The Impact of Familiarity Network on Equilibrium Bidding in Auctions

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Motivation

Old roots, new branches

- **Information asymmetry** The existence of familiarity networks is a common feature in most auctions e.g., local-tourists structure. These networks can provide (some) bidders with both informational and strategic advantages.
- Example: At a typical summer weekend country auction in New England, over half those in attendance are likely to be tourists and another quarter nonprofessional locals.

Introduction

- **Auctions** the first-price sealed bid auction and the second-price sealed bid auction, with zero reserve price.
- **Two key ingredients of the familiarity network**
  - **Local knowledge** Each bidder possesses perfect information about the valuations of a specific subset of other bidders, but lacks complete knowledge regarding the valuations or actions of participants beyond their direct connections.
  - **Private partition structure** Each bidder is aware only of the bidders within their own network but has no knowledge of bidders outside of their relatives.
- **Potential applications** EU Emissions Trading System (EU ETS), procurement auctions, treasury auctions, spectrum allocations, to online ad auctions.

Research Questions

- **Equilibrium characterization** How the possibility of knowing the rivals' information affects the competition of auctions as well as bidding strategies in equilibrium?
- **Revenue implication** How does the presence of familiarity networks impact resource allocation efficiency, expected revenue, and bidders' surplus.
- **Revenue Equivalence Theorem** Do both first-price and second-price auctions produce the same expected revenue, as stated by the Revenue Equivalence Theorem?
- **From a technical perspective** When do Bayesian Nash Equilibrium (BNE) in weakly undominated strategies exist, and if they do, are equilibrium payoffs unique? (Auctions are modeled as Bayesian games with continuous type and action spaces.)

Literature Review

- **Information asymmetry** P. R. Milgrom and Weber (1982); Maskin and Tirole (1992); Perry and Reny (2002); Esö and White (2004); Kim and Che (2004); Andreoni, Che, and Kim (2007); Garetto (2016); Kim (2016); Kim and Koh (2020).
- **Signals** Bergemann and Välimäki (2006); Fang and Morris (2006); Tian and Xiao (2010); Kozlovskaya (2018).
- **Recent contributions** Kim and Koh (2020), McClellan (2023).

The Game

Setup

- A finite set of risk-neutral bidders $N \equiv \{1, 2, \cdots, n\}$, $n \geq 3$.
- One object the auctioneer wants to sell.
- Bidder i's valuation V_i with a typical realization v_i, drawn from the interval $[\underline{v}, \bar{v}]$, $0 \leq \underline{v} < \bar{v} < +\infty$. $\mathbf{v} \equiv (v_1, v_2 \cdots, v_n)$ within the valuation space $\mathbf{V} \equiv [\underline{v}, \bar{v}]^n$.
- A common continuous distribution function F, $supp(\mathrm{F}) \subseteq \mathbb{R}_+$ whose density function f is bounded away from zero.
- Bidders are asked to submit sealed bids $\mathbf{b} \equiv (b_1, b_2 \cdots, b_n)$ simultaneously, where $B_i \ni b_i$ is the pure strategy set of player $i \in N$, $B_i \subseteq \mathbb{R}_+$.
- Bidder i observes v_i, and possibly other valuations in $\mathbf{v}$, given her familiarity network.

The Game

Familiarity network

- Assume bidders possess *local knowledge*.
- Define $G_i \subseteq N$ as the familiarity set for bidder $i \in N$. Let $v^{G_i} \subseteq [\underline{v}, \bar{v}]^n$ as i's view of the bidding vector $\mathbf{v}$.

$$v^{G_i} \equiv \begin{cases} v_j & \text{if } j \in G_i \\ V_j & \text{if } j \notin G_i \end{cases} \text{for all } i, j \in N \tag{1}$$

- Familiarity structure $\mathbf{G} \equiv (G_1, G_2, \cdots, G_n) \subseteq N^n$ forms an open cover on N (under the discrete topology) i.e., $N \subseteq \bigcup_{i \in N} G_i$.
- $\mathbf{G}$ is said to be *symmetric* if, for all $i, j \in N$, $i \in G_j$, indicates $j \in G_i$; $\mathbf{G}$ is *asymmetric* if there exist $i, j \in N$ such that $i \in G_j$ but $j \notin G_i$.

The Game

Auction format

- In the second-price auction, the bidder who submits the highest bid wins the object but pays an amount equal to the second-highest bid. Payoff function $\tilde{u}_i$ is given by:

$$\tilde{u}_i(\mathbf{b}, V_i) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i \geq \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases} \tag{2}$$

- In the first-price auction, the bidder who submits the highest bid wins the object and paying the amount of their bid. The payoff function is given by:

$$\tilde{u}_i(\mathbf{b}, V_i) = \begin{cases} v_i - b_i & \text{if } b_i \geq \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases} \tag{3}$$

The Game

- The auction is played as the following:

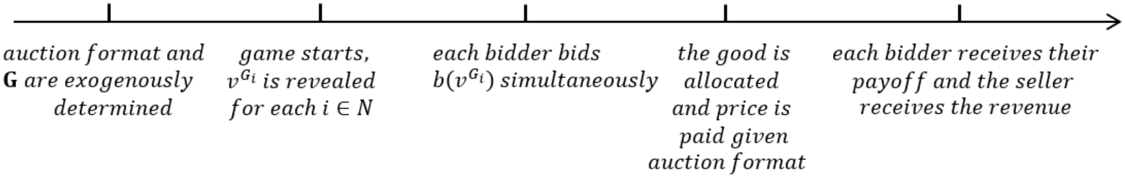


Figure: The game

The Game

Efficiency and revenue

- An auction is said to be *efficient* in equilibrium if and only if the object is allocated to the bidder with the highest valuation in every realization of $\mathbf{v}$.
- The seller's *revenue* is the amount paid by the winner. In a first-price auction, this corresponds to the highest bid, whereas in a second-price auction, it is the second-highest bid.

A symmetric example
Setup

- First-price auction with 4 players, $N = \{1, 2, 3, 4\}$.
- $\mathbf{G} = \{\{1, 2\}, \{1, 2\}, \{3, 4\} \{3, 4\}\}$.
- $\mathrm{F} \equiv \mathrm{Unif}[0, 1]$.

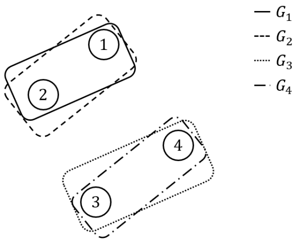


Figure: The game

A symmetric example

Strategy profile

- The equilibrium bidding strategy is given by

$$b_i^* = \min\{v_i, \max\{\frac{2}{3}v_i, v_j\}\},\ j \in G_i \tag{4}$$

- Intuition: consider bidders initially competing within her familiarity set, then competing outside.
  - In equilibrium, the front-runner strategically submits an unconstrained bid: $b_i^* = \max\{\frac{2}{3}v_i, v_j\}$.
  - Competitive pressure is reduced when $\frac{2}{3}v_i \geq v_j$, and increased when $\frac{2}{3}v_i < v_j$.

A symmetric example

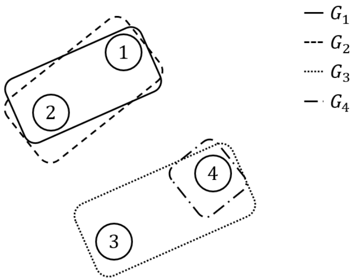
Inefficiency

- Consider a scenario where the valuations follow $v_1 > v_3 > v_4 > \frac{2}{3}v_1 > \frac{2}{3}v_3 > v_2$, which occurs with positive probability.
- In equilibrium, the bids would be $b_1^* = \frac{2}{3}v_1, b_2^* = v_2, b_3^* = v_4, b_4^* = v_4$.
- **Inefficiency**: bidder 1 values the good the most whereas bidder 3 obtain the good (a tie is broken in favor of a bidder with a higher valuation if there are multiple highest bidders).

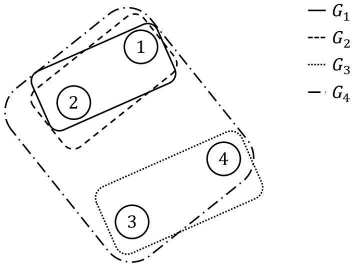
Two asymmetric examples

Setup

- Same setup, with the only difference being a change in bidder 4.



(a) Case 1.
$\mathbf{G} = \{\{1, 2\}, \{1, 2\}, \{3, 4\} \{4\}\}$



(b) Case 2.
$\mathbf{G} = \{\{1, 2\}, \{1, 2\}, \{3, 4\} \{1, 2, 3, 4\}\}$

Two asymmetric examples

Inefficiency

- Case 1. when the valuations follow $v_1 > v_3 > v_4 > \frac{2}{3}v_1 > \frac{2}{3}v_3 > \frac{3}{4}v_4 > v_2$. Bidder 1 values the good the most, does not obtain it—instead, bidder 3 does.
- Case 2. when the valuations follow $v_1 > v_4 > \frac{2}{3}v_1 > v_3 > \frac{2}{3}v_3 > v_2$. Bidder 1 values the good the most, but bidder 4 obtains it.

Expected revenue comparison

- In the symmetric case, the expected revenue is 0.584. In the asymmetric cases, the expected revenues are 0.597 for Case 1 and 0.602 for Case 2.
- **Nonmonotonicity** As bidder 4 gradually getting to know more, the expected revenue first decreases and then increases.

Results

Second-price auction

- **Proposition 1.** In a second-price auction, "truthful-telling" (bidding one's own valuation) is a weakly undominated strategy, i.e., $b_i(v^{G_i}) = v_i$ for all $i \in N$. The auction allocation is efficient, and the expected revenue is $E[r^{2nd}] = n(n-1)\int_{supp(\mathrm{F})}(\mathrm{F}^{n-2} - \mathrm{F}^{n-1})f \cdot vdv$.
- **Hint.** Considering $|G_i| = n$ or $|G_i| < n$; and comparing v_i with $\max_{j \in N \setminus \{i\}} v_j$ cases. Truthful-telling remains weakly dominant.
- e.g., when $\{V_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} \mathrm{Unif}[0,1]$, $E[r^{2nd}] = \frac{n-1}{n+1}$.

Results

First-price auction

- **Theorem 1.** The first-price auction possesses a pure strategy equilibrium in which the bidders employ non-decreasing bidding functions.
- **Hint.** Follows the approach in Reny (1999) and the corrigendum by Ewerhart and Reny (2022).
    1. For any non-equilibrium bidding strategy, there exists some player whose strategy yields a payoff strictly higher than the non-equilibrium payoff, even if others slightly deviate from this nonequilibrium strategy.
    2. There must be a point in strategy space where no bidder can profitably deviate.
    3. All mixed equilibria are in fact pure.

See: Appendix B

Results

First-price auction

- **Proposition 2.**
  1. In equilibrium, a front-runner bids at least a second-highest valuation in $G_\cdot$ and wins against all bidders in her familiarity set $G_\cdot$.
  2. The equilibrium allocation is inefficient with positive probability when there are at least two bidders $k, l \in N$ whose familiarity sets satisfies (i) $1 < |G_{k,l}| < n$ and (ii) $|G_k \setminus G_l| \geq 1$, $|G_l \setminus G_k| \geq 2$.
  3. The equilibrium is not unique.
- **Hint.** 2.1 the front-runner can slightly raise her infimum bid to increase her probability of winning by outbidding bidders within her familiarity network; 2.2 the front-runner with a lower valuation outbids the other occurs when the former has a strong second-highest bidder within her familiarity network; 2.3 the optimization is not necessarily strictly concave or having a global maximum.

See: Appendix C

Revenue implications in a first-price auction

Proposition 3.

1. **Revenue Implication** Under mild assumptions, a first-price auction yields a lower expected revenue compared to a second-price auction.
2. **Equity-Efficiency Tradeoff** If further assuming $\frac{1-F(v_i)}{f(v_i)}$ is strictly decreasing on its support, bidders obtain a greater surplus in a first-price auction than in a second-price auction. However, this first-price auction also results in inefficient allocations with positive probability.
3. **Nonmonotonicity** The revenue (and total surplus) in a first-price auction exhibits nonmonotonic behavior as bidders' familiarity network becomes more dense.

See: Appendix E

The impact of familiarity network

Setup

- Restrict the analysis to $\mathrm{F} \equiv \mathrm{Unif}[0,1]$, and **G** is *symmetric*.
- The equilibrium strategy for bidder $i \in N$ now is

$$b_i^*(v^{G_i}) = \min\{v_i, \max\{\frac{n-|G_i|}{n-|G_i|+1} v_i, \max_{j \in G_i \setminus \{i\}} v_j\}\} \tag{5}$$

- $b_i^*(v^{G_i})$ and F satisfies all regularity conditions and assumptions thus Proposition 2, 3 holds.

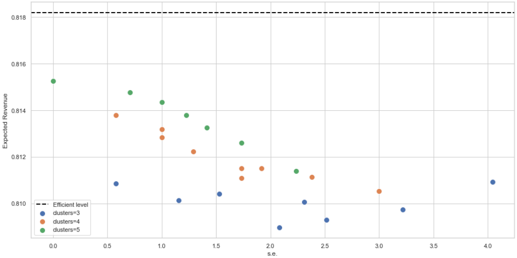
The impact of familiarity network

Size effect

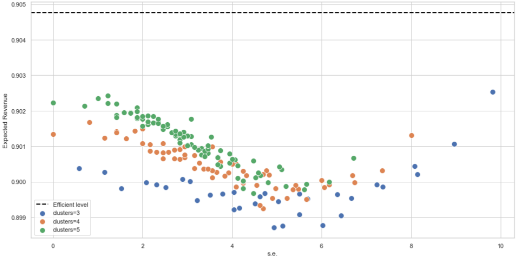
- **Number of bidders vs. familiarity networks** An increase in the number of bidders intensifies competition, whereas familiarity networks tend to lead to inefficiencies and reduce expected revenue.
- **Corollary 1.** The former outweighs the latter; the expected revenue increases monotonically as more players join the auction, regardless of the structure of the familiarity network.

The impact of familiarity network

Skewness effect



(a) $n = 10$



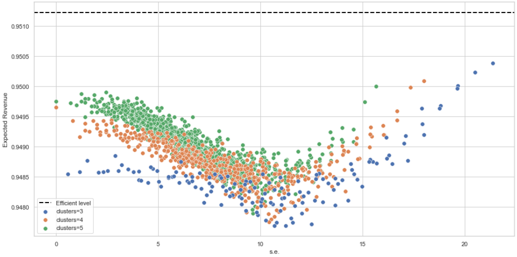
(b) $n = 20$

Figure: Expected revenue for different **G** and n

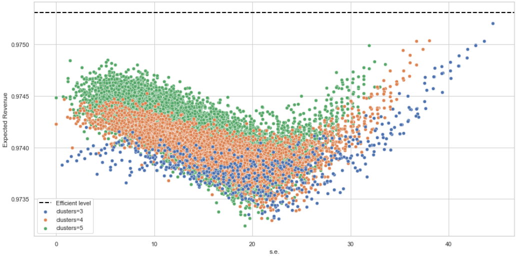
Illustrated in the simulation. Typo: in Figures all should be ”s.d.”.

The impact of familiarity network

Skewness effect



(a) $n = 40$



(b) $n = 80$

Figure: Expected revenue for different **G** and n

Illustrated in the simulation.

The impact of familiarity network

Skewness effect

- **Number of players** Expected revenue generally increases as the number of players grows, assuming a similar evenness of cluster sizes.
- **Skewness effect** The impact of skewness in the familiarity network is also nonmonotonic. As the familiarity network becomes more unbalanced, expected revenue forms a U-shaped curve.
- **Number of clusters** More clusters generally lead to higher expected revenue, controlling for the evenness of the familiarity network distribution.

Results

The impact of familiarity network

Corollary 2. When $\mathrm{F} \equiv \mathrm{Unif}[0, 1]$, and **G** is *symmetric*, in a first-price auction,

1. **Size Effect** The overall effect of increasing the number of bidders is to intensify competition. Bidders do not reduce their bids as more participants join the auction.
2. **Skewness effect** The impact of skewness in the familiarity network is nonmonotonic, forms a U-shaped curve. More clusters generally lead to higher expected revenue, controlling for the evenness of the familiarity network.

Discussions

- **Failure of Revenue Equivalence Theorem** Allocation in the second-price auction remains efficient. A first-price auction yields lower expected revenue than a second-price auction when bidders possess partial local information about others' valuations.
- **Equity-Efficiency Tradeoff** Bidders obtain a greater surplus in a first-price than in a second-price auction with the presence of familiarity network. However, this first-price auction also results in inefficient allocations with positive probability.
- **Future Studies** This includes, but is not limited to, equilibrium concept refinement (e.g., Bayesian correlated equilibrium), or collective learning (in a collective deliberation framework).

Discussions

Collective learning framework (envisaged)

- n bidders, 1 good, $t = 0, 1, \cdots$ the auction starts at latest T. Before the auction, bidders are free to catch up with other bidders (e.g., with strangers), each conversation can be either good or bad, where a good communication are likely to be informative, where a bad one only results in no news.
- To be specific, an informative conversation brings other's value ("I'll be very honest with you, I believe this good is worth 100 EUR."), whereas the latter maintains the previous belief ("Hmm, the snack food is so good!").
- Conversation is costly (time/ effort), each round of conversation cost everyone c. At each time period bidders jointly decide whether to keep communicating. If no winning coalition want to talk anymore/ reaching period T, the first-price auction begins.

Takeaways

- **Robustness of Second-Price and Inefficiency of First-Price Auctions** In the presence of familiarity networks, the Revenue Equivalence Theorem may fail.
- **Nonmonotonic Influence of Familiarity Networks** In first-price auctions, expected revenue follows a U-shape curve.
- **Positive Size Effect and Nonmonotonic Skewness Effect** Increasing the number of bidders in first-price auctions leads to higher expected revenue. As the network becomes more unbalanced, expected revenue initially decreases before subsequently increasing.

Bottomline

Thanks for your attention.

Q&A

Similar to any fixed-point theorem, to ensure the existence of an equilibrium is equivalent to show the following (1) B_i is a compact Hausdorff space; (2) u_i is bounded and measurable; and (3) $\bar{\mathbf{A}}$ is *better-reply secure* as in Corollary 5.2, Reny (1999).

1. Followed by Observation 2, I restrict bidding strategy to be within $[0, \bar{v}]$. B_i is thus defined over a closed interval of the real line, which is compact (by Heine-Borel theorem). Furthermore, B_i is a Hausdorff space because $\mathbb{R}$ is Hausdorff when endowed with the Euclidean topology. In metric spaces like $\mathbb{R}$, distinct points can always be separated by disjoint open sets, thus ensuring the Hausdorff property.

Back: Theorem 1.

Appendix B

2. The payoff function for bidder i is given by $u_i = (v_i - b_i)Pr(b_i \geq b_{-i})$. By construction, $0 \leq u_i \leq v_i - b_i \leq \bar{v}$, thus u_i is bounded. Additionally, since u_i depends on continuous, measurable functions of the bids, it is measurable.
3. Suppose that m is not an equilibrium, and let (m^*, u^*) be an element of mixed extension's vector payoff function. By definition, $\lim u(m^t) = u^*$ for some sequence of mixed strategies $\{m^t\}$ converging to m^*. Given m^*_{-i} and any small $\varepsilon > 0$, bidder i can achieve a payoff within ε of her supremum by employing a strictly increasing bid function $b_i(\cdot)^{\varepsilon}$. Moreover, her payoff is continuous in the other bidders' mixed strategies at $(b_i^{\varepsilon}, m^*_{-i})$.

Back: Theorem 1.

- If ties occur with zero probability under m^*, then payoffs are continuous at the limit, i.e., $u^* = u(m^*)$. Since m^* is not an equilibrium, there exists some bidder i and a small enough $\varepsilon > 0$ such that $u_i(b_i^\varepsilon, m_{-i}^*) > u(m^*) = u^*$. The continuity of $u_i(b_i^\varepsilon, \cdot)$ at m_{-i}^* implies that bidder i can secure a payoff strictly above u^* at m^*.
- If ties occur with positive probability under m^*, then all bidders who tie with positive probability would strictly prefer to win with probability one. Along the sequence $\{m^t\}$, at least one of those bidders who ties in the limit loses with a probability bounded away from zero. This bidder i can then achieve a strictly higher payoff than the limiting payoff $u_i^* = \lim u_i(m^t)$ by employing b_i^ε for a sufficiently small ε. In other words, $u_i(b_i^\varepsilon, m_{-i}^*)$ is strictly greater than $u_i(m^t)$ for large t. Hence, again by the continuity of $u_i(b_i^\varepsilon, \cdot)$ at m_{-i}^*, we get $u_i(b_i^\varepsilon, m_{-i}^*) > u^*$, implying that bidder i can secure a strictly higher payoff above u^* at m^*. This establishes better-reply security.

Back: Theorem 1.

Appendix C

For any front-runner $i \in N$, denote the second-highest valuation bidder (if one exists, which requires $|G_i| > 1$) in G_i as j, with valuation v_j, here j can also be a front-runner in G_j. Let $b_i(v^{G_i})$ represent an arbitrary equilibrium bid by bidder i, given v^{G_i}, and let $\underline{b}_i$ be the infimum of her equilibrium bids. The goal is to show that $\underline{b}_i \geq v_j$.
First, note that bidder i must receive a strictly positive expected payoff in equilibrium. For instance, she could bid $\tilde{b}_i = v_j + \varepsilon$ where $\varepsilon > 0$ is small enough, and win the auction with positive probability. Given Observation 3, i would outbid all other bidders in G_i.
Now, assume for the sake of contradiction that $\underline{b}_i < v_j$.

Back: Proposition 2.

Case 1. If j is not a front-runner in G_j, denote the front-runner in G_j as k. Similarly, let $b_k(v^{G_j})$ represent an arbitrary equilibrium bid by bidder $k \in G_j$, given v^{G_j}, and let $\underline{b}_k$ be the infimum of her equilibrium bids. By the similar logic, bidder j earns a strictly positive payoff in equilibrium if $\tilde{b}_j \geq \underline{b}_k$, where bidding $\tilde{b}_j = \underline{b}_i + \epsilon$ with $\epsilon > 0$, or a zero payoff otherwise. In latter case, a weakly undominated strategy for j is to bid $b_j(v^{G_j}) = v_j$, then $\underline{b}_i < v_j$ would make i lose the auction and get zero payoff in equilibrium, leading to a contradiction because i should have a strict positive payoff. In the formal case, for both bidders to earn positive payoffs, they must be tied in their bidding. This implies that their infimum bids must coincide, and each bidder must place a mass point at this infimum. However, bidder i can always raise her mass point slightly, increasing her probability of winning (by outbidding j) and improving her payoff conditional on winning only slightly. This leads to a contradiction. Thus, it follows that $b_i(v^{G_i}) \geq \underline{b}_i \geq v_j$.

Back: Proposition 2.

Appendix C

Case 2. If j is also a front-runner in G_j, by similar logic as front-runner i in G_i, bidder j must also earn a strictly positive payoff in equilibrium. The contradiction is been shown by the same argument as above.
In both cases, the assumption that $\underline{b}_i < v_j$ leads to contradictions. Therefore, it must be that $b_i(v^{G_i}) \geq \underline{b}_i \geq v_j$. By Observation 3, bidder i will outbid all other bidders within G_i.

Back: Proposition 2.

(2) Equipped with (1), for every bidder who is not a front-runner, they lose the auction with probability 1. A weakly dominated strategy for such a bidder is to bid truthfully i.e., bid exactly v_i and and receive a zero payoff. Now, given (i), (ii), there exists $i, j, k \in N$ s.t., $k \in G_i \setminus G_j$, $i \notin G_j$, $j \notin G_i$. What I want to show is within the category of $v_j > v_i > v_k > \max_{l \in N \setminus \{i,j,k\}} v_m$, which happens with positive probability, as F is continuous. In this scenario, i and j are front-runners. What remains to be shown is that $b_i > b_j$, leading to inefficiency, as bidder j, who holds the highest valuation, loses the auction while i wins.

Step 1. $b_i \geq v_k$.

This follows directly from (1), as i is the front-runner and $k \in G_i$.

Back: Proposition 2.

Step 2. $b_j < v_j - \frac{1}{|N-G_j|\bar{f}}$, where $\bar{f} := \max_{b \in (\underline{v}, v_j]} \frac{f(b)}{\mathrm{F}(b)}$.

This follows from solving the optimization problem for bidder j. Since $\mathrm{F}(b) > 0$ for all $b \in (\underline{v}, v_j]$, $\frac{f(b)}{\mathrm{F}(b)}$ is finite, which implies the existence of $\bar{f}$. Given j is also a front runner, bidding $\underline{v}$ will never allow her to win the auction. Furthermore, by (1) and given the support of her bidding strategy lies within$(\underline{v}, v_j]$:

$$\begin{aligned}
&\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_{-j}) \\
\overset{(a)}{\Longrightarrow}\ &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) Pr(b_j \geq b_{l \in G_j}) Pr(b_j \geq b_{l \in N \setminus G_j}) \\
\Longrightarrow\ &\max_{b_j \in (\underline{v}, v_j]} (v_j - b_j) \mathrm{F}(b_j)^{|N-G_j|}
\end{aligned} \tag{6}$$

(a) The probability that bidder j overbids everyone is equivalent to overbid both bidders in G_j and not in G_j.

Back: Proposition 2.

Appendix C

To solve (6), take the first-order condition (F.O.C.). A corner solution is easily ruled out: either bidding too little (close to infimum), which would result in winning with zero probability, or bidding too high, e.g., truthful telling, would lead to receiving zero revenue with probability 1.

$$
\begin{aligned}
&(v_j - b_j)\frac{f(b_j)}{\mathrm{F}(b_j)} = \frac{1}{|N - G_j|} \\
\xRightarrow{\bar{f} \geq \frac{f(b)}{\mathrm{F}(b)}}\ & b_j < v_j - \frac{1}{|N - G_j| \cdot \bar{f}}
\end{aligned} \tag{7}
$$

Back: Proposition 2.

Step 3. $b_i > b_j$, and this occurs with positive probability.
In addition to the condition $v_j > v_i > v_k > \max_{l \in N \setminus \{i,j,k\}} v_m$, I further let $v_k > v_j - \frac{1}{|N-G_j|\bar{f}}$.
Since each valuation is drawn from a continuous interval, this event happens with positive probability, i.e., $Pr(v_j > v_i > v_k > \max_{l \in N \setminus \{i,j,k\}} v_m; v_k > v_j - \frac{1}{|N-G_j|\bar{f}}) > 0$, for

$\{v_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} \mathrm{F}$.

This indeed leads to $b_i > b_j$ because, $v_j \overset{(a)}{>} v_i \overset{(b)}{\geq} b_i \overset{(c)}{\geq} v_k \overset{(d)}{>} v_j - \frac{1}{|N-G_j|\bar{f}} \overset{(e)}{>} b_j$, where (a) and (d) are by construction, (b) follows from Observation 3, (c) is derived from Step 1, and (e) comes from Step 2.

Back: Proposition 2.

Appendix C

(3) By similar logic as in Appendix A, without further constraints, the equilibrium is not unique. For example, uniqueness would require condition (4.3) to attain a global maximum, which is difficult to satisfy in the general case. Further constraints could be imposed by specifying a particular common distribution function, or by restricting the analysis to symmetric equilibria with linear bidding functions.

Back: Proposition 2.

Appendix E

NTS. In any equilibrium of a first-price auction, the expected payoff from the first price auction has a lower bound:

$$u_i(\mathbf{v}^{G_i})] \geq \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds, \;\; v_i \in [\underline{v}, \bar{v}] \tag{8}$$

The equality holds if **G** is *symmetric*.

Back: Proposition 3.

Case 1. If i is not a front-runner i.e., [1] $v_i \leq \max_{(2)} \mathbf{v}^{G_\cdot}$ or if there exists at least one familiarity set that contains i, but i does not have the highest bid in that set i.e., $\exists j \neq i$, s.t., $i \in G_j, v_i \leq \max_{(2)} \mathbf{v}^{G_j}$. Define $\tilde{v} := \max\{\max_{(2)} \mathbf{v}^{G_i}, \max_{(2)} \mathbf{v}^{G_j}\}, i \in G_j$. One can observe that the two conditions mentioned above can actually be combined into a single statement: $i \in G_j, v_i < \max_{(2)} \mathbf{v}^{G_j}, j \in N$. Then $v_i \in [\underline{v}, \tilde{v}]$ would be the support of Case 1.
Given Proposition 2(1), it is clear that bidder i would never win the auction. This is because there is always a front-runner who can bid at least the second-highest bid, which will exceed any bid from the interval $[\underline{v}, v_i]$ i.e., $Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}}) \equiv 0, \forall s \in [\underline{v}, v_i]$.
Consequently, $u_i(\mathbf{v}^{G_i}) = 0 = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}}) ds, \ \forall v_i \in [\underline{v}, \tilde{v}]$.
The equality holds if **G** is *symmetric*.

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[1] Define $\max_{(2)} \mathbf{v}^{G_\cdot}$ as the second-highest value within $\mathbf{v}^{G_\cdot}$ for the sake of convenience, a formal definition can be found in Section 2.

Case 2. If $v_i = \tilde{v}$, suppose for the sake of contradiction that bidder i earns a strictly positive payoff. Then, her infimum bid must be less than v_i (Observation 3) but weakly greater than the infimum bid of the second-highest bidder. In this case, the second-highest bidder can profitably deviate by outbidding i's infimum bid (moving from 0 to a strictly positive value). If both bidders' infimum bids coincide, then both must place mass points there, which creates a contradiction, as shown in the proofs in Proposition 2(1). This leads to the conclusion that bidder i cannot earn a strictly positive payoff in this case, thus, $u_i(\mathbf{v}^{G_i}) = 0 = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds, \ \forall v_i = \tilde{v}$

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Appendix E

Case 3. If $v_i > \tilde{v}$. Let $p^{out}(b)$ denote the probability that bidder i outbids all bidders in G_j which does not contain i, this condition ensures that this probability does not depend on bidder i's valuation (as they cannot observe). By Proposition 2(1), bidder i will win with probability $p^{out}(b)$ by bidding $b \geq \tilde{v}$ because she'll outbid all players in G_i which contains i (Observation 3). Furthermore, by Proposition 2(1), bidder i never bids below $\max \mathbf{v}^{G_i}$. (i) if $\tilde{v} = \max_{(2)} \mathbf{v}^{G_i}$, it degenerates to the case in Kim, Che (2006), which leads to $u_i(\mathbf{v}^{G_i}) = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}}) ds, \ \forall v_i \in [v_i, \max_{(2)} \mathbf{v}^{G_i}]$.

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Appendix E

(ii) if $\tilde{v} = \max_{(2)} \mathbf{v}^{G_i}$, $i \in G_j$, $j \in N \setminus \{i\}$, in other words, $\max_{(2)} \mathbf{v}^{G_j} \geq \max_{(2)} \mathbf{v}^{G_i}$. Bidder i's bid might fall in the interval $[\max_{(2)} \mathbf{v}^{G_i}, \max_{(2)} \mathbf{v}^{G_j}]$. By Proposition 2(1), $u_i(\mathbf{v}^{G_i}) = \max\limits_{b \geq \max_{(2)} \mathbf{v}^{G_i}} Pr(b \geq b_{-i})(v_i - b) \geq \max\limits_{b \geq \tilde{v}} p^{out}(b)(v_i - b)$. The inequality holds because bidder i optimizes over a narrower interval. Similarly, using (i), we have: $u_i(\mathbf{v}^{G_i}) \geq \int_{\underline{v}}^{\max_{(2)} \mathbf{v}^{G_j}} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds + \int_{\max_{(2)} \mathbf{v}^{G_j}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds = \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds$.
When **G** is *symmetirc*, (ii) in Case 3 occurs with zero probability. Consequently, $u_i(\mathbf{v}^{G_i})] \geq \int_{\underline{v}}^{v_i} Ep_i(s, \mathbf{v}^{G_i \setminus \{s\}})ds$, $v_i \in [\underline{v}, \bar{v}]$ holds as an equality (Note that in all other integration parts, the equality always holds).

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