Mar 09/02/21

Teorema de Green

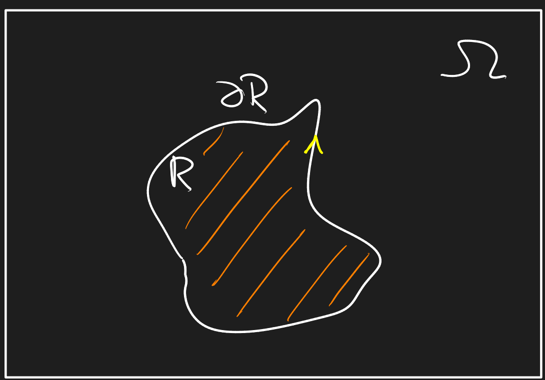
Tenemos :

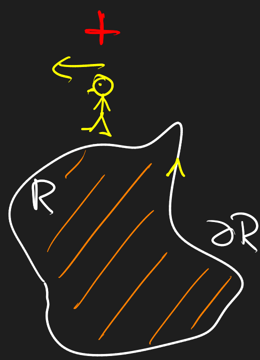
$\Omega \subset \mathbb{R}^2$

$F : \Omega \to \mathbb{R}^2$ de Clase $C^1(\Omega)$

$R \subset \Omega$ región tipo III

de borde ∂R suave, dif a trozos





Teo : Si $\mathcal{C} = \partial R^+$

está orientado positivamente

(curva cerrada simple)

y $F(x,y) = \big(P(x,y), Q(x,y)\big)$

entonces :

$$\iint_R Q_x - P_y \, dx\,dy = \int_{\mathcal{C}} P\,dx + Q\,dy$$

($\mathcal{C}$ es el borde de R (∂R) orientado positivamente)

Obs:

Si cambio $\mathcal{C}$ por $\mathcal{C}^-$ (contrario),

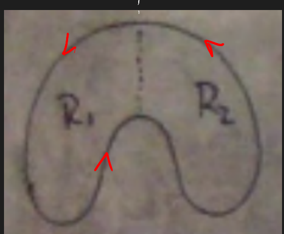
cambia el signo

$$\iint_R Q_x - P_y \, dx\,dy = -\int_{\mathcal{C}} P\,dx + Q\,dy$$

Obs:

Vale para regiones más generales (incluso con agujeros)

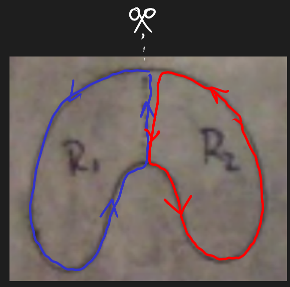
ej



$R = R_1 \cup R_2$

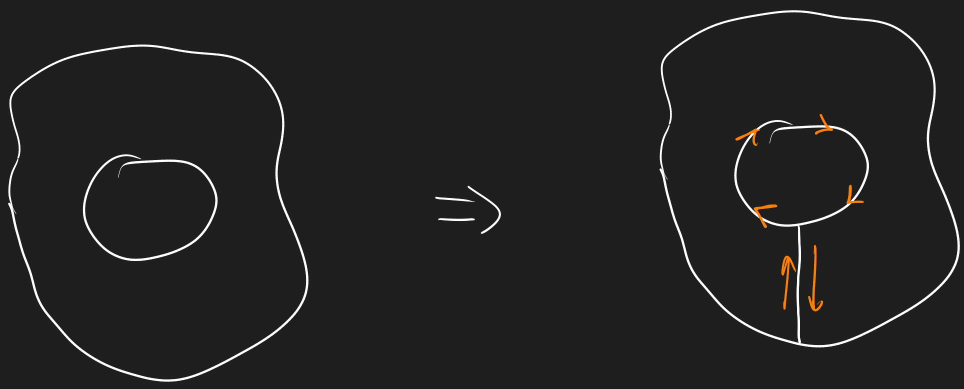
$$\iint_R Q_x - P_y \, dx\,dy = \int_{\partial R^+} P\,dx + Q\,dy$$

$$\overbrace{\iint_{R_1} \boxed{} + \iint_{R_2} \boxed{}} = \int_{\partial R_1^+} F \cdot d\vec{s} + \int_{\partial R_2^+} F \cdot d\vec{s}$$

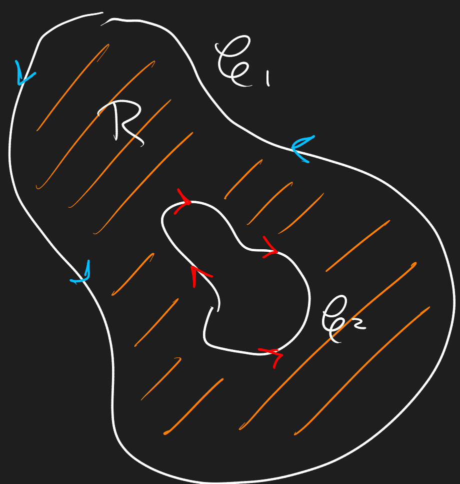


Se cancelan las rectas

Otro ejemplo



Si R tiene uno o varios agujeros



$$\partial R = \mathcal{C}_1 \cup \mathcal{C}_2$$

Green nos dice que:

$$\iint_R Q_x - P_y \, dx\, dy = \int_{\mathcal{C}_1} P\,dx + Q\,dy + \int_{\mathcal{C}_1} P\,dx + Q\,dy$$

(orientada antihorario) (orientada horario)

Aplicaciones / ejemplos

$$F(x,y) = \frac{1}{2}(-y, x)$$

una constante

$$Q_x - P_y = \frac{1}{2}\left(1-(-1)\right) = 1$$

Green dice

$$\text{Área}(R) = \iint_R Q_x - P_y \, dx = \int_{\partial R^+} -y\,dx + x\,dy$$

Obs:

Si la región R tiene ∂R descripto en polares por $r = r(\theta)$

$$\partial R = \operatorname{Im}(\sigma)$$

$$\sigma : [a,b] \to \mathbb{R}^2$$

$$\sigma(\theta) = \left(r(\theta) \cdot \cos\theta \,,\; r(\theta) \cdot \sin\theta \right)$$

$$\sigma'(\theta) = \left(r' \cdot \cos\theta - r \cdot \sin\theta \,, \right.$$
$$\left. r' \cdot \sin\theta + r \cdot \cos\theta \right)$$

$$2\,A(R) = \int_{\partial R^+} -y \cdot dx + x\,dy$$

$$= \int_{\partial R^+} (-y, x)\, d\vec{s}$$

$$= \int_a^b \left\langle (-r \cdot \sin\theta \,,\, r \cdot \cos\theta) \,,\, \begin{pmatrix} r' \cdot \cos\theta - r \cdot \sin\theta \,, \\ r' \cdot \sin\theta + r \cdot \cos\theta \end{pmatrix} \right\rangle d\theta$$

$$= \int -r' \cdot r \cdot \cancel{\sin\theta} \cdot \cos\theta + r^2 \cdot \sin^2\theta + r' \cdot r \cdot \cancel{\sin\theta} \cdot \cos\theta + r^2 \cos\theta \; d\theta$$

$$\Rightarrow A(R) = \frac{1}{2} \int_a^b r(\theta)^2 \, d\theta$$

Notar que en la circunferencia: $r(\theta) = R$

$$A(\text{Circ}) = \frac{1}{2} \int_0^{2\pi} R^2 \, d\theta$$

$$= \pi \cdot R^2 \quad //$$

Práctica: 1° Green directamente y comparar

2° Parte Green para casos que no son (a priori) problemas de Green.

Aplicación

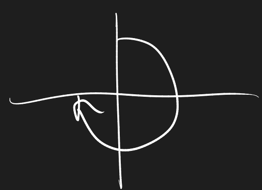
Calculamos integrales curvilíneas peludas usando Green.

Problema:

Sea $\mathcal{C}$ la curva dada y orientada por:

$$\sigma : \left[0, \tfrac{3}{2}\pi\right] \to \mathbb{R}^2$$

$$\sigma(t) = (\sin t, \cos t)$$



Sea F el campo dado por

$$F(x,y) = \big(\underbrace{2 \cdot \cos(x^2 y) \cdot xy}_{\text{Feo}} \underbrace{- 2y}_{\text{simple}},\ \underbrace{x^2 \cdot \cos(x^2 y)}_{\text{Feo}} \underbrace{+ 3x}_{\text{Simple}}\big)$$

Calcular la integral

$$\int_{\mathcal{C}} F \cdot d\vec{s} = \text{?}$$ Horrible la composición!

Observemos que

$$Q_x(x,y) = 2x \cdot \cos(x^2 y) - x^2 \cdot \cos(x^2 y) \cdot 2xy + 3$$

$$P_y(x,y) = -2\sin(x^2 \cdot y) \cdot x^2 \cdot xy + 2\cos(x^2 y) \cdot x - 2$$

$$Q_x - P_y = 3 - (-2) = 5$$

- Si $\mathcal{C}$ fuera el borde de R

$\mathcal{C} = \partial R$,

Green dice que:

$$\iint_R 5 \, dx \, dy = \int_{\partial R^+} F \cdot d\vec{s}$$

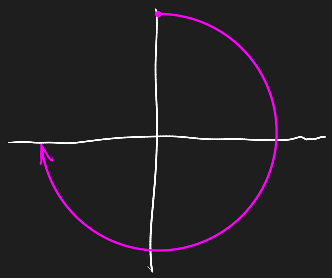
Pero la curva no es cerrada!

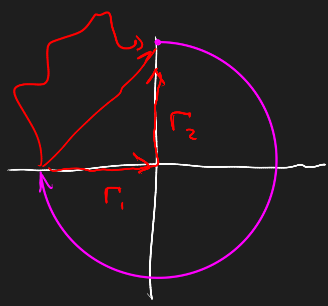
Solución: la cerramos :)

$$\mathcal{C}: \quad \sigma: [0, \tfrac{3}{2}\pi] \to \mathbb{R}^2$$

$$\sigma(t) = (\sin t, \cos t)$$

$$\sigma'(t) = (\cos t, -\sin t)$$





Γ: gamma

Dos trocitos

$$\Gamma_1 : \sigma_1 \, [0,1] \to \mathbb{R}^2$$

$$\sigma_1(t) = (0, t)$$

$$\sigma_1'(t) = (0, 1)$$

$$\Gamma_2 : \sigma_2 : [-1, 0] \to \mathbb{R}^2$$

$$\sigma_2(t) = (t, 0)$$

$$\sigma_2'(t) = (1, 0)$$

R = Pacman

$$\partial R = \mathcal{C} \cup \Gamma_1 \cup \Gamma_2$$

orientada por σ, σ_2, σ_1

Green dice

$$\iint_R 5 \cdot dx \cdot dy = \int_{\partial R^+} F \cdot d\vec{s}$$

$$-\iint_R 5\,dx\,dy = \underbrace{\int_{\mathcal{C}} F\,d\vec{s}}_{\text{Incógnita}} + \int_{\Gamma_1} F\cdot d\vec{s} + \int_{\Gamma_2} F\cdot d\vec{s}$$

sentido de recorrido deja área a derecha

Calculamos

$$-\iint_R 5\,dx\,dy = -5\cdot\frac{3}{4}\pi$$

$$\int_{\Gamma_1} P\,dx + Q\,dy = \int_0^1 \langle (P(\sigma_1(t)), Q(\sigma_1(t))), (0,1)\rangle\, dt$$

$$F(x,y) = \Big(\overbrace{\underbrace{2\cdot\cos(x^2y)\cdot xy}_{\text{Feo}} - \underbrace{2y}_{\text{Simple}}}^{P},\ \overbrace{\underbrace{x^2\cdot\cos(x^2y)}_{\text{Feo}} + \underbrace{3x}_{\text{Simple}}}^{Q}\Big)$$

$$= \int_0^1 0\,dt = 0 \,//$$

$$\int_{\Gamma_2} P\,dx + Q\,dy = \int_0^1 \langle (P(\sigma_2(t)), Q(\sigma_2(t))), (1,0)\rangle\, dt$$

$$= \int_{-1}^{0} 0\,dt = 0 \,//$$

Junto todo

$$-\iint_R 5\,dx\,dy = \int_{\mathcal{C}} F\,d\vec{s} + \int_{\Gamma_1} F\cdot d\vec{s} + \int_{\Gamma_2} F\cdot d\vec{s}$$

$$\underbrace{-\iint_R 5\,dx\,dy}_{-\frac{15}{4}\pi} = \underbrace{\int_{\mathcal{C}} F\,d\vec{s}}_{\text{Incógnita}} + \underbrace{\int_{\Gamma_1} F\cdot d\vec{s}}_{=0} + \underbrace{\int_{\Gamma_2} F\cdot d\vec{s}}_{=0}$$

$$\Rightarrow \int_{\mathcal{C}} F\cdot d\vec{s} = -\frac{15}{4}\pi$$

Obs :

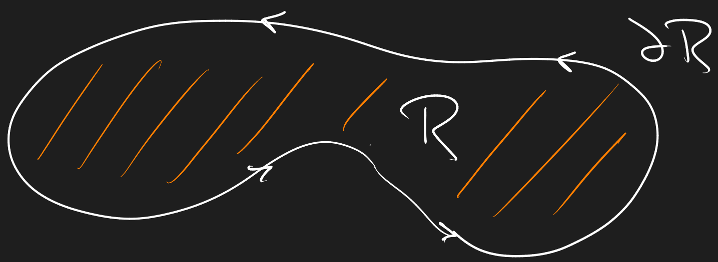
Sea $F = \nabla f \quad f \in C^2$

$f : \mathbb{R} \to \mathbb{R}$

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = (f_x, f_y)$

Sea $\mathcal{C}$ una curva cerrada simple



Por Green :

$$\iint_R Q_x - P_y \, dx\,dy = \int_{\mathcal{C}} F \cdot d\vec{s} = \int_{\mathcal{C}} \nabla f = 0$$

$$\|$$

$$\iint_R \underbrace{f_{yx} - f_{xy}}_{\text{iguales pues } C^2} \, dx\,dy = 0$$

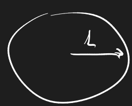
Considerar:

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (P, Q)$$

Verificar que

$$Q_x - P_y = 0 \quad \forall (x,y) \in \mathbb{R}^2, \quad (x,y) \neq (0,0)$$

Considerar $\mathcal{C} = \partial B(0,1)$ orientada positivamente



$$\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$$

Si valiera Green:

$$0 = \iint_{B(0,1)} Q_x - P_y = \int_{\mathcal{C}} F \, d\vec{s}$$

Sin embargo:

$$\sigma(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \int_0^{2\pi} \langle (-\sin t, \cos t), (-\sin t, \cos t) \rangle \, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

Si vale Green:

$$\underbrace{0 = \iint_{B(0,1)} Q_x - P_y}_{\text{esto vale}} \neq \underbrace{\int_{\mathcal{C}} F\, d\vec{s} = 2\pi}_{\text{esto vale}}$$

dónde falla esta igualdad?

Obs:

- Si $F = \nabla f \Rightarrow \int_{\mathcal{C}} F\, d\vec{s} = 0 \quad \forall\, \mathcal{C}$ cerrada
- Si $F = \nabla f = (P, Q)$

$$\Rightarrow Q_x - P_y \equiv 0$$

$$\Rightarrow \text{(Green)} \int_{\mathcal{C}} F\, d\vec{s} = 0$$

- Si: $F = (P, Q)$

  y $Q_x - P_y \equiv 0$ en Ω

  $\Rightarrow F = \nabla f$ en Ω ??

  No!