REMARKS ON GW/PT UNDER DEL PEZZO TRANSITIONS

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Abstract. A projective threefold transition $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ is del Pezzo if ϕ contracts a smooth del Pezzo surface to a point. We show that the GW/PT correspondence holds on Y implies that it holds on X. In particular, a hypersurface of degree 6 in $\mathbb{P}(3,2,1,1,1)$ gives a new example to the correspondence. The main tools are (i) the explicit semistable reduction as a double point degeneration and (ii) deformations of del Pezzo surfaces into toric surfaces (Proposition 3.12). Careful and repeated applications of the degeneration formulas in GW and PT theories then reduce the problem to known cases.

1. Introduction

Two smooth projective threefolds X and Y are related by a *geometric transition* if there exists a crepant contraction $\phi\colon Y \to \bar{Y}$ followed by a smoothing $\mathfrak{X} \to \Delta$ of $\bar{Y} = \mathfrak{X}_0$ with the fiber $X = \mathfrak{X}_t$ for some $t \neq 0$. We write $Y \searrow X$ or $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ for this process.

When the singular variety $\bar{Y}$ has only ordinary double points, this is known as a conifold transition. Comparison of geometric invariants under conifold transitions has been intensively studied in the literature. For example, the Gromov–Witten/Pandharipande–Thomas (GW/PT) correspondence (Conjecture 3.7) holds on Y implies that it holds on X [11].

Here we are interested in more general transitions. A transition $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ is a *del Pezzo transition of degree* d if the ϕ exceptional set E is a smooth divisor with $E^3 = d$ and $\phi(E)$ is a point. In particular $E \cong S_d$ is a smooth del Pezzo surface of degree d. The purpose of this note is to compare certain geometric invariants under del Pezzo transitions.

The main tool used is the double point degenerations we have found in [5, §4] (see (2.3) and (2.6)). In simple terms, to conclude that the semistable model of the smoothing $\mathfrak{X} \to \Delta$ is of double point type, we control the base change degree for $\mathfrak{X} \to \Delta$ so that we may first perform the base change and then perform only one weighted blow-up to achieve the semistable model (see (2.4)). The double point degenerations then allow us to employ various existing degeneration formulas to study the comparison by reducing the problem to its local models $Y_d \searrow X_d$ for $d \in \{1,2,3,4,5,6\mathrm{I},6\mathrm{II},7,8\}$. Here X_d (resp. S_d) is a smooth del Pezzo threefold (resp. surface) of degree d , $Y_d = \mathbb{P}_{S_d}(K_{S_d} \oplus \mathscr{O})$. Note that for $d = 6$ there are indeed two possible smoothings (see Example 2.1).

2020 *Mathematics Subject Classification.* Primary 14N35; Secondary 57R77.

Key words and phrases. Gromov–Witten, Pandharipande–Thomas, del Pezzo transition.

As a first application, we prove in Theorem 2.2 that if φ is a group homomorphism from the complex cobordism group to a group, then

$$\varphi(Y) - \varphi(X) = \varphi(Y_d) - \varphi(X_d).$$

In particular, up to complex cobordism, a del Pezzo transition is equivalent to its local model.

Secondly, we study the descendent GW/PT correspondence (Conjecture 3.7) under del Pezzo transitions. We show in Theorem 3.19 that if Y satisfies Conjecture 3.7, then so does X for descendent insertions (3.7). To apply degeneration formulas for GW and PT-invariants we need to prove that the relative descendent GW/PT correspondence (Conjecture 3.11) holds for the pairs (Y_d, E) and (Y_d, H), where E (resp. H) is the zero (resp. infinity) section of Y_d (see Theorem 3.16). The trick is to show that that every del Pezzo surface is deformation equivalent to a toric surface (Proposition 3.12) and therefore Y_d is deformation equivalent to a toric threefold whose GW/PT correspondence is known by [16].

As a byproduct, we also prove Conjecture 3.7 for certain (weak) Fano threefolds (Theorem 3.14). In particular we obtain new cases which are not contained in [11, 16, 17], e.g. the del Pezzo threefold of degree one X_1, which is a hypersurface of degree 6 in $\mathbb{P}(3,2,1,1,1)$ (see Remarks 3.15 and 3.22).

Acknowledgments. C.-L. is supported by NSTC, Taiwan with grant number NSTC 113-2115-M-002-004-MY3, and by the Core Research Group of National Taiwan University. S.-Y. is supported by the PhD fund in C.-L.'s NSTC grant above. S.-S. is supported by NSTC, Taiwan with grant numbers NSTC 111-2115-M-A49-019-MY3 and 114-2115-M-A49-007-MY3.

2. Double Point Degenerations

2.1. **Del Pezzo Transitions.** We briefly review the basics of del Pezzo transitions (see [5, §1 & §4] and references therein for more details) and set notations for the rest of this paper.

Given a del Pezzo transition $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ of degree d, we know that the exceptional divisor E is a smooth del Pezzo surface of degree d (cf. [19, Proposition 2.13]). Moreover, E is not isomorphic to $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_1$ because the deformation space $\mathrm{Def}(\bar{Y}, p)$ must contain a smoothing component (cf. [5, Remark 1.8]), where $\phi(E) = \{p\}$. In particular we have $1 \le d \le 8$.

We shall recall the construction of a (standard) local model $Y_d \searrow X_d$ of del Pezzo transitions degree of d in the following example.

Example 2.1. For $1 \le d \le 8$, let S_d denote a smooth del Pezzo surface of degree d which is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $8 - d$ points.

Let us denote by α the weight

$$\alpha = \begin{cases} (3,2,1,1) & \text{if } d = 1, \\ (2,1,1,1) & \text{if } d = 2, \\ (1,\dots,1) & \text{if } 3 \le d \le 8, \end{cases} \tag{2.1}$$

where the last sequence of 1 is repeated $d+1$ times. Then we have the anti-canonical embedding $S_d \hookrightarrow \mathbb{P}(\alpha)$. Let $\bar{Y}_d$ be the projective cone over S_d with vertex p in the weighted projective space $\mathbb{P}(\alpha, 1)$ and

$$Y_d = \mathbb{P}_{S_d}(K_{S_d} \oplus \mathscr{O}). \tag{2.2}$$

It is immediate that Y_d is the weighted blow-up of $\bar{Y}_d$ at the point $p = [0 : \cdots : 0 : 1] \in \mathbb{P}(\alpha, 1)$ with the weight α. Therefore S_d is the exceptional divisor of $Y_d \to \bar{Y}_d$. Note that $Y_d \to \bar{Y}_d$ is the restriction of the weighted blow-up $\mathbb{P}_{\mathbb{P}(\alpha)}(\mathscr{O}(-1) \oplus \mathscr{O}) \to \mathbb{P}(\alpha, 1)$ at p with the weight $(\alpha, 1)$. To summarize, we have

$$\begin{array}{ccc} Y_d & \hookrightarrow & \mathbb{P}_{\mathbb{P}(\alpha)}(\mathscr{O}(-1) \oplus \mathscr{O}) \\ \downarrow & & \downarrow \\ \bar{Y}_d & \hookrightarrow & \mathbb{P}(\alpha, 1). \end{array}$$

We denote by $(X_d, \mathscr{O}_{X_d}(1))$ a smooth del Pezzo threefold of degree d. By the classification of Fujita and Iskovskikh (see [2], [5, Appendix A] and the references therein), we have the anti-canonical embedding $X_d \hookrightarrow \mathbb{P}(\alpha, 1)$. Moreover, X_d is one of the following:

(1) $d = 1$ and X_1 is a hypersurface of degree 6 in $\mathbb{P}(3, 2, 1, 1, 1)$.
(2) $d = 2$ and X_2 is a double cover $X \to \mathbb{P}^3$ ramified along a surface in $\mathbb{P}^3$ of degree 4.
(3) $d = 3$ and $X_3 \hookrightarrow \mathbb{P}^4$ is a hypersurface of degree 3.
(4) $d = 4$ and $X_4 \hookrightarrow \mathbb{P}^5$ is a complete intersection of two quadrics.
(5) $d = 5$ and $X_5 \hookrightarrow \mathbb{P}^6$ is a linear section of Plücker embedding of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ by a codimension 3 subspace.
(6I) $d = 6$ and $X_{6\mathrm{I}} \hookrightarrow \mathbb{P}^7$ is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, and $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ by Segre embedding.
(6II) $d = 6$ and $X_{6\mathrm{II}} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ by Segre embedding.
(7) $d = 7$ and $X_7 \hookrightarrow \mathbb{P}^8$ is a blow-up of $\mathbb{P}^3$ at a point.
(8) $d = 8$ and $X_8 = \mathbb{P}^3 \hookrightarrow \mathbb{P}^9$ with $\mathscr{O}_X(1) = \mathscr{O}_{\mathbb{P}^3}(2)$.

Then X_d is a smoothing of the singular del Pezzo threefold $\bar{Y}_d$ and therefore we get the desired del Pezzo transition $Y_d \searrow X_d$ for $d \in \{1, 2, 3, 4, 5, 6\mathrm{I}, 6\mathrm{II}, 7, 8\}$. Here we adopt the convention that $Y_{6\mathrm{I}} = Y_{6\mathrm{II}} := Y_6$.

Let $\mathfrak{X} \to \Delta$ be the corresponding smoothing of the del Pezzro transition $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$. Note that ϕ is the weighted blow-up at the unique singularity p of $\bar{Y}$ with weight α (2.1) (cf. [19, Theorem 2.11]).

To use the local model $Y_d \searrow X_d$ to study $Y \searrow X$, we need two double point degenerations. Let us review the construction of such degenerations in [5, §4.1].

The *Kähler degeneration* $\mathcal{Y} := \mathrm{Bl}_{E \times \{0\}}(Y \times \Delta) \to \Delta$ is the deformation to the normal cone. Since E has codimension one in Y and $E|_E = K_E$, the special fiber $\mathcal{Y}_0 = Y \cup Y_d$ is a simple normal crossing divisor with $Y_d = \mathbb{P}_E(K_E \oplus \mathscr{O})$. The intersection $E = Y \cap Y_d$ is understood as the infinity divisor (or relative hyperplane section) of $Y_d \to E$. We also denote this degeneration by

$$Y \rightsquigarrow Y \cup_E Y_d. \tag{2.3}$$

The *complex degeneration* is $\mathcal{X} \to \Delta$ is the semistable reduction of the smoothing $\mathfrak{X} \to \Delta$. Set $n_d = 6, 4, 3$ for $d = 1, 2, 3$ respectively, and $n_d = 2$ for $d \geq 4$. It is obtained by a degree n_d base change $\mathcal{X}' \to \Delta$ allowed by the weighted blow-up at $p \in \mathcal{X}'$ with weight $(\alpha, 1)$:

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathfrak{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ \Delta \arrow[r, equal] & \Delta \arrow[r] & \Delta. \end{tikzcd} \tag{2.4}$$

For the control of the base change degree n_d, see the proof of Proposition 1.10 in [5]. The special fiber $\mathcal{X}_0 = Y \cup X_d$ is a simple normal crossing divisor with X_d being a smooth del Pezzo threefold of degree d. The intersection $E = Y \cap X_d$ in X_d is now understood as a general member of the linear system $|-K_{X_d}|$, and the normal bundle of the intersection E in Y is K_E. In particular, we have

$$N_{E/Y} \otimes N_{E/X} \cong \mathscr{O}(K_E) \otimes \mathscr{O}(-K_E) \cong \mathscr{O}_E. \tag{2.5}$$

We also denote this degeneration by

$$X \rightsquigarrow Y \cup_E X_d. \tag{2.6}$$

Notice that the local model $Y_d \searrow X_d$ appears in the special fibers $\mathcal{X}_0$ and $\mathcal{Y}_0$. For $d = 6$, $X_{6\mathrm{I}}$ and $X_{6\mathrm{II}}$ are distinguished by the irreducible component of $\mathrm{Def}(\bar{Y}, p)$ that contains the image of the holomorphic map $\Delta \to \mathrm{Def}(\bar{Y}, p)$ induced by the smoothing $\mathfrak{X} \to \Delta$.

2.2. **Topology: Chern Numbers.** Applying the double point degeneration (2.6) to del Pezzo transitions of degree d, we can identify them with the local model $Y_d \searrow X_d$ in the complex cobordism ring Ω_*^U. Recall that Ω_*^U is generated by all stable almost complex manifolds, and it is a polynomial ring over $\mathbb{Z}$. Two manifolds determine the same element in Ω_*^U if and only if their Chern numbers coincide.

Theorem 2.2. *Given a del Pezzo extremal transition $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ of degree d , we let Y_d and X_d be as in Example 2.1. If $\varphi\colon \Omega_6^U \to \Lambda$ is a group homomorphism, then*

$$\varphi(Y) - \varphi(X) = \varphi(Y_d) - \varphi(X_d)$$

for $d \in \{1, 2, 3, 4, 5, 6\mathrm{I}, 6\mathrm{II}, 7, 8\}$.

Proof. We will apply algebraic cobordism theory to prove this theorem, see [6, 7] for further details and references. Indeed, we use the double point cobordism ring $\omega_*(\mathbb{C})$ of Levine-Pandharipande [7, Theorem 1] to give a simple description of the difference $[X] - [Y]$.

By (2.5), (2.6) and double point relations [7, Definition 0.1], we get

$$[X] - [Y] - [X_d] + [\mathbb{P}(N_{E/Y} \oplus \mathscr{O})] = 0 \tag{2.7}$$

in $\omega_3(\mathbb{C})$. According to (2.2) and that ϕ is crepant, it follows that $\mathbb{P}(N_{E/Y} \oplus \mathscr{O}) = Y_d$ by the adjunction formula. Then the theorem follows from (2.7), the isomorphism

$$\omega_*(\mathbb{C}) \cong \Omega^U_{2*}$$

(see [6, Lemma 4.3.1, Theorem 4.3.7]) and that φ is a group homomorphism. □

We give some examples of group homomorphims φ.

Example 2.3. If the group Λ is Ω^U_6, then the identity map $\varphi = \mathrm{id}$ is a trivial example, and we get (2.7).

Let $\mathsf{Z}(M, q)$ be the partition function for degree 0 Donaldson–Thomas invariants on a smooth projective threefold M. It gives another example of group homomorphisms. In fact, by the degeneration formula in Donaldson–Thomas theory, a group homomorphism (see [7, §13])

$$\varphi\colon \Omega^U_6 \cong \omega_3(\mathbb{C}) \to \mathbb{Q}[\![q]\!]^*$$

is defined by $\varphi(M) \coloneqq \mathsf{Z}(M, q)$, where $\mathbb{Q}[\![q]\!]^* \subseteq \mathbb{Q}[\![q]\!]$ is the multiplicative group of power series with constant term 1.

Example 2.4. Let R be a commutative ring containing $\mathbb{Q}$. We recall the Hirzebruch R-genus φ (see [1, §1]). By definition, it is a ring homomorphism $\varphi\colon \Omega^U_* \otimes \mathbb{Q} \to R$, which depends only on Chern numbers. To each series of the form $Q(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots \in R[\![x]\!]$ there corresponds the Hirzebruch R-genus $\varphi_Q(M) \coloneqq \int_M \prod_i Q(x_i)$, where x_i's are the Chern roots of a stable almost complex manifold M. If M has complex dimension three, then

$$\varphi_Q(M) = (a_1^3 - a_1a_2 + a_3)c_3 + (a_1a_2 - 3a_3)c_1c_2 + a_3c_1^3, \tag{2.8}$$

where $c_i = c_i(M)$ is the Chern number.

Since φ_Q is also a group homomorphism, we can apply Theorem 2.2 to compute the difference of the R-genus of a del Pezzo transition $Y \searrow X$ of degree d. It suffices to compute the Chern numbers of the local model $Y_d \searrow X_d$. By (2.2) and Euler sequence for the projective bundle $\pi\colon Y_d \to S_d$,

$$0 \to \mathscr{O}_{Y_d} \to \pi^*(K_{S_d} \oplus \mathscr{O}) \otimes \mathscr{O}_{Y_D}(1) \to T_{Y_d} \to \pi^*T_{S_d} \to 0,$$

it is easily seen that $c_1(Y_d)^3 = 8d$, $c_1(Y_d)c_2(Y_d) = 24$ and

$$c_3(Y_d) = 2(12 - d) = 2c_2(S_d),$$

where S_d is a smooth del Pezzo surface of degree d. On the other hand, since X_d is a del Pezzo threefold of degree d, we have that $c_1(X_d)^3 = 8d$. From standard arguments using the Riemann–Roch, Serre duality, and Kodaira vanishing [2, Corollary 2.1.14], we find that

$$d + 2 = \chi(\mathscr{O}(1)) = d + \frac{1}{12}c_1(X_d)c_2(X_d),$$

i.e., $c_1(X_d)c_2(X_d) = 24$. The top Chern number of X_d is given in Table 1 by the classification of del Pezzo threefolds, see for example [5, Appendix A] and references therein. Set $\Delta\chi_{\mathrm{top}} \coloneqq$

$\chi_{\mathrm{top}}(Y_d)-\chi_{\mathrm{top}}(X_d)$, where $\chi_{\mathrm{top}}(-)$ is the topological Euler number. We also list the topological difference $\Delta\chi_{\mathrm{top}}$ of $Y \searrow X$ in Table 1 (cf. [5, Remark 1.12]). Therefore

$$\begin{aligned}\varphi_Q(Y)-\varphi_Q(X) &= (a_1^3-a_1a_2+a_3)(2\chi_{\mathrm{top}}(S_d)-\chi_{\mathrm{top}}(X_d))\\ &= (a_1^3-a_1a_2+a_3)\Delta\chi_{\mathrm{top}}\end{aligned}$$

by (2.8) and $\chi_{\mathrm{top}}(Y_d)=\chi_{\mathrm{top}}(\mathbb{P}^1)\chi_{\mathrm{top}}(S_d)$. As a byproduct of 2.7 and that X_d and Y_d have same $c_1c_2=24$, we also obtain that $c_1(Y)c_2(Y)=c_1(X)c_2(X)$.

d	1	2	3	4	5	6I	6II	7	8
$c_3(X_d)$	-38	-16	-6	0	4	6	8	6	4
$\Delta\chi_{\mathrm{top}}$	60	36	24	16	10	6	4	4	4

Table 1. Topological numbers of $Y_d \searrow X_d$.

3. Quantum: GW/PT Correspondence

In this section, we will use the double point degenerations (2.3) and (2.6) to relate descendent GW/PT correspondence under del Pezzo transitions (Theorem 3.19). We also prove correspondences for certain (weak) Fano threefolds (Theorems 3.14 and 3.16). As §3.1 and 3.2 are primarily to review the basics of GW and PT theories and to set notations, the exposition is condensed. See [16, 17, 11] and references therein for more information.

3.1. **Absolute Theories.** Let M be a smooth projective threefold. Fix a curve class $0\neq\beta\in\mathrm{NE}(M)$, integers $r\in\mathbb{Z}_{\geq 0}$ and $g\in\mathbb{Z}$. Set $\mathsf{c}_\beta := (c_1(T_V),\beta)$.

Let $\overline{\mathcal{M}}'_{g,r}(M,\beta)$ denote be the moduli space of r-marked genus g degree β stable maps $C\to M$, where the stable map is required to have positive degree on each connected component of the (possibly disconnected) domain C. The moduli space $\overline{\mathcal{M}}'_{g,r}(M,\beta)$ is equipped with a virtual fundamental class and its virtual dimension is $\mathsf{c}_\beta + r$.

Definition 3.1. Let ψ_j be the first Chern class of cotangent line bundle associated to the j-th marked point for $j=1,\ldots,r$. Then the disconnected descendent GW-invariant is defined as

$$\langle\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_r-1}(\gamma_r)\rangle'_{g,\beta} = \int_{[\overline{\mathcal{M}}'_{g,r}(M,\beta)]^{\mathrm{vir}}}\prod_{j=1}^r \psi_j^{\alpha_j-1}\cup \mathrm{ev}_j^*(\gamma_j)$$

where ev_j is the evaluation map given by the j-th marked point and $\gamma_j\in H^*(M,\mathbb{Q})$. We define the following associated partition function

$$\mathrm{Z}'_{\mathrm{GW}}\left(M;u\,\middle|\,\prod_{j=1}^r\tau_{\alpha_j-1}(\gamma_j)\right)_\beta = \sum_{g\in\mathbb{Z}}\left\langle\prod_{j=1}^r\tau_{\alpha_j-1}(\gamma_j)\right\rangle'_{g,\beta} u^{2g-2}. \tag{3.1}$$

Note that $\overline{\mathcal{M}}'_{g,r}(M,\beta)$ is empty for g sufficiently negative. Therefore (3.1) is a Laurent series in $\mathbb{Q}(\!(u)\!)$.

To define PT-invariants, we consider the moduli space of stable pairs. A *stable pair*

$$(F, s\colon \mathscr{O}_M \to F)$$

on M consists of a pure one-dimensional sheaf F on M and a section s with zero-dimensional cokernel. Given $n \in \mathbb{Z}$, let $P_n(M,\beta)$ be the moduli space of stable pairs with $\mathrm{ch}_2(F) = \beta$ and $\chi(F) = n$. Then $P_n(M,\beta)$ is fine and projective, and it admits a virtual fundamental class of virtual dimension c_β. Let $\mathbb{F}$ be the universal sheaf of $P_n(M,\beta)$. Consider the k-th descendent insertion

$$\tau_k(\gamma) \coloneqq \pi_{P*}(\pi_M^*(\gamma)\cdot \mathrm{ch}_{2+k}(\mathbb{F})) \in H^*(P_n(M,\beta),\mathbb{Q})$$

of a class $\gamma \in H^p(M,\mathbb{Q})$ where π_P and π_M are projections on $P_n(M,\beta)\times M$.

Definition 3.2. Given $\alpha_j \in \mathbb{N}$ and $\gamma_j \in H^*(M,\mathbb{Q})$ for $1 \le j \le r$, the corresponding descendent PT-invariant is

$$\langle \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_r-1}(\gamma_r)\rangle_{n,\beta} = \int_{[P_n(M,\beta)]^{\mathrm{vir}}} \prod_{j=1}^r \tau_{\alpha_j-1}(\gamma_j).$$

We define the following associated partition function

$$\mathrm{Z}_{\mathrm{PT}}\left(M;q \,\middle|\, \prod_{j=1}^r \tau_{\alpha_j-1}(\gamma_j)\right)_\beta = \sum_{n\in\mathbb{Z}} \left\langle \prod_{j=1}^r \tau_{\alpha_j-1}(\gamma_j)\right\rangle_{n,\beta} q^n. \tag{3.2}$$

Note that the moduli space $P_n(M,\beta)$ is empty for n sufficiently negative. Therefore (3.2) is a Laurent series $\mathbb{Q}(\!(q)\!)$ as well.

For the case of primary insertions, Pardon [18] proved the conjecture of Maulik–Nekrasov–Okounkov–Pandharipande [12, 13] for all complex threefolds with nef anti-canonical bundle.

Theorem 3.3 ([18, Theorem 1.6])**.** *Let M be a smooth projective threefold with nef anti-canonical bundle. Then, for classes $(\gamma_1,\dots,\gamma_r) \in H^*(M,\mathbb{Q})^{\oplus r}$, the partition function $\mathrm{Z}_{\mathrm{PT}}(M;q \mid \gamma_1\dots\gamma_r)_\beta \in \mathbb{Q}(q)$ and*

$$(-q)^{-\mathsf{c}_\beta/2}\mathrm{Z}_{\mathrm{PT}}(M;q \,|\, \gamma_1\cdots\gamma_r)_\beta = (-iu)^{\mathsf{c}_\beta}\mathrm{Z}'_{\mathrm{GW}}(M;u \,|\, \gamma_1\cdots\gamma_r)_\beta$$

under the variable change $-q = e^{iu}$.

To relate descendent GW and PT-invariants, we need the correspondence matrices found by Pandharipande and Pixton [16, 17]. The matrices relating them were predicted in [13, Conjecture 4].

Let $\widehat{\alpha} = (\widehat{\alpha}_1,\dots,\widehat{\alpha}_{\widehat{\ell}})$, with $\widehat{\alpha}_1 \ge \cdots \ge \widehat{\alpha}_{\widehat{\ell}} \ge 1$, be a partition of length $\ell(\widehat{\alpha}) \coloneqq \widehat{\ell}$ and size $|\widehat{\alpha}| \coloneqq \sum \widehat{\alpha}_j$. Let $\iota_\Delta\colon \Delta \to M^{\widehat{\ell}}$ be the inclusion of the small diagonal in the product $M^{\widehat{\ell}}$. For $\gamma \in H^*(M,\mathbb{Q})$, we write

$$\gamma\cdot\Delta \coloneqq \iota_{\Delta*}(\gamma) \in H^*(M^{\widehat{\ell}},\mathbb{Q}).$$

Let $\{\theta_j\}$ be a basis of $H^*(M,\mathbb{Q})$. By Künneth formula, we have

$$\gamma \cdot \Delta = \sum_{j_1,\ldots,j_{\widehat{\ell}}} c^{\gamma}_{j_1,\ldots,j_{\widehat{\ell}}} \theta_{j_1} \otimes \cdots \otimes \theta_{j_{\widehat{\ell}}}.$$

The descendent insertion $\tau_{[\widehat{\alpha}]}(\gamma)$ is defined by [17, (3)]

$$\tau_{[\widehat{\alpha}]}(\gamma) = \sum_{j_1,\ldots,j_{\widehat{\ell}}} c^{\gamma}_{j_1,\ldots,j_{\widehat{\ell}}} \tau_{\widehat{\alpha}_1-1}(\theta_{j_1}) \cdots \tau_{\widehat{\alpha}_{\widehat{\ell}}-1}(\theta_{j_{\widehat{\ell}}}).$$

The key construction in [16, §0.5] is a universal correspondence matrix $\widetilde{\mathsf{K}}$ indexed by partitions α and $\widehat{\alpha}$ of positive size with

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[\sqrt{-1}, c_1, c_2, c_3](\!(u)\!)$$

and $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} = 0$ if $|\alpha| < |\widehat{\alpha}|$. By specializing the formal variables c_i to $c_i(T_M)$, the elements of $\widetilde{\mathsf{K}}$ act by cup product on $H^*(M,\mathbb{Q})$ with $\mathbb{Q}[i](\!(u)\!)$-coefficients.

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a partition and P a partition of $\{1, \ldots, \ell\}$. For each $S \in P$, a subset of $\{1, \ldots, \ell\}$, let α_S be the subpartition consisting of the parts α_j for $j \in S$ and

$$\gamma_S = \prod\nolimits_{j\in S} \gamma_j.$$

Definition 3.4 ([16])**.** For even cohomology classes $\gamma_j \in H^{2*}(M,\mathbb{Q})$, let

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{\substack{P \text{ set partitions} \\ \text{of } \{1,\ldots,\ell\}}} \prod_{S\in P} \sum_{0<|\widehat{\alpha}|\leqslant|\alpha_S|} \tau_{[\widehat{\alpha}]}\left(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S\right).$$

Example 3.5. If $\alpha = (1, \ldots, 1)$ then $\overline{\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)} = \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)$. If $\gamma = \mathrm{pt}$, the class of a point, then

$$\tau_{[\alpha]}(\mathrm{pt}) = \tau_{\alpha_1-1}(\mathrm{pt})\cdots\tau_{\alpha_\ell-1}(\mathrm{pt}).$$

If $\alpha = (\alpha_1)$, then $\tau_{[\alpha]}(\gamma) = \tau_{\alpha_1-1}(\gamma)$.

Notation 3.6. Given even cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(M,\mathbb{Q})$, integers $\alpha_i \in \mathbb{N}$ for $1 \le j \le r$, we set $A = \prod_{i=1}^r \tau_{\alpha_i-1}(\gamma_i)$.

We are now in a position to state the conjectural GW/PT correspondence [16].

Conjecture 3.7. *Let A be as in Notation 3.6. Then:*

(1) The $\mathrm{Z}_{\mathrm{PT}}(M; q \mid A)_\beta$ is the Laurent expansion of a rational function in q.
(2) We have

$$(-q)^{-\mathsf{c}_\beta/2}\mathrm{Z}_{\mathrm{PT}}(M; q \mid A)_\beta = (-iu)^{\mathsf{c}_\beta}\mathrm{Z}'_{\mathrm{GW}}\left(M; u \,\middle|\, \overline{A}\right)_\beta$$

under the variable change $-q = e^{iu}$.

Note that the variable change in (2) is well-defined assuming (1), and (1) is called the *rationality conjecture.*

3.2. **Relative Theories.**

Definition 3.8. Let D be a smooth (connected) divisor on M, and let $\mathcal{B}$ be a basis of $H^*(D,\mathbb{Q})$. A *cohomology weighted partition* η with respect to $\mathcal{B}$ is a set of pairs

$$\{(a_1,\delta_1),\ldots,(a_r,\delta_r)\},\quad \text{where } \delta_j\in\mathcal{B} \text{ and } a_1\geq\cdots\geq a_r\geq 1,$$

such that $\vec{\eta}:=(a_j)\in\mathbb{N}^r$ is a partition of size $|\eta|=\sum a_j$ and length $\ell(\eta)=r$. Its *dual partition* $\eta^\vee$ is the cohomology weighted partition $\{(a_j,\delta_j^\vee)\}_j$ (with respect to the dual basis $\mathcal{B}^\vee$ of $\mathcal{B}$). We write $\eta=(1,\delta)^r$ when $a_j=1$ and $\delta_j=\delta$ for all j.

The automorphism group $\operatorname{Aut}(\eta)$ consists of $\sigma\in\mathfrak{S}_{\ell(\eta)}$ such that $\eta^\sigma=\eta$. We set

$$\mathfrak{z}(\eta)=|\operatorname{Aut}(\eta)|\cdot\prod\nolimits_{j=1}^{\ell(\eta)}a_j.$$

Definition 3.9. Let A be as in Notation 3.6 and η a cohomology weighted partition. The relative descendent GW-invariant [8, p.240] associated to A and η is defined as

$$\langle A\mid\eta\rangle'_{g,\beta}=\frac{1}{|\operatorname{Aut}(\eta)|}\int_{[\overline{\mathcal{M}}'_{g,r}(M/D,\beta,\eta)]^{\mathrm{vir}}}\prod_{j=1}^{r}\left(\psi_j^{\alpha_j-1}\cup\operatorname{ev}_j^*(\gamma_j)\right)\cup\operatorname{ev}_D^*(\delta_j).$$

The associated partition function is the Laurent series

$$\mathrm{Z}'_{\mathrm{GW}}\left(M/D;u\mid A\mid\eta\right)_\beta=\sum_{g\in\mathbb{Z}}\langle A\mid\eta\rangle'_{g,\beta}\,u^{2g-2}\tag{3.3}$$

In relative PT-theory, we consider the moduli space $P_n(M/D,\beta)$ introduced by Li-Wu [9] (cf. [13, §3.2]) which parametrizes stable pairs (F,s) relative to D, such that $\chi(F)=n\in\mathbb{Z}$ and $\mathrm{ch}_2(F)=\beta$. We have the intersection map

$$\epsilon\colon P_n(M/D,\beta)\to\operatorname{Hilb}(D,|\eta|)\tag{3.4}$$

to the Hilbert scheme of $|\eta|=(D,\beta)$ points of the connected divisor D. Fix $d\in\mathbb{N}$ and let $\eta=\{(a_j,\delta_j)\}_j$ be a cohomology weighted partition of size d with respect to $\mathcal{B}$. Let

$$C_\eta=\frac{1}{\mathfrak{z}(\eta)}P_{\delta_1}[a_1]\ldots P_{\delta_{\ell(\eta)}}[a_{\ell(\eta)}]\cdot\mathbf{1}\in H^*(\operatorname{Hilb}(D,d),\mathbb{Q}),$$

see [13, §3.2.2]. Here $\mathbf{1}$ is the vacuum vector $|0\rangle=1\in H^0(\operatorname{Hilb}(D,0),\mathbb{Q})$. Then $\{C_\eta\}_{|\eta|=d}$ is the Nakajima basis of $H^*(\operatorname{Hilb}(D,d),\mathbb{Q})$ with Poincaré pairing

$$\int_{\operatorname{Hilb}(D,d)}C_\eta\cup C_\nu=\begin{cases}\frac{(-1)^{d-\ell(\eta)}}{\mathfrak{z}(\eta)} & \text{if } \nu=\eta^\vee,\\ 0 & \text{otherwise.}\end{cases}$$

Definition 3.10. With notation as in Definition 3.9, the relative descendent PT-invariant associated to A and η is

$$\langle A\mid\eta\rangle_{n,\beta}=\int_{[P_n(M/D,\beta)]^{\mathrm{vir}}}\left(\prod_{i=1}^{r}\tau_{\alpha_i-1}(\gamma_i)\right)\cup\epsilon^*(C_\eta).$$

The associated partition function is the Laurent series

$$\mathrm{Z}_{\mathrm{PT}}\left(M/D; q \mid A \mid \eta\right)_\beta = \sum_{n\in\mathbb{Z}} \langle A \mid \eta\rangle_{n,\beta}\, q^n. \tag{3.5}$$

Now, we can state the conjectural relative descendent GW/PT correspondence [13, 16, 17].

Conjecture 3.11. *With notation as in Definition 3.9, we have:*

(1) The $\mathrm{Z}_{\mathrm{PT}}\left(M/D; q \,|\, A \,|\, \eta\right)_\beta$ is the Laurent expansion of a rational function in q.
(2) Under the variable change $e^{iu} = -q$,

$$(-q)^{-\mathsf{c}_\beta^M/2}\mathrm{Z}_{\mathrm{PT}}\left(M/D; q \,|\, A \,|\, \eta\right)_\beta = (-iu)^{\mathsf{c}_\beta^M+\ell(\eta)-|\eta|}\mathrm{Z}'_{\mathrm{GW}}\left(M/D; u \,\middle|\, \overline{A} \,\middle|\, \eta\right)_\beta.$$

To study Conjectures 3.7 and 3.11 for $Y_d = \mathbb{P}_{S_d}(K_{S_d} \oplus \mathscr{O})$ over a nontoric surface S_d, we need the following proposition.

Proposition 3.12. *For each $1 \le d \le 8$, there exists a toric surface S_d^0 such that it is deformation equivalent to S_d.*

Proof. For $d \in \{6, 7, 8\}$, S_d is already a toric surface. Hence, we assume that $d \le 5$. Consider the points

$$\begin{aligned}
&\sigma_1(t) = [1 : t^5 : 0], \qquad &&\sigma_2(t) = [t^5 : 1 : 0], \qquad &&\sigma_3(t) = [0 : 0 : 1]\\
&\sigma_4(t) = [1 : t^2 : t], \qquad &&\sigma_5(t) = [t^2 : 1 : t], \qquad &&\sigma_6(t) = [t : t^2 : 1]\\
&\sigma_7(t) = [1 : t^3 : 2t], \qquad &&\sigma_8(t) = [t^4 : 1 : 2t].
\end{aligned}$$

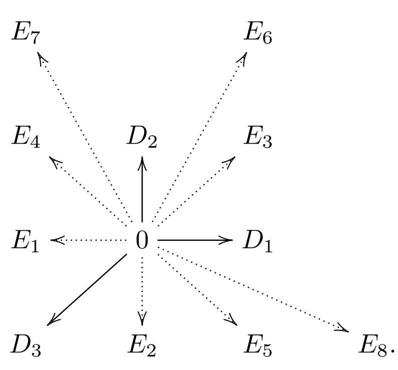
It is checked that for generic $t \neq 0$, the points $\{\sigma_i(t)\}_{1\le i\le 8} \subseteq \mathbb{P}^2$ lie in a general position, i.e., (i) no three are collinear; (ii) no six on the same conic; (iii) no eight on a cubic with a double point at one of them. Consider the deformation family defined by a sequence of blow-ups

$$\mathcal{S} = \mathcal{S}''' \xrightarrow{\pi''} \mathcal{S}'' \xrightarrow{\pi'} \mathcal{S}' \xrightarrow{\pi} \mathbb{P}^2 \times \mathbb{C} \xrightarrow{t} \mathbb{C},$$

where

- π is the blow-up along the graphs of σ_1, σ_2, σ_3;
- π' is the blow-up along the strict transform of the graphs of σ_4, σ_5, σ_6;
- π'' is the blow-up along the strict transform of the graphs of σ_7, σ_8.

It follows that the central fiber of $\mathcal{S}$ is the toric surface defined by the rays



Also, if we only blow-up the strict transforms of the graphs $\sigma_1, \ldots, \sigma_k$, then the general fiber is a del Pezzo surface of degree $d = 9 - k$ and the central fiber is the toric surface S^0_{9-k} defined by the rays $D_1, D_2, D_3, E_1, \ldots, E_k$. $\square$

Remark 3.13. For $d \geq 3$, one can compare Proposition 3.12 with [4, Proposition 4.1].

3.3. **Correspondence.** First, we apply Proposition 3.12 to prove the descendent GW/PT correspondence for certain threefolds.

Theorem 3.14. *Conjecture 3.7 holds for M being one the following threefolds:*

(1) the projective bundle $Y_d = \mathbb{P}_{S_d}(K_{S_d} \oplus \mathscr{O})$;
(2) a smooth Fano threefolds with Picard number $\rho \geq 6$.

Proof. By Proposition 3.12, we can deform S_d to a toric surface S^0_d. Thus $Y_d = \mathbb{P}_{S_d}(K_{S_d} \oplus \mathscr{O})$ and $Y^0_d = \mathbb{P}_{S^0_d}(K_{S^0_d} \oplus \mathscr{O})$ are also deformation equivalent. On the other hand, if M satisfies (2), then $M = \mathbb{P}^1 \times S_{\rho-2}$ by the Mori–Mukai classification [14]. Then the theorem follows from [16, Theorem 7] and the deformation invariance of GW and PT-invariants. $\square$

Remark 3.15. The Fano threefolds $\mathbb{P}^1 \times S_1$ and $\mathbb{P}^1 \times S_5$ are not complete intersections in product of projective spaces, and thus they are not treated in [17].

We also establish the relative descendent GW/PT correspondence for Y_d, which will be used in Theorem 3.19.

Theorem 3.16. *For $1 \leq d \leq 8$, let E and H be the zero section $\mathbb{P}_{S_d}(\mathscr{O})$ and the infinity section $\mathbb{P}_{S_d}(K_{S_d})$ of Y_d respectively. Then Conjecture 3.11 holds for Y_d/E and Y_d/H.*

Proof. As before, S_d and S^0_d are deformation equivalent by Proposition 3.12. Let E^0 (resp. H^0) be the zero section $\mathbb{P}_{S^0_d}(\mathscr{O})$ (resp. infinity section $\mathbb{P}_{S^0_d}(K_{S^0_d})$) of $Y^0_d = \mathbb{P}_{S^0_d}(K_{S^0_d} \oplus \mathscr{O})$. It follows from [17, Theorem 2] that Conjecture 3.11 holds for the pairs Y^0_d/E^0 and Y^0_d/H^0. Since both relative GW-invariants and relative PT-invariants are invariant under deformations of pairs, Conjecture 3.11 also holds for Y_d/E and Y_d/H. $\square$

To use the double point degenerations (2.3) and (2.6), we need the following degeneration formulas for GW and PT-invariants. To save notations, we denote these two degenerations by $\mathcal{W} \to \Delta$. It has a smooth fiber $M \coloneqq \mathcal{W}_t$ $(t \neq 0)$, a special fiber $\mathcal{W}_0 = M_0 \cup_D M_\infty$ and $D = M_0 \cap M_\infty$ a smooth divisor. Let $\iota\colon M \hookrightarrow \mathcal{W}$, $\iota_0\colon M_0 \hookrightarrow \mathcal{W}$ and $\iota_\infty\colon M_\infty \hookrightarrow \mathcal{W}$ be the inclusion maps.

Theorem 3.17. *Suppose that $\gamma_1, \ldots, \gamma_r$ are even cohomology classes on the total space $\mathcal{W}$, and let $A = \tau_{\alpha_1-1}(\gamma_1) \ldots \tau_{\alpha_r-1}(\gamma_r)$. For a nonzero class $\beta' \in \mathrm{NE}(\mathcal{W})$, we have*

$$\sum_{\substack{\beta\in\mathrm{NE}(M)\\ \iota_*\beta=\beta'}} \mathrm{Z}'_{\mathrm{GW}}\left(M; u \mid \overline{A}\right)_\beta$$
$$= \sum \mathfrak{z}(\eta) u^{2\ell(\eta)} \mathrm{Z}'_{\mathrm{GW}}\left(M_0/D; u \mid \overline{A_{I_0}} \mid \eta\right)_{\beta_0} \mathrm{Z}'_{\mathrm{GW}}\left(M_\infty/D; u \mid \overline{A_{I_\infty}} \mid \eta^\vee\right)_{\beta_\infty},$$

where the summation on the second line runs over

(a) *splittings*

$$\iota_{0*}\beta_0 + \iota_{\infty*}\beta_\infty = \beta' = \iota_*\beta \tag{3.6}$$

such that $(D, \beta_0) = (D, \beta_\infty)$,

(b) *partitions* $I_0 \sqcup I_\infty = \{1, 2, \ldots, r\}$, *and*

(c) *cohomology weighted partition* η *such that* $|\eta| = (D, \beta_0)$ *with respect to a fixed basis of* $H(D, \mathbb{Q})$.

See for example [8] and [17, p.403]. Similarly, the formula without bars, namely without applying the universal transformation to descendent insertions, also holds.

Theorem 3.18. *With notation as in Theorem 3.17, we have*

$$\sum_{\substack{\beta \in \mathrm{NE}(M) \\ \iota_*\beta = \beta'}} \mathrm{Z}_{\mathrm{PT}}\left(M; q \mid A\right)_\beta$$
$$= \sum (-1)^{\ell(\eta)} \mathfrak{z}(\eta)(-q)^{-|\eta|} \mathrm{Z}_{\mathrm{PT}}\left(M_0/D; q \mid A_{I_0} \mid \eta\right)_{\beta_0} \mathrm{Z}_{\mathrm{PT}}\left(M_\infty/D; q \mid A_{I_\infty} \mid \eta^\vee\right)_{\beta_\infty},$$

where the summation on the second line runs over the same index set in Theorem 3.17.

See for example [9, 13], [16, p.2761] and [10, Theorem 6.12].

Now, we are ready to prove the descendent GW/PT correspondence for del Pezzo transitions.

Theorem 3.19. *Let* $Y \xrightarrow{\phi} \bar{Y} \rightsquigarrow X$ *be a del Pezzo transition with the smoothing* $\mathfrak{X} \to \Delta$. *Suppose that* $\beta \in \mathrm{NE}(X)$ *is a nonzero class and* $\alpha = (\alpha_1, \ldots, \alpha_r)$ *a fixed partition. Assume* $\gamma_i \in H^*(\mathfrak{X}, \mathbb{Q})$, $i = 1, \ldots, r$, *are fixed even cohomology classes and if* $\gamma_i \in H^0(\mathfrak{X}, \mathbb{Q})$, *then* $\alpha_i = 1$.

(a) *If Conjecture 3.7* (1) *holds for* Y, *then it holds for* X *and descendent insertions*

$$\gamma_{i|X}, \quad i = 1, \ldots, r. \tag{3.7}$$

(b) *Set* $A = \prod_{i=1}^r \tau_{\alpha_i - 1}(\gamma_i|_X)$. *If furthermore Conjecture 3.7* (2) *holds for* Y, *then it holds for* X *with descendent insertions* (3.7), *i.e.,*

$$(-q)^{-\mathsf{c}_\beta/2} \mathrm{Z}_{\mathrm{PT}}\left(X; q \,|\, A\right)_\beta = (-iu)^{\mathsf{c}_\beta} \mathrm{Z}'_{\mathrm{GW}}\left(X; u \,\middle|\, \overline{A}\right)_\beta.$$

Proof. By the string equation, we may assume that $\gamma_i \in H^{>0}(\mathfrak{X}, \mathbb{Q})$ for all i. Let $d = E^3$ be the degree of the transition $Y \searrow X$. Set $S_{\mathrm{loc}} = S_d$, $X_{\mathrm{loc}} = X_d$, and $Y_{\mathrm{loc}} = Y_d$. We denote by $h = c_1(\mathscr{O}_{X_{\mathrm{loc}}}(1))$, and by E (resp. H) the zero section $\mathbb{P}_{S_{\mathrm{loc}}}(\mathscr{O})$ (resp. the infinity section $\mathbb{P}_{S_{\mathrm{loc}}}(K_{S_{\mathrm{loc}}})$) of $\pi \colon Y_{\mathrm{loc}} \to S_{\mathrm{loc}}$.

Applying the degeneration formulas, Theorems 3.17 and 3.18, to the double point degeneration (2.6), and a virtual dimension counting, we get

$$\mathrm{Z}'_{\mathrm{GW}}\left(X; u \mid \overline{A}\right)_\beta = \sum_\rho \rho! u^{2\rho} \mathrm{Z}'_{\mathrm{GW}}\left(Y/E; u \mid \overline{\phi^* A} \mid (1,1)^\rho\right)_{\beta_0} \mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}/h; u \mid \varnothing \mid (1, \mathrm{pt})^\rho\right)_{\beta_\infty},$$

$$\mathrm{Z}_{\mathrm{PT}}\left(X; q \mid A\right)_\beta = \sum_\rho \rho! q^{-\rho} \mathrm{Z}_{\mathrm{PT}}\left(Y/E; u \mid \phi^* A \mid (1,1)^\rho\right)_{\beta_0} \mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}/h; q \mid \varnothing \mid (1, \mathrm{pt})^\rho\right)_{\beta_\infty},$$

where $(E,\beta_0)=(h,\beta_\infty)=\rho$ and $\phi_*\beta_0=\beta$.

For $r\geq 0$ and a cohomology weighted partition η, we define

$$\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}/h;u\mid A\mid\eta\right)_r := \sum_{(h,\beta_{\mathrm{loc}})=r}\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}/h;u\mid A\mid\eta\right)_{\beta_{\mathrm{loc}}},$$
$$\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}/h;q\mid A\mid\eta\right)_r := \sum_{(h,\beta_{\mathrm{loc}})=r}\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}/h;q\mid A\mid\eta\right)_{\beta_{\mathrm{loc}}}.$$

Then it suffices to show that the conjecture holds for Y/E with any insertion and for X_{loc}/h with trivial insertion, i.e., for $(E,\tilde\beta)=r$,

$$(-iu)^{\mathsf{c}_\beta}\mathrm{Z}'_{\mathrm{GW}}\left(Y/E;u\mid\overline{\phi^*A}\mid(1,1)^r\right)_{\tilde\beta}=(-q)^{\mathsf{c}_\beta/2}\mathrm{Z}_{\mathrm{PT}}\left(Y/E;q\mid\phi^*A\mid(1,1)^r\right)_{\tilde\beta}, \tag{3.8}$$
$$(-iu)^{2r}\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}/h;u\mid\varnothing\mid(1,\mathrm{pt})^r\right)_r=(-q)^r\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}/h;q\mid\varnothing\mid(1,\mathrm{pt})^r\right)_r, \tag{3.9}$$

and are rational in q.

For (3.8), we apply the degeneration formulas to the double point degeneration (2.3):

$$\mathrm{Z}'_{\mathrm{GW}}\left(Y;u\mid\overline{\phi^*A}\right)_{\tilde\beta}=\sum_\rho\rho!u^{2\rho}\mathrm{Z}'_{\mathrm{GW}}\left(Y/E;u\mid\overline{\phi^*A}\mid(1,1)^\rho\right)_{\tilde\beta_0}\mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/H;u\mid\varnothing\mid(1,\mathrm{pt})^\rho\right)_{\tilde\beta_\infty},$$
$$\mathrm{Z}_{\mathrm{PT}}\left(Y;q\mid\phi^*A\right)_{\tilde\beta}=\sum_\rho\rho!q^{-\rho}\mathrm{Z}_{\mathrm{PT}}\left(Y/E;q\mid\phi^*A\mid(1,1)^\rho\right)_{\tilde\beta_0}\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/H;q\mid\varnothing\mid(1,\mathrm{pt})^\rho\right)_{\tilde\beta_\infty},$$

where $(E,\tilde\beta_0)=(H,\tilde\beta_\infty)=\rho$ and $\tilde\beta_0+\pi_*\tilde\beta_\infty=\tilde\beta$. Let $\mathfrak{f}$ be the fiber class of $\pi\colon Y_{\mathrm{loc}}\to S_{\mathrm{loc}}$. Note that when $\tilde\beta_0=\tilde\beta$, we must have $\rho=r$ and $\beta_\infty=r\mathfrak{f}$.

By induction on the order of $\tilde\beta_0$, we only need to show that

(i) the conjecture holds for Y_{loc}/H;
(ii) the terms $\mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/H;u\mid\varnothing\mid(1,\mathrm{pt})^r\right)_{r\mathfrak{f}}$, $\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/H;q\mid\varnothing\mid(1,\mathrm{pt})^r\right)_{r\mathfrak{f}}$ are not zero.

The (i) follows from Theorem 3.16. For (ii), we apply [15, Proposition 6], which gives

$$\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/H;q\mid\varnothing\mid(1,\mathrm{pt})^r\right)_{r\mathfrak{f}}=\int_{\mathrm{Hilb}(S,r)}C_{(1,\mathrm{pt})^r}\neq 0,$$

and, by (i),

$$(-iu)^{2r}\mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/H;u\mid\varnothing\mid(1,\mathrm{pt})^r\right)_{r\mathfrak{f}}=(-q)^r\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/H;q\mid\varnothing\mid(1,\mathrm{pt})^r\right)_{r\mathfrak{f}}\neq 0.$$

For (3.9), we apply the degeneration formula to $X_{\mathrm{loc}}\rightsquigarrow Y_{\mathrm{loc}}\cup_E X_{\mathrm{loc}}$: if we let

$$\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}};u\mid A\right)_r := \sum_{(h,\beta_{\mathrm{loc}})=r}\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}};u\mid A\right)_{\beta_{\mathrm{loc}}},$$
$$\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}};q\mid A\right)_r := \sum_{(h,\beta_{\mathrm{loc}})=r}\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}};q\mid A\right)_{\beta_{\mathrm{loc}}},$$

then

$$\mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}; u \mid \mathrm{pt}^r\right)_r = \sum_{\rho} \rho! u^{2\rho} \mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/E; u \mid \mathrm{pt}^r \mid (1,1)^{\rho}\right)_{\beta_0} \mathrm{Z}'_{\mathrm{GW}}\left(X_{\mathrm{loc}}/h; u \mid \varnothing \mid (1,\mathrm{pt})^{\rho}\right)_{\rho},$$

$$\mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}; q \mid \mathrm{pt}^r\right)_r = \sum_{\rho} \rho! q^{-\rho} \mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/E; q \mid \mathrm{pt}^r \mid (1,1)^{\rho}\right)_{\beta_0} \mathrm{Z}_{\mathrm{PT}}\left(X_{\mathrm{loc}}/h; q \mid \varnothing \mid (1,\mathrm{pt})^{\rho}\right)_{\rho},$$

where $(E, \beta_0) = \rho$ and $(h, \phi_*\beta_0) = r$. Since $H - E = \pi^* c_1(\mathscr{O}_{S_d}(1))$ is nef on Y_{loc}, the condition implies that

$$0 \leq (H - E, \beta_0) = (h, \phi_*\beta_0) - (E, \beta_0) = r - \rho.$$

By induction on r, it suffices to prove

(1) the conjecture holds for X_{loc} with no ψ-classes;
(2) the conjecture holds for Y_{loc}/E;
(3) the terms $\mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/E; u \mid \mathrm{pt}^r \mid (1,1)^r\right)_{r\mathfrak{f}}$, $\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/E; q \mid \mathrm{pt}^r \mid (1,1)^r\right)_{r\mathfrak{f}}$ are not zero.

The (1) and (2) follow from Theorems 3.3 and 3.16 respectively. For (3), we apply [15, Proposition 6] again, which gives

$$\mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/E; q \mid \mathrm{pt}^r \mid (1,1)^r\right)_{r\mathfrak{f}} = \int_{\mathrm{Hilb}(S,r)} (\tau_0(\mathrm{pt}))^r \cdot C_{(1,1)^r} \neq 0,$$

and, by (2),

$$(-iu)^{2r} \mathrm{Z}'_{\mathrm{GW}}\left(Y_{\mathrm{loc}}/E; u \mid \mathrm{pt}^r \mid (1,1)^r\right)_{r\mathfrak{f}} = (-q)^r \mathrm{Z}_{\mathrm{PT}}\left(Y_{\mathrm{loc}}/E; q \mid \mathrm{pt}^r \mid (1,1)^r\right)_{r\mathfrak{f}} \neq 0.$$

This completes the proof. $\square$

Remark 3.20. Given a del Pezzo transition $Y \to \bar{Y} \rightsquigarrow X$, we assume that $h^i(\mathscr{O}_X) = h^i(\mathscr{O}_Y) = 0$ for $i = 1, 2$. Then $\rho(\bar{Y}) = \rho(X)$ (cf. [3, Proposition 3.1]) and thus one may infer that the restriction map $H^{\mathrm{ev}}(\mathfrak{X}, \mathbb{Q}) \cong H^{\mathrm{ev}}(\bar{Y}, \mathbb{Q}) \to H^{\mathrm{ev}}(X, \mathbb{Q})$ is surjective (cf. [5, Proposition 1.13]).

In the proof of Theorem 3.19, we have used Pardon's result (Theorem 3.3) on X_d for primary insertions. Applying Theorem 3.19, we can further prove that Conjecture 3.7 holds for X_d with stationary descendent insertions.

Corollary 3.21. *Every smooth del Pezzo threefold X_d satisfies the descendent* GW/PT *correspondence with stationary descendent insertions, i.e., all descendent insertions are even classes of positive degree.*

Proof. For each $1 \leq d \leq 8$, consider the del Pezzo trasnition $Y_d \searrow X_d$ construced in Example 2.1. By Theorem 3.14, Y_d satisfies satisfies Conjecture 3.7. Since $h^i(\mathscr{O}_{X_d}) = h^i(\mathscr{O}_{Y_d}) = 0$ for $i = 1, 2$, every even cohomology class of X has a lifting in $H^{\mathrm{ev}}(\bar{Y}, \mathbb{Q}) \cong H^{\mathrm{ev}}(\mathfrak{X}, \mathbb{Q})$, see Remark 3.20. Therefore X_d satisfies the descendent GW/PT correspondence with stationary descendent insertions by Theorem 3.19. $\square$

Remark 3.22. The X_1 is a new case which is not contained in [11, 17]. For $d \in \{2, 5\}$, X_d is treated in [11, Corollaries 4.2 & 4.5] and other cases are contained in [16, 17].

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