11. Analizar la consistencia y la estabilidad de uno de los métodos anteriores

a) $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \rightarrow$ Upwind: $\underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t}}_{A} + \underbrace{u\,\frac{T_i^n - T_{i-1}^n}{\Delta x}}_{B} = 0$

$$T_i^{n+1} = T(t+\Delta t, x) = T_i^n + \left(\frac{\partial T}{\partial t}\right)_i^n \Delta t + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_i^n + \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_i^n$$

$$T_{i-1}^n = T(t, x+\Delta x) = T_i^n - \Delta x\left(\frac{\partial T}{\partial x}\right)_i^n + \frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n - \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n + \frac{\Delta x^4}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_i^n$$

A: $$\frac{1}{\Delta t}\left[\cancel{T_i^n} + \Delta t\left(\frac{\partial T}{\partial t}\right)_i^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{\Delta t^3}{6}\left(\frac{\partial^2 T}{\partial t^3}\right)_i^n + \frac{\Delta t^4}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_i^n - \cancel{T_i^n}\right] =$$

$$= \left(\frac{\partial T}{\partial t}\right)_i^n + \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_i^n + \frac{\Delta t^3}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_i^n$$

B: $$\frac{u}{\Delta x}\left[\cancel{T_i^n} - \cancel{T_i^n} + \Delta x\left(\frac{\partial T}{\partial x}\right)_i^n - \frac{\Delta x^2}{2}\left(\frac{\partial T}{\partial x^2}\right)_i^n + \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n - \frac{\Delta x^4}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_i^n\right] =$$

$$= u\left[\left(\frac{\partial T}{\partial x}\right)_i^n - \frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + \frac{\Delta x^2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right) - \frac{\Delta x^3}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)\right]$$

Sustituyendo en la ecuación inicial: $A + B = 0$

$$\left(\frac{\partial T}{\partial t}\right)_i^n + \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_i^n + \frac{\Delta t^3}{24}\left(\frac{\partial^4 T}{\partial t^4}\right)_i^n + u\left[\left(\frac{\partial T}{\partial x}\right)_i^n - \frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{\Delta x^2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n - \frac{\Delta x^3}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_i^n\right] = 0$$

La ecuación de convección se cumple, entonces:

$$\underbrace{\left(\frac{\partial T}{\partial t}\right)_i^n + u\left(\frac{\partial T}{\partial x}\right)_i^n}_{\substack{\parallel \\ 0}} + E_i^n = 0 \quad / \quad E_i^n = \frac{\Delta t}{2}\left(\frac{\partial T}{\partial t}\right)_i^n + \frac{\Delta t^2}{2}\left(\frac{\partial^3 T}{\partial t^2}\right)_i^n - u\,\frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + u\,\frac{\Delta x^2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n$$

porque la ecuación de convección se cumple siempre

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} \Rightarrow \frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = -u\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) = u^2\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial^3 T}{\partial t^3} = \frac{\partial^2}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}(-u)\left(\frac{\partial^2 T}{\partial t^2}\right) = -u^3\frac{\partial^3 T}{\partial x^3}$$

Entonces, el error E_i^n viene dado por:

$$E_i^n = u^2 \frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n - u^3 \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n - u\frac{\Delta x}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + u\frac{\Delta x^2}{2}\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n =$$

$$= \left(\frac{\Delta t}{2}u^2 - \frac{\Delta x}{2}u\right)\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + \left(u^2\frac{\Delta x^2}{6} - \frac{\Delta t^2}{6}u^3\right)\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n =$$

$$\underset{C = \frac{u\Delta t}{\Delta x}}{\overset{\downarrow}{=}} -\left(\frac{u\Delta x}{2}\right)(1-c)\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + \frac{u\Delta x^2}{6}(1-c^2)\left(\frac{\partial^3 T}{\partial x^3}\right)_i^n$$

Como la ecuación de convección se cumple siempre $E_i^n \to 0$, esto ocurre cuando $\Delta x \to 0$, no obstante podemos ajustar los parámetros $\Delta x, \Delta t, u$ tal que $(1-c^2) \ll 1$

b) Estabilidad.

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = 0 \to \text{upwind: } \frac{T_i^{n+1} - T_i^n}{\Delta t} + u\frac{T_i^n - T_{i-1}^n}{\Delta x} - \alpha\frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} = 0$$

Despejamos T_i^{n+1} en función de los parámetros $c = u\frac{\Delta t}{\Delta x}$ y $s = \alpha\frac{\Delta t}{\Delta x^2}$

$$T_i^{n+1} = T_i^n + c(T_{i-1}^n - T_i^n) + s(T_{i-1}^n - 2T_i^n + T_{i+1}^n)$$

El error cometido $\xi_i^n = T_i^n - {}^*T_i^n$ / ${}^*T_i^n$ es el resultado dado por el ordenador y T_i^n la solución sin aprox

este error satisface la ecuación de transporte:

$$\xi_j^{n+1} = \xi_j^n + c(\xi_{j-1}^n - \xi_j^n) + s(\xi_{j-1}^n - 2\xi_j^n + \xi_{j+1}^n) \quad / \quad \xi_j^n = G^n e^{i\sigma j}$$

$$G^{n+1}e^{i\sigma j} = G^n e^{i\sigma j} + c(G^n e^{i\sigma(j-1)} - G^n e^{i\sigma j}) + s(G^n e^{i\sigma(j-1)} - 2G^n e^{i\sigma j} + G^n e^{i\sigma(j+1)})$$

$$G = 1 + c(e^{-i\sigma} - 1) + s(e^{-i\sigma} - 2 + e^{i\sigma}) = 1 + c(\cos\sigma - i\sin\sigma - 1) + s(2\cos\sigma - 2)$$

$$G = 1 - 2s(1-\cos\sigma) - c(1 - \cos\sigma + i\sin\sigma) = 1 - (2s+c)(1-\cos\sigma) - ic\sin\sigma$$

Para que el esquema sea estable $|G| \leq 1\ \forall\sigma \to \underset{A}{-1 \leq} 1 - (2s+c)(1-\cos\sigma) - ic\sin\sigma \underset{B}{\leq 1}$

A: $-1 \leq 1 - (2s+c)(1-\cos\sigma) - ic\sin\sigma$; $\quad 2 \geq (2s+c)(1-\cos\sigma) + ic\sin\sigma$

separando en parte Re e Im: $2 \geq (2s+c)(1-\cos\sigma)$; $1 \geq (2s+c)\underbrace{\left(\frac{1-\cos\sigma}{2}\right)}_{\text{siempre} \leq 1 :\ \cos\sigma \in [-1,1]} \to \boxed{1 \geq (2s+c)}$

$$0 \geq c\sin\sigma$$

B: $\cancel{1} - (2s+c)(1-\cos\sigma) - i\,c\sin\sigma \leq \cancel{1}$

$-(2s+c)\underbrace{(1-\cos\sigma)} \leq 0 \longrightarrow$ se cumple siempre

$\cos\sigma \in [-1,1] \rightarrow 1-\cos\sigma \in [0,2]$

$c\sin\sigma \leq 0$

Entonces $\boxed{1 \geq (2s+c) \;/\; c \in \mathbb{R}e}$