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2 Week; 행렬식, 기본 행렬, 행렬 분해

2주 학습 목표

- ❖ 선형방정식 시스템의 행렬방정식 표현 이해
- ❖ 인버터블 선형 시스템의 경우에, 역행렬을 이용한 행렬 방정식 풀이
- ❖ 동차 선형 시스템 해 경우 수 이해
- ❖ 행렬식(determinat) 계산 학습
- ❖ 기본 행렬 및 행조작 학습
- ❖ 기본 행렬에 의한 행렬의 분해(factorization) 이해
- ❖ 기본 행렬에 의한 행렬의 LU 분해(factorization) 이해

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1.5 Matrix Equations

1.6 Determinants

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1.8 Applications of Systems of Linear Equations

1.5 Matrix Equations

학습 목표

- ❖ 선형방정식 시스템의 행렬방정식 표현 이해
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- ❖ 동차 선형 시스템 해 경우 수 이해
- ❖ 행렬식(determinat) 정의 및 2x2 행렬의 행렬식 정의 이해
- ❖ 대각 행렬의 행렬식 연산 이해
- ❖ 행렬식 계산을 위한 행렬의 행조작 이론 이해
- ❖ 대형 행렬의 행렬식 계산 연습

Matrix form of the Linear system

To illustrate the process, consider the linear system

$$\begin{cases} x - 6y - 4z = -5 \\ 2x - 10y - 9z = -4 \\ -x + 6y + 5z = 3 \end{cases}$$

The matrix of coefficients is given by

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix}$$

Now let $\mathbf{x}$ and $\mathbf{b}$ be the vectors

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

Then the original linear system can be rewritten as

$$A\mathbf{x} = \mathbf{b}$$

We refer to this equation as the **matrix form** of the linear system and $\mathbf{x}$ as the **vector form** of the solution.

In certain cases we can find the solution of a linear system in matrix form directly by matrix multiplication. In particular, if A is invertible, we can multiply both sides of the previous equation on the left by A^{-1}, so that

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

Since matrix multiplication is associative, we have

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

therefore,

$$\mathbf{x} = A^{-1}\mathbf{b}$$

역행렬을 이용한 행렬 방정식 풀이 예제

For the example above, the inverse of the matrix

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix} \quad \text{is} \quad A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, the solution to the linear system in vector form is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

That is,

$$x = -1 \qquad y = 2 \qquad \text{and} \qquad z = -2$$

We have just seen that if the matrix A has an inverse, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. This fact is recorded in Theorem 10.

THEOREM 10 If the $n \times n$ matrix A is invertible, then for every vector $\mathbf{b}$, with n components, the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

역행렬을 이용한 선형방정식 시스템 풀이 예제

EXAMPLE 1 Write the linear system in matrix form and solve.

$$\begin{cases} 2x + y = 1 \\ -4x + 3y = 2 \end{cases}$$

Homogeous Linear System

DEFINITION 1 **Homogeneous Linear System** A **homogeneous linear system** is a system of the form $A\mathbf{x} = \mathbf{0}$.

The vector $\mathbf{x} = \mathbf{0}$ is always a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, and is called the **trivial solution**.

동차 선형 시스템 예제

EXAMPLE 2 Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

동차 선형 시스템 해 경우 수

EXAMPLE 3 Show that if $\mathbf{x}$ and $\mathbf{y}$ are distinct solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} + c\mathbf{y}$ is a solution for every real number c.

Solution Using the algebraic properties of matrices, we have that

$$\begin{aligned} A(\mathbf{x} + c\mathbf{y}) &= A(\mathbf{x}) + A(c\mathbf{y}) \\ &= A\mathbf{x} + cA\mathbf{y} \\ &= \mathbf{0} + c\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Hence, $\mathbf{x} + c\mathbf{y}$ is a solution to the homogeneous system.

The result of Example 3 shows that if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has two distinct solutions, then it has infinitely many solutions. That is, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ either has one solution (the trivial solution) or has infinitely many solutions. The same result holds for the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$. To see this, let $\mathbf{u}$ and $\mathbf{v}$ be distinct solutions to $A\mathbf{x} = \mathbf{b}$ and c a real number. Then

$$\begin{aligned} A(\mathbf{v} + c(\mathbf{u} - \mathbf{v})) &= A\mathbf{v} + A(c(\mathbf{u} - \mathbf{v})) \\ &= A\mathbf{v} + cA\mathbf{u} - cA\mathbf{v} \\ &= \mathbf{b} + c\mathbf{b} - c\mathbf{b} = \mathbf{b} \end{aligned}$$

These observations are summarized in Theorem 11.

THEOREM 11 If A is an $m \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has no solutions, one solution, or infinitely many solutions.

1.5 절 요약

Fact Summary

Let A be an $m \times n$ matrix.

1. If A is invertible, then for every $n \times 1$ vector $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.
2. If A is invertible, then the only solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.
3. If $\mathbf{u}$ and $\mathbf{v}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then the vector $\mathbf{u} + c\mathbf{v}$ is another solution for every scalar c.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, infinitely many solutions, or no solution.

Homework #1-5

- Exercise Set 1.5
  - 5,15,23,29

1.6 Determinants

학습 목표

- ❖행렬식(determinat) 정의 및 2x2 행렬의 행렬식 정의 이해
- ❖대각 행렬의 행렬식 연산 이해
- ❖행렬식 계산을 위한 행렬의 행조작 이론 이해
- ❖대형 행렬의 행렬식 계산 연습

2x2 행렬의 행렬식

DEFINITION 1 **Determinant of a 2×2 Matrix** The **determinant** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denoted by $|A|$ or $\det(A)$, is given by

$$|A| = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Using this terminology a 2×2 matrix is invertible if and only if its determinant is nonzero.

EXAMPLE 1 Find the determinant of the matrix.

a. $A = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$ **c.** $A = \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}$

Solution

a. $|A| = \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} = (3)(2) - (1)(-2) = 8$

b. $|A| = \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = (3)(2) - (5)(4) = -14$

c. $|A| = \begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix} = (1)(0) - (0)(-3) = 0$

3x3 행렬의 행렬식

DEFINITION 2 **Determinant of a 3×3 Matrix** The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The computation of the 3×3 determinant takes the form

$$|A| = a_{11} \begin{vmatrix} * & * & * \\ * & a_{22} & a_{23} \\ * & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} * & * & * \\ a_{21} & * & a_{23} \\ a_{31} & * & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} * & * & * \\ a_{21} & a_{22} & * \\ a_{31} & a_{32} & * \end{vmatrix}$$

where the first 2×2 determinant is obtained by deleting the first row and first column, the second by deleting the first row and second column, and the third by deleting the first row and third column.

3x3 행렬식 계산 예제

EXAMPLE 2 Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$$

Solution By Definition 2, the determinant is given by

$$\begin{aligned} \det(A) = |A| &= 2\begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix} \\ &= (2)[3-(-12)] - (1)(9-20) + (-1)(-9-5) \\ &= 30 + 11 + 14 \\ &= 55 \end{aligned}$$

Minor(소행렬식) 와 Cofactor(여인수)

DEFINITION 3 **Minors and Cofactors of a Matrix** If A is a square matrix, then the **minor** M_{ij}, associated with the entry a_{ij}, is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j from the matrix A. The **cofactor** of a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

For the matrix of Example 2, several minors are

$$M_{11} = \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} \qquad M_{12} = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} \qquad \text{and} \qquad M_{13} = \begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix}$$

Using the notation of Definition 3, the determinant of A is given by the *cofactor expansion*

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-1)^2(15) + 1(-1)^3(-11) - 1(-1)^4(-14) \\ &= 30 + 11 + 14 = 55 \end{aligned}$$

nxn 행렬의 행렬식

DEFINITION 4 **Determinant of a Square Matrix** If A is an $n \times n$ matrix, then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \sum_{k=1}^{n} a_{1k}C_{1k}$$

Similar to the situation for 3×3 matrices, the determinant of any square matrix can be found by expanding along any row or column.

THEOREM 12 Let A be an $n \times n$ matrix. Then the determinant of A equals the cofactor expansion along any row or any column of the matrix. That is, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}$$

삼각행렬(Triangular Matrx)

DEFINITION 5 **Triangular Matrices** An $m \times n$ matrix is **upper triangular** if $a_{ij} = 0$, for all $i > j$, and is **lower triangular** if $a_{ij} = 0$, for all $i < j$. A square matrix is a **diagonal** matrix if $a_{ij} = 0$, for all $i \neq j$.

Some examples of upper triangular matrices are

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and some examples of lower triangular matrices are

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

삼각 행렬의 행렬식

THEOREM 13 If A is an $n \times n$ triangular matrix, then the determinant of A is the product of the terms on the diagonal. That is,

$$\det(A) = a_{11} \cdot a_{22} \cdots a_{nn}$$

Proof We present the proof for an upper triangular matrix. The proof for a lower triangular matrix is identical. The proof is by induction on n. If $n = 2$, then $\det(A) = a_{11}a_{22} - 0$ and hence is the product of the diagonal terms.

Assume that the result holds for an $n \times n$ triangular matrix. We need to show that the same is true for an $(n+1) \times (n+1)$ triangular matrix A. To this end let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & a_{2,n+1} \\ 0 & 0 & a_{33} & \cdots & a_{3n} & a_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} & a_{n,n+1} \\ 0 & 0 & 0 & \cdots & 0 & a_{n+1,n+1} \end{bmatrix}$$

Using the cofactor expansion along row $n+1$, we have

$$\det(A) = (-1)^{(n+1)+(n+1)} a_{n+1,n+1} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

Since the determinant on the right is $n \times n$ and upper triangular, by the inductive hypothesis

$$\begin{aligned} \det(A) &= (-1)^{2n+2}(a_{n+1,n+1})(a_{11}a_{22}\cdots a_{nn}) \\ &= a_{11}a_{22}\cdots a_{nn}a_{n+1,n+1} \end{aligned}$$

행렬식 성질

THEOREM 14

Let A be a square matrix.

1. If two rows of A are interchanged to produce a matrix B, then $\det(B) = -\det(A)$.
2. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
3. If a row of A is multiplied by a real number α to produce a matrix B, then $\det(B) = \alpha\det(A)$.

행렬식 계산 예제

EXAMPLE 3 Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ -2 & -4 & 1 & -3 \end{bmatrix}$$

행렬식 추가 성질 (1)

THEOREM 15

Let A and B be $n \times n$ matrices and α a real number.

1. The determinant computation is multiplicative. That is,

$$\det(AB) = \det(A)\det(B)$$

2. $\det(\alpha A) = \alpha^n \det(A)$
3. $\det(A^t) = \det(A)$
4. If A has a row (or column) of all zeros, then $\det(A) = 0$.
5. If A has two equal rows (or columns), then $\det(A) = 0$.
6. If A has a row (or column) that is a multiple of another row (or column), then $\det(A) = 0$.

행렬식 추가 성질 (2)

THEOREM 16 A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof If the matrix A is invertible, then by Theorem 15,

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

Since the product of two real numbers is zero if and only if at least one of them is zero, we have $\det(A) \neq 0$ [also $\det(A^{-1}) \neq 0$].

To establish the converse, we will prove the contrapositive statement. Assume that A is not invertible. By the remarks at the end of Sec. 1.4, the matrix A is row equivalent to a matrix R with a row of zeros. Hence, by Theorem 14, there is some real number $k \neq 0$ such that $\det(A) = k\det(R)$, and therefore by Theorem 15, part 4,

$$\det(A) = k\det(R) = k(0) = 0$$

COROLLARY 1 Let A be an invertible matrix. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof If A is invertible, then as in the proof of Theorem 16, $\det(A) \neq 0$, $\det(A^{-1}) \neq 0$, and

$$\det(A)\det(A^{-1}) = 1$$

Therefore,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

인버터블 정방행렬의 성질

THEOREM 17

Let A be a square matrix. Then the following statements are equivalent.

1. The matrix A is invertible.
2. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$.
3. The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. The matrix A is row equivalent to the identity matrix.
5. The determinant of the matrix A is nonzero.

크래머 공식(Cramer's Rule)

THEOREM 18

Cramer's Rule Let A be an $n \times n$ invertible matrix, and let $\mathbf{b}$ be a column vector with n components. Let A_i be the matrix obtained by replacing the ith column of A with $\mathbf{b}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the unique solution to the linear system $A\mathbf{x} = \mathbf{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad \text{for} \qquad i = 1, 2, \ldots, n$$

Proof Let I_i be the matrix obtained by replacing the ith column of the identity matrix with $\mathbf{x}$. Then the linear system is equivalent to the matrix equation

$$AI_i = A_i \qquad \text{so} \qquad \det(AI_i) = \det(A_i)$$

By Theorem 15, part 1, we have

$$\det(A)\det(I_i) = \det(AI_i) = \det(A_i)$$

Since A is invertible, $\det(A) \neq 0$ and hence

$$\det(I_i) = \frac{\det(A_i)}{\det(A)}$$

Expanding along the ith row to find the determinant of I_i gives

$$\det(I_i) = x_i \det(I) = x_i$$

where I is the $(n-1) \times (n-1)$ identity. Therefore,

$$x_i = \frac{\det(A_i)}{\det(A)}$$

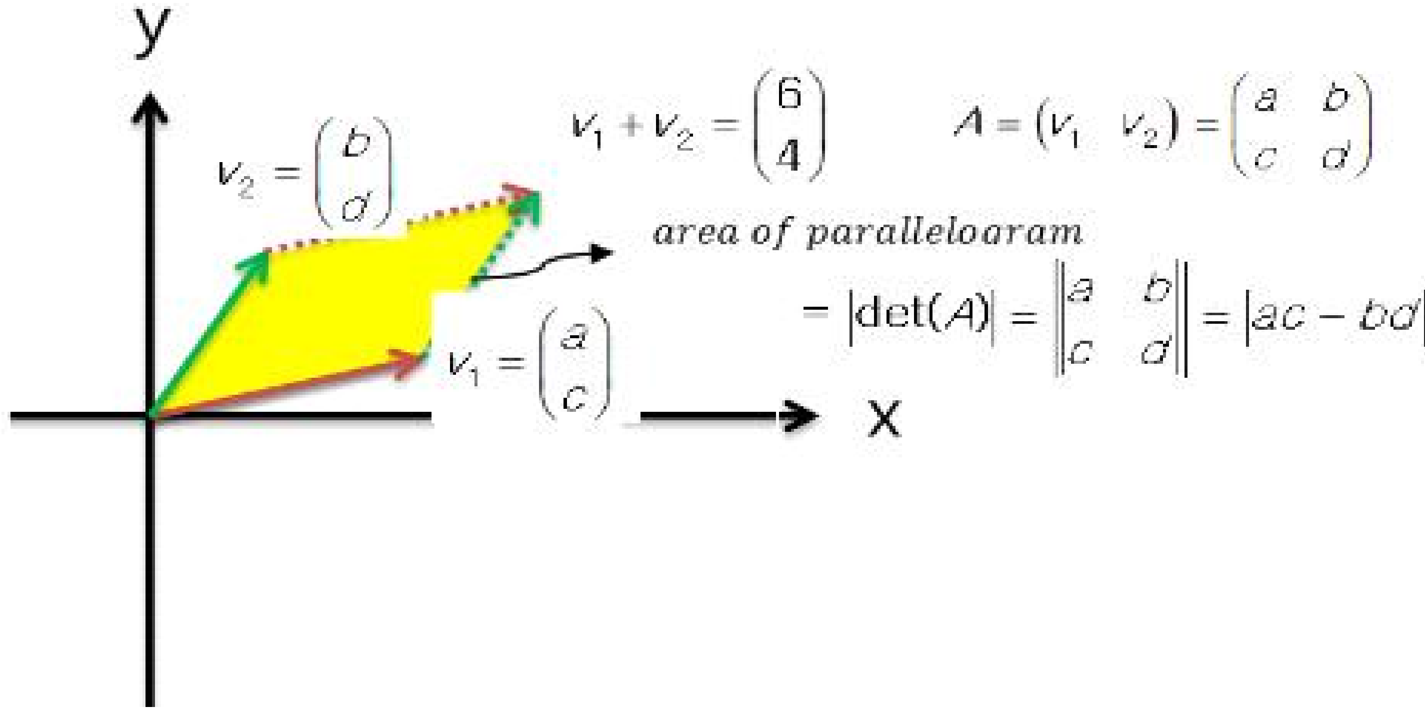
크래머 공식을 이용한 풀이 예제

EXAMPLE 7 Solve the linear system.

$$\begin{cases} 2x+3y-z=2 \\ 3x-2y+z=-1 \\ -5x-4y+2z=3 \end{cases}$$

행렬식 응용 사례 – 행렬식의 기하학적 해석

❖ 면적 및 체적 계산
https://angeloyeo.github.io/2019/08/06/determinant.html



1.6 절 요약 (1)

Fact Summary

Let A and B be $n \times n$ matrices.

1. $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$
2. The determinant of A can be computed by expanding along any row or column provided that the signs are adjusted using the pattern

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3. The matrix A is invertible if and only if $\det(A) \neq 0$.

1.6 절 요약 (2)

4. If A is a triangular matrix, then the determinant of A is the product of the diagonal terms.
5. If two rows of A are interchanged, the determinant of the resulting matrix is the negative of the determinant of A.
6. If a multiple of one row of A is added to another row, the determinant is not altered.
7. If one row of A is multiplied by a scalar c, the determinant of the resulting matrix is c times the determinant of A.
8. $\det(AB) = \det(A)\det(B)$, $\det(cA) = c^n \det(A)$, $\det(A^t) = \det(A)$
9. If A has a row or column of zeros, then $\det(A) = 0$.
10. If one row or column of A is a multiple of another row or column, then $\det(A) = 0$.
11. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Homework #1-6

- Exercise Set 1.6
  - 1,9,25,35,51

1.7 Elementary Matrices and *LU Factorization*

학습 목표

- ❖ (행) 기본 행렬 정의 및 행조작과의 관련 이해
- ❖ (행) 기본 행렬을 행렬 A 왼쪽에 곱했을 때, 행렬 A 의 행에 대한 오퍼레이션 결과 이해
- ❖ (행) 기본 행렬에 의한 행렬의 분해(factorization) 이해
- ❖ (행) 기본 행렬에 의한 행렬의 LU 분해(factorization) 이해

행렬 분해 활용

- ❖ 행렬의 분해(factorization/decomposition)는 정수의 소인수 분해와 유사한 점이 많다.
- ❖ 정수(integer) 의 소인수(prime number) 분해(factorization) 응용
  - ➢ 약수, 최대 공약수, 최소 공배수 등을 구하는 데 유용
  - ➢ 암호 알고리즘 (RSA) 에 유용하게 활용
- ❖ 행렬 분해 응용
  - ➢ 행렬 연산에 유용 ; Diagonalization(6장), Jordan Normal Form
  - ➢ LU 분해는 행렬 방정식 풀이에 활용
  - ➢ SVD 분해 (6장)

    추천 알고리즘 (recommendation by matrix factorization)

LU 분해 필요성 사례

$$\begin{cases} x_1 & = 3 \\ -x_1 + x_2 & = -1 \\ 2x_1 - x_2 + x_3 & = 5 \end{cases}$$

we obtain the solution $x_1 = 3$, $x_2 = 2$, and $x_3 = 1$. Thus, from a computational perspective, to find the solution of a linear system, it is desirable that the corresponding matrix be either upper or lower triangular.

In this section we show how, in certain cases, an $m \times n$ matrix A can be written as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix. We call this an **LU factorization of A**. For example, an LU factorization of the matrix $\begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix}$ is given by

$$\begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

with $L = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$. We also show in this section that when such a factorization of A exists, a process that involves both forward and back substitution can be used to find the solution to the linear system $A\mathbf{x} = \mathbf{b}$.

기본 행렬 및 행 조작

DEFINITION 1 **Elementary Matrix** An elementary matrix is any matrix that can be obtained from the identity matrix by performing a single elementary row operation.

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

행교환

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad E_1 A = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

어떤 행의 **k** 배 를 현 행에 더하기

$$E_2 A = \begin{bmatrix} 1 & 2 & 3 \\ 4+k & 5+2k & 6+3k \\ 7 & 8 & 9 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

한 행을 **c** 배함

$$E_3 A = \begin{bmatrix} 1 & 2 & 3 \\ 4c & 5c & 6c \\ 7 & 8 & 9 \end{bmatrix}$$

기본 행렬과 열 조작

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ 열교환

$$A\,E_1 = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 어떤 열의 **k** 배 를 현 열에 더하기

$$A\,E_2 = \begin{bmatrix} 1+2k & 2 & 3 \\ 4+5k & 5 & 6 \\ 7+8k & 8 & 9 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 한 열을 **c** 배 함

$$A\,E_3 = \begin{bmatrix} 1 & 2c & 3 \\ 4 & 5c & 6 \\ 7 & 8c & 9 \end{bmatrix}$$

행조작 과 기본 행렬 곱의 등가

THEOREM 19 Let A be an $m \times n$ matrix and E the elementary matrix obtained from the $m \times m$ identity matrix I by a single row operation $\mathcal{R}$. Denote by $\mathcal{R}(A)$ the result of performing the row operation on A. Then $\mathcal{R}(A) = EA$.

By repeated application of Theorem 19, a sequence of row operations can be performed on a matrix A by successively multiplying A by the corresponding elementary matrices. Specifically, let E_i be the elementary matrix corresponding to the row operation $\mathcal{R}_i$ with $1 \leq i \leq k$. Then

$$\mathcal{R}_k \cdots \mathcal{R}_2(\mathcal{R}_1(A)) = E_k \cdots E_2 E_1 A$$

기본행렬에 의한 행조작 예

EXAMPLE 1 Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Use elementary matrices to perform the row operations $\mathcal{R}_1: \ R_2 - 3R_1 \longrightarrow R_2$, $\mathcal{R}_2: \ R_3 + R_1 \longrightarrow R_3$, and $\mathcal{R}_3: \ R_3 + 3R_2 \longrightarrow R_3$.

Solution The elementary matrices corresponding to these row operations are given by

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

respectively, so that

$$E_3E_2E_1A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

The reader should check that the matrix on the right-hand side is equal to the result of performing the row operations above on A in the order given.

행 등가 행렬 필요충분조건

THEOREM 21 Let A and B be $m \times n$ matrices. The matrix A is row equivalent to B if and only if there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A$.

In light of Theorem 21, if A is row equivalent to B, then B is row equivalent to A. Indeed, if A is row equivalent to B, then

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

for some elementary matrices $E_1, E_2, \ldots, E_k$. Successively multiplying both sides of this equation by $E_k^{-1}, E_{k-1}^{-1}, \ldots,$ and E_1^{-1}, we obtain

$$A = E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B$$

Since each of the matrices $E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$ is an elementary matrix, B is row equivalent to A.

Theorem 22 uses elementary matrices to provide a characterization of invertible matrices.

기본 행렬의 인버터빌리티

THEOREM 20 Let E be an $n \times n$ elementary matrix. Then E is invertible. Moreover, its inverse is also an elementary matrix.

역행렬 존재 필요충분조건

THEOREM 22 An $n \times n$ matrix A is invertible if and only if it can be written as the product of elementary matrices.

LU 분해(Fctorization)

LU factorization. The details for solving a linear system using an LU factorization are presented later in this section. If A is an $n \times n$ matrix with an LU factorization given by $A = LU$, then L and U are also $n \times n$. See Fig. 1. Then by Theorem 13 of Sec. 1.6, the determinant of A is given by

$$\det(A) = (\ell_{11} \cdots \ell_{nn})(u_{11} \cdots u_{nn})$$

where ℓ_{ii} and u_{ii} are the diagonal entries of L and U, respectively. If this determinant is not zero, then by Theorem 9 of Sec. 1.4 the inverse of the matrix A is given by

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

To describe the process of obtaining an LU factorization of an $m \times n$ matrix A, suppose that A can be reduced to an upper triangular matrix by a sequence of row operations which correspond to lower triangular elementary matrices. That is, there exist lower triangular elementary matrices $L_1, L_2, \ldots, L_k$ such that

$$L_k L_{k-1} \cdots L_1 A = U$$

Since each of the matrices L_i with $1 \le i \le k$ is invertible, we have

$$A = L_1^{-1} L_2^{-1} \cdots L_k^{-1} U$$

By Theorem 20, $L_1^{-1}, L_2^{-1}, \ldots,$ and L_k^{-1} are elementary matrices. They are also lower triangular. Now let $L = L_1^{-1} L_2^{-1} \cdots L_k^{-1}$. Observe that L is lower triangular as it is the product of lower triangular matrices. The desired factorization is thus given by $A = LU$.

LU 분해 예제

EXAMPLE 2 Find an LU factorization of the matrix

$$A = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix}$$

행렬의LU 분해 충분조건

THEOREM 23 Let A be an $m \times n$ matrix that can be reduced to the upper triangular matrix U without row interchanges by means of the $m \times m$ lower triangular matrices L_1, $L_2, \ldots L_k$. If $L = L_1^{-1} L_2^{-1} \cdots L_k^{-1}$, then A has the LU factorization $A = LU$.

LU 분해 예제

EXAMPLE 3 Find an LU factorization of the matrix

$$A = \begin{bmatrix} 1 & -3 & -2 & 0 \\ 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & 2 \end{bmatrix}$$

LU 분해를 이용한 선형 시스템 풀이

The following steps summarize the procedure for solving the linear system $A\mathbf{x} = \mathbf{b}$ when A admits an LU factorization.

1. Use Theorem 23 to write the linear system $A\mathbf{x} = \mathbf{b}$ as $L(U\mathbf{x}) = \mathbf{b}$.
2. Define the vector $\mathbf{y}$ by means of the equation $U\mathbf{x} = \mathbf{y}$.
3. Use forward substitution to solve the system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
4. Use back substitution to solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Note that $\mathbf{x}$ is the solution to the original linear system.

PLU 분해(factorization)

We have seen that a matrix A has an LU factorization provided that it can be row-reduced without interchanging rows. We conclude this section by noting that when row interchanges are required to reduce A, a factorization is still possible. In this case the matrix A can be factored as $A = PLU$, where P is a *permutation matrix*, that is, a matrix that results from interchanging rows of the identity matrix. As an illustration, let

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

The matrix A can be reduced to

$$U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

by means of the row operations $\mathcal{R}_1$: $R_1 \leftrightarrow R_3$, $\mathcal{R}_2$: $-R_1 + R_2 \longrightarrow R_2$, and $\mathcal{R}_3$: $-R_2 + R_3 \longrightarrow R_3$. The corresponding elementary matrices are given by

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

PLU 분해 예제

Observe that the elementary matrix E_1 is a permutation matrix while E_2 and E_3 are lower triangular. Hence,

$$\begin{aligned} A &= E_1^{-1}\left(E_2^{-1}E_3^{-1}\right)U \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{bmatrix} \\ &= PLU \end{aligned}$$

1.7 절 요약

Fact Summary

1. A row operation on a matrix A can be performed by multiplying A by an elementary matrix.
2. An elementary matrix is invertible, and the inverse is an elementary matrix.
3. An $n \times n$ matrix A is invertible if and only if it is the product of elementary matrices.
4. An $m \times n$ matrix A has an LU factorization if it can be reduced to an upper triangular matrix with no row interchanges.
5. If $A = LU$, then L is invertible.
6. An LU factorization of A provides an efficient method for solving $A\mathbf{x} = \mathbf{b}$.

Homework #1-7

- Exercise Set 1.7
  - 19,23,29

1.8 Applications of Systems of Linear Equations

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응용 사례

- 선형대수는 공학 전분야에 걸쳐 응용되고 있음.
- 회로이론, 데이터 사이언스, AI, 빅데이터 분야에서 많은 응용 사례가 있음.
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- 선형 회귀 (6장)
- 직교 함수 급수 근사화 (6장)
- 푸리에 급수 근사화 (6장)

Homework #1-8

- Exercise Set 1.8
  - 17,19,21