Iván

Feb 10

REC: TEOREMA DE GREEN

Se pide cuando se trabaja con derivadas. (No comerse el coco con esto)

SEA $\Omega \in \mathbb{R}^2$ ABIERTO, $F: \Omega \longrightarrow \mathbb{R}^2$,

$F = (P, Q)$, $P, Q: \Omega \longrightarrow \mathbb{R}$ DE CLASE C^1

SEA $R \in \Omega$ UNA REGIÓN (O UNIÓN FINITA DE REGIONES) DE TIPO III, $C = \partial R$ CURVE CERRADA, SIMPLE, SUAVE A TROZOS, ORIENTADA POSITIVAMENTE

LUEGO $\displaystyle \underbrace{\int_C F\, d\vec{s}}_{= \int_a^b \langle F(\sigma(t)),\, \sigma'(t) \rangle dt} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

P3 E12) Calcular

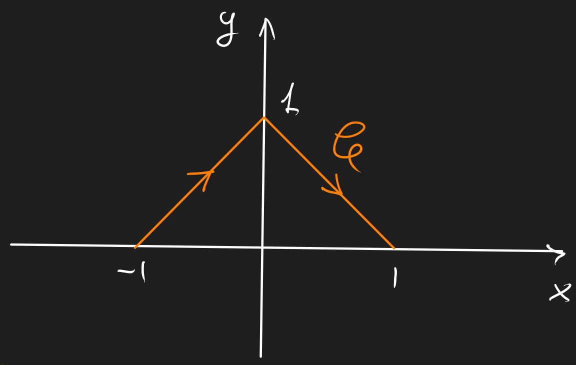
$$\int_{\mathcal{C}} F\, d\vec{s} \qquad \text{con } F = (P, Q)$$

$$P(x,y) = \frac{x \cdot \sin\left(\frac{\pi}{2(x^2+y^2)}\right) - y\,(x^2+y^2)}{(x^2+y^2)^2}$$

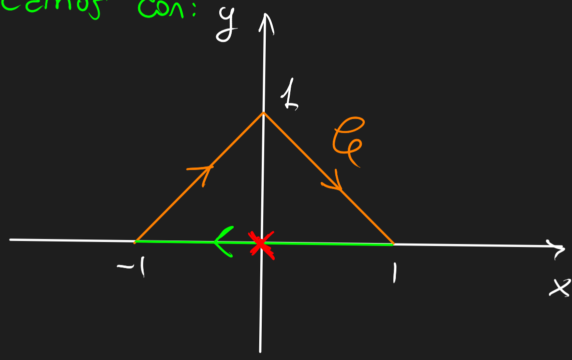
$$Q(x,y) = \frac{y \cdot \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) + x\,(x^2+y^2)}{(x^2+y^2)^2}$$

Sobre:

$$\mathcal{C} = \begin{cases} y = x + 1 & -1 \leq x \leq 0 \\ y = 1 - x & 0 \leq x \leq 1 \end{cases}$$ Orientada de $(-1,0)$ a $(1,0)$



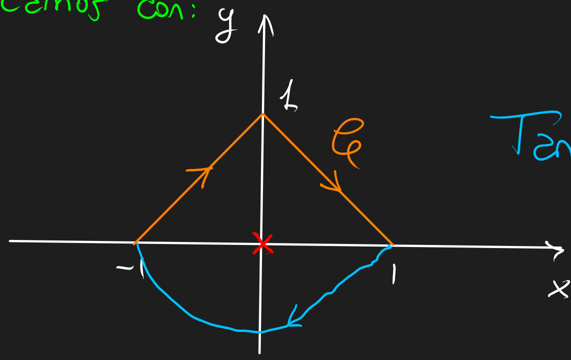
Veamos con:



$B \to \{(x,0),\ -1 \leq x \leq 1\}$

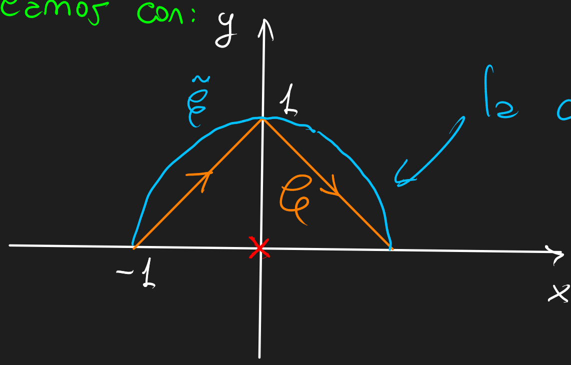
$\hookrightarrow F$ no es C^1 en $(0,0)$

Veamos con:



Tampoco! lo incluye!

Veamos con:



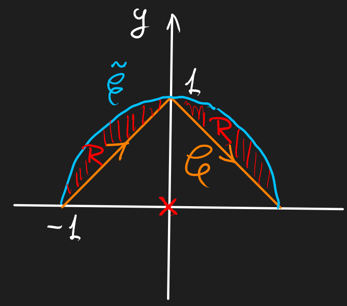
la cierro así! :)

Me conviene el semicírculo porque al componer me desaparecen los x^2+y^2 pues son iguales al radio $= 1$.

Planteo

$$\int_{\mathcal{C}} F\,d\vec{s} + \int_{\tilde{\mathcal{C}}} F\,ds \overset{\text{Teo G.}}{=} \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$\underbrace{\phantom{\int_{\mathcal{C}} F\,d\vec{s}}}_{\text{Incógnita}}$



$\mathcal{C} \cup \tilde{\mathcal{C}}$ está orientada positivamente

Calculemos integrales:

Teníamos:

$$P(x,y) = \frac{x \cdot \sin\left(\frac{\pi}{2(x^2+y^2)}\right) - y\,(x^2+y^2)}{(x^2+y^2)^2}$$

$$Q(x,y) = \frac{y \cdot \operatorname{sen}\left(\frac{\pi}{2(x^2+y^2)}\right) + x(x^2+y^2)}{(x^2+y^2)^2}$$

Llamo:

$$K(x,y) = x^2+y^2$$

$$g(t) = \frac{\sin\left(\frac{\pi}{2t}\right)}{t^2}$$

Reescribo

$$P(x,y) = x \cdot g \circ k(x,y) - \frac{y}{k(x,y)}$$

$$Q(x,y) = y \cdot g \circ k(x,y) + \frac{x}{k(x,y)}$$

$$\frac{\partial Q}{\partial x} = y \cdot g' \circ k(x,y) . 2x + \frac{k(x,y) - x . 2x}{k^2(x,y)}$$

$$\frac{\partial P}{\partial y} = x \cdot g' \circ \cancel{k(x,y)} . 2y - \frac{k(x,y) - y . 2y}{k^2(x,y)}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{2k(x,y) - 2x^2 - 2y^2}{k^2(x,y)} = 0$$

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy = 0 \,//$$

Calculo integral curvilínea

Parametrizo a

$$\tilde{\mathcal{C}} \to \sigma : [0, \pi] \to \mathbb{R}^2$$

$$\sigma(t) = (\cos t, \sin t)$$

Chequeo orientación

$$\sigma(0) = (1, 0)$$

$$\sigma(\pi) = (-1, 0)$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\int_{\tilde{\mathcal{C}}} F\,ds = \int_0^{\pi} \langle F(\sigma(t)), \sigma'(t)\rangle\, dt =$$

$$= \int_0^{\pi} \left\langle \left(\frac{\cos \cdot \operatorname{sen}\left(\cancel{\frac{\pi}{2\cdot 1}}\right) - \operatorname{sen} t \cdot \cancel{1}}{\cancel{1^2}}, \frac{\operatorname{sen} t \cdot \operatorname{sen}\left(\cancel{\frac{\pi}{2\cdot 1}}\right) + \cos t \cdot \cancel{1}}{\cancel{1^2}} \right), (-\operatorname{sen} t, \cos t) \right\rangle dt$$

$\vdots$

$$= \int_0^{\pi} 1 \cdot dt = \pi$$

Vuelvo a

$$\int_{\mathcal{C}} F\,ds + \pi = 0 \Rightarrow \boxed{\int_{\mathcal{C}} F\,ds = -\pi}$$

Ej:

Hallar $\int_{\mathcal{C}} F\,d\vec{s}$

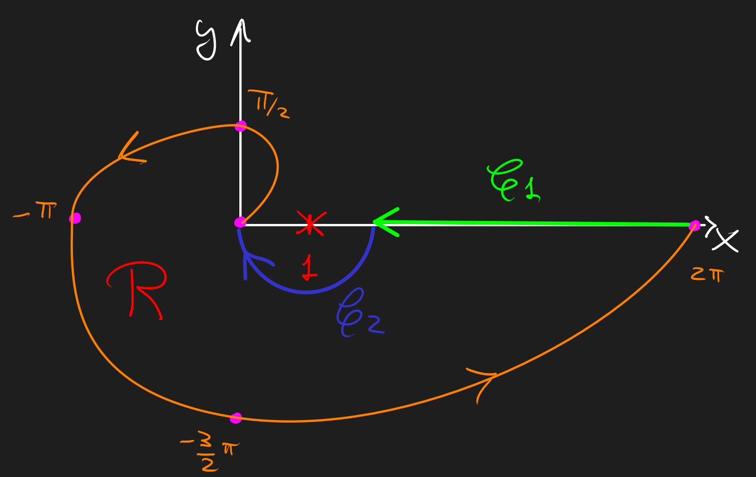
$\mathcal{C} \to$ dada en forma Polar por

$r(\theta) = \theta \quad , \quad 0 \leq \theta \leq 2\pi$

orientada de

$(0,0)$ a $(2\pi, 0)$

$$F(x,y) = \left(\frac{-y}{(x-1)^2 + y^2} + 3y \; , \; \frac{x-1}{(x-1)^2 + y^2} \right)$$



$$\underbrace{\int_{\mathcal{C}} F \, d\vec{s}}_{?} + \int_{\mathcal{C}_1} F \, d\vec{s} + \int_{\mathcal{C}_2} F \, d\vec{s} =$$

green

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy$$

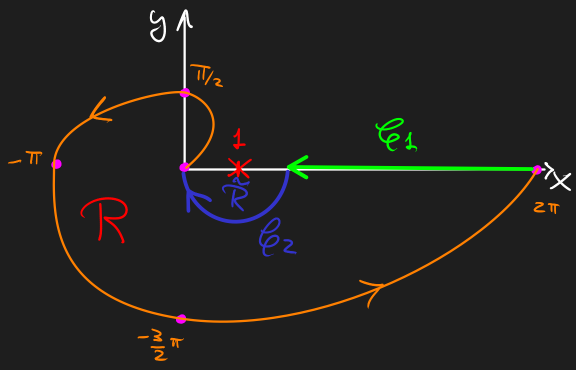
Calculo

$$\frac{\partial Q}{\partial x}(x,y) = \frac{(x-1)^2 + y^2 - (x-1)\cdot 2(x-1)}{\left((x-1)^2 + y^2\right)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-1\left((x-1)^2+y^2\right) + y \cdot 2y}{\left((x-1)^2+y^2\right)^2} + 3$$

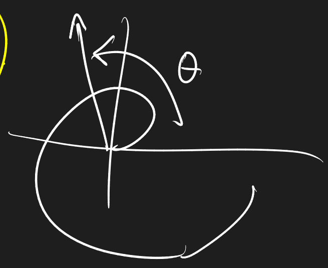
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 3 = -3$$

$$\iint_R (-3)\, dx\, dy = -3 \text{ Área}(R)$$



$$\text{Área}(R) = A(R\cup\tilde{R}) - \underbrace{A(\tilde{R})}_{\frac{\pi\cdot R^2}{2} = \frac{\pi}{2}}$$

$R\cup\tilde{R})$



en Polares:

$$R\cup\tilde{R} = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq \theta \end{cases}$$

Polares ↓ Jacobiano

$$\iint_{R\cup\tilde{R}} 1\,dx\,dy = \int_0^{2\pi}\int_0^{\theta} r\,dr\,d\theta$$

$$A(R) = \frac{4}{3}\pi^3 - \frac{\pi}{2}$$

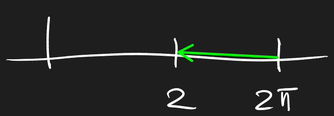
Entonces

$$\iint_{R} (-3)\,dx\,dy = -3\cdot\left(\frac{4}{3}\pi^3 - \frac{\pi}{2}\right)$$

$$= \frac{3}{2}\pi - 4\pi^3$$

Calculo curvilineas

$\mathcal{C}_1$:



$$\sigma(t) = (t, 0) \qquad 2 \leq t \leq 2\pi$$

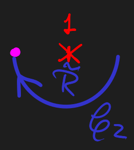
↑ orientación opuesta!

$$\int_{\mathcal{C}_1} F\,ds = -\int_2^{2\pi} \langle F(\sigma(t)), \sigma'(t)\rangle\,dt$$

$$= -\int_{2}^{2\pi}$$

$$= 0$$

$\mathcal{C}_2$:



$$\sigma(t) = (1 + \cos t, \sin t) \qquad \pi \leq t \leq 2\pi$$

↑ orienta en sentido contrario al de $\mathcal{C}_2$

$$\int_{\mathcal{C}_2} F\, d\vec{s} = -\int_{\pi}^{2\pi} \langle F(\sigma(t)), \sigma'(t) \rangle\, dt$$

$$= -\int_{\pi}^{2\pi} \left\langle \left(-\frac{\sin t}{1} + 3\sin t, \frac{\cos t}{1} \right), (-\sin t, \cos t) \right\rangle dt$$

$$= -\int_{\pi}^{2\pi} -2\sin^2 t + \cos^2 t = \ldots = \frac{\pi}{2}$$

Vuelvo

$$\int_{\mathcal{C}} F\, ds + \frac{\pi}{2} + 0 = -4\pi^3 + \frac{3}{2}\pi$$

Podría separar el campo F en dos !