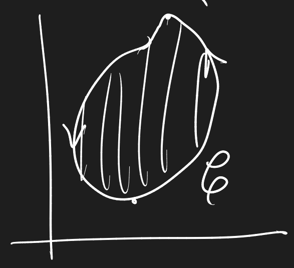
Teoremas de Stokes y Gauss

Feb 16

Green

$\mathbb{R}^2$

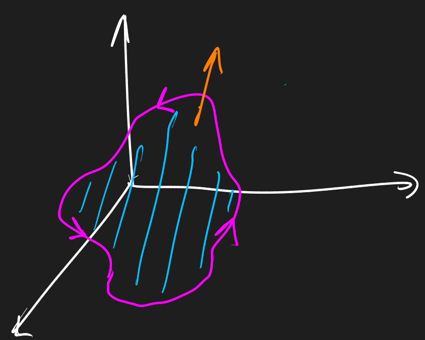


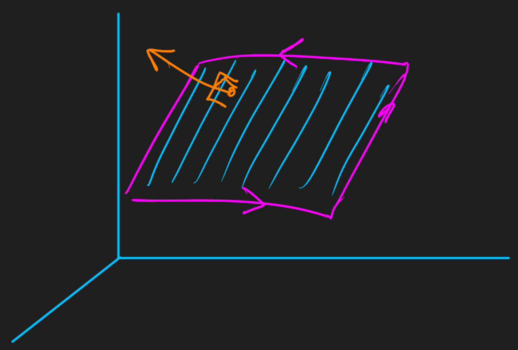
$$\int_{\mathcal{C}} F \cdot d\vec{s} \overset{\text{Green}}{=} \iint_D Q_x - P_y \, dx\,dy$$

$F \in \mathbb{R}^2$

$F(x,y) = (P(x,y), Q(x,y))$

Stokes $\mathbb{R}^3$





$$\boxed{\int_{\substack{\partial S \\ \cap \\ \mathcal{C}}} F \cdot d\vec{s} = \iint_S \nabla \times F \cdot d\vec{S}}$$

$$\nabla \times F = \det \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix}$$

$F = (F_1, F_2, F_3)$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \, , \, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) , \; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \, , \, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \, , \, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Gauss : Divergencia

Diver. negat. Divergencia positiva

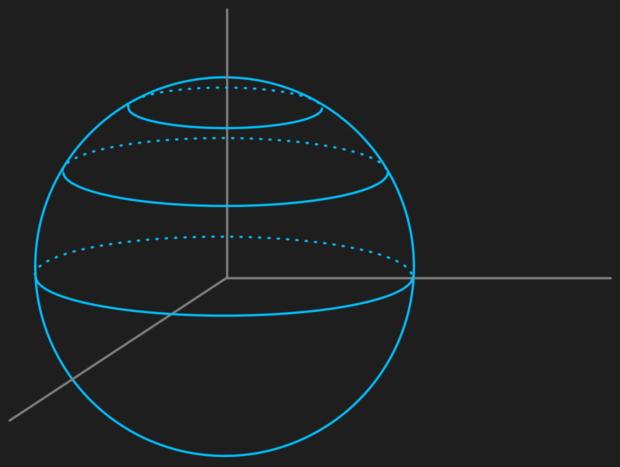
$$\underset{\underset{\mathbb{R}}{\cap}}{\operatorname{div}(F)} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)$$

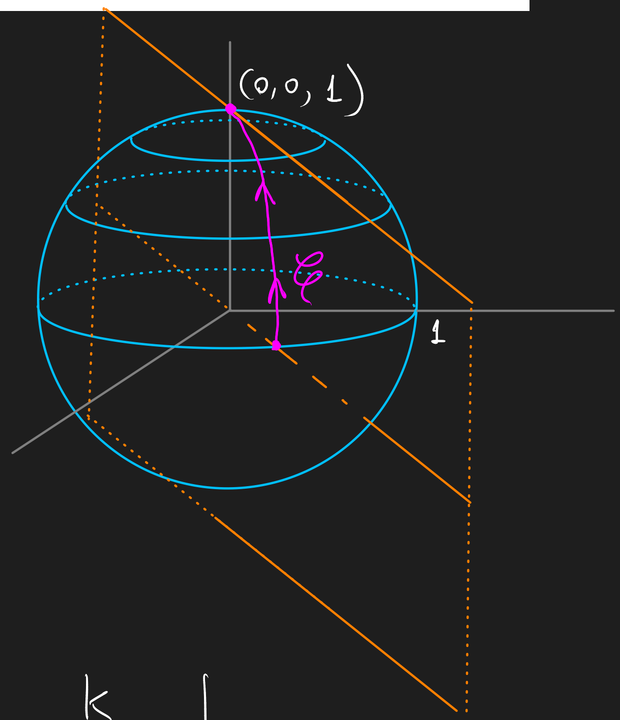
$$\iint_S F \cdot \eta \, ds = \iiint_\Omega \overbrace{\operatorname{div}(F)}^{\in \mathbb{R}} \, dV$$

Ejercicio 10. Calcular la integral de línea $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ donde $\mathbf{F}$ es el campo vectorial definido por

$$\mathbf{F}(x, y, z) = \left(2xy + z^2, x^2 - 2yz, 2xz - y^2\right)$$

y $\mathcal{C}$ es la curva que está contenida en la esfera $x^2 + y^2 + z^2 = 1$ y el plano de ecuación $y = x$ recorrida desde el punto $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ al polo norte.





$$\nabla \times F = \det \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right), \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \left(-2y - (-2y), -(2z - 2z), 2x - 2x \right)$$

$$= (0, 0, 0)$$

$$\int_{\mathcal{C}} F \cdot ds = f(q) - f(p)$$

$$= f(0,0,1) - f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$\mathbf{F}(x,y,z) = (\underbrace{2xy+z^2}_{\frac{\partial f}{\partial x}},\ \underbrace{x^2-2yz}_{\frac{\partial f}{\partial y}},\ \underbrace{2xz-y^2}_{\frac{\partial f}{\partial z}})$$

$$\left.\begin{aligned} f_x(x,y,z) &= 2xy+z^2 \\ f_y &= x^2-2yz \\ f_z &= 2xz-y^2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} f &= yx^2 + xz^2 + C(y,z) \\ f &= yx^2 - zy^2 + C(x,z) \\ f &= xz^2 - zy^2 + C(x,y) \end{aligned}\right.$$

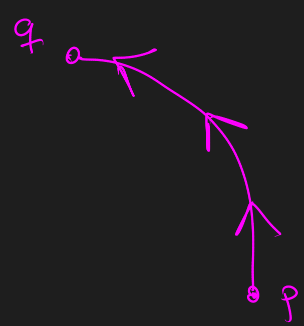
$$\boxed{f(x,y,z) = yx^2 + xz^2 - zy^2}$$

$$f(0,0,1) = 0$$

$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \quad //$$

$$\boxed{\int_{\mathcal{C}} F\cdot ds = -\frac{\sqrt{2}}{4}}$$

$\approx -0,3\ldots$



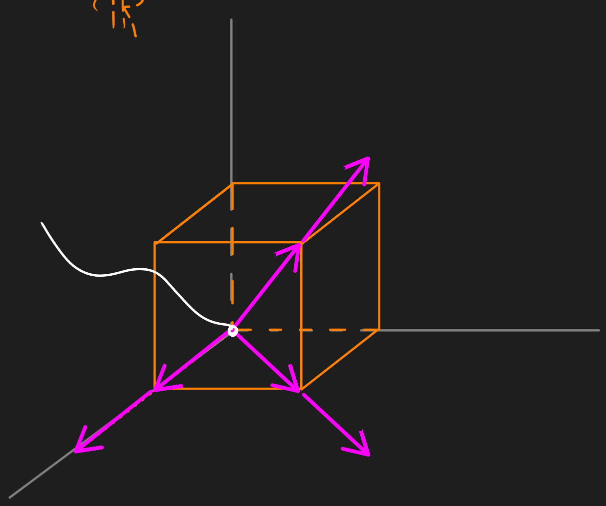
Feb 17
Miercoles.

Ejercicio 11. Rehacer el ejercicio 17 de la práctica 2 usando el teorema de Gauss.

- Guía Práctica 2:

↳ **Ejercicio 17.** Evaluar el flujo saliente del campo $\mathbf{F}(x, y, z) = (x, y, z)$ a través del borde del cubo $[0,1] \times [0,1] \times [0,1]$.

$\mathbb{R}^3$



$$\iint_S F \cdot \hat{n}\, d\vec{S} = \iiint_\Omega \operatorname{div}(F)\, dV$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 1 = 3$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 3\, dV = 3 \quad //$$

Ejercicio 12. Calcular $\int_S (x+y+z)\,dS$ donde S es el borde de la bola unitaria, es decir

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1\}.$$

$$f(x,y,z) = (x+y+z) \in \mathbb{R}$$

$$\int_S f\,dS =$$

$$= \int_u \int_v f(T(u,v)) \cdot \|T_u \times T_v\|\,du\,dv$$

Esféricas

$$x = r \cdot \cos\theta \cdot \sin\varphi$$

$$y = r \cdot \sin\theta \cdot \sin\varphi$$

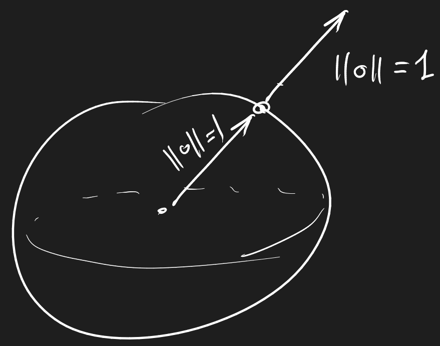
$$z = r \cdot \cos\varphi$$

$$\iint_S \langle F \cdot \eta \rangle\,dt = \iiint_\Omega \operatorname{div}(F)$$

$$\langle F, \eta \rangle = f(x,y,z)$$

$$\langle (F_1, F_2, F_3), (\quad\quad) \rangle = x+y+z$$

$$\left\langle \left(\;,\;,\; \right), (x, y, z) \right\rangle =$$



$$\left\langle \underbrace{(1, 1, 1)}_{F}, \underbrace{(x, y, z)}_{\eta} \right\rangle =$$

$$\iint_S F \cdot \eta \, dS \overset{\text{Gauss}}{=} \iiint_\Omega \operatorname{div} F \, dV$$

$$\operatorname{div} F = 0$$

$$= \iint_S x + y + z \, dS = 0$$

Otra forma de verlo:

$$f(x,y,z) = (x + y + z) \in \mathbb{R}$$