Computer Lab 3

General Notes

1. You are expected to do your own work in these computer labs.
2. Save all m–files. Some of them will come in handy later in the course.
3. Be sure that you hand in all the required results in the correct format. Use the checklist provided to make sure you have everything.
4. Do not ignore Hints and Notes. The Hints will help you work efficiently. The Notes will help understanding the material.
5. Practice good programming techniques
   (a) Avoid infinite loops
   (b) For complicated programs, make a logic flowchart in advance
   (c) Read and understand any error messages
   (d) Use the (excellent) Matlab help menu as needed
   (e) Prepare professional plots:
      i. label all axes clearly
      ii. make sure multiple curves on the same plot are clearly labeled
      iii. use different colors and/or plotting symbols and styles to distinguish different curves.
   (f) Test your code by trying simple validation cases
   (g) Be sure to clear all arrays at the start of every m–file (except function m-files)

There are several important calculations that should be done in advance. Read the description and prepare carefully before you attempt this lab.

Problem 1.

Consider the problem of steady state heat conduction in a rod with fixed end temperatures in which there is a spatially dependent heat source. The ODE describing this problem (for a particular form of the heat source) is:

$$\frac{d^2T}{dx^2} + 12x^2 = 0, \qquad T(0) = T(1) = 0. \tag{1}$$

We are going to solve this equation in three ways: analytically, by finite differences, and by Fourier series.

a. Solve the problem analytically by integrating twice and using the boundary conditions to determine the constants of integration. **Do this before you come to the lab. You will need this solution and doing it during the lab will be a waste of time.**

b. Solve this problem using finite difference methods.
   i. Begin by imagining a mesh with N interior points each separated by a constant mesh spacing, h.
   ii. Then use the second order accurate finite difference formula for the second derivative to convert the differential equation into a set of algebraic equations for the temperature, T_i, at the mesh point i. ***Hint***: The finite difference equations written at point i will involve the temperatures T_{i+1}, T_i, and T_{i-1}. Why?
   iii. Apply the boundary conditions at the end points to get N equations in N unknowns (the value of the temperature at each interior point).

iv. Write these equations in matrix/vector form as $Ax = b$. ***Note***: Since the original differential equation is non-homogeneous, i.e. there is a non-zero 'right hand side', the column vector b will be full. **Do this before you come to the lab. You will need to form the matrix A and the vector b, and doing the algebra during the lab will be a waste of time.**

v. Write an m–file that solves the finite difference equations. Use your m–file from the last computer lab as a guide. Set N and compute the mesh size h. Form the matrix A (which will be tridiagonal) ***Hint***: Be sure to initialize A to zero before beginning. Form the right hand side, b. Solve the algebraic equations using backslash. Use $N = 5$.

c. Solve this equation using Fourier Sine series. Since you did this problem in Assignment 3, all you have to do here is to double check your work against the solutions posted on line. Evaluate the first 3 non–zero terms of the Fourier series.

Prepare a plot that compares the exact, the finite difference, and the Fourier series solution.

Hand in

- ⋆ Your m–files
- ⋆ Your analytical solution
- ⋆ The plot comparing the exact, the finite difference, and the Fourier series solution.

Problem 2.

Solve the heat equation using finite differences.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{2}$$

$$\begin{array}{llll} \text{IC:} & T = 1, \quad t = 0 & \text{and} \quad 0 < x < 1 & \text{initial temperature distribution} \\ \text{BC:} & T = 0, \quad t > 0 & \text{and} \quad x = 0, 1 & \text{fixed end temperatures} \end{array} \tag{3}$$

It was shown in class that using a first order forward difference in time and a second order centered difference in space resulted in the following equation for the temperature at time t_i and position x_j:

$$T_{i+1,j} = T_{i,j} + \frac{\Delta t}{(\Delta x)^2}[T_{i,j+1} - 2T_{i,j} + T_{i,j-1}] \tag{4}$$

Note: Since this is an equation for the temperature at the next time step in terms of known quantities (none of the terms on the right hand side involve time level $i + 1$), this is an *explicit* algorithm.

a. Set $\Delta x = 0.1$, i.e. take 11 mesh points across the interval from $x = 0$ to $x = 1$, and take $\Delta t = 0.001$. Write an m–file that computes the temperature as a function of time and space from t = 0 to t = 1, i.e. take 1000 time steps. Store the temperature as a function of x at the times t = 0, 0.05, 0.1, 0.2 and 1.0. Store the temperature at all the mesh points at the given times above.

b. Compare your numerical solution at given times with the exact solution obtained in class and given in the books. This will involve summing the Fourier series. Take 100 non–zero terms.

Hand in

- ⋆ Your m–files
- ⋆ A single graph of temperature vs. x at specified times that compares your numerical solution with the exact solution obtained in class. Plot the exact solution as a continuous curve and your numerical solution using points.