Homework 2

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Problem description

Denote $W_{i,j}$ as the edge weight between vertex x_i and x_j in the graph.

$D_{i,j}$ is a diagonal matrix where $D_{i,i} = \sum_{j=1}^{N} w_{i,j}$

Prove that Graph Laplacian $L = D - W$ is a positive semi-definite matrix.

Proof

To prove L is a positive semi-definite matrix, we want to show that For each possible $\boldsymbol{x}$, $\boldsymbol{x}(D-W)\boldsymbol{x}^T \geq 0$ always holds.

$$
\begin{aligned}
\boldsymbol{x}(D-W)\boldsymbol{x}^T &= \boldsymbol{x}D\boldsymbol{x}^T - \boldsymbol{x}W\boldsymbol{x}^T \\
&= \sum_{i=1}^{N} x_i^2 \sum_{j=1}^{N} w_{i,j} - \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j w_{i,j} \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}x_i^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}x_j^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}x_i x_j \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}(x_i^2 + x_j^2 - 2x_i x_j) \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{i,j}(x_i - x_j)^2
\end{aligned}
$$

$$\because w_{i,j} \geq 0 \quad \therefore \boldsymbol{x}(D-W)\boldsymbol{x}^T \geq 0 \quad \therefore L \text{ is the positive semi-definite matrix.}$$