Computer Lab 4

General Notes

1. You are expected to do your own work in these computer labs.
2. Save all m–files. Some of them will come in handy later in the course.
3. Be sure that you hand in all the required results in the correct format. Use the checklist provided to make sure you have everything.
4. Do not ignore Hints and Notes. The Hints will help you work efficiently. The Notes will help understanding the material.
5. Practice good programming techniques
   (a) Avoid infinite loops
   (b) For complicated programs, make a logic flowchart in advance
   (c) Read and understand any error messages
   (d) Use the (excellent) Matlab help menu as needed
   (e) Prepare professional plots:
      i. label all axes clearly
      ii. make sure multiple curves on the same plot are clearly labeled
      iii. use different colors and/or plotting symbols and styles to distinguish different curves.
   (f) Test your code by trying simple validation cases
   (g) Be sure to clear all arrays at the start of every m–file (except function m-files)

There are several important calculations that should be done in advance. Read the description and prepare carefully before you attempt this lab.

Problem 1.

Objective: *To observe the instability of the explicit finite difference method for the heat equation.*

As discussed in class, the explicit finite difference method in Problem 2 of Computer Lab 3 is unstable if the time step, Δt, is too large. Specifically, for stability, one needs $\Delta t < (\Delta x)^2/2$, or $r < 1/2$. For small Δx, this can require prohibitively small time steps. You will notice that the specified time steps in Computer Lab 3 were small and met this criterion. Using your m–file from Computer Lab 3 integrate the heat equation for $\Delta t = 0.006$ and $\Delta x = 0.1$. Take 15 time steps and observe the numerical instability that takes place by plotting the results at each time step.

Prepare plots that show the temperature as a function of x for the times $t = 0.012,\ 0.024,\ 0.048$ and 0.072.

Hand in

⋆ A plot of T vs. x for the four different times given above..

Problem 2.

Objective: *To apply explicit finite difference methods for the heat equation for more complex boundary conditions and heat sinks.*

One of the advantages of finite difference methods is that they can handle arbitrary time dependent boundary conditions, arbitrary initial conditions, source and sink terms and simple types of nonlinearities.

Consider the heat equation with a heat source, $S(t, x, u)$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + S(t, x, T) \tag{1}$$

$$\begin{array}{llllll} \text{IC:} & T = f(x), & t = 0 & \text{and} & 0 < x < 1 & \text{initial temperature distribution} \\ \text{BC:} & T = g(t), & t > 0 & \text{and} & x = 0 & \text{time dependent end temperature} \\ \text{BC:} & T = h(t), & t > 0 & \text{and} & x = 1 & \text{time dependent end temperature} \end{array} \tag{2}$$

Note: f(x), g(t) and h(t) are assumed known.

Using a first order forward difference in time and a second order centered difference in space results in the following equation for the temperature at time t_{i+1} and position x_j:

$$T_{i+1,j} = T_{i,j} + \frac{\Delta t}{(\Delta x)^2}[T_{i,j+1} - 2\,T_{i,j} + T_{i,j-1}] + \Delta t\, S(t_i, x_j, T_{i,j}) \tag{3}$$

or with $r = \Delta t/(\Delta x)^2$

$$T_{i+1,j} = r\,T_{i,j+1} + (1 - 2r)T_{i,j} + r\,T_{i,j-1} + \Delta t\, S(t_i, x_j, T_{i,j}) \tag{4}$$

Note: Since this is an equation for the temperature at the next time step in terms of known quantities, (none of the terms on the right hand side involve time level $i + 1$), this continues to be an *explicit* algorithm.

Set $\Delta x = 0.1$, i.e. take 9 interior mesh points across the interval from $x = 0$ to $x = 1$, and take $\Delta t = 0.001$. Adapt your m–file from Computer Lab 3 to solve the problem for two cases: in each case, compute the temperature as a function of time and space from $t = 0$ to $t = 1$, i.e. take 1000 time steps.

Case 1: $S = 0$, $f(x) = 1 - x$, $g(t) = 1 + 0.5 \sin(2\pi t)$ and $h(t) = 0$. This corresponds to the physical situation where the initial state is the state of conduction and then one starts to oscillate the left hand boundary temperature in a sinusoidal fashion. We might be interested in how far the 'temperature wave' penetrates into the rod as a result. Plot the temperature as a function of x for $t = 0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$ and answer this question: does the oscillation in temperature imposed at the left damp out as one goes farther from the boundary, or does the temperature oscillate with the same amplitude everywhere? Answer this question by plotting the temperature at $x = 0.5$ as a function of time and comparing the amplitude of oscillation with 0.5.

Case 2: $S = -5\,T^4$, $f = 1$, $g = 1$ and $h = 1$. This corresponds physically to the case where the rod is initially at uniform temperature, is held at that temperature at both ends, but loses heat to the environment through thermal radiation. One might be interested in the degree to which radiation will cool the bar. Plot the temperature as a function of x for $t = 0,\ 0.01,\ 0.03,\ 0.06,\ 1$. Because of the nature of the boundary conditions, one expects a symmetric temperature profile. Does the temperature at the midpoint of the rod reach a steady state? If so, what is the temperature there, i.e. by what percentage does radiation cool the rod at its center?

Hand in

- ⋆ Your m–files.

Case 1:

- ⋆ The plot of T vs. x for the times given above.
- ⋆ The plot of temperature vs. time at $x = 0.5$.
- ⋆ The answer to the question of whether the 'temperature wave' diminishes with distance or not.

Case 2:

- ⋆ The plot of temperature vs. x for the times given above.
- ⋆ The answer to the question about reaching steady state.
- ⋆ The percentage radiative cooling that takes place at the center of the rod.

Problem 3.

Objective: *To solve Laplace's equation using finite differences and relaxation methods.*

Note: This problem involves preliminary calculations – be sure to do them before the computer lab.

Consider the potential problem discussed in class and in the assignment:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{5}$$

$$\begin{aligned} &x = 0 \quad \phi = 0, \\ &x = 1 \quad \phi = f(y) = \begin{cases} y & 0 \le y \le 1/2 \\ 1 - y & 1/2 < y \le 1 \end{cases} \\ &y = 0, 1 \quad \phi = 0 \end{aligned} \tag{6}$$

This problem was solved by Fourier sine series in y in Assignment 4 for the case of $f(y)$ being a "triangle wave". Refer to the solution sheets for the details. We are going to solve this by finite differences and relaxation techniques.

Assume a uniform mesh on the square domain with $h = 0.2$. Accordingly there will be $4 \times 4 = 16$ interior points where the solution must be generated.

a. Standard finite differences using second order differences results in a large matrix problem of the form $A\,\Phi = b$, where the matrix A is sparse and the vector Φ is the vector of unknown values of the potential at the mesh points. Vector Φ is defined as follows:

$$\Phi = [\phi_{11},\ \phi_{12}, \ldots,\ \phi_{16},\ \phi_{21},\ \phi_{22}, \ldots,\ \phi_{61},\ \phi_{62}, \ldots,\ \phi_{66}] \tag{7}$$

where $\phi_{ij} = \phi(x_i, y_j)$, $x_i = (i-1)\Delta x$ and $y_j = (j-1)\Delta y$. The row $(6(j-1)+i)$ of A corresponds to the mesh point at (x_i, y_j). When this is a point on the boundary, the corresponding row of A is used to apply the boundary condition at that point. Otherwise, when (x_i, y_j) is an interior point, the discretized form of the Laplace's operator at that point is used to prescribe the corresponding row of A. The matrix A is available on Vista (file name: A.mat).

Form the vector b by applying the boundary conditions using the standard five point stencil for Laplace's equation for the case of the boundary conditions given above. Solve the finite difference equations *exactly* directly using backslash. Save the results in an array for later plotting.

b. Write an m–file that solves the finite difference equations iteratively using relaxation techniques, i.e. the Gauss–Seidel method. Start sweeping from the bottom left interior mesh point and move first in the x direction and then in the y direction; i.e. update the values at mesh points in the following order: $\phi_{22},\ \phi_{23}, \ldots, \phi_{25},\ \phi_{32},\ \phi_{33}, \ldots$. Make 7 sweeps over the mesh and save the results in an array for plotting.

c. Evaluate the exact solution at all the mesh points by taking 20 terms in the Fourier series and save the results in an array for plotting.

d. Produce a single contour plot that compares the exact Fourier series solution, the exact solution of the finite difference equations (which will be different because of truncation error) and the results of the iterative method. For help with contour plotting, refer to the example available on Vista (filename: MATLABTutorial–Surf.m). For contour levels use these values: [0.01, 0.05, 0.1, 0.2, 0.3].

e. Prepare a short one paragraph discussion of the differences, if any, between the three solutions and the reasons for them.

Hand in

- ⋆ Your m–files.
- ⋆ The contour plot comparing the exact Fourier series solution, the exact solution of the finite difference equations and the results of the iterative method
- ⋆ The answer to the question in part **e**.