

Mars 09/02/21

# Teorema de Green

Tenemos :

$$\Omega \subset \mathbb{R}^2$$

$$F : \Omega \rightarrow \mathbb{R}^2 \text{ de Clase } C^1(\Omega)$$

$$R \subset \Omega \text{ región tipo III}$$

de borde  $\partial R$

suave,  
dif = trozos



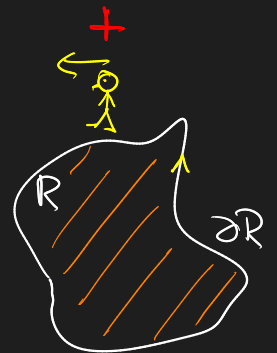
$$\underline{\text{Teo}} : \text{Si } \ell = \partial R^+$$

está orientado **positivamente**

(curva cerrada simple)

$$\text{y } F(x,y) = (P(x,y), Q(x,y))$$

entonces :



$$\iint_R Q_x - P_y \, dx \, dy = \int_{\mathcal{C}} P \, dx + Q \, dy$$

( $\mathcal{C}$  es el borde de  $R$  ( $\partial R$ )  
orientado positivamente)

Obs:

Si cambio  $\mathcal{C}$  por  $\mathcal{C}^-$ ,  
cambia el signo

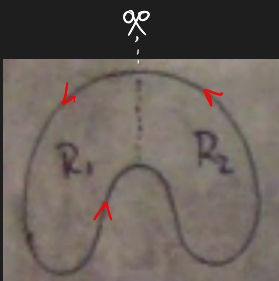
↖ contrario

$$\iint_R Q_x - P_y \, dx \, dy = - \int_{\mathcal{C}} P \, dx + Q \, dy$$

Obs:

Vale para regiones más generales (incluso con agujeros)

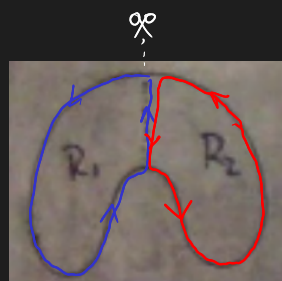
ej



$$R = R_1 \cup R_2$$

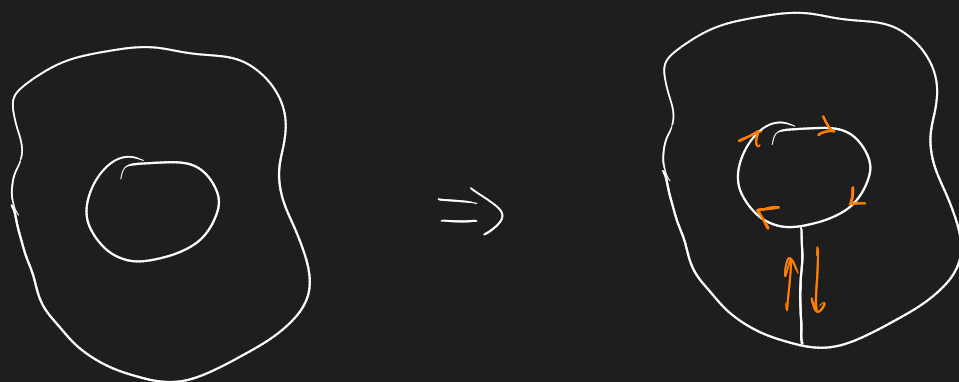
$$\iint_R Q_x - P_y \, dx \, dy = \int_{\partial R^+} P \, dx + Q \, dy$$

$$\overbrace{\iint_{R_1} + \iint_{R_2}} = \int_{\partial R_1^+} \mathbf{F} \cdot d\vec{s} + \int_{\partial R_2^+} \mathbf{F} \cdot d\vec{s}$$

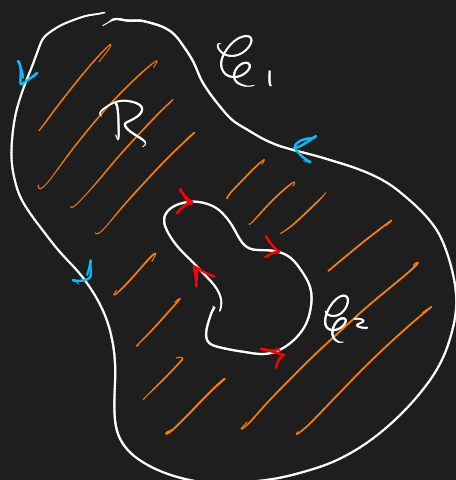


Se cancelan las rectas

Otro ejemplo



Si  $R$  tiene uno o varios agujeros



$$\partial R = C_1 \cup C_2$$

Green nos dice que:

$$\iint_R Q_x - P_y \, dx \, dy = \int_{\mathcal{C}_1} P \, dx + Q \, dy + \int_{\mathcal{C}_2} P \, dx + Q \, dy$$

$\mathcal{C}_1$  (orientada anti horario)       $\mathcal{C}_2$  (orientada horario)

Aplicaciones / ejemplos



$$F(x, y) = \frac{1}{2}(-y, x)$$

$$Q_x - P_y = \frac{1}{2}(1 - (-1)) = 1$$

una constante

Green dice

$$\underline{\text{Área}}(R) = \iint_R Q_x - P_y \, dx \, dy = \frac{1}{2} \int_{\partial R^+} -y \, dx + x \, dy$$

Obs:

Si la región  $R$  tiene  $\partial R$  descrito en polares por  $r = r(\theta)$

$$\partial R = \text{Im}(\sigma)$$

$$\sigma : [a, b] \rightarrow \mathbb{R}^2$$

$$\sigma(\theta) = (r(\theta) \cdot \cos \theta, r(\theta) \cdot \sin \theta)$$

$$\sigma'(\theta) = \begin{pmatrix} r' \cdot \cos \theta - r \cdot \sin \theta, \\ r' \cdot \sin \theta + r \cdot \cos \theta \end{pmatrix}$$

$$2 A(R) = \int_{\partial \mathbb{R}^+} -y \cdot dx + x \cdot dy$$

$$= \int_{\partial \mathbb{R}^+} (-y, x) d\vec{s}$$

$$= \int_a^b \left\langle (-r \cdot \sin \theta, r \cdot \cos \theta), \begin{pmatrix} r' \cdot \cos \theta - r \cdot \sin \theta, \\ r' \cdot \sin \theta + r \cdot \cos \theta \end{pmatrix} \right\rangle d\theta$$

$$= \int -r' \cdot r \cdot \cancel{\sin \theta} \cdot \cos \theta + r^2 \cdot \sin^2 \theta + r' \cdot r \cdot \cancel{\sin \theta} \cdot \cos \theta + r^2 \cos^2 \theta d\theta$$

$$\Rightarrow A(R) = \frac{1}{2} \int_a^b r(\theta)^2 d\theta$$

Notar que en la circunferencia:  $r(\theta) = R$

$$A(\text{Circ}) = \frac{1}{2} \int_0^{2\pi} R^2 d\theta$$

$$= \pi \cdot R^2 //$$

Práctica: 1° Green directamente y comparar

2° Parte Green para casos que no son (a priori) problemas de Green.

## Aplicación

Calcular los integrales curvilineales pedidos usando Green.

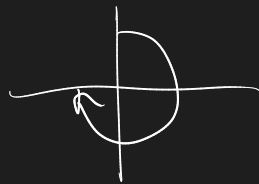
Problema:

Sea  $C$  la curva dada y orientada

por:

$$\sigma: [0, \frac{3}{2}\pi] \rightarrow \mathbb{R}^2$$

$$\sigma(t) = (\sin t, \cos t)$$



Sea  $F$  el campo dado por

$$F(x,y) = \left( \underbrace{2 \cdot \cos(x^2y)}_{F_{xy}} \cdot \underbrace{xy}_{\text{simple}} - \underbrace{2y}_{F_{xy}}, \underbrace{x^2 \cdot \cos(x^2y)}_{F_{xy}} + \underbrace{3x}_{\text{simple}} \right)$$

Calcular la integral

$$\int_C F \cdot d\vec{s} = ? \text{ Horrible la composición!}$$

Observemos que

$$Q_x(x,y) = 2x \cdot \cos(x^2 y) - x^2 \cdot \cos(x^2 y) \cdot 2xy + 3$$

$$P_y(x,y) = -2 \sin(x^2 y) \cdot x^2 \cdot xy + 2 \cos(x^2 y) \cdot x - 2$$

$$Q_x - P_y = 3 - (-2) = 5$$

• Só  $C$  recorre el borde de  $R$

$$C = \partial R,$$

Green dice que:

$$\iint_R 5 \cdot dx \cdot dy = \int_{\partial R^+} F \cdot d\vec{s}$$

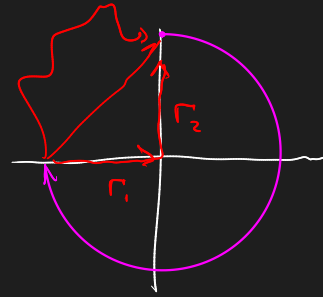
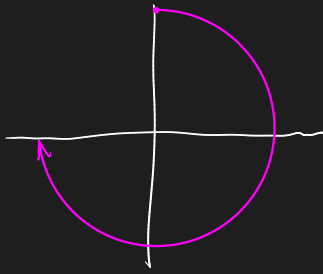
Pero la curva no es cerrada!

Solución: la cerramos  $\ddot{\cup}$

$$C: \sigma: [0, \frac{3}{2}\pi] \rightarrow \mathbb{R}^2$$

$$\sigma(t) = (\sin t, \cos t)$$

$$\sigma'(t) = (\cos t, -\sin t)$$



$\Gamma_2$  e  $\Gamma_1$

Dos trociscos

$$\Gamma_1 : \sigma_1 : [0, 1] \rightarrow \mathbb{R}^2$$

$$\sigma_1(t) = (0, t)$$

$$\sigma_1'(t) = (0, 1)$$

$$\Gamma_2 : \sigma_2 : [-1, 0] \rightarrow \mathbb{R}^2$$

$$\sigma_2(t) = (t, 0)$$

$$\sigma_2'(t) = (1, 0)$$

$$R = \text{Rechen}$$

$$\partial R = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

orientada por  $\sigma_1, \sigma_2, \sigma_3$

Green dice

$$\iint_R \vec{F} \cdot d\vec{x} \cdot d\vec{y} = \int_{\partial R^+} \vec{F} \cdot d\vec{z}$$



$$-\iint_R 5 \, dx \, dy = \underbrace{\int_C F \, d\vec{s}}_{\text{Inógnita}} + \int_{\Gamma_1} F \cdot d\vec{s} + \int_{\Gamma_2} F \cdot d\vec{s}$$

sentido de recorrido  
deja áreas a derecha

Calculamos

$$-\iint_R 5 \, dx \, dy = -5 \cdot \frac{3}{4} \pi$$

$$\int_{\Gamma_1} P \, dx + Q \, dy = \int_0^1 \left\langle (P(\sigma_1(t)), Q(\sigma_1(t))), (0, 1) \right\rangle dt$$

$$F(x, y) = \left( \underbrace{2 \cdot \cos(x^2 y)}_{\text{Feo}} \cdot \underbrace{xy}_{\text{simple}} - 2y, \underbrace{x^2 \cdot \cos(x^2 y)}_{\text{Feo}} + \underbrace{3x}_{\text{simple}} \right)$$

$$= \int_0^1 0 \, dt = 0 //$$

$$\int_{\Gamma_2} P \, dx + Q \, dy = \int_0^1 \left\langle (P(\sigma_2(t)), Q(\sigma_2(t))), (1, 0) \right\rangle dt$$

$$= \int_{-1}^0 0 \, dt = 0 //$$

Junto todo

$$\underbrace{- \iint_{\mathbb{R}} 5 \cdot dx \cdot dy}_{-\frac{15}{4}\pi} = \underbrace{\int_{\mathcal{C}} F \cdot d\vec{s}}_{\text{Incongnita}} + \underbrace{\int_{\Gamma_1} F \cdot d\vec{s}}_{=0} + \underbrace{\int_{\Gamma_2} F \cdot d\vec{s}}_{=0}$$

$$\Rightarrow \int_{\mathcal{C}} F \cdot d\vec{s} = -\frac{15}{4}\pi$$

Obs :

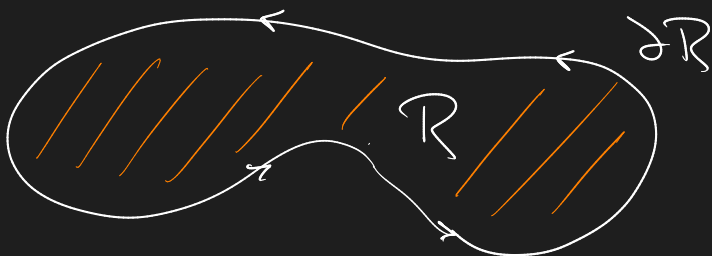
$$\text{Sez } F = \nabla f \quad f \in C^2$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$F(x, y) = (f_x, f_y)$$

Sez  $\mathcal{C}$  una curva cerrada simple



Por Green :

$$\iint_{\mathbb{R}} Q_x - P_y \, dx \, dy = \int_{\mathcal{C}} F \cdot d\vec{s} = \int_{\mathcal{C}} \nabla f = 0$$

||

$$\iint_{\mathbb{R}} \underbrace{f_{yx} - f_{xy}}_{\text{iguales por } C^2} \, dx \, dy = 0$$

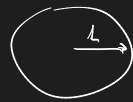
Considerar :

$$F(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) = (P, Q)$$

Verificar que

$$Q_x - P_y = 0 \quad \forall (x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0)$$

Considerar  $\mathcal{C} = \partial B(0, 1)$  orientado positivamente



$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

Si valiers Green :

$$0 = \iint_{B(0,1)} Q_x - P_y = \int_{\mathcal{C}} F \, d\vec{s}$$

Sin embargo:

$$\sigma(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$$

$$\sigma'(t) = (-\sin t, \cos t)$$

$$\begin{aligned} \int_{\mathcal{C}} P dx + Q dy &= \int_0^{2\pi} \langle (-\sin t, \cos t), (-\sin t, \cos t) \rangle dt \\ &= \int_0^{2\pi} 1 dt = 2\pi \end{aligned}$$

Si vale Green:

$$\underbrace{0 = \iint_{B(0,1)} Q_x - P_y}_{\text{esto vale}} \quad \neq \quad \underbrace{\int_{\mathcal{C}} F d\vec{s}}_{\text{esto vale}} = 2\pi$$

donde falla esta igualdad?

Obs:

• Si  $F = \nabla f \Rightarrow \int_{\mathcal{C}} F d\vec{s} = 0 \quad \forall \mathcal{C} \text{ cerrada}$

• Si  $F = \nabla f = (P, Q)$

$$\Rightarrow Q_x - P_y \equiv 0$$

$$\Rightarrow (\text{Green}) \int_{\partial} F d\vec{s} = 0$$

$$\circ S: F = (P, Q)$$

$$\text{et } Q_x - P_y \equiv 0 \text{ en } \Omega$$

$$\Rightarrow F = \nabla f \text{ en } \Omega \quad ??$$

No!

