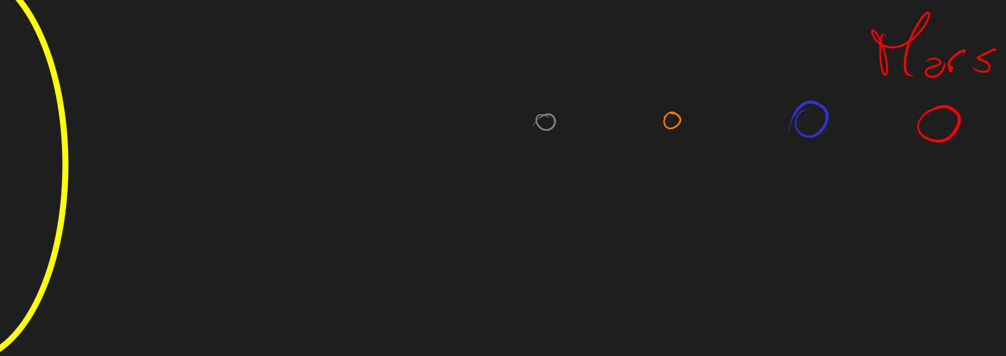
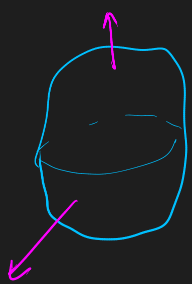
Feb 23/2021



Teorema de Gauss

$\Omega \subset \mathbb{R}^3$ tipo IV

"Burbujoide"



$S = \partial \Omega$

S superficie cerrada

orientada η_{ext}

$F \in C^1(\Omega)$ (revisar Green)

C^1 en todo Ω!

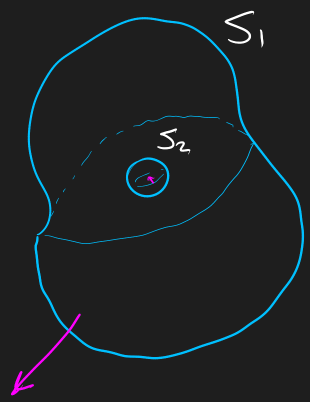
$$\int_{\Omega} \operatorname{div} F \, dV = \int_{\partial \Omega_{ext}} F \cdot \eta_{ext} \, dS$$

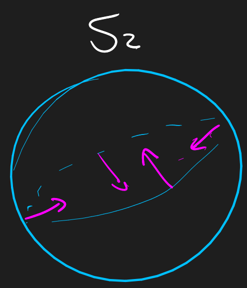
triple! dado por el dominio

$$= \int_{\partial \Omega_{ext}} F \, d\vec{S}$$

Extensiones / observaciones

1) Regiones con huecos





$$\partial\Omega = S_1 \cup S_2$$

S_1: hacia afuera de sí misma

S_2: hacia adentro de sí misma

$$\int_{\Omega} \operatorname{div} F \, dV = \int_{S_1^{ext}} F \, d\vec{S} + \int_{S_2^{int}} F \, d\vec{S}$$

Notar:

$$\partial\Omega = S_1 \cup S_2$$

cuando la oriento:

$$(\partial\Omega)^{ext} = S_1^{ext} \cup S_2^{int}$$

Obs:

$\Omega \subset \mathbb{R}^3$

$\partial\Omega = S$ cerrada, η ext

$F \in C^2(\Omega)$

$$\int_S \nabla \times F \, d\vec{S} = \int_\Omega \mathrm{div}(\nabla \times F)\, dV = 0$$

Puedo separarla en dos cáscaras con borde

S_2 S_1

Stokes

$$\int_{\mathcal{C}^-} F\, d\vec{s} + \int_{\mathcal{C}} F\, d\vec{s} = 0$$

$$\mathcal{C}^- = \partial S_1 \qquad \mathcal{C} = \partial S_2$$

Divergencia, fuentes y sumideros

Interpretar $\mathrm{div}\, F(p)$, $p \in \mathbb{R}^3$

"campo de velocidades"

$F : \mathbb{R}^3 \to \mathbb{R}^3$

$\mathrm{div}\, F : \mathbb{R}^3 \to \mathbb{R}$

$p \in \mathbb{R}^3$

$B_r(p)$: bola centro p, radio $r>0$

Lo pienso como velocidad

$$\int_{B_r(p)} \operatorname{div} F(p)\, dV = \int_{\partial B_r(p)_{ext}} F\, d\vec{S}$$

$$\frac{1}{|B_r(p)|} \cdot \int_{B_r(p)} \operatorname{div} F(p)\, dV = \frac{1}{|B_r(p)|} \cdot \int_{\partial B_r(p)_{ext}} F\, d\vec{S}$$

$|B_r(p)|$ = volumen (3D) Bola

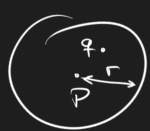
Por Teorema de Valor integral :

$$\operatorname{div} F(q) = \frac{1}{|B_r(p)|} \cdot \int_{B_r(p)} \operatorname{div} F\, dv$$

$$= \frac{1}{|B_r(p)|} \cdot \int_{\partial B_r(p)_{ex}} F\, d\vec{S}$$

Promedio del flujo saliente a través de la superficie :

$q \in B_r(p)$



$$\operatorname{div} F(p) = \lim_{r \to 0} \operatorname{div} F(q)$$

$$= \lim_{r \to 0} \frac{1}{|B_r(p)|} \int F \, d\vec{s}$$

Ley de Gauss

$\vec{r} : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \vec{r}(x,y,z) = (x,y,z)$

$r : \mathbb{R}^3 \to \mathbb{R} \qquad r(x,y,z) = \|(x,y,z)\|_2$

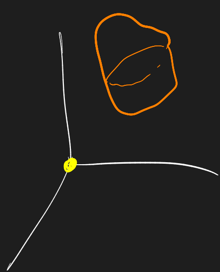
$F : \mathbb{R}^3 \setminus \vec{0} \to \mathbb{R}$

$$F(x,y,z) = \frac{\vec{r}}{r^3}$$

$\Omega \subset \mathbb{R}^3$ tipo $\underline{\text{IV}}$ / $\vec{0} \notin \partial\Omega$

A

B



C si $\vec{0}$ está en la cáscara no puedo hacer nada

$$\int_{\partial\Omega_{ext}} F \, d\vec{s} = \int_{\partial\Omega_{ext}} \left\langle \frac{\vec{r}}{r^3}, \eta \right\rangle ds = \begin{cases} 4\pi & \text{si engloba el origen} \\ 0 & \text{si no} \end{cases}$$

$\operatorname{div} F = 0$ (cuentas)

(B) $$\int_{\partial\Omega} F\, d\vec{S} = \int_{\Omega} \operatorname{div} F\, dV = 0$$

(A) $B_\varepsilon(\vec{0})$, $\varepsilon > 0$

$$\int_{\partial\Omega_{ext}} F\, d\vec{S} + \int_{\partial B_\varepsilon(0)_{int}} F\, d\vec{S} = 0$$

Recordar!

$$\int_{\partial\Omega_{ext}} F\, d\vec{S} = \int_{\partial B_\varepsilon(0)_{ext}} F\, dS$$

η a la esfera

$$= \int_{\partial B_\varepsilon(0)} \left\langle \frac{\vec{r}}{r^3}, \frac{\vec{r}}{r} \right\rangle dS$$

$$= \int_{\partial B_\varepsilon(0)} \frac{1}{r^4} \cdot r^2\, dS$$

$r = \varepsilon$

$$= \frac{1}{\varepsilon^2} \int_{\partial B_\varepsilon(0)} 1 \cdot dS$$

$$= \frac{1}{\varepsilon^2} \cdot 4\pi \cdot r^2$$

$$= \frac{\cancel{\varepsilon^2}}{\cancel{\varepsilon^2}} \cdot 4\pi$$

$$= 4\pi \quad //$$