

Práctica #2

Mars 02/02/21

Integración sobre curvas en $\mathbb{R}^2, \mathbb{R}^3$

Longitud de una curva $\mathcal{C} \subset \mathbb{R}^3 (\mathbb{R}^2)$

parametrizada por $\sigma: [a, b] \rightarrow \mathcal{C}$ regular
(a trozos).

$$\text{Long}(\mathcal{C}) := \int_a^b \|\sigma'(t)\| \cdot dt$$

↑
; No depende de la parametrización!

Parámetro de Longitud de Curva

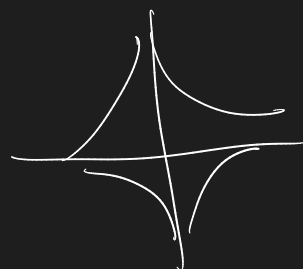
$$s: [a, b] \rightarrow [0, \text{Long} \mathcal{C}]$$

$$s(t) = \int_a^t \|\sigma'(r)\| \cdot dr$$

Ejemplo:

$$\sigma: [0, 2\pi] \rightarrow \mathbb{R}^2$$

$$\sigma(t) = (\cos^3 t, \sin^3 t)$$

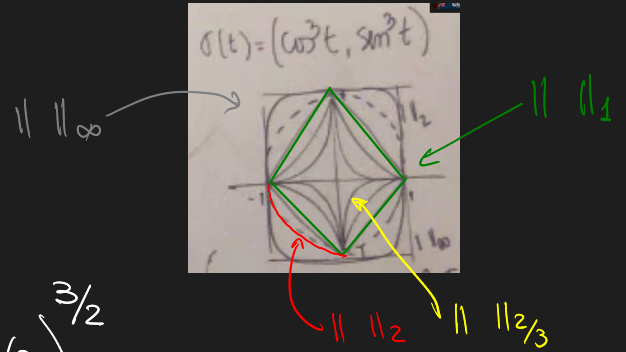


+

Recordar

$$\left((\cos^3 t)^{2/3} + (\sin^3 t)^{2/3} \right)^{3/2} = 1$$

Bola unitaria en $\| \cdot \|_{2/3}$



Ej:

$$\sigma'(t) = (-3 \cos^2 t \cdot \sin t, 3 \sin^2 t \cdot \cos t)$$

$$\begin{aligned} \|\sigma'(t)\| &= 3 \left(\cos^4 t + \sin^4 t \right)^{1/2} \\ &= 3 \left(\cos^2 t \cdot \sin^2 t \cdot (\cos^2 t + \sin^2 t) \right)^{1/2} \\ &= 3 |\cos t \cdot \sin t| \end{aligned}$$

$$\text{Long}(\ell) = 3 \int_0^{2\pi} |\cos t \cdot \sin t| dt$$

$$\text{divido en } \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \dots + \int + \int$$

$$= 3 \cdot 4 \cdot \int_0^{\pi/2} \cos t \sin t$$

=

$$= 6 \quad \text{Notas que es mayor que } 4\sqrt{2}$$

Longitud de Curvas en Polares

Para $\theta \in [a, b]$ tenemos

la curva dada por $r = r(\theta)$

$$\text{Long}(\ell) = \int_a^b \underbrace{\sqrt{r'(\theta)^2 + r(\theta)^2}}_{= \|\sigma'(\theta)\|} d\theta$$

Partiendo de

$$\sigma(\theta) = (r(\theta) \cdot \cos \theta, r(\theta) \cdot \sin \theta)$$

Cardioides:

$$r(\theta) = 1 + \cos \theta$$

$$r'(\theta) = -\sin \theta, \quad \theta \in [0, 2\pi]$$

$$L(\ell) = \int_0^{2\pi} \left(\sin^2 \theta + (1 + \cos \theta)^2 \right)^{1/2} d\theta$$

$$= \int_0^{2\pi} (2 + 2 \cos \theta)^{1/2} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos \theta} d\theta$$



Identidad Trig y triguito:

$$\begin{aligned}\cos \theta &= \cos \left(2 \cdot \frac{\theta}{2} \right) \\ &= \cos^2 \left(\frac{\theta}{2} \right) - \sin^2 \left(\frac{\theta}{2} \right) \\ &= 2 \cos^2 \left(\frac{\theta}{2} \right) - 1\end{aligned}$$

$$\textcircled{*} = (\sqrt{2})^2 \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta$$

$$= 2 \cdot \int_0^{\pi} \cos \frac{\theta}{2} d\theta - 2 \int_{\pi}^{2\pi} \cos \frac{\theta}{2}$$

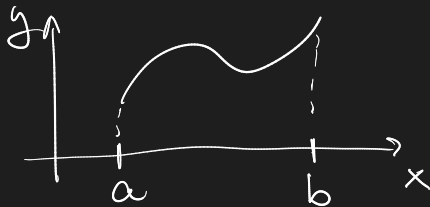
$$= 4 \cdot \sin \frac{\theta}{2} \Big|_0^{\pi} - 4 \sin \frac{\theta}{2} \Big|_{\pi}^{2\pi}$$

$$= 4 + 4 = 8 //$$

Un caso particular:

$$\sigma: [a, b] \rightarrow \mathbb{R}^2$$

$$\sigma(x) = (x, f(x))$$



$$\text{Long} = \int_a^b \sqrt{1 + f'(x)^2} dx$$

$\parallel \sigma'(x) \parallel$
↓

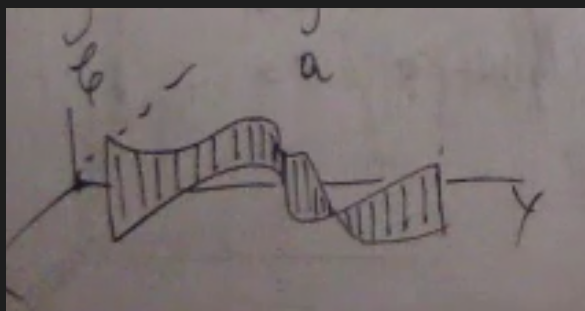
Integración sobre \mathcal{C}

$$\mathcal{C} \subset \mathbb{R}^2$$

$$\sigma : [a, b] \rightarrow \mathcal{C} \text{ regular}$$

$$f : \mathcal{C} \rightarrow \mathbb{R} \text{ continua}$$

$$(f \circ \sigma : [a, b] \rightarrow \mathbb{R})$$



$$\int_{\mathcal{C}} f \cdot ds = \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\| \cdot dt$$

Ejemplo de Centro de Masa

$$\mathcal{C}, \sigma : [a, b] \rightarrow \mathcal{C} \text{ regular}$$

$$\rho : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0} \text{ densidad continua}$$

$$Masa(\mathcal{C}) = \int_{\mathcal{C}} \rho \, ds$$

$$\text{Centro de masa } c(\mathcal{C}) = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\int_{\mathcal{C}} x \, ds}{\int_{\mathcal{C}} \rho \, ds}$$

$$\bar{y} = \frac{\int_{\mathcal{C}} y \, ds}{\int_{\mathcal{C}} \rho \, ds}$$

$$\bar{z} = \text{idem}$$

$$\mathcal{C} = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{array}{l} x^2 + y^2 + (z-1)^2 = 1, \\ z = 1, \\ y \geq 0 \end{array} \right\}$$



$$\sigma : [0, \pi] \rightarrow \mathbb{R}^3$$

$$\sigma(t) = (\cos t, \sin t, 1)$$

$$\|\sigma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$M_{252} :$$

$$\int_0^\pi \|\sigma'(t)\| dt = \pi$$

$$\bar{x} = \underbrace{\int_0^\pi x ds}_{M_{252}}$$

$$= \underbrace{\int_0^\pi \cos t dt}_{\pi} = 0 \quad \begin{array}{l} \swarrow \text{centro en } x=0 \\ \parallel \end{array}$$

$$\bar{z} = \underbrace{\int_C z \, ds}_\pi = \frac{1}{\pi} \int_0^\pi 1 \, dt$$

$$= 1 \quad \leftarrow \text{centro en } z = 1 \text{ (trivial)}$$

$$\bar{y} = \frac{1}{\pi} \int_C y \, ds$$

$$= \frac{1}{\pi} \int_0^\pi \sin t \, dt$$

$$= \frac{1}{\pi} \left(-\cos t \Big|_0^\pi \right)$$

$$= \frac{2}{\pi} < 1$$

centro de masa en $(0, \frac{2}{\pi}, 1)$

