Análisis I - Análisis Matemático I - Matemática I - Análisis II (C)

1er. cuatrimestre 2020

Simulacro Segundo Parcial

1. Sea $f\colon \mathbb{R}^2 \to \mathbb{R}$ de clase $\mathcal{C}^2$ tal que su polinomio de Taylor de orden 2 en $(-1,1)$ es

$$p(x,y) = 2x^2 - xy + 5x - y + 5.$$

(a) Decidir si f tiene un extremo local en $(-1,1)$.

(b) Calcular

$$\lim_{(x,y)\to(-1,1)} \frac{f(x,y)-2}{\|(x,y)-(-1,1)\|}$$

a) Calculo $\nabla p(x,y) = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$

- $\frac{\partial p}{\partial x}(x,y) = 4x - y + 5$
- $\frac{\partial p}{\partial y}(x,y) = -x - 1$

$\frac{\partial p}{\partial x}(-1,1) = \frac{\partial f}{\partial x}(-1,1) = -4 - 1 + 5 = 0$

$\frac{\partial p}{\partial y}(-1,1) = \frac{\partial f}{\partial y}(-1,1) = 0$

$\nabla f(-1,1) = (0,0)$

$\therefore$ $(-1,1)$ es punto crítico.

- $P_{xx}(x,y) = 4$
- $P_{yy}(x,y) = 0$
- $P_{xy}(x,y) = P_{yx}(x,y) = -1$

El Hessiano de f en $(-1,1)$ es

$$Hf(-1,1) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$$

Criterio del Hessiano

1) $\left.\begin{array}{l} \det Hf(x_0,y_0) > 0 \\ f_{xx}(x_0,y_0) > 0 \end{array}\right\} \Rightarrow (x_0,y_0)$ es mínimo

2) $\left.\begin{array}{l} \det Hf(x_0,y_0) > 0 \\ f_{xx}(x_0,y_0) < 0 \end{array}\right\} \Rightarrow (x_0,y_0)$ es máximo

3) $\det Hf(x_0,y_0) < 0 \Rightarrow (x_0,y_0)$ es punto silla

4) $\det Hf(x_0,y_0) = 0 \Rightarrow$ el criterio no me sirve.

Entonces

$$Hf(-1,1) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$$

$\det Hf(-1,1) = -1 \Rightarrow (-1,1)$ es punto silla

$\therefore$ f no tiene extremo local en $(-1,1)$.

(b) Calcular

$$\lim_{(x,y)\to(-1,1)} \frac{f(x,y)-2}{\|(x,y)-(-1,1)\|}$$

Uso Prop. de P. de Taylor

Polinomio de Taylor de grado n de f en (x_0,y_0)

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-P_n(x,y)}{\|(x,y)-(x_0,y_0)\|^n} = 0$$

Calculo polin. de Taylor de orden 1 de f a partir de $p(x,y)$

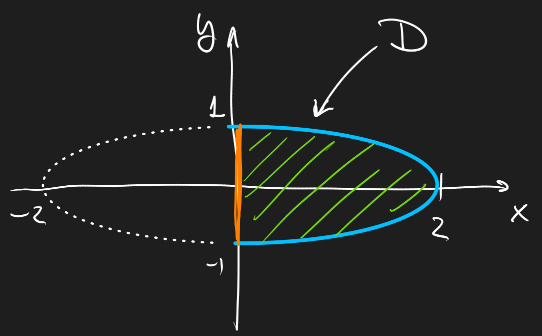
$$P_1(x,y) = \overbrace{p(-1,1)}^{=2} + \overbrace{p_x(-1,1)}^{=0}(x+1) + \overbrace{p_y(-1,1)}^{=0}(y-1)$$

$$P_1(x,y) = 2$$

Por prop., el límite es cero. //

2. Sea $f\colon \mathbb{R}^2 \to \mathbb{R}$ definida por $f(x,y) = xy^2 + 2y^2 + 1$. Hallar los máximos y mínimos absolutos de f en

$$D = \left\{ (x,y) \mid \frac{x^2}{4} + y^2 \leq 1,\ x \leq 0 \right\}.$$



• Veo el Interior $\mathring{D}$:

Calculo Hessiano de f :

$$f_x = y^2$$

$$f_y = 2xy + 4y$$

$$\Rightarrow \nabla f(x,y) = (y^2, 2xy + 4y) \overset{\text{quiero}}{=} (0,0)$$

$$\Leftrightarrow \begin{cases} y^2 = 0 \\ 2xy + 4y = 0 \end{cases} \Rightarrow \text{vale } \forall x$$

$$PCs = \{ (x,0) : x \in [0,2] \}$$

- $f_{xx} = 0$
- $f_{yy} = 2x + 4$
- $f_{xy} = f_{yx} = 2y$

$$Hf(x,y) = \begin{bmatrix} 0 & 2y \\ 2y & 2x+4 \end{bmatrix}$$

$$Hf(x,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2x+4 \end{bmatrix}$$

el criterio no me dice nada ! pues $\det Hf(x,0) = 0$

Pero no importa , pues

$f(x,0) = 1$, como candidato a max/min .

- Veo el Borde ∂D

1) segmento $(0,y)$ con $y \in [-1,1]$

Lo parametrizo como

$$\sigma(t) = (0, t) \quad \text{con } t \in [-1,1]$$

$f(x,y) = xy^2 + 2y^2 + 1$

$$f(\sigma(t)) = f(0,t)$$

$$= 2t^2 + 1$$

$$f'(\sigma(t)) = 4t = 0 \iff t = 0$$

$$\sigma(0) = (0,0)$$

PC en $(0,0)$.

2) Elipse :

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) \end{cases} \quad \text{con} \quad \underbrace{\frac{x^2}{4} + y^2 = 1}_{g(x,y)}$$

$$\begin{cases} \text{I} & y^2 = \lambda \cdot \frac{x}{2} \\ \text{II} & 2xy + 4y = \lambda \cdot 2y \Rightarrow 2xy + 2y(2-\lambda) = 0 \\ \text{III} & \frac{x^2}{4} + y^2 = 1 \end{cases}$$

$$2y(x + 2 - \lambda) = 0$$

$\hookrightarrow y=0$

$\hookrightarrow x = \lambda - 2$

- Si $y=0$

$$0 = \lambda \cdot \frac{x}{2} \begin{cases} \nearrow x=0 \quad \times \text{ Abs por III} \\ \searrow \lambda = 0 \end{cases}$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 2^2$$

$$x = \begin{cases} \nearrow 2 \\ \searrow -2 \end{cases}$$

$$PCs = \{(2,0), (-2,0)\} \text{ con } \lambda = 0$$

Si $\lambda = x+2$ en I :

$$y^2 = (x+2)\frac{x}{2}$$

$$y^2 = x^2 + x$$

$$y^2 = x(x+1)$$

en III

$$\frac{x^2}{4} + x^2 + x = 1 \Rightarrow \frac{5}{4}x^2 + x - 1 = 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{5}{4}}}{\cancel{2} \cdot \frac{5}{\cancel{4}\,2}} =$$

$$= \frac{-1 \pm \sqrt{6}}{\frac{5}{2}}$$

$$= \frac{2(-1 \pm \sqrt{6})}{5}$$

$$= \frac{2}{5}(-1 \pm \sqrt{6})$$

$$x = \begin{cases} \frac{2}{5}(\sqrt{6} - 1) \\ \frac{2}{5}(-\sqrt{6} - 1) \end{cases}$$

Meto x en III

$$\frac{1}{\cancel{4}}\left(\frac{\cancel{4}}{25} \cdot \left(6 - 2\sqrt{6} + 1\right)\right)$$

$$\frac{x^2}{4} + y^2 = 1$$

$\boxed{x_1}$

$$y^2 = 1 - \frac{x^2}{4}$$

$$= \left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{2}\right)$$

$$= \left(1 + \frac{1}{5}(\sqrt{6} - 1)\right)\left(1 - \frac{1}{5}(\sqrt{6} - 1)\right)$$

Abandono, sigo der de enter

(III) $\frac{x^2}{4} + y^2 = 1$

$x = \lambda - 2 \Rightarrow (\lambda-2)^2 = \lambda^2 - 4\lambda + 4$

$\frac{\lambda^2}{4} - \lambda + \cancel{1} + y^2 = \cancel{1}$

$y^2 = \lambda - \frac{\lambda^2}{4}$

$y^2 = \lambda\left(1 - \frac{\lambda}{4}\right)$

$\lambda - 2$

(I) $y^2 = \lambda \cdot \frac{x}{2}$

$\lambda - \frac{\lambda^2}{4} = \lambda \cdot \frac{(\lambda-2)}{2}$

$\lambda - \frac{\lambda^2}{4} = \frac{\lambda^2}{2} - \lambda$

$0 = \frac{3}{4}\lambda^2 - 2\lambda$

$0 = \lambda\left(\frac{3}{4}\lambda - 2\right)$

$\hookrightarrow \lambda = 0 \Rightarrow \begin{cases} x = \lambda - 2 \overset{\lambda=0}{=} -2 & \checkmark \\ y^2 = \lambda - \frac{\lambda^2}{4} = 0 & \checkmark \end{cases}$ Caso anterior

$\hookrightarrow \frac{3}{4}\lambda = 2$

$\hookrightarrow \lambda = \frac{8}{3}$

$\hookrightarrow x = \frac{8}{3} - 2 = \frac{2}{3}$

$y^2 = \frac{8}{9} \Rightarrow y = \frac{4}{3}, \; -\frac{4}{3}$

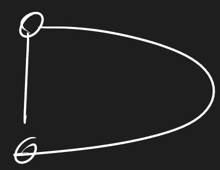
de antes $PC_S = \{(2,0), (-2,0)\}$ con $\lambda = 0$

ahora

$$PC_S = \left\{\left(\frac{2}{3}, \frac{4}{3}\right), \left(\frac{2}{3}, -\frac{4}{3}\right)\right\}$$

- Veo esquinas

$$PC_S = \{(0,-1), (0,1)\}$$



Evaluo todo

$f(x,0) = 1$ Mín abs : $(x,0)$ con $x \in [0,2]$

$f(0,0) = 1$

$f(2,0) = 1$
$f(-2,0) = 1$

$f\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{155}{27} \approx 5{,}74$
$f\left(\frac{2}{3}, -\frac{4}{3}\right) = \frac{155}{27}$

Máximos absolutos $\left(\frac{2}{3}, \frac{4}{3}\right)$ y $\left(\frac{2}{3}, -\frac{4}{3}\right)$

$f(0,1) = 3$
$f(0,-1) = 3$

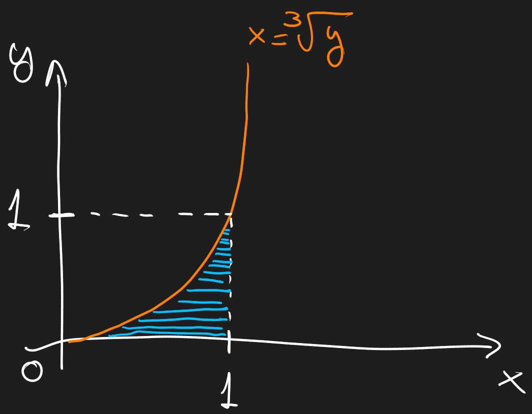
3. Calcular las siguientes integrales

(a) $\displaystyle\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{1}{1+x^4}\,dx dy.$

(b) $\displaystyle\iiint_E xz^2 \; dV$ donde E es el sólido debajo de la superficie $z = x^2$ y arriba del rectángulo $R = [0,1] \times [2,3]$ en el plano xy.

$$\int_{y=0}^{y=1} \int_{x=\sqrt[3]{y}}^{x=1} \frac{1}{1+x^4}\,dx\,dy = \star$$

$$\underbrace{x=\sqrt[3]{y}}_{x^3=y}$$

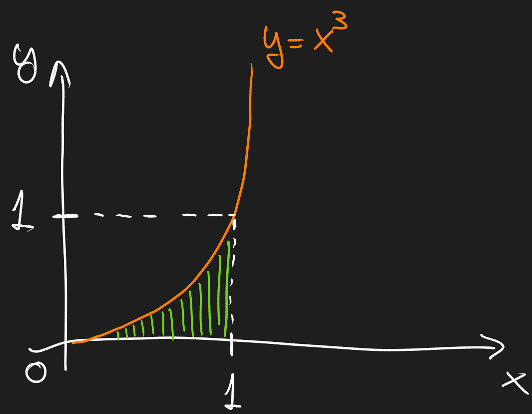


$$0 \leq y \leq 1$$

$$0 \leq \sqrt[3]{y} \leq x \leq 1$$

$$0 \leq y \leq x^3 \leq 1$$

$$\Rightarrow \text{Tengo} \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq x^3 \end{cases}$$



$$\star = \int_{x=0}^{x=1} \int_{y=0}^{y=x^3} \frac{1}{1+x^4}\,dy\,dx$$

$$= \int_{x=0}^{x=1} \frac{1}{1+x^4} \cdot x^3 \cdot dx$$

CA :

$$\left(\ln(1+x^4)\right)' = \frac{1}{1+x^4} \cdot 4x^3$$

$$\left(\frac{1}{4}\ln(1+x^4)\right)' = \frac{1}{\cancel{4}} \cdot \frac{1}{1+x^4} \cdot \cancel{4}\, x^3$$

$$= \frac{1}{4}\ln(1+x^4)\Big|_{x=0}^{1}$$

$$= \frac{1}{4}\ln 2$$

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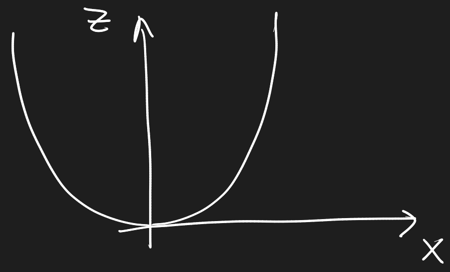
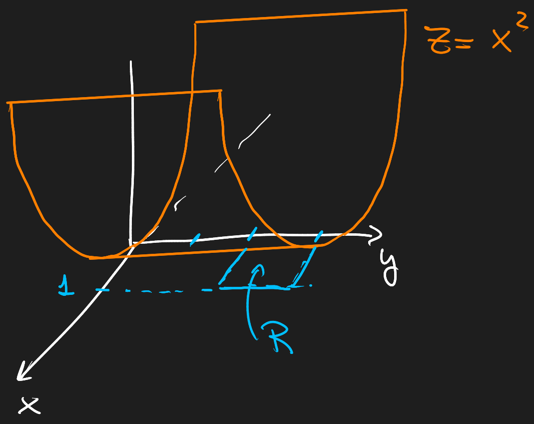
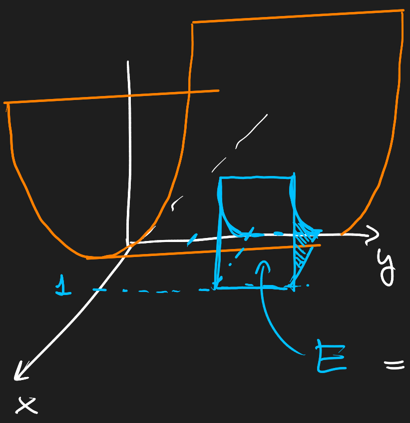
(b) $\iiint_E xz^2\, dV$ donde E es el sólido debajo de la superficie $z = x^2$ y arriba del rectángulo $R = [0,1]\times[2,3]$ en el plano xy.







$$E = \left\{(x,y,z)\in\mathbb{R}^3 : \begin{array}{l} 0\leq x\leq 1, \\ 2\leq y\leq 3, \\ 0\leq z\leq x^2 \end{array}\right\}$$

$$\int_{x=0}^{1}\int_{z=0}^{x^2}\int_{y=2}^{3} x z^2 \, dy\, dz\, dx = \int_{x=0}^{1}\int_{z=0}^{x^2} x z^2 \int_{y=2}^{3} 1 \, dy\, dz\, dx$$

$$= \int_{x=0}^{1} x \int_{z=0}^{x^2} z^2 \, dz\, dx$$

$$= \int_{x=0}^{1} x \cdot \left[\frac{z^3}{3}\right]_0^{x^2} dx$$

$$= \int_{x=0}^{1} x \cdot \frac{x^6}{3}\, dx$$

$$= \frac{1}{3} \int_0^1 x^7 \, dx$$

$$= \frac{1}{3} \frac{x^8}{8} \Big]_0^1$$

$$= \frac{1}{24}$$

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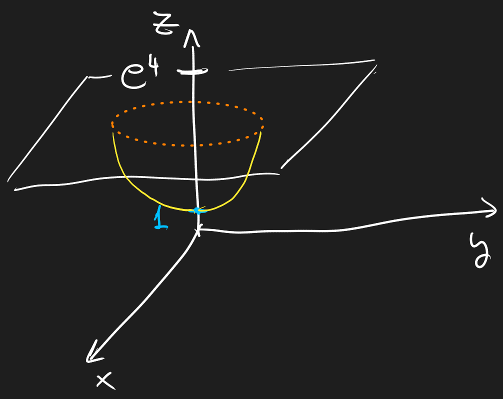
4. Hallar el volumen del sólido acotado por las superficies

$$z = e^{4x^2+4y^2} \quad y \quad z = e^4.$$

$$z = e^{4(x^2+y^2)}$$

$$z = e^4$$



Si igualo

$$e^4 = e^{4(x^2+y^2)} \iff x^2+y^2 = 1$$

Uso polares

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ z = z \end{cases} \qquad \text{con} \quad \begin{array}{l} r \in [0,1] \\ \theta \in [0, 2\pi) \\ e^{4(x^2+y^2)} \leq z \leq e^4 \\ e^{4r^2} \leq z \leq e^4 \end{array}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=e^{4r^2}}^{e^4} 1 \cdot r \cdot dz\, dr\, d\theta =$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \left(e^4 - e^{4r^2} \right) dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{r^2}{2} e^4 \Big|_0^1 d\theta - \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \cdot e^{4r^2}\, dr\, d\theta$$

I $\qquad\qquad\qquad\qquad$ II

$$\text{I} \quad \int_{\theta=0}^{2\pi} \frac{r^2}{2} e^4 \Big|_0^1 \, d\theta = 2 \cdot \pi \cdot \frac{1}{2} e^4$$

$$= \pi \cdot e^4$$

$$\text{II} \quad \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \cdot e^{4r^2} \, dr \, d\theta =$$

CA:

$$\left(e^{4r^2}\right)' = e^{4r^2} \cdot 8r$$

$$\left(\frac{1}{8} e^{4r^2}\right)' = \frac{1}{\cancel{8}} \cdot e^{4r^2} \cdot \cancel{8} r$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{8} \cdot e^{4r^2} \Big|_{r=0}^{1} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}\left(e^4 - 1\right) d\theta$$

$$= \frac{\pi}{4}\left(e^4 - 1\right)$$

Finalmente

$$\text{I} - \text{II} = \pi \cdot e^4 - \overbrace{\frac{\pi}{4}\left(e^4 - 1\right)}^{-\frac{\pi}{4} e^4 + \frac{\pi}{4}}$$

$$= \frac{3}{4}\pi \cdot e^{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{4}\left(3e^{4} + 1\right)$$

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