反向傳播作業

$$\underset{(B,4)}{a_1} \longrightarrow \underset{(B,4)}{z} \longrightarrow \underset{(B,3)}{a_2} \longrightarrow \underset{(B,3)}{y} \longrightarrow L$$

$a_1 = xW_1 + b_1$, z (sigmoid), $a_2 = zW_2 + b_2$, y (softmax), $L = -\sum_k t_k \ln y_k$

有批次 B 時，$L = -\frac{1}{B}\sum_b \sum_k t_{bk} \ln y_{bk}$

$[n_1, n_2, n_3] = [5, 4, 3] \Rightarrow$ $x : (B,5)$, $W_1 : (5,4)$, $b_1 : (4,)$
$z : (B,4)$, $W_2 : (4,3)$, $b_2 : (3,)$

softmax : $y_k = \dfrac{\exp(a_i)}{\sum_{i=1}^{n} \exp(a_i)}$, sigmoid : $z(x) = \dfrac{1}{1+e^{-x}}$

①

$$\underset{(B,3)}{\frac{\partial L}{\partial y}} = \left(\frac{\partial L}{\partial y_1} \quad \frac{\partial L}{\partial y_2} \quad \frac{\partial L}{\partial y_3} \right) = -\frac{1}{B}\begin{pmatrix} \frac{t_{11}}{y_{11}} & \frac{t_{12}}{y_{12}} & \frac{t_{13}}{y_{13}} \\ \vdots & \vdots & \vdots \\ \frac{t_{b1}}{y_{b1}} & \frac{t_{b2}}{y_{b2}} & \frac{t_{b3}}{y_{b3}} \end{pmatrix}_{B\times 3} \#$$

②

$$\frac{\partial y_1}{\partial (a_2)_1} = \begin{pmatrix} \frac{\partial y_{11}}{\partial a_{211}} & \frac{\partial y_{11}}{\partial a_{212}} & \frac{\partial y_{11}}{\partial a_{213}} \\ \frac{\partial y_{12}}{\partial a_{211}} & \frac{\partial y_{12}}{\partial a_{212}} & \frac{\partial y_{12}}{\partial a_{213}} \\ \frac{\partial y_{13}}{\partial a_{211}} & \frac{\partial y_{13}}{\partial a_{212}} & \frac{\partial y_{13}}{\partial a_{213}} \end{pmatrix} = \begin{pmatrix} y_{11}(y_{12}+y_{13}) & -y_{11}y_{12} & -y_{11}y_{13} \\ -y_{11}y_{12} & y_{12}(y_{11}+y_{13}) & -y_{12}y_{13} \\ -y_{11}y_{13} & -y_{12}y_{13} & y_{13}(y_{11}+y_{12}) \end{pmatrix}$$

$$\frac{\partial y_b}{\partial (a_2)_b} = \begin{pmatrix} \frac{\partial y_{b1}}{\partial a_{2b1}} & \frac{\partial y_{b1}}{\partial a_{2b2}} & \frac{\partial y_{b1}}{\partial a_{2b3}} \\ \frac{\partial y_{b2}}{\partial a_{2b1}} & \frac{\partial y_{b2}}{\partial a_{2b2}} & \frac{\partial y_{b2}}{\partial a_{2b3}} \\ \frac{\partial y_{b3}}{\partial a_{2b1}} & \frac{\partial y_{b3}}{\partial a_{2b2}} & \frac{\partial y_{b3}}{\partial a_{2b3}} \end{pmatrix} = \begin{pmatrix} y_{b1}(y_{b2}+y_{b3}) & -y_{b1}y_{b2} & -y_{b1}y_{b3} \\ -y_{b1}y_{b2} & y_{b2}(y_{b1}+y_{b3}) & -y_{b2}y_{b3} \\ -y_{b1}y_{b3} & -y_{b2}y_{b3} & y_{b3}(y_{b1}+y_{b2}) \end{pmatrix}$$

$$\underset{(B,3)}{\frac{\partial L}{\partial a_2}} = \underset{(B,3)}{\frac{\partial L}{\partial z}} \cdot \underset{(3,3)}{\frac{\partial z}{\partial a_2}} = \begin{pmatrix} \boxed{\text{—} \ \frac{\partial L}{\partial z_1} \ \text{—}} \\ \vdots \\ \boxed{\text{—} \ \frac{\partial L}{\partial z_b} \ \text{—}} \end{pmatrix}_{B\times 3} \ \begin{matrix} \xrightarrow{\text{dot}} \frac{\partial z_1}{\partial (a_2)_1} \\ \frac{\partial z_2}{\partial (a_2)_2} \\ \vdots \\ \xrightarrow{\text{dot}} \frac{\partial z_b}{\partial (a_2)_b} \end{matrix}$$

#

簡化 $\frac{\partial L}{\partial a_2}$

Consider 單筆資料

$$\frac{\partial L}{\partial a_2} = \frac{\partial L}{\partial z} \cdot \frac{\partial z}{\partial a_2}$$

$$= \left(-\frac{t_1}{z_1} \quad -\frac{t_2}{z_2} \quad -\frac{t_3}{z_3} \right) \begin{pmatrix} z_1(z_2+z_3) & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_2(z_1+z_3) & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & z_3(z_1+z_2) \end{pmatrix}$$

$$= \Big(-t_1 \underbrace{(z_2+z_3)}_{(1-z_1)} + t_2 z_1 + t_3 z_1 \,,\ t_1 z_2 - t_2 \underbrace{(z_1+z_3)}_{(1-z_2)} + t_3 z_2 \,,\ t_1 z_3 + t_2 z_3 - t_3(z_1+z_2) \Big)$$

$$= \Big(-t_1 + z_1 \underbrace{(t_1+t_2+t_3)}_{=1} \,,\ -t_2 + z_2(t_1+t_2+t_3) \,,\ -t_3 + z_3(t_1+t_2+t_3) \Big)$$

$$= (z_1 - t_1 \,,\ z_2 - t_2 \,,\ z_3 - t_3)$$

Hence, 批次 B 時,

$$\frac{\partial L}{\partial a_2} = \frac{1}{B} \begin{pmatrix} z_{11} - t_{11} & z_{12} - t_{12} & z_{13} - t_{13} \\ \vdots & \vdots & \vdots \\ z_{b1} - t_{b1} & z_{b2} - t_{b2} & z_{b3} - t_{b3} \end{pmatrix}_{B\times 3}$$

#

③

$$\frac{\partial L}{\partial b_2} = \underbrace{[1, \cdots, 1]}_{B\text{個}}{}_{1\times B} \cdot \frac{\partial L}{\partial a_2}$$

$(3,)$ $\quad (B,3)$ #

(上次作業的結果)

④

$$\frac{\partial L}{\partial W_2} = \frac{\partial L}{\partial a_2} \cdot \frac{\partial a_2}{\partial W_2} = Z^T \cdot \frac{\partial L}{\partial a_2}$$

$(4,3)$ $\quad (4,B)$ $\quad (B,3)$ #

(上次作業的結果)

⑤

$$\frac{\partial L}{\partial Z} = \frac{\partial L}{\partial a_2} \cdot \frac{\partial a_2}{\partial Z} = \frac{\partial L}{\partial a_2} \cdot W_2^T$$

$(B,4)$ $\quad (B,3)$ $\quad (3,4)$ #

$$\frac{\partial a_2}{\partial Z} = \begin{pmatrix} \frac{\partial a_{21}}{\partial Z_1} & \frac{\partial a_{21}}{\partial Z_2} & \frac{\partial a_{21}}{\partial Z_3} & \frac{\partial a_{21}}{\partial Z_4} \\ \frac{\partial a_{22}}{\partial Z_1} & \frac{\partial a_{22}}{\partial Z_2} & \frac{\partial a_{22}}{\partial Z_3} & \frac{\partial a_{22}}{\partial Z_4} \\ \frac{\partial a_{23}}{\partial Z_1} & \frac{\partial a_{23}}{\partial Z_2} & \frac{\partial a_{23}}{\partial Z_3} & \frac{\partial a_{23}}{\partial Z_4} \end{pmatrix} = \begin{pmatrix} W_{211} & W_{221} & W_{231} & W_{241} \\ W_{212} & W_{222} & W_{232} & W_{242} \\ W_{213} & W_{223} & W_{233} & W_{243} \end{pmatrix}$$

$$= W_2^T$$

⑥

$$\frac{\partial Z_1}{\partial (a_1)_1} = \begin{pmatrix} \frac{\partial Z_{11}}{\partial a_{111}} & & & 0 \\ & \frac{\partial Z_{12}}{\partial a_{112}} & & \\ & & \frac{\partial Z_{13}}{\partial a_{113}} & \\ 0 & & & \frac{\partial Z_{14}}{\partial a_{114}} \end{pmatrix} = \begin{pmatrix} Z_{11}(1-Z_{11}) & 0 & 0 & 0 \\ 0 & Z_{12}(1-Z_{12}) & 0 & 0 \\ 0 & 0 & Z_{13}(1-Z_{13}) & 0 \\ 0 & 0 & 0 & Z_{14}(1-Z_{14}) \end{pmatrix}$$

$$
\frac{\partial z_b}{\partial (a_1)_b} = \begin{pmatrix} \frac{\partial z_{b1}}{\partial a_{1b1}} & & & \\ & \frac{\partial z_{b2}}{\partial a_{1b2}} & & \\ & & \frac{\partial z_{b3}}{\partial a_{1b3}} & \\ & & & \frac{\partial z_{b4}}{\partial a_{1b4}} \end{pmatrix} = \begin{pmatrix} z_{b1}(1-z_{b1}) & 0 & 0 & 0 \\ 0 & z_{b2}(1-z_{b2}) & 0 & 0 \\ 0 & 0 & z_{b3}(1-z_{b3}) & 0 \\ 0 & 0 & 0 & z_{b4}(1-z_{b4}) \end{pmatrix}
$$

$$
\underset{(B,4)}{\frac{\partial L}{\partial a_1}} = \underset{(B,4)}{\frac{\partial L}{\partial z}} \cdot \underset{(4,4)}{\frac{\partial z}{\partial a_1}} = {\begin{pmatrix} \boxed{\text{——} \ \frac{\partial L}{\partial z_1} \ \text{——}} \\ \vdots \\ \boxed{\text{——} \ \frac{\partial L}{\partial z_b} \ \text{——}} \end{pmatrix}}_{B\times 4}
\quad
\begin{matrix} \text{dot} & \frac{\partial z_1}{\partial (a_1)_1} \\ & \frac{\partial z_2}{\partial (a_1)_2} \\ & \vdots \\ \text{dot} & \frac{\partial z_b}{\partial (a_1)_b} \end{matrix}
$$

#

⑥ (寫法 2)

$$
\underset{(B,4)}{\frac{\partial L}{\partial a_1}} = \underset{(B,4)}{\frac{\partial L}{\partial z}} \cdot \underset{(4,4)}{\frac{\partial z}{\partial a_1}} = {\begin{pmatrix} \frac{\partial L}{\partial z_{11}} z_{11}(1-z_{11}) & \cdots & \frac{\partial L}{\partial z_{14}} z_{14}(1-z_{14}) \\ \vdots & \ddots & \vdots \\ \frac{\partial L}{\partial z_{b1}} z_{b1}(1-z_{b1}) & \cdots & \frac{\partial L}{\partial z_{b4}} z_{b4}(1-z_{b4}) \end{pmatrix}}_{B\times 4}
$$

#

⑦

$$\frac{\partial L}{\partial b_1} = \underbrace{[1, \cdots, 1]}_{B\text{個}}{}_{1\times B} \cdot \frac{\partial L}{\partial a_1}$$

(上次作業的結果)

$(4,)$ $(B,4)$

⑧

$$\frac{\partial L}{\partial W_1} = \frac{\partial L}{\partial a_1} \cdot \frac{\partial a_1}{\partial W_1} = x^T \cdot \frac{\partial L}{\partial a_1}$$

(上次作業的結果)

$(5,4)$ $(5,B)$ $(B,4)$ #

###我李健立自己思考解決、絕無抄襲###