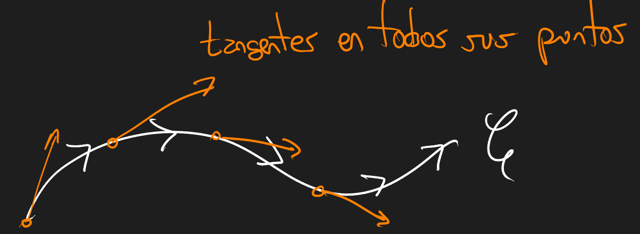
Mercury 03/02/21

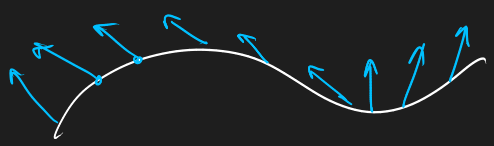
Práctica #3

Integrar Campos Vectoriales sobre Curvas Orientadas

$\mathcal{C} \subset \mathbb{R}^2, \mathbb{R}^3$ curva orientada



Existe $\eta : \mathcal{C} \to \mathbb{R}^2, \mathbb{R}^3$ un campo continuo de vectores tangentes (unitarios) a $\mathcal{C}$.



$F : \mathcal{C} \to \mathbb{R}^2, \mathbb{R}^3$ campo continuo.

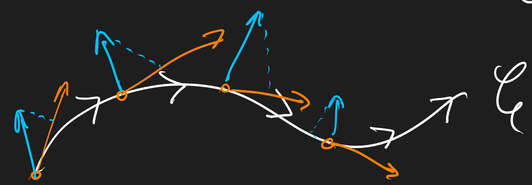
Si $\sigma : [a,b] \to \mathcal{C}$ es una parametrización regular de $\mathcal{C}$ que preserva la orientación,

entonces :

$$\int_{\mathcal{C}} F \cdot d\vec{s} = \int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

notar que

$$= \int_a^b \underbrace{\left\langle F(\sigma(t)), \frac{\sigma'(t)}{\|\sigma'(t)\|} \right\rangle}_{\text{Componente tangencial de } F \text{ en } \sigma(t)} \cdot \|\sigma'(t)\| \, dt$$



Obs: Trivial pero útil

Si $\mathcal{C}$ curva orientada y F un campo

tal que

$$F \perp \mathcal{C} \Rightarrow \int_{\mathcal{C}} F \cdot d\vec{s} = 0$$

si $p \in \mathcal{C}$,

$\eta(p)$ vector tg. a $\mathcal{C}$ en p

$$\Rightarrow \langle F(p), \eta(p) \rangle = 0 \quad \forall p \in \mathcal{C}$$

$$\Rightarrow \int_{\mathcal{C}} F \, d\vec{s} = \int \underbrace{\langle F(\sigma'(t)), \sigma'(t) \rangle}_{=0} = 0$$

Importante

Si $\mathcal{C}$ curva orientada

y $\sigma : [a,b] \to \mathcal{C}$ param. regular que invierte la orientación,

$$\int_{\mathcal{C}} F \, d\vec{s} = - \int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

A veces decimos $\mathcal{C}$ una curva orientada por $\sigma : [a,b] \to \mathcal{C}$.

Ejemplos

$$\mathcal{C} = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{array}{l} y = 1 - x^2, \\ x + y + z = 1, \\ x, y \geq 0 \end{array} \right\}$$

$(|x| \leq 1)$

Parametrización

dibujarlo es difícil

$x + \ldots$

$z = x^2 - x$

$\sigma : [0,1] \to \mathbb{R}$

$\sigma(x) = (x, 1 - x^2, x^2 - x)$

Consideremos $\mathcal{C}$ orientada por σ,

$$F(x,y,z) = (2x,\, y,\, -x)$$

$$\int_{\mathcal{C}} F\, d\vec{s} = \int_0^1 \langle F(\sigma(x)), \sigma'(x) \rangle \, dx$$

$$= \int_0^1 \langle (2x, 1-x^2, -x), (1, -2x, 2x-1) \rangle \, dx$$

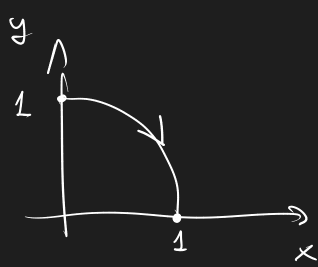
$$= \int_0^1 \cancel{2x} - \cancel{2x} + 2x^3 - 2x^2 + x \; dx$$

$$= \int \dots \text{ hacer}$$

Ejemplos

$\mathcal{C}$ curva orientada como en el dibujo

$$\mathcal{C} = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1,\; x, y \geq 0 \right\}$$



$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

$$F(x,y) = (-y, x)$$

Calcular $\displaystyle\int_{\mathcal{C}} F \, d\vec{s} = ?$

Param:

$$\sigma : [0, \tfrac{\pi}{2}]$$

$$\sigma(t) = (\sin t, \cos t)$$

$$\sigma'(t) = (\cos t, -\sin t)$$

$$\int_{\mathcal{C}} F \, d\vec{s} = \int_0^{\pi/2} \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

$$= \int_0^{\pi/2} \langle (-\cos t, \sin t), (\cos t, -\sin t) \rangle \, dt$$

$$= \frac{\pi}{2} \; //$$

Importante:

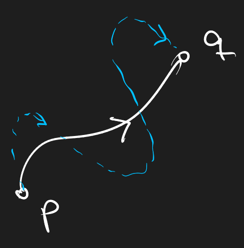
$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

$$F = \nabla f$$

función potencial del campo

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$\mathcal{C}$ curva orientada



$$\int_{\mathcal{C}} F \, d\vec{s} = f(q) - f(p)$$

¡No depende del camino!

Sea $\sigma : [a,b] \to \mathcal{C}$ regular que respeta la orientación

$$\sigma(a) = p$$
$$\sigma(b) = q$$

$$\int_{\mathcal{C}} F \, d\vec{s} = \int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle \, dt$$

$$= \int_a^b \langle \nabla f(\sigma(t)), \sigma'(t) \rangle \, dt$$

$$= \int_a^b (f \circ \sigma)'(t) \, dt$$

$$= (f \circ \sigma)(b) - (f \circ \sigma)(a)$$

$$= f(q) - f(p) \quad \blacksquare$$

Ejercicio

Tiene que ver con la pregunta:

¿Si tengo ∇f, cómo conseguir f?

Sea f tal que

$$\nabla f(x,y) = \left(2x \cdot e^{x^2+y}, e^{x^2+y}\right)$$

$f:\mathbb{R}^2\to\mathbb{R}$

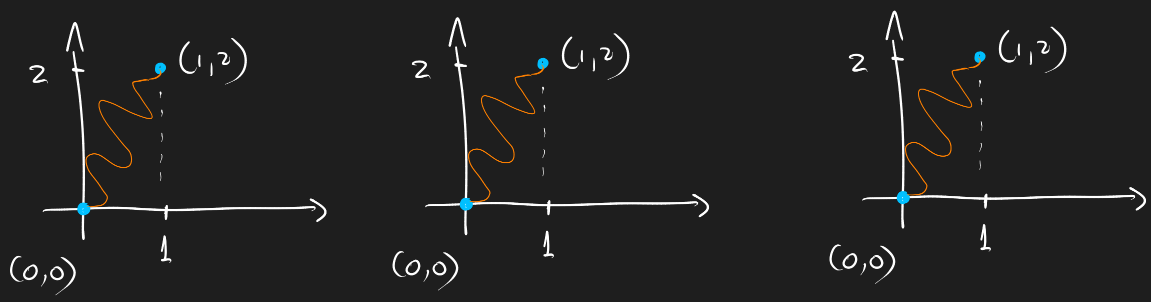
$f(0,0)=1$

Cuanto vale $f(1,2)$?

Idea:

Solo me importan los extremos:

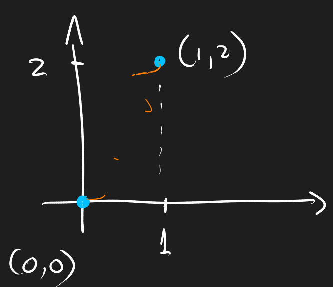
elijo una curva



$$f(1,2)-f(0,0)=\int_{\mathcal{C}}\nabla f\, d\vec{s}$$

cualquier curva

A)



$$\mathcal{C}=\sigma_1\cup\sigma_2$$

$\sigma_1:[0,1]\to\mathbb{R}^2$

$\sigma_1(t)=(t,0)$

$\sigma_1'(t)=(1,0)$

$\sigma_2:[0,2]\to\mathbb{R}^2$

$\sigma_2(t)=(1,t)$

$\sigma_2'(t)=(0,1)$

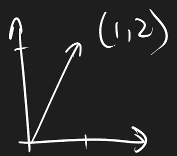
$$\int_{\mathcal{C}} \nabla f \, d\vec{s} = \int_0^1 \left\langle \left(2t \cdot e^{t^2}, e^{1+t}\right) (1,0) \right\rangle +$$

("Mate" words!)

$$+ \int_0^2 \left\langle \left(2t^{1+t}, e^{1+t}\right), (0,1) \right\rangle dt$$

$$=$$

$$= e^3 - 1 \Rightarrow f(1,2) = e^3$$

B) 

$$\sigma : [0,1] \to \mathbb{R}^2$$

$$\sigma(t) = (t, 2t)$$

$$\sigma'(t) = (1, 2)$$

$$\int_{\mathcal{C}} \nabla f \, d\vec{s} =$$

$$= \int_0^2 \left\langle \left(2t \cdot e^{t^2+2t}, e^{t^2+2t}\right) \right\rangle dt$$

$$= \int_0^1 e^{t^2+2t} \cdot (2t+1)$$

$u = t^2 + 2t$

$du = 2t + 2$

$$= \int_0^3 e^u \, du = e^3 - 1$$

$$\Rightarrow \quad f(1,2) = e^3$$