Mars 02/02/21

Práctica #2

Integración sobre curvas en $\mathbb{R}^2$, $\mathbb{R}^3$

Longitud de una curva $\mathcal{C} \subset \mathbb{R}^3$ $(\mathbb{R}^2)$

parametrizada por $\sigma : [a,b] \to \mathcal{C}$ regular
(a trozos).

$$\text{Long}(\mathcal{C}) := \int_a^b \|\sigma'(t)\| . dt$$

¡ No depende de la parametrización !

Parámetro de Longitud de Curva

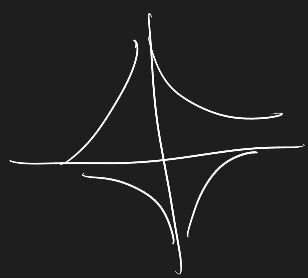
$$S : [a,b] \longrightarrow [0, \text{Long}\,\mathcal{C}]$$

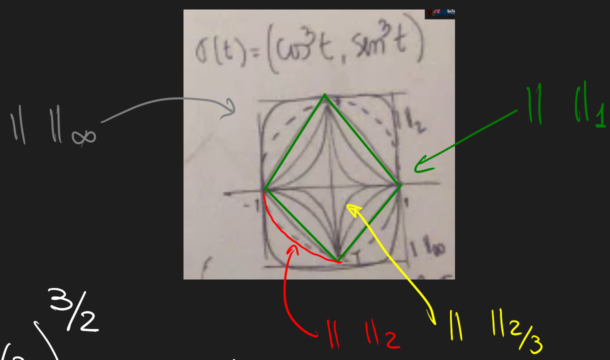
$$s(t) = \int_a^t \|\sigma'(r)\| . dr$$

Ejemplo:

$$\sigma : [0, 2\pi] \to \mathbb{R}^2$$

$$\sigma(t) = (\cos^3 t, \text{sen}^3 t)$$





Recordar

$$\left((\cos^3 t)^{2/3} + (\sin^3 t)^{2/3} \right)^{3/2} = 1$$

Bola unitaria en $\| \ \|_{2/3}$

Ej:

$\sigma'(t) = (-3\cos^2 t \cdot \sin t \ , \ 3\sin^2 t \cdot \cos t$

$\|\sigma'(t)\| = 3\left(\cos^4 \qquad \right)^{1/2}$

$= 3\left(\cos^2 t \cdot \sin^2 t \cdot (\cos^2 t + \sin^2 t) \right)^{1/2}$

$= 3 |\cos t \cdot \sin t|$

$$\text{Long}(\mathcal{C}) = 3 \int_0^{2\pi} |\cos t \cdot \sin t| \, dt$$

divido en $\int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \dots + \int + \int$

$= 3 \cdot 4 \cdot \int_0^{\pi/2} \cos t \sin t$

$=$

$= 6$ // Notar que es mayor que $4 \cdot \sqrt{2}$

Longitud de Curvas en Polares

Para $\theta \in [a,b]$ tenemos la curva dada por $r = r(\theta)$

$$\mathrm{Long}(\mathcal{C}) = \int_a^b \underbrace{\sqrt{r'(\theta)^2 + r(\theta)^2}}_{= \|\sigma'(\theta)\|} \, d\theta$$

Partiendo de

$$\sigma(\theta) = \left(r(\theta) \cdot \cos\theta \, , \, r(\theta) \cdot \sin\theta\right)$$

Cardioide:

$$r(\theta) = 1 + \cos\theta$$

$$r'(\theta) = -\sin\theta \qquad , \quad \theta \in [0, 2\pi]$$

$$L(\mathcal{C}) = \int_0^{2\pi} \left(\sin^2\theta + (1+\cos\theta)^2\right)^{1/2} d\theta$$

$$= \int_0^{2\pi} (2 + 2\cos\theta)^{1/2} \, d\theta$$

$$= \sqrt{2} \cdot \int_0^{2\pi} \sqrt{1+\cos\theta} \, d\theta$$

⊛

Identidad Trig y truquito:

$$\begin{aligned}\cos\theta &= \cos\left(2\cdot\frac{\theta}{2}\right)\\ &= \cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)\\ &= 2\cos^2\left(\frac{\theta}{2}\right)-1\end{aligned}$$

$$\circledast = \left(\sqrt{2}\right)^2 \int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta$$

$$= 2\cdot\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta - 2\int_{\pi}^{2\pi}\cos\frac{\theta}{2}$$

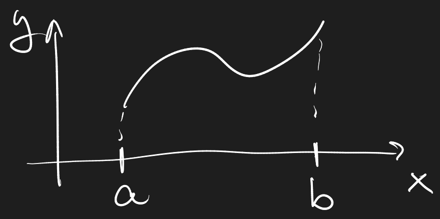
$$= 4\cdot\sin\frac{\theta}{2}\Big|_0^{\pi} - 4\sin\frac{\theta}{2}\Big|^{2\pi}$$

$$= 4+4 = 8 \quad //$$

Un caso particular:

$$\sigma : [a,b] \to \mathbb{R}^2$$

$$\sigma(x) = (x, f(x))$$



$$\text{Long} = \int_a^b \overbrace{\sqrt{1+f'(x)^2}}^{\|\sigma'(x)\|}$$

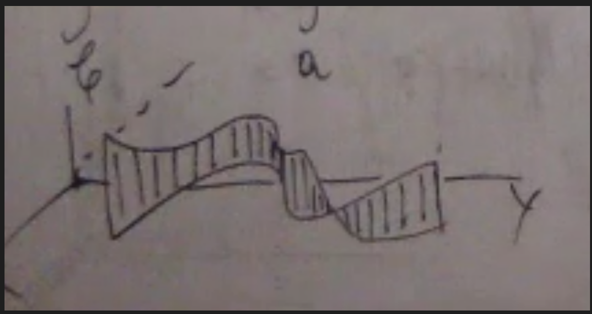
Integración sobre $\mathcal{C}$

$\mathcal{C} \subset \mathbb{R}^2$

$\sigma : [a,b] \to \mathcal{C}$ regular

$f : \mathcal{C} \to \mathbb{R}$ continua

$(f \circ \sigma : [a,b] \to \mathbb{R})$



$$\int_{\mathcal{C}} f \cdot ds = \int_{\mathcal{C}} f(\sigma(t)) \cdot \| \sigma'(t) \| \cdot dt$$

Ejemplo de Centro de Masa

$\mathcal{C}$, $\sigma : [a,b] \to \mathcal{C}$ regular

$\rho : \mathcal{C} \to \mathbb{R}_{\geq 0}$ densidad continua

$$\text{Masa}(\mathcal{C}) = \int_{\mathcal{C}} \rho \, ds$$

Centro de masa $c(\mathcal{C}) = (\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int_{\mathcal{C}} x \, ds}{\int_{\mathcal{C}} \rho \, ds} \qquad \bar{y} = \frac{\int_{\mathcal{C}} y \, ds}{\int_{\mathcal{C}} \rho \, ds} \qquad \bar{z} = \text{idem}$$

$$\mathcal{C} = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{array}{r} x^2 + y^2 + (z-1)^2 = 1, \\ z = 1, \\ y \geq 0 \end{array} \right\}$$

$\sigma : [0, \pi] \to \mathbb{R}^3$

$\sigma(t) = (\cos t, \sin t, 1)$

$$\| \sigma'(t) \| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

Masa :

$$\int_0^\pi \| \sigma'(t) \| \, dt = \pi$$

$$\bar{x} = \frac{\int_0^\pi x \, ds}{\text{Masa}}$$

$$= \frac{\int_0^\pi \cos t \, dt}{\pi} = 0$$

centro en $x=0$

$$\bar{z} = \frac{\int_{\mathcal{C}} z\, ds}{\pi} = \frac{1}{\pi}\int_0^{\pi} 1\, dt$$

$$= 1$$ ← centro en $z = 1$ (trivial)

$$\bar{y} = \frac{1}{\pi}\int_{\mathcal{C}} y\, ds$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin t\, dt$$

$$= \frac{1}{\pi}\left(-\cos t \Big|_0^{\pi}\right)$$

$$= \frac{2}{\pi} < 1$$

centro de masa en $\left(0, \frac{2}{\pi}, 1\right)$