Eze

Luna Mar 15

Diagramas de Fase
(autovalores complejos)

Problema

$$X'(t) = A\,X(t)$$

donde $\det(\lambda I - A)$ tiene raices complejas conjugadas

Auto valor

$$\lambda = \alpha + \beta i$$

con $\beta i < 0$!

$$\bar{\lambda} = \alpha - \beta i$$

Auto vector

$$\xi = V_1 + i\,V_2$$

Escribo

$$X(t) = \underbrace{y_1(t)}_{\text{Tomo}} \cdot V_1 + \underbrace{y_2(t)} \cdot V_2$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t} \cdot r \cdot \cos(\theta - \beta t) \\ e^{\alpha t} \cdot r \cdot \sin(\theta - \beta t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad \begin{aligned} C_1 &= r \cdot \cos\theta \\ C_2 &= r \cdot \sin\theta \end{aligned}$$

Un ejemplo concreto

$$X'(t) = \begin{pmatrix} 4 & -2 \\ 5 & 2 \end{pmatrix} X(t)$$

$$\det(\lambda I - A) = (\lambda - 4)(\lambda - 2) + 10$$

$\lambda_1 = 3 - 3i$ ⟸ uso ésta!

puer signo menos

$\lambda_2 = 3 + 3i$

Trabajamos con $\lambda_1 = 3 - 3i$

V_1 : autovector asociado

$$\begin{pmatrix} 1+3i & -2 \\ 5 & -1+3i \end{pmatrix} \begin{pmatrix} ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Ya sé que son ld puer $\det(\;\cdot\;) = 0$

(Pasa siempre que buscamos autovectores)

Recordar

$$(1+3i)(1-3i) = 1 + 9 = 10$$

↓

$$\begin{pmatrix} 1+3i & -2 \\ 5 & -1+3i \end{pmatrix} \begin{pmatrix} 1-3i \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1+3i & -2 \\ 5 & -1+3i \end{pmatrix} \begin{pmatrix} 1-3i \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 5 \end{pmatrix}} + i \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}$$

$$= V_1 + i \cdot V_2$$

Escribo

$$X(t) = e^{(3-3i)t} \cdot \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$$

$$= e^{3t} \cdot \Big(\overbrace{\cos 3t}^{\cos -3t = \cos 3t} - \overbrace{i \sin 3t}^{\sin -3t = -\sin 3t} \Big) \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$$

Parte Real $\qquad\qquad$ Parte Im

$$= e^{3t} \begin{pmatrix} \cos 3t - 3 \sin 3t \\ 5 \cos 3t \end{pmatrix} + i \cdot e^{3t} \begin{pmatrix} -3 \cos 3t - \sin 3t \\ -5 \sin 3t \end{pmatrix}$$

Sol: $\left\{ e^{3t} \begin{pmatrix} \cos 3t - 3 \sin 3t \\ 5 \cos 3t \end{pmatrix}, \; e^{3t} \begin{pmatrix} -3 \cos 3t - \sin 3t \\ -5 \sin 3t \end{pmatrix} \right\}$

Obs

$$\alpha = Re(\alpha + \beta i) = 3 > 0$$

Son flechas que se alejan del $(0,0)$

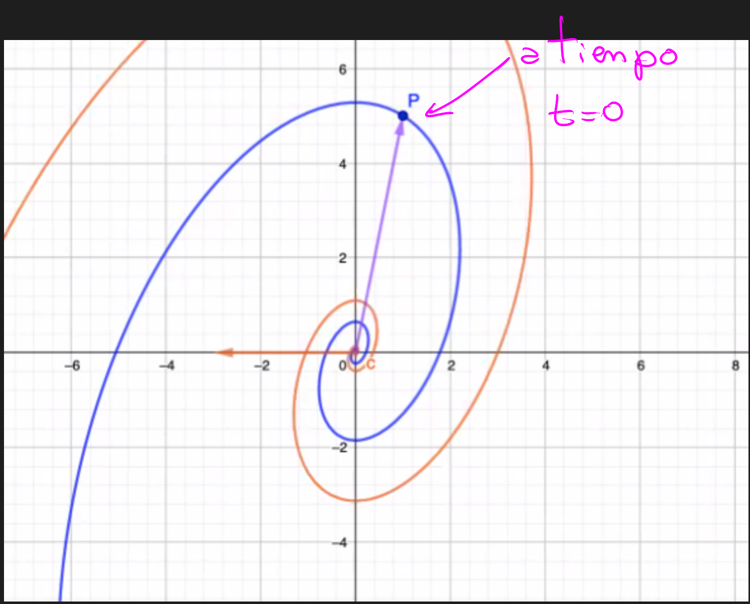
pues $e^{\alpha t} \longrightarrow \infty$ cuando $t \to \infty$

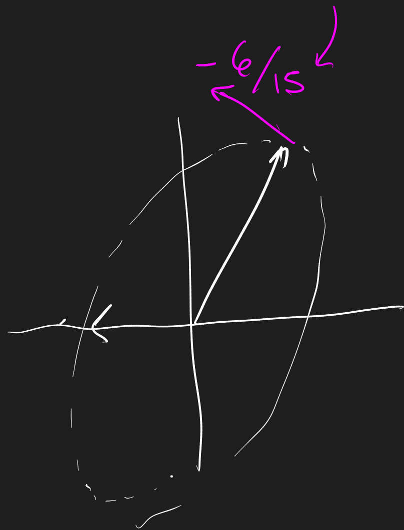
Obs :

$$X(0) = p \Rightarrow X'(0) = A.p$$

$$X(0) = \begin{pmatrix} 1 \\ 5 \end{pmatrix} = V_1$$

$$X'(0) = \begin{pmatrix} 4 & -2 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \frac{-6}{15}$$





Para parcial, saber

- Sentido de giro
- Hacia adentro / afuera

Linealización

$X' = F(X)$ con F <u>no</u> lineal

(no es AX)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = F(x, y)$$

Ejemplo

$$F(x,y) = \begin{pmatrix} x(2-x-y) \\ y(-1+x-y) \end{pmatrix}$$

$$= \begin{pmatrix} 2x - x^2 - xy \\ -y + xy - y^2 \end{pmatrix}$$

Idea:

Si $X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Big/ F(X_0) = 0$

$$\Rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

"Puntos de Equilibrio"

Obs:

Si escribo $y = x - x_0$

Taylor orden 1.

$\Rightarrow$ usando que $F(x) \cong DF(x_0)(x - x_0)$

$$y' = x' = F(x) \cong \underbrace{DF(x_0)}_{A}(x - x_0) = Ay$$

Miramos $\boxed{y' = Ay}$ que sabemos que se aproxima por el Teorema de abajo

Teorema:

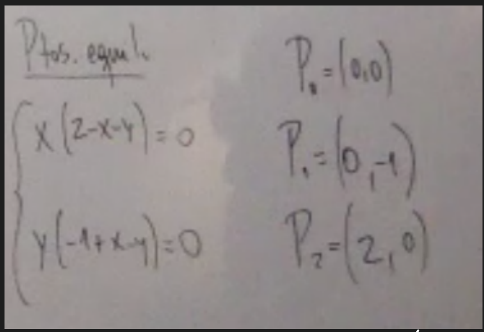
Cerca de x_0 (equilibrio) las soluciones de

$$x' = F(x)$$

se parecen a las soluciones de

$$y' = DF(x_0)\,y$$

Ejemplo


Ptos. equil.

$$\begin{cases} x(2-x-y)=0 \\ y(-1+x-y)=0 \end{cases}$$

$P_0=(0,0)$

$P_1=(0,-1)$

$P_2=(2,0)$

$P_3 = ($?

$$\begin{cases} x+y=2 \\ x-y=1 \end{cases} \Rightarrow x = 1+y$$

$$1+2y = 2$$
$$2y = 1$$
$$y = \frac{1}{2}$$

$$P_3 = \left(\frac{3}{2}, \frac{1}{2}\right)$$

$$DF(x,y) = \begin{pmatrix} 2-2x-y & -x \\ y & -1+x-2y \end{pmatrix}$$

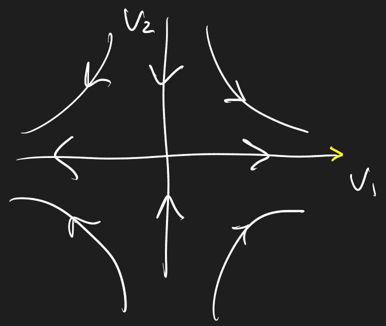
$P_0 = (0,0)$:

$$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$ ← ya está diagonalizada

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

$$v_1 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$v_2 = \begin{pmatrix}0\\1\end{pmatrix}$$



$$P_1 = (0,-1)$$

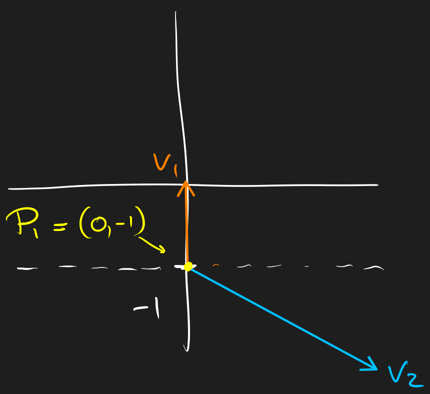
$$DF(0,-1) = \begin{pmatrix}3 & 0\\ -1 & 1\end{pmatrix}$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

$$v_1 = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$v_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

Importante! <u>no</u> estamos en el $(0,0)$!

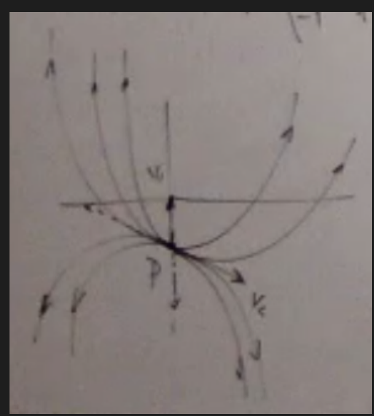


más grande

$$y_1 = e^{3t}\cdot v_1$$
$$y_2 = e^{t}\cdot v_2$$

pienso que

$$y_1 = (y_2)^3$$



concav. posit.

$P_2 = (2,0)$:

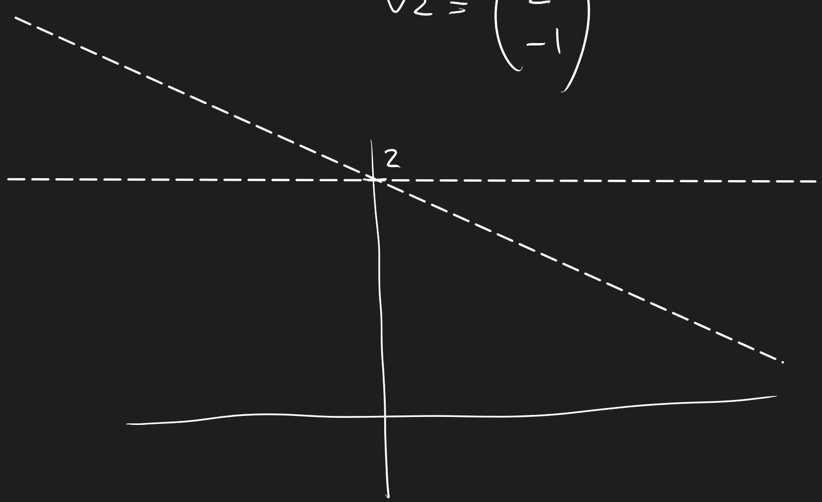
$$DF(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$$

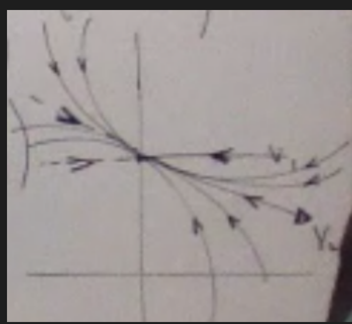
$\lambda_1 = -2$

$\lambda_2 = -1$

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$





$P_3 = \left(\frac{3}{2}, \frac{1}{2}\right)$

$$DF\left(\frac{3}{2}, \frac{1}{2}\right) = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -3 & -3 \\ 1 & -1 \end{pmatrix}$$

$$\det \begin{pmatrix} \lambda+3 & 3 \\ -1 & \lambda+1 \end{pmatrix} = (\lambda+3)(\lambda+1) + 3$$

$$= \lambda^2 + 4\lambda + 6$$

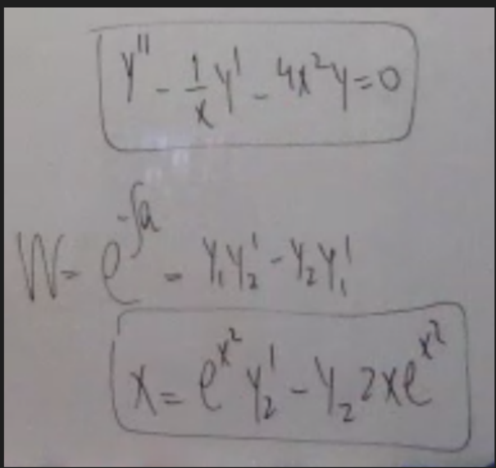
$$\frac{-4 \pm \sqrt{16-24}}{2}$$ complejos

Linealización

Puntos de equilibrio

Estabilidad : Diagrama de Fase

De la 6, 12 :

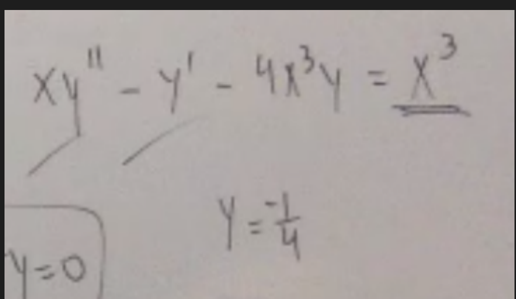


$$\boxed{y'' - \frac{1}{x}y' - 4x^2 y = 0}$$

$$W = e^{-\int a} = y_1 y_2' - y_2 y_1'$$

$$\boxed{x = e^{x^2} y_2' - y_2 2x e^{x^2}}$$

$$y_2' - 2x y_2 = x e^{-x^2}$$



$$xy'' - y' - 4x^3 y = \underline{x^3}$$

$$y = \frac{-1}{4}$$

$$\boxed{y = 0}$$