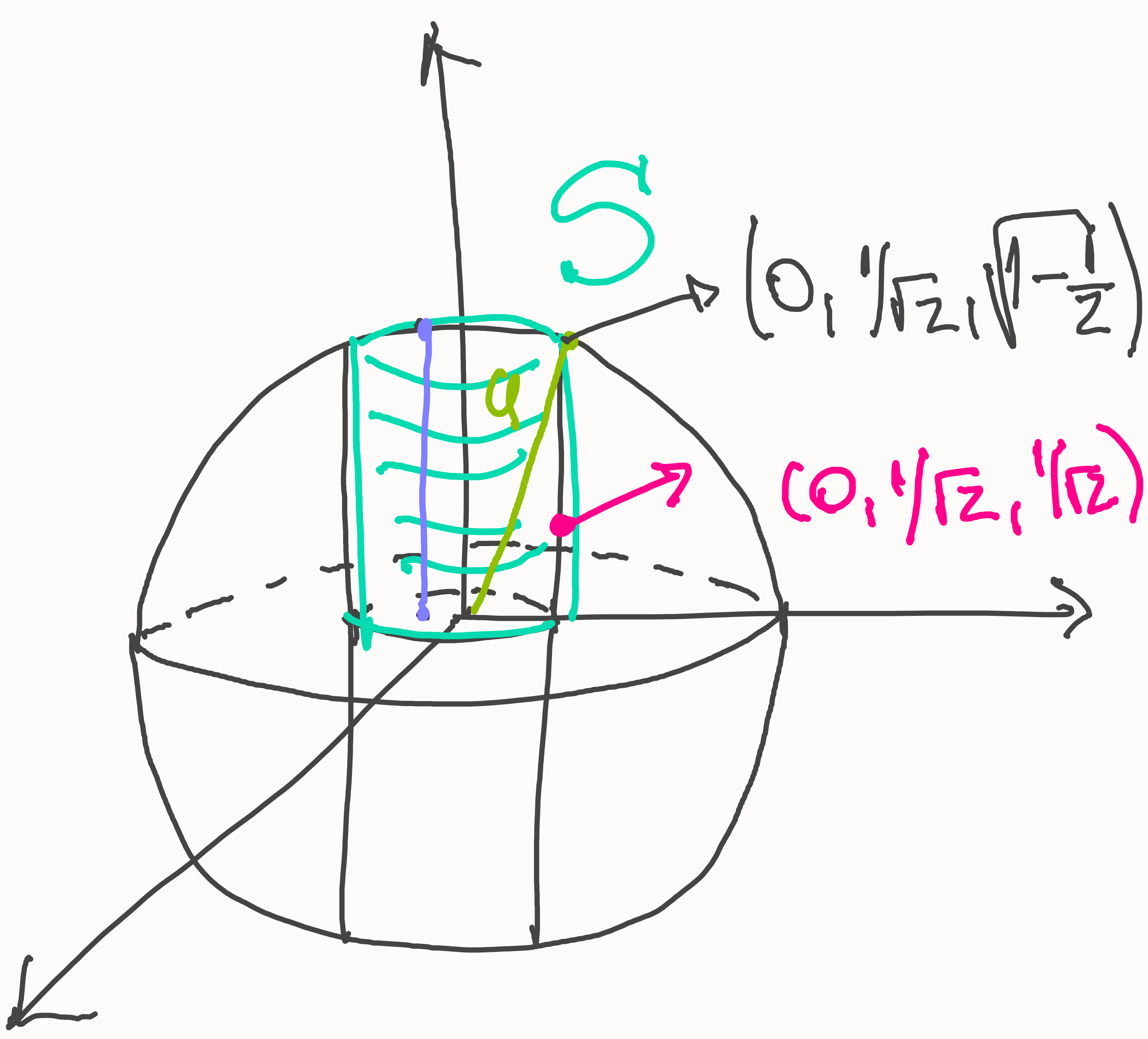
Ej Parcial:

Sea $S = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{x^2+y^2+z^2=1}_{\text{esfera}},\ \underbrace{z \geq 0}_{\text{"casco de arriba"}}$

$\underbrace{x^2+y^2 \leq 1/2\}}_{\text{cilindro.}}$

a) Hallar una param. de S de forma tal que la normal en $(0, 1/\sqrt{2}, 1/\sqrt{2})$ sea $(0, 1/\sqrt{2}, 1/\sqrt{2})$

b) Hallar el área de S.



Esféricas

$$\begin{cases} x = \cos\theta \sin\varphi \\ y = \sin\theta \sin\varphi \\ z = \cos\varphi \end{cases}$$

$\overset{\shortparallel}{m}$

Cilíndricas

$x = \frac{1}{\sqrt{2}} \cos\theta$ $\qquad \theta \in [0, 2\pi]$

$y = \frac{1}{\sqrt{2}} \sin\theta$

$z = z$ $\qquad z \in [0, 1-$

$0 \leq z \leq \sqrt{1-x^2-y^2}$ $\quad \overset{\shortparallel}{m}.$

$(x,y) \in D = \{x^2+y^2 \leq 1/2\}$

$0 \leq z \leq \sqrt{1-x^2-y^2}$

$T(x,y) = (x, y, \sqrt{1-x^2-y^2})$, $T: D \to S$

Veamos que T es param. $\operatorname{Im}(T) = S$.

$\subseteq)$ $(x,y) \in D \Rightarrow T(x,y) \in S$ pues:

- $x^2 + y^2 + \left(\sqrt{1-x^2-y^2}\right)^2 = 1$ ✓
- $\sqrt{1-x^2-y^2} \geq 0$ ✓
- $x^2+y^2 \leq 1/2$ pues $(x,y) \in D$ ✓

$\supseteq)$ $(x,y,z) \in S \Rightarrow z \geq 0$

$x^2+y^2+z^2=1 \Rightarrow z^2 = 1-x^2-y^2$

$x^2+y^2 \leq 1/2 \Longrightarrow (x,y) \in D$

$\left.\begin{array}{l} z \geq 0 \\ z^2 = 1-x^2-y^2 \end{array}\right\} \Rightarrow z = \sqrt{1-x^2-y^2}$

$\Rightarrow T(x,y) = (x, y, \sqrt{1-x^2-y^2}) = (x,y,z) \Rightarrow (x,y,z) \in \mathrm{Im}(T)$.

Como T cont. $\Rightarrow$ T es param de S.

$T_x = \left(1, 0, \dfrac{-x}{\sqrt{1-x^2-y^2}}\right) \qquad T_y = \left(0, 1, \dfrac{-y}{\sqrt{1-x^2-y^2}}\right)$

$T_x \times T_y = \det \begin{pmatrix} i & j & k \\ 1 & 0 & -x/\sqrt{\ } \\ 0 & 1 & -y/\sqrt{\ } \end{pmatrix} = \left(\dfrac{x}{\sqrt{\ }}, +\dfrac{y}{\sqrt{\ }}, 1\right)$

¿T orienta hacia afuera?

$T(0, 1/\sqrt{2}) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$T_x \times T_y (0, 1/\sqrt{2}) = \left(0, \dfrac{1/\sqrt{2}}{1/\sqrt{2}}, 1\right) = (0,1,1)$ ✓

orienta hacia afuera !!.

$\left(\frac{x}{\sqrt{\ }}, \frac{y}{\sqrt{\ }}, 1\right)$

b) $\text{área}(S) = \iint_D \| T_x \times T_y (x,y) \| \, dx\,dy$.

$= \iint_{x^2+y^2 \leq 1/2} \left(\dfrac{x^2}{1-x^2-y^2} + \dfrac{y^2}{1-x^2-y^2} + 1 \right)^{1/2} dx\,dy$

$$= \iint_{x^2+y^2\leq 1/2} \left(\frac{\cancel{x^2}+\cancel{y^2}+1-\cancel{x^2}-\cancel{y^2}}{1-x^2-y^2} \right)^{1/2} dx\,dy$$

$$\underset{\text{polares}}{=} \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\frac{1}{1-r^2}\right)^{1/2} \cdot r\,dr\,d\theta = 2\pi \cdot \int_0^{1/\sqrt{2}} \frac{r}{(1-r^2)^{1/2}}\,dr$$

$x = r\cos\theta$
$y = r\sin\theta$
$r \in [0, 1/\sqrt{2}]$
$\theta \in [0, 2\pi]$.

$$= 2\pi \int_1^{1/2} \frac{-1/2\,dt}{\sqrt{t}}$$

$t = 1-r^2$
$dt = -2r\,dr$

$$= \pi \int_{1/2}^1 \frac{1}{\sqrt{t}}\,dt = \pi \; 2\sqrt{t}\Big|_{1/2}^{1} = \boxed{2\pi\left(1-1/\sqrt{2}\right)}$$

$$\left(\frac{t^{-1/2+1}}{-1/2+1}\right)$$

—o—

. Dar una param. de la curva dada en coord. polares como $r = \cos\theta$, $\theta \in [-\pi/2, \pi/2]$.
Mostrar que la curva es suave y hallar su long.

Sol: $x = r\cos\theta$ $\qquad r = \cos\theta$
$y = r\sin\theta$

$\mathcal{C}$ = curva $\Rightarrow$ $\alpha(\theta) = (\cos^2\theta, \cos\theta\sin\theta)$
$\theta \in [-\frac{\pi}{2}, \pi/2]$. Veamos que α es param. de $\mathcal{C}$.

. α continua ✓

. $\mathrm{Im}(\alpha) = \mathcal{C}$

5) $\theta \in [-\pi/2, \pi/2]$, $(\underline{\cos^2\theta}, \cos\theta\,\text{sen}\,\theta) \in \mathcal{C}$?

$\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{array}, \; r = \cos\theta, \; \theta \in [-\pi/2, \pi/2] \}$

Si pues tomo $\begin{array}{l} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{array}$ con $r = \cos\theta$

$\Rightarrow (x,y) = (\cos^2\theta, \text{sen}\,\theta\cos\theta) \in \mathcal{C}$ ✓

2) $(x,y) \in \mathbb{R}^2 \Rightarrow \exists\, \theta \in [-\pi/2, \pi/2] \,/\, \begin{array}{l} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{array}$ $r = \cos\theta$

$\Rightarrow (x,y) = \alpha(\theta) \Rightarrow$ ✓

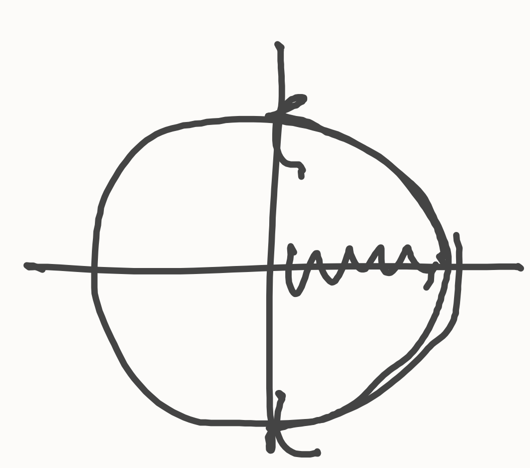
$\alpha(-\pi/2) = (0,0) = \alpha(\pi/2)$

$\mathcal{C}$ cerrada.

Veamos que $\mathcal{C}$ es suave.

¿α es regular?

regular:
- α es inyectiva en $(-\pi/2, \pi/2)$
- α $\mathcal{C}^1$
- $\alpha'(\theta) \neq (0,0)$ $\forall\, \theta \in (-\pi/2, \pi/2)$



- α es $\mathcal{C}^1$ ✓
- α es inyectiva?

$\alpha(\theta) = \alpha(\tilde\theta) \Rightarrow$ (1) $\cos^2\theta = \cos^2\tilde\theta$ $\wedge$

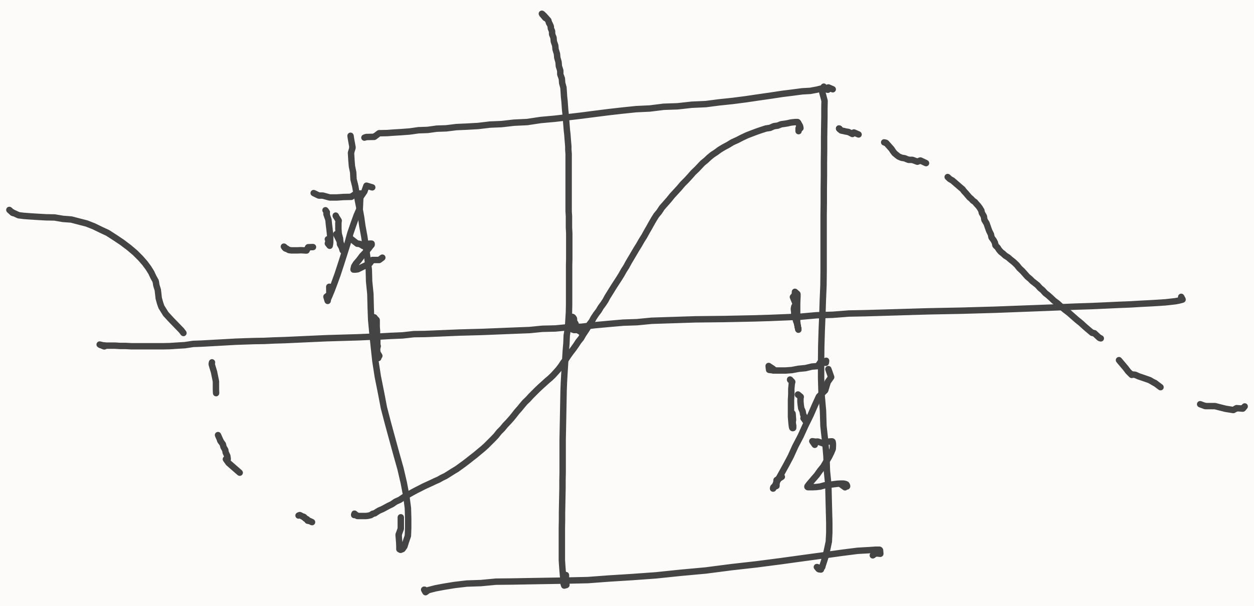
$\theta, \tilde\theta \in (-\pi/2, \pi/2)$ (2) $\cos\theta\,\text{sen}\,\theta = \cos\tilde\theta\,\text{sen}\,\tilde\theta$

$\Rightarrow$ de (1) $|\cos\theta| = |\cos\tilde\theta|$ pero como $\theta, \tilde\theta \in (-\pi/2, \pi/2)$

$\Rightarrow \cos\theta = \cos\tilde\theta \neq 0 \overset{(2)}{\Rightarrow} \underbrace{\text{sen}\,\theta = \text{sen}\,\tilde\theta}$.

$\Downarrow$

como sen es inyectiva entre $(-\pi/2, \pi/2) \Rightarrow \underline{\theta = \tilde\theta}$.



$\Rightarrow \alpha$ es inyectiva en $(-\pi/2, \pi/2)$

• $\alpha'(\theta) = (-2\cos\theta \operatorname{sen}\theta, -\operatorname{sen}^2\theta + \cos^2\theta) \overset{?}{\neq} (0,0) \ \forall \theta \in (-\pi/2, \pi/2)$

$\cos\theta \neq 0 \ \forall \theta \in (-\pi/2, \pi/2)$

$-2\cos\theta\operatorname{sen}\theta = 0 \underset{\theta \in (-\pi/2, \pi/2)}{\Leftrightarrow} \theta = 0.$

miro la segunda coord en $\theta = 0$

$-\operatorname{sen}^2\theta + \cos^2\theta = 1$

$\alpha'(\theta) \neq (0,0)$ en $(-\pi/2, \pi/2)$.

$\Rightarrow \alpha$ es param regular y $\therefore$ $\mathcal{C}$ es una curva suave.

Longitud de $\mathcal{C}$.

$$\alpha'(\theta) = (\underbrace{-2\cos\theta\operatorname{sen}\theta}_{-\operatorname{sen}(2\theta)}, \underbrace{\cos^2\theta - \operatorname{sen}^2\theta}_{\cos(2\theta)})$$

$$\cdot \operatorname{long}(\mathcal{C}) = \int_{-\pi/2}^{\pi/2} \|\alpha'(\theta)\| \, d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} (\underbrace{\operatorname{sen}^2(2\theta) + \cos^2(2\theta)}_{=1})^{1/2} \, d\theta = \pi.$$

—— o ——

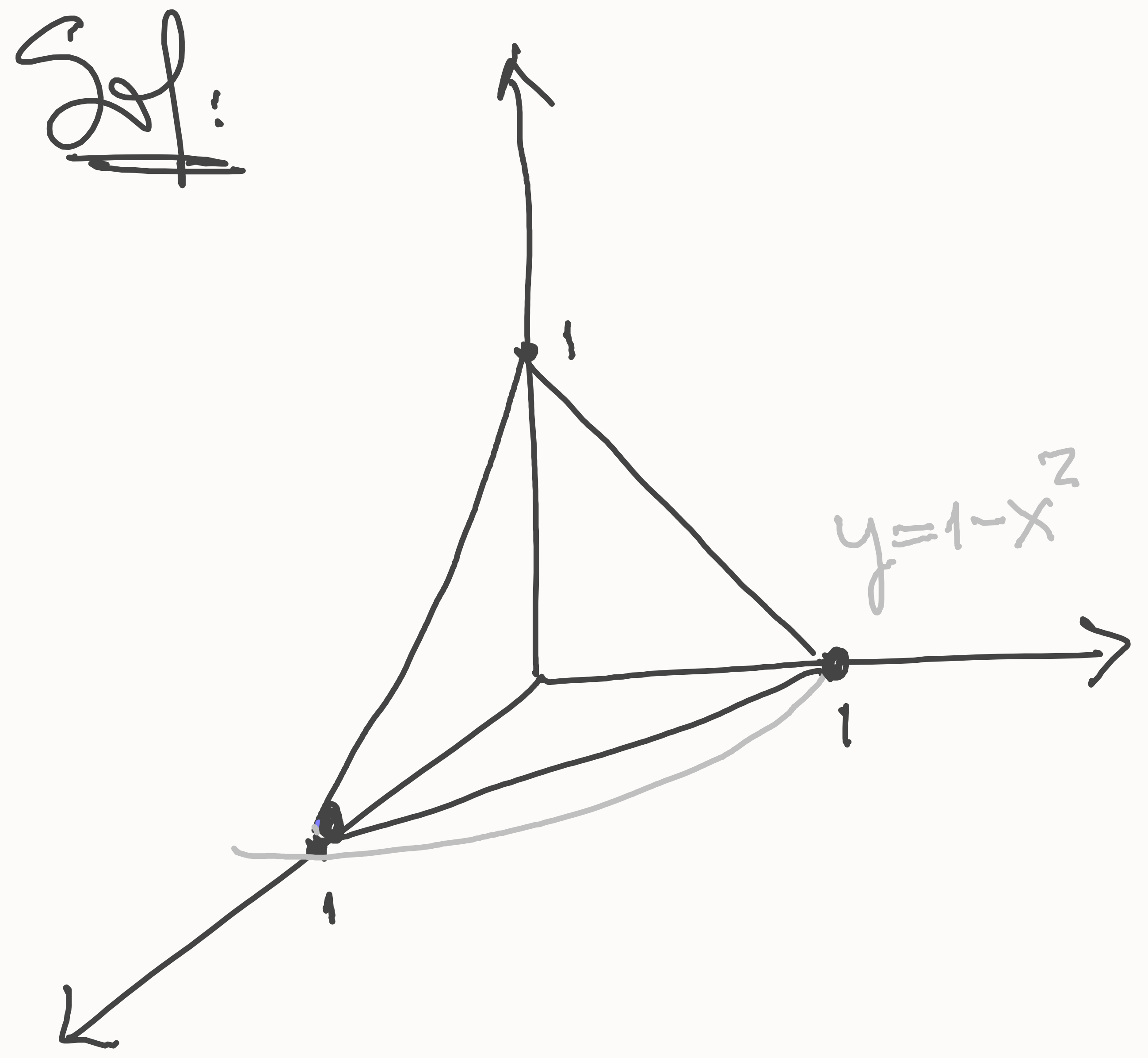
Considerar la curva $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 : y = 1 - x^2,$
$x + y + z = 1, \ x, y \geq 0\}$

a)

a) obtener una param reg. de $\mathscr{C}$ que empiece en $(0,1,0)$ y termine en $(1,0,0)$

b) Calcular $\int_{\mathscr{C}} F \cdot ds$. con $\mathscr{C}$ orientado como en a) donde $F(x,y,z) = (2x, y, -z)$.

Sol:



$y = 1 - x^2$

$x + y + z = 1$

$x, y \geq 0$.

$z = 1 - x - y$

$= 1 - x - 1 + x^2$

$z = x^2 - x$

$$\alpha(t) = (x(t), y(t), z(t))$$

$$= (t, 1 - t^2, t^2 - t) \qquad t \in [0,1]$$

$\begin{cases} \alpha(0) = (0,1,0) \leftarrow \\ \alpha(1) = (1,0,0) \end{cases}$ Ej: Verq! α es param de $\mathscr{C}$

Veamos que α es regular:

- α inyectiva en $[0,1]$
- $\alpha \in \mathscr{C}^1$
- $\alpha'(t) \neq (0,0,0)$ $\forall t \in [0,1]$.

1) ¿α inyectiva? $\alpha(t) = \alpha(s) \underset{1^{\circ} \text{coord}}{\Longrightarrow} t = s \Rightarrow \alpha$ inyec

2) ¿α es $\mathscr{C}^1$? Si pues c/ coord es un polinomio.

3) $\alpha'(t) = (\underset{\substack{\neq \\ 0}}{1}, \ -2t, \ 2t-1) \neq (0,0,0)$

$\Rightarrow$ α es regular y da la orientación correcta.

b) $F(x,y,z) = (2x, y, -z)$,

$$\int_{\mathcal{C}} F \cdot ds \underset{\substack{\downarrow \\ \alpha \text{ orienta} \\ \text{a } \mathcal{C}.}}{=} \int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle \, dt$$

$$= \int_0^1 \langle (2t, 1-t^2, -t^2+t), (1, -2t, 2t-1) \rangle \, dt$$

$$= \int_0^1 2t - (1-t^2)2t + (t-t^2)(2t-1) \, dt.$$

$$= \int_0^1 \cancel{2t} - \cancel{2t} + \cancel{2t^3} + 2t^2 - t - \cancel{2t^3} + t^2 \, dt$$

$$= \int_0^1 3t^2 - t \, dt = t^3 - \frac{t^2}{2} \Big|_0^1 = 1 - \frac{1}{2} = \boxed{1/2}$$

$\boxtimes$