Análisis I - Análisis Matemático I - Matemática I - Análisis II (C)

1er. cuatrimestre 2020

Simulacro Segundo Parcial

1. Sea $f\colon \mathbb{R}^2 \to \mathbb{R}$ de clase $\mathcal{C}^2$ tal que su polinomio de Taylor de orden 2 en $(-1,1)$ es

$$p(x,y) = 2x^2 - xy + 5x - y + 5.$$

(a) Decidir si f tiene un extremo local en $(-1,1)$.

(b) Calcular

$$\lim_{(x,y)\to(-1,1)} \frac{f(x,y)-2}{\|(x,y)-(-1,1)\|}$$

a) • Las derivadas segundas de f coinciden con el Polin. de Taylor de orden 2

• Calculo

$$f_x(x,y) \overset{\text{en } (-1,1)}{=} 4x - y + 5 \Rightarrow f_x(-1,1) = 0$$

$$f_y(x,y) = -x - 1 \Rightarrow f_y(-1,1) = 0$$

$\underbrace{\qquad\qquad}_{\text{es Punto Crítico.}}$

$$f_{xx}(x,y) \overset{\text{en } (-1,1)}{=} 4$$

$$f_{yy}(x,y) = 0$$

$$f_{xy}(x,y) \underset{f \in \mathcal{C}^2}{=} f_{yx}(x,y) = -1$$

Armo Hessiano :

$$Hf(x,y) \overset{(-1,1)}{=} \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$$

- $\det Hf = -1 < 0 \Rightarrow (-1,1)$ es Punto Silla.

$\therefore (-1,1)$ no es extremo local.

Recuerdo : Criterio del Hessiano

Concavidad en $\mathbb{R}$

$\cup$ $\cap$

$f''>0$ $f''<0$

- $\left.\begin{array}{l} \overset{\text{concavidad}}{f_{xx}} > 0 \\ \det Hf > 0 \end{array}\right\} \Rightarrow$ PC. es mínimo

- $\left.\begin{array}{l} f_{xx} < 0 \\ \det Hf > 0 \end{array}\right\} \Rightarrow$ PC. es máximo

- $\det Hf < 0 \Rightarrow$ PC es punto silla.

- $\det Hf = 0 \Rightarrow$ el Criterio no sirve

b) Quiero usar Prop. de Polinomio de Taylor

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - P_n(x,y)}{\|(x,y)-(a,b)\|^n} = 0$$

Como tengo P_2 y quiero P_1

$$P_1(x,y) = ?$$

$$P_2(x,y) = \underbrace{f(-1,1)}_{p(-1,1)=2} + \underbrace{f_x(-1,1)}_{=0}(x+1) + \underbrace{f_y(-1,1)}_{=0}(y-1)$$
$$+ \frac{1}{2}\underbrace{f_{xx}(-1,1)}_{=4}(x+1)^2 + \frac{1}{2}\underbrace{f_{yy}(-1,1)}_{=0}(y-1)^2$$
$$+ \underbrace{f_{xy}(-1,1)}_{=-1}(x+1)(y-1)$$

$$\Rightarrow P_1(x,y) = 2$$

$\therefore$ estoy en el caso

$$\lim_{(x,y)\to(-1,1)} \frac{f(x,y) - \overbrace{P_1(x,y)}^{=2}}{\|(x,y)-(-1,1)\|^1} \overset{\text{Prop. Polin. de Taylor.}}{=} 0$$

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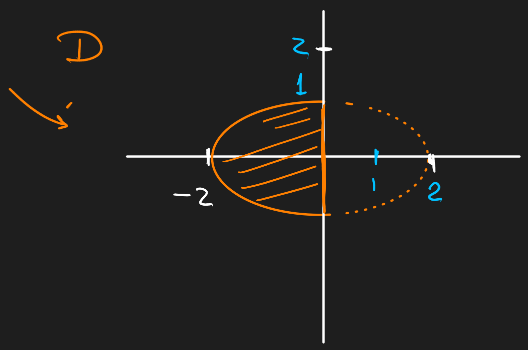
2. Sea $f\colon \mathbb{R}^2 \to \mathbb{R}$ definida por $f(x,y) = xy^2 + 2y^2 + 1$. Hallar los máximos y mínimos absolutos de f en

$$D = \left\{ (x,y) \mid \frac{x^2}{4} + y^2 \leq 1,\ x \leq 0 \right\}.$$

Dibujo D

Elipse



Busco en el interior $\mathring{D}$

- $f_x = y^2 = 0 \iff y = 0$

  $f_y = 2xy + 4y = 0 \iff xy = -2y$

  si $y=0 \Rightarrow x.y = -2y \ \forall x$

PCs en $(x,0)$ $\forall x \in (-1,0)$

Además

$f(x,0) = 1 \quad \forall x \in \mathbb{R}$

Crit:

$$\begin{bmatrix} 0 & 2y \\ 2y & 2x+4 \end{bmatrix} \Rightarrow \det Hf = -4y^2 \Big|_{y=0} = 0 \ \text{!!}$$

Después comparo con otros candidatos.

Borde ∂D

Sobre $x = 0$:

$$\left.\begin{aligned} f_x &= y^2 \\ f_y &= \underbrace{\cancel{2xy}}_{=0} + 4y \end{aligned}\right\} = (0,0) \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

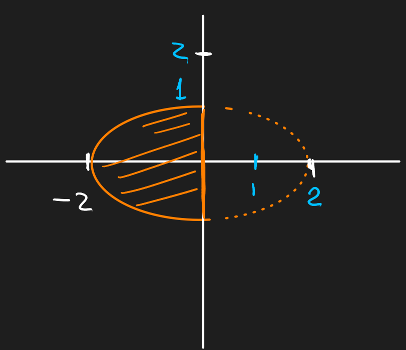
mismo que arriba.

Sobre la semi-elipse

- $\dfrac{x^2}{4} + y^2 = 1$ con $x \leq 0$

Puedo parametrizar

$$\sigma(t) = (2 \cdot \cos t \ , \ \sin t) \qquad t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$$



$$(0 \ , \ 1) \checkmark \Big|_{\frac{\pi}{2}}$$

$$(-2 \ , \ 0) \checkmark \Big|_{\pi}$$

Compongo :

$$g(t) = 2\cos t \cdot \sin^2 t + 2\sin^2 t + 1$$

$$= 2 \cdot \sin^2 t \, (\cos t + 1) + 1$$

$$\underbrace{}_{\substack{\geq 0 \\ \leq 1}} \qquad \underbrace{}_{\substack{\geq 0 \\ \leq 1}}$$

$1 \leq g(t) \leq 3$

$$g(t) = 1 \iff \sin^2 t = 0 \quad \text{ó} \quad \cos t + 1 = 0$$

$$\sin^2 t = 0 \iff \sin t = 0 \qquad t \in \left[\tfrac{\pi}{2}, \tfrac{3}{2}\pi\right]$$

$$\iff t = \pi$$

$$\cos t + 1 = 0 \iff \cos t = -1$$

$$\iff t = \pi$$

Evalúo sobre la curva

$$\sigma(t) = (2 \cdot \cos t, \sin t)$$

$$\sigma(\pi) = (-2, 0)$$

- PC en $(-2,0)$

Vértices : $\{(0,1), (0,-1)\}$

Evalúo en f los PCs

- $f(x,0) = 1 \quad \forall x \in (-2,0)$
- $f(0,0) = 1$

- $f(-2,0) = 1$

- $f(0,1) = 3$

- $f(0,-1) = 3$

$$\boxed{\text{máximos absolutos} = \{(0,1),(0,-1)\}}$$

$$\boxed{\text{mínimos absolutos} = \{(x,0) \mid x \in [-2,0]\}}$$

3. Calcular las siguientes integrales

(a) $\displaystyle\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{1}{1+x^4}\,dx\,dy.$

(b) $\displaystyle\iiint_E xz^2\, dV$ donde E es el sólido debajo de la superficie $z = x^2$ y arriba del rectángulo $R = [0,1] \times [2,3]$ en el plano xy.

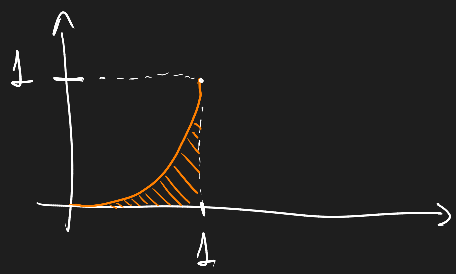
a) $\displaystyle\int \frac{1}{1+x^4}\,dx$ es complicada o no existe

$\therefore$ reescribo región de integración D

$$D = \{0 \leq y \leq 1 \,,\, \sqrt[3]{y} \leq x \leq 1\}$$

$$\sqrt[3]{y} \leq x \leq 1$$

$$0 \leq \underbrace{y \leq x^3} \leq 1$$



$$\tilde{D} = \{0 \leq x \leq 1 \,,\, 0 \leq y \leq x^3\}$$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^3} \frac{1}{1+x^4}\,dx = \int_{x=0}^{x=1} \frac{1}{1+x^4} \cdot y \Big|_{y=0}^{y=x^3} dx$$

$$= \int_{x=0}^{x=1} \frac{x^3}{1+x^4}\,dx$$

$$= \frac{1}{4} \ln(1+x^4) \Big|_{x=0}^{x=1}$$

$$= \frac{1}{4}\left(\ln(2) - 1\right)$$

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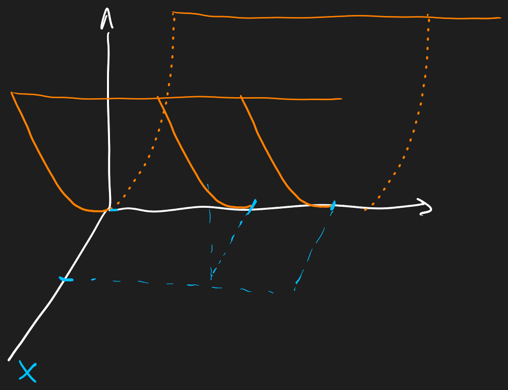
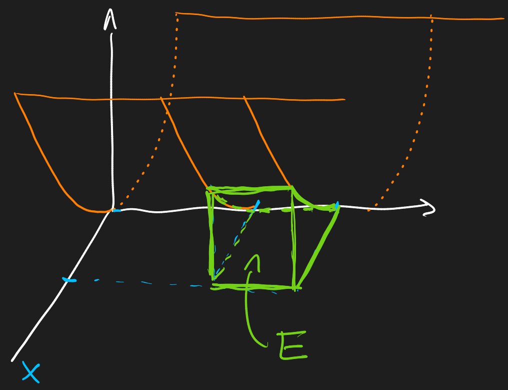
b)





$$E = \left\{ (x,y,z) : 0 \leq x \leq 1 \, , \; 2 \leq y \leq 3 \; , \; 0 \leqslant z \leqslant x^2 \right\}$$

$$\iiint_E x z^2 \, dV = \int_{y=2}^{y=3} \int_{x=0}^{1} \int_{z=0}^{z=x^2} x z^2 \, dz \, dx \, dy$$

$$= \int_{y=2}^{y=3} \int_{x=0}^{1} x \frac{z^3}{3} \Big|_0^{x^2} \, dx \, dy$$

$$= \int_{y=2}^{y=3} \int_{x=0}^{1} - x \cdot \frac{x^6}{3} \, dx \, dy$$

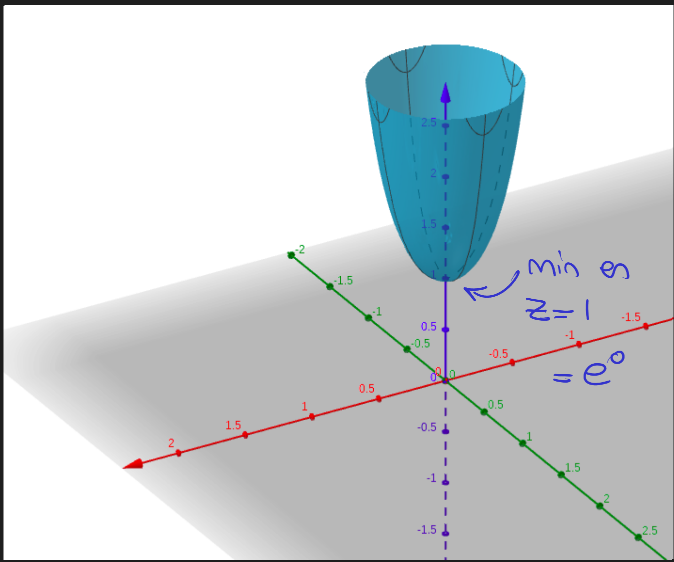
$$= \int_{y=2}^{3} - \frac{x^8}{3.8} \Big|_0^1 \, dy$$

$$= \int_{y=2}^{3} \frac{1}{24} \, dy$$

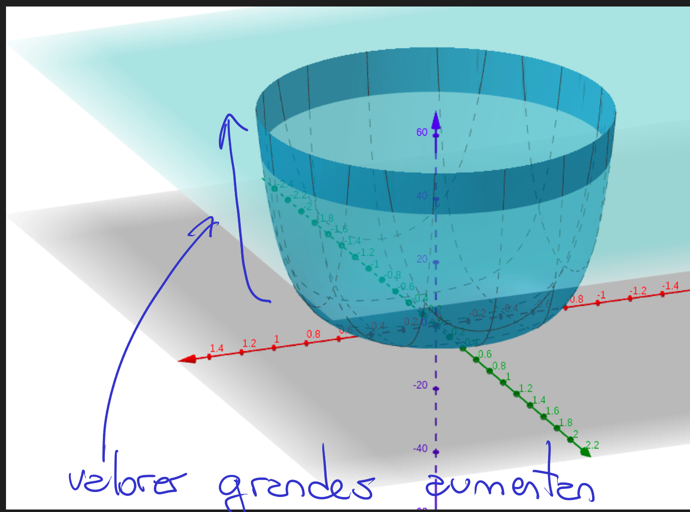
$$= \frac{1}{24} \Bigg]$$

4. Hallar el volumen del sólido acotado por las superficies

$$z = e^{4x^2+4y^2} \quad y \quad z = e^4. \approx 54{,}6$$



min en $z = 1$ $= e^0$



valores grandes aumentan

pendiente rápidamente: $z = e^{\text{Paraboloide}}$

exponencial

Integrar sobre $e^{4x^2+4y^2}$ no me permite separar en integrales de 1 variable

∴ tengo que hacer cambio de variables o parametrizar.

Nota que $4x^2+4y^2$ se simplifica si reemplazo x e y por cos y sin.

Uso Cilíndricas

$$\begin{cases} x = r \cdot \cos\theta & r \in [0, \phi(z)] \\ y = r \cdot \sin\theta & \theta \in [0, 2\pi] \\ z = z \quad \text{con} \quad z \in [1, e^4] \end{cases}$$

donde

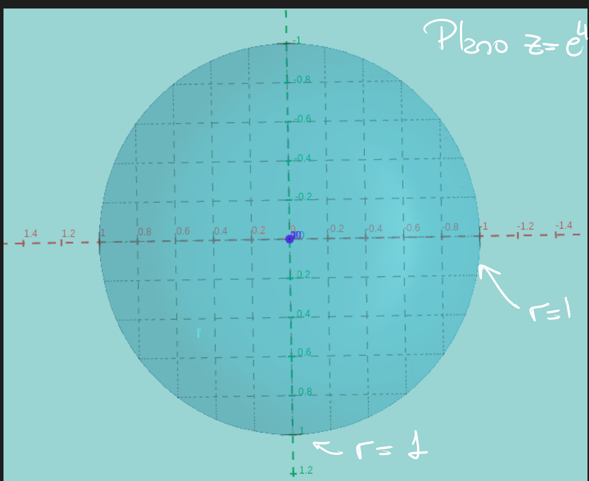
$$\phi(z)\Big|_{z=1} = 0$$

$$z = e^{4\underbrace{(x^2+y^2)}_{r^2\cdot\cos^2\theta + r^2\cdot\sin^2\theta}}$$

$$z = e^{4r^2} \quad \Rightarrow \quad \log \overset{z \geq 1}{z} = 4r^2$$

$$|r| = \frac{1}{2}\sqrt{\log \overset{z\geq 1}{z}}$$

$$r \overset{r \geq 0}{=} \frac{1}{2}\sqrt{\log z}$$



Si $z = e^4$ (intersec con el plano)

$\Rightarrow r = 1$

Que se da cuando

$$e^{4x^2+4y^2} = e^4$$

$$x^2 + y^2 = 1$$

$$\int_x \int_y^{\sqrt{z}} f(x,y)\, dv =$$

Por qué está mal ?

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{\frac{1}{2}\sqrt{\log z}} 1 \cdot r \, dr\, d\theta\, dz$$

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \frac{r^2}{2} \Big|_0^{\frac{1}{2}\sqrt{\log z}}$$

$$= \int_{z=1}^{e^4} \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} \cdot \frac{1}{4} \cdot \log z$$

$$= \int_{z=1}^{e^4} + \frac{\cancel{2}\pi}{\cancel{2}\cdot 4} \log z$$

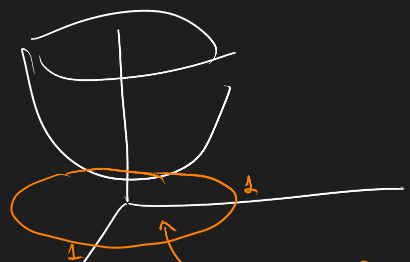
$$= \frac{\pi}{4} \log e^4 - \frac{\pi}{4} \cdot \log 1$$

$$= \pi$$

$$\int e^{-r^2} \cdot r \, dr = \int -e^u \cdot \frac{1}{2} \cdot du = -\frac{1}{2} e^u + C$$

$$u = -r^2 \qquad\qquad = -\frac{1}{2} e^{-r^2}$$

$$du = -2r \cdot dr$$



$D = \{ (x,y) : x^2 + y^2 \leq 1 \}$

y además

$$e^{4(x^2+y^2)} \leq e^4 \qquad \forall (x,y) \in D$$

$$\Rightarrow \quad E = \left\{ (x,y,z) : e^{4(x^2+y^2)} \leq z \leq e^4 , \ (x,y) \in D \right\}$$

$$\therefore \ Vol(E) = \iint_D e^4 \, dA - \iint_D e^{4(x^2+y^2)} \, dA \geq 0$$

$$= \iint_D e^4 - e^{4(x^2+y^2)} \, dA$$

Polares

$$\left.\begin{array}{l} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{array}\right\} \Rightarrow x^2 + y^2 = r^2 \cdot (\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

$$= \underbrace{\int_{\theta=0}^{2\pi} \int_{r=0}^{1}}_{D'} (e^4 - e^{4r^2}) \cdot r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} e^4 . r \, dr \, d\theta - \int_{\theta=0}^{2\pi}\int_{r=0}^{1} e^{4r^2} . r \, dr \, d\theta$$

$$u = 4r^2$$
$$du = 8r \, dr$$

$$= e^4 \int_0^{2\pi} \frac{r^2}{2}\Big|_0^1 - \int_0^{2\pi}\int_{u=0}^{u=4} e^u . \frac{du}{8} \, d\theta$$

$$= \pi . e^4 - \frac{1}{8}\int_0^{2\pi} e^u \Big|_0^4 \, d\theta$$

$$= \pi . e^4 - \frac{1}{8} . \int_0^{2\pi} (e^4 - 1) \, d\theta$$

$$= \boxed{\pi . e^4 - \frac{\pi}{4} . (e^4 - 1)}$$