Teorema de Green

Tenemos:

$$\Omega \subset \mathbb{R}^2$$

$$F: \Omega \to \mathbb{R}^2 \text{ de Clase } C^1(\Omega)$$

$$R \subset \Omega \text{ región tipo III}$$

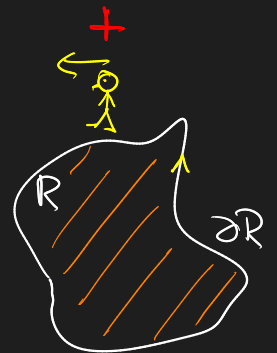de borde ∂R suave, dif a trozos

Teo: Si $C = \partial R^+$

está orientado **positivamente** (curva cerrada simple)

y $F(x,y) = \left(P_{(x,y)}, Q_{(x,y)} \right)$

entonces:

$$\iint\limits_{R} Q_x - P_y \, dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy$$

($\mathcal{C}$ es el borde de R (∂R)
orientado positivamente)

Obs:

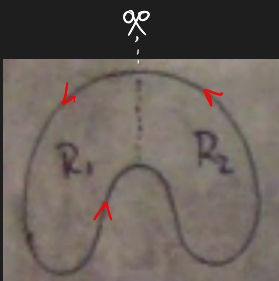Si cambio $\mathcal{C}$ por $\mathcal{C}^-$, ← contrario

cambia el signo

$$\iint\limits_{R} Q_x - P_y \, dx\,dy = - \int_{\mathcal{C}} P\,dx + Q\,dy$$

Obs:
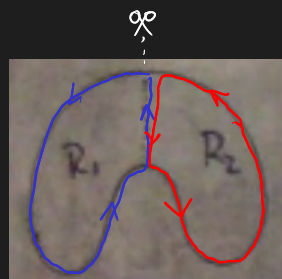
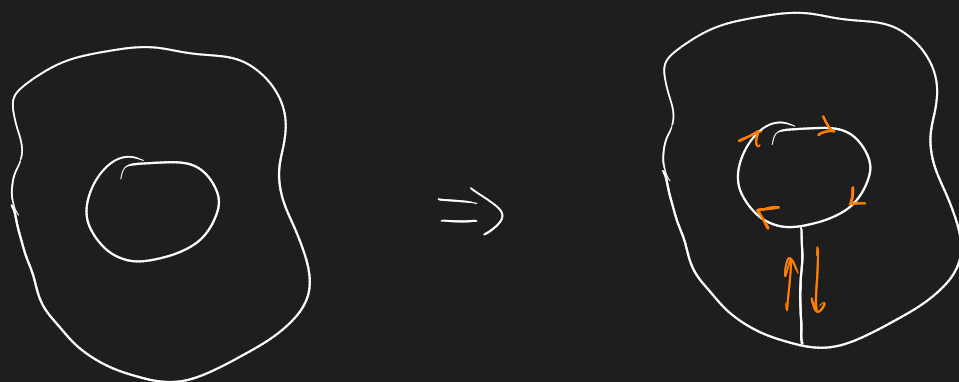Vale para regiones más generales (incluso con agujeros)

ej

$R = R_1 \cup R_2$

$$\iint\limits_{R} Q_x - P_y \, dx\,dy = \int_{\partial R^+} P\,dx + Q\,dy$$

$$\iint_{R_1} \boxed{} + \iint_{R_2} \boxed{} = \int_{\partial R_1^+} F \cdot d\vec{s} + \int_{\partial R_2^+} F \cdot d\vec{s}$$
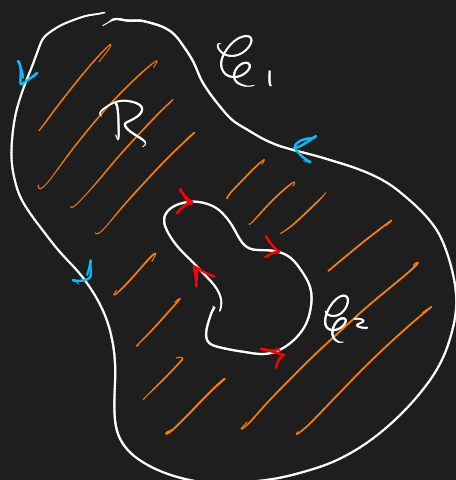

Se cancelan las rectas

Otro ejemplo

Si R tiene uno o varios agujeros

$$\partial R = \ell_1 \cup \ell_2$$

Green nos dice que:

$$\iint_R Q_x - P_y \, dx \, dy = \int_{\mathcal{E}_1} P \, dx + Q \, dy + \int_{\mathcal{E}_1} P \, dx + Q \, dy$$

$\mathcal{E}_1$
(orientada
antihorario)

$\mathcal{E}_1$
(orientada
horario)

Aplicaciones / ejemplos

$$F(x,y) = \frac{1}{2}(-y, x)$$

$$Q_x - P_y = \frac{1}{2}(1 - (-1)) = 1$$

una constante

Green dice

$$\text{Área}(R) = \iint_R Q_x - P_y \, dx = \frac{1}{2}\int_{\partial R^+} -y \, dx + x \, dy$$

Obs:

Si la región R tiene ∂R descripto
en polares por $r = r(\theta)$

$$\partial R = Im(\sigma)$$

$$\sigma : [a,b] \to \mathbb{R}^2$$

$$\sigma(\theta) = \left(r(\theta) \cdot \cos\theta, \; r(\theta) \cdot \sin\theta \right)$$

$$\sigma'(\theta) = \left(r' \cdot \cos\theta - r \cdot \sin\theta, \; r' \cdot \sin\theta + r \cdot \cos\theta \right)$$

$$2\, A(R) = \int_{\partial R^+} -y \cdot dx + x \, dy$$

$$= \int_{\partial R^+} (-y, x) \, d\vec{s}$$

$$= \int_a^b \left\langle (-r\cdot\sin\theta, \; r\cdot\cos\theta), \; \left(\begin{matrix} r' \cdot \cos\theta - r\cdot\sin\theta, \\ r' \cdot \sin\theta + r\cdot\cos\theta \end{matrix} \right) \right\rangle d\theta$$

$$= \int -r'\cdot r \cdot \cancel{\sin\theta \cdot \cos\theta} + r^2 \cdot \sin^2\theta + r' \cdot r \cdot \cancel{\sin\theta \cdot \cos\theta} + r^2 \cos\theta \; d\theta$$

$$\Rightarrow A(R) = \frac{1}{2} \int_a^b r(\theta)^2 \, d\theta$$

Notar que en la circunferencia: $r(\theta) = R$

$$A(\text{Circ}) = \frac{1}{2} \int_0^{2\pi} R^2 \, d\theta$$

$$= \pi \cdot R^2 \quad /\!/$$

Práctica : 1° Green directamente y comparar

2° Parte Green para casos que no son (a priori) problemas de Green.

Aplicación
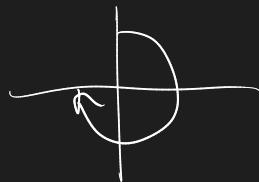
Calculemos integrales curvilíneas peludas usando Green.

Problema :

Sea C la curva dada y orientada por :

$$\sigma : \left[0, \frac{3}{2}\pi \right] \longrightarrow \mathbb{R}^2$$

$$\sigma(t) = \left(\sin t , \cos t \right)$$

Sea F el campo dado por

$$F(x,y) = \left(\underbrace{2 \cdot \cos(x^2 y) \cdot xy}_{Feo} \underbrace{- 2y}_{Simple} , \underbrace{x^2 \cdot \cos(x^2 y)}_{Feo} + \underbrace{3x}_{Simple} \right)$$

Calcular la integral

$$\int_{\mathscr{C}} \vec{F} \cdot d\vec{s} = ?$$ Horrible la composición !

Observemos que

$$Q_x(x,y) = 2x \cdot \cos(x^2 y) - x^2 \cdot \cos(x^2 y) \cdot 2xy + 3$$

$$P_y(x,y) = -2\sin(x^2 \cdot y) \cdot x^2 \cdot xy + 2\cos(x^2 y) \cdot x - 2$$

$$Q_x - P_y = 3 - (-2) = 5$$

Si $\mathscr{C}$ fuera el borde de R

$$\mathscr{C} = \partial R,$$

Green dice que:

$$\iint_R 5 \cdot dx \cdot dy = \int_{\partial R^+} \vec{F} \cdot d\vec{s}$$

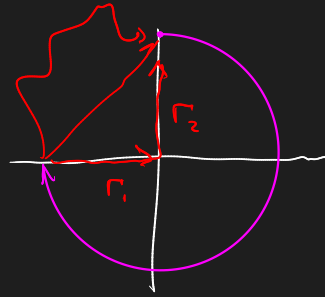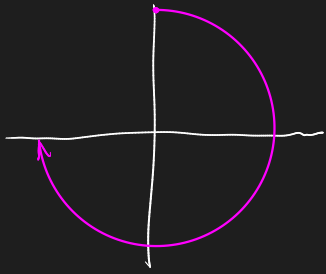Pero la curva <u>no</u> es cerrada!

Solución: la cerramos ☺

$$\mathscr{C} : \quad \sigma : [0, \tfrac{3}{2}\pi] \to \mathbb{R}^2$$

$$\sigma(t) = (\sin t, \cos t)$$

$$\sigma'(t) = (\cos t, -\sin t)$$

Γgamma

Dos trocitos

Γ_1 : σ, $[0,1] \to \mathbb{R}^2$

$\sigma_1(t) = (0, t)$

$\sigma_1'(t) = (0, 1)$

Γ_2 : $\sigma_2 : [-1, 0] \to \mathbb{R}^2$

$\sigma_2(t) = (t, 0)$

$\sigma_2'(t) = (1, 0)$

R = Pacman

$\partial R = \mathcal{C} \cup \Gamma_1 \cup \Gamma_2$

orientada por $\sigma, \sigma_2, \sigma_1$

Green dice

$$\iint_R 5 \, dx \, dy = \int_{\partial R^+} F \cdot d\vec{s}$$

$$-\iint\limits_{R} 5 \cdot dx \cdot dy = \int_{e} F \, d\vec{s} + \int_{\Gamma_1} F \cdot d\vec{s} + \int_{\Gamma_2} F \cdot d\vec{s}$$

↑

sentido de recorrido
deja área a derecha

Incógnita

Calculamos

$$-\iint\limits_{R} 5 \, dx \, dy = -5 \cdot \frac{3}{4}\pi$$

$$\int_{\Gamma_1} P \, dx + Q \, dy = \int_{0}^{1} \left\langle (P(\sigma_1(t)), Q(\sigma_1(t))), (0,1) \right\rangle dt$$

$$P \qquad\qquad Q$$

$$F(x,y) = \left(2 \cdot \cos(x^2 y) \cdot xy - 2y \;,\; x^2 \cdot \cos(x^2 y) + 3x \right)$$

Feo   simple   Feo   simple

$$= \int_{0}^{1} 0 \, dt = 0$$

$$\int_{\Gamma_2} P \, dx + Q \, dy = \int_{0}^{1} \left\langle (P(\sigma_2(t)), Q(\sigma_2(t))), (1,0) \right\rangle dt$$

$$= \int_{-1}^{0} 0 \, dt = 0$$

Junto todo

$$-\iint_R 5 \cdot dx \cdot dy = \underbrace{\int_{\mathscr{C}} F \, d\vec{s}}_{\text{Incógnita}} + \underbrace{\int_{\Gamma_1} F \cdot d\vec{s}}_{=0} + \underbrace{\int_{\Gamma_2} F \cdot d\vec{s}}_{=0}$$

$$\underbrace{}_{-\frac{15}{4}\pi}$$

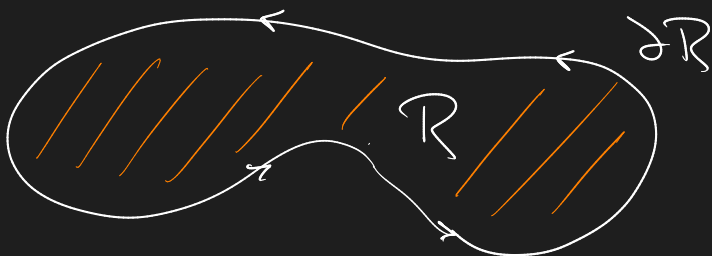$$\Rightarrow \quad \int_{\mathscr{C}} F \cdot d\vec{s} = -\frac{15}{4}\pi$$

Obs :

Sea $F = \nabla f \qquad f \in C^2$

$$f : \mathbb{R} \to \mathbb{R}$$
$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$F(x,y) = (f_x, f_y)$$

Sea $\mathscr{C}$ una curva cerrada simple

Por Green :

$$\iint_R Q_x - P_y \, dxdy = \int_C F \cdot d\vec{s} = \int_C \nabla f = 0$$

$$\|$$

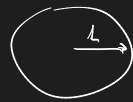$$\iint_R \underbrace{f_{yx} - f_{xy}}_{\text{iguales por } C^2} dxdy = 0$$

Considerar:

$$F(x,y) = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right) = (P, Q)$$

Verificar que

$$Q_x - P_y = 0 \qquad \forall (x,y) \in \mathbb{R}^2 , \quad (x,y) \neq (0,0)$$

Considerar $C = \partial B(0,1)$ orientada positivamente

$$C = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}$$

Si valiera Green:

$$0 = \iint_{B(0,1)} Q_x - P_y = \int_C F \, d\vec{s}$$

Sin embargo:

$$\sigma(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \int_0^{2\pi} \langle (-\sin t, \cos t), (-\sin t, \cos t) \rangle \, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

Si vale Green:

$$0 = \underbrace{\iint_{B(0,1)} Q_x - P_y}_{\text{esto vale}} \;\cancel{=}\; \underbrace{\int_{\mathcal{C}} F\, d\vec{s}}_{\text{esto vale}} = 2\pi$$

¿dónde falla esta igualdad?

Obs:

• Si $F = \nabla f \Rightarrow \displaystyle\int_{\mathcal{C}} F\, d\vec{s} = 0 \quad \forall\, \mathcal{C}$ cerrada

• Si $F = \nabla f = (P, Q)$

$$\Rightarrow Q_x - P_y \equiv 0$$

$$\Rightarrow \text{(Green)} \int_{\mathscr{C}} \vec{F} \, d\vec{s} = 0$$

○ S: $\vec{F} = (P, Q)$

y $\quad Q_x - P_y \equiv 0 \quad$ en Ω

$$\Rightarrow \vec{F} = \nabla f \text{ en } \Omega \text{ ??}$$

$$\underline{\text{No}} \, !$$