Computer Lab 5

Problem 1.

Objective: *to solve Laplace's equation using successive over–relaxation (SOR) and to observe the speed up over the previous computer lab.*

Consider the potential problem discussed in class and in the assignment:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{1}$$

$$x=0,\quad \phi=0,\quad \text{and}\quad y=0,1,\quad \phi=0 \tag{2}$$

$$x=1,\quad \phi=f(y)=\begin{cases} y & 0\le y\le 1/2 \\ 1-y & 1/2<y\le 1 \end{cases} \tag{3}$$

This problem was solved by Fourier sine series in y in Assignment 4 for the case of $f(y)$ being a "triangle wave". Refer to the solution sheets for the details. We are going to solve this by finite differences and successive over-relaxation.

Assume a uniform mesh on the square domain with $h=0.1$. Accordingly there will be $9\times 9=81$ interior points where the solution must be generated. Modify your m–file from Problem 3, Computer lab 4 to handle 81 points.

a. Using the five point stencil, apply 15 iterations of the Gauss-Seidel method, using $\phi=0$ as the first guess. Save the results for plotting and comparison.

b. The *SOR* method applies over-relaxation to the results. If $\phi^{*(m+1)}$ is the result from Gauss–Seidel at iteration $m+1$, SOR computes the next iterate as

$$\phi^{(m+1)}=w\phi^{*(m+1)}+(1-w)\phi^{(m)} \tag{4}$$

where w is known as the *relaxation factor*. If $w>1$, the method is called "over–relaxation".
Apply 15 iterations of SOR to the same problem with the same initial guess. Use $w=1.5$.

c. Evaluate the exact solution at all the mesh points by taking 20 terms in the Fourier series and save the results in an array for plotting (Modify your m–file from Problem 3, Computer lab 4 to handle 81 points).

d. Compare the SOR results with the Gauss–Seidel results for the same number of iterations, and with the exact Fourier series solution. Do this comparison by a contour plot and discuss the results. For contour levels use [0.01, 0.05, 0.1, 0.2, 0.3]. Which method (SOR or Gauss-Seidel) gives the more accurate results after 15 iterations?

Hand in

- ⋆ Your m–files.
- ⋆ The contour plot comparing the SOR results, Gauss–Seidel results and the exact Fourier series solution.
- ⋆ Your answer to the question in part **d**.

Problem 2.

Objective: *To use numerical methods to solve a fairly complicated heat transfer design problem.*

Consider the steady heating of a square flat plate that is placed in an air stream flowing in the x direction (see figure 1). All four sides of the plate are held at fixed but different temperatures, T_1 through T_4. The plate loses heat by convective heat transfer and by radiation. The resulting boundary value

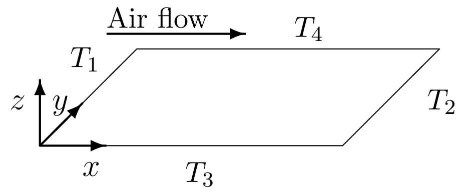

Figure 1: Schematic illustration of problem 2

problem that results is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial T^2}{\partial y^2} - U(x)(T - T_\infty) - \sigma T^4 = 0 \tag{5}$$

$$\begin{array}{llll} T = T_1, & x = 0 & \text{and} & T = T_2 = 300, \quad x = 1 \\ T = T_3 = 240, & y = 0 & \text{and} & T = T_4 = 280, \quad y = 1 \end{array} \tag{6}$$

Here, σ is related to the Stefan–Boltzmann constant, T_∞ is the temperature far from the plate, and $U(x)$ is the spatially varying convective heat transfer coefficient (usually called h, but called U here to avoid confusion with the mesh spacing, h). Take $T_\infty = 100$, $\sigma = 10^{-8}$ and $U(x) = 5.0/\sqrt{0.1 + x}$, appropriate for laminar boundary layers.

The problem is to determine the temperature T_1 such that every point on the plate is at 200 degrees or higher.

The finite difference version of this equation using second order centered differences on a square mesh with equal mesh spacing h in each direction results in a form that is convenient for an iterative solution:

$$T_{i,j} = \frac{1}{4}(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}) - \frac{h^2}{4}[U(x_i)(T_{i,j} - T_\infty) + \sigma T_{i,j}^4]. \tag{7}$$

Write an m–file that solves these equations iteratively to find the steady state for a given choice of the temperature T_1. Use $h = 0.05$, i.e. there will be $19 \times 19 = 361$ interior nodes. Find the minimum temperature T_1 such that every point on the plate is above 200 degrees. Do this by picking a value of T_1, solving the resulting finite difference equations by the Gauss–Seidel method, finding the minimum temperature on the plate, and picking another value of T_1, etc. A very helpful Matlab command is `min`. After you have converged a solution for a given choice of T_1 and you have the temperatures in a matrix T, you can use the command `Tmin = min(min(T))` to very simply find the minimum value of the temperature within matrix T.

Thus there are two iterations involved: solving the equations for a given value of T_1, and then iteratively changing T_1 until the minimum temperature is 200 degrees. You can approach this second iteration by trial and error or you can use some better logic and form or conceptualize a graph of the minimum temperature vs. the choice of T1 and make it a root-finding problem.

a. For an initial guess take $T_1 = 300$.

b. Iterate the equations using Gauss–Seidel until the successive relative error at all the mesh points is less than 0.01%.

c. Prepare a plot of the minimum value of the temperature on the plate as a function of the value of T_1 for $200 \le T_1 \le 300$.

d. Find the required value of T_1 to an accuracy of 1%.

Hand in

- ⋆ Your m–files.
- ⋆ Your value of T_1.
- ⋆ A plot of the minimum temperaturein the domain vs. T_1.
- ⋆ Both a surface and a contour plot of your solution for the final value of T_1. For contour levels use $[210,\ 230, \ldots,\ 290]$.