

1. Sea $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ de clase \mathcal{C}^2 tal que su polinomio de Taylor de orden 2 en $(-1, 1)$ es

$$p(x, y) = 2x^2 - xy + 5x - y + 5.$$

(a) Decidir si f tiene un extremo local en $(-1, 1)$.

(b) Calcular

$$\lim_{(x,y) \rightarrow (-1,1)} \frac{f(x,y) - 2}{\|(x,y) - (-1,1)\|}$$

a) • Las derivadas segundas de f coinciden con el Polin. de Taylor de orden 2

• Calculo

$$f_x(x,y) \stackrel{\text{en } (-1,1)}{\downarrow} = 4x - y + 5 \Rightarrow f_x(-1,1) = 0$$

$$f_y(x,y) \stackrel{\downarrow}{=} -x - 1 \Rightarrow f_y(-1,1) = 0$$

$$f_{xx}(x,y) \stackrel{\text{en } (-1,1)}{\downarrow} = 4$$

$$f_{yy}(x,y) = 0$$

$$f_{xy}(x,y) = f_{yx}(x,y) = -1$$

\uparrow
 $f \in \mathcal{C}^2$

$\underbrace{\hspace{10em}}$
es Punto Crítico.

Armo Hessiano :

$$Hf(x,y) \overset{(-1,1)}{\downarrow} = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$$

• $\det Hf = -1 < 0 \Rightarrow (-1,1)$ es Punto Silla.

∴ $(-1,1)$ no es extremo local.

Recordo : Criterio del Hessiano

Concavidad en \mathbb{R}

• $\left. \begin{array}{l} f'' > 0 \\ \det Hf > 0 \end{array} \right\} \Rightarrow PC, \text{ es m\u00ednimo}$

$$\bigcup_{f'' > 0} \bigcap_{f'' < 0}$$

• $\left. \begin{array}{l} f'' < 0 \\ \det Hf > 0 \end{array} \right\} \Rightarrow PC, \text{ es m\u00e1ximo}$

• $\det Hf < 0 \Rightarrow PC \text{ es punto silla,}$

• $\det Hf = 0 \Rightarrow$ el Criterio no sirve ☹

b) Quiero usar Prop. de Polinomio de Taylor

$$\lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y) - P_n(x,y)}{\| (x,y) - (a,b) \|^n} = 0$$

Como tengo P_2 y quiero P_1

$$P_1(x,y) = ?$$

$$\begin{aligned} P_2(x,y) = & \underbrace{f(-1,1)}_{=2} + \underbrace{f_x(-1,1)}_{=0} (x+1) + \underbrace{f_y(-1,1)}_{=0} (y-1) \\ & + \underbrace{\frac{1}{2} f_{xx}(-1,1)}_{=4} (x+1)^2 + \underbrace{\frac{1}{2} f_{yy}(-1,1)}_{=0} (y-1)^2 \\ & + \underbrace{f_{xy}(-1,1)}_{=-1} (x+1)(y-1) \end{aligned}$$

$$\Rightarrow P_1(x,y) = 2$$

∴ estoy en el caso

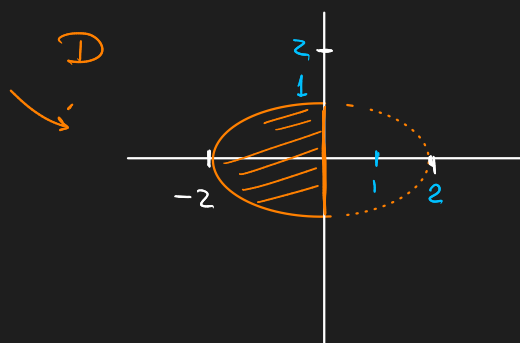
$$\lim_{(x,y) \rightarrow (-1,1)} \frac{f(x,y) - \underbrace{P_1(x,y)}_{=2}}{\| (x,y) - (-1,1) \|^1} \stackrel{\text{Prop. Polin. de Taylor.}}{=} 0 //$$

2. Sea $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ definida por $f(x, y) = xy^2 + 2y^2 + 1$. Hallar los máximos y mínimos absolutos de f en

$$D = \left\{ (x, y) \mid \frac{x^2}{4} + y^2 \leq 1, x \leq 0 \right\}.$$

Dibujó D

Elipse



Busca en el interior \mathring{D}

$$\begin{aligned} \bullet \quad f_x &= y^2 = 0 \Leftrightarrow y=0 \\ f_y &= 2xy + 4y = 0 \Leftrightarrow xy = -2y \\ &\quad \text{si } y=0 \Rightarrow xy = -2y \quad \forall x \end{aligned}$$

PCs en $(x, 0) \quad \forall x \in (-1, 0)$

Además

$$f(x, 0) = 1 \quad \forall x \in \mathbb{R}$$

Crit:

$$\begin{bmatrix} 0 & 2y \\ 2y & 2x+4 \end{bmatrix} \Rightarrow \det Hf = -4y^2 \Big|_{y=0} = 0 \quad \parallel$$

Después comparo con otros candidatos.

Border ∂D

Sobre $x=0$:

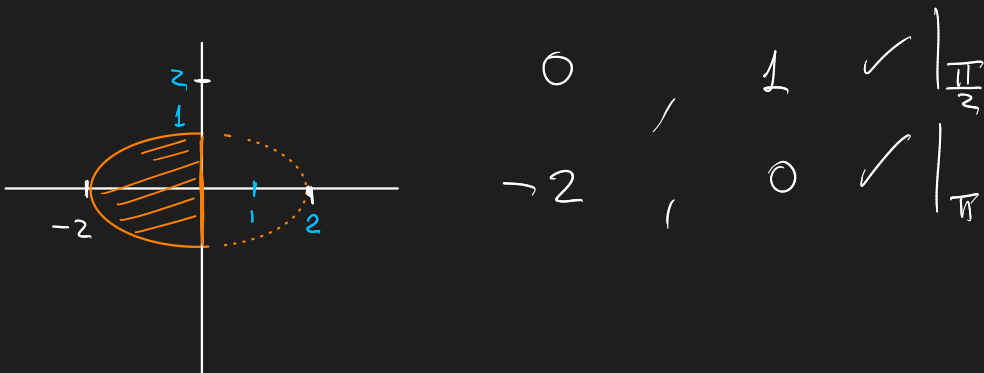
$$\left. \begin{array}{l} f_x = y^2 \\ f_y = \cancel{2xy} + 4y \\ \quad = 0 \end{array} \right\} = (0,0) \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad \text{mismo que arriba.}$$

Sobre la semi-elipse

$$\bullet \quad \frac{x^2}{4} + y^2 = 1 \quad \text{con } x \leq 0$$

Puedo parametrizar

$$\sigma(t) = (2 \cdot \cos t, \sin t) \quad t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$$



Compongo:

$$\begin{aligned} g(t) &= 2 \cos t \cdot \sin^2 t + 2 \sin^2 t + 1 \\ &= 2 \cdot \sin^2 t (\cos t + 1) + 1 \end{aligned}$$

$$1 \leq g(t) \leq 3$$

$$g(t) = 1 \Leftrightarrow \sin^2 t = 0 \quad \text{ó} \quad \cos t + 1 = 0$$

$$\sin^2 t = 0 \Leftrightarrow \sin t = 0 \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$\Leftrightarrow t = \pi$$

$$\cos t + 1 = 0 \Leftrightarrow \cos t = -1$$

$$\Leftrightarrow t = \pi$$

Evaluó sobre la curva

$$\sigma(t) = (2 \cdot \cos t, \sin t)$$

$$\sigma(\pi) = (-2, 0)$$

• PC en $(-2, 0)$

$$\text{Vértices} : \{(0, 1), (0, -1)\}$$

Evaluó en f los PCs

$$\bullet f(x, 0) = 1 \quad \forall x \in (-2, 0)$$

$$\bullet f(0, 0) = 1$$

$$\bullet f(-2, 0) = 1$$

$$\bullet f(0, 1) = 3$$

$$\bullet f(0, -1) = 3$$

$$\left. \begin{array}{l} \bullet f(0, 1) = 3 \\ \bullet f(0, -1) = 3 \end{array} \right\} \text{máximos absolutos} = \{(0, 1), (0, -1)\}$$

$$\text{mínimos absolutos} = \{(x, 0) \mid x \in [-2, 0]\}$$

3. Calcular las siguientes integrales

(a) $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{1}{1+x^4} dx dy.$

(b) $\iiint_E xz^2 dV$ donde E es el sólido debajo de la superficie $z = x^2$ y arriba del rectángulo $R = [0, 1] \times [2, 3]$ en el plano xy .

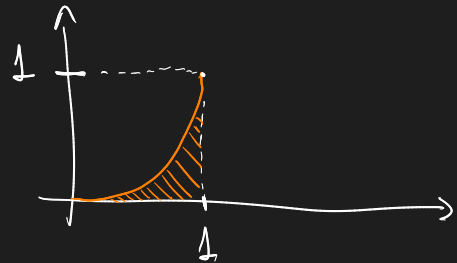
a) $\int \frac{1}{1+x^4} dx$ es complicada o no existe

°. reescribo región de integración D

$$D = \{ 0 \leq y \leq 1, \sqrt[3]{y} \leq x \leq 1 \}$$

$$\sqrt[3]{y} \leq x \leq 1$$

$$0 \leq \underbrace{y \leq x^3}_{\text{}} \leq 1$$



$$\tilde{D} = \{ 0 \leq x \leq 1, 0 \leq y \leq x^3 \}$$

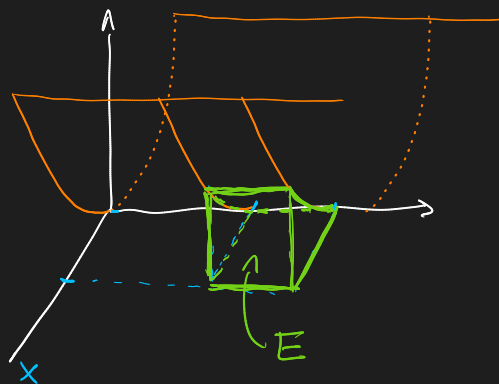
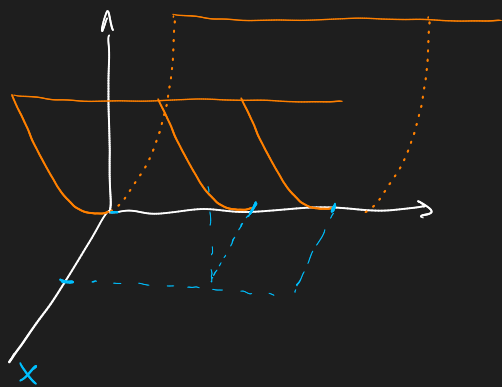
$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^3} \frac{1}{1+x^4} dx = \int_{x=0}^{x=1} \frac{1}{1+x^4} \cdot y \bigg|_{y=0}^{y=x^3} dx$$

$$= \int_{x=0}^{x=1} \frac{x^3}{1+x^4} dx$$

$$= \frac{1}{4} \ln(1+x^4) \bigg|_{x=0}^{x=1}$$

$$= \frac{1}{4} (\ln(2) - 1) //$$

b)



$$E = \left\{ (x, y, z) : 0 \leq x \leq 1, 2 \leq y \leq 3, 0 \leq z \leq x^2 \right\}$$

$$\iiint_E x z^2 dV = \int_{y=2}^{y=3} \int_{x=0}^1 \int_{z=0}^{z=x^2} x z^2 dz dx dy$$

$$= \int_{y=2}^{y=3} \int_{x=0}^1 x \left. \frac{z^3}{3} \right|_0^{x^2} dx dy$$

$$= \int_{y=2}^{y=3} \int_{x=0}^1 x \cdot \frac{x^6}{3} dx dy$$

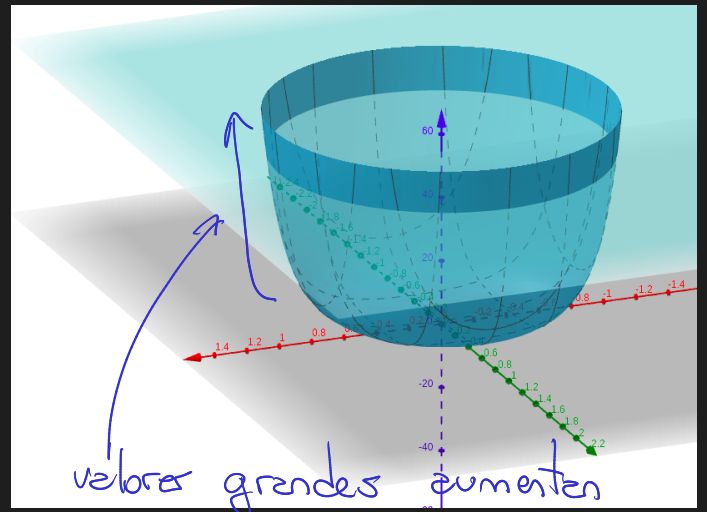
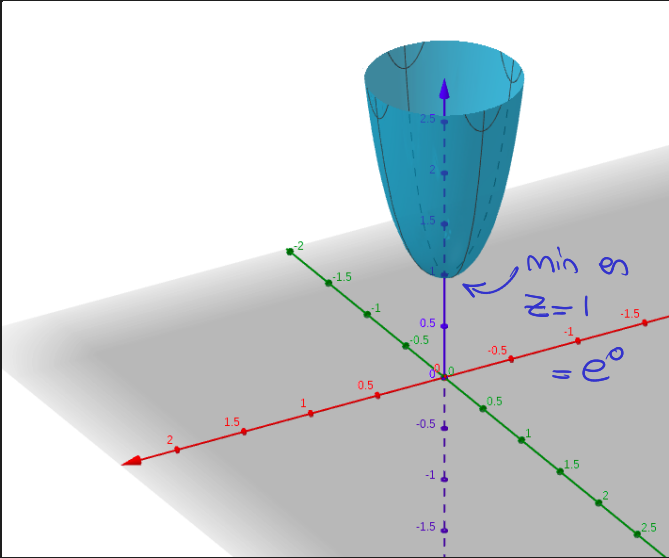
$$= \int_{y=2}^3 \left. \frac{x^8}{3 \cdot 8} \right|_0^1 dy$$

$$= \int_{y=2}^3 \frac{1}{24} dy$$

$$= \frac{1}{24} \int$$

4. Hallar el volumen del sólido acotado por las superficies

$$z = e^{4x^2+4y^2} \quad y \quad z = e^4 \approx 54,6$$



pendiente rápidamente:
 $z = e^{\text{Paraboloide}}$
↑ exponencial

Integrar sobre $e^{4x^2+4y^2}$ no me permite
separar en integrales de 1 variable
so tengo que hacer cambio de variables o
parametrizar.

Notar que $4x^2 + 4y^2$ se simplifica si reemplazo
x e y por cos y sin.

Uso Cilindrico

$$\begin{cases} x = r \cdot \cos \theta & r \in [0, \phi(z)] \\ y = r \cdot \sin \theta & \theta \in [0, 2\pi] \\ z = z & \text{con } z \in [1, e^4] \end{cases}$$

donde

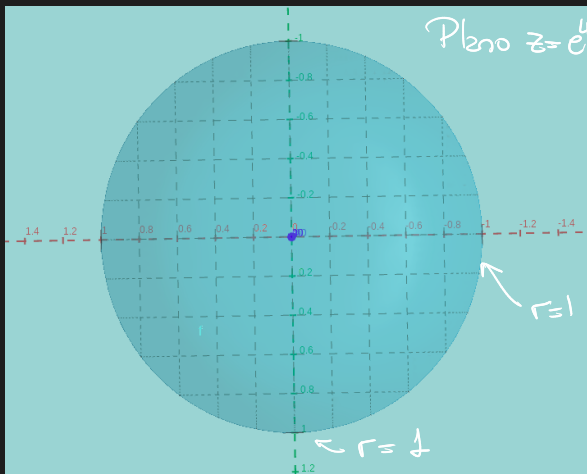
$$\phi(z) \Big|_{z=1} = 0$$

$$z = e^{\underbrace{4(x^2+y^2)}_{r^2 \cos^2 \theta + r^2 \sin^2 \theta}}$$

$$z = e^{4r^2} \Rightarrow \log z = 4r^2 \quad \begin{matrix} \swarrow z \geq 1 \\ \searrow z \geq 1 \end{matrix}$$

$$|r| = \frac{1}{2} \sqrt{\log z}$$

$$r \stackrel{r \geq 0}{=} \frac{1}{2} \sqrt{\log z}$$



Si $z = e^4$ (intersec con el plano)

$$\Rightarrow r = 1$$

Que se da cuando

$$e^{4x^2+4y^2} = e^4$$

$$x^2 + y^2 = 1$$

$$\int_x \int_y \int_z f(x,y) dv =$$

Por qué está mal?

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{\frac{1}{2} \sqrt{\log z}} 1 \cdot r dr d\theta dz$$

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \left. \frac{r^2}{2} \right|_0^{\frac{1}{2} \sqrt{\log z}}$$

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} \cdot \frac{1}{4} \cdot \log z$$

$$= \int_{z=1}^{e^4} + \frac{\pi}{2 \cdot 4} \log z$$

$$= \frac{\pi}{4} \log e^4 - \frac{\pi}{4} \cdot \log 1$$

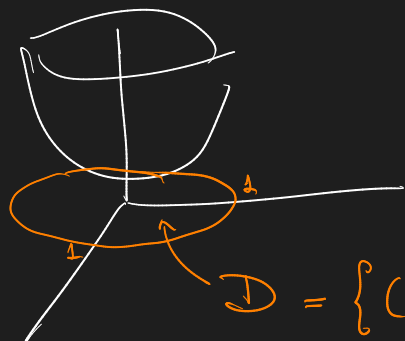
$$= \pi$$

$$\int e^{-r^2} \cdot r dr = \int -e^u \cdot \frac{1}{2} \cdot du = -\frac{1}{2} e^u + C$$

$$u = -r^2$$

$$du = -2r \cdot dr$$

$$= -\frac{1}{2} e^{-r^2}$$



$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

z edemár

$$e^{4(x^2+y^2)} \leq e^4 \quad \forall (x, y) \in D$$

$$\Rightarrow E = \{(x, y, z) : e^{4(x^2+y^2)} \leq z \leq e^4, (x, y) \in D\}$$

$$\therefore \text{Vol}(E) = \iint_D e^4 dA - \iint_D e^{4(x^2+y^2)} dA \geq 0$$

$$= \iint_D e^4 - e^{4(x^2+y^2)} dA$$

Polaris

$$\left. \begin{aligned} x &= r \cdot \cos \theta \\ y &= r \cdot \sin \theta \end{aligned} \right\} \Rightarrow x^2 + y^2 = r^2 \cdot (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$= \underbrace{\int_{\theta=0}^{2\pi} \int_{r=0}^1}_{D'} (e^4 - e^{4r^2}) \cdot r dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^1 e^4 \cdot r \, dr \, d\theta - \int_{\theta=0}^{2\pi} \int_{r=0}^1 e^{4r^2} \cdot r \, dr \, d\theta$$

$$u = 4r^2$$

$$du = 8r \, dr$$

$$= e^4 \int_0^{2\pi} \left. \frac{r^2}{2} \right|_0^1 - \int_0^{2\pi} \int_{u=0}^{u=4} e^u \cdot \frac{du}{8} \, d\theta$$

$$= \pi \cdot e^4 - \frac{1}{8} \int_0^{2\pi} e^u \Big|_0^4 \, d\theta$$

$$= \pi \cdot e^4 - \frac{1}{8} \cdot \int_0^{2\pi} (e^4 - 1) \, d\theta$$

$$= \pi \cdot e^4 - \frac{\pi}{4} \cdot (e^4 - 1)$$