Práctica #4

Jupiter 04/02/21

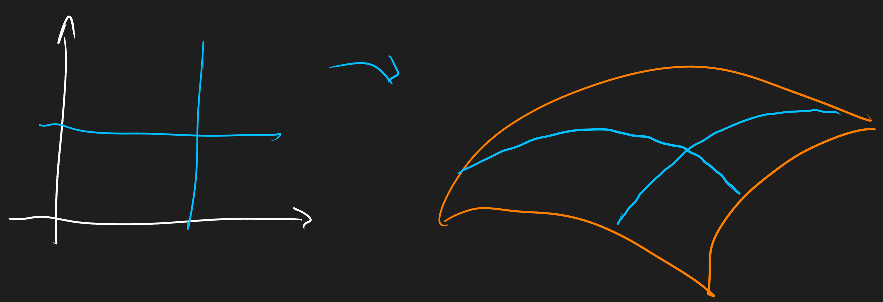
Superficies Parametrizadas

Superficie $S \subseteq \mathbb{R}^3$ es un conjunto /

$\exists\ T : D \subseteq \mathbb{R}^2 \to S$ continua

y sobreyectiva

Preguntar: Puede una curva ocupar todo $D \subset \mathbb{R}^2$?



$T(u,v) = (x$

Ejemplos:

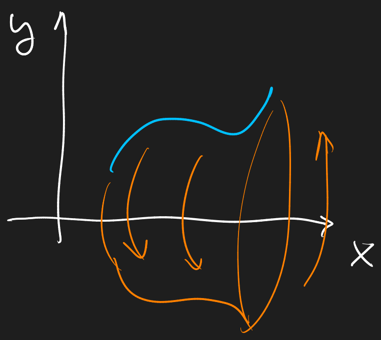
1) $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$

$T : D \to S$

$$T(x,y) = (x, y, f(x,y))$$

Superficie de Revolución

$y = f(x)$, rotamos en eje x



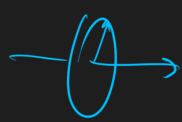
Dos parámetros : x identifica el punto en el gráfico

θ identifica el ángulo de giro

$$T(x,\theta) = (\underset{\substack{\uparrow \\ \text{fijo}}}{x}, \underbrace{f(x)\cos\theta, f(x).\sin\theta}_{\text{circunferencia de radio } f(x)})$$

$x \in [a,b]$

$\theta \in [0, 2\pi)$



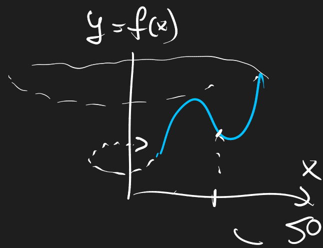
2) $f : [a,b] \to \mathbb{R}$

$y = f(x)$

rotamos en el eje y

Dos parámetros

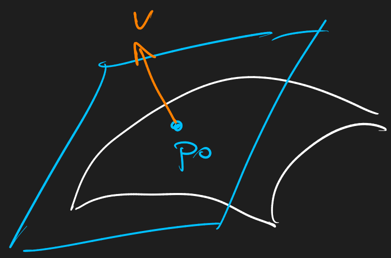
x y θ



no determina una única $f(x)$

$$T(x,\theta) = \left(\underset{\uparrow \text{radio}}{x} \cdot \cos\theta ,\ f(x) ,\ x \cdot \sin x \right)$$

Suavidad y tangentes



Def. geom. de vector normal en la teórico de hoy.

Si $T : D \to S$ es regular $\left(\begin{array}{l} T \text{ dif} \\ T_u, T_v \neq \vec{0} \\ T_u \not\parallel T_v \end{array} \right)$

$$T_u(u,v) = \left(\frac{\partial X}{\partial u}(u,v) ,\ \frac{\partial Y}{\partial u}(u,v) ,\ \frac{\partial Z}{\partial u}(u,v) \right)$$

$$T_v(u,v) \quad \left(\frac{\partial X}{\partial v}(u,v) ,\ \frac{\partial Y}{\partial v}(u,v) ,\ \frac{\partial Z}{\partial v}(u,v) \right)$$

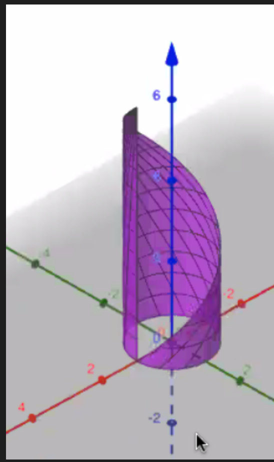
Vector normal en $p_0 \in S$

$$p_0 = T(u_0, v_0)$$

$$\eta(p_0) = T_u \times T_v$$

$$= \det \begin{bmatrix} i & j & k \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{bmatrix}$$

Ej: Escalera Caracol



Dos parámetros

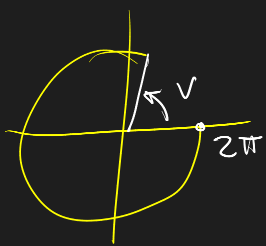
v : dice la altura

$v \in [0, 2\pi]$

u : describe la curva a altura v

↙ normalizo con $2\pi - v$

$u \in [0, 1]$



$$T : [0,1) \times [0, 2\pi) \longrightarrow S$$

Armo:

$$T(u,v) = \Big(\cos\big(v + (2\pi - v)u\big) , \sin\big(v + (2\pi - v)u\big) , v \Big)$$

evalúo

$$T(u,0) = (\cos 2\pi u , \sin 2\pi , 0)$$ ↙ circ. en $z=0$

$$T(u, 2\pi) = (1, 0, 2\pi)$$ ← "pico" de la cima.

↑ se degenera: Pierde la propiedad de "Longitud Positiva"

Parametrizando con cambios de coord.

Cilíndricas:

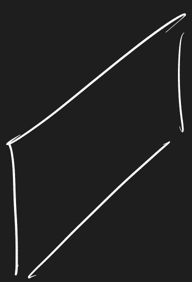
$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$
$$z = z$$

Si fijamos $r = R > 0$, tenemos

$$T(\theta, z) = (R \cdot \cos\theta,\ R \cdot \sin\theta,\ z)$$

tengo cilindros

Si fijamos θ



tengo planos

Esféricas

Fijamos algún $R \geq 0$

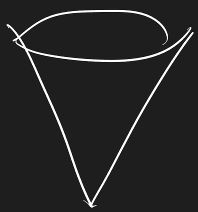
$$T(\theta, \varphi) = (R \cdot \cos\theta \cdot \sin\varphi,\ R \cdot \sin\theta \cdot \sin\varphi,\ R \cdot \cos\varphi)$$

$\theta \in [0, 2\pi]$

$\varphi \in [0, \pi]$

Fijamos $\varphi = \varphi_0$

$$T(r,\theta) = \big(r \cdot \cos\theta \cdot \sin\varphi_0 ,\ r \cdot \sin\theta \cdot \sin\varphi_0 ,\ r \cdot \cos\varphi_0 \big)$$



Cono

$$x^2 + y^2 = a \cdot z^2$$

- Fijamos θ :

(Semi) plano vertical en dirección θ.

Área de una superficie

$S \subset \mathbb{R}^3$, $T : D \to S$ para un

$$A(S) = \int_D \| T_u \times T_v \| \, du\, dv$$

Ejemplos

1) $S = \text{Graf}(f) \quad , f : D \to \mathbb{R}$

$T_x(x,y) =$

$T_y(x,y) =$

$$\| T_x \times T_y \| = \sqrt{1 + f_x^2 + f_y^2}$$

$$\boxed{A(\text{Graf}(f)) = \int_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy}$$

2) Área de la esfera de Radio $R > 0$

$f : D \to \mathbb{R}$

$f(x,y) = \sqrt{R^2 - x^2 - y^2}$

$D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \in R \}$

$$f_x(x,y) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$f_y(x,y) = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

$$\|T_x \times T_y\| = \left(\frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2} + 1\right)^{1/2}$$

$$= \left(\frac{R^2}{R^2-x^2-y^2}\right)^{1/2}$$

$$= \frac{R}{\sqrt{R^2-x^2-y^2}}$$

- Calculamos

$$\int_D \frac{R}{\sqrt{R^2-(x^2+y^2)}}\,dx\,dy =$$

polares

$$= R \cdot \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr\,d\theta$$

$$= R \cdot 2\pi \int_0^R \frac{r}{\sqrt{}}$$

$u = R^2 - r^2$

$du = -2r\,dr$

$$= -R \cdot \pi \int_{R^2}^{0} \frac{1}{\sqrt{u}}$$

$$= R \cdot \pi \left(2\sqrt{u} \right|_0^{R^2}$$

$$= 2\pi R^2 = \text{Área de } \tfrac{1}{2} \text{ esfera}$$

$$\text{Área de la esfera} \overset{2\times}{=} 4\pi \cdot R^2$$

- Mismo ejercicio con esféricas

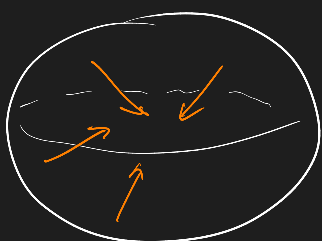
$$T(\theta,\varphi) = (R \cdot \cos\theta \cdot \sin\varphi,\ R \sin\theta \cdot \sin\varphi,\ R \cdot \cos\varphi)$$

$$T_\theta = (-R \cdot \sin\theta \cdot \sin\varphi,\ \dots$$

$$T_\varphi = (\dots$$

$$T_\theta \times T_\varphi = \underbrace{-R^2 \sin\varphi}_{\in \mathbb{R}} \cdot (\cos\theta \cdot \sin\varphi,\ \sin\theta \cdot \sin\varphi,\ \cos\varphi)$$

$$= \underbrace{-R \sin\varphi}_{\in \mathbb{R}} \cdot \underbrace{(R \cdot \cos\theta \cdot \sin\varphi,\ R \cdot \sin\theta \cdot \sin\varphi,\ R \cdot \cos\varphi)}_{\text{punto en esfera de radio } R}$$



So

$$\|T_\theta \times T_\varphi\| = R^2 . \sin\varphi$$

$$A(S) = \int_0^{2\pi}\int_0^{\pi} R^2 . \sin\varphi . d\theta\, d\varphi$$

$$= 2\pi R^2 \int_0^{\pi} \sin\varphi . d\varphi$$

$$= 4\pi R^2$$