

Prácticas #4

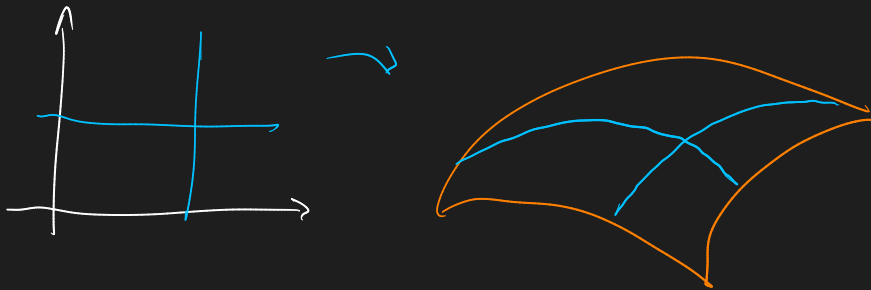
Jupiter 04/02/21

Superficies Parametrizadas

Superficie $S \subseteq \mathbb{R}^3$ es un conjunto /

$\exists T: D \subseteq \mathbb{R}^2 \rightarrow S$ continua
y sobreyectiva

Preguntar: Puede una curva ocupar todo $D \subseteq \mathbb{R}^2$?



$$T(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Ejemplos:

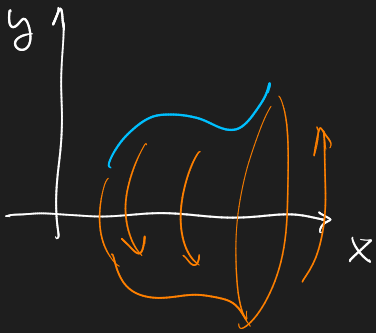
1) $D \subseteq \mathbb{R}$, $f: D \rightarrow \mathbb{R}$

$$T: D \rightarrow S$$

$$T(x, y) = (x, y, f(x, y))$$

Superficie de Revolución

$y = f(x)$, rotamos en eje x



Dos parámetros : x identifica el punto en el gráfico
 θ identifica el ángulo de giro

$$T(x, \theta) = \left(\underset{\substack{\uparrow \\ \text{fijo}}}{x}, \underbrace{f(x) \cos \theta, f(x) \sin \theta}_{\substack{\text{circunferencia de radio } f(x) \\ \rightarrow \theta}} \right)$$

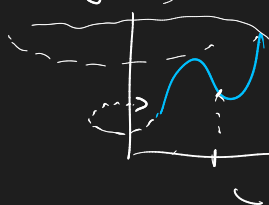
$$x \in [a, b]$$

$$\theta \in [0, 2\pi)$$

$$2) f: [a, b] \rightarrow \mathbb{R}$$

$$y = f(x)$$

rotamos en el eje y
 $y = f(x)$

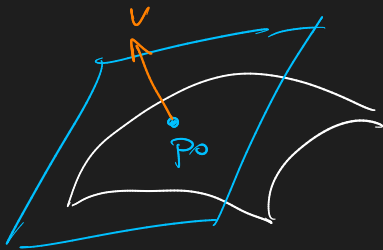


Dos parámetros
 x y θ

solo el x me determina una única $f(x)$

$$T(x, \theta) = \left(\underset{\substack{\uparrow \\ \text{radio}}}{x} \cdot \cos \theta, f(x), x \cdot \sin x \right)$$

Superficies y tangentes



Def. geom. de vector normal en la
teoría de hoy.

Si $T: D \rightarrow S$ es regular $\left(\begin{array}{l} T \text{ dif} \\ T_u, T_v \neq \vec{0} \\ T_u \times T_v \end{array} \right)$

$$T_u(u, v) = \left(\frac{\partial X}{\partial u}(u, v), \frac{\partial Y}{\partial u}(u, v), \frac{\partial Z}{\partial u}(u, v) \right)$$

$$T_v(u, v) = \left(\frac{\partial X}{\partial v}(u, v), \frac{\partial Y}{\partial v}(u, v), \frac{\partial Z}{\partial v}(u, v) \right)$$

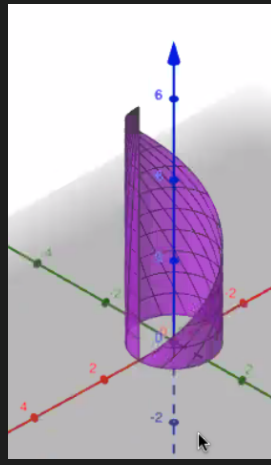
Vector normal en $p_0 \in S$

$$p_0 = T(u_0, v_0)$$

$$\eta(p_0) = T_u \times T_v$$

$$= \det \begin{bmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix}$$

Ej. Escalera Caracol



Dos parámetros

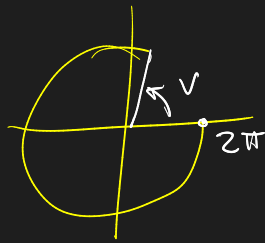
v : describe la altura

$$v \in [0, 2\pi]$$

μ : describe la curva a altura v

↙ normalizo con $2\pi - v$

$$\mu \in [0, 1]$$



$$T : [0, 1) \times [0, 2\pi) \rightarrow S$$

Armo:

$$T(\mu, v) = (\cos(v + (2\pi - v)\mu), \sin(v + (2\pi - v)\mu), v)$$

evaluó

$$T(\mu, 0) = (\cos 2\pi\mu, \sin 2\pi\mu, 0) \quad \swarrow \text{circ. en } z=0$$

$$T(\mu, 2\pi) = (1, 0, 2\pi) \quad \swarrow \text{"pico" de la cima.}$$

↙ se degenera : Pierde la propiedad de
"Longitud Positiva"

Parametrizando con cambios de coord.

Cilíndricas:

$$x = r \cdot \cos \theta$$

$$y = r \cdot \sin \theta$$

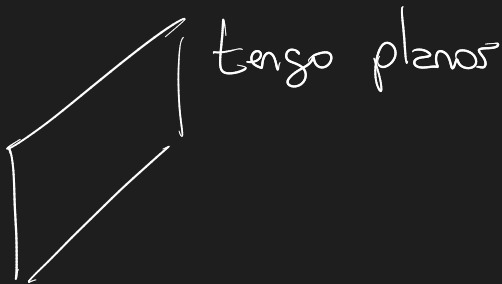
$$z = z$$

Si fijamos $r = R > 0$, tenemos

$$T(\theta, z) = (R \cdot \cos \theta, R \cdot \sin \theta, z)$$

tengo cilindros

Si fijamos θ



Esféricas

Fijamos algún $R \geq 0$

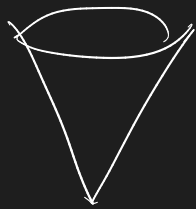
$$T(\theta, \varphi) = (R \cdot \cos \theta \cdot \sin \varphi, R \cdot \sin \theta \cdot \sin \varphi, R \cdot \cos \varphi)$$

$$\theta \in [0, 2\pi]$$

$$\varphi \in [0, \pi]$$

Fijamos $\varphi = \varphi_0$

$$T(r, \theta) = \begin{pmatrix} r \cdot \cos \theta \cdot \sin \varphi_0, & r \cdot \sin \theta \cdot \sin \varphi_0, \\ & r \cdot \cos \varphi_0 \end{pmatrix}$$



cono

$$x^2 + y^2 = a \cdot z^2$$

si fijamos θ :

(semi) plano vertical en dirección θ .

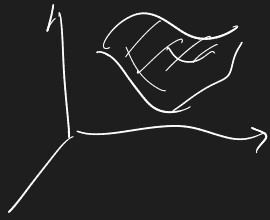
Área de una superficie

$$S \subset \mathbb{R}^3, T: D \rightarrow S \quad \text{para un}$$

$$A(S) = \int_D \|T_u \times T_v\| \, du \, dv$$

Ejemplos

$$1) S = \text{Gr} f(f) \quad , \quad f: D \rightarrow \mathbb{R}$$



$$T_x(x, y) =$$

$$T_y(x, y) =$$

$$\|T_x \times T_y\| = \sqrt{1 + f_x^2 + f_y^2}$$

$$A(\text{Gr} f(f)) = \int_D \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$$

2) Área de la esfera de Radio $R \geq 0$



$$f: D \rightarrow \mathbb{R}$$

$$f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

$$f_x(x, y) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$f_y(x, y) = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

$$\begin{aligned}
 \|T_x \times T_y\| &= \left(\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1 \right)^{1/2} \\
 &= \left(\frac{R^2}{R^2 - x^2 - y^2} \right)^{1/2} \\
 &= \frac{R}{\sqrt{R^2 - x^2 - y^2}}
 \end{aligned}$$

• Calculamos

$$\begin{aligned}
 \int_D \frac{R}{\sqrt{R^2 - (x^2 + y^2)}} dx dy &= \\
 &\stackrel{\text{polares}}{=} R \cdot \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{R^2 - r^2}} dr d\theta \\
 &= R \cdot 2\pi \int_0^R \frac{r}{\sqrt{R^2 - r^2}} \\
 &\quad \left(\begin{array}{l} u = R^2 - r^2 \\ du = -2r dr \end{array} \right) \\
 &= -R \cdot \pi \int_{R^2}^0 \frac{1}{\sqrt{u}}
 \end{aligned}$$

$$= R \cdot \pi \left(2\sqrt{u} \right) \Big|_0^{R^2}$$

$$= 2\pi R^2 = \text{Área de } \frac{1}{2} \text{ esfera}$$

$$\text{Área de la esfera} = \overset{1/2 \times}{4\pi} \cdot R^2 //$$

• Mismo ejercicio con esféricas

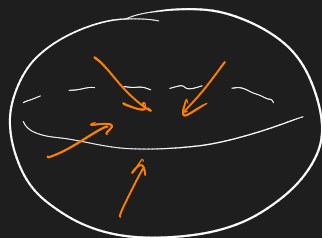
$$T(\theta, \varphi) = (R \cdot \cos \theta \cdot \sin \varphi, R \cdot \sin \theta \cdot \sin \varphi, R \cdot \cos \varphi)$$

$$T_\theta = (-R \cdot \sin \theta \cdot \sin \varphi, \dots)$$

$$T_\varphi = (\dots)$$

$$T_\theta \times T_\varphi = \underbrace{-R^2 \sin \varphi}_{\in \mathbb{R}} \cdot (\cos \theta \cdot \sin \varphi, \sin \theta \cdot \sin \varphi, \cos \varphi)$$

$$= \underbrace{-R \sin \varphi}_{\in \mathbb{R}} \cdot \underbrace{(R \cdot \cos \theta \cdot \sin \varphi, R \cdot \sin \theta \cdot \sin \varphi, R \cdot \cos \varphi)}_{\text{punto en esfera de radio } R}$$



• So

$$\|T_\theta \times T_\varphi\| = R^2 \cdot \sin \varphi$$

$$A(s) = \int_0^{2\pi} \int_0^\pi R^2 \cdot \sin \varphi \cdot d\theta d\varphi$$

$$= 2\pi R^2 \int_0^\pi \sin \varphi \cdot d\varphi$$

$$= 4\pi R^2 //$$

