Análisis I - Análisis Matemático I - Matemática I - Análisis II (C)

1er. cuatrimestre 2020

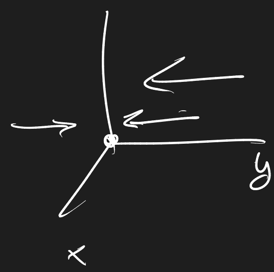
Segundo Recuperatorio del Segundo Parcial - 18/08/2020

1. Calcular el siguiente límite

$$\lim_{(x,y)\to(0,0)} \frac{\cos(y)e^{2x} - 1 - 2x - 2x^2 + \frac{1}{2}y^2}{x^2 + y^2}.$$

$\boxed{x=0}$:

$$\lim_{y\to 0} \frac{\cos(y) - 1 + \frac{1}{2}y^2}{y^2}$$



$\boxed{y=0}$:

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^2}$$

$$\lim_{x\to 0} \frac{\overbrace{e^{2x}}^{\to 1} \overbrace{- 1 - 2x - 2x^2}^{\to 0}}{\underbrace{x^2}_{\to 0}} \overset{L'H}{=} \lim_{x\to 0} \frac{2e^{2x} - 2 - 4x}{2x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{4e^{2x} - 4}{2} = 0$$

$\boxed{x=0}$

$$\lim_{y\to 0} \frac{\cos(y) - 1 + \frac{1}{2}y^2}{y^2} \overset{L'H}{=} \lim_{y\to 0} \frac{-\sin(y) + y}{2y}$$

$$\overset{L'H}{=} \lim_{y\to 0} \frac{-\cos(y) + 1}{2} = 0$$

Acoto:

$$\left| \frac{\cos(y)\cdot e^{2x} - 1 - 2x \overbrace{- 2x^2}^{\leq +2x^2} + \overbrace{\tfrac{1}{2}y^2}^{\leq y^2 \leq 2y^3}}{\underbrace{x^2+y^2}_{\|(x,y)-(0,0)\|^2 \xrightarrow[(x,y)\to(0,0)]{} 0}} - 0 \right| \leq$$

Sospechoso

Si $y = 2x$

$$\Rightarrow -2x^2 + \tfrac{1}{2}y^2 = -2x^2 + 2x^2 = 0$$

$$\Rightarrow \lim_{x\to 0} \frac{\cos(2x)\cdot e^{2x} - 1 - 2x}{x^2 + 4x^2} =$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\overbrace{-2\sin(2x)}^{\to 0}\cdot e^{2x} + \overbrace{\cos(2x)}^{\to 1}\cdot 2\cdot \overbrace{e^{2x}}^{\to 1} - 2}{10\cdot x}$$

$$= \lim_{x\to 0} \frac{\cos(2x)\cdot 2\cdot e^{2x} - 2}{10x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-4\cdot \overbrace{\sin(2x)}^{\to 0}\cdot e^{2x} + 4\overbrace{\cos(2x)}^{\to 1}\cdot \overbrace{e^{2x}}^{\to 1}}{10} = \frac{4}{10} \neq 0$$

$\therefore$ el límite no existe.

2. Sea $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ el campo vectorial dado por $F(x,y) = (3x^2 - 4y, 4y - 4x)$.

(a) Probar que F es un campo vectorial gradiente.

(b) Hallar los extremos relativos y los puntos silla de su función potencial f.

a) $f_x(x,y) = 3x^2 - 4y$

$f_y(x,y) = 4y - 4x$

$$\Rightarrow f(x,y) = x^3 - 4xy + \phi(y) + C$$

$$f(x,y) = 2y^2 - 4xy + \varphi(x) + C$$

$$\Rightarrow f(x,y) = x^3 + 2y^2 - 4xy \qquad \text{con } C = 0$$

Como encontré la función potencial f de F => F es un campo gradiente

b) Veo $\nabla f(x,y) = \vec{0}$

$$(3x^2 - 4y,\ 4y - 4x) \overset{\text{quiero}}{=} (0,0)$$

$$\begin{cases} 3x^2 - 4y = 0 \\ 4y - 4x = 0 \end{cases} \Rightarrow y = x$$

$\hookrightarrow 3x^2 - 4x = 0$

$x(3x - 4) = 0$

$\hookrightarrow x = 0$

$\hookrightarrow 3x - 4 = 0$

$x = \frac{4}{3}$

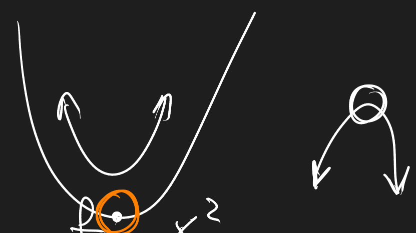
Candidatos :

$$(x,y) = (0,0)$$

$$(x,y) = \left(\tfrac{4}{3}, \tfrac{4}{3}\right)$$

Criterio del Hessiano :

1) $\left.\begin{array}{l} \det > 0 \\ f_{xx} > 0 \end{array}\right\}$ (x_0, y_0) es mínimo



$f = x^2$

$f_x = 2x$

$f_{xx} = 2$

2) $\left.\begin{array}{l} \det > 0 \\ f_{xx} < 0 \end{array}\right\}$ (x_0, y_0) es máx

3) $\det < 0 \Rightarrow$ es Punto silla

4) $\det = 0 \Rightarrow$ el criterio no me sirve.

Calculo Hessiano

$$Hf(x,y) = \begin{bmatrix} 6x & -4 \\ -4 & 4 \end{bmatrix}$$

$$Hf(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 4 \end{bmatrix}$$

$$\det Hf(0,0) = -16 < 0$$

$$\boxed{(0,0) \text{ es Punto silla}}$$

$$Hf\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = \begin{bmatrix} 6 \cdot \frac{4}{3} & -4 \\ -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

$$\det Hf\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = 32 - 16$$

$$= 16 > 0$$

Como $f_{xx} > 0$

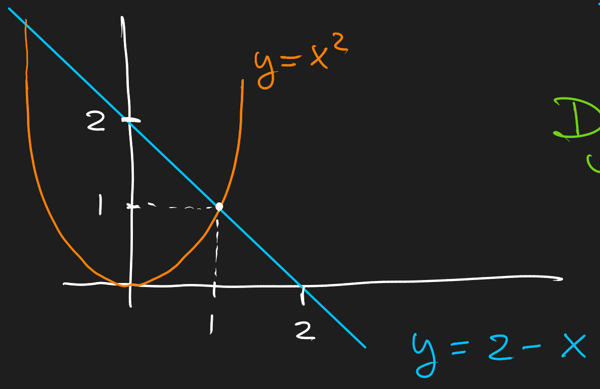
$$\Rightarrow \boxed{\left(\frac{4}{3}, \frac{4}{3}\right) \text{ es mínimo local}}$$

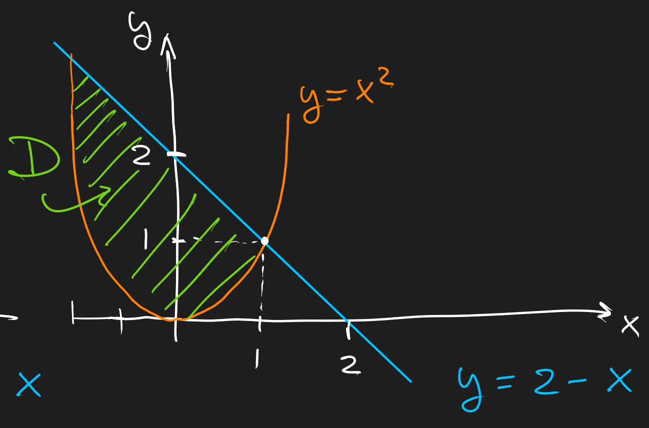
3. Calcular las siguientes integrales

(a) $\iint_D (2x+1)dA$ donde D es la región encerrada entre la curva $y = x^2$ y la recta $x + y = 2$,

(b) $\iiint_E x\, dV$ donde E es el sólido encerrado entre las superficies $z = e^{x^2}$ y $z = -y$ para (x, y) en el rectángulo $R = [1, 2] \times [0, 2]$ del plano xy.

a)





Veo extremos de x

$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x-a) = 0$$

$$x^2 - ax - x + a$$

$$x^2 + x\underbrace{(-a-1)}_{=1} + \underbrace{a}_{=-2} = 0$$

$\therefore a = -2$

Raíces

$x = 1$

$x = -2$

$$\iint_D 2x+1\, dA = \int_{x=-2}^{x=1} \int_{y=x^2}^{y=2-x} 2x+1\, dy\, dx$$

$$= \int_{x=-2}^{x=1} (2x+1)\cdot \underbrace{\left(2-x-x^2\right)} \, dx$$

$$= \int_{-2}^{1} 4x - 2x^2 - 2x^3 + 2 - x - x^2 \, dx$$

$$= \int_{-2}^{1} -2x^3 - 3x^2 + 3x + 2 \quad dx$$

$$= \left[-2\cdot\frac{x^4}{4} - 3\cdot\frac{x^3}{3} + 3\cdot\frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \left[-\frac{x^4}{2} - x^3 + \frac{3}{2}\cdot x^2 + 2x \right]_{-2}^{1}$$

$$= \underbrace{-\frac{1}{2} - 1 + \frac{3}{2} + 2}_{1+1=2} - \left(\underbrace{-8 + 8 + 6 - 4}_{2}\right)$$

$$= 0$$

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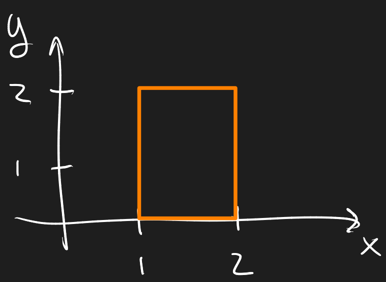
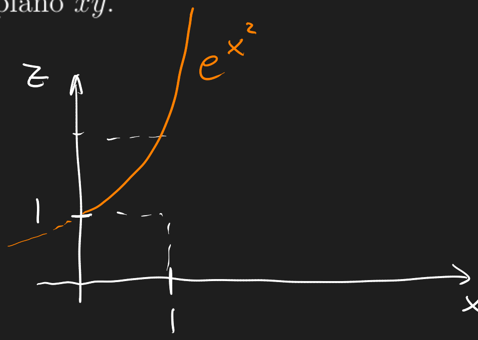
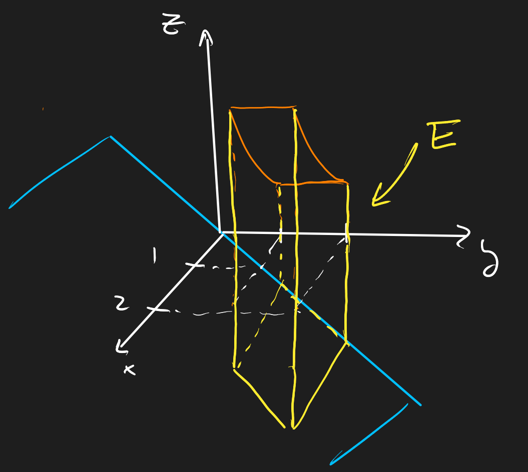
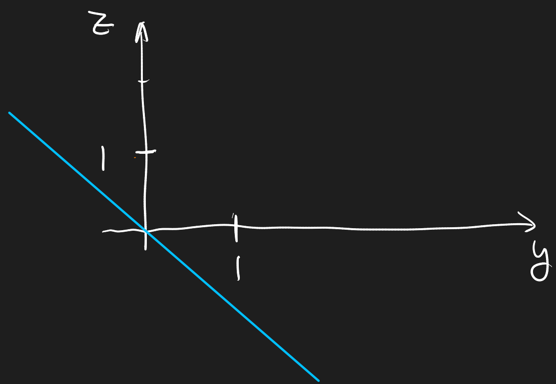
(b) $\iiint_E x\,dV$ donde E es el sólido encerrado entre las superficies $z = e^{x^2}$ y $z = -y$ para (x, y) en el rectángulo $R = [1, 2] \times [0, 2]$ del plano xy.









$$\int_{x=1}^{x=2}\int_{y=0}^{y=2}\int_{z=-y}^{z=e^{x^2}} x\,dz\,dy\,dx =$$

$$= \int_{x=1}^{x=2} x \cdot \int_{y=0}^{y=2} e^{x^2}+y\,dy\,dx$$

$$= \int_{x=1}^{x=2} x \cdot \left[y\cdot e^{x^2}+\frac{y^2}{2}\right]_{y=0}^{y=2} dx$$

$$= \int_{x=1}^{x=2} x\left(2\cdot e^{x^2}+2-0\right)dx$$

$$= \int_1^2 2x\cdot e^{x^2}+2x\,dx$$

$$= e^{x^2} + x^2 \Big|_{x=1}^{x=2}$$

$$= e^4 + 4 - e - 1$$

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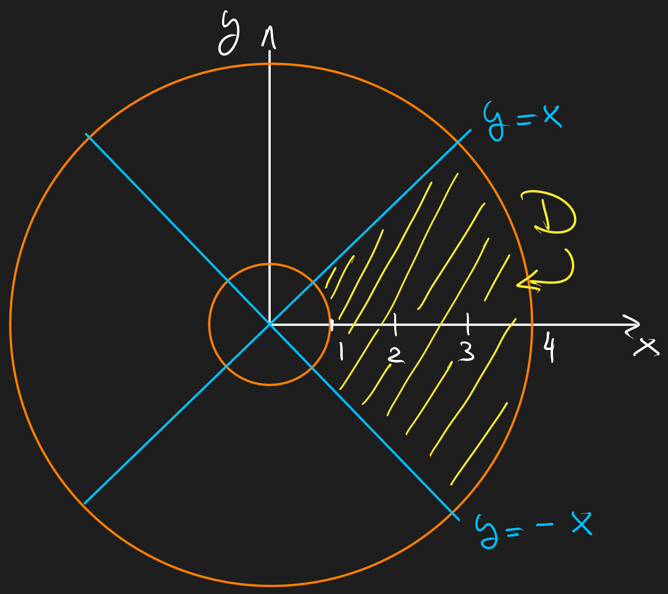
4. Calcular
$$\iint_D \frac{1}{x^2+y^2+1}\,dA$$
donde $D = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x^2+y^2 \le 16,\ -x \le y \le x\}$.



Quiero usar polares

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$$

con $r \in [1,4]$

$\theta \in [0, \frac{\pi}{4}] \cup [\frac{7}{4}\pi, 2\pi)$

$$\iint_D f(x,y)\,dA = \int_{r=1}^{4}\int_{\theta=0}^{\pi/4} \frac{1}{r^2+1}\cdot r\,.\,d\theta\,dr \;+$$

$$+\int_{r=1}^{4}\int_{\theta=\frac{7}{4}\pi}^{2\pi} \frac{1}{r^2+1}\cdot r\,.\,d\theta\,dr$$

Calculo primitiva de

$$\int \frac{r}{r^2+1}\,dr = \frac{1}{2}\ln(r^2+1)$$

Vuelvo

$$\int_{r=1}^{4}\int_{\theta=0}^{\pi/4} \frac{1}{r^2+1}\cdot r\,.\,d\theta\,dr = \int_{r=1}^{4} \frac{r}{r^2+1}\cdot\frac{\pi}{4}\,dr$$

$$= \frac{\pi}{4}\cdot\frac{1}{2}\cdot\left[\ln(r^2+1)\right]_1^4$$

$$= \frac{\pi}{8} \cdot \left(\ln(17) - \ln(2) \right)$$

$$\int_{r=1}^{4} \int_{\theta = \frac{7}{4}\pi}^{2\pi} \frac{1}{r^2+1} \cdot r \, d\theta \, dr = \int_{r=1}^{4} \frac{r}{r^2+1} \cdot \left(2\pi - \frac{7}{4}\pi\right) dr$$

$$= \frac{1}{4}\pi \cdot \frac{1}{2} \cdot \left(\ln(17) - \ln(2) \right)$$

Finalmente

$$\iint_D f(x,y)\, dA = \frac{\pi}{4} \cdot \left(\ln(17) - \ln(2) \right)$$

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