7.- Estudia la consistencia y la estabilidad de uno de los métodos del boletín

Ejercicio 3. $$\frac{T_j^{n+1} - T_j^n}{\Delta t} = \frac{\alpha}{\Delta t}\left(-\frac{1}{12}T_{j-2}^n + \frac{4}{3}T_{j-1}^n - 2'5\,T_j^n + \frac{4}{3}T_{j+1}^n - \frac{1}{12}T_{j+2}^n\right)$$

Vamos a hacer el desarrollo de Taylor:

$$T_j^{n+1} = T(x, t+\Delta t) = T_j^n + \Delta t\left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{6}\left(\frac{\partial^3 T}{\partial t^3}\right)_j^n$$

$$T_{j-2}^n = T(x-2\Delta x, t) = T_j^n - 2\Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{4\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n - \frac{8\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n$$

$$T_{j-1}^n = T(x-\Delta x, t) = T_j^n - \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n - \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n$$

$$T_{j+1}^n = T(x+\Delta x, t) = T_j^n + \Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n$$

$$T_{j+2}^n = T(x+2\Delta x, t) = T_j^n + 2\Delta x\left(\frac{\partial T}{\partial x}\right)_j^n + \frac{4\Delta x^2}{2}\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n + \frac{8\Delta x^3}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n$$

Sustituyendo en la ecuación principal y operando tenemos:

$$\underbrace{\left(\frac{\partial T}{\partial t}\right)_j^n - \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)_j^n}_{\text{ecuación difusión}} + \underbrace{\frac{\Delta t}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + \frac{\Delta t^2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_j^n}_{\text{Error } E_j^n} = 0$$

Para que el método sea consistente $\lim_{\Delta t, \Delta x \to 0} E_j^n = 0$

Vemos claramente que esto es cierto.

Podemos expresarlo en función de derivadas espaciales:

$$\left(\frac{\partial^2 T}{\partial t^2}\right) = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}\alpha\left(\frac{\partial T}{\partial t}\right) = \frac{\partial^4 T}{\partial x^4}\alpha^2$$

$$E_n = \frac{\alpha^2\Delta t}{2}\left(\frac{\partial^4 T}{\partial x^4}\right)_j^n + O(\Delta t, \Delta x^4)$$

2) Estabilidad

Estudiamos si la acumulación de errores conlleva a que la solución diverja o converja

$E_j^n = G^n e^{i\theta j}$ → sustituimos para cada término

$$G^{n+1} e^{i\theta j} = G^n e^{i\theta j} - s\left(-\frac{1}{12} G^n e^{i\theta(j-2)} + \frac{4}{3} G^n e^{i\theta(j-1)} - 2'5\, e^{i\theta j} G^n + \frac{4}{3} G^n e^{i\theta(j+1)} - \frac{1}{12} G^n e^{i\theta(j+2)} \right)$$

$$G = 1 - s\left(\frac{8}{3}\cos\theta - 2'5 - \frac{1}{6}\cos(2\theta) \right)$$

Para que el sistema sea estable los módulos de los errores deben de ser $|G| \leq 1$

$$-1 \leq 1 - s\left(\frac{8}{3}\cos\theta - 2'5 - \frac{1}{6}\cos(2\theta) \right) \leq 1 \rightarrow -1 \leq 1 - s(x) \leq 1$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \Rightarrow \frac{8}{3}\cos\theta - 2'5 - \frac{1}{3}\cos^2\theta + \frac{1}{6} = \frac{1}{3}\left(\cos^2\theta - 8\cos\theta + 7\right) = \frac{f(\theta)}{3}$$

Calculamos los extremos de $f(\theta)$:

$$\frac{df}{d\theta} = \frac{df}{d(\cos\theta)}\frac{d\cos\theta}{d\theta} = -\sin\theta(2\cos\theta - 8) = 0 \Rightarrow \begin{cases} \sin\theta = 0 \rightarrow \theta = m\pi \ / \ m \in \mathbb{N} \\ \cos\theta = 4 \rightarrow \theta = \arccos(4) \end{cases}$$

$$\left.\frac{d^2 f}{d\theta^2}\right|_{\theta_0} > 0 \rightarrow \text{mínimo} \quad y \quad \left.\frac{d^2 f}{d\theta^2}\right|_{\theta_0} < 0$$

$f(\theta = 2m\pi) = 1 - 8 + 7 = 0$ (mínimo)

$f(\theta = (2m+1)\pi) = 1 + 8 + 7 = 16$ (máximo)

1) $G \leq 1 \rightarrow \cancel{1} - \frac{s}{3} f(\theta) \leq \cancel{1}$ $\quad f(\theta) \in [0, 16] \Rightarrow -\frac{s}{3} f(\theta)$ siempre es menor que 1 $\forall s > 0$

2) $G \geq -1 \rightarrow -1 \leq 1 - \frac{s}{3} f(\theta) \quad -2 \leq -\frac{s}{3} f(\theta) \quad s f(\theta) \geq 6$

$f(\theta)_{máx} = 16 \rightarrow \quad s\,16 \geq 6 \quad s \geq \frac{6}{16} = \frac{3}{8} \Rightarrow s = \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{3}{8} = 0'375$

$$0 \leq s \leq 0'375$$