Homework 3

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Problem description

Denote between-class scatter matrix S_B as: $S_B = \sum\limits_{i=1}^{c} n_i(\boldsymbol{\mu_i} - \boldsymbol{\mu})(\boldsymbol{\mu_i} - \boldsymbol{\mu})^T$

Where $\boldsymbol{\mu_i} = \frac{1}{n_i} \sum\limits_{\boldsymbol{x} \in \omega_i} \boldsymbol{x}, \quad \boldsymbol{\mu} = \frac{1}{n} \sum \boldsymbol{x}$

Prove that the rank of S_B is a most $c-1$.

Lemma

If A, B are two matrices, then $rk(A \times B) = \min(rk(A), rk(B))$

It is one of the basic knowledge in linear algebra, so I just skip the proof.

Proof

We know $\boldsymbol{\mu} = \frac{1}{n} \sum \boldsymbol{x} = \frac{1}{n} \sum\limits_{i=1}^{c} n_i \boldsymbol{\mu}_i$

So that $\boldsymbol{\mu}_c = n\boldsymbol{\mu} - \sum\limits_{i=1}^{c-1} \boldsymbol{\mu}_i, \boldsymbol{\mu}_c - \boldsymbol{\mu} = \sum\limits_{i=1}^{c-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_i) = -\sum\limits_{i=1}^{c-1} (\boldsymbol{\mu_i} - \boldsymbol{\mu})$.

That's to say, $\boldsymbol{\mu_c} - \boldsymbol{\mu}$ can be represented by $\boldsymbol{\mu}_1 - \boldsymbol{\mu}, \boldsymbol{\mu}_2 - \boldsymbol{\mu}, \ldots, \boldsymbol{\mu_{c-1}} - \boldsymbol{\mu}$

Now we rewrite S_B as the following form:

$$S_B = [\boldsymbol{\mu}_1 - \boldsymbol{\mu} \quad \boldsymbol{\mu}_2 - \boldsymbol{\mu} \quad \ldots \quad \boldsymbol{\mu_c} - \boldsymbol{\mu}] \begin{bmatrix} n_1 & 0 & \ldots & 0 \\ 0 & n_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & n_c \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_1 - \boldsymbol{\mu} \\ \boldsymbol{\mu}_2 - \boldsymbol{\mu} \\ \ldots \\ \boldsymbol{\mu_c} - \boldsymbol{\mu} \end{bmatrix}$$

According to the lemma, $rk(S_B) \le rk([\boldsymbol{\mu}_1 - \boldsymbol{\mu}, \boldsymbol{\mu}_2 - \boldsymbol{\mu}, \ldots, \boldsymbol{\mu_c} - \boldsymbol{\mu}])$.

We have already known that $\boldsymbol{\mu}_c - \boldsymbol{\mu} = -\sum\limits_{i=1}^{c-1} (\boldsymbol{\mu_i} - \boldsymbol{\mu})$, so $rk(S_B) \le c-1$