Análisis I - Análisis Matemático I - Matemática I - Análisis II (C)

2do. cuatrimestre 2020

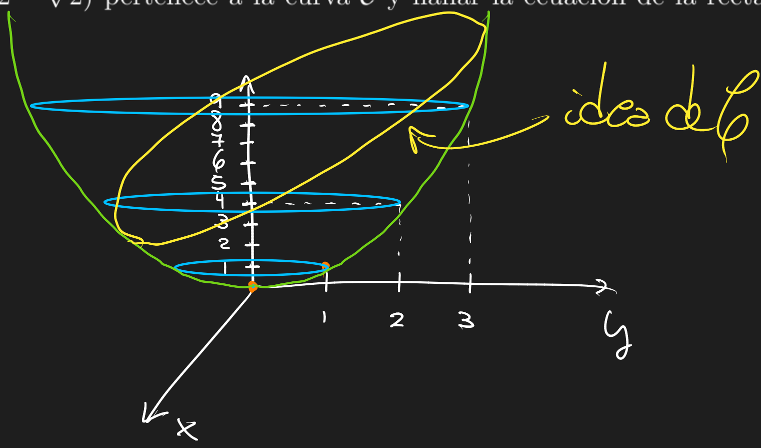
Simulacro del Primer Parcial - 14/10/2020

Ejercicio 1: Sea $\mathcal{C}$ la curva que se obtiene como la intersección de las superficies

$$x^2 + y^2 - z = 0 \qquad y \qquad x^2 - 4x + y^2 + z = 0$$

(a) Hallar una función $r(t)$ cuya imagen describa la curVa $\mathcal{C}$

(b) Verificar que el punto $P = (1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2 - \sqrt{2})$ pertenece a la curva $\mathcal{C}$ y hallar la ecuación de la recta tangente a $\mathcal{C}$ en el punto P



- $x^2 + y^2 = 1$
- $x^2 + y^2 = 4 = 2^2$
- $x^2 + y^2 = 9 = 3^2$

a) $\begin{cases} z = x^2 + y^2 \\ z = 4x - x^2 - y^2 \end{cases}$

Resto

$$0 = 2x^2 + 2y^2 - 4x$$

$$0 = x^2 - 2x + y^2 \qquad (x-1)^2 = x^2 - 2x + 1$$

$$0 = (x^2 - 2x + 1) - 1 + y^2$$

$$1 = (x-1)^2 + y^2$$

Paso a polares

$$\begin{cases} x-1 = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases} \qquad \begin{array}{l} r = 1 \\ \theta \in [0, 2\pi) \end{array}$$

$$\begin{cases} x = 1 + \cos\theta \\ y = \sin\theta \end{cases}$$

Falta z :

$$z = x^2 + y^2 \implies z = (1+\cos\theta)^2 + \sin^2\theta$$
$$= 1 + 2\cos\theta + \left(\cos^2\theta + \sin^2\theta\right)$$
$$= 2 + 2\cos\theta$$

Con la otra (verifico)

$$z = -(x^2+y^2) + 4x$$
$$= -2 - 2\cos\theta + 4(1+\cos\theta)$$
$$= 2 + 2\cos\theta$$

Obtuve

$$\begin{cases} x = 1 + \cos\theta \\ y = \sin\theta \\ z = 2 + 2\cos\theta \end{cases}$$

$$\sigma(t) = \left(1 + \cos t,\ \sin t,\ 2 + 2\cos t\right) \quad t \in [0, 2\pi)$$

$P = \left(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2 - \sqrt{2}\right)$

$$\begin{cases} 1 + \cos t \overset{\text{quiero}}{\downarrow}= 1 - \frac{\sqrt{2}}{2} \\ \sin t = \frac{\sqrt{2}}{2} \\ 2 + 2\cos t = 2 - \sqrt{2} \end{cases}$$

$$\cancel{1} + \cos t = \cancel{1} - \frac{\sqrt{2}}{2}$$

$$\cos t = -\frac{\sqrt{2}}{2}$$

$$t = \arccos\left(-\frac{\sqrt{2}}{2}\right)$$

$$t = \frac{3}{4}\pi \in [0, 2\pi) \checkmark$$

$$\sin\left(\frac{3}{4}\pi\right) = \frac{\sqrt{2}}{2} \checkmark$$

- P existe en $\mathscr{C}$ y se da cuando

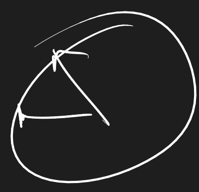
$t = \frac{3}{4}\pi$ en $\sigma(t)$

Derivo

$$\sigma(t) = (1 + \cos t, \sin t, 2 + 2\cos t)$$

$$\sigma'(t) = \left(-\sin t,\ \cos t,\ -2\sin t\right)$$

$$\sigma'\left(\tfrac{3}{4}\pi\right) = \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ -\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\right)$$



$$= \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ -\sqrt{2}\right)$$

Armo recta L

$$\boxed{L : (x,y,z) = P + \alpha\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ -\sqrt{2}\right) \qquad \alpha \in \mathbb{R}}$$

Otra forma: Primero paso cada ecuación a polares, después busco su intersección.

Polares $\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ z = z \end{cases}$

$\boxed{1}$ $z = x^2 + y^2 \Rightarrow z = r^2$ $\quad r \in [0, +\infty)$

$x = r \cdot \cos\theta$ $\quad \theta \in [0, 2\pi)$

$y = r \cdot \sin\theta$

$\boxed{2}$ $z = 4x - (x^2 + y^2) \Rightarrow z = 4 \cdot r \cdot \cos\theta - r^2$

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

$r \in [0, +\infty)$

$\theta \in [0, 2\pi)$

Igualo $\boxed{1}$ con $\boxed{2}$

$x = r \cdot \cos\theta$ ← Coinciden en $\boxed{1}$ y $\boxed{2}$

$y = r \cdot \sin\theta$

y además

$\begin{cases} z = r^2 \\ z = 4 \cdot r \cdot \cos\theta - r^2 \end{cases}$

$\Rightarrow r^2 = 4 \cdot r \cdot \cos\theta - r^2$

$$0 = 4r \cdot \cos\theta - 2r^2$$

$$0 = 2r\left(2\cos\theta - r\right)$$

$r > 0$

$$0 = 2\cos\theta - r$$

$$\boxed{r = 2\cos\theta}$$

Volviendo

$$x = r \cdot \cos\theta \quad \Rightarrow \quad 2\cos^2\theta$$

$$y = r \cdot \sin\theta \quad \Rightarrow \quad 2\cos\theta \cdot \sin\theta$$

$$z = r^2 \quad \Rightarrow \quad 4\cos^2\theta$$

Finalmente, puedo definir

$$\sigma(t) = \left(2\cos^2 t,\ 2\cos t \cdot \sin t,\ 4\cos^2 t\right)$$

con $t \in [0, 2\pi)$

b) $\sigma(t) = \left(2\cos^2 t,\ 2\cos t \cdot \sin t,\ 4\cos^2 t\right)$

Iguab coord. a coord.

$$\begin{cases} 1-\frac{\sqrt{2}}{2} = 2\cos^2 t & \text{I} \\ \frac{\sqrt{2}}{2} = 2\cos t \cdot \sin t & \text{II} \\ 2-\sqrt{2} = 4\cos^2 t & \text{III} \end{cases}$$

Si:

$$1-\frac{\sqrt{2}}{2} = 2\cos^2 t \Rightarrow \cos^2 t = \frac{1}{2} - \frac{\sqrt{2}}{4}$$

$$|\cos t| = \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^{1/2}$$

$$\cos t = \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^{1/2} \quad \text{I}$$

$$\cos t = -\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^{1/2} \quad \text{II}$$

I $\quad t_1 = \arccos\left(\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^{1/2}\right)$

calculadora

$$t_1 \overset{\downarrow}{=} \frac{3}{8}\pi$$

$$\cos t_1 \approx 0{,}38 \in [0, 2\pi) \checkmark$$

II $\quad t_2 = \arccos\left(-\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^{1/2}\right)$

calc

$$t_2 \overset{\downarrow}{=} \frac{5}{8}\pi$$

$$\cos t_2 = -0{,}38 \notin [0, 2\pi)$$

$\therefore$ el punto P pertenece a la curva $\mathcal{C}$ cuando $t = \frac{3}{8}\pi$

Busco recta tangente a $\mathcal{C}$ en P

Derivo $\sigma(t)$:

$$\sigma(t) = \left(2\cos^2 t,\ 2\cos t \cdot \sin t,\ 4\cos^2 t\right)$$

$$\sigma'(t) = \left(-4 \cdot \cos t \cdot \sin t,\ -2\sin^2 t + 2\cos^2 t,\ -8\cos t \cdot \sin t\right)$$

Evalúo en $t = \frac{3}{8}\pi$

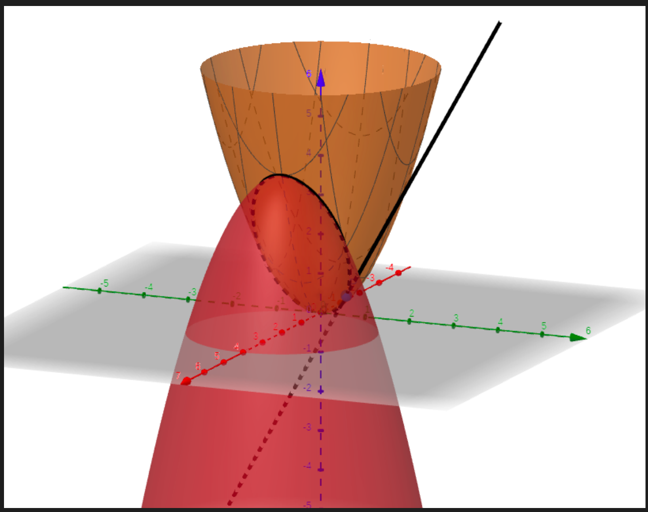
$$\sigma'\left(\tfrac{3}{8}\pi\right) = \left(-\sqrt{2},\ -\sqrt{2},\ -2\sqrt{2}\right)$$

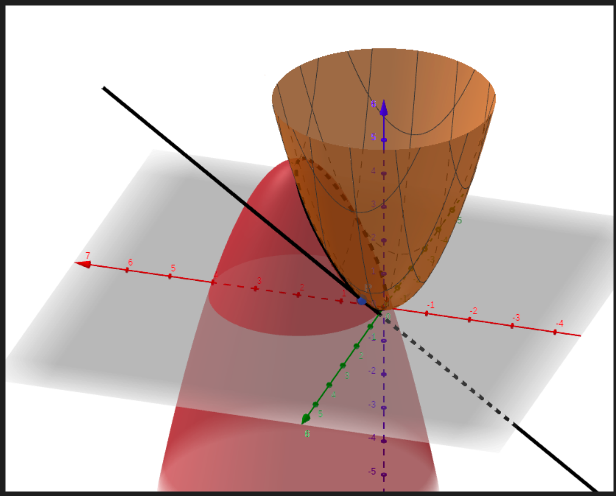
$\uparrow$ dirección tangente a la curva

Defino recta L como

$$\boxed{L : (x, y, z) = P + \alpha\left(-\sqrt{2},\ -\sqrt{2},\ -2\sqrt{2}\right)}$$

$\alpha \in \mathbb{R}$





Ejercicio 2: Calcular los siguientes límites:

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{(x-1)^2\,\mathrm{sen}(x^2)y}{x^2+y^4}$.

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x^2+y^2}$.

a) Acoto

$$\left| \frac{(x-1)^2 \cdot \sin(x^2) \cdot y}{x^2+y^4} - 0 \right| \leq \frac{(x-1)^2 \cdot \overbrace{|\sin x^2|}^{\leq x^2} \cdot |y|}{x^2+y^4}$$

$$\leq \frac{(x-1)^2 \cdot x^2 \cdot |y|}{\underbrace{x^2+y^4}_{\geq x^2}}$$

$$\Rightarrow \frac{x^2}{x^2+y^4} \leq 1$$

$$\leq \underbrace{(x-1)^2}_{\to 1} \cdot \underbrace{|y|}_{\to 0} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\therefore$ el límite es cero //

a) Sospecho que <u>no</u> existe por ese denominador

Si $x = y^2$ $\qquad \sigma(t) = (\underbrace{t^2}_{x}, \underbrace{t}_{y})$

$$\lim_{y\to 0} \frac{(y^2-1)^2 \cdot \sin(y^4) \cdot y}{y^4 + y^4} =$$

$$= \lim_{y\to 0} \frac{(y^2-1)^2 \cdot \sin(y^4) \cdot y}{2 \cdot y^4} \to 1$$

Sabemos que

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1$$

$$= \lim \frac{(y^4 - 2y^2 + 1) \cdot y}{2} = 0 \quad !!$$

Otra curva

$y = x$

$$\lim_{x\to 0} \frac{(x-1)^2 \cdot \sin(x^2) \cdot x}{x^2 + x^4} =$$

$$= \lim_{x \to 0} \frac{(x-1)^2 \cdot \sin(x^2) \cdot x}{x^2(1+x^2)} = \frac{(x-1)^2 \cdot \sin(x^2)}{x + x^3} \overset{L'H}{=} \frac{2(x-1)\cdot \sin x^2 + (x-1)^2 \cdot \cos x^2 \cdot 2x}{1+3x^2}$$

$$= \lim_{x \to 0} \frac{(x-1)^2 \cdot x}{1+x^2}$$

$$(x^2 - 2x + 1)x$$

$$\frac{x^3 - 2x^2 + x}{x^2+1}$$

$y = m \cdot x$

$x = my$

$$\lim_{x \to 0} \frac{(x-1)^2 \cdot \sin(x^2) \cdot m x}{x^2 + m^4 \cdot x^4}$$

$$= \lim \frac{\overbrace{(x-1)^2}^{\to 1} \cdot \sin(x^2) \cdot m \cancel{x}}{\cancel{x}\,(x + m^4 \cdot x^3)}$$

$$\overset{L'H}{=} \lim \frac{m \cdot \cos(x^2) \cdot \overbrace{2x}^{\to 0}}{\underbrace{1}_{=1} + \underbrace{2m^4 \cdot x^2}_{\to 0}} = 0$$

$$\lim_{y \to 0} \frac{(my-1)^2 \cdot \sin(m^2 y^2) \cdot y}{m^2 y^2 + y^4}$$

$$= (m-1) \cdot \lim_{y \to 0} \frac{\sin(m^2 y^2)}{m^2 \cdot y + y^3}$$

$$\overset{L'H}{=} (m-1) \lim_{y \to 0} \frac{\cos(m^2 \cdot y^2) \cdot m^2 \cdot \overbrace{2y}^{\to 0}}{\underbrace{m^2 + 3y}_{\to m^2}}$$

$$= 0$$

$y = x^a$

$$\lim_{x \to 0} \frac{(x-1)^2 \cdot \sin(x^2) \cdot x^a}{x^2 + x^{4a}} \overset{1.}{=} \lim \frac{\sin x^2}{2x^{3/2}} \overset{L'H}{=} l \frac{\cos x^2 \cdot 2x}{\cancel{2} \cdot \frac{3}{\cancel{2}} \cdot x^{1/2}}$$

$2x^2 \qquad 2x^{\frac{3}{2}}$

$$= l \frac{2x \cdot \cos x^2}{3x^{1/2}}$$

$$\overset{L'H}{=} l \frac{\underbrace{2 \cdot \cos x^2}_{=2} - \underbrace{4x^2 \cdot \sin x^2}_{=0}}{\frac{3}{2} \cdot x^{-\frac{1}{2}}}$$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{x^2+y^2}$.

Si
$y = x$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{2 \cdot x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{e^{x^2}\cdot 2\cancel{x}}{2\cdot 2\cancel{x}}$$

$$= \lim_{x\to 0} \frac{e^{x^2}}{2} = \frac{1}{2}$$

Si $y = 0$

$$\lim_{x\to 0} \frac{e^{0}-1}{x^2} = 0$$

Como encontré 2 límites distintos $\Rightarrow$ el límite no existe //

Ejercicio 3: Sea $f : \mathbb{R}^2 \to \mathbb{R}$ definida por:

$$f(x,y) = \begin{cases} \frac{x^2y^2 - \operatorname{sen}(x^4)}{x^2+\frac{1}{3}y^2} + 2 & \text{si } (x,y) \neq (0,0), \\ a & \text{si } (x,y) = (0,0). \end{cases}$$

Hallar, si es posible, un valor de $a \in \mathbb{R}$ para que $f(x,y)$ sea continua en todo $\mathbb{R}^2$. ¿Es f diferenciable para algún a?

Veo $x = 0$

$$f(0,y) = \frac{0 - \sin(0)}{0^2 + \frac{1}{3}y^2} + 2 = \underset{\substack{\uparrow \\ \text{Candidato a } a}}{2}$$

$$\left| \frac{x^2 \cdot y^2 - \sin(x^4)}{\underbrace{x^2}_{\geq 0} + \frac{1}{3}\underbrace{y^2}_{\geq 0}} + 2 - \overset{a}{\overset{\downarrow}{2}} \right| \leq \left| \frac{x^2 \cdot y^2 - \sin(x^4)}{\frac{1}{3}x^2 + \frac{1}{3}y^2} \right|$$

$$\frac{1}{3}x^2 \underset{x \geq 0}{\leq} x^2$$

Desig T.

$$\leq \frac{x^2 \cdot y^2}{\frac{1}{3}(x^2+y^2)} + \frac{\overbrace{|\sin(x^4)|}^{\leq x^4}}{\frac{1}{3}(x^2+y^2)}$$

$$\leq \frac{\|(x,y)\|^4}{\frac{1}{3}\|(x,y)\|^2} + \frac{\|(x,y)\|^4}{\frac{1}{3}\|(x,y)\|^2}$$

$$= 3\|(x,y)\|^2 + 3\|(x,y)\|^2$$

$$= 6\|(x,y)\|^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\therefore$ Si $a = 2 \Rightarrow f$ es continua en todo $\mathbb{R}^2$.

Es diferenciable? a debe ser 2 para que f sea continua.

Solo si

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left(\overset{2}{\widetilde{f(0,0)}} + f_x(0,0)\cdot(x-0) + f_y(0,0)(y-0)\right)}{\|(x,y)\|} \overset{?}{=} 0$$

Calculo derivadas parciales

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - \overset{=2}{\widetilde{f(0,0)}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{h^2\cdot 0} - \sin(h^4)}{h^2} + 2 - 2}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{\sin h^4}{h^2} = \lim_{h\to 0} \frac{\sin h^4}{h^3}$$

$$\overset{L'H}{\downarrow} \lim_{h\to 0} \frac{\cos(h^4)\cdot 4h^3}{3h^2} = \lim_{h\to 0} \frac{4\cdot\cos(h^4)\cdot h}{3}$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - \overbrace{f(0,0)}^{=2}}{h}$$

$$= \lim_{h\to 0} \left(\underbrace{\frac{0 - \overbrace{\sin(0)}^{=0}}{0 + \frac{1}{3}h^2}}_{=0} + 2 \quad - 2 \right) \frac{1}{h}$$

$$f_y(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left(\overbrace{f(0,0)}^{=2} + \overbrace{f_x(0,0)}^{0} \cdot (x-0) + \overbrace{f_y(0,0)}^{=0} (y-0) \right)}{\|(x,y)\|}$$

$$\lim_{(x,y)\to(0,0)} \frac{1}{\|(x,y)\|} \cdot \left(f(x,y) - \overbrace{f(0,0)}^{=2} \right)$$

<u>Acoto</u>

$$\left| \frac{1}{\|(x,y)\|} \left(\frac{x^2 \cdot y^2 - \sin x^4}{x^2 + \frac{1}{3}y^2} + 2 \quad - 2 \right) \right| =$$

$$= \frac{1}{\|(x,y)\|} \underbrace{\left| \frac{x^2 \cdot y^2 - \sin x^4}{x^2 + \frac{1}{3}y^2} \right|}_{\text{lo acoté arriba}} \leq \frac{1}{\|(x,y)\|} \left(\frac{x^2 \cdot y^2}{\frac{1}{3}(x^2+y^2)} + \frac{|\sin(x^4)|}{\frac{1}{3}(x^2+y^2)} \right)$$

$$\leq \frac{\|(x,y)\|^4}{\frac{1}{3}\|(x,y)\|^3} + \frac{\|(x,y)\|^4}{\frac{1}{3}\|(x,y)\|^3}$$

$$= 6\,\|(x,y)\| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\therefore$ $f(x,y)$ es diferenciable en $(0,0)$

con $a = 2$,

Además $f(x,y)$ es diferenciable $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

pues es suma, resta, producto, composición y cociente (con denominador distinto de cero) de funciones diferenciables (polinomios y la función seno).

$\therefore$ $f(x,y)$ es diferenciable $\forall (x,y) \in \mathbb{R}^2$ //

Ejercicio 4: Sea $f : \mathbb{R}^2 \to \mathbb{R}$ una función diferenciable tal que el plano tangente a su gráfico en el punto $(1, 2, f(1, 2))$ es

$$-x - 2y + z = -1.$$

Si $x = 3s + t^2$ e $y = 2s^2 + 2t$ y definimos $F(s, t) = f(x, y)$, calcular la ecuación del plano tangente al gráfico de F en el punto $(0, 1, F(0, 1))$.

$$\begin{cases} x = 3s + t^2 \\ y = 2s^2 + 2t \end{cases}$$

$$F(s,t) = f(3s + t^2,\ 2s^2 + 2t)$$

Plano tangente a F

$$P_1(0,1) = F(0,1) + \frac{\partial F}{\partial s}(s,t)\Big|_{(0,1)}(s-0) + \frac{\partial F}{\partial t}(s,t)\Big|_{(0,1)}(t-1)$$

$$F(0,1) = ?$$

$$\begin{cases} x = 3s + t^2 \,\Big|_{(0,1)} \\ y = 2s^2 + 2t \,\Big|_{(0,1)} \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$$

$$F(0,1) = f(1,2)$$

Veo su plano tangente en el $(1,2)$ pues coinciden

$$f(1,2) = x + 2y - 1\Big|_{(1,2)}$$

$$= 1 + 4 - 1$$

$$F(0,1) = 4$$

Calculo derivadas parciales

$$\frac{\partial}{\partial s} F(s,t) = \frac{\partial f}{\partial x}(x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial s}(s,t)$$

$$\frac{\partial}{\partial t} F(s,t) = \frac{\partial f}{\partial x}(x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial t}(s,t)$$

Coinciden con las derivadas del plano pues es su polinomio de grado 1.

Plano : $z = x + 2y - 1$

- $\dfrac{\partial f}{\partial x} = 1$
- $\dfrac{\partial f}{\partial y} = 2$

$$\frac{\partial x}{\partial s}(s,t) = \frac{\partial}{\partial s}\left(3s + t^2\right) = 3$$

$$\frac{\partial x}{\partial t}(s,t) = 2t$$

$$\frac{\partial x}{\partial t}(0,1) = 2$$

$$\frac{\partial y}{\partial s}(s,t) = \frac{\partial}{\partial t}\left(2s^2 + 2t\right) = 4s$$

$$\frac{\partial y}{\partial s}(0,1) = 0$$

$$\frac{\partial y}{\partial t}(s,t) = 2$$

Finalmente

$$\frac{\partial}{\partial s}F(s,t) = \underbrace{\frac{\partial f}{\partial x}(x(s,t),y(s,t))}_{=1} \cdot \underbrace{\frac{\partial x}{\partial s}(s,t)}_{=3} + \underbrace{\frac{\partial f}{\partial y}(x,y)}_{=2} \cdot \underbrace{\frac{\partial y}{\partial s}(s,t)}_{=0}$$

$$\frac{\partial}{\partial t}F(s,t) = \frac{\partial f}{\partial x}(x(s,t),y(s,t)) \cdot \underbrace{\frac{\partial x}{\partial t}(s,t)}_{=2} + \frac{\partial f}{\partial y}(x,y) \cdot \underbrace{\frac{\partial y}{\partial t}(s,t)}_{=2}$$

$$\frac{\partial}{\partial s}F(0,1) = 3$$

$$\frac{\partial}{\partial t}F(0,1) = 2+4 = 6$$

Volviendo a la ecuación del plano

$$P_1(0,1) = \underbrace{F(0,1)}_{=4} + \underbrace{\frac{\partial F}{\partial s}(s,t)\Big|_{(0,1)}}_{=3}(s-0) + \underbrace{\frac{\partial F}{\partial t}(s,t)\Big|_{(0,1)}}_{=6}(t-1)$$

$$P_1(0,1) = 4 + 3s + 6(t-1)$$

$$= 4 + 3s + 6t - 6$$

$$P_1(0,1) = 3s + 6t - 2$$

Como F es diferenciable por ser composición de funciones diferenciables (f, x, y)

$\Rightarrow$ Su plano tangente existe en $(0,1,F(0,1))$ y es

$$\boxed{\Pi : \ \omega = 3s + t - 2}$$

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