Simulation of an n-link pendulum

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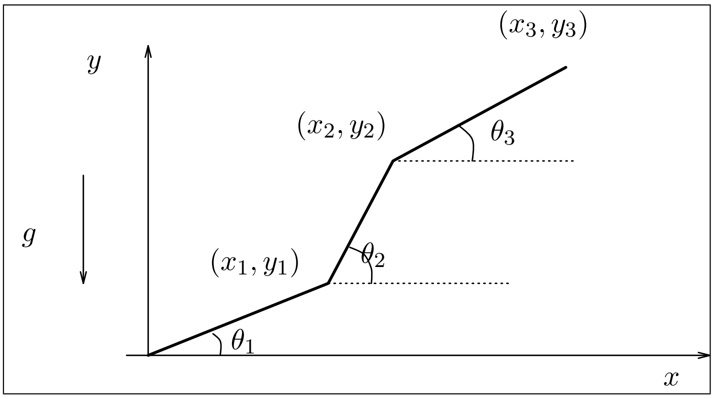

Figure 1: 3-link Pendulum

1 Equation of motion of an N-link pendulum

The n-link pendulum is described on picture (1). For the i-link :

$$E_i^c = 1/2m_i\left((\dot{x}_{i-1} - l_i/2\sin(\theta_i)\dot{\theta}_i)^2 + (\dot{y}_{i-1} + l_i/2\cos(\theta_i)\dot{\theta}_i)^2\right) + 1/2J_i\dot{\theta}_i^2 \tag{1}$$

with :

$$J_i = m_i l_i^2/12 \tag{2}$$

$$E_i^p = m_i g\,(y_{i-1} + l_i/2\sin(\theta_i)) \tag{3}$$

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With the added point $(x_0, y_0) = (0, 0)$. The n-link pendulum has the following Lagrangian :

$$
\begin{aligned}
L(q, \dot{q}) &= \sum_{i=1}^{n} \Big\{ 1/2 m_i \left((\dot{x}_{i-1} - l_i/2 \sin(\theta_i)\dot{\theta}_i)^2 + (\dot{y}_{i-1} + l_i/2 \cos(\theta_i)\dot{\theta}_i)^2 \right) + 1/2 J_i \dot{\theta}_i^2 \qquad (4) \\
&- \left(m_i g \left(y_{i-1} + l_i/2 \sin(\theta_i) \right) \right) \Big\} \qquad (5)
\end{aligned}
$$

And is subject to the following set of constraints :

$$
i = 1, \ldots, n \quad \left\{ \begin{array}{lcl} x_i - x_{i-1} &=& l_i \cos(\theta_i) \\ y_i - y_{i-1} &=& l_i \sin(\theta_i) \end{array} \right. \tag{6}
$$

A time derivative of equation (7) leads to :

$$
i = 1, \ldots, n \quad \left\{ \begin{array}{l} \dot{x}_i - \dot{x}_{i-1} + l_i \sin(\theta_i)\dot{\theta}_i = 0 \\ \dot{y}_i - \dot{y}_{i-1} - l_i \cos(\theta_i)\dot{\theta}_i = 0 \end{array} \right. \tag{7}
$$

Which is of the form $A'(q)\dot{q} = 0$. If we solve the linear system $A'(q)\tilde{q} = 0$ it's easy to see that the solution depend on n-parameters $(\pi_i)_{i=1,n}$

$$
\tilde{x}_i = -\sum_{k \le i} l_k \sin(\theta_k)\pi_k \tag{8}
$$

$$
\tilde{y}_i = \sum_{k \le i} l_k \cos(\theta_k)\pi_k \tag{9}
$$

$$
\tilde{\theta}_i = \pi_k \tag{10}
$$

In a matrix form equation (10) gives $\tilde{q} = S(q)\pi$ and $A(q)'S(q) = 0$

Th Euler-Lagrange equations for the n-link pendulum can be derived from The lagrangian and the constraints :

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q_i}(q(t), \dot{q}(t)) = (A(q)\lambda)_i \\
&\frac{d}{dt} q = \dot{q}
\end{aligned} \tag{11}
$$