Feb 15

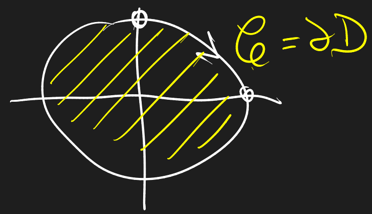
Guia 3 - Teo. de Green

$$\int_{\mathcal{C}} F \cdot ds = \int_{t=a}^{t=b} \left\langle F(\sigma(t)), \sigma'(t) \right\rangle dt$$

$$\mathcal{C} = \partial D$$

$\sigma : [a,b] \to \mathcal{C}$

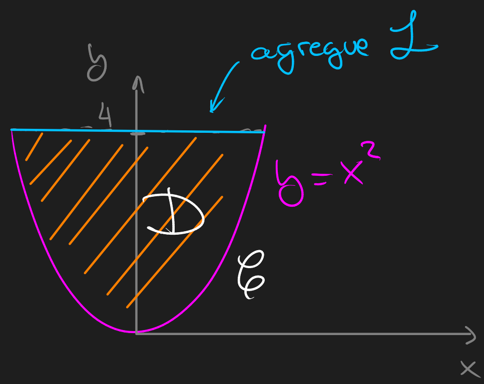
$\sigma(t) = (\underset{1}{\sin t}, \underset{0}{\cos t}) \quad t \in [0, 2\pi]$



$$= \iint_D Q_x - P_y \, dx\, dy$$

donde

$F(x,y) = (Q, P)_{(x,y)}$



$$\int_{\mathcal{C}} F \cdot ds = ?$$

$\sigma(t) = (t, t^2) \quad t \in [-2, 2]$

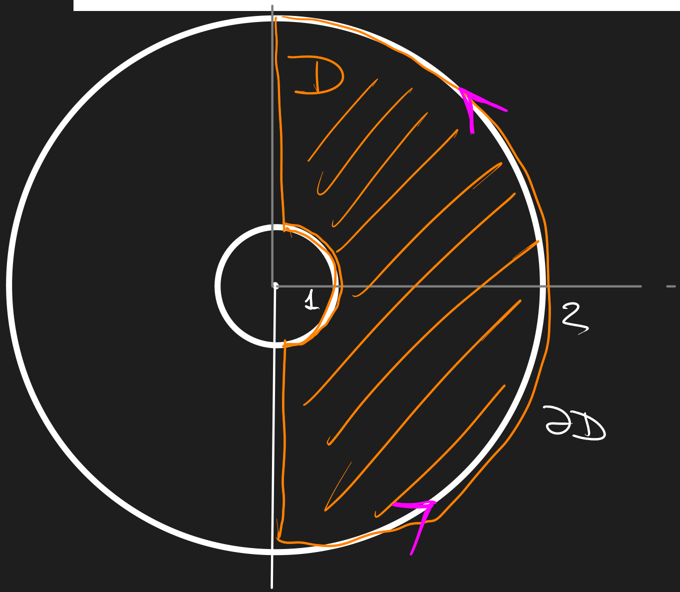
$$\int_{L \cup \mathcal{C}} F ds = \iint_D Q_x - P_y \, dx\, dy$$

$$= \int_{x=-2}^{2} \int_{y=x^2}^{y=4} Q_x - P_y \, dx \, dy$$

Ejercicio 9. Sea $D = \{(x,y) : 1 \le x^2+y^2 \le 4,\ x \ge 0\}$. Calcular

$$\int_{\partial D} x^2 y\, dx - xy^2\, dy.$$

Como siempre, ∂D está recorrido en sentido directo (el contrario a las agujas del reloj).



$$\int_{\partial D} x^2 y\, dx - xy^2\, dy =$$

$$F = (P, Q)$$

$$F(x,y) = (x^2 y, -xy^2) \in C^1$$

$$= \int_{\partial D} F \cdot d\vec{s} =$$

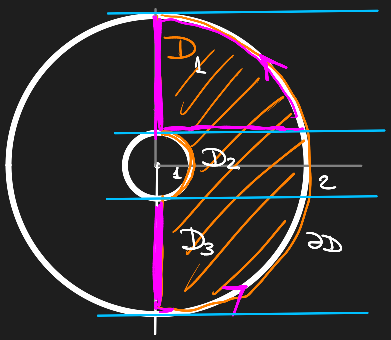
$$\partial D = \mathcal{C}$$

$$= \iint_D \underbrace{Q_x - P_y}_{=?}\, dx\, dy$$

$$F(x,y) = (x^2 y, -xy^2)$$

$$\left.\begin{aligned} Q_x &= -y^2 \\ P_y &= x^2 \end{aligned}\right\} Q_x - P_y = -y^2 - x^2 = -(x^2+y^2)$$

$$= \iint_{D} \underbrace{-(x^2+y^2)}_{-B(x,y)} \, dx\, dy$$



$$= \iint_{D_1} -B(x,y) + \iint_{D_2} -B(x,y) + \iint_{D_3} -B(x,y)\, dxy$$

$$\longmapsto \quad \underbrace{}$$

$x(y) = y$

$y \in [1,2]$

$0 \leq x \leq \sqrt{4-y^2}$

$$\left[\begin{array}{l} x^2+y^2 = 2^2 \\ x^2 = 2^2 - y^2 \\ |x| = \sqrt{4-y^2} \\ x>0 \\ x = \sqrt{4-y^2} \end{array}\right.$$

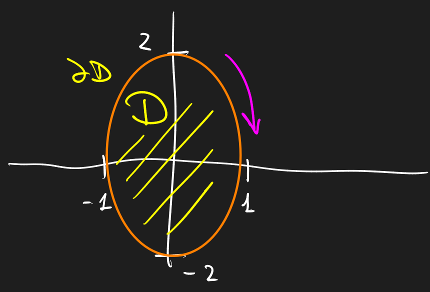
$$\int_{y=1}^{2} \int_{x=0}^{x=\sqrt{4-y^2}} -x^2-y^2 \, dx\, dy =$$

$$= \int_{y=1}^{2} -y^2 \cdot \sqrt{4-y^2} - \left[\frac{x^3}{3} \right]_0^{\sqrt{4-y^2}} dy$$

$$= \int_{y=1}^{2} -y^2 \cdot \sqrt{4-y^2} - \frac{1}{3}(4-y^2)^{3/2} \, dy$$

Ejercicio 10. Calcular el trabajo efectuado por el campo de fuerzas $\mathbf{F}(x, y) = (y + 3x, 2y - x)$ al mover una partícula rodeando una vez la elipse $4x^2 + y^2 = 4$ en el sentido de las agujas del reloj.

$$\frac{x^2}{\underset{r_x}{1^2}} + \frac{y^2}{\underset{r_y^2}{2^2}} = 1$$



$Q_x = -1$

$P_y = 1$

$$\int_{\underset{\partial D}{\mathcal{C}\,=}} F \cdot d\vec{s} = -\iint_D \underbrace{Q_x - P_y}_{\substack{= -1-1 \\ = -2}} \, dx\,dy$$

$$= \iint_D 2 \, dx\,dy$$

$$= 2 \cdot \iint_D 1 \, dx\,dy$$

$$= 2 \cdot \text{Área}(D)$$

$$= 2 \cdot \pi \cdot a \cdot b$$

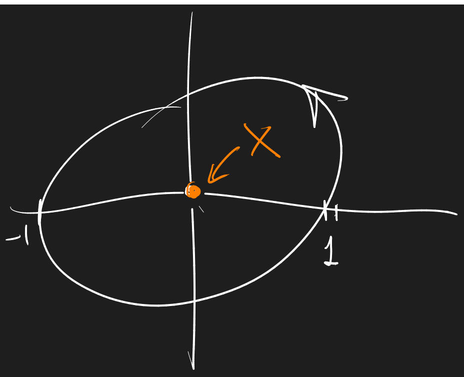
Centroides:

$a = 1$

$b = 2$

$$= 4 \cdot \pi \;//$$

Ejercicio 11. Sea $\mathbf{F}(x,y) = \big(P(x,y), Q(x,y)\big) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$. Calcular $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ donde $\mathcal{C}$ es la circunferencia unitaria centrada en el origen orientada positivamente. Calcular $Q_x - P_y$. ¿Se satisface en este caso el teorema de Green?



$$\int_{\mathcal{C}}$$

$$Q_x = \frac{-1(x^2+y^2) + x\,2x}{(x^2+y^2)^2} = \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$P_y = \frac{1(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q_x - P_y = 0$$

$$Q_x - P_y = 0 \Rightarrow$$

Pero! no es conservativo!

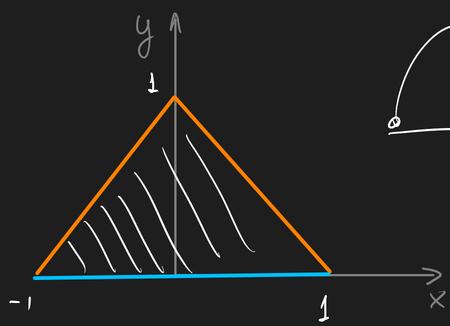
Ejercicio 12. Calcular $\int_{\mathcal{C}} f_1 dx + f_2 dy$ siendo

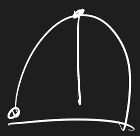
$$f_1(x,y) = \frac{x \operatorname{sen}\left(\dfrac{\pi}{2(x^2+y^2)}\right) - y\,(x^2+y^2)}{(x^2+y^2)^2}, \quad f_2(x,y) = \frac{y \operatorname{sen}\left(\dfrac{\pi}{2\,(x^2+y^2)}\right) + x(x^2+y^2)}{(x^2+y^2)^2},$$

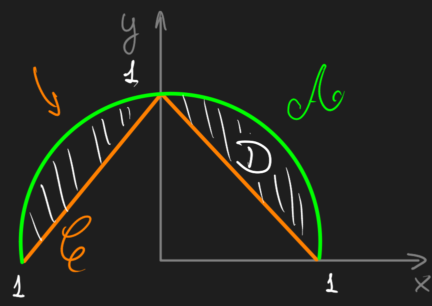
y $\mathcal{C}$ la curva

$$\mathcal{C} = \begin{cases} y = x+1 & \text{si } -1 \le x \le 0, \\ y = 1-x & \text{si } \quad 0 \le x \le 1, \end{cases}$$

recorrida del $(-1,0)$ al $(1,0)$.







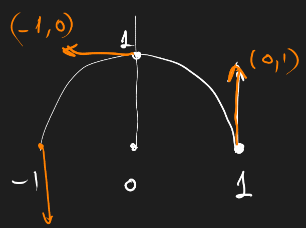
$$\int_{\mathcal{C}} F \cdot ds + \underbrace{\int_{\mathcal{A}} F \cdot ds}_{(x^2+y^2)=1} = \iint_D \underbrace{Q_x - P_y}_{cte = a} \, dxdy$$

$$= a \iint_D 1 \, dxdy$$

$$\int_{\mathcal{A}} \Big\langle (x-y, y+x), (-y, x) \Big\rangle \, dxdy \qquad = a \cdot \text{Área}(D)$$

$$= a \cdot \left(\frac{\pi}{2} - 1 \right)$$

$$\boxed{(x,y) \xrightarrow{\text{tans}} (-y, x)}$$



$$\iint_{\tilde{D}} -xy + y^2 + yx + x^2 \, dxdy = \iint_{\tilde{D}} \underbrace{x^2+y^2}_{=1} \, dxdy = \frac{\pi}{2} \Big]$$

Ejercicio 13. Determinar todas las circunferencias $\mathcal{C}$ en el plano $\mathbb{R}^2$ sobre las cuales vale la igualdad

$$\int_{\mathcal{C}} -y^2\,dx + 3x\,dy = 6\pi.$$

$$F(x,y) = \left(-y^2,\ 3x\right)$$

$$Q_x = 3$$

$$P_y = -2y$$

$$\sigma(r,\theta) = \left(r\cdot\cos\theta,\ r\cdot\sin\theta\right)$$

$$r \in \mathbb{R}$$

$$\theta \in [0, 2\pi]$$

$$Q_x - P_y = 3 + 2y$$

$$\int_{\mathcal{C}} F\cdot ds = \iint_{D} 3 + 2y\ dx\,dy = 6\pi$$