2150691501 선형 대수

강의 안내

강의 안내 (1/2)

2150691501/2150691502 Linear Algebra (선형대수)

- 강의 웹사이트
  - 학교 스마트캠퍼스

- 강의 목표
  - 공학에서 가장 중요하고 유용하게 사용되는 선형 시스템 기반 수학 이론 및 파이썬으로 선형 대수 문제 풀이 방법 학습
    - 행렬, 행렬식의 특성 이해
    - 벡터공간의 기본 성질과 그 응용방법 습득
    - 기저벡터, 좌표, 좌표 변환, 선형 변환 의 행렬 표현 이해
    - 전공분야 관련 문제 풀이에 선형대수 지식을 적용할 수 있는 능력 배양
    - 파이썬으로 선형대수 문제 풀는 방법 학습

강의 안내 (2/2)

2150691501 Linear Algebra (선형대수)

- 교재 및자료
  - 주교재 ;
    "Introduction to Linear Algebra with Application,"
    Jim Defranza, Daniel Gagliardi, McGraw-Hill
  - 부교재
    https://pabloinsente.github.io/intro-linear-algebra

- 수업 진행 방법
  - (오프라인)
    - 빔프로젝터를 이용한 강의, 질의/응답 위주의 강의를 통한 이론 이해
    - 발표
  - (온라인) 동영상 강의를 통해, 문제 풀이 실습, 파이썬 문제 풀이

- 평가방법
  - 중간 평가 30%, 기말평가 40%
    - 발표 및 수업 참여 10%, 과제 20%

2150691501 선형 대수

1 Week; 선형 방정식과 행렬

1주 학습 목표

- ❖ 선형 방정식 시스템 이해
- ❖ 선형 방정식 시스템 체계적 풀이 방법 학습 – 가우스 소거법 이해
- ❖ 확장 행렬 및 에켈론 형식 이해
- ❖ 행 조작을 통한 확장 행렬의 에켈론 형식으로의 변환 및 이를 이용한 선형 방정식 시스템 풀이 방법 학습
- ❖ 행렬 연산 정리 학습
- ❖ 전치 행렬 이해
- ❖ 정방 행렬의 역행렬 정의 및 구하는 방법 학습

Contents of 1 week Lecture

1.1 Systems of Linear Equations

학습 목표

- ❖ 선형 방정식 시스템 이해 및 해 의미 이해
- ❖ 각 선형 방정식의 기하학적 의미
- ❖ 선형 방정식 시스템 해의 기하학적 의미
- ❖ 선형 시스템 해 경우수 이해 (해가 유일하다, 무수히 많다, 해가 없다) 및 해 경우수의 기하학적 해석 이해
- ❖ 선형 시스템의 시스템적 풀이 방법(가우스 소거법) 이해

선형 방정식 시스템 정의

❖ 다음 형태의 방정식 모임을 n 개 변수의 m 개 선형 방정식의 시스템이라 함. 또는 mxn 선형 시스템이라고도 함 (또는 n 원 1차 연립방정식).

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

예1 ; 2개 변수의 2개 선형 방정식 시스템

$$\begin{cases} x + y = 1 \\ -x + y = 1 \end{cases}$$

❖ 예2 ; 4개 변수의 3개 선형 방정식 시스템(3x4 선형 시스템)

$$\begin{cases} -2x_1 + 3x_2 + x_3 - x_4 = -2 \\ x_1 + x_3 - 4x_4 = 1 \\ 3x_1 - x_2 - x_4 = 3 \end{cases}$$

선형 방정식 해 정의 및 기하학적 해석

❖ $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ (*) 에 대해 $x_1 = s_1,\ x_2 = s_2,\ \cdots,\ x_n = s_n$ 를 선형방정식 (*) 에 대입했을 때,
$a_1s_1 + a_2s_2 + \cdots a_ns_n = b$ 을 만족하면, $(s_1,\ s_2,\ \cdots,\ s_n)$ 을 식(*) 의 해라 한다.

예; 2x+3y=8 에서 x=1, y=2 는 2*1+3*2=8 을 만족하므로 해이다.

❖ $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ (*) 해 집합의 기하학적 해석

선형 방정식 (*) 을 만족하는 해 $x_1 = s_1,\ x_2 = s_2,\ \cdots,\ x_n = s_n$ 의 집합은 n 차원 실수 벡터 공간에서 벡터 $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ 에 수직인 초평면(hyperplane) 이다.

(1.3절에서 자세히 설명)

초평면 (hyperplane) 정의

- ❖ $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ **를 만족하는 해** $(x_1, \ x_2, \cdots, x_n)$ **집합은** n **차원 실수 벡터 공간에서의 초평면**(hyperplane)**이다**(1.3**절에 자세한 설명**).
- ❖ **초평면**(hyperplane)**은 기하학적 개체**(geometric entity)**로 이 개체가 위치하는 공간**(ambient space) **보다 차원이 하나 작은 부분 공간이다**.

  예1; 3차원 공간의 초평면은 2차원 평면

  ➢ 예2; 2차원 공간의 초평면은 1차원 직선.

$$\begin{cases} x + y = 1 \\ -x + y = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} x + y = 1 \\ 2y = 2 \end{cases}$$

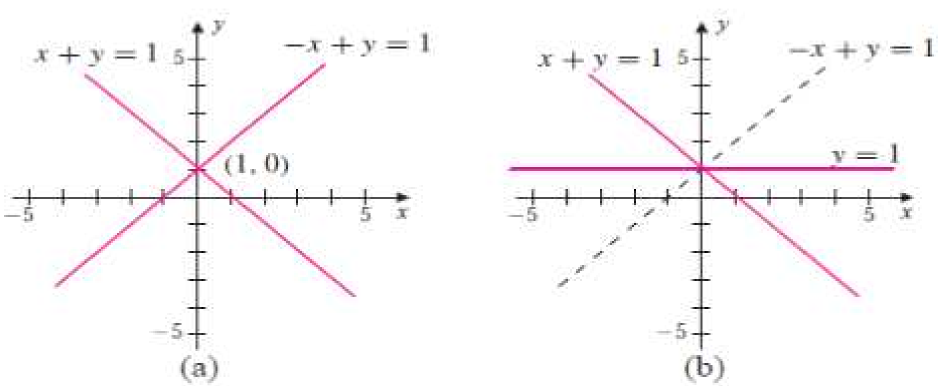


Figure 2

선형 방정식 시스템의 해 정의

❖ 선형 방정식 시스템 (식 (1))

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

에 대해 $x_1 = s_1,\ x_2 = s_2,\ \cdots,\ x_n = s_n$ 를
식 (1) 의 각 선형 방정식에 대입했을 때 해당 방정식을 모두 만족하면, 즉,

$$\begin{cases} a_{11}s_1 + a_{12}s_2 + \cdots + a_{1n}s_n = b_1 \\ a_{21}s_1 + a_{22}s_2 + \cdots + a_{2n}s_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad = \ \vdots \\ a_{m1}s_1 + a_{m2}s_2 + \cdots a_{mn}s_n = b_m \end{cases}$$

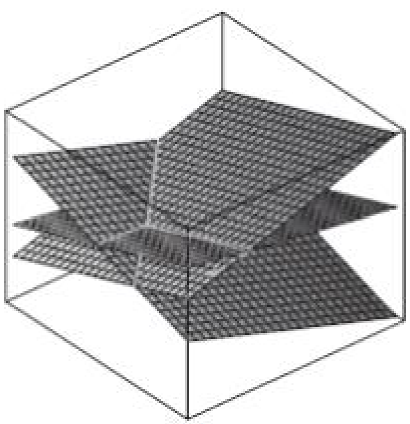
$(s_1,\ s_2,\ \cdots,\ s_n)$ 을 선형 방정식 시스템 (식(1)) 의 해라 한다.

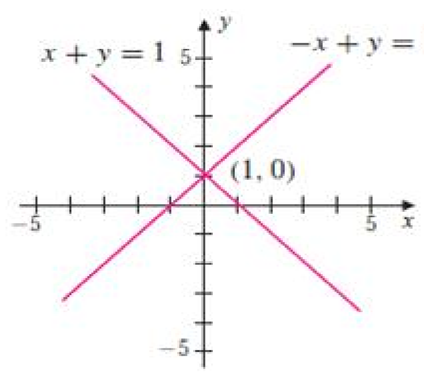
선형 방정식 시스템 해의 기하학적 의미

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

❖ 상기 식(1) 선형방정식 시스템의 각 선형 방정식의 해 집합은 n 차원 실수 벡터 공간에서의 (n-1) 차원 초평면이다. 따라서, 식(1)의 선형방정식 시스템의 해는 각 선형방정식을 만족해야 하므로, 각 선형 방정식 해 집합인 초평면들의 교집합이다.

$$\begin{cases} x + y = 1 \\ -x + y = 1 \end{cases}$$





선형 방정식 시스템 해 경우수의 기하학적 이해

❖ 선형방정식 시스템 (식 (1)) 을 만족하는 해 집합은 식 (1)의 각 선형 방정식의 해집합인 초평면들의 교집합이다.

- ➢ 면이거나, 공집합이다.변수가 n 인 선형 방정식의 해집합은 (n-1) 차원 초평면이다.
- ➢ 2개의 (n-1) 차원 초평면들의 교집합은 (n-2) 차원 초평면이거나, 공집합이거나, (n-1) 차원 초평면이다.

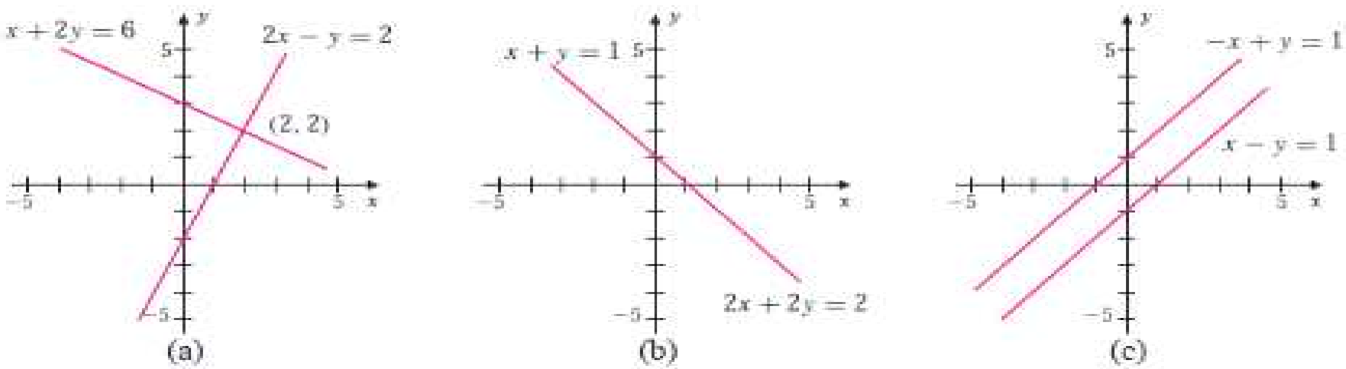


Figure 1

❖ n 개 변수 선형 방정식 시스템이 유일한 해 (n 차원 공간의 벡터(한 점))를 갖기 위해 필요조건은 n 개 선형 방정식을 가져야 한다. 즉, m=n

선형 방정식 시스템의 풀이 방법 복습

❖ 2원(2 변수) 1차 연립방정식 경우 (이미 중학교에서 학습)
(https://calcproject.tistory.com/45)

[중2-1] 1. 연립방정식의 풀이 : 가감법, 대입법 (개념+수학문제)

1. 가감법 : 두 식에 적당한 수를 곱하거나 나누어 한 문자를 소거하는 방법

$$\begin{cases} x+y=5 \ldots\ldots (ㄱ) \\ 2x-y=8 \ldots\ldots (ㄴ) \end{cases}$$ 이라면,

(ㄱ)식의 양변에 2를 곱합니다.

$$\begin{cases} 2x+2y=10 \ldots\ldots 2\times(ㄱ) \\ 2x-y=8 \ldots\ldots (ㄴ) \end{cases}$$

이때, $2\times$(ㄱ) $-$ (ㄴ)을 계산하면,

$(2x+2y)-(2x-y)=10-8$

$3y=2$ (ㄷ)

$\therefore y=\frac{2}{3}$ 입니다.

y를 (ㄱ)에 대입하면,

$x+\frac{2}{3}=5$

$x=5-\frac{2}{3}=\frac{15}{3}-\frac{2}{3}=\frac{13}{3}$

$\therefore x=\frac{13}{3}, \; y=\frac{2}{3}$

선형대수; Gauss Elimination)

$$\begin{cases} x+y=5 \ldots\ldots (ㄱ) \\ 2x-y=8 \ldots\ldots (ㄴ) \end{cases}$$

계단식행태로 변환

$$\begin{cases} x+y=5 \quad \ldots (ㄱ) \\ 3y=2 \quad \ldots (ㄷ) \end{cases}$$

후방대입법

(Back Substitution)

(ㄷ)에서

$y=2/3$ .... (ㄹ)

이를 (ㄱ)에 대입하면,

선형 방정식 시스템의 시스템적 풀이 - 가우스 소거법

The elimination method, also called *Gaussian elimination*, is an algorithm used to solve linear systems. To describe this algorithm, we first introduce the *triangular form* of a linear system.

An $m \times n$ linear system is in **triangular form** provided that the coefficients $a_{ij} = 0$ whenever $i > j$. In this case we refer to the linear system as a **triangular system**. Two examples of triangular systems are

$$\begin{cases} x_1 - 2x_2 + x_3 = -1 \\ x_2 - 3x_3 = 5 \\ x_3 = 2 \end{cases} \quad \text{and} \quad \begin{cases} x_1 + x_2 - x_3 - x_4 = 2 \\ x_2 - x_3 - 2x_4 = 1 \\ 2x_3 - x_4 = 3 \end{cases}$$

When a linear system is in triangular form, then the solution set can be obtained using a technique called **back substitution**. To illustrate this technique, consider the linear system given by

$$\begin{cases} x_1 - 2x_2 + x_3 = -1 \\ x_2 - 3x_3 = 5 \\ x_3 = 2 \end{cases}$$

등가 선형 시스템

DEFINITION 2 **Equivalent Linear Systems** Two linear systems are **equivalent** if they have the same solutions

For example, the system

$$\begin{cases} x_1 - 2x_2 + x_3 = -1 \\ 2x_1 - 3x_2 - x_3 = 3 \\ x_1 - 2x_2 + 2x_3 = 1 \end{cases}$$

has the unique solution $x_1 = 19$, $x_2 = 11$, and $x_3 = 2$, so the linear systems

$$\begin{cases} x_1 - 2x_2 + x_3 = -1 \\ \qquad x_2 - 3x_3 = 5 \\ \qquad\qquad x_3 = 2 \end{cases} \quad \text{and} \quad \begin{cases} x_1 - 2x_2 + x_3 = -1 \\ 2x_1 - 3x_2 - x_3 = 3 \\ x_1 - 2x_2 + 2x_3 = 1 \end{cases}$$

are equivalent.

등가 선형 시스템 을 위한 방정식 오퍼레이션

THEOREM 1 Let

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

be a linear system. Performing any one of the following operations on the linear system produces an equivalent linear system.

1. Interchanging any two equations.
2. Multiplying any equation by a nonzero constant.
3. Adding a multiple of one equation to another.

가우스 소거법 예제 (1)

EXAMPLE 1 Use the elimination method to solve the linear system.

$$\begin{cases} x + y = 1 \\ -x + y = 1 \end{cases}$$

Solution Adding the first equation to the second gives the equivalent system

$$\begin{cases} x + y = 1 \\ 2y = 2 \end{cases}$$

From the second equation, we have $y = 1$. Using back substitution gives $x = 0$. The graphs of both systems are shown in Fig. 2. Notice that the solution is the same in both, but that adding the first equation to the second rotates the line $-x + y = 1$ about the point of intersection.

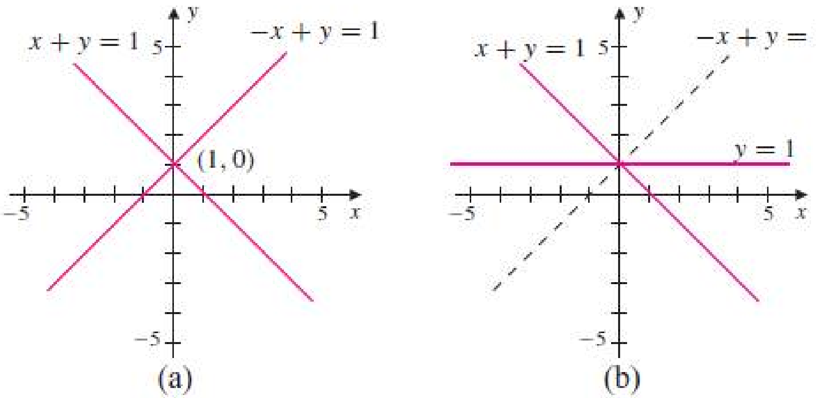


Figure 2

가우스 소거법 예제 (2)

EXAMPLE 2 Solve the linear system.

$$\begin{cases} x + y + z = 4 \\ -x - y + z = -2 \\ 2x - y + 2z = 2 \end{cases}$$

Solution To convert the system to an equivalent triangular system, we first eliminate the variable x in the second and third equations to obtain

가우스 소거법 예제 (3)

EXAMPLE 3 Solve the linear system.

$$\begin{cases} 4x_1 - 8x_2 - 3x_3 + 2x_4 = 13 \\ 3x_1 - 4x_2 - x_3 - 3x_4 = 5 \\ 2x_1 - 4x_2 - 2x_3 + 2x_4 = 6 \end{cases}$$

가우스 소거법 예제 (4)

EXAMPLE 5 Solve the linear system.

$$\begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 2x_1 + x_2 = 2 \\ x_1 + 8x_2 - x_3 = 3 \\ -x_1 - 5x_2 - 12x_3 = 4 \end{cases}$$

Solution To convert the linear system to an equivalent triangular system, we will eliminate the first terms in equations 2 through 4, and then the second terms in equations 3 and 4, and then finally the third term in the fourth equation. This is accomplished by using the following operations.

$$\begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 2x_1 + x_2 = 2 \\ x_1 + 8x_2 - x_3 = 3 \\ -x_1 - 5x_2 - 12x_3 = 4 \end{cases} \quad \begin{array}{r} -2E_1 + E_2 \to E_2 \\ -E_1 + E_3 \to E_3 \\ E_1 + E_4 \to E_4 \end{array} \quad \to \begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 3x_2 - 4x_3 = -8 \\ 9x_2 - 3x_3 = -2 \\ -6x_2 - 10x_3 = 9 \end{cases}$$

$$\begin{array}{r} -3E_2 + E_3 \to E_3 \\ 2E_2 + E_4 \to E_4 \end{array} \quad \to \begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 3x_2 - 4x_3 = -8 \\ 9x_3 = 22 \\ -18x_3 = -7 \end{cases}$$

$$2E_3 + E_4 \to E_4 \quad \to \begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 3x_2 - 4x_3 = -8 \\ 9x_3 = 22 \\ 0 = -37 \end{cases}$$

The last equation of the final system is an impossibility, so the original linear system is inconsistent and has no solution.

1.1 절 요약

Fact Summary

1. A $m \times n$ linear system has a unique solution, infinitely many solutions, or no solutions.
2. Interchanging any two equations in a linear system does not alter the set of solutions.
3. Multiplying any equation in a linear system by a nonzero constant does not alter the set of solutions.
4. Replacing an equation in a linear system with the sum of the equation and a scalar multiple of another equation does not alter the set of solutions.
5. Every linear system can be reduced to an equivalent triangular linear system.

Homework #1-1

- Exercise Set 1.1
  - 1,11,25,37,41

1.2 Matrices and Elementary Row Operations

SOONGSIL UNIVERSITY

학습 목표

- ❖ 선형 방정식 시스템의 확장 행렬 표현
- ❖ 확장행렬 행조작에 의한 선형 방정식 시스템 해 풀이 학습
- ❖ 행 에켈론 형식 이해
- ❖ 행 에켈론 형식에 의한 선형 방정식 시스템 해 풀이

선형방정식 시스템과 확장행렬

$$\begin{cases} -4x_1 + 2x_2 \qquad\quad - 3x_4 = 11 \\ 2x_1 - x_2 - 4x_3 + 2x_4 = -3 \\ \qquad 3x_2 \qquad\quad - x_4 = 0 \\ -2x_1 \qquad\qquad\quad + x_4 = 4 \end{cases}$$

$$\left[\begin{array}{cccc|c} -4 & 2 & 0 & -3 & 11 \\ 2 & -1 & -4 & 2 & -3 \\ 0 & 3 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 & 4 \end{array}\right]$$

Augmented matrix of the linear system

$$\begin{bmatrix} -4 & 2 & 0 & -3 \\ 2 & -1 & -4 & 2 \\ 0 & 3 & 0 & -1 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

Coefficient matrix

가우스 소거법의 동등 확장행렬에서의 행조작

The method of elimination on a linear system is equivalent to performing similar operations on the rows of the corresponding augmented matrix. The relationship is illustrated below:

Linear system	Corresponding augmented matrix
$\begin{cases} x+y-z=1 \\ 2x-y+z=-1 \\ -x-y+3z=2 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & -1 \\ -1 & -1 & 3 & 2 \end{array}\right]$
Using the operations $-2E_1+E_2 \to E_2$ and $E_1+E_3 \to E_3$, we obtain the equivalent triangular system	Using the operations $-2R_1+R_2 \to R_2$ and $R_1+R_3 \to R_3$, we obtain the equivalent augmented matrix
$\begin{cases} x+y-z=1 \\ -3y+3z=-3 \\ 2z=3 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 2 & 3 \end{array}\right]$

확장행렬 행조작에 의한 선형방정식 시스템 해 풀이

THEOREM 2 Any one of the following operations performed on the augmented matrix, corresponding to a linear system, produces an augmented matrix corresponding to an equivalent linear system.

1. Interchanging any two rows.
2. Multiplying any row by a nonzero constant.
3. Adding a multiple of one row to another.

Solving Linear Systems with Augmented Matrices

The operations in Theorem 2 are called **row operations**. An $m \times n$ matrix A is called **row equivalent** to an $m \times n$ matrix B if B can be obtained from A by a sequence of row operations.

The following steps summarize a process for solving a linear system.

1. Write the augmented matrix of the linear system.
2. Use row operations to reduce the augmented matrix to triangular form.
3. Interpret the final matrix as a linear system (which is equivalent to the original).
4. Use back substitution to write the solution.

Example 1 illustrates how we can carry out steps 3 and 4.

확장행렬 동등 선형방정식 시스템 해

EXAMPLE 1 Given the augmented matrix, find the solution of the corresponding linear system.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ **b.** $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$ **c.** $\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 0 & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution

a. Reading directly from the augmented matrix, we have $x_3 = 3, x_2 = 2$, and $x_1 = 1$. So the system is consistent and has a unique solution.

b. In this case the solution to the linear system is $x_4 = 3, x_2 = 1 + x_3$, and $x_1 = 5$. So the variable x_3 is free, and the general solution is $S = \{(5, 1+t, t, 3) \mid t \in \mathbb{R}\}$.

c. The augmented matrix is equivalent to the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 1 \\ \quad\;\; 3x_2 - x_3 \qquad\;\; = 1 \end{cases}$$

Using back substitution, we have

$$x_2 = \frac{1}{3}(1 + x_3) \qquad \text{and} \qquad x_1 = 1 - 2x_2 - x_3 + x_4 = \frac{1}{3} - \frac{5}{3}x_3 + x_4$$

So the variables x_3 and x_4 are free, and the two-parameter solution set is given by

$$S = \left\{ \left(\frac{1}{3} - \frac{5s}{3} + t, \frac{1}{3} + \frac{s}{3}, s, t \right) \middle| \, s, t \in \mathbb{R} \right\}$$

선형방정식 시스템의 확장행렬 행조작에 의한 해 풀이 예제

EXAMPLE 2 Write the augmented matrix and solve the linear system.

$$\begin{cases} x - 6y - 4z = -5 \\ 2x - 10y - 9z = -4 \\ -x + 6y + 5z = 3 \end{cases}$$

Solution To solve this system, we write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{array}\right]$$

where we have shaded the entries to eliminate. Using the procedure described above, the augmented matrix is reduced to triangular form as follows:

$$\left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{array}\right] \quad \begin{array}{r} -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array} \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The equivalent triangular linear system is

$$\begin{cases} x - 6y - 4z = -5 \\ 2y - z = 6 \\ z = -2 \end{cases}$$

which has the solution $x = -1$, $y = 2$, and $z = -2$.

Echelon Form

DEFINITION 2 **Echelon Form** An $m \times n$ matrix is in **row echelon form** if

1. Every row with all 0 entries is below every row with nonzero entries.
2. If rows $1, 2, \ldots, k$ are the rows with nonzero entries and if the leading nonzero entry (**pivot**) in row i occurs in column c_i, for $1, 2, \ldots, k$, then $c_1 < c_2 < \cdots < c_k$.

The matrix is in **reduced row echelon form** if, in addition,

3. The first nonzero entry of each row is a 1.
4. Each column that contains a pivot has all other entries 0.

The process of transforming a matrix to reduced row echelon form is called *Gauss-Jordan elimination.*

Here are three additional matrices that *are* in reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and two that are *not* in reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & -5 & 6 \\ 0 & 1 & 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

축약 행에켈론 형식 이용 풀이 예제(1)

EXAMPLE 3 Solve the linear system by transforming the augmented matrix to reduced row echelon form.

$$\begin{cases} x_1 - x_2 - 2x_3 + x_4 = 0 \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -6 \\ -x_1 + 2x_2 + x_3 + 3x_4 = 2 \\ x_1 + x_2 - x_3 + 2x_4 = 1 \end{cases}$$

축약 행에켈론 형식 이용 풀이 예제(2)

EXAMPLE 4 Solve the linear system.

$$\begin{cases} 3x_1 - x_2 + x_3 + 2x_4 = -2 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 - 3x_2 + 2x_3 - 4x_4 = -6 \end{cases}$$

축약 행에켈론 형식 이용 풀이 예제(3)

EXAMPLE 5 Solve the linear system.

$$\begin{cases} x+y+z=4 \\ 3x-y-z=2 \\ x+3y+3z=8 \end{cases}$$

행에켈론 형식에 의한 해 경우 수 설명

▶ mxn 선형방정식 시스템의 확장 행렬이 에켈론 형식으로 정리된 상태에서

1) 유일해의 경우
   - 에켈론 형식 확장 행렬의 계수 행렬에서 0 행벡터 가 아닌 행의 갯수가 n 임.

2) 해가 없는 경우
   - 행에켈론 형식의 확장행렬의 계수행렬의 0 행벡터 인 확장행렬 행의 마지막 요소가 0 이 아님.

3) 해가 무수히 많은 경우
   - 행에켈론 형식의 확장행렬에서 0 행벡터가 있음.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2 절 요약

Fact Summary

1. Altering an augmented matrix by interchanging two rows, or multiplying a row by a nonzero constant, or replacing a row with the sum of the same row and a scalar multiple of another row does not alter the set of solutions of the corresponding linear system.
2. If an augmented matrix is row-reduced to triangular form, the coefficient matrix has a row of zeros, and the corresponding augmented term is not zero, then the linear system has no solutions.
3. Every matrix has a unique reduced row echelon form.
4. If the augmented matrix of an $n \times n$ linear system is row-reduced to triangular form and the coefficient matrix has no rows of zeros, then the linear system has a unique solution.
5. If the augmented matrix of an $n \times n$ linear system is row-reduced to triangular form, the coefficient matrix has rows of zeros, and each corresponding augmented term is 0, then the linear system has infinitely many solutions.

Homework #1-2

- Exercise Set 1.2
  - 1,9,25,35,49

1.3 matrix Algebra

SOONGSIL UNIVERSITY

학습 목표

- 행렬 정의, 행렬 행벡터 및 열벡터 정의 이해
- 행렬 덧셈, 스칼라 곱셈 연산 이해
- 벡터 스칼라 곱 정의 이해
- 벡터의 크기(norm) 이해
- 벡터 스칼라곱의 기하학적 해석 이해
- 행렬 곱셈 연산 학습
- 행렬 연산 및 그 성질 이해
- 행렬의 전치 및 벡터의 전치 이해
- 선형 방정식의 기하학적 해석 이해

행렬(matrix) 정의 및 표기

❖ m x n 행렬은 m*n 개의 요소 (숫자 나 식) 를 가지면서 요소가 m 개의 행, n 개의 열로 배치된 배열을 말한다.

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 5 & 7 & 11 \\ 2 & 3 & 22 \end{bmatrix}$$

$$\begin{array}{lll} a_{11} = -2 & a_{12} = 1 & a_{13} = 4 \\ a_{21} = 5 & a_{22} = 7 & a_{23} = 11 \\ a_{31} = 2 & a_{32} = 3 & a_{33} = 22 \end{array}$$

$$\begin{array}{cc} & \text{Column } j \\ & \downarrow \\ \text{Row } i \longrightarrow & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \end{array}$$

Figure 1

행렬의 요소(entry) 는 Figure 1 에서 처럼 , 해당 위치의 행 및 열의 인덱스를 이용하여 표기

- 소문자 : a_{ij} (행렬 A에서 i 행, j 열에 있는 요소)

행렬 내 특정 행, 열 표기

- i 번째 행 ; $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$
- e번째 열 ; $A^j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdots \\ a_{mj} \end{bmatrix}$

벡터, 열벡터, 행벡터, 동등 행렬

A **vector** is an $n \times 1$ matrix. The entries of a vector are called its **components**. For a given matrix A, it is convenient to refer to its *row vectors* and its *column vectors*. For example, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & -1 & 2 \end{bmatrix}$$

Then the column vectors of A are

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

while the row vectors of A, written vertically, are

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Two $m \times n$ matrices A and B are **equal** if they have the same number of rows and columns and their corresponding entries are equal. Thus, $A = B$ if and only if $a_{ij} = b_{ij}$, for $1 \le i \le m$ and $1 \le j \le n$. Addition and scalar multiplication of matrices are also defined componentwise.

행렬 덧셈 및 스칼라 곱셈

DEFINITION 1 **Addition and Scalar Multiplication** If A and B are two $m \times n$ matrices, then the **sum** of the matrices $A + B$ is the $m \times n$ matrix with the ij term given by $a_{ij} + b_{ij}$. The **scalar product** of the matrix A with the real number c, denoted by cA, is the $m \times n$ matrix with the ij term given by ca_{ij}.

EXAMPLE 1 Perform the operations on the matrices

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & -1 \\ -3 & 6 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & -1 \\ 3 & 5 & 6 \\ 4 & 2 & 1 \end{bmatrix}$$

a. $A + B$ b. $2A - 3B$

행열 덧셈 및 스칼라 곱셈의 성질

THEOREM 4 **Properties of Matrix Addition and Scalar Multiplication** Let A, B, and C be $m \times n$ matrices and c and d be real numbers.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $c(A + B) = cA + cB$
4. $(c + d)A = cA + dA$
5. $c(dA) = (cd)A$
6. The $m \times n$ matrix with all zero entries, denoted by $\mathbf{0}$, is such that $A + \mathbf{0} = \mathbf{0} + A = A$.
7. For any matrix A, the matrix $-A$, whose components are the negative of each component of A, is such that $A + (-A) = (-A) + A = \mathbf{0}$.

Scalar product of vectors

DEFINITION 2 **Dot Product of Vectors** Given two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

the **dot product** is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

❖ 벡터의 도트 프로덕트는 스칼라곱(scalar product) 이라 고도 하며, $\langle \mathbf{u},\mathbf{v} \rangle$ 라고 표기하기도 한다. 즉, $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u},\mathbf{v} \rangle$

Observe that the dot product of two vectors is a scalar. For example,

$$\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix} = (2)(-5) + (-3)(1) + (-1)(4) = -17$$

Length(norm) of a Vector

Length of a Vector in $\mathbb{R}^n$ The **length** (or **norm**) of a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in $\mathbb{R}^n$, denoted by $\|\mathbf{v}\|$, is defined as

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= \sqrt{\mathbf{v} \cdot \mathbf{v}} \end{aligned}$$

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

Distance between Vectors

Distance Between Vectors in $\mathbb{R}^n$ Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

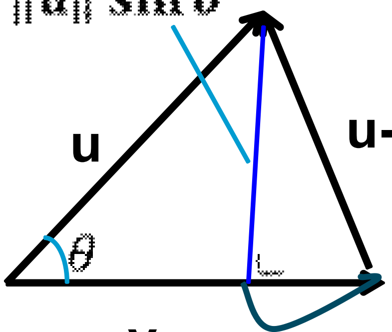
be vectors in $\mathbb{R}^n$. The distance between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

벡터 스칼라곱의 기하학적 해석(6장 참조)

❖ 앞의 벡터간 거리 식에서

$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}$$
$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$



❖ 왼쪽 그림에서 피타고라스 정리를 적용하면

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2\sin^2\theta + \|\,\|\mathbf{v}\| - \|\mathbf{u}\|\cos\theta\,\|^2$$
$$= \|\mathbf{u}\|^2\sin^2\theta + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2\cos^2\theta$$
$$- 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

$$\boxed{\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}$$

행렬 곱셈 정의

DEFINITION 3 **Matrix Multiplication** Let A be an $m \times n$ matrix and B an $n \times p$ matrix; then the product AB is an $m \times p$ matrix. The ij term of AB is the dot product of the ith row vector of A with the jth column vector of B, so that

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \\ -4 & 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(3)+(3)(-1)+(0)(1) & -2+12+0 & 5-6+0 \\ 6-1-3 & -4+4+0 & 10-2-9 \\ -12-6+2 & 8+24+0 & -20-12+6 \end{bmatrix} = \begin{bmatrix} 0 & 10 & -1 \\ 2 & 0 & -1 \\ -16 & 32 & -26 \end{bmatrix}$$

행렬 곱셈은 교환법칙이 성립하지 않음

EXAMPLE 2 Verify that the matrices

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

do not satisfy the commutative property for multiplication.

Solution The products are

$$AB = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$$

so that $AB \neq BA$.

행렬 곱은 분배법칙 성립

EXAMPLE 4 Perform the operations on the matrices

$$A = \begin{bmatrix} -3 & 1 \\ 2 & 2 \\ -1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 & -1 & 3 \\ 2 & 5 & -3 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 2 & -2 & 1 \\ 1 & 6 & -2 & 4 \end{bmatrix}$$

a. $A(B+C)$

b. $AB + AC$

행렬 곱셈의 성질

THEOREM 5 **Properties of Matrix Multiplication** Let A, B, and C be matrices with sizes so that the given expressions are all defined, and let c be a real number.

1. $A(BC) = (AB)C$
2. $c(AB) = (cA)B = A(cB)$
3. $A(B + C) = AB + AC$
4. $(B + C)A = BA + CA$

We have already seen that unlike with real numbers, matrix multiplication does not commute. There are other properties of the real numbers that do not hold for matrices. Recall that if x and y are real numbers such that $xy = 0$, then either $x = 0$ or $y = 0$. This property does not hold for matrices. For example, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

행렬의 전치(전치 행렬)

DEFINITION 4 **Transpose** If A is an $m \times n$ matrix, the **transpose** of A, denoted by A^t, is the $n \times m$ matrix with ij term

$$(A^t)_{ij} = a_{ji}$$

where $1 \le i \le n$ and $1 \le j \le m$.

For example, the transpose of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{is} \quad A^t = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

THEOREM 6 Suppose A and B are $m \times n$ matrices, C is an $n \times p$ matrix, and c is a scalar.

1. $(A + B)^t = A^t + B^t$
2. $(AC)^t = C^t A^t$
3. $(A^t)^t = A$
4. $(cA)^t = cA^t$

DEFINITION 5 **Symmetric Matrix** An $n \times n$ matrix is **symmetric** provided that $A^t = A$.

대칭 행렬 정의

EXAMPLE 5 Find all 2×2 matrices that are symmetric.

Solution Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is symmetric if and only if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^t$$

which holds if and only if $b = c$. So a 2×2 matrix is symmetric if and only if the matrix has the form

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

벡터의 전치 및 스칼라 곱의 행렬 표현

❖ 행벡터의 전치는 각 성분의 순서는 변경없이 열벡터가 되며, 마찬가지로 열벡터의 전치는 각 성분의 순서는 변경없이 행벡터가 됨.

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^t = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^t = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

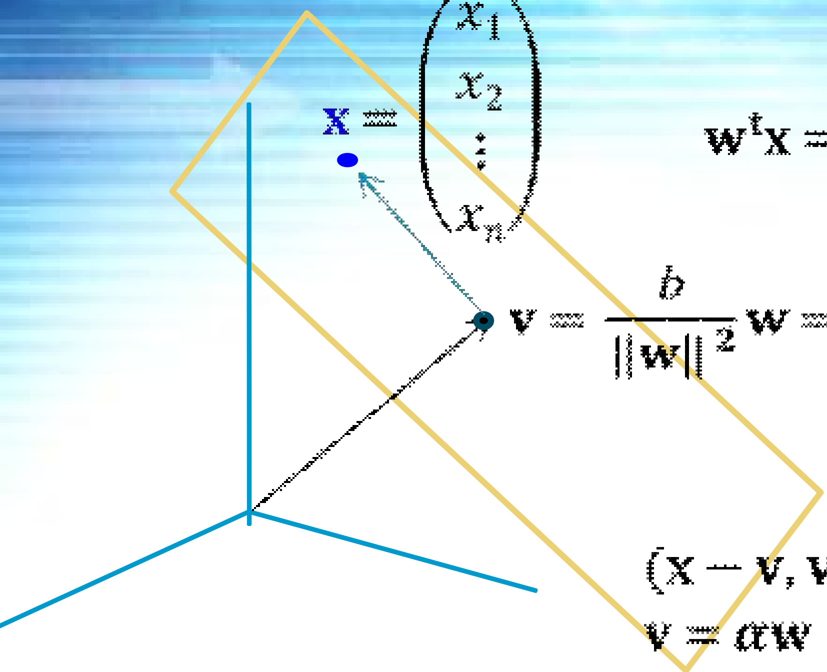
❖ 따라서, 벡터의 스칼라 곱 $\mathbf{u} \cdot \mathbf{v}$ 은 $\mathbf{u}^t\mathbf{v}$ 과 같다.

즉, $\boxed{\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t\mathbf{v}}$

선형 방정식 해의 기하학적 의미

- $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ (*) ($\mathbf{w}^t\mathbf{x} = b$ (**))
  식 (*)/(**) 해 집합은 n 차원 실수 벡터 공간에서의 벡터 $\mathbf{w}$ 에 수직인 초평면(hyperplane)이다. (다음장 그림 참조)

선형방정식(*)의 해 집합은 **n** 차원 실수 벡터 공간에서 벡터 **w** 에 수직인 초평면



$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{w}^t\mathbf{x} = b$$

$$\mathbf{v} = \frac{b}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{b}{(a_1^2 + a_2^2 + \cdots + a_n^2)} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$(\mathbf{x} - \mathbf{v}, \mathbf{v}) = (\mathbf{x} - \mathbf{v})^t\mathbf{v} = 0 \;\rightarrow\; \mathbf{x}^t\mathbf{v} = \mathbf{v}^t\mathbf{v}$$

$$\mathbf{v} = \alpha\mathbf{w} \qquad \rightarrow \quad \mathbf{x}^t\mathbf{w} = \alpha\mathbf{w}^t\mathbf{w} = b$$

$$\mathbf{w} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \;\rightarrow\; \mathbf{w}^t\mathbf{w} = a_1^2 + a_2^2 + \cdots + a_n^2 = \|\mathbf{w}\|^2$$

$$\rightarrow\; \alpha\mathbf{w}^t\mathbf{w} = b$$

$$\rightarrow\; \alpha = \frac{b}{\mathbf{w}^t\mathbf{w}} = \frac{b}{\|\mathbf{w}\|^2} = \frac{b}{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$\rightarrow\; \mathbf{v} = \alpha\mathbf{w} = \frac{b}{\|\mathbf{w}\|^2}\mathbf{w}$$

1.3 절 요약

Fact Summary

Let A, B, and C be matrices.

1. The definitions of matrix addition and scalar multiplication satisfy many of the properties enjoyed by real numbers. This allows algebra to be carried out with matrices.
2. When AB is defined, the ij entry of the product matrix is the dot product of the ith row vector of A with the jth column vector of B.
3. Matrix multiplication does not in general commute. Even when AB and BA are both defined, it is possible for $AB \neq BA$.
4. The distributive properties hold. That is, $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$.
5. $(A+B)^t = A^t + B^t$, $(AB)^t = B^t A^t$, $(A^t)^t = A$, $(cA)^t = cA^t$
6. The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is symmetric if and only if $b = c$.

Homework #1-3

- Exercise Set 1.3
  - 3,11,21,25,31,37,41

1.4 The Inverse of a Square Matrix

SOONGSIL UNIVERSITY

학습 목표

❖ 역정방 행열의 역행렬 정의

❖ 2x2 행렬의 역행렬 이해

❖ ■⌧☎≥3) 행렬의 역행렬 계산 방법 학습

❖ 행렬곱의 역행렬 계산 이해

항등행렬(Identity matrix)

In the real number system, the number 1 is the multiplicative identity. That is, for any real number a,

$$a \cdot 1 = 1 \cdot a = a$$

We also know that for every number x with $x \neq 0$, there exists the number $\frac{1}{x}$, also written x^{-1}, such that

$$x \cdot \frac{1}{x} = 1$$

We seek a similar relationship for square matrices. For an $n \times n$ matrix A, we can check that the $n \times n$ matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is the **multiplicative identity**. That is, if A is any $n \times n$ matrix, then

$$AI = IA = A$$

This special matrix is called the **identity matrix**. For example, the 2×2, 3×3, and 4×4 identity matrices are, respectively,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

정방 행렬의 역행렬 정의

DEFINITION 1 **Inverse of a Square Matrix** Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that

$$AB = I = BA$$

then the matrix B is a (multiplicative) **inverse** of the matrix A.

EXAMPLE 1 Find an inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

We refer to the unique inverse as *the inverse of* A and denote it by A^{-1}. When the inverse of a matrix A exists, we call A **invertible**. Otherwise, the matrix A is called **noninvertible**.

역행렬 유일성

THEOREM 7 The inverse of a matrix, if it exists, is unique.

Proof Assume that the square matrix A has an inverse and that B and C are both inverse matrices of A. That is, $AB = BA = I$ and $AC = CA = I$. We show that $B = C$. Indeed,

$$B = BI = B(AC) = (BA)C = (I)C = C$$

2x2 행렬의 역행렬

THEOREM 8 The inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if $ad - bc \neq 0$. In this case the inverse is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

대형 정방행렬(n$\geq$3)의 역행렬 계산 방법

To find the inverse of larger square matrices, we extend the method of augmented matrices. Let A be an $n \times n$ matrix. Let B be another $n \times n$ matrix, and let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$ denote the n column vectors of B. Since $A\mathbf{B}_1, A\mathbf{B}_2, \ldots, A\mathbf{B}_n$ are the column vectors of AB, in order for B to be the inverse of A, we must have

$$A\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad A\mathbf{B}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

That is, the *matrix equations*

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

must all have unique solutions. But all n linear systems can be solved simultaneously by row-reducing the $n \times 2n$ augmented matrix

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right]$$

On the left is the matrix A, and on the right is the matrix I. Then A **will have an inverse if and only if it is row equivalent to the identity matrix.** In this case, each of the linear systems can be solved. If the matrix A does not have an inverse, then the row-reduced matrix on the left will have a row of zeros, indicating at least one of the linear systems does not have a solution.

Example 2 illustrates the procedure.

대형 정방 행렬($n \geq 3$)의 역행렬 계산 예(1)

EXAMPLE 2 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution To find the inverse of this matrix, place the identity on the right to form the 3×6 matrix

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Now use row operations to reduce the matrix on the left to the identity, while applying the same operations to the matrix on the right. The final result is

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

so the inverse matrix is

$$A^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

The reader should check that $AA^{-1} = A^{-1}A = I$.

대형 정방 행렬(n ≥ 3)의 역행렬 계산 예(2)

EXAMPLE 3 Use the method of Example 2 to determine whether the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 1 \\ 3 & -3 & 1 \end{bmatrix}$$

is invertible.

Solution Following the procedure described above, we start with the matrix

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 & 1 & 0 \\ 3 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

행렬곱의 역행렬

THEOREM 9 Let A and B be $n \times n$ invertible matrices. Then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof Using the properties of matrix multiplication, we have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = BB^{-1} = I$$

Since, when it exists, the inverse matrix is unique, we have shown that the inverse of AB is the matrix $B^{-1}A^{-1}$.

1.4 절 요약

Fact Summary

Let A and B denote matrices.

1. The inverse of a matrix, when it exists, is unique.
2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
3. A matrix A is invertible if and only if it is row equivalent to the identity matrix.
4. If A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Homework #1-4

- Exercise Set 1.4
  - 1,15,21,27,31,35,39