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MASTER OF ECONOMICS

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THE IMPACT OF FAMILIARITY NETWORK ON EQUILIBRIUM BIDDING IN AUCTIONS

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Affidavit

I hereby confirm that the Master thesis entitled “THE IMPACT OF FAMILIARITY NETWORK ON EQUILIBRIUM BIDDING IN AUCTIONS” has been written independently and without any other sources than cited.

Moscow, ____________________ ______________________________

Name

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Abstract: Information asymmetry among bidders is prevalent across various auction formats, whether implicit or explicit. I employ familiarity networks to investigate a semi-perfect information auction framework characterized by information asymmetry, ex ante learning, and arbitrary network structures. In this model, risk-neutral bidders compete for the allocation of a good by submitting sealed bids according to specified allocation rules (first- or second-price), where each bidder possesses local knowledge about a subset of others' values, determined by the underlying network structure. Key findings include: (1) In second-price auctions, truthful-telling (i.e., bidding one's true valuation) constitutes an equilibrium for any structure of the familiarity network. In first-price, a pure strategy equilibrium exists, where the front-runner (i.e., the bidder with the highest valuation within her familiarity set) bids between the second-highest valuation and her own, with non-unique equilibria in both formats. (2) The Revenue Equivalence Theorem fails under mild assumptions; second-price auctions are efficient, while first-price auctions yield lower expected revenue with partial local information. (3) Bidders gain more surplus in first-price, but this leads to inefficient allocations with positive probability. (4) In first-price, expected revenue is nonmonotonic as bidders' familiarity increases; it initially decreases with partial information but increases as bidders gain full information. (5) Under mild regularity conditions, increasing bidders intensifies competition in first-price auctions, though the impact of network skewness is nonmonotonic. These results provide insights into the equity-efficiency tradeoff in mechanism design, and is applicable in contexts like emissions trading, enterprise competition, and online bidding algorithms.

Key words: first-price auction, second-price auction, familiarity network, information asymmetry, equity-efficiency tradeoff

Notice: Generative AI models were utilized in this paper for grammar correction and code debugging.

Chapter 1

Introduction

> *"The results of the auction show that the market is ready: there was a high level of participation and competition, with 132 bids from 17 different European countries... It will connect European off-takers with hydrogen supplies both from Europe and outside Europe and prepare the ground for future auctions under the European Hydrogen Bank."*
>
> *– Kadri Simson, European Commissioner, 2024* [1]

On July 14, 2021, the European Commission introduced a comprehensive set of legislative proposals setting out how it intends to achieve climate neutrality in the European Union (EU) by 2050 [2]. Central to this ambition is an intermediate goal of reducing net greenhouse gas emissions by at least 55% by 2030. This legislative package, which is part of the European Green Deal, outlines revisions to existing EU climate policies, including the EU Emissions Trading System (EU ETS), the Effort Sharing Regulation, and regulations governing transport and land use. These revisions provide a concrete framework for how the EU intends to meet its

[1] Opening remarks by Commissioner Simson at the EU-Japan High-Level Hydrogen Business Forum

[2] Energy, Climate change, Environment, European Commission

climate targets.

One key element of the EU ETS, particularly from Phase 3 (2013-2020) onward, is the shift toward auctioning as the primary method for allocating emission allowances. Auctioning is recognized as the most transparent allocation method and operationalizes the "polluter pays" principle, requiring businesses within the EU ETS to purchase an increasing share of their emission allowances through auctions. This mechanism is crucial for driving emission reductions and ensuring that the costs of pollution are borne by those responsible for it. The Renewable Energy Procurement process, also discussed in Anatolitis, Azanbayev, and Fleck 2022, serves as a real-world application of auction theory. It effectively translates the abstract concepts of auction theory into significant practical outcomes, further demonstrating the theory's value beyond theoretical contexts (List and Shogren 1999).

Another significant aspect of the EU ETS is its collaboration with non-EU countries, such as Japan (consider a more closely aligned Kyoto Protocol). This partnership aims to strengthen ties and facilitate the exchange of innovative ideas between European and e.g., Japanese business leaders, contributing to the shared goal of achieving climate neutrality by 2050. However, information asymmetry presents a challenge; EU member states may have a deeper understanding of each other's systems compared to newly joined countries. This asymmetry could potentially lead to inefficiencies and waste under certain auction mechanisms (Kim and Koh 2020).

The starting point of my analysis on the presence of familiarity network in auctions stems from observations about information asymmetry among bidders as illustrated above. In practical scenarios, achieving symmetrical knowledge among bidders about their rivals is uncommon. Typically, bidders possess varying degrees of information about their rivals, creating an asymmetry in their understanding. For example, in a procurement auction involving both domestic and foreign firms, domestic entities are more likely to be acquainted with the technical capabilities of their fellow domestic bidders compared to their knowledge of foreign rivals. In auctions for government assets, such as mineral rights, timber harvesting contracts, and spectrum allocations, the participant pool often comprises established incumbents with a substantial industry history alongside relative newcomers. A similar distinction may exist with respect to the institutional buyers and non-institutional buyers in art auctions as well as treasury auctions (Kim and Che 2004).

The following story illustrates the presence of information asymmetry more clearly: At a typical summer weekend country auction in New England, over half those in attendance are likely to be tourists and another quarter nonprofessional locals Smith 2002. This intriguing context is shaped by the distinctive composition of the audience, where locals benefit from a sense of community cohesiveness and a familiarity network. The close-knit relationships among locals fostered by their shared geographic and social context contribute to know each other's valuations more. In contrast, tourists, unfamiliar with the local community and each other, may lack the same degree of interconnection. The more closed to outside buyers a dealer-dominated auction is, the more it operates like an exchange auction; the more open it is, the more it resembles a sales auction.

What's more, this context can be extended to the fashion online world, where it's typically considered anonymous and symmetrical for bidders. In such a setting, bidders may not necessarily be human; we face rivals such as auto-bidding robots equipped with specific algorithms. For instance, in online ad auctions, robots strive to maximize their value, focusing on metrics like the number of clicks/conversions, while adhering to a minimum admissible ratio based on the return-on-investment constraint, using similar algorithms (Deng et al. 2022). Thus, despite humans being unfamiliar with each other, algorithms themselves may share some similarities in valuations and bidding strategy, akin to the earlier example of locals and tourists.

A bidder's belief about the informational set of rival's emerges as a crucial determinant of bidding strategy (consider the story of signaling). Knowing other bidder's information proves pivotal for bidders, providing a dual advantage. First, bidder with more information can estimate the value of an auctioned object more precisely, gaining an informational advantage; second, they can better predict their rival's bidding strategy (which depends on the latter's information), gaining a strategic advantage.

My research goal is to develop a new theoretical framework that both integrates and extends existing literature, allowing for a tractable analysis of the impact of familiarity networks on bidding strategies within the equilibrium framework of standard auctions. I seek to answer key questions: (1) How the possibility of knowing the rivals' information affects the competition of auctions as well as bidding strategies in equilibrium? (2) How does the presence of a familiarity network affect the efficiency of resource allocation, expected revenue, and bidders' surplus in first-price and second-price auctions? (3) Do both first-price and second-price auctions produce the same expected revenue, as stated by the Revenue Equivalence Theorem? The answers to these questions will be fundamentally influenced by the key parameters of my model—specifically, the extent of familiarity—through their effects on informational and strategic advantages.

My contribution is to introduce familiarity network into the existing literature on auction theory, enabling me to analyze how informational and strategic advantages influence outcomes in standard auction formats with risk-neutral bidders. This approach broadens the scope of inquiry by raising new questions and examining the interactions between key components of the auction process. In particular, I study how bidders with asymmetric information operate in first- and second-price auctions, where valuations are drawn independently. The local knowledge introduces an additional layer of analysis for evaluating the performance of these auctions. I explore how information asymmetries among bidders may lead to a failure of the Revenue Equivalence Theorem. In first-price auctions, local bidders may strategically under-bid to mitigate the "winner's curse", thereby reducing expected auction revenue and potentially resulting in inefficient allocations. Lastly, I examine the overall impact of familiarity networks, analyzing whether the presence of social networks exacerbate or alleviate competition. I address these issues by providing the following answers:

- **Existence of Equilibria.** In a second-price auction, truthful-telling (i.e., bidding one's true valuation) is a weakly undominated strategy and thus constitutes an equilibrium for any structure of the familiarity network. In a first-price auction, a pure strategy equilibrium exists for any structure of the familiarity network, where the front-runner (i.e., the bidder with the highest valuation within her familiarity set) bids at least as much as the second-highest valuation and at most her own valuation. Additionally, in both types of auctions, the equilibria are not unique.

- **Revenue Equivalence Theorem.** The Revenue Equivalence Theorem does not hold under mild assumptions. Allocation in the second-price auction remains efficient. A first-price auction yields lower expected revenue than a second-price auction when bidders possess partial local information about others' valuations.

- **Equity-Efficiency Tradeoff.** Bidders obtain a greater surplus in a first-price than in a second-price auction with the presence of familiarity network. However, this first-price auction also results in inefficient allocations with positive probability.

- **Nonmonotonicity.** In a first-price auction, expected revenue (and total surplus) is nonmonotonic as bidders' familiarity increases. Revenue initially drops with partial information about rivals but rises again as bidders gain full information.

- **Size and Skewness Effect.** Under mild regularity conditions, in a first-price auction, increasing the number of bidders intensifies competition, leading to higher expected revenue, irrespective of the network's structure. Conversely, the impact of skewness in the familiarity network is nonmonotonic; as the network becomes more unbalanced, expected revenue initially decreases before subsequently increasing. In general, a higher number of clusters tends to result in greater expected revenue, controlling for the evenness of the familiarity network.

Related literature. While information acquisition in auctions has been an important issue in existing literature, few studies investigate the acquisition of information about other bidders (Kim and Koh 2020). There is a strand of literature in auctions that studies how information asymmetry alters bidding behavior is presented in auctions with interdependent valuations, where a bidder's signal contains information about other bidders' valuations (see P. R. Milgrom and Weber 1982; Maskin and Tirole 1992; Esö and White 2004; Perry and Reny 2002). The theoretical significance of this issue was not fully recognized until the seminal work by Kim and Che 2004, who explored how bidders respond differently to knowledge of other bidders' types across various auction formats. In their setting, each bidder possesses either complete knowledge of their rival's type or no knowledge at all. The existence of equilibrium in first-price auction was provided in Kim 2016. What's more, Andreoni, Che, and Kim 2007 presented experimental evidence that supports Kim and Che 2004. In my setting, I aim to investigate scenarios involving imperfect knowledge of other bidder's valuation, guided by insights derived from the familiarity network. In specific applications related to firms' pricing, Garetto 2016, leveraging the assumption of information asymmetry (Kim and Che 2004), formulated a model for trade and international price-setting. In the model, firms exhibit heterogeneity, and the market for each good resembles an international oligopoly with imperfect information. The environment considered involves the coexistence of both informed and uninformed producers. The study demonstrates that, in the presence of cross-sectional heterogeneity in information, uninformed producers tend to exhibit lower pass-through rates compared to their informed counterparts.

Another strand of literature examines setups similar to mine, where bidders are initially privately informed and observe signals related to their rivals' information. However, these studies primarily focus on costly learning processes, whereas my approach simplifies this aspect by treating the learning process as free and instead emphasizes the level of familiarity rather than binary signals. Bergemann and Välimäki 2006 discuss the possibility of bidders engaging in costly "espionage" in a private value first-price auction, referring to the activity of learning other bidders' information. In a related application, Kozlovskaya 2018 investigates industrial espionage in a duopoly market, where firms can "spy" on a rival firm to gain information about market demand. Additionally, Fang and Morris 2006 explore setups with private values, studying how standard auctions are affected when bidders observe their rivals' values or signals correlated with them. They assume that information about rivals is exogenously given, while later Tian and Xiao 2010 extend this by endogenizing bidders' information acquisition, ultimately establishing the same revenue ranking.

Several recent studies have also emphasized the role of bidder asymmetries. A typical finding is that revenue is more sensitive to asymmetries (and is lower) in the second-price auction than in the first-price auction, providing an argument for the superiority of the first-price auction. For example, Kim and Koh 2020 studied costly learning rival's information in the first-price or second-price auction, in which two bidders with each observes a binary signal. Only when the learning cost is low or values are sufficiently interdependent, there's a efficiency/revenue trade-off, otherwise first-price auction is better. As an improvement, McClellan 2023 focuses on information disclosure, studied second-price common-value auctions in which one bidder may be an expert with access to more precise information about the value of the good.

Thus, while one body of literature addresses how bidders respond to knowledge of other bidders' types, it typically considers only extreme scenarios. Another strand examines the role of bidder asymmetries, but primarily in the context of signaling or information disclosure. There remains a gap in understanding the mechanisms within a social network context. My research addresses this gap by considering a free learning process—where bidders possess prior knowledge—and examining a private auction setting with bidders experiencing information asymmetry.

Structure of the paper. The structure of the paper is outlined as follows: Chapter 2 defines the general auction framework, introduces the equilibrium concepts, explains the familiarity network, and provides preliminary observations. Chapter 3 characterizes the equilibrium and revenue implications in both first- and second-price auctions in the general framework. It also presents with an example to illustrate the main intuitions and further explores the role of the familiarity network in a simplified setting, supported by simulations. Finally, Chapter 4 summarizes the key observations and presents findings on the robustness of second-price auctions to the familiarity network, as well as the influence of the familiarity network in first-price auctions, including its effects on inefficiency and bidders' surplus. The chapter concludes with suggestions for potential improvements. The Appendix contains the formal proofs and code omitted from the main text.

Chapter 2

General framework

2.1 Semi-perfect information auction framework

In light of the discussion above, existing auction models do not sufficiently account for the influence of social networks. Consequently, these models and their outcomes cannot simply be "taken off the shelf" for my research, making the development of a new framework imperative. Specifically, I introduce a Semi-perfect Information Auction as an extension of the Independent Private Value (IPV) model (Myerson 1981; Riley and Samuelson 1981), where each bidder has information about the values of only a subset of other bidders, determined by the structure of her underlying network. Drawing on key insights from prior works by Kim and Che 2004; Kim and Koh 2020, my model is structured as follows:

2.1.1 The game

Setup. A seller has a single indivisible good to sell to $n \geq 3$ risk neutral bidders. The seller is assumed to put no value on the good. Bidders have independent private values. Bidder i's valuation V_i with a typical realization v_i is drawn from the interval $[\underline{v}, \bar{v}]$, $0 \le \underline{v} < \bar{v} < +\infty$, following a common continuous distribution function $\mathrm{F}, supp(\mathrm{F}) \subseteq \mathbb{R}_+$ whose density function f is bounded away from zero. F is assumed to be common knowledge. Letting $N \equiv \{1, 2, \cdots, n\}$ denote the set of all bidders, the profile of bidders' valuations is a vector $\mathbf{v} \equiv (v_1, v_2 \cdots, v_n)$ within the valuation space $\mathbf{V} \equiv [\underline{v}, \bar{v}]^n$, has the joint density $f_N(\mathbf{v}) = \prod_{i \in N} f(v_i)$. Bidders are asked to submit sealed bids $\mathbf{b} \equiv (b_1, b_2 \cdots, b_n)$ simultaneously, where $B_i \ni b_i$ is the pure strategy set of player $i \in N$, $B_i \subseteq \mathbb{R}_+$. A payoff function (utility) for player i is given by $\tilde{u}_i : \mathbf{B} \times [\underline{v}, \bar{v}] \to \mathbb{R}$, where $\mathbf{B} \equiv \times_{i=1}^{N} B_i$. The expected payoff for i is given by the product of the payoff and the probability of her winning the auction i.e., $u_i(\mathbf{v}) \equiv \tilde{u}_i \cdot Pr(i \text{ wins the good})$.

Auctions. I examine two classical auction formats: the first-price sealed bid auction and the second-price sealed bid auction, both incorporating a zero reserve price. The auction is played as the following:

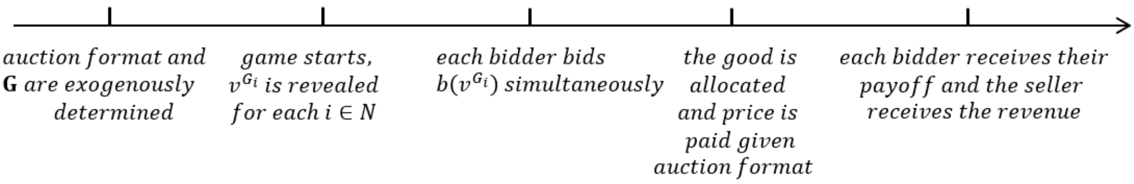


Figure 2.1: The game

In the second-price auction, the bidder who submits the highest bid wins the object but pays an amount equal to the second-highest bid. Under the IPV setting, second-price auctions are analogous to English auctions, with tie-breaking conducted arbitrarily. The payoff function is given by:

$$\tilde{u}_i(\mathbf{b}, V_i) = \begin{cases} v_i - \max\limits_{j \neq i} b_j & if\ b_i \ge \max\limits_{j \neq i} b_j \\ 0 & if\ b_i < \max\limits_{j \neq i} b_j \end{cases} \tag{2.1}$$

In contrast, the first-price sealed-bid auction entails simultaneous bidding, and the participant with the highest bid wins, paying the amount of their bid. A crucial aspect of this format is the tie-breaking rule, which plays a key role in ensuring the existence of an equilibrium, as highlighted by Kim and Che 2004. For example, in a standard Bertrand competition between two firms with heterogeneous costs, a Nash equilibrium (NE) exists only if ties are broken in favor of the lower-cost firm.

Following this rationale, I adopt the following rules for consideration of efficiency:

- a tie is broken in favor of a bidder with a higher valuation if there are multiple highest bidders; and
- if there are multiple highest bidders with the same valuation, then the object is assigned randomly with equal probability among those bidders.

While the tie-breaking rule is endogenous, it can be effectively implemented by conducting an auxiliary second-price auction among the bidders who submitted the highest bid in the first-price auction. The payoff function is given by:

$$\tilde{u}_i(\mathbf{b}, V_i) = \begin{cases} v_i - b_i & if\ b_i \geq \max_{j \neq i} b_j \\ 0 & if\ b_i < \max_{j \neq i} b_j \end{cases} \tag{2.2}$$

Familiarity network. Assume bidders possess *local knowledge*, meaning that each bidder has perfect information regarding the valuations of a specified subset of other bidders, but lacks complete information about the valuations or actions of participants outside of their direct connections. To formalize this concept, I introduce a familiarity structure **G** $\equiv (G_1, G_2, \cdots, G_n) \subseteq N^n$, where **G** represents a collection of sets that forms an open cover on N (under the discrete topology) i.e., $N \subseteq \bigcup_{i \in N} G_i$. Define $G_i \subseteq N$ as the familiarity set for bidder $i \in N$, where bidder i has perfect information of the valuations (or realizations) of all bidders in G_i, while the valuations of bidders outside of G_i remain unknown to i. Specifically, define $v^{G_i} \subseteq [\underline{v}, \bar{v}]^n$ as i's view of the bidding vector. A bidder's local knowledge within G_i is:

$$v^{G_i} \equiv \begin{cases} v_j & if\ j \in G_i \\ V_j & if\ j \notin G_i \end{cases} for\ all\ i, j\ \in\ N \tag{2.3}$$

I thus incorporate the setting from Kim and Che 2004, where **G** represents a collection of disjoint groups i.e., $G_i \cap G_j = \emptyset$ for any $i \neq j$. What's more, a familiarity structure **G** is said to be *symmetric* if, for all $i, j \in N$, whenever $i \in G_j$, it also holds that $j \in G_i$. Conversely, a familiarity structure **G** is *asymmetric* if there exist $i, j \in N$ such that $i \in G_j$ but $j \notin G_i$. While Kim and Che 2004 focuses primarily on the symmetric case, I extend the analysis to a broader setting in network structure. It is not necessarily true that mutual knowledge exists between any pair of bidders who are familiar with each other, as information flow can be unidirectional (Cha, Mislove, and Gummadi 2009).

One immediate observation is that G_i is non-empty, as $i \in G_i$ for all $i \in N$ e.g., each bidder is at least familiar with their own valuation. Additionally, the partition structure is assumed to be private knowledge and exogenously determined. For instance, bidder i only knows G_i, but has no knowledge of the familiarity sets of other bidders i.e., [1]G_{-i}. This aligns more closely with the familiarity network setting, where the information available to each bidder is limited to their own network, without insight into how others are connected or informed.

Equilibrium concept. I restrict attention to strategies which satisfy a version of weakly undominated (Bergemann, Brooks, and Morris 2017). In particular, for every structure of **G**, the support of the possibly randomized bid does not exceed the largest possible valuation in the support of bidder i's familiarity set $G_i, i \in N$, or $supp(b_i(v^{G_i})) \subseteq [\underline{v}, \max supp(v^{G_i})]$. This restriction is weaker than weakly undominated; in the case of complete information, it allows bidder i to bid up to and including her valuation v_i, whereas the conventional definition of weakly undominated would

[1]Similar to the notation used for strategies, I denote $-i$ as the familiarity set excluding bidder i i.e., $G_j, \forall j \neq i$.

also rule out bidding v_i, and thereby exclude the unique equilibrium strategy in which two (or more) bidders who share the highest valuation bid their valuation.

In both games, I look for Bayesian Nash Equilibrium (BNE) in weakly undominated strategies. A BNE is a NE of a Bayesian game $< N, \{B_i\}, \{u_i\}, \{V_i\}, \mathrm{F} >$. For any player $i \in N$, a (non-negative) bidding strategy b_i : **V** $\rightarrow \mathbb{R}_+$ is defined such that $b_i(v^{G_i})$ is the bid submitted by player i given her corresponding valuation v^{G_i} according to her familiarity set G_i.

Efficiency and revenue. Under the assumption of risk-neutral bidders and within the IPV framework, the equilibrium efficiency is straightforward. Following Birulin and Izmalkov 2011, an auction is considered *efficient* in equilibrium if and only if the object is allocated to the bidder with the highest valuation in every realization of valuations **v**. Formally, the auction is efficient if the winner i of the auction satisfy $i \in \underset{j \in N}{\operatorname{argmax}}\, v_j$.

Since there is zero reservation price for both bidder and seller, the seller's revenue, as discussed by Klemperer 1999, is simply the amount the winner pays. In a first-price auction, this is the highest bid, while in a second-price auction, it is the second-highest bid. Denote $|\cdot|$ as the cardinality of a set,

- In a first-price auction, the revenue is $r^{1st} := \max_{i \in N} b_i$
- In a second-price auction, consider the case of multiple bidders tied for the highest valuation, let $\mathrm{B} := \underset{j \in N}{\operatorname{argmax}}\, b_j$, the revenue is defined as

$$r^{2nd} := \begin{cases} \underset{i \,\in\, N \backslash \mathrm{B}}{\max}\, b_i & if \;\; |\mathrm{B}| = 1 \\ \underset{i \,\in\, \mathrm{B}}{\max} b_i & if \;\; |\mathrm{B}| > 1 \end{cases}$$

The expected revenue, denoted as $E[r^{1st/2nd}]$, will be formally defined in a later section. Further analysis will simplify this definition and help avoid unnecessary complications.

2.1.2 Preliminary observation

To facilitate comparison, my model includes two extreme cases of familiarity structures. In one scenario, every player is a singleton i.e., each bidder is aware only of their own valuation. This corresponds to the standard assumption in auction literature (Myerson 1981). In the other scenario, there is a perfect information within groups i.e., all bidders have knowledge of all others' valuations. This results in a game equivalent to the Bertrand competition. As is well known, revenue equivalence holds for these two partition structures. To be specific,

Observation 1. If $G_i = \{i\}$ or $G_i = N$ for all $i \in N$, then first-price and second-price auctions allocate the good efficiently and are revenue equivalent.

Proof. For any $i \in N$, if $|G_i| = 1$, the result directly follows from Myerson 1981; Riley and Samuelson 1981, as the familiarity network has no effect in this case. Similarly, if $|G_i| = n$, it's an auction where values are common knowledge, in equilibrium the bidder with the highest valuation bids the second-highest valuation (and wins the auction) in equilibrium.

Hence, I constraint on the group size become imperative to focus on more interesting cases. Assume the following:

Assumption 1. There exists player $i \in N$ such that her familiarity set G_i satisfies $1 < |G_i| < n$.

Chapter 3

Results

3.1 Characterization of equilibrium and revenue implications in a second-price auction

In a second-price auction, it is a weakly dominant strategy for each bidder to bid their true value. The optimality of this strategy is independent of any information the bidder may have about their network structure. As a result, the equilibrium, as well as the expected revenue, is unaffected by player's' familiarity structure. This idea is formalized in the following proposition:

Proposition 1. In a second-price auction, "truthful-telling" (bidding one's own valuation) is a weakly undominated strategy, i.e., $b_i(v^{G_i}) = v_i$ for all $i \in N$. The auction allocation is efficient, and the expected revenue is $E[r^{2nd}] = n(n-1)\int_{supp(\mathrm{F})}(\mathrm{F}^{n-2} - \mathrm{F}^{n-1})f \cdot vdv$.

A formal proof is provided in Appendix A, following a similar approach as in Vickrey or second-price IPV auctions. The proof proceeds by contradiction: any deviation from truthful-telling, either by overbidding or underbidding, would not increase the bidder's expected payoff. Further, it has been shown that the equilibrium is not unique unless additional restrictions are placed on the equilibrium concept or on the form of the distribution F.

Given the expression, the revenue implication can be demonstrated, for example, when F is a uniform distribution over the interval $[0,1]$, the expected revenue is $E[r^{2nd}] = \frac{n-1}{n+1}$. This result aligns with the standard findings in auction theory literature related to revenue equivalence theorem (Riley and Samuelson 1981; Ivanova-Stenzel and Salmon 2008).

Here, the efficient allocation naturally arises in the second-price auction. This robustness stems from the fact that any familiarity structure does not affect the equilibrium bidding strategy. However, the equilibrium in the first-price auction is not as straightforward, requiring a more nuanced analysis. Therefore, I focus on the first-price auction in the remainder.

3.2 Characterization of equilibrium in a first-price auction

3.2.1 The existence of equilibrium

Standard existence and characterization of equilibrium regarding the existence of equilibrium (e.g., Vickrey 1961; Riley and Samuelson 1981; Lebrun 1996) do not directly apply in my framework. This is due to the following factors: (1) multi-dimensional signals: each bidder knows valuation within her familiarity set; (2) asymmetric environment: I allow for familiarity structure to be arbitrary; and (3) non-quasiconcavity: payoffs need not be quasiconcave in bidder's own strategy. Given these complexities, I begin by demonstrating the existence of a pure strategy equilibrium.

Observation 2. In first-price auctions, for all $i \in N$, any bidding strategy $b_i > \bar{v}$

is strictly dominated.

Proof. In words, no bidder will bid above the maximum possible value of the good in equilibrium. It suffices to show that $\forall b_i > \bar{v}$ is strictly dominated by truthful-telling. This is because $\forall b_i > \bar{v}$, $u_i(b_i) := (v_i - b_i)Pr(b_i \geq b_{-i}) \leq (v_i - \bar{v})Pr(b_i \geq b_{-i}) < 0 \leq (v_i - v_i)Pr(v_i \geq b_{-i}) =: u_i(v_i)$, where $Pr(b_i \geq b_{-i})$ is the probability of player i wins the auction.

Observation 2 can be strengthened to establish a stricter upper bound on the bidding strategies of each bidder, leading to the following observation:

Observation 3. In first-price, for all $i \in N$, any bidding strategy $b_i > v_i$ is strictly dominated.

Proof. Similarly to Observation 2, no bidder would implement a bidding strategy to over bid her values of the good. By similar logic as in Observation 2, for all $b_i > v_i$, $u_i(b_i) := (v_i - b_i)Pr(b_i \geq b_{-i}) < (v_i - v_i)Pr(b_i \geq b_{-i}) \leq (v_i - v_i)Pr(v_i \geq b_{-i}) =: u_i(v_i)$.

Equipped with Observation 2 and 3, I show the existence of equilibrium in the first-price semi-perfect information framework, with the only requirement of assuming bidding strategy $b_i(\cdot)$ being non-decreasing.

Theorem 1. The first-price auction possesses a pure strategy equilibrium in which the bidders employ non-decreasing bidding functions.

A formal proof is provided in Appendix B, drawing on the framework established by Reny 1999 and the subsequent corrigendum by Ewerhart and Reny 2022, which does not affect the main results. The proof proceeds by first demonstrating the existence of equilibrium in mixed strategies. Specifically, for any non-equilibrium bidding strategy, there exists some player whose strategy yields a payoff strictly higher than the non-equilibrium payoff, even if others slightly deviate from this non-equilibrium strategy. Additionally, since the strategy space is compact, there must be a point where no bidder can profitably deviate. Finally, all mixed equilibria are in fact pure, due to the assumption for a nondecreasing bidding function and the strategy space is Hausdorf.

3.2.2 Properties of the equilibrium

After demonstrating the existence of an equilibrium, I now turn to several properties of the equilibrium. Denote the bidder with the highest valuation among a her familiarity set as the *highest valuation bidder in* $G.$ i.e., $\underset{j \in G_i}{\operatorname{argmax}}\ v_j$. Since the size of each familiarity set $|G.|$ is finite, there always exists at least one highest valuation bidder in each $G.$. Furthermore, I define a bidder as *front-runner*, if she has the highest valuation among her familiarity set, i.e., $i \in \underset{j \in G_i}{\operatorname{argmax}}\ v_j$. One immediate observation is the existence of front-runner, to be specific:

Observation 4. There exists at least one front-runner for any familiarity structure **G**.

Proof. Since N is finite, there must exist at least one bidder with the highest valuation among all bidders in N. Let this bidder be i. Because bidder i has the global maximum valuation, she must also have the highest valuation within her familiarity set G_i ($G_i \subseteq N$). Thus, bidder i is the front-runner in G_i, which proves the existence.

The proposition of equilibrium is stated as follows:

Proposition 2.

1. In equilibrium, a front-runner bids at least a second-highest valuation in $G.$ and wins against all bidders in her familiarity set $G.$.

2. The equilibrium allocation is inefficient with positive probability when there are at least two bidders $k, l \in N$ whose familiarity sets satisfies (i) $1 < |G_{k,l}| < n$ and (ii) $|G_k \setminus G_l| \geq 1$, $|G_l \setminus G_k| \geq 2$.

3. The equilibrium is not (necessary) unique.

Proposition 2 presents a similar argument to Proposition 1 in Kim and Che 2004. Although the procedure is analogous, I must demonstrate that it applies within my model, which incorporates *asymmetric* cases. A formal proof is provided in Appendix C. The idea behind the proof of Proposition 2(1) arises from the fact that any front-runner has a strictly positive expected payoff. In equilibrium, the front-runner can slightly raise her infimum bid to increase her probability of winning by outbidding bidders within her familiarity network, thereby improving her expected payoff slightly. For Proposition 2(2), the argument demonstrates that under the assumptions of the familiarity structure, there exist at least two different front-runners. The case where the front-runner with a lower valuation outbids the other occurs when the former has a strong second-highest bidder within her familiarity network, which pushes up her infimum bid. Finally, Proposition 2(3) results from the fact that the optimization problem is not necessarily strictly concave or having a global maximum.

It's clear that the assumptions in Proposition 2(2) implicitly imply Assumption 1. The interpretation of Proposition 2 is that, in equilibrium, a front-runner must bid at least the second-highest valuation she is aware of to avoid the risk of losing and receiving zero payoff. This imposes a lower bound on her bidding strategy. Consequently, this constraint can sometimes (with positive probability) cause the second-highest valuation bidder to overbid the highest valuation bidder under mild constraints on the structure of $\mathbf{G}$, leading to inefficiency in the auction's outcome.

The following example illustrates the above argument, offering a clearer understanding of how Proposition 2 operates. Despite the overall allocation potentially being inefficient with positive probability, it remains efficient within each bidder's familiarity set e.g., the good is always allocated to the front-runner.

3.2.3 Examples

A symmetric case

Here I consider a simple example of $n = 4$, based on Kim and Che 2004 for consistency, but modified the notation to reflect the semi-perfect information framework. Notably, the main result in Kim and Che 2004 remains applicable in this context. Suppose that there are four bidders with the following symmetric familiarity structure **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{3,4\}\}$ (see Figure 3.1). Suppose also that each bidder i draws his valuation uniformly from $[0,1]$. Intuitively, each bidder thus compete with two unknown rivals (bid exactly $\frac{2}{3}v_i$), which follows the standard bidding strategy in Riley and Samuelson 1981. And compete with one bidder whose valuation is known by the bidder $j, j \neq i$. It is then an equilibrium for each bidder to bid $b_i = \min\{v_i, \max\{\frac{2}{3}v_i, v_j\}\},\ j \in G_i$. This is an equilibrium that maximize the expected payoff for each bidder i (See Appendix D for a formal proof and see Kim and Che 2004 for an intuitive explanation).

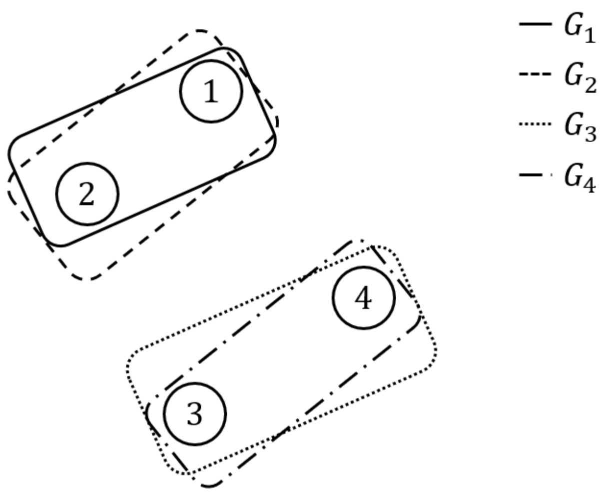


Figure 3.1: $n = 4$ case, **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{3,4\}\}$

In equilibrium, the front-runner strategically adopts an unconstrained bid: $b_i =$

$\max\{\frac{2}{3}v_i, v_j\}$. When $\frac{2}{3}v_i \geq v_j$ the bidder with higher valuation behaves as if facing minimal competition from the lower-valuation bidder j within the set, alleviating competitive pressure. Conversely, when $\frac{2}{3}v_i < v_j$, the second-highest valuation functions as a constraining factor, intensifying competition between bidders.

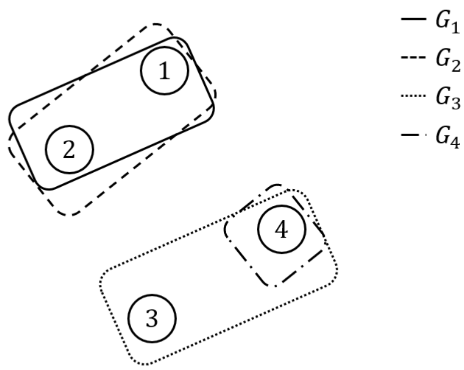
However, these equilibrium strategies may lead to inefficient allocations. Consider a scenario where the valuations follow $v_1 > v_3 > v_4 > \frac{2}{3}v_1 > \frac{2}{3}v_3 > v_2$. This configuration can occur with positive probability. In equilibrium, the bids would be $b_1 = \frac{2}{3}v_1, b_2 = v_2, b_3 = v_4, b_4 = v_4$. Thus, the inefficiency emerges as bidder 1 values the good the most whereas bidder 3 obtain the good. As elucidated, a lower v_2 permits bidder 1 to strategically bid more aggressively than bidder 3, who faces constraints imposed by a higher v_4. Consequently, the latter bidder manages to outbid the former due to the influence of this constraint. This highlights a scenario where strategic bidding based on familiarity structures and competition constraints can lead to inefficient auction outcomes.

Two asymmetric cases

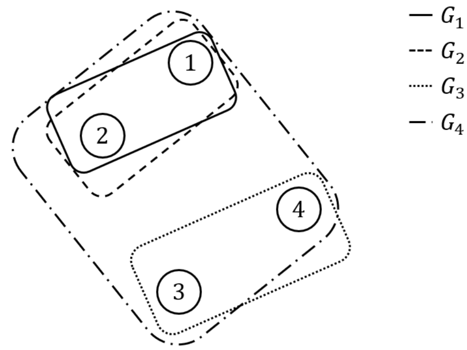
Slightly modifying the above case where $n = 4$, suppose bidder 4 now has one of two information scenarios: Case 1. Bidder 4 has no information about other bidders' values i.e., **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{4\}\}$ (see Figure 3.2(a)) or Case 2. Bidder 4 has full information about other players i.e., **G** $= \{\{1,2\},\{1,2\},\ \{3,4\}\,\{1,2,3,4\}\}$ (see Figure 3.2(b)). Need to show that the main intuition from the previous example remains applicable in asymmetric cases.

Following similar logic and assuming a linear, non-decreasing bidding strategy, bidder $i = 1,2,3$ maintains $b_i^* = \min\{v_i, \max\{\frac{2}{3}v_i, v_j\}\},\ j \in G_i$. The remaining is to determine one equilibrium bidding strategy for bidder 4.

- Case 1. **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{4\}\}$. Since she lacks additional information (e.g., about other bidders or latent network structures), the optimal bid for her



(a) **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{4\}\}$



(b) **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{1,2,3,4\}\}$

Figure 3.2: $n = 4$, asymmetry case

is identical to the private IPV framework i.e., $b_4^* = \frac{3}{4}v_4$.

- Case 2. **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{1,2,3,4\}\}$. With perfect information in this auction, bidder 4 can either bid the second-highest value (as per Proposition 2(1)) if she is the front-runner, or bid exactly $b_4^* = v_4$ otherwise.

Similar to Appendix D, it is straightforward to show that the above constitutes an equilibrium. The inefficiency arises, for example,

- Case 1. when the valuations follow $v_1 > v_3 > v_4 > \frac{2}{3}v_1 > \frac{2}{3}v_3 > \frac{3}{4}v_4 > v_2$, which occurs with positive probability. In equilibrium, the bids would be $b_1 = \frac{2}{3}v_1, b_2 = v_2, b_3 = v_4, b_4 = \frac{3}{4}v_4$. Here, inefficiency emerges as bidder 1, who values the good the most, does not obtain it—instead, bidder 3 does.

- Case 2. when the valuations follow $v_1 > v_4 > \frac{2}{3}v_1 > v_3 > \frac{2}{3}v_3 > v_2$, which also occurs with positive probability. The equilibrium bids would be $b_1 = \frac{2}{3}v_1, b_2 = v_2, b_3 = v_3, b_4 = v_4$. Again, inefficiency arises, this time bidder 1 values the good the most, but bidder 4 obtains it.

The above examples intuitively illustrate Proposition 2, demonstrating that inefficiency arises with positive probability due to information asymmetry. The next step is to analyze how revenue changes accordingly. Based on the aforementioned examples, in the symmetric case, the expected revenue is 0.584 (aligned with Kim and Che 2004). In the asymmetric cases, the expected revenues are 0.597 for Case 1 and 0.602 for Case 2.

Intuitively, without strictly defining the "density" of the familiarity network, we can infer that the network is getting more dense in the following order: asymmetric Case 1, symmetric case, and then asymmetric Case 2 e.g., how bidder 4 gradually learns about more bidders while keeping other factors unchanged (similar to a newcomer gradually getting to know more people in a neighborhood). This analysis implicitly points out that the expected revenue first decreases and then increases. One might consider that the competition-reduction effect is gradually outweighed by the increasing desire to win the auction as bidders gain more information. Further analysis will be provided in the following section.

3.3 Revenue implications in a first-price auction

The allocation inefficiency identified above could lead to a revenue characterization, if the well-known Revenue Equivalence Theorem (see Vickrey 1961; Myerson 1981) holds.

Before proceeding, recall that the Revenue Equivalence Theorem relies on four general conditions: (1) Each bidder's type is drawn from a "well-behaved" distribution, i.e., F must be strictly increasing and continuous. (2) Bidders are risk neutral. (3) The bidder with the highest type wins. (4) The bidder with the lowest possible valuation has an expected payoff of zero.

It is evident from Proposition 2 that the inefficiency in allocation violates condition (3), as the bidder with the highest type does not always win. This suggests that the Revenue Equivalence Theorem may not hold under my model.

Further, I define a vector $\mathbf{v}^{G_i} := (v_j)_{j \in G_i}$ to represent the valuations within bidder i's familiarity set, removing randomness. Bidder i wins the good with probability $p_i(\mathbf{v}) := Pr(b_i(v^{G_i}) \geq b_{-i}(v^{G_{-i}}))$. Consequently, bidder i's average probability of winning $Ep_i(\mathbf{v}^{G_i})$ is given by:

$$Ep_i(\mathbf{v}^{G_i}) \equiv \int_{\mathbf{V}^{G_{-i}}} p_i(\mathbf{v}^{G_i}, \mathbf{v}^{G_{-i}}) f_{G_{-i}}(\mathbf{v}^{G_{-i}}) d\mathbf{v}^{G_{-i}} \tag{3.1}$$

It can be shown that (see a formal proof in Appendix E), in any equilibrium of a first-price auction, the expected payoff (Note that I slightly abuse notation here by defining $u_i(\cdot)$ on $\mathbf{v}^{G_i}$) from the first price auction has a lower bound:

$$u_i(\mathbf{v}^{G_i})] \geq \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}}) ds, \ v_i \in [\underline{v}, \bar{v}] \tag{3.2}$$

The equality holds if **G** is *symmetric* (as in Kim and Che 2004). In turn, the expected revenue has an upper bound:

$$E[r^{1st}](\mathbf{v}^{G_i}) \leq \int_{\mathbf{V}} \left(\sum_i J(v_i) p_i(\mathbf{v}) \right) f_N(\mathbf{v}) d\mathbf{v} \tag{3.3}$$

where $J(v_i) := v_i - \frac{1-\mathrm{F}(v_i)}{f(v_i)}$. (3.3) holds in equality if the envelope theorem (P. Milgrom and Segal 2002) holds, which provide an easy method for comparing revenues across different auction formats.

Assumption 2. $J(\cdot)$ is strictly increasing on its support.

In particular, if assume as the above that $J(\cdot)$ is strictly increasing on its support, an auction format that induces an efficient allocation would yield strictly higher expected revenue than one that induces an inefficient allocation. This is because the bidders' surplus can be calculated as the total surplus minus the seller's revenue i.e., $\int_{\mathbf{V}} \left(\sum_i \frac{1-\mathrm{F}(v_i)}{f(v_i)} p_i(\mathbf{v}) \right) f_N(\mathbf{v}) d\mathbf{v}$ (Kim and Che 2004). What's more, if $\frac{1-\mathrm{F}(v_i)}{f(v_i)}$ is strictly decreasing on its support, bidders receive greater surplus in a first-price auction than in a second-price auction.

The above assumption, along with (3.2-3.3), leads to the following: First, recall that the allocation is efficient in a first-price auction when each bidder observes only their own valuation, or completely connected (see Observation 1). In contrast, after giving assumptions in Proposition 2(2), the allocation is inefficient with positive probability. Under Assumption 2, the expected revenue is lower when allocation is inefficient. The revenue (and total surplus) from a first-price auction exhibits non-monotonic behavior as bidders gain more information about others. Specifically, expected revenue decreases when bidders transition from having no information about other bidders to having partial information about a subset of bidders. However, revenue increases back to its original level as bidders acquire full information about all players.

Second, note that the expected revenue remains invariant in a second-price auction across all familiarity structures (Proposition 1), and revenue equivalence holds between first- and second-price auctions under the singleton structure (see Observation 1). Therefore, this implies that the expected revenue is lower in a first-price auction than in a second-price auction when assumptions in Proposition 2(2) and Assumption 2 holds.

I conclude the above as the following proposition:

Proposition 3.

1. **Revenue Implication.** Given assumptions in Proposition 2(2) and Assumption 2, a first-price auction yields a lower expected revenue compared to a second-price auction.

2. **Equity-Efficiency Tradeoff.** If further assuming $\frac{1-\mathrm{F}(v_i)}{f(v_i)}$ is strictly decreasing on its support, bidders obtain a greater surplus in a first-price auction than in a second-price auction. However, this first-price auction also results in inefficient allocations with positive probability.

3. **Nonmonotonicity.** The revenue (and total surplus) in a first-price auction exhibits nonmonotonic behavior as bidders' familiarity network becomes more dense. Expected revenue initially decreases when bidders acquire partial information about some bidders, but increases again as they gain complete information about all competitors.

3.4 The impact of familiarity network

So far, I have demonstrated the nonmonotonicity of revenue as the familiarity network becomes more dense in first-price auctions, alongside the irrelevance result for second-price auctions. In this section, I will further analyze the role of the familiarity network in the first-price case. To provide clearer results and make them comparable to the literature, I restrict the analysis to the case where the distribution of valuations follows $\mathrm{F} \equiv \mathrm{Unif}[0,1]$, and focusing on *symmetric* cases. Now, (3.2-3.3) hold as equalities due to such symmetry (see Appendix E).

3.4.1 Equilibrium bidding strategy in closed-form

One advantage of these simplifications is that the equilibrium strategy can now be expressed in closed form. By applying similar logic as in the example in Section 3.2.3, the equilibrium strategy for bidder $i \in N$ is given by:

$$b_i(v^{G_i}) = \min\{v_i, \max\{\frac{n-|G_i|}{n-|G_i|+1}v_i, \max_{j \in G_i \setminus \{i\}} v_j\}\} \tag{3.4}$$

The proof follows the same approach as in Appendix D and is therefore omitted. The main adjustment is replacing $\frac{2}{3}v_i$ with $\frac{n-|G_i|}{n-|G_i|+1}v_i$, after which the rest of the proof holds. The intuition is that bidders are now competing against two types of players: those with perfect information and those with no information. When competing against the former, bidder $i \in N$ must submit at least the second-highest bid to win against all rivals within her familiarity set. Against the latter, she follows strategies typical of a symmetric, linear bidding strategy in the standard IPV framework, where she bids to maximize expected payoff against $\max_{j \in G_i \setminus \{i\}} v_j$. Finally, to ensure non-negative (one can either be a front-runner or not) expected payoffs, the bidding strategy has an upper bound at v_i. It is clear that this bidding strategy satisfies all the general results from previous sections (namely, Observation 2-3 and Proposition 2). Moreover, the distribution $\mathrm{F} \equiv \mathrm{Unif}[0,1]$ satisfies Assumption 2, as $J(v) \equiv 2v - 1$ is strictly increasing on $[0,1]$, and $\frac{1-\mathrm{F}(v)}{f(v)} \equiv 1 - v$ is strictly decreasing on $[0,1]$. Therefore, Proposition 3 is satisfied when **G** satisfies the assumptions in Proposition 2(2) (mild regularity conditions for familiarity network).

3.4.2 Size effect of familiarity

Intuitively, as the number of bidders increases in an auction, the expected revenue generally rises as well (Smith 2002). This is because the expected maximum valuation, determined by the first-order statistic, increases as the number of participants grows. Given that I assume a non-decreasing bidding strategy, the highest bid also rises in expectation, and since the revenue is equivalent to the maximum bid, the expected revenue increases accordingly. However, when we introduce familiarity networks, where some bidders have information about others' values, this added information may lead to inefficiencies in the allocation, thereby reducing expected revenue (see Proposition 3). To better understand the interaction of these two forces—more participants versus the effect of familiarity—let's first examine their overall impact.

Observation 5. Given equation (3.4), bidding will not decrease for any existing players when additional players join the auction. i.e., $b_i' \geq b_i(v^{G_i})$, where b_i' represents the bidding strategy for player i when extra players (≥ 1) are added to the existing familiarity structure **G**.

A formal proof is provided in Appendix F. The idea is that, without loss of generality, even the addition of one extra player will not decrease the bids of all existing bidders, regardless of whether the new player is within anyone's familiarity set or not. Furthermore, the new bidder also adheres to the equilibrium bidding strategy outlined in (3.4), ensuring the iteration.

With Observation 5 in mind, I now present the following corollary to examine the size effect in first-price auction. This result naturally holds to the second-price case, as supported by Proposition 1.

Corollary 1. The expected revenue increases monotonically as more players join the auction, regardless of the structure of the familiarity network.

See Appendix G for a formal proof. This is because bidders do not lower their bids when more participants join the auction (as stated in Observation 5). With additional players, the likelihood of higher valuations increases, raising the expected highest bid (the first-order statistic), which in turn leads to higher overall revenue for the auctioneer.

Despite the potential inefficiencies introduced by the presence of a familiarity network, the overall size effect remains positive. Specifically, as more players enter the auction, the expected revenue for the auctioneer tends to increase. This increase in revenue implies that the surplus available to bidders may decrease, as a larger share of the total revenue is allocated to the seller.

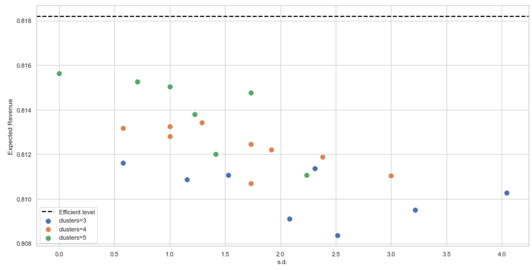
3.4.3 Skewness effect of familiarity

Another aspect of the familiarity network is the evenness in the size of each cluster (i.e., $G_i \equiv G_j$ for all $i, j \in N$ within the same cluster). Suppose there are 6 bidders and 3 clusters. A more "even" structure would be a "2-2-2" distribution of bidders across the clusters, as opposed to a "1-1-4" distribution, which is more unbalanced. A natural measure of this evenness is the standard deviation (s.d.) of the number of bidders in each cluster. The smaller the s.e., the less skewed the division of players is.

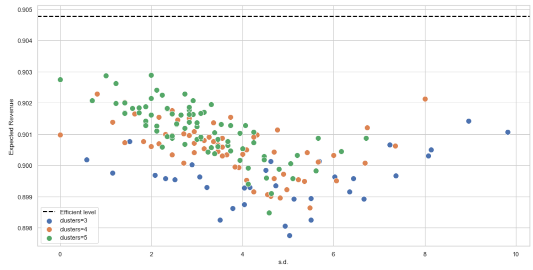
As seen in Proposition 3, the availability of information about other bidders reduces the effective number of competitors a front-runner faces, which softens competition. Simultaneously, the familiarity network can raise the front-runner's bid (see Proposition 2(1)) in a way not possible in the standard case. However, the revenue result suggests that the reduction in competition outweighs the increase in bids, leading to a net effect where the front-runner tends to bid less. Intuitively, if the familiarity structure shifts towards either extreme—fully connected or no connection (both efficient scenarios)—overall competition intensifies, and the expected revenue rises accordingly.

While this point cannot be proven in general, it is illustrated in the simulation results. The following figure presents a simulation of expected revenue for varying numbers of clusters and players. The results provide a visual representation of how expected revenue shifts depending on the structure of familiarity networks and the number of participants. The dashed line represents the expected revenue in an efficient auction (e.g., $E[r^{1st}] = \frac{n-1}{n+1}$). On the far left is the case with an evenly divided familiarity network, while the far right shows the most skewed division possible under a given number of clusters (e.g., for 3 clusters, a distribution like "(n-2)-1-1". All combinations illustrated in the figure result in inefficiency and lower revenue due to the conditions stated in Proposition 2(2).

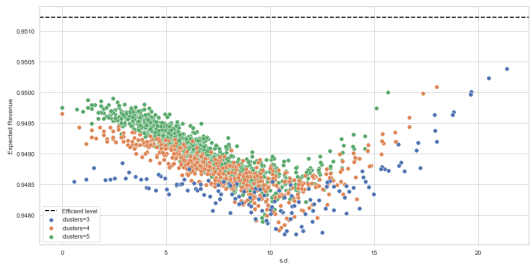
Several observations emerge: (1) **Number of players.** Expected revenue generally increases as the number of players grows, assuming a similar evenness of cluster sizes. This is because as the number of players increases, the expected first-order statistics improve while maintaining a comparable familiarity structure. (2) **Skewness effect.** The impact of skewness in the familiarity network is also non-



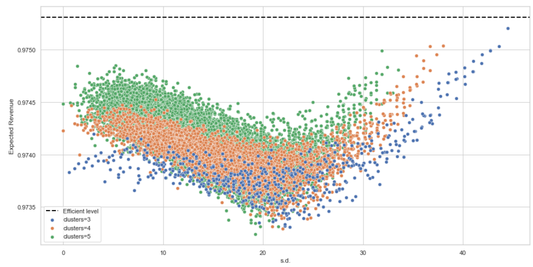
(a) $n = 10$



(b) $n = 20$



(c) $n = 40$



(d) $n = 80$

Figure 3.3: Expected revenue for different familiarity networks with varying numbers of bidders

monotonic. As the familiarity network becomes more unbalanced, expected revenue initially decreases, then increases, forming a U-shaped curve. (3) **Number of clusters.** More clusters generally lead to higher expected revenue, controlling for the evenness of the familiarity network distribution.

I record the implications above as the following:

Corollary 2. When $\mathrm{F} \equiv \mathrm{Unif}[0,1]$, and **G** is *symmetric*, in a first-price auction. (1) **Size Effect.** The overall effect of increasing the number of bidders is to intensify competition. Bidders do not reduce their bids as more participants join the auction. Consequently, the expected revenue strictly increases, regardless of the structure of the familiarity network. (2) **Skewness effect.** The impact of skewness in the familiarity network is nonmonotonic. As the familiarity network becomes more unbalanced, expected revenue initially decreases, then increases. More clusters generally lead to higher expected revenue, controlling for the evenness of the familiarity network.

Chapter 4

Discussion and perspectives

I analyze a semi-perfect information auction model in the presence of a familiarity network, featuring an information asymmetry, zero reservation price, ex ante learning, and arbitrary network structures. The auction format considered is either first-price or second-price. I establish the existence of equilibria (pure strategy BNEs) in both formats. This analysis extends existing auction models by incorporating information asymmetry through social networks, providing new insights into how bidding strategies evolve in the presence of local knowledge, and how this, in turn, impacts auctions efficiency, expected revenue and bidder surplus.

The equilibria identified in this model differ across the auction formats. In the second-price auction, truthful-telling is a weakly undominated strategy, ensuring equilibrium for any familiarity network structure. In contrast, in the first-price auction, equilibria are bounded within an interval: the front-runner bids at least as much as the second-highest valuation and at most their own valuation. Additionally, in both auction formats, the equilibria are not unique.

The equilibria implies that in a second-price auction, the allocation is efficient, generating the highest possible expected revenue and remaining robust to any familiarity network structure. However, in a first-price auction, the equilibrium allocation is inefficient with positive probability, leading to lower expected revenue compared to the second-price auction. Nonetheless, in terms of consumer surplus, bidders receive greater surplus in the first-price auction than in the second-price auction. Moreover, the revenue in a first-price auction exhibits non-monotonic behavior as the bidders' familiarity network becomes denser. Expected revenue initially decreases when bidders acquire partial information about some rivals, but rises again as they gain complete information about all competitors.

My results provide three key contributions to the literature: First, I extend the model of Kim and Che 2004 to accommodate more general familiarity networks, allowing bidders to process local knowledge. Specifically, I identify BNEs in a first-price auction under mild conditions of familiarity structures, where inefficiency occurs with positive probability, leading to revenue loss. Second, I show that the Revenue Equivalence Theorem, as presented in Riley and Samuelson 1981 and Ivanova-Stenzel and Salmon 2008, generally not hold in my model. This is due to inefficiency, where the highest-type bidder does not always win, violating one condition of the theorem. Third, I further examine the role of familiarity networks and demonstrate both the size and skewness effects. Analytically, increasing the number of bidders intensifies competition, consistent with much of the prevailing auction theory literature (Riley and Samuelson 1981). Simulation results show that skewness in the network's structure leads to nonmonotonic revenue outcomes, consistent with findings by Kim and Che 2004 and Kim 2016. As the familiarity network becomes more unbalanced, expected revenue initially decreases but then increases. Overall, having more clusters tends to result in higher expected revenue, controlling for the evenness of the familiarity network.

It may seem natural to prefer a second-price auction when designing mechanisms that prioritize both efficiency and revenue. After all, second-price auctions generate higher revenue for the auctioneer and lead to efficient outcomes. However, my findings highlight an important equity-efficiency tradeoff. While the second-price auction maximizes revenue and allocates the goods efficiently, bidders tend to obtain a greater surplus in a first-price auction under mild assumptions. Thus, from the perspective of bidder welfare, a first-price auction may be a better fit.

In the context of the EU ETS (Hepburn et al. 2006), if a second-price auction is been implemented, it would allocate emissions allowances to the firms that value them the most, typically those with the highest marginal abatement cost, resulting in an efficient allocation. Since this format maximizes revenue, it could also lead to more effective emissions reductions and greater compliance. Conversely, a first-price auction could allow companies to secure allowances at lower prices by underbidding, maximizing their surplus. While this might lead to inefficiencies—where firms that don't need as many permits win—these firms could use the cost savings to invest in green technologies or other compliance strategies, generating long-term benefits despite short-term inefficiency (Fazekas 2008). To make the long story short, the most suitable mechanism remains debatable, and my findings pave the way for further analysis.

Indeed, this model serves as a foundational step in introducing novel concepts to the literature by integrating familiarity networks into auction theory. I conclude the paper by outlining several potential directions for future research:

Bayesian Correlated Equilibrium. If bidders' valuations are correlated but private, the revenue and efficiency rankings established in this paper still hold. Inefficient allocation arises in any equilibrium of a first-price auction, and as in the current model, this inefficiency results in poor revenue performance compared to the second-price auction. However, my model is not immediately generalizable to the affiliated or interdependent valuations case. In such scenarios, correlated knowledge will render allocations inefficient, even in a second-price auction. While the linkage effect will still favor the second-price auction, the precise comparison between the two formats remains unclear. A new refinement of equilibrium in such cases is the Bayesian correlated equilibrium (Bergemann and Morris 2016; Bichler, Fichtl, and Oberlechner 2023), which allows players to observe the given information structure and additional signals. This refinement might provide a more nuanced framework for analyzing auctions with interdependent valuations.

Signals. Another avenue for extension involves relaxing the dichotomous nature of a bidder's knowledge by incorporating finite signals rather than relying solely on perfect valuations. In the current setting, each bidder operates under semi-perfect information, either possessing complete knowledge of others' valuations or having no information at all. These imperfect signals would influence bidders' strategies, potentially disrupting efficient allocation in the first-price auction. In contrast, the weak dominance argument indicates that these signals would have no effect on bidding behavior in the second-price auction. Consequently, generalizing the model to account for imperfect knowledge of rivals' types requires further research.

Costly Learning. An additional aspect that merits extension is the introduction of a costly learning process, which may extend the familiarity network as well to a dynamic setting. In practice, knowledge of other bidders' valuations may not be readily accessible and often requires a costly learning process (Kim and Koh 2020), such as investing time and effort to engage in conversations and better understand the strategies and valuations of competitors. Broader one's social network allows a bidder to refine their assessments of competitors' strategies and adjust their own strategy accordingly. While this introduces some modeling complexities—since different types of bidders have varying incentives to learn about their rivals—certain types of bidders are likely to engage in learning in equilibrium if the cost is sufficiently low. However, the inclusion of learning costs increases the overall social cost, further reinforcing the inefficiency of first-price auctions and potentially exacerbating their revenue underperformance compared to second-price auctions.

Dynamic. The final perspective considers the dynamics of repeated auctions, such as those occurring during summer weekend country auctions. In this context, bidders would gradually update their information about other participants based on historical interactions (consider Bayesian learning), and new entrants may also join the auction. These factors would alter the static equilibrium and align more closely with the legislative bargaining literature (e.g., Anesi and Seidmann 2015; Anesi and Duggan 2018). In this framework, it is essential to consider stationary Markov perfect equilibria (Maskin and Tirole 1992) along with appropriate refinements (e.g., simple solutions in Anesi and Seidmann 2015) as discussed in the bargaining literature.

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Appendices

Appendix A

NTS. In a second-price auction, "truthful-telling" (bidding one's own valuation) is a weakly undominated strategy, i.e., $b_i(v^{G_i}) = v_i$ for all $i \in N$. The auction allocation is efficient, and the expected revenue is $E[r^{2nd}] = n(n-1)\int_{supp(\mathrm{F})}(\mathrm{F}^{n-2} - \mathrm{F}^{n-1}) f \cdot v dv$.

Proof. The proof is following the standard IPV model of second-price auctions, with an additional focus on how information asymmetry may influence the outcome. For any $i \in N$, her familiarity set is G_i. Suppose i's valuation is v_i. There are two cases for her familiarity set:

(1) If $|G_i| = n$, from i's perspective there is no randomness, $v^{G_i} = (v_1, v_2, \cdots, v_n)$. And i knows that $b_j = v_j$ for all $j \neq i$ in equilibrium.

- If $\max_{j \in N} v_j = v_i$, let $\bar{v}_{-i} := \max_{j \neq i} v_j$ be the highest valuation among the other bidders. If $\bar{v}_{-i} = v_i$, bidder i bidding either v_i or v_{-i} results in a payoff of 0, and bidder i cannot achieve a positive payoff. If $\bar{v}_{-i} < v_i$, by bidding v_i, bidder i wins the auction, pays $\bar{v}_{-i}$ and receives a positive payoff of $v_i - \bar{v}_{-i}$.

- If $\max_{j \in N} v_j > v_i$, then by bidding v_i, bidder i loses the auction and receives a payoff of 0. Any bid that wins the auction would need to no lower than $\max_{j \in N} v_j$, requiring bidder i to overbid and pay more than v_i, resulting in a non-positive payoff.

(2) If $|G_i| < n$. Let $\bar{b}_{-i}$ denote the highest bid of the other bidders $j \neq i$ i.e., $\bar{b}_{-i} := \max_{j \neq i} b_j$ (from i's perspective this is a random variable since i do not know every player's value).

- If $\bar{b}_{-i} < v_i$, any bid b_i greater than $\bar{b}_{-i}$ will win the auction and the bidder will pay $\bar{b}_{-i}$, resulting in a positive payoff of $v_i - \bar{b}_{-i} > 0$. This includes bidding v_i, which ensures winning and receiving the same positive payoff.
- If $\bar{b}_{-i} = v_i$, then:
  - bidding $b_i = v_i$ will give bidder i a payoff of 0, either winning or losing depending on the tie-breaking allocation.
  - bidding b_i more than v_i guarantees winning the auction, but will result in a negative payoff since the payment exceeds the valuation.
  - bidding b_i less than v_i leads to losing the auction and a payoff of 0.
- If $\bar{b}_{-i} > v_i$, the best payoff bidder i can achieve is 0. Winning would require a bidding $b_i > \bar{b}_{-i}$, but since $v_i - b_i < v_i - \bar{b}_{-i} < 0$, it would lead to a negative payoff. Hence, it is better to lose the auction and receive the outside option of 0, which can be ensured by bidding v_i.

Since bidding v_i is never worse and sometimes (strictly) better than bidding b_i, it follows that bidding v_i weakly dominates bidding b_i, and this holds for all $v_i \neq b_i$, bidding v_i is a weakly dominant strategy. Therefore, it forms a BNE in the second-price auction.

One immediate observation from the above is that the equilibrium is not necessarily unique. For example, one can easily construct other weakly dominated strategies by modifying the losing-auction scenario. Since this is a BNE, I do not account for refinements like trembling-hand equilibrium. For example, an alternative weakly dominated equilibrium, other than truthful-telling, is as follows: if a bidder is not a front-runner, she submits the second-highest bid among her familiarity set; if she is a front-runner, she bids truthfully.

$$b_i(v^{G_i}) = \begin{cases} v_i & if\ i \in \underset{v_j \in G_i}{\text{argmax}}\ \mathbf{v} \\ \underset{v_k \in G_i \backslash \{\underset{v_j \in G_i}{\max}\ \mathbf{v}\}}{\max} \mathbf{v} & otherwise \end{cases} \tag{4.1}$$

Furthermore, since $b_i \equiv v_i$ for all $i \in N$, it is immediate that the auction is efficient. The expected revenue is solely determined by the second-highest value, which is drawn from the common distribution F irrespective to any structure of the familiarity network. Specifically, the revenue corresponds to the second-order statistic of the independently and identically distributed (i.i.d.) values drawn from F.

For a valuation V, the k-th order statistic $V_{(k)}$, where $k = 1, 2, \cdots, n$ is defined as the k-th smallest value in the ordered sample of valuations. Formally, the k-th order statistic is given by: $\mathrm{F}_{V_{(k)}}(v) := Pr(V_{(k)} \leq v) = \sum_{\ell=k}^{n} \binom{n}{\ell} \mathrm{F}(v)^{\ell}(1 - \mathrm{F}(v))^{n-\ell}$. Then for $k = 2$, $\mathrm{F}_{V_{(2)}}(v) = n\mathrm{F}^{n-1} - (n-1)\mathrm{F}^{n}$.

The expected value of the revenue in the second-price auction $E[r^{2nd}]$ is given by the following integral: $E[r^{2nd}] := \int_{supp(\mathrm{F})} v f_{V_{(2)}}(v) dv$. The expression for the expected revenue is thus:

$$\begin{aligned} E[r^{2nd}] &= \int_{supp(\mathrm{F})} v f_{V_{(2)}}(v) dv \\ &= \int_{supp(\mathrm{F})} v[n(n-1)\mathrm{F}^{n-2} f - n(n-1)\mathrm{F}^{n-1} f] dv \\ &= n(n-1) \int_{supp(\mathrm{F})} (\mathrm{F}^{n-2} - \mathrm{F}^{n-1}) f \cdot v dv \end{aligned} \tag{4.2}$$

This completes the proof of the expected revenue for the second-price auction.

Appendix B

NTS. The first-price auction possesses a pure strategy equilibrium in which the bidders employ non-decreasing bidding functions.

Proof. Reny 1999; Ewerhart and Reny 2022 have demonstrated the existence of a pure strategy equilibrium in nondecreasing bid functions for asymmetric first-price auctions with risk neutral bidders and multi-dimensional signals in the independent private values case. My remaining task is to establish that the model in question fits within their general setting.

To this end, I first aim to demonstrate that the auction admits a mixed strategy NE in which bidders adopt nondecreasing bidding strategies. Moreover, these non-decreasing bidding functions lead to a pure strategy equilibrium, as established in the appendix of Reny 1999; Ewerhart and Reny 2022.

To proceed, I will first define the relevant terms and notations as in Reny 1999. Here, I tailor their notation to align with my model:

Definition B.1. A game **A** $:= (B_i, u_i)_{i=1}^n$ is a compact, Hausdorf game if strategy set B_i is a compact Hausdorf space, and payoff u_i is bounded and measurable.

Definition B.2. Denote the mixed extension of **A** as $\bar{\mathbf{A}} := (M_i, u_i)_{i=1}^n$, where M_i denotes the set of (regular, countably additive) probability measures on the Borel subsets of B_i. u_i is extended to **M** $\equiv \times_{i=1}^N M_i$ by defining $u_i := \int_{\mathbf{B}} u_i(b) d\mu$ for all $\mu \in$ **M**.

Definition B.3. A game **A** $:= (B_i, u_i)_{i=1}^n$ is *better-reply secure* if whenever (b^*, u^*) is in the closure of the graph of its vector payoff function and b^* is not an equilibrium, some player i secure a payoff strictly above u^* at b^* i.e., there exists $\bar{b}_i \in B_i$, such that $u_i(\bar{b}_i, b'_{-i}) \geq u^*$ for all b'_{-i} in some open neighborhood of b^*_{-i}.

Corollary 5.2 in Reny 1999. Suppose that **A** $= (B_i, u_i)_{i=1}^n$ is a compact, Hausdorf game. Then **A** possesses a mixed strategy NE if its mixed extension, $\bar{\mathbf{A}}$, is *better-reply secure*.

Similar to any fixed-point theorem, the remaining steps to ensure the existence of an equilibrium are to show the following (1) B_i is a compact Hausdorf space; (2) u_i is bounded and measurable; and (3) $\bar{\mathbf{A}}$ is *better-reply secure*.

1. Followed by Observation 2, I restrict bidding strategy to be within $[0, \bar{v}]$. B_i is thus defined over a closed interval of the real line, which is compact (by Heine-Borel theorem). Furthermore, B_i is a Hausdorff space because $\mathbb{R}$ is Hausdorff when endowed with the Euclidean topology. In metric spaces like $\mathbb{R}$, distinct points can always be separated by disjoint open sets, thus ensuring the Hausdorff property.

2. The payoff function for bidder i is given by $u_i = (v_i - b_i)Pr(b_i \geq b_{-i})$. By construction, $0 \leq u_i \leq v_i - b_i \leq \bar{v}$, thus u_i is bounded. Additionally, since u_i depends on continuous, measurable functions of the bids, it is measurable.

3. Suppose that m is not an equilibrium, and let (m^*, u^*) be an element of mixed extension's vector payoff function. By definition, $\lim u(m^t) = u^*$ for some sequence of mixed strategies $\{m^t\}$ converging to m^*. Given m^*_{-i} and any small $\varepsilon > 0$, bidder i can achieve a payoff within ε of her supremum by employing a strictly increasing bid function $b_i(\cdot)^\varepsilon$. Moreover, her payoff is continuous in the other bidders' mixed strategies at $(b_i^\varepsilon, m^*_{-i})$.

   If ties occur with zero probability under m^*, then payoffs are continuous at the limit, i.e., $u^* = u(m^*)$. Since m^* is not an equilibrium, there exists some bidder i and a small enough $\varepsilon > 0$ such that $u_i(b_i^\varepsilon, m^*_{-i}) > u(m^*) = u^*$. The continuity of $u_i(b_i^\varepsilon, \cdot)$ at m^*_{-i} implies that bidder i can secure a payoff strictly above u^* at m^*.

   If ties occur with positive probability under m^*, then all bidders who tie with positive probability would strictly prefer to win with probability one. Along the sequence $\{m^t\}$, at least one of those bidders who ties in the limit loses with a probability bounded away from zero. This bidder i can then achieve a strictly higher payoff than the limiting payoff $u_i^* = \lim u_i(m^t)$ by employing b_i^ε for a sufficiently small ε. In other words, $u_i(b_i^\varepsilon, m_{-i}^*)$ is strictly greater than $u_i(m^t)$ for large t. Hence, again by the continuity of $u_i(b_i^\varepsilon, \cdot)$ at m_{-i}^*, we get $u_i(b_i^\varepsilon, m_{-i}^*) > u^*$, implying that bidder i can secure a strictly higher payoff above u^* at m^*. This establishes better-reply security.

Consequently, $\mathbf{A}$ satisfies the conditions required for the existence of a mixed strategy NE, the framework is in the realm of Reny 1999; Ewerhart and Reny 2022. This implies the existence of a pure strategy equilibrium (to Theorem 3.1 and Appendix in Reny 1999) in which the bidders employ non-decreasing bidding functions. Thus, I conclude that the auction possesses a pure strategy equilibrium.

One immediate observation is that the existence of an equilibrium does not require any specific structure of the familiarity network (it can be viewed as a multi-dimensional signal scenario).

Appendix C

NTS. Proposition 2. In first-price auction, (1) In equilibrium, a front-runner bids at least a second-highest valuation in $G_\cdot$ and wins against all bidders in $G_\cdot$. (2) The equilibrium allocation is inefficient with positive probability when there are at least two bidders $k, l \in N$ whose familiarity sets satisfies (i) $1 < |G_{k,l}| < n$ and (ii) $|G_k \setminus G_l| \geq 1$, $|G_l \setminus G_k| \geq 2$. (3) The equilibrium is not unique.

Proof. This statement is closely related to Proposition 1 in Kim and Che 2004. While the procedure is similar, I need to demonstrate that it holds in my model, which includes asymmetric cases.

(1) For any front-runner $i \in N$, denote the second-highest valuation bidder (if one exists, which requires $|G_i| > 1$) in G_i as j, with valuation v_j, The major difference to the proof in Kim and Che 2004 is that here j can also be a front-runner in G_j. Let $b_i(v^{G_i})$ represent an arbitrary equilibrium bid by bidder i, given v^{G_i}, and let $\underline{b}_i$ be the infimum of her equilibrium bids. The goal is to show that $\underline{b}_i \geq v_j$.

First, note that bidder i must receive a strictly positive expected payoff in equilibrium. For instance, she could bid $\tilde{b_i} = v_j + \varepsilon$ where $\varepsilon > 0$ is small enough, and win the auction with positive probability. Given Observation 3, i would outbid all other bidders in G_i.

Now, assume for the sake of contradiction that $\underline{b}_i < v_j$.

- Case 1. If j is not a front-runner in G_j, denote the front-runner in G_j as k. Similarly, let $b_k(v^{G_j})$ represent an arbitrary equilibrium bid by bidder $k \in G_j$, given v^{G_j}, and let $\underline{b}_k$ be the infimum of her equilibrium bids. By the similar logic, bidder j earns a strictly positive payoff in equilibrium if $\tilde{b_j} \geq \underline{b}_k$, where bidding $\tilde{b_j} = \underline{b}_i + \epsilon$ with $\epsilon > 0$, or a zero payoff otherwise. In latter case, a weakly undominated strategy for j is to bid $b_j(v^{G_j}) = v_j$, then $\underline{b}_i < v_j$ would make i lose the auction and get zero payoff in equilibrium, leading to a contradiction because i should have a strict positive payoff. In the formal case, for both bidders to earn positive payoffs, they must be tied in their bidding. This implies that their infimum bids must coincide, and each bidder must place a mass point at this infimum. However, bidder i can always raise her mass point slightly, increasing her probability of winning (by outbidding j) and improving her payoff conditional on winning only slightly. This leads to a contradiction. Thus, it follows that $b_i(v^{G_i}) \geq \underline{b}_i \geq v_j$.

- Case 2. If j is also a front-runner in G_j, by similar logic as front-runner i in G_i, bidder j must also earn a strictly positive payoff in equilibrium. The contradiction is been shown by the same argument as above.

In both cases, the assumption that $\underline{b}_i < v_j$ leads to contradictions. Therefore, it must be that $b_i(v^{G_i}) \geq \underline{b}_i \geq v_j$. By Observation 3, bidder i will outbid all other bidders within G_i.

(2) Equipped with (1), for every bidder who is not a front-runner, they lose the auction with probability 1. A weakly dominated strategy for such a bidder is to bid truthfully i.e., bid exactly v_i and and receive a zero payoff. Now, given (i), (ii), there exists $i, j, k \in N$ s.t., $k \in G_i \setminus G_j$, $i \notin G_j$, $j \notin G_i$. What I want to show is within the category of $v_j > v_i > v_k > \max_{l \in N \setminus \{i,j,k\}} v_m$, which happens with positive probability, as F is continuous. In this scenario, i and j are front-runners. What remains to be shown is that $b_i > b_j$, leading to inefficiency, as bidder j, who holds the highest valuation, loses the auction while i wins.

Step 1. $b_i \geq v_k$.

This follows directly from (1), as i is the front-runner and $k \in G_i$.

Step 2. $b_j < v_j - \frac{1}{|N-G_j|\bar{f}}$, where $\bar{f} := \max_{b \in (\underline{v}, v_j]} \frac{f(b)}{\mathrm{F}(b)}$.

This follows from solving the optimization problem for bidder j. Since $\mathrm{F}(b) > 0$ for all $b \in (\underline{v}, v_j]$, $\frac{f(b)}{\mathrm{F}(b)}$ is finite, which implies the existence of $\bar{f}$. Given j is also a front runner, bidding $\underline{v}$ will never allow her to win the auction. Furthermore, by (1) and given the support of her bidding strategy lies within$(\underline{v}, v_j]$:

$$
\begin{aligned}
&\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_{-j}) \\
\overset{(a)}{\Longrightarrow} &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_{l \in G_j}) Pr(b_j \geq b_{l \in N \setminus G_j}) \\
\overset{(b)}{\Longrightarrow} &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_{l \in N \setminus G_j}) \\
\overset{(c)}{\Longrightarrow} &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_l)^{|N-G_j|} \\
\Longrightarrow &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) \mathrm{F}(b_j)^{|N-G_j|}
\end{aligned}
\tag{4.3}
$$

(a) The probability that bidder j overbids everyone is equivalent to overbid both bidders in G_j and not in G_j. (b) j is assumed to be a front runner. (c) Valuations is drawn i.i.d. and, $b_l \sim \mathrm{F}$.

To solve (4.3), take the first-order condition (F.O.C.). A corner solution is easily ruled out: either bidding too little (close to infimum), which would result in winning with zero probability, or bidding too high, e.g., truthful telling, would lead to receiving zero revenue with probability 1.

$$
\begin{aligned}
(v_j - b_j)\frac{f(b_j)}{\mathrm{F}(b_j)} &= \frac{1}{|N - G_j|} \\
\overset{\bar{f} \geq \frac{f(b)}{\mathrm{F}(b)}}{\Longrightarrow} b_j &< v_j - \frac{1}{|N - G_j| \cdot \bar{f}}
\end{aligned}
\tag{4.4}
$$

Step 3. $b_i > b_j$, and this occurs with positive probability.

In addition to the condition $v_j > v_i > v_k > \max\limits_{l \in N \setminus \{i,j,k\}} v_m$, I further let $v_k > v_j - \frac{1}{|N-G_j|\bar{f}}$. Since each valuation is drawn from a continuous interval, this event happens with positive probability, i.e., $Pr(v_j > v_i > v_k > \max\limits_{l \in N \setminus \{i,j,k\}} v_m; v_k > v_j - \frac{1}{|N-G_j|\bar{f}}) > 0$, for $\{v_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} \mathrm{F}$.

This indeed leads to $b_i > b_j$ because, $v_j \overset{(a)}{>} v_i \overset{(b)}{\geq} b_i \overset{(c)}{\geq} v_k \overset{(d)}{>} v_j - \frac{1}{|N-G_j|\bar{f}} \overset{(e)}{>} b_j$, where (a) and (d) are by construction, (b) follows from Observation 3, (c) is derived from Step 1, and (e) comes from Step 2.

(3) By similar logic as in Appendix A, without further constraints, the equilibrium is not unique. For example, uniqueness would require condition (4.3) to attain a global maximum, which is difficult to satisfy in the general case. Further constraints could be imposed by specifying a particular common distribution function (Lebrun 2006), or by restricting the analysis to symmetric equilibria with linear bidding functions (Riley and Samuelson 1981).

The above completes the proof of part 2 of Proposition 2.

Appendix D

NTS. For a first-price auction, suppose $n = 4$, $\mathrm{F} \equiv \mathrm{Unif}[0,1]$, **G** $= \{\{1,2\},\{1,2\},\{3,4\}\,\{3,4\}\}$. For all $i \in N$, $b_i = \min\{v_i, \max\{\frac{2}{3}v_i, v_j\}\}$, $j \in G_i$ in the example is indeed an equilibrium.

Proof. This is the same example as in Example 1 of Kim and Che 2004 for proofreading and the baseline for the two extra asymmetric examples. While they present the proof more intuitively, I will demonstrate it from scratch. Consider bidder 1 and 2. Without loss of generality, assume $v_1 \geq v_2$. For bidder 2, her optimal strategy is $b_2^* = v_2$, given that bidder 1's strategy $b_1^* \geq v_2$ results in 0 utility for her. To justify this:

- For all $b_2 > v_2$, then her expected utility is $u_2 = (v_2 - b_2)Pr(b_2 \geq b_{-2}) \leq 0$, which would not result in a profitable deviation.
- For all $b_2 < v_2$, she will always bid less than bidder 1 and will never win, resulting in 0 utility.

For bidder 1, to overbid bidder 2, she needs to bid at least v_2. This ensures there exist some bid e.g., $\exists 0 \leq \varepsilon \leq v_1 - v_2$, $b_1 = v_2 + \varepsilon$, such that bidder 1's utility is non-negative and overbid bidder 2, as shown: $(v_1 - b_1)Pr(b_1 \geq b_{-1}) \geq (v_1 - v_2 - \varepsilon)Pr(b_1 \geq b_{-1}) \geq 0$.

Let's now examine bidder 1's optimal strategy. Assume that the expected bidding strategy for all individuals outside G_1 follows the same functional form: $Eb : [0,1] \to \mathbb{R}$, where $b_i = Eb(v_i)$ represents the bidding strategy of any bidder $i \notin G_1$. From bidder 1's perspective, all (homogeneous) bidders $i \notin G_1$ are engaging in a standard first-price sealed-bid auction.

Bidder i's optimization problem is to maximize her expected utility given by

$$\max_{b_i}(v_i - b_i)Pr(b_{-i} = Eb(v_{-i}) \leq b_i) \tag{4.5}$$

To solve this, take the F.O.C.,

$$Eb'(v_i) + (n-1)\frac{f(v_i)}{\mathrm{F}(v_i)}Eb(v_i) - v_i(n-1)\frac{f(v_i)}{\mathrm{F}(v_i)} = 0 \tag{4.6}$$

where

- n is number of players.
- F is the common knowledge of distribution of valuation.
- f is the corresponding probability density function (pdf).

This leads to the following solution for $Eb(v_i)$, the expected bid function:

$$Eb(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} \mathrm{F}^{n-1}(\tilde{v})d\tilde{v}}{\mathrm{F}^{n-1}(v_i)} \tag{4.7}$$

For the case where valuations are i.i.d. from $\mathrm{Unif}[0,1]$ i.e., $v \overset{i.i.d.}{=} \mathrm{Unif}[0,1]$, $Eb(v_i) = \frac{n-1}{n}v_i$ (Riley and Samuelson 1981).

Thus, given $v_1 \geq v_2$, bidder 1 faces the following maximization problem in a first-price auction setting:

$$\begin{aligned}
&\max_{b_1}(v_1 - b_1)Pr(b_1 \geq b_{-1}) \\
\Longrightarrow &\max_{b_1}(v_1 - b_1)Pr(b_1 \geq max\{b_2, b_3, b_4\}) \\
\Longrightarrow &\max_{b_1}(v_1 - b_1)Pr(b_1 \geq b_2, b_1 \geq b_3, b_1 \geq b_4) \\
\overset{b_i = Eb(v_i) for\ i=\{3,4\}, b_2 = v_2}{\Longrightarrow} &\max_{b_1}(v_1 - b_1)Pr(b_1 \geq v_2, b_1 \geq Eb(v_3), b_1 \geq Eb(v_4)) \\
\overset{i.i.d. of v}{\Longrightarrow} &\max_{b_1}(v_1 - b_1)Pr(v_2 \leq b_1)Pr(Eb(v_3) \leq b_1)Pr(Eb(v_4) \leq b_1) \\
\overset{Eb(v_i) = \frac{n-1}{n}v_i}{\Longrightarrow} &\max_{b_1}(v_1 - b_1)Pr(v_2 \leq b_1)Pr(\frac{3}{4}v_3 \leq b_1)Pr(\frac{3}{4}v_4 \leq b_1) \\
\overset{v_2 \leq b_1}{\Longrightarrow} &\max_{b_1}(v_1 - b_1) \cdot 1 \cdot Pr(v_3 \leq \frac{4}{3}b_1)Pr(v_4 \leq \frac{4}{3}b_1)
\end{aligned} \tag{4.8}$$

For $Pr(v_3 \leq \frac{4}{3}b_1)$, by $v \overset{i.i.d.}{=} \text{Unif}[0,1]$, $Pr(v_3 \leq \frac{4}{3}b_1) = \begin{cases} \frac{4}{3}b_1 & if \frac{4}{3} \leq 1 \\ 1 \quad if & \frac{4}{3} > 1 \end{cases} = \min\{\frac{4}{3}b_1, 1\}$.

The above leads to $\max\limits_{b_1}(v_1 - b_1)\min\{\frac{4}{3}b_1, 1\}^2$. Bidder 1 faces two possible situations,

1. If $\min\{\frac{4}{3}b_1, 1\} = \frac{4}{3}b_1 \Leftrightarrow b_1 \leq \frac{3}{4}$, $\max\limits_{b_1}(v_1 - b_1)(\frac{4}{3}b_1)^2$. By taking F.O.C., $b_1^* = \frac{2}{3}v_1$ if interior solution exists (corner solution if $\frac{2}{3}v_1 < v_2$, then F.O.C < 0, $b_1^* = v_2$). Combine together, $b_1^* = \max\{\frac{2}{3}v_1, v_2\}$.

2. If $\min\{\frac{4}{3}b_1, 1\} = 1 \Leftrightarrow b_1 \geq \frac{3}{4}$, In this scenario, bidder 1 believes she can win with probability 1. Hence, her utility is simply $\max\limits_{b_1}(v_1 - b_1)$, and the utility decreases with increasing b_1. $b_1^* = \begin{cases} \frac{3}{4} & if v_1 \geq \frac{3}{4} \\ 0 \quad if & v_1 < \frac{3}{4} \end{cases}$. the second scenario is bidder 1's corner solution such that she turns to her outside option to get 0 instead of a negative utility, but this cannot be true under $b_1 \geq \frac{3}{4}$. So finally, $b_1^* = \frac{3}{4}\ if\ v_1 \geq \frac{3}{4}$.

When $v_1 \geq \frac{3}{4}$, bidder 1 much choose between the above two scenarios: either ensure to get the item, or slightly decrease the bid to gain a maximum expected return. Compare the payoff between the two:

1. Payoff in case 1: $(v_1 - \frac{2}{3}v_1)(\frac{4}{3} \cdot \frac{2}{3}v_1)^2$ and payoff in case 2: $v_1 - \frac{3}{4}$ when $\frac{2}{3}v_1 \geq v_2$;

2. Payoff in case 1: $(v_1 - v_2)(\frac{4}{3} \cdot v_2)^2$ and payoff in case 2: $v_1 - \frac{3}{4}$ when $\frac{2}{3}v_1 < v_2$.

The first is always true by taking the difference: $(v_1 + \frac{9}{4})(v_1 - \frac{9}{8})^2 > 0$ for $\forall v_i \in [0, 1]$. Then the second is also true: $(v_1 - v_2)(\frac{4}{3} \cdot v_2)^2 > (v_1 - \frac{2}{3}v_1)(\frac{4}{3} \cdot \frac{2}{3}v_1)^2 > v_1 - \frac{3}{4}$ for a subset of v_i interval i.e., $\frac{2}{3}v_1 < v_2$. Ergo, I've shown that the second case ($b_1 = \frac{3}{4}$) will never be selected. The optimal strategy for bidder 1 is thus: $b_1^* = \max\{\frac{2}{3}v_1, v_2\}$ for any realization v_1.

Finally, recalling the assumption $v_1 \geq v_2$, and by symmetry, the equilibrium strategies for bidder 1 and bidder 2 (and similarly for bidder 3 and bidder 4) can be characterized as follows:

$$b_1^* = \begin{cases} \max\{\frac{2}{3}v_1, v_2\} & if v_1 \geq v_2 \\ v_1 & if v_1 < v_2 \end{cases}$$
$$\xrightarrow{b_1 - v_1 \geq 0} b_i = \min\{v_i, \max\{\frac{2}{3}v_i, v_j\}\},\ j \in G_i \tag{4.9}$$

which is indeed a BNE.

Appendix E

NTS. In any equilibrium of a first-price auction, the expected payoff from the first price auction has a lower bound:

$$u_i(\mathbf{v}^{G_i})] \geq \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds,\ v_i \in [\underline{v}, \bar{v}] \tag{4.10}$$

The equality holds if **G** is *symmetric* (degenerate to Lemma 1 in Kim and Che 2004).

Proof. Differ to the proof of Lemma 1 presented in Kim and Che 2004, it is important to discuss an additional Case 2 and Case 3 (ii), which pertains to ties and the presence of one front-runner within the familiarity structure of others. Cases 1 and 3 employ similar ideas as those found in Cases 1 and 2 of Kim and Che 2004 (can be omitted, but its better to show rigorously). For $\forall i \in N$ consider the following situations:

Case 1. If i is not a front-runner i.e., [1]$v_i \leq \max_{(2)} \mathbf{v}^{G_\cdot}$ or if there exists at least one familiarity set that contains i, but i does not have the highest bid in that set i.e.,

[1]Define $\max_{(2)} \mathbf{v}^{G_\cdot}$ as the second-highest value within $\mathbf{v}^{G_\cdot}$ for the sake of convenience, a formal definition can be found in Section 2.

$\exists j \neq i$, s.t., $i \in G_j, v_i \leq \max_{(2)} \mathbf{v}^{G_j}$. Define $\tilde{v} := \max\{\max_{(2)} \mathbf{v}^{G_i}, \max_{(2)} \mathbf{v}^{G_j}\}$, $i \in G_j$. One can observe that the two conditions mentioned above can actually be combined into a single statement: $i \in G_j, v_i < \max_{(2)} \mathbf{v}^{G_j}$, $j \in N$. Then $v_i \in [\underline{v}, \tilde{v}]$ would be the support of Case 1.

Given Proposition 2(1), it is clear that bidder i would never win the auction. This is because there is always a front-runner who can bid at least the second-highest bid, which will exceed any bid from the interval $[\underline{v}, v_i]$ i.e., $Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}}) \equiv 0$, $\forall s \in [\underline{v}, v_i]$. Consequently, $u_i(\mathbf{v}^{G_i}) = 0 = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds,\ \forall v_i \in [\underline{v}, \tilde{v}]$.

Case 2. If $v_i = \tilde{v}$, suppose for the sake of contradiction that bidder i earns a strictly positive payoff. Then, her infimum bid must be less than v_i (Observation 3) but weakly greater than the infimum bid of the second-highest bidder. In this case, the second-highest bidder can profitably deviate by outbidding i's infimum bid (moving from 0 to a strictly positive value). If both bidders' infimum bids coincide, then both must place mass points there, which creates a contradiction, as shown in the proofs in Proposition 2(1). This leads to the conclusion that bidder i cannot earn a strictly positive payoff in this case, thus, $u_i(\mathbf{v}^{G_i}) = 0 = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds,\ \forall v_i = \tilde{v}$

Case 3. If $v_i > \tilde{v}$. Let $p^{out}(b)$ denote the probability that bidder i outbids all bidders in G_j which does not contain i, this condition ensures that this probability does not depend on bidder i's valuation (as they cannot observe). By Proposition 2(1), bidder i will win with probability $p^{out}(b)$ by bidding $b \geq \tilde{v}$ because she'll outbid all players in $G_{\cdot}$ which contains i (Observation 3). Furthermore, by Proposition 2(1), bidder i never bids below $\max \mathbf{v}^{G_i}$. (i) if $\tilde{v} = \max_{(2)} \mathbf{v}^{G_i}$, it degenerates to the case in P. Milgrom and Segal 2002; Kim and Che 2004, which leads to $u_i(\mathbf{v}^{G_i}) = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds,\ \forall v_i \in [v_i, \max_{(2)} \mathbf{v}^{G_i}]$. (ii) if $\tilde{v} = \max_{(2)} \mathbf{v}^{G_j}$, $i \in G_j$, $j \in N \setminus \{i\}$, in other words, $\max_{(2)} \mathbf{v}^{G_j} \geq \max_{(2)} \mathbf{v}^{G_i}$. Bidder i's bid might fall in the interval $[\max_{(2)} \mathbf{v}^{G_i}, \max_{(2)} \mathbf{v}^{G_j}]$. By Proposition 2(1), $u_i(\mathbf{v}^{G_i}) = \max\limits_{b \geq \max_{(2)} \mathbf{v}^{G_i}} Pr(b \geq b_{-i})(v_i - b) \geq \max\limits_{b \geq \tilde{v}} p^{out}(b)(v_i - b)$. The inequality holds because bidder i optimizes over a narrower interval. Similarly, using (i), we have: $u_i(\mathbf{v}^{G_i}) \geq \int_{\underline{v}}^{\max_{(2)} \mathbf{v}^{G_j}} Ep_i(s, \mathbf{v}^{G_i\setminus\{s\}})ds + \int_{\max_{(2)} \mathbf{v}^{G_j}}^{v_i} Ep_i(s, \mathbf{v}^{G_i\setminus\{s\}})ds = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i\setminus\{s\}})ds$.

When **G** is *symmetirc*, (ii) in Case 3 occurs with zero probability. Consequently, $u_i(\mathbf{v}^{G_i})] \geq \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i\setminus\{s\}})ds,\ v_i \in [\underline{v}, \bar{v}]$ holds as an equality (Note that in all other integration parts, the equality always holds).

Combining the above three cases completes the proof of Appendix E.

Appendix F

NTS. Assume $\mathrm{F} \equiv \mathrm{Unif}[0,1]$, and **G** is *symmetric*. Given the equilibrium strategy (3.4), the bidding behavior of any existing player will not decrease when additional players join the auction. In other words, $b_i' \geq b_i(v^{G_i})$, where b_i' represents the bidding strategy for player i represents the bidding strategy for player i when extra players (≥ 1) are added to the existing **G**.

Proof. It suffices to show that the argument hold in one extra player case. Suppose a new player $m \notin N$ joins the auction for the set N. For any existing player $i \in N$, their bidding strategy follows (3.4) $b_i(v^{G_i}) = \min\left\{v_i, \max\{\frac{n-|G_i|}{n-|G_i|+1}v_i, \max\limits_{j\in G_i\setminus\{i\}} v_j\}\right\}$. Now, consider two possible scenarios based on whether player i is familiar with player m:

Case 1. Suppose $m \in G_i$. Let us denote the updated bidding strategy for player i as follows: $b_i(v'^{G_i'}) = \min\left\{v_i, \max\{\frac{(n+1)-|G_i'|}{(n+1)-|G_i'|+1}v_i, \max\limits_{j\in G_i'\setminus\{i\}} v_j\}\right\}$, where $G_i' \equiv G_i \cup \{m\}$. In this case, observe that $|G_i'| = |G_i| + 1$, thus $\max\limits_{j\in G_i'\setminus\{i\}} v_j = \max\{\max\limits_{j\in G_i\setminus\{i\}} v_j, v_m\} \geq \max\limits_{j\in G_i\setminus\{i\}} v_j$. Combining the above, $b_i(v'^{G_i'}) \geq b_i(v^{G_i})$.

Case 2. Suppose $m \notin G_i$. Now the updated bidding strategy for player i would be: $b_i(v'^{G_i}) = \min\left\{v_i, \max\{\frac{(n+1)-|G_i|}{(n+1)-|G_i|+1}v_i, \max\limits_{j\in G_i\setminus\{i\}} v_j\}\right\}$. Since $\frac{(n+1)-|G_i|}{(n+1)-|G_i|+1} = \frac{1}{1+\frac{1}{(n+1)-|G_i|}}$ is strictly increasing in n, and all other components remain unchanged, we have $b_i(v'^{G_i'}) \geq b_i(v^{G_i})$.

For cases involving more than one player, assume they join the auction sequentially, with all existing players adapting their strategies according to (3.4) as each new player enters. This completes the proof presented in Appendix F.

Appendix G

NTS. Corollary 1.

Proof. Following a similar argument as in Appendix F, it suffices to demonstrate that the addition of one extra player strictly increases the expected revenue. Specifically, need to show that $E'[r^{1st}] > E[r^{1st}]$, where $E'[r^{1st}]$ is the expected revenue with $n+1$ players. By definition, before player m joins, the expected revenue is given by $E[r^{1st}] = E\max\{b_1, b_2, \cdots, b_n\}$, where b_i satisfies (3.4) for $i \in N$. Furthermore, player $n+1$ has a valuation drawn from the support $[0,1]$. This implies that the bidding support remains $[0,1]$ by Observation 3. The remainder of the proof proceeds as follows, with the following steps needing to be shown:

Step 1. $E'[r^{1st}] \geq E\max\{b_1, b_2, \cdots, b_{n+1}\}$.

$$\begin{aligned} E'[r^{1st}] &:= E\max\{b'_1, b'_2, \cdots, b'_n, b_{n+1}\} \\ &= E\max\left\{\max\{b'_1, b'_2, \cdots, b'_n\}, b_{n+1}\right\} \\ &\overset{Observation\ 5}{\geq} E\max\left\{\max\{b_1, b_2, \cdots, b_n\}, b_{n+1}\right\} \\ &\geq E\max\left\{b_1, b_2, \cdots, b_n, b_{n+1}\right\} \end{aligned} \tag{4.11}$$

where b'_i is the bidding strategy for each $i \in N$ after the $n+1$-th player joins, as described in Appendix F.

Step 2. $E\max\{b_1, b_2, \cdots, b_{n+1}\} > E\max\{b_1, b_2, \cdots, b_n\}$.

The above is equivalent to showing that

$$E\max\{\max\{b_1, b_2, \cdots, b_n\}, b_{n+1}\} > E\max\{b_1, b_2, \cdots, b_n\} \tag{4.12}$$

Since $\max\{\max\{b_1, b_2, \cdots, b_n\}, b_{n+1}\} = \frac{\max\{b_1,b_2,\cdots,b_n\}+b_{n+1}+|\max\{b_1,b_2,\cdots,b_n\}-b_{n+1}|}{2}$ by the definition of the maximum function, what needs to be shown is further equivalent to:

$$\begin{aligned} &E\max\{b_1, b_2, \cdots, b_{n+1}\} > E\max\{b_1, b_2, \cdots, b_n\} \\ \Longleftrightarrow &E\left[|\max\{b_1, b_2, \cdots, b_n\} - b_{n+1}| - (\max\{b_1, b_2, \cdots, b_n\} - b_{n+1})\right] > 0 \end{aligned} \tag{4.13}$$

By Observation 3 and Appendix C(1), for there to be a positive expected payoff in equilibrium for front-runner, it must hold that $\max\{b_1, b_2, \cdots, b_n\} < \max_{i \in N} v_i$. Additionally, according to equation (3.4),

$$\begin{aligned} b_{n+1}(v^{G_{n+1}}) &= \min\{v_{n+1}, \max\{\frac{(n+1) - |G_{n+1}|}{(n+1) - |G_{n+1}| + 1} v_{n+1}, \max_{j \in G_{n+1} \setminus \{n+1\}} v_j\}\} \\ &\geq \frac{(n+1) - |G_{n+1}|}{(n+1) - |G_{n+1}| + 1} v_{n+1} \\ &\overset{(a)}{\geq} \frac{1}{2} v_{n+1} \end{aligned} \tag{4.14}$$

(a) This is because $\frac{(n+1)-|G_{n+1}|}{(n+1)-|G_{n+1}|+1}$ is a decreasing function of $|G_{n+1}|$, and $\max|G_{n+1}| = n + 1$.

Furthermore, since $\frac{1}{2} v_{n+1} > \max_{i \in N} v_i$ happens with positive probability (recall that the valuations are i.i.d. draws from the distribution F), it follows that

$$b_{n+1} \geq \frac{1}{2} v_{n+1} > \max_{i \in N} v_i > \max\{b_1, b_2, \cdots, b_n\} \tag{4.15}$$

This inequality also holds with positive probability. Finally, equation (4.13) can be derived from equation (4.15).

Combining Step 1 and Step 2, we obtain (4.16), which completes the proof.

$$E'[r^{1st}] \geq E\max\{b_1, b_2, \cdots, b_{n+1}\} > E\max\{b_1, b_2, \cdots, b_n\} =: E[r^{1st}] \tag{4.16}$$

Simulation code

```
# for examples
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import random

def b_1(v_1, v_2):
    x = 0
    if v_2 >= v_1:
        x = v_1
    elif v_2 < 2/3 * v_1:
        x = 2/3 * v_1
    else:
        x = v_2
    return x

revenue_list = []
# for n = 4, two groups 2-2 with perfect information
for _ in range(1000000):
    [val_1, val_2, val_3, val_4]= [random.uniform(0, 1) for _ in range
        (4)]
    bid_1 = b_1(val_1, val_2)
```

```
    bid_2 = b_1(val_2, val_1)
    bid_3 = b_1(val_3, val_4)
    bid_4 = b_1(val_4, val_3)

    revenue_list.append(max(bid_1, bid_2, bid_3, bid_4))

# get 0.584, alien with Kim 2006
pd.DataFrame(revenue_list).mean()

revenue_list = []
# for n = 4, case 1
for _ in range(1000000):
    [val_1, val_2, val_3, val_4]= [random.uniform(0, 1) for _ in range
        (4)]
    bid_1 = b_1(val_1, val_2)
    bid_2 = b_1(val_2, val_1)
    bid_3 = b_1(val_3, val_4)
    bid_4 = 3/4 * val_4
    revenue_list.append(max(bid_1, bid_2, bid_3, bid_4))

# get 0.597
pd.DataFrame(revenue_list).mean()

revenue_list = []
# for n = 4, case 2
for _ in range(1000000):
    [val_1, val_2, val_3, val_4]= [random.uniform(0, 1) for _ in range
        (4)]
    bid_1 = b_1(val_1, val_2)
    bid_2 = b_1(val_2, val_1)
    bid_3 = b_1(val_3, val_4)
    bid_4 = [max(val_1, val_2, val_3) if max(val_1, val_2, val_3,
        val_4) == val_4 else val_4][0]
```

```
    revenue_list.append(max(bid_1, bid_2, bid_3, bid_4))

# get 0.602
pd.DataFrame(revenue_list).mean()
```

```
# for part 3.4.3
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import random
from itertools import combinations_with_replacement
import statistics

# find all combinations of ‘k‘ numbers >= 1 that add up to ‘n‘.
def find_combinations(n, clusters):
    results = []

    # subtract 1 from each number to use 0-based combinations with replacement
    for combo in combinations_with_replacement(range(1, n - clusters + 2), clusters):
        if sum(combo) == n:
            results.append(tuple(sorted(combo)))  # avoid duplicate permutations

    # remove duplicates by converting to set and then back to list
    unique_results = list(set(results))

    return sorted(unique_results)

def highest_except_i(lst, i):
    # remove the i’th element from the list
```

```
    lst_except_i = lst[:i] + lst[i+1:]
    # find the maximum value in the modified list
    try:
        max_value = max(lst_except_i)
    except ValueError:
        max_value = 0
    return max_value


# equilibrium bidding strategy
def bid4i(value_list, n, i):
    return min(value_list[i], max((n- len(value_list))/(n- len(
        value_list) + 1)* value_list[i], highest_except_i(value_list,
        i)))

# get exp revenue and s.e.
def get_df(n, clusters):
    data = {
        "Expected␣Revenue": [],
        "s.e.": []
    }
    df = pd.DataFrame(data)

    # get all combinations given n and num of clusters
    combinations = find_combinations(n, clusters)

    # iterate
    for bidders in combinations:
        # standear deviaition of such structure
        sd = statistics.stdev(bidders)
        # for expect revenue
        revenue_list = []
        # for each combination do the simulation
```

```
        for _ in range(20000):
            # get v^{G_i} for each group, e.g., valuation_G = [[0.7, 0.2], [0,3, 0,5, 0,4], [0.8]]
            valuation_G = [[random.uniform(0, 1) for _ in range(structure)] for structure in bidders]
            bid_list = []
            highest_bid = 0
            for each in valuation_G:
                bid_list += [bid4i(each, n, i) for i in range(len(each))]
            revenue_list.append(max(bid_list))
        # write into df
        # df = df.append({"Expected Revenue": statistics.mean(revenue_list), "s.e.": sd}, ignore_index=True)
        df = pd.concat([df, pd.DataFrame([{"Expected Revenue": statistics.mean(revenue_list), "s.e.": sd}])], ignore_index=True)

        print("1 row added")
    return df

# the following is an demo for n=80, readers are free to alter with n=10, 20, 40 to generate whole results in Figure 3.2
# efficient exp revenue
n = 80

eff_revenue = (n-1)/(n+1)

# Plotting
sns.set(style="whitegrid")
plt.figure(figsize=(16, 8))

# Efficient exp revenue line
```

```
plt.axhline(y=eff_revenue, color="black", linestyle="--", linewidth=2, label="Efficient level")

# Scatter plots - plot "clusters=5" first to make it the bottom layer
sns.scatterplot(x="s.e.", y="Expected Revenue", s=40, data=df_80_5, color=sns.color_palette()[2], label="clusters=5")  # Bottom layer
sns.scatterplot(x="s.e.", y="Expected Revenue", s=40, data=df_80_4, color=sns.color_palette()[1], label="clusters=4")
sns.scatterplot(x="s.e.", y="Expected Revenue", s=40, data=df_80_3, color=sns.color_palette()[0], label="clusters=3")

# Customizing legend order
handles, labels = plt.gca().get_legend_handles_labels()
order = [0, 3, 2, 1]  # Reordering to keep clusters=3, clusters=4, clusters=5
plt.legend([handles[idx] for idx in order], [labels[idx] for idx in order])

# Display plot
plt.show()
```