\int_0^z -GmM \cdot \frac{x}{(x^2+y^2+t^2)^{3/2}}\,dt}_{G_2}, 0\Bigg)$$

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$$G_1 = -GmM.y\int_0^z \frac{1}{(x^2+y^2+t^2)^{3/2}}\,dt + GmMz\int_0^y \frac{1}{(x^2+t^2+z^2)^{3/2}}\,dt = \cdots = -GmMy.\frac{z}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} - GmMz\frac{3y}{(x^2+y^2)^{5/2}}$$

$$G_1 = \frac{-GmM\cdot yz\cdot\left((x^2+y^2)^{3/2}+3\sqrt{x^2+y^2+z^2}\right)}{(x^2+y^2)^{5/2}\cdot\sqrt{x^2+y^2+z^2}}$$

$$G_2 = GmMx\int_0^z \frac{1}{(x^2+y^2+t^2)^{3/2}}\,dt = \cdots = \frac{GmM\,xz}{(x^2+y^2)\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow G(x,y,z) = \left(-\frac{GmMyz\left((x^2+y^2)^{3/2}+\sqrt{x^2+y^2+z^2}\right)}{(x^2+y^2)^{5/2}\sqrt{x^2+y^2+z^2}},\ \frac{GmM\,xz}{(x^2+y^2)\sqrt{x^2+y^2+z^2}},\ 0\right)$$

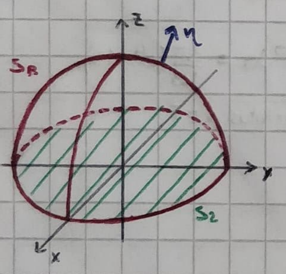
(19) a) $F=(x,y,z) \Rightarrow \nabla.F = (1+1+1) = 3 \neq 0 \Rightarrow$ No $\exists\, G \,/\, F = \nabla\times G$

b) $F = (x^2+1,\ x-2xy,\ y) \Rightarrow \nabla.F = (2x-2x+0) = 0 \Rightarrow \exists\, G\,/\,F=\nabla\times G$

$$\Rightarrow G(x,y,z) = \left(\int_0^z x-2xy\,dt - \int_0^y t\,dt,\ -\int_0^z (x^2+1)\,dt,\ 0\right) = \left(xz-2xyz-\frac{y^2}{2},\ x^2z+z,\ 0\right)$$

(20) $S_R = \{(x,y,z)\,/\,x^2+y^2+z^2=R^2,\ z\geq 0\} \Rightarrow \phi(\varphi,\theta) = (R\cos\theta\,\mathrm{sen}\varphi,\ R\,\mathrm{sen}\theta\,\mathrm{sen}\varphi,\ R\cos(\varphi))$ con $\theta\in[0,2\pi]$, $\varphi\in[0,\pi/2]$

$F(x,y,z) = (xz - x\cos z,\ -yz+y\cos z,\ 4-x^2-y^2) \Rightarrow \nabla.F = (z-\cos z - z + \cos z + 0) = 0$



$$\Rightarrow \iiint_\Omega \nabla.F\,dxdydz = \iint_{\partial\Omega = S_R\cup S_2} F\,ds \Rightarrow \iint_{S_R} F\,ds = \iiint_\Omega 0.\,dxdydz - \iint_{S_2} F\,ds$$

$\Rightarrow S_2 = \{(x,y,z)\,/\,x^2+y^2\leq R^2,\ z=0\} \Rightarrow \phi_2(\theta,r) = (r\cos\theta,\ r\,\mathrm{sen}\theta,\ 0)$ con $r\in[0,R]$, $\theta\in[0,2\pi]$

$\Rightarrow \phi_{2r} = (\cos\theta,\ \mathrm{sen}\theta,\ 0)$ $\Rightarrow \phi_{2\theta}\times\phi_{2r} = (0,0,-r) \rightarrow$ Orientado "hacia afuera"

$\phi_{2\theta} = (-r\,\mathrm{sen},\ r\cos\theta,\ 0)$

$$\Rightarrow \iint_{S_2} F\,ds = \int_0^R\int_0^{2\pi} \left(-r\cos\theta,\ r\,\mathrm{sen}\theta,\ 4-(r\cos\theta)^2-(r\,\mathrm{sen}\theta)^2\right)(0,0,-r)\,d\theta dr = \int_0^R\int_0^{2\pi}(4-r^2)(-r)\,dr d\theta = 2\pi\left(\int_0^R -4r\,dr + \int_0^R r^3\,dr\right)$$

$$= 2\pi\left(-\frac{4r^2}{2}\Big|_0^R + \frac{r^4}{4}\Big|_0^R\right) = 2\pi\left(\frac{R^4}{4} - 2R^2\right)$$

$$\Rightarrow \iint_{S_R} F\,ds = -\iint_{S_2} F\,ds = 2\pi\left(2R^2 - \frac{R^4}{4}\right)$$

$\Rightarrow$ Para que el flujo sea máximo: $\frac{d(2\pi(2R^2 - R^4/4))}{dR} = 0 \Rightarrow 8\pi R - 2\pi R^3 = 0$

$(8\pi - 2\pi R^2).R = 0 \Rightarrow 8\pi - 2\pi R^2 = 0 \Rightarrow R^2 = 4 \Rightarrow R = 2$

$\Rightarrow$ El flujo es máximo en $R=2$

(21) $V = (x, y, xy - z) \Rightarrow \nabla . F = (1 + 1 - 1) = 1$

$\Rightarrow \iint_S F ds = \iiint_\Omega \nabla . F \, dx dy dz = \iiint_\Omega 1 . \, dx dy dz = Vol(\Omega) > 0 \Rightarrow$ El flujo será positivo por lo que se está expandiendo

(22) $T(x, y, z, t)$ temperatura en $\underbrace{(x, y, z)}_{\bar{x}}$ en $t \Rightarrow T = 30 - t - x^2 - y^2 - z^2 \Rightarrow T: \mathbb{R}^3 \times (0,1) \rightarrow \mathbb{R}$

$\Rightarrow F(\bar{x}, t) = -K \nabla T(\bar{x}, t) = -K(-2x, -2y, -2z)$

$\Rightarrow S = [0,4] \times [0,5] \times [0,3]$

$\Rightarrow \underbrace{\iint_S (F(\bar{x}, s), n) ds}_{\text{Flujo a través de } S} = \iiint_\Omega div(F(\bar{x}, s)) d\bar{x}$

$\Rightarrow \iiint_\Omega u(\bar{x}, t_0 + h) d\bar{x} - \iiint_\Omega u(\bar{x}, t_0) d\bar{x} = \int_{t_0}^{t_0+h} \frac{\partial}{\partial t} \iiint_\Omega u(\bar{x}, s) d\bar{x} ds = \underbrace{\int_{t_0}^{t_0+h} \iiint_\Omega \frac{\partial u}{\partial t}(\bar{x}, s) d\bar{x} ds}_{\text{Flujo de } u \text{ a través de } \partial\Omega = S \text{ entre } t_0 \text{ y } t_0+h}$

Nos interesa lo que entra en el cuerpo así que orientamos la superficie con la normal interior

$\Rightarrow \int_{t_0}^{t_0+h} \iiint_\Omega \frac{\partial u}{\partial t}(\bar{x}, s) d\bar{x} ds = -\int_{t_0}^{t_0+h} \iiint_\Omega div(F(\bar{x}, s)) d\bar{x} ds \Rightarrow \frac{\partial u}{\partial t}(\bar{x}, t) = -div(F(\bar{x}, t))$

$\Rightarrow \frac{\partial u}{\partial t}(\bar{x}, t) = div(\underbrace{K}_{1} \nabla T(\bar{x}, t)) = (-2 + (-2) + (-2)) = -6$

$\Rightarrow \int_0^1 \int_0^4 \int_0^5 \int_0^3 -6 \, dx dy dz dt = \int_0^1 \int_0^4 \int_0^5 -6x \Big|_0^3 dy dz dt = \int_0^1 \int_0^4 -18 y \Big|_0^5 dz dt = \int_0^1 -90 z \Big|_0^4 dt = -360 t \Big|_0^1 = -360$

(23)

1° identidad: $\iint_{\partial\Omega} f \nabla g . n \, ds = \iiint_\Omega (f \Delta g + \nabla f . \nabla g) dx dy dz$

$\Rightarrow$ Por Gauss $\iint_{\partial\Omega} F n \, dS = \iiint_\Omega \nabla . F \, dx dy dz \Rightarrow$ Si $F = f \nabla g \Rightarrow \nabla . F = \left(\frac{\partial}{\partial x}\left(f \frac{\partial g}{\partial x}\right), \frac{\partial}{\partial y}\left(f \frac{\partial g}{\partial y}\right), \frac{\partial}{\partial z}\left(f \frac{\partial g}{\partial z}\right)\right)$

$= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f \frac{\partial^2 g}{\partial z^2}$

$= \langle \nabla f, \nabla g \rangle + f \Delta g$

$\Rightarrow$ Entonces: $\iint_{\partial\Omega} f \nabla g . n \, ds = \iiint_\Omega (\nabla f . \nabla g + f \Delta g) dx dy dz$

2° identidad: $\iint_{\partial\Omega} (f\nabla g - g\nabla f) \cdot n \, ds = \iiint_{\Omega} (f\Delta g - g\Delta f) \, dx\,dy\,dz$

$\Rightarrow$ Por Gauss: $\iint_{\partial\Omega} F n \, ds = \iiint_{\Omega} \nabla \cdot F \, dx\,dy\,dz \Rightarrow$ Si $F = P - Q$ / $P = f\nabla g$, $Q = g\nabla f$: $\iint_{\partial\Omega} F n \, ds = \iint_{\partial\Omega} (P-Q) n \, ds = \iint_{\partial\Omega} (f\nabla g - g\nabla f) n \, ds$

Utilizando la 1° identidad

$\Rightarrow \iint_{\partial\Omega} (f\nabla g - g\nabla f) n \, ds = \iiint_{\Omega} (f\Delta g + \nabla f \nabla g - (g\Delta f + \nabla g \nabla f)) \, dx\,dy\,dz = \iiint_{\Omega} (f\Delta g - g\Delta f) \, dx\,dy\,dz$

(24) $f \in C^2(\Omega)$ y $C^1(\partial\Omega)$ con $f=0$ en $\partial\Omega$, $f \not\equiv 0 / \Delta f = \lambda f$ en $\Omega \Rightarrow \lambda$ autovalor de Δ
f autofunción asociada a λ

$\Rightarrow \lambda \neq \mu$ autovalores de Δ en Ω, f y g autofunciones asociadas $\Rightarrow \Delta f = \lambda f$
$\Delta g = \mu g$

$\Rightarrow$ 2° identidad de Green: $\int_\Omega (f\underbrace{\Delta g}_{\mu g} - g\underbrace{\Delta f}_{\lambda f})\,dV = \int_{\partial\Omega} (\underbrace{f\nabla g}_{0} - \underbrace{g\nabla f}_{0})\,n\,ds = 0 \Rightarrow \int_\Omega (f\mu g - g\lambda f)\,dV = \int_\Omega \underbrace{(\mu-\lambda)}_{\neq 0} gf\,dV = 0$

$\Rightarrow \underbrace{(\mu-\lambda)}_{\neq 0} \cdot \int_{\partial\Omega} gf\,dV = 0 \Rightarrow \iiint_\Omega fg\,dV = 0$

(25) Supongo $f \not\equiv 0 / \Delta f = 0$ en B y $f=0$ en ∂B

$\Rightarrow$ 1° identidad de Green: $\int_{\partial\Omega} \underbrace{f}_{0} \nabla f \cdot n\,ds = \int_\Omega (f\underbrace{\Delta f}_{0} + \nabla f \nabla f)\,dV \Rightarrow 0 = \int_\Omega \nabla f \nabla f\,dV = \int_\Omega (f_x, f_y, f_z)(f_x, f_y, f_z)\,dV$

$\Rightarrow 0 = \int_\Omega (\underbrace{f_x^2}_{\geq 0} + \underbrace{f_y^2}_{\geq 0} + \underbrace{f_z^2}_{\geq 0})\,dV \Rightarrow \nabla f \cdot \nabla f = 0 \Rightarrow \nabla f = 0$ en B

$\Rightarrow$ Por lo visto en (9), si $\nabla f = 0$ en B $\Rightarrow f = cte \Rightarrow$ Pero $f=0$ en $\partial\Omega$, por lo tanto $f \equiv 0$

(26) $\int_C E\,ds = -\frac{1}{c}\frac{d}{dt}\iint_S H\,ds \Rightarrow \frac{d}{dt}\iint_S H\,ds + c\int_C E\,ds = 0 \xRightarrow{\text{Por Stokes}} \iint_S \frac{\partial H}{\partial t}\,ds + c\iint_S \nabla \times E\,ds = 0$

$\Rightarrow$ Tomo S como un disco ~~D~~ de radio r (D_r): $\iint_{D_r} \frac{\partial H}{\partial t}\,ds + c\iint_S \nabla \times E\,ds = 0$

$\Rightarrow$ Divido ambos lados por el area del disco: $\frac{1}{Area(D_r)}\left(\iint_{D_r} \frac{\partial H}{\partial t}\,ds + c\iint_{D_r} \nabla \times E\,ds\right) = 0$

$\Rightarrow$ Si $r \to 0$: $\lim_{r\to 0} \frac{1}{Area(D_r)}\left(\iint_{D_r} \frac{\partial H}{\partial t}\,ds + c\iint_{D_r} \nabla \times E\,ds\right) = 0 \Rightarrow$ La integral converge a lo que vale la función en el centro del disco

$\Rightarrow H_t(P) + c\,\nabla \times E(P) = 0 \Rightarrow$ Como D_r era arbitrario, por lo que vale para cualquier otra superficie

$\Rightarrow H_t + c\,\nabla \times E = 0$