4.7 FIRST APPLICATIONS OF DEFINITE INTEGRALS

The development of calculus by Newton and Leibniz was a vital step in the advancement of pure mathematics, but Newton also advanced the applied sciences and mathematics. Not only did he discover theoretical results, but he immediately used those results to answer important applied questions about gravity and motion. The success of these applications of mathematics to the physical sciences helped establish what we now take for granted: mathematics can and should be used to answer questions about the world.

Newton applied mathematics to the outstanding problems of his day, problems primarily in the field of physics. In the intervening 300 years, thousands of people have continued these theoretical and applied traditions and have used mathematics to help develop our understanding of all of the physical and biological sciences as well as the behavioral sciences and business. Mathematics is still used to answer new questions in physics and engineering, but it is also important for modeling ecological processes, for understanding the behavior of DNA, for determining how the brain works, and even for devising strategies for voting effectively. The mathematics you are learning now can help you become part of this tradition, and you might even use it to add to our understanding of different areas of life.

It is important to understand the successful applications of integration in case you need to use those particular applications. It is also important that you understand the **process** of building models with integrals so you can apply it to new problems. Conceptually, converting an applied problem to a Riemann sum (Fig. 1) is the most valuable step. Typically, it is also the most difficult.

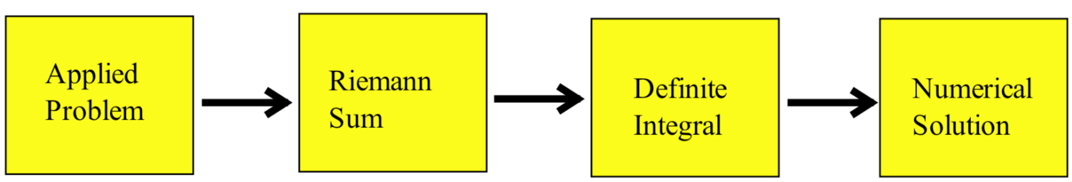

Fig. 1

Area between f and g

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs.

If $f(x) \ge g(x)$ for all x in $[a,b]$, then we can approximate the area between f and g by partitioning the interval $[a,b]$ and forming a Riemann sum (Fig. 2). The height of each rectangle is $f(c_i) - g(c_i)$ so the area of the i^{th} rectangle is $(\text{height})\cdot(\text{base}) = \{f(c_i) - g(c_i)\}\cdot\Delta x_i$. This approximation of the total area is

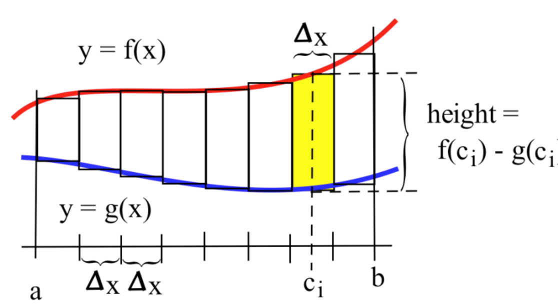

Fig. 2

$$\text{area} \approx \sum_{i=1}^{n} \{f(c_i) - g(c_i)\}\cdot\Delta x_i \quad , \quad \text{a Riemann sum.}$$

The limit of this Riemann sum, as the mesh of the partitions approaches 0, is the definite integral

$$\int_a^b \{ f(x) - g(x) \} \, dx \quad .$$

We will sometimes use an arrow to indicate "the limit of the Riemann sum as the mesh of the partitions approaches zero," and will write

$$\sum_{i=1}^{n} \{f(c_i) - g(c_i)\} \cdot \Delta x_i \longrightarrow \int_a^b \{ f(x) - g(x) \} \, dx \quad .$$

If **f(*x*) ≥ g(*x*)** on the interval [a,b],

then $\left\{ \begin{array}{l} \text{area bounded by the graphs} \\ \text{of f and g and vertical} \\ \text{lines at } x = a \text{ and } x = b \end{array} \right\} = \int_a^b \{ f(x) - g(x) \} \, dx \quad .$

Example 1: Find the area bounded between the graphs of $f(x) = x$ and $g(x) = 3$ for $1 \le x \le 4$. (Fig. 3)

Solution: It is clear from the figure that the area between f and g is 2.5 square inches. Using the theorem,

area between f and g for $1 \le x \le 3$ is

$$\int_1^3 \{ 3 - x \} \, dx = 3x - \frac{x^2}{2} \Big|_1^3 = \left(\frac{9}{2}\right) - \left(\frac{5}{2}\right) = 2 \text{ , and}$$

area between f and g for $3 \le x \le 4$ is

$$\int_3^4 \{ x - 3 \} \, dx = \frac{x^2}{2} - 3x \Big|_3^4 = \left(\frac{-8}{2}\right) - \left(\frac{-9}{2}\right) = \frac{1}{2} \quad .$$

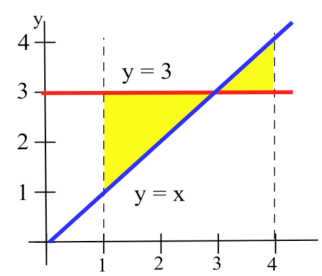

Fig. 3

The two integrals also tell us that the total area between f and g is 2.5 square inches.

The single integral $\int_1^4 \{ 3 - x \} \, dx = 1.5$ which is not the **area** we want in this problem. The value of the **integral is 1.5**, and the value of the **area is 2.5** .

Practice 1: Use integrals and the graphs of $f(x) = 1 + x$ and $g(x) = 3 - x$ to determine the area between the graphs of f and g for $0 \le x \le 3$.

Example 2: Two objects start from the same location and travel along the same path with velocities $v_A(t) = t + 3$ and $v_B(t) = t^2 - 4t + 3$ meters per second (Fig. 4).

How far ahead is A after 3 seconds? After 5 seconds?

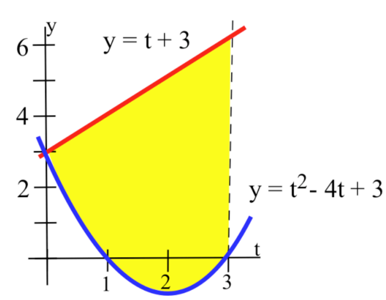

Fig.4

Solution: Since $v_A(t) \ge v_B(t)$, the "area" between the graphs of v_A and v_B represents the distance between the objects.

After 3 seconds, the distance apart $= \int_0^3 v_A(t) - v_b(t)\, dt = \int_0^3 (t+3) - (t^2-4t+3)\, dt$

$$= \int_0^3 5t - t^2\, dt = \frac{5}{2}t^2 - \frac{t^3}{3} \Big|_0^3 = \left(\frac{5}{2}\cdot 9 - \frac{27}{3}\right) - \left(\frac{5}{2}\cdot 0 - \frac{0}{3}\right) = 13\frac{1}{2} \text{ meters.}$$

After 5 seconds, the distance apart $= \int_0^5 v_A(t) - v_b(t)\, dt = \frac{5}{2}t^2 - \frac{t^3}{3} \Big|_0^5 = 20\frac{5}{6}$ meters.

If $f(x) \ge g(x)$, we can use the simpler argument that the area of region A is $\int_a^b f(x)\, dx$ and the area of region B is $\int_a^b g(x)\, dx$, so the area of region C, the area between f and g, is

$$\text{area of C} = (\text{area of A}) - (\text{area of B}) = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b f(x) - g(x)\, dx\ .$$

If the same function is not always greater, then we need to be very careful and find the intervals where $f \ge g$ and the intervals where $g \ge f$.

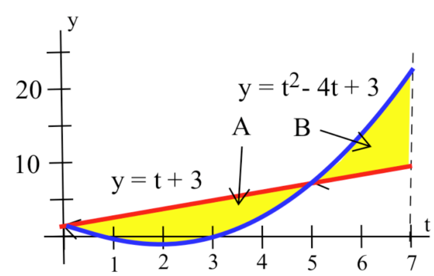

Fig. 5

Example 3: Find the **area** of the shaded region in Fig. 5.

Solution: For $0 \le x \le 5$, $f(x) \ge g(x)$ so the **area** of A is

$$\int_0^5 f(x) - g(x)\, dx = \int_0^5 (x+3) - (x^2-4x+3)\, dx$$

$$= \int_0^5 5x - x^2\, dx = \frac{5}{2}x^2 - \frac{x^3}{3} \Big|_0^5 = 20\frac{5}{6}\ .$$

For $5 \le x \le 7$, $g(x) \ge f(x)$ so the **area** of B is

$$\int_5^7 g(x) - f(x)dx = \int_5^7 (x^2-4x+3) - (x+3)\, dx = \int_5^7 x^2 - 5x\, dx = \frac{x^3}{3} - \frac{5}{2}x^2 \Big|_5^7 = 12\frac{4}{6}\ .$$

Altogether, the total **area** between f and g for $0 \le x \le 7$ is

$$\int_0^5 f(x) - g(x)\, dx + \int_5^7 g(x) - f(x)\, dx = 20\frac{5}{6} + 12\frac{4}{6} = 33\frac{1}{2}\ .$$

Average Value of a Function

We know the average of n numbers, $a_1, a_2, \ldots, a_n$, is their sum divided by n: average (mean) $= \frac{1}{n} \sum_{k=1}^{n} a_k$.

Finding the average of a function on an interval, an infinite number of values, requires an integral.

To find a Riemann sum approximation of the average value of f on the interval [a,b], we can partition [a,b] into n equally long subintervals of length $\Delta x = (b-a)/n$, pick a value c_i of x in each subinterval, and find the average of the numbers $f(c_i)$. Then

$$\text{average of } f \approx \frac{f(c_1)+f(c_2)+\ldots+f(c_n)}{n} = \frac{1}{n} \sum_{k=1}^{n} f(c_i) = \sum_{k=1}^{n} f(c_i) \cdot \frac{1}{n}$$

This last sum is not a Riemann sum since it does not have the form $\sum f(c_i) \cdot \mathbf{\Delta x_i}$, but it can be manipulated into one:

$$\sum_{i=1}^{n} f(c_i) \cdot \frac{1}{n} = \sum_{i=1}^{n} f(c_i) \cdot \frac{\mathbf{b-a}}{n} \cdot \frac{1}{\mathbf{b-a}} = \frac{1}{\mathbf{b-a}} \sum_{i=1}^{n} f(c_i) \cdot \frac{\mathbf{b-a}}{n} = \frac{1}{b-a} \sum_{i=1}^{n} f(c_i) \cdot \Delta x.$$

$$\text{Then } \{\text{average of } f\} \approx \frac{1}{b-a} \sum_{k=1}^{n} f(c_i) \cdot \Delta x \longrightarrow \frac{1}{b-a} \int_a^b f(x)\,dx = \{\text{average of } f\}$$

as the number of points n gets larger and the mesh, $(b-a)/n$, approaches 0.

Definition: **Average (Mean) Value of a Function**

For an integrable function f on the interval [a,b],

the average value of f on [a,b] is $\frac{1}{b-a} \int_a^b f(x)\,dx$

The average value of a positive f has a nice geometric interpretation. Imagine that the area under f (Fig. 6a) is a liquid that can "leak" through the graph to form a rectangle with the same area (Fig. 6b). If the height of the rectangle is H, then the area of the rectangle is $H \cdot (b-a)$. We know the area of the rectangle is the same as the area under f so

$$H \cdot (b-a) = \int_a^b f(x)\,dx. \text{ Then}$$

$$H = \frac{1}{b-a} \int_a^b f(x)\,dx, \text{ the average value of } f \text{ on } [a,b].$$

The average value of positive f is the height H of the rectangle whose area is the same as the area under f.

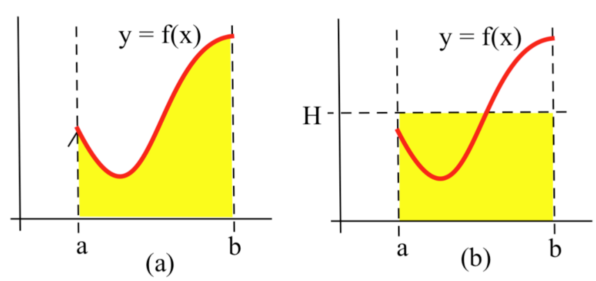

Fig. 6

Example 4: Find the average value of $f(x) = \sin(x)$ on the interval $[0,\pi]$. (Fig. 7)

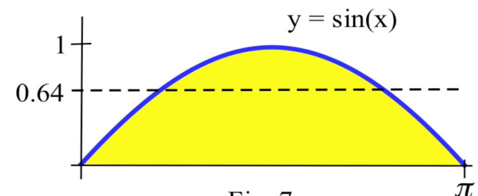

Fig. 7

Solution: Average value $= \frac{1}{\pi-0}\int_0^{\pi} \sin(x)\,dx$

$$= \frac{1}{\pi}(-\cos(x))\Big|_0^{\pi} = \frac{1}{\pi}\{-(-1)-(-1)\} = \frac{2}{\pi} \approx 0.6366 \ .$$

A rectangle with height $2/\pi \approx 0.64$ on the interval $[0,\pi]$ encloses the same area as one arch of the sine curve. The average value of $\sin(x)$ on the interval $[0, \mathbf{2\pi}]$ is 0 since $\frac{1}{2\pi}\int_0^{2\pi}\sin(x)dx = 0$.

Practice 2: During a 9 hour work day, the production rate at time t hours was $r(t)= 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate.

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals. In these situations, the original function f is replaced by some "average of f." If f is rather jagged time data, then the ten year average of f is the integral $g(x) = \frac{1}{10}\int_{x-5}^{x+5} f(t)\,dt$ .an average of f over 5 units on each side of x. For example, Fig. 9 shows the graphs of a Monthly Average (rather "noisy" data) of surface temperature data, an Annual Average (still rather "jagged), and a Five Year Average (a much smoother function). Typically the average function reveals the pattern much more clearly than the original data. This use of a "moving average" value of "noisy" data (weather information, stock prices) is a very common.

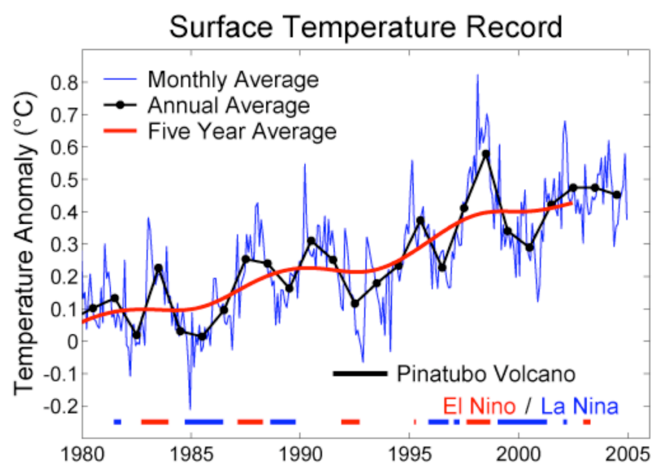

Fig. 9

Work

The amount of work done on an object is the force applied to the object times the distance the object is moved while the force is applied (Fig. 10) or, more succinctly, **work = (force)·(distance)** .

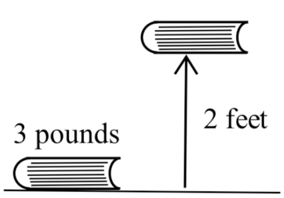

Fig. 10

If you lift a 3 pound book 2 feet, then the force is 3 pounds, the weight of the book, and the distance moved is 2 feet, so you have done (3 pounds)·(2 feet) = 6 foot–pounds of work. When the applied force and the distance are both constants, then calculating work is simply a matter of multiplying.

Practice 3: How much work is done lifting a 10 pound object from the ground to the top of a 30 foot building (assume the cable is weightless).

If either the force or the distance is variable, then integration is needed.

Example 5: How much work is done lifting a 10 pound object from the ground to the top of a 30 foot building if the cable weighs 2 pounds per foot. (Fig. 11)

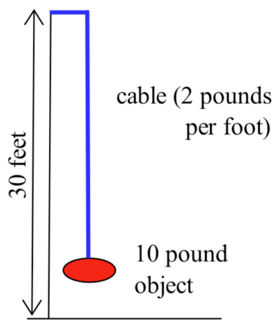

Fig. 11

Solution: This is more difficult than the Practice problem. When the object is at ground level, a force of 70 pounds (10 pounds plus the weight of 30 feet of cable) must be applied, but when the object is 29 feet above the ground, only 12 pounds of force are needed. In general, if the object is x feet above the ground (Fig. 12), then $30 - x$ feet of cable, weighing $2(30-x)$ pounds, is used so the required force is

$$f(x) = 10 + 2(30-x) = 70 - 2x \text{ pounds.}$$

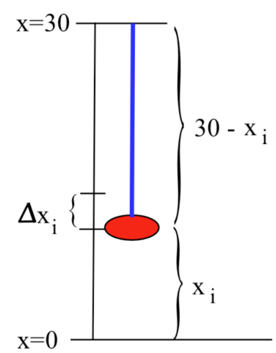

Fig. 12

Let's partition the height of the building into small increments so the force needed in each subinterval does not change much. The force in the i^{th} subinterval will be approximately $f(c_i)$ for some c_i in the subinterval, and the distance moved will be the length of the subinterval, Δx_i. The work done to move the object through the subinterval will be $f(c_i) \cdot \Delta x_i$, and the total work will be the sum of the work on each subinterval:

$$\text{work} \approx \sum (\text{subinterval work}) = \sum f(c_i)\, \Delta x_i = \sum \{70 - 2c_i\}\, \Delta x_i \qquad \text{(a Riemann sum)}$$

$$\longrightarrow \int_0^{30} \{70 - 2x\}\, dx \qquad \text{as the mesh approaches } 0.$$

We have approximated the solution to an applied problem by a Riemann sum, and obtained an exact solution by taking the limit of the Riemann sum to get a definite integral. Now we just need to evaluate the definite integral:

$$\int_0^{30} (70 - 2x)\, dx = 70x - x^2 \Big|_0^{30} = \{70\cdot 30 - (30)^2\} - \{70\cdot 0 - (0)^2\} = (2100-900) - (0) = 1200 \text{ foot-pounds.}$$

Practice 4: Suppose the building in Example 5 is 50 feet tall and the cable weighs 3 pounds per foot.

(a) How much force is needed when the 10 pound object is x feet above the ground?

(b) Write an integral for the work done raising the object from the ground to a height of 10 feet. From a height of 10 feet to a height of 20 feet.

In the previous Example and Practice problem, the force was variable and the distance was Δx. In later sections we will examine situations where the force is constant and the distance changes.

Summary

These area, average and work applications simply introduce a few of the applications of definite integrals and illustrate the pattern of going from an applied problem to a Riemann sum, on to a definite integral and, finally, to a number. More applications will be explored in Chapter 5. The rest of this chapter will examine additional ways of finding antiderivatives and of finding the values of definite integrals when an antiderivative can not be found.

PROBLEMS

x	f(x)	g(x)	h(x)
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	5	2

Table 1

In problems 1 – 4, use the values in Table 1 to estimate the areas.

1. Estimate the **area** between f and g for $1 \le x \le 4$.

2. Estimate the **area** between f and g for $1 \le x \le 6$.

3. Estimate the **area** between f and h for $0 \le x \le 4$.

4. Estimate the **area** between g and h for $0 \le x \le 6$.

5. Estimate the area of the island in Fig. 13.

6. Estimate the area of the island in Fig. 13 if the distances between the lines is 50 feet instead of 40 feet,.

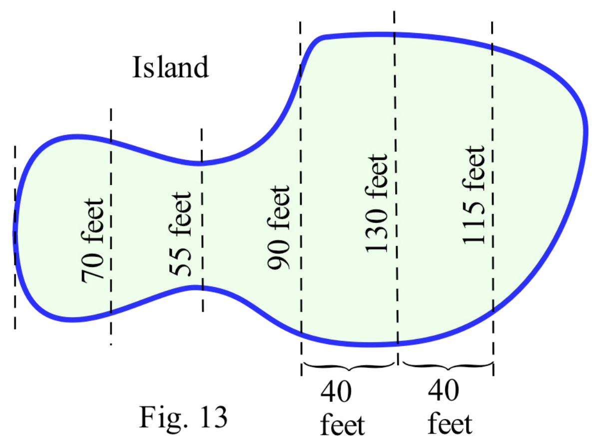

Fig. 13

In problems 7 – 18, sketch the graph of each function and find the **area** between the graphs of f and g for x in the given interval.

7. $f(x) = x^2 + 3$, $g(x) = 1$ and $-1 \le x \le 2$.

8. $f(x) = x^2 + 3$, $g(x) = 1 + x$ and $0 \le x \le 3$.

9. $f(x) = x^2$, $g(x) = x$ and $0 \le x \le 2$.

10. $f(x) = 4 - x^2$, $g(x) = x + 2$ and $0 \le x \le 2$.

11. $f(x) = \frac{1}{x}$, $g(x) = x$ and $1 \le x \le e$.

12. $f(x) = \sqrt{x}$, $g(x) = x$ and $0 \le x \le 4$.

13. $f(x) = x + 1$, $g(x) = \cos(x)$ and $0 \le x \le \pi/4$.

14. $f(x) = (x-1)^2$, $g(x) = x + 1$ and $0 \le x \le 3$.

15. $f(x) = e^x$, $g(x) = x$ and $0 \le x \le 2$.

16. $f(x) = \cos(x)$, $g(x) = \sin(x)$ and $0 \le x \le \pi/4$.

17. $f(x) = 3$, $g(x) = \sqrt{1 - x^2}$ and $0 \le x \le 1$.

18. $f(x) = 2$, $g(x) = \sqrt{4 - x^2}$ and $-2 \le x \le 2$.

In problems 19 – 22, use the values in Table 1 to estimate the average values.

19. Estimate the average value of f on the interval $[0.5, 4.5]$.

20. Estimate the average value of f on the interval $[0.5, 6.5]$.

21. Estimate the average value of f on the interval $[1.5, 3.5]$.

22. Estimate the average value of f on the interval $[3.5, 6.5]$.

In problems 23 – 32, find the **average value** of f on the given interval.

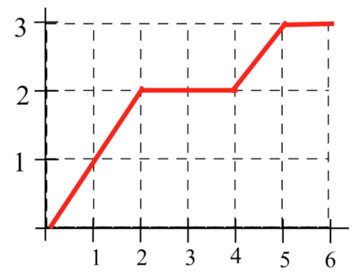

Fig. 14

23. $f(x)$ in Fig. 14 for $0 \le x \le 2$.

24. $f(x)$ in Fig. 14 for $0 \le x \le 4$.

25. $f(x)$ in Fig. 14 for $1 \le x \le 6$.

26. $f(x)$ in Fig. 14 for $4 \le x \le 6$.

27. $f(x) = 2x + 1$ for $0 \le x \le 4$.

28. $f(x) = x^2$ for $0 \le x \le 2$.

29. $f(x) = x^2$ for $1 \le x \le 3$.

30. $f(x) = \sqrt{x}$ for $0 \le x \le 4$.

31. $f(x) = \sin(x)$ for $0 \le x \le \pi$.

32. $f(x) = \cos(x)$ for $0 \le x \le$ π.

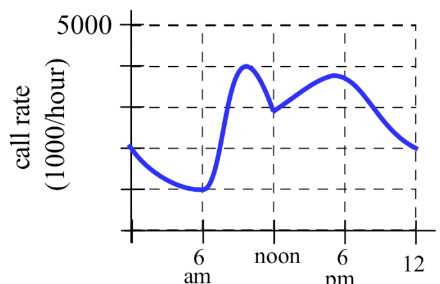

Fig. 15

33. Calculate the average value of $f(x) = \sqrt{x}$ on the interval $[0, C]$ for $C = 1, 9, 81, 100$. What is the pattern?

34. Calculate the average value of $f(x) = x$ on the interval $[0, C]$ for $C = 1, 10, 80, 100$. What is the pattern?

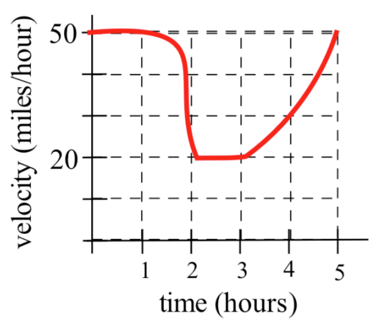

Fig. 16

35. Fig. 15 shows the number of telephone calls per minute at a large company. (a) Estimate the average number of calls per minute from 8 am to 5 pm. (b) From 9 am to 1 pm.

36. Fig. 16 shows the velocity of a car during a 5 hour trip.

(a) Estimate how far the car traveled during the 5 hours.

(b) At what **constant** velocity should you drive in order to travel the same distance in 5 hours?

37. (a) How much work is done lifting a 20 pound bucket from the ground to the top of a 30 foot building with a cable which weighs 3 pounds per foot?

(b) How much work is done lifting the same bucket from the ground to a height of 15 feet with the same cable?

38. (a) How much work is done lifting a 60 pound chair from the ground to the top of a 20 foot building with a cable which weighs 1 pound per foot?
    (b) How much work is done lifting the same chair from the ground to a height of 5 feet with the same cable?

39. (a) How much work is done lifting a 10 pound calculus book from the ground to the top of a 30 foot building with a cable which weighs 2 pounds per foot?
    (b) From the ground to a height of 10 feet? (c) From a height of 10 feet to a height of 20 feet?

40. How much work is done lifting an 80 pound injured child to the top of a 20 foot hole using a stretcher weighing 14 pounds and a cable which weighs 1 pound per foot?

41. How much work is done lifting an 60 pound injured child to the top of a 15 foot hole using a stretcher weighing 10 pounds and a cable which weighs 2 pound per foot?

42. How much work is done lifting an 120 pound injured adult to the top of a 30 foot hole using a stretcher weighing 10 pounds and a cable which weighs 2 pound per foot?

Section 4.7 **PRACTICE Answers**

Practice 1: Using geometry (Fig. 17): $A = \frac{1}{2}(2)(1) = 1$ and $B = \frac{1}{2}(4)(2) = 4$ so total area = A + B = **5**.

Using integrals:

$$A = \int_0^1 (3-x) - (1+x)\,dx = \int_0^1 (2-2x)\,dx = 2x - x^2 \Big|_0^1 = (2-1) - (0) = 1.$$

$$B = \int_1^3 (1+x) - (3-x)\,dx = \int_1^3 (2x-2)\,dx = x^2 - 2x \Big|_1^3 = (9-6) - (1-2) = 4.$$

The single integral $\int_0^3 (1+x) - (3-x)\,dx$ is **not** correct: $\int_0^3 (1+x) - (3-x)\,dx = 3$.

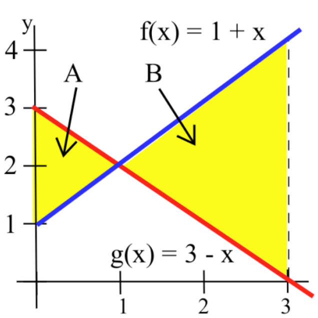

Fig.17

Practice 2: Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{9-0}\int_0^9 5+\sqrt{t}\;dt$

$$= \frac{1}{9}\int_0^9 5+t^{1/2}\;dt = \frac{1}{9}\left(5t + \frac{2}{3}t^{3/2}\right)\Big|_0^9$$

$$= \frac{1}{9}\left(45+\frac{2}{3}9^{3/2}\right) - \frac{1}{9}(0) = \frac{1}{9}(45+18) = \mathbf{7}$$ **cars per hour.**

Practice 3: Work = (force)·(distance) = (10 pounds)·(30 feet) = **300 foot·pounds**.

Practice 4: (a) force = (force for cable) + (force for object)

= (length of cable)(density of cable) + 10 pounds

= (50 – x feet)(3 pounds/foot) + 10 pounds = **160 – 3x pounds**

(b) "from the ground to a height of 10 feet:"

Work $\approx \sum$ (subinterval work) $= \sum f(c_i)\,\Delta x_i = \sum \{160-3c_i\}\,\Delta x_i$ (a Riemann sum)

$$\longrightarrow \int_0^{10} \{160-3x\}\,dx$$ as the mesh approaches 0.

"From a height of 10 feet to a height of 20 feet:" work $= \int_{10}^{20}\{160-3x\}\,dx$.