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001

GLOBAL IGUSA ZETA FUNCTIONS AND K-EQUIVALENCE*

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Abstract. Let $\mathfrak{X}$ and $\mathfrak{X}'$ be two smooth projective varieties over the ring of integers of a p-adic field $\mathbf{K}$ with general fibers being X and X'. We introduce a (family of) good s-norms on the pluricanonical spaces of X and X', called global Igusa zeta functions in s, and show that if the r-canonical maps send X and X' birationally to their images respectively, then any isometry between $H^0(X, rK_X)$ and $H^0(X', rK_{X'})$ with respect to this s-norm (for some $s > 0$ and $s \neq 1/r$) induces K-equivalence on the $\mathbf{K}$-points between X and X'.

Key words. Igusa zeta function; K-equivalence; p-adic integration.

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0. Introduction. In Royden's study of isometries on Teichmüller spaces, he proved that a compact Riemann surface M of genus $g \geq 2$ is determined by the normed space $(H^0(M, 2K_M), \|-\|)$, where

$$\|\alpha\| = \int_M |\alpha|,$$

and $|\alpha| = |f|\, dx \wedge dy$ if $\alpha = f(z)\,(dz)^{\otimes 2}$ locally. Namely, the following Torelli type theorem holds.

THEOREM 0.1 ([Roy71, Theorem 1]). *Let M and M' be compact Riemann surfaces of genus $g \geq 2$, and let*

$$T\colon \big(H^0(M', 2K_{M'}), \|-\|\big) \longrightarrow \big(H^0(M, 2K_M), \|-\|\big)$$

be a complex linear isometry between the spaces of quadratic differentials. Then there is a conformal map $f\colon M \to M'$ and a complex constant u with $|u| = 1$ such that $T = u \cdot f^*$.

We are interested in studying the p-adic analogue of Royden's theory. Let $\mathbf{K}$ be a fixed p-adic (local) field, i.e., a finite extension of $\mathbf{Q}_p$, and $\mathcal{O}$ its ring of integers. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$, π a uniformizer of $\mathcal{O}$, and $\mathbf{F}_q = \mathcal{O}/\mathfrak{m}$ the residue field of $\mathcal{O}$. For an n-dimensional projective variety $\mathfrak{X}$ over $\mathcal{O}$, there are two parts of $\mathfrak{X}$ — the general fiber $X = \mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathbf{K}$ and the special fiber $X_0 = \mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathbf{F}_q$.

Now assume that X is smooth over $\mathbf{K}$. We can view $X(\mathbf{K})$ as a $\mathbf{K}$-analytic n-dimensional manifold (note that we used the smoothness of X here). Since $X(\mathbf{K})$ is compact, Serre's theorem shows that $X(\mathbf{K})$ is $\mathbf{K}$-bianalytic to a finite disjoint union (of copies) of $\mathcal{O}^n$ (see [Ser65], [Igu00, Sec. 7.5]). Let us define a quasinorm $\|-\|_{1/r}$ on the r-pluricanonical space $H^0(X, rK_X)$ as follows.

DEFINITION 0.2. For any $\alpha \in H^0(X, rK_X)$, we can write it locally as $a(u)\,(du)^r$ on a chart $U \cong \mathcal{O}^n$ with coordinate $u = (u_1, \ldots, u_n)$. We define

$$\int_U |\alpha|^{1/r} = \int_{\mathcal{O}^n} |a(u)|^{1/r} d\mu_{\mathcal{O}},$$

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where $|-|$ is the normalized norm on $\mathbf{K}$ so that $|u| = q^{-v(u)}$, and $\mu_{\mathcal{O}}$ is the normalized Haar measure on the locally compact abelian group $\mathbf{K}^n$ so that $\mu_{\mathcal{O}}(\mathcal{O}^n) = 1$. Also, we define the p-adic norm of α on X by

$$\|\alpha\|_{1/r} = \int_{X(\mathbf{K})} |\alpha|^{1/r} := \sum_{U} \int_{U} |\alpha|^{1/r},$$

which is independent of the choice of atlas $\{U\}$ by the change of variables formula (see [Igu00, Sec. 7.4]).

Under this analogous p-adic setting, in author's bachelor thesis [Lee21], Royden's method was generalized to curves over $\mathbf{K}$ via p-adic integrals, and indeed the estimates are easier than those in [Roy71]. It is natural for us to consider the higher dimensional case. Actually the higher dimensional case over $\mathbf{C}$, conjectured by Yau, was generalized by Chi for r being sufficiently large and sufficiently divisible [Chi16]. Indeed, M can be determined up to a birational map by the (pseudo-)norm

$$\|\alpha\| = \int_{M} |\alpha|^{2/r}$$

on $H^0(M, rK_M)$. Later on, the same statement for general r was proven by Antonakoudis in [Ant14, 5.2]. Therefore, we shall do the p-adic work by adopting Antonakoudis' approach in the present paper.

Throughout this paper, we will use Fraktur font to denote varieties over $\mathcal{O}$ (e.g. $\mathfrak{X}$, $\mathfrak{X}'$, $\mathfrak{Y}$), normal font to denote their general fibers over $\mathbf{K}$ (e.g. X, X', Y), and normal font with subscript 0 to denote their special fibers over $\mathbf{F}_q$ (e.g. X_0, X_0', Y_0).

From now on, let us assume that X is smooth over $\mathbf{K}$. Strictly speaking, we should also require that $X(\mathbf{K})$ is nonempty so that the integrals are nontrivial. In fact, under a more stringent assumption that $\mathfrak{X}$ is smooth over $\mathcal{O}$, there exists a good reduction map

$$h_1 \colon X(\mathbf{K}) \longrightarrow X_0(\mathbf{F}_q),$$

and $X(\mathbf{K}) \neq \varnothing$ is equivalent to $X_0(\mathbf{F}_q) \neq \varnothing$. We shall see that the latter condition can be achieved when q is sufficiently large by using the Weil conjectures (cf. Remark 1.1).

The first result in this paper can be stated as follows (cf. Theorem 2.1).

THEOREM 0.3. *Let X and X' be smooth projective varieties over $\mathbf{K}$ of dimension n, and let r be a positive integer. Suppose there is an isometry*

$$T \colon \big(H^0(X', rK_{X'}), \|-\|_{1/r}\big) \longrightarrow \big(H^0(X, rK_X), \|-\|_{1/r}\big).$$

Then the images of the r-canonical maps $\Phi_{|rK_X|}$ and $\Phi_{|rK_{X'}|}$ are birational to each other.

The proof of the complex case given by Antonakoudis was based on Fourier transforms over $\mathbf{C}$ used by Rudin in [Rud08, 7.5.2]. However, some properties of Fourier transforms used in the proof may not hold in the p-adic setting. To achieve the p-adic statement, we shall use cut-off functions to bypass the difficulty and this in turn simplifies the argument.

Recall that two birational $\mathbf{Q}$-Gorenstein varieties X_1 and X_2 are K-equivalent if there exists a common resolution $\phi_1 \colon Y \to X_1$ and $\phi_2 \colon Y \to X_2$ such that $\phi_1^* K_{X_1} = \phi_2^* K_{X_2}$. For smooth varieties, K-equivalence is the same as c_1-equivalence.

K-equivalent varieties share many properties; for example, V. Batyrev [Bat99] and C.-L. Wang [Wan98] showed that K-equivalent smooth varieties over $\mathbf{C}$ have the same Betti numbers.

In Theorem 0.3, assuming that the r-canonical maps are birational to their images, X and X' are birational. Therefore, it is of interest to determine whether the birational map gives rise to a K-equivalence.

In [Bat99] and [Wan98], the *canonical measure* $\mu_{\mathfrak{X}}$ was intensively used, and we shall recall its construction here for reader's convenience. By Hensel's lemma, each fiber $h_1^{-1}(\bar{x})$ of the good reduction map h_1 is $\mathbf{K}$-bianalytic to $\pi\mathcal{O}^n$ (canonically up to a linear transformation, see [Liu02, Section 10.1]). This gives a canonical choice of the decomposition of $X(\mathbf{K})$ in Serre's theorem, i.e.,

$$X(\mathbf{K}) = \bigsqcup_{\bar{x}\in X_0(\mathbf{F}_q)} h_1^{-1}(\bar{x}).$$

Using the canonical measure on $\pi\mathcal{O}^n$ (defined as the normalized Haar measure $\mu_{\mathcal{O}}$ on the locally compact abelian group $\mathbf{K}^n$ so that $\mu_{\mathcal{O}}(\mathcal{O}^n) = 1$), we obtain a measure on each $h_1^{-1}(\bar{x})$ which can be glued into a measure $\mu_{\mathfrak{X}}$ on $X(\mathbf{K})$. Note that this measure is independent of the choice of the bianalytic map $h_1^{-1}(\bar{x}) \xrightarrow{\sim} \pi\mathcal{O}^n$.

To further characterize K-equivalence, we will construct a new norm on $H^0(X, rK_X)$ by mixing the canonical measure $d\mu_{\mathfrak{X}}$ and the p-adic norm $|\alpha|^{1/r}$; this construction is crucial in our approach. For a positive number s, the s-norm of $\alpha \in H^0(X, rK_X)$ is defined as

$$\|\alpha\|_s = \int_{X(\mathbf{K})} |\alpha|^s \, d\mu_{\mathfrak{X}}^{1-rs}.$$

Here, the integral is defined similarly to the one in Definition 0.2: locally, if $\alpha = a(u)\,(du)^r$ on a chart $U \cong \pi^k\mathcal{O}^n$ in $h^{-1}(\bar{x}) \cong \pi\mathcal{O}^n$, we define

$$\int_U |\alpha|^s \, d\mu_{\mathfrak{X}}^{1-rs} = \int_{\pi^k\mathcal{O}^n} |a(u)|^s \, d\mu_{\mathcal{O}}.$$

This is independent of the choice of atlas $\{U\}$.

It is clear that $\|-\|_s$ defines a norm on the space $H^0(X, rK_X)$. Since the Igusa zeta function associated to an analytic function $f\colon \mathcal{O}^n \to \mathbf{K}$ is defined by

$$s \longmapsto Z(s, f) = \int_{\mathcal{O}^n} |f|^s \, d\mu_{\mathcal{O}},$$

we shall call the assignment

$$s \longmapsto \|\alpha\|_s = \int_{X(\mathbf{K})} |\alpha|^s \, d\mu_{\mathfrak{X}}^{1-rs}, \quad s \in \mathbf{C} \text{ and } \operatorname{Re}(s) \geq 0,$$

the *global Igusa zeta function* associated to α. The term "global" is used to emphasize that they are defined over projective varieties, whereas the original (local) Igusa zeta functions are defined over the affine space $\mathcal{O}^n$. As in the case of local Igusa zeta functions, global Igusa zeta functions are related to the Poincaré series and are rational in $t = q^{-s}$ (see [Igu78], [Meu81], [Den84]).

For later reference, we introduce the following birationality assumption:

(♠) $\Phi_{|V|}\colon X \dashrightarrow \mathbf{P}(V)^\vee$ and $\Phi_{|V'|}\colon X' \dashrightarrow \mathbf{P}(V')^\vee$ are birational to their images.

With the above setup, we now state the second result (cf. Corollary 3.3).

THEOREM 0.4. *Let $\mathfrak{X}$ and $\mathfrak{X}'$ be smooth projective varieties over $\mathcal{O}$ of relative dimension n and let r be a positive integer. Suppose the general fibers X and X' satisfy the birationality assumption (♠), with $V = H^0(X, rK_X)$ and $V' = H^0(X', rK_{X'})$, and there is an isometry*

$$T\colon (V', \|-\|_s) \longrightarrow (V, \|-\|_s)$$

for some positive number $s \neq 1/r$. Then X and X' are K-equivalent on the $\mathbf{K}$-points, i.e., there exists a smooth projective variety Y over $\mathbf{K}$ with birational morphisms $\phi\colon Y \to X$ and $\phi'\colon Y \to X'$ such that $\phi^* K_X = \phi'^* K_{X'}$ on $Y(\mathbf{K})$.

In [Kaw02], Kawamata proved that D-equivalence (i.e., equivalence between derived categories of coherent sheaves) implies K-equivalence for general type smooth projective varieties over $\mathbf{C}$. As a p-adic normed space analogue, the second result of this paper is to show that isometry on the p-adic normed spaces implies K-equivalence on the $\mathbf{K}$-points when X and X' satisfy (♠).

The main idea of the proof is to determine the Jacobian function by the norms. More precisely, for a resolution $\phi\colon Y \to X$, we use the norms on $H^0(X, rK_X)$ and $H^0(Y, rK_Y)$ to determine $J_\phi = \phi^* d\mu_{\mathfrak{X}} / d\mu_Y$, based on the proof of Theorem 0.3.

The same proof applies to the case where $\mathfrak{X}$ is replaced by a klt pair $(\mathfrak{X}, \mathfrak{D})$ so that the canonical measure $d\mu_{\mathfrak{X},\mathfrak{D}}$ (defined in Section 1) is finite and the space $H^0(X, rK_X)$ is replaced by a linear subspace V of $H^0(X, \lfloor r(K_X + D + L)\rfloor)$ for some $\mathbf{R}$-divisor L in order that the map $\Phi_{|V|}$ maps X birationally into its image.

I have to remark that Chi posted a similar result to Theorem 0.3 in [Chi22] later on. In fact, he was in my bachelor thesis oral examination committee although he resigned after reviewing my manuscript. The paper is organized as follows. In Section 1, we will construct the global Igusa zeta function using the p-adic norm and the canonical measure. In Section 2, we will give a proof of Theorem 0.3. In Section 3, we will prove Theorem 0.4, our result on K-equivalence.

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1. Global Igusa zeta functions. As in the introduction, for a p-adic (local) field $(\mathbf{K}, |-|)$, let

- $\mathcal{O} = \{x \in \mathbf{K} \mid |x| \leq 1\}$ be the ring of integers,
- $\mathfrak{m} = \{x \in \mathbf{K} \mid |x| < 1\} = (\pi)$, the maximal ideal of $\mathcal{O}$,
- $v\colon \mathbf{K} \to \mathbf{Z} \cup \{\infty\}$, the valuation on $\mathbf{K}$, and
- $\mathbf{F}_q = \mathcal{O}/\mathfrak{m}$, the residue field.

Here we scale the norm so that $|u| = q^{-v(u)}$ for all $u \in \mathbf{K}^\times$.

Fix a positive integer r. Let $\mathfrak{X}$ be an n-dimensional projective variety over $\mathcal{O}$ with the following assumption: $\operatorname{Sm} X(\mathbf{K})$ (as a $\mathbf{K}$-analytic manifold) is nonempty, where $\operatorname{Sm} X = X \setminus \operatorname{Sing} X$ is the smooth part of X. It follows from the valuative criterion for properness that $\mathfrak{X}(\mathcal{O}) = \mathfrak{X}(\mathbf{K}) = X(\mathbf{K})$.

When X is smooth, we have defined the norm $\|-\|_{1/r}$ on the space of pluricanonical r-forms $H^0(X, rK_X)$ via the p-adic integral:

$$\|\alpha\|_{1/r} = \int_{X(\mathbf{K})} |\alpha|^{1/r}.$$

This integral is also defined when X has at worst klt singularities. The assumption that $\operatorname{Sm} X(\mathbf{K})$ is nonempty ensures that the norm is not trivial (by implicit function theorem). Another reason to make this assumption is that we want $X(\mathbf{K})$ to be Zariski dense in X. Indeed, $X(\mathbf{K})$ contains a $\mathbf{K}$-analytic open subset that is bianalytic to an n-dimensional polydisc Δ with the result that the dimension of the Zariski closure of $X(\mathbf{K})$ is at least n.

In order to resolve the non-general type case, we shall extend the definition of the norm to a Kawamata log-terminal (klt for short) pair (X, D):

$$\|\alpha\|_{1/r,D} := \int_{X(\mathbf{K})} |\alpha|^{1/r}, \quad \alpha \in H^0(X, \lfloor r(K_X + D)\rfloor).$$

When $\mathfrak{X}$ is smooth over $\mathcal{O}$, we have defined for each positive number s the s-norm

$$\|\alpha\|_s = \int_{X(\mathbf{K})} |\alpha|^s \, d\mu_{\mathfrak{X}}^{1-rs},$$

where $d\mu_{\mathfrak{X}}$ is the canonical measure on $X(\mathbf{K})$ defined by the good reduction

$$h_1 \colon \mathfrak{X}(\mathcal{O}) \longrightarrow X_0(\mathbf{F}_q),$$

and X_0 is the special fiber of $\mathfrak{X}$ over $\mathbf{F}_q$. The smooth assumption here is crucial: it provides a canonical decomposition

$$X(\mathbf{K}) = X(\mathcal{O}) = \coprod_{\bar{x} \in X_0(\mathbf{F}_q)} h_1^{-1}(\bar{x}) \cong \coprod_{\bar{x} \in X_0(\mathbf{F}_q)} \pi\mathcal{O}^n.$$

Note that $d\mu_{\mathfrak{X}}$ is locally equal to the p-adic norm $|\omega|$ of a generator $\omega \in H^0(\mathfrak{U}, K_{\mathfrak{X}})(\mathcal{O})$, where $\mathfrak{U} \subseteq \mathfrak{X}$ is an open subset such that $K_{\mathfrak{X}}|_{\mathfrak{U}}$ is free. This allows us to generalize the definition of canonical measure as in [Wan98, Section 2.10].

For a pair $(\mathfrak{X}, \mathfrak{D})$, we say it is klt if $\mathfrak{X}$ admits a $\mathcal{O}$-resolution $\phi \colon \mathfrak{Y} \to \mathfrak{X}$ and its general fiber (X, D) is klt. In this case, we define $\mu_{\mathfrak{X},\mathfrak{D}}$ locally as follows:

$$\mu_{\mathfrak{X},\mathfrak{D}}(S) = \int_{\phi^{-1}(S)} |\phi^*\omega|^{1/r_{\mathfrak{D}}},$$

where $r_{\mathfrak{D}}$ is a positive integer and $\mathfrak{U} \subseteq \mathfrak{X}$ is an open subset such that $r_{\mathfrak{D}}(K_{\mathfrak{X}} + \mathfrak{D})$ is Cartier and free on $\mathfrak{U}$, and $\omega \in H^0(\mathfrak{U}, r_{\mathfrak{D}}(K_{\mathfrak{X}} + \mathfrak{D}))$ is a local generator. Note that this p-adic norm does not depend on the choice of ω, $r_{\mathfrak{D}}$ and $\mathfrak{U}$, and hence glues into a measure $\mu_{\mathfrak{X},\mathfrak{D}}$ on $X(\mathbf{K})$.

When $\mathfrak{X}$ is smooth over $\mathcal{O}$, then $d\mu_{\mathfrak{X},\mathfrak{D}}$ is locally the p-adic norm $|\omega|^{1/r_{\mathfrak{D}}}$ of a generator ω of $r_{\mathfrak{D}}(K_{\mathfrak{X}} + \mathfrak{D})$.

For any $\mathbf{R}$-divisor L and any positive integer r, we define $\|-\|_{s,\mathfrak{D}}$ similarly on the space $H^0(X, \lfloor r(K_X + D + L)\rfloor)$ by

$$\|\alpha\|_{s,\mathfrak{D}} = \int_{X(\mathbf{K})} |\alpha|^s \, d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs}$$

when $\mathrm{Re}(s) \geq 0$ is small enough.

Indeed, replacing L by L', which is determined by the equation

$$r(K_X + D + L') = \lfloor r(K_X + D + L) \rfloor,$$

we may assume that $r(K_X + D + L)$ is Cartier. Let F be the fixed locus divisor of the linear system $|r(K_X + D + L)|$ and consider a log-resolution $\phi\colon Y \to X$ such that

$$\phi^* D = \sum_{E \in \mathcal{E}} d_E E, \quad \phi^* L = \sum_{E \in \mathcal{E}} \ell_E E, \quad \phi^* F = \sum_{E \in \mathcal{E}} f_E E, \quad K_Y = \phi^* K_X + \sum_{E \in \mathcal{E}} m_E E$$

with $\sum E$ a smooth normal crossing divisor of the K-analytic manifold Y. Then $\|-\|_{s,\mathfrak{D}}$ is defined on $H^0(X, r(K_X + D + L))$ when

$$\mathrm{Re}\big(s \cdot (f_E + r(m_E - d_E - \ell_E)) + (1 - rs) \cdot (m_E - d_E)\big) > -1, \quad \forall E \in \mathcal{E},$$

which is equivalent to

$$\mathrm{Re}(s) < s_r(\mathfrak{X}, \mathfrak{D}, L) := \inf_{E \in \mathcal{E}^+} \left(\frac{m_E - d_E + 1}{r\ell_E - f_E} \right),$$

where $\mathcal{E}^+ = \{E \in \mathcal{E} \mid r\ell_E > f_E\}$. Note that the klt assumption on the pair $(\mathfrak{X}, \mathfrak{D})$ is used here so that $s_r(\mathfrak{X}, \mathfrak{D}, L) > 0$.

Since $s_r(\mathfrak{X}, \mathfrak{D}, L)$ is defined to be the largest number such that the integral of $|\alpha|^s \, d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs}$ is finite for all $0 \leq \mathrm{Re}(s) < s_r(\mathfrak{X}, \mathfrak{D}, L)$ and $\alpha \in H^0(X, r(K_X + D + L))$, it is independent of the choice of the resolution ϕ.

For general L and a linear subspace $V \subseteq H^0(X, \lfloor r(K_X + D + L) \rfloor)$, $s_r(\mathfrak{X}, \mathfrak{D}, L)$ is defined to be $s_r(\mathfrak{X}, \mathfrak{D}, L')$, where L' is determined by the equation

$$r(K_X + D + L') = \lfloor r(K_X + D + L) \rfloor;$$

$s_r(\mathfrak{X}, \mathfrak{D}, V) \geq s_r(\mathfrak{X}, \mathfrak{D}, L)$ is defined to be the largest number such that the integral of $|\alpha|^s \, d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs}$ is finite for all $0 \leq \mathrm{Re}(s) < s_r(\mathfrak{X}, \mathfrak{D}, V)$ and $\alpha \in V$.

In the smooth case, the assumption on $X(\mathbf{K})$ being nonempty is equivalent to $X_0(\mathbf{F}_q)$ being nonempty.

Remark 1.1. By the Weil conjectures [Del74], we have

$$\# X_0(\mathbf{F}_q) = \sum_{i=0}^{2n} \sum_{j=1}^{h_i} (-1)^i \alpha_{ij} \geq q^n + 1 - \sum_{i=1}^{2n-1} h^i q^{i/2},$$

where α_{ij} are algebraic integers satisfying $|\alpha_{ij}| = q^{i/2}$, and $h^i = h^i(X)$ are the Betti numbers. Hence, the condition $\# X_0(\mathbf{F}_q) > 0$ could be achieved by imposing some bounds on q and h^i's. For example,

$$q \geq \left(\sum_{i=1}^{2n-1} h^i \right)^2$$

will do. We denote by $q_0(\mathfrak{X})$ to be the smallest positive integer such that

$$q^n + 1 \geq \sum_{i=1}^{2n-1} h^i q^{i/2}, \quad \forall q \geq q_0(\mathfrak{X}).$$

In particular, when $\mathfrak{X}$ is a curve (over $\mathcal{O}$) of genus $g \geq 1$, $q_0(\mathfrak{X}) = 4g^2 - 2$.

For each $k \geq 1$, consider the mod-$\mathfrak{m}^k$ reductions

$$\begin{array}{ccc} \mathfrak{X}(\mathcal{O}) \xrightarrow{h_k} \mathfrak{X}(\mathcal{O}/\mathfrak{m}^k) & \text{and} & H^0(\mathfrak{X}, rK_{\mathfrak{X}})(\mathcal{O}) \longrightarrow H^0(\mathfrak{X}, rK_{\mathfrak{X}})(\mathcal{O}/\mathfrak{m}^k) \\ x \longmapsto \bar{x}^{(k)} & & \alpha \longmapsto \bar{\alpha}^{(k)}, \end{array}$$

where $H^0(\mathfrak{X}, rK_{\mathfrak{X}})(R)$ denotes the R-points of $H^0(\mathfrak{X}, rK_{\mathfrak{X}})$ for $R = \mathcal{O}$ or $\mathcal{O}/\mathfrak{m}^k$. Applying the method in [Igu77], the norm of $\alpha \in H^0(\mathfrak{X}, rK_{\mathfrak{X}})(\mathcal{O})$ can be computed by counting its zeros on each $\mathfrak{X}(\mathcal{O}/\mathfrak{m}^k)$:

Proposition 1.2. *Suppose that $\mathfrak{X}$ is smooth. For a nonzero element $\alpha \in H^0(\mathfrak{X}, rK_{\mathfrak{X}})(\mathcal{O})$, we have*

$$\|\alpha\|_s = \frac{\#X_0(\mathbf{F}_q)}{q^n} - (q^s - 1) \sum_{k=1}^{\infty} \frac{N_k}{q^{k(n+s)}},$$

where

$$N_k = \# \left\{ \bar{x}^{(k)} \in \mathfrak{X}(\mathcal{O}/\mathfrak{m}^k) \,\middle|\, \bar{\alpha}^{(k)}\left(\bar{x}^{(k)}\right) = 0 \right\}$$

is the cardinality of the zero set of $\bar{\alpha}^{(k)} \in H^0(\mathfrak{X}, rK_{\mathfrak{X}})(\mathcal{O}/\mathfrak{m}^k)$ on $\mathfrak{X}(\mathcal{O}/\mathfrak{m}^k)$.

Proof. Since $\mathfrak{X}$ is smooth over $\mathcal{O}$, each fiber of h_1 is $\mathbf{K}$-bianalytic to $\pi\mathcal{O}^n$ with measure preserved. For $\bar{x}^{(1)} \in X_0(\mathbf{F}_q)$, let $u = (u_1, \ldots, u_n)$ be a local coordinate of $h_1^{-1}(\bar{x}^{(1)})$. Then

$$\int_{h_1^{-1}(\bar{x}^{(1)})} |\alpha|^s = \int_{\pi\mathcal{O}^n} |a(u)|^s \, du$$

for some analytic function $a(u) \in \mathcal{O}[[u]]$. Let

$$N_k\left(\bar{x}^{(1)}\right) = \# \left\{ \bar{y}^{(k)} \in h_k(h_1^{-1}(\bar{x}^{(1)})) \,\middle|\, \bar{\alpha}^{(k)}\left(\bar{y}^{(k)}\right) = 0 \right\}$$

be the cardinality of the zero set of $\bar{\alpha}^{(k)}$ on the fiber of $\bar{x}^{(1)}$. Then it follows from

$$\mu_{\mathcal{O}}\left(|a|^{-1}\left([0, q^{-k}]\right)\right) = \mu_{\mathcal{O}}\left(\left\{ u \mid \bar{a}^{(k)}\left(\bar{u}^{(k)}\right) = 0 \right\}\right) = \frac{N_k\left(\bar{x}^{(1)}\right)}{q^{kn}}$$

that

$$\int_{\pi\mathcal{O}^n} |a(u)|^s \, du = \sum_{k=0}^{\infty} q^{-ks} \cdot \mu_{\mathcal{O}}(|a|^{-1}(q^{-k})) = \frac{1}{q^n} - (q^s - 1) \sum_{k=1}^{\infty} q^{-ks} \cdot \frac{N_k\left(\bar{x}^{(1)}\right)}{q^{kn}}.$$

Summing the above equation over $\bar{x}^{(1)} \in X_0(\mathbf{F}_q)$, we get

$$\|\alpha\|_s = \sum_{\bar{x}^{(1)}} \left(\frac{1}{q^n} - (q^s - 1) \sum_{k=1}^{\infty} \frac{N_k\left(\bar{x}^{(1)}\right)}{q^{k(n+s)}} \right) = \frac{\#X_0(\mathbf{F}_q)}{q^n} - (q^s - 1) \sum_{k=1}^{\infty} \frac{N_k}{q^{k(n+s)}}.$$

$\square$

In particular, we have [Wei82, Theorem 2.2.5]

$$\|\alpha\|_0 = \int_{X(\mathbf{K})} d\mu_{\mathfrak{X}} = \frac{\#X_0(\mathbf{F}_q)}{q^n}, \quad \|\alpha\|_\infty := \lim_{s\to\infty} \|\alpha\|_s = \frac{\#X_0(\mathbf{F}_q) - N_1}{q^n}.$$

We see that

$$\|\alpha\|_s = \frac{\#X_0(\mathbf{F}_q)}{q^n} - (1-t)\sum_{k=1}^{\infty} \frac{N_k}{q^{kn}} t^{k-1}$$

is a holomorphic function in $t = q^{-s}$ near $t = 0$. As (local) Igusa zeta functions are rational functions in t [Igu78], [Meu81], the similar result also holds for global Igusa zeta functions:

PROPOSITION 1.3. *The holomorphic function $t \mapsto \|\alpha\|_s$ is a rational function in $t = q^{-s}$ of the form*

$$\frac{P_\alpha(t)}{\prod_E (q^{m_E+1} - t^{a_E})},$$

where P_α is a polynomial with coefficients in $\mathbf{Z}[q^{-1}]$ and $\{(a_E, m_E)\}_{E\in\mathcal{E}}$ are the discrepancies associated to the divisor (α).

In fact, it holds for the pair case:

PROPOSITION 1.4. *Let L be an $\mathbf{R}$-divisor on a projective klt pair $(\mathfrak{X}, \mathfrak{D})$. For a nonzero element $\alpha \in H^0(X, \lfloor r(K_X+D+L)\rfloor)$, the function $t_{\mathfrak{D}} \mapsto \|\alpha\|_{s,\mathfrak{D}}$ is a rational function in $t_{\mathfrak{D}} = q^{-s/r_{\mathfrak{D}}}$ of the form*

$$\frac{P_\alpha(t_{\mathfrak{D}})}{t_{\mathfrak{D}}^e \prod_E \big(q^{m_E-d_E+1} - t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a_E - r d_E)}\big)},$$

where $r_{\mathfrak{D}}$ is the least positive integer such that $r_{\mathfrak{D}}(K_X + D)$ is Cartier, P_α is a polynomial with coefficients in $\mathbf{Z}[q^{-1}]$, e is a nonnegative integer and $\{(a_E, d_E, m_E)\}_{E\in\mathcal{E}}$ are the discrepancies associated to the divisors (α) and D.

For the original ($\mathfrak{D} = 0$) case, $r_{\mathfrak{D}} = 1$, and notice that $\|\alpha\|_s$ is bounded as t tends to 0, so the exponent e could be chosen to be 0.

Proof. The proof is essentially the same as in [Meu81], using resolution of singularities. Consider a log resolution $\phi\colon Y \to X$ such that

$$\phi^*(\alpha) = \sum_E a_E E, \quad \phi^* D = \sum_E d_E E, \quad K_Y = \phi^* K_X + \sum_E m_E E$$

with $\sum E$ a normal crossing divisor.

It follows from the definition that

$$\|\alpha\|_{s,\mathfrak{D}} = \int_{X(\mathbf{K})} |\alpha|^s \, d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs} = \int_{Y(\mathbf{K})} |\phi^*\alpha|^s \cdot \phi^* d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs}.$$

Decompose $Y(\mathbf{K})$ into disjoint compact charts Y_i such that in coordinate we have

$$\phi^*\alpha = a_i(u) \cdot u^{A_i + rM_i} (du)^{\otimes r}, \quad \phi^* d\mu_{\mathfrak{X},\mathfrak{D}}(u) = m_i(u) \cdot |u|^{M_i - D_i} du,$$

where $a_i(u) \neq 0$ and $m_i(u) \neq 0$ for all $u \in Y_i$. After further decomposing Y_i, we may assume that each Y_i is a polydisc and that $|a_i(u)| \equiv q^{-a}$ and $m_i(u) \equiv q^{-m}$ are constants on Y_i. Then

$$\int_{Y(\mathbf{K})} |\phi^*\alpha|^s \cdot \phi^* d\mu_{\mathfrak{X},\mathfrak{D}}^{1-rs} = \sum_i \int_{Y_i} q^{-as}|u|^{sA_i} \cdot q^{-m(1-rs)}|u|^{M_i-(1-rs)D_i}\, du.$$

Say Y_i is the polydisc $\pi^k\mathcal{O}^n$. It is easy to calculate that

$$\begin{aligned}
&\int_{\pi^k\mathcal{O}^n} q^{-as}|u|^{sA_i} \cdot q^{-m(1-rs)}|u|^{M_i-(1-rs)D_i}\, du \\
&\qquad = q^{-m} t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a-rm)} \prod_j \int_{\pi^k\mathcal{O}^n} |u_j|^{sa_{i,j}+m_{i,j}-(1-rs)d_{i,j}}\, du \\
&\qquad = q^{-m} t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a-rm)} \prod_j \frac{(1-q^{-1})q^{-k(sa_{i,j}+m_{i,j}-(1-rs)d_{i,j}+1)}}{1-q^{-(sa_{i,j}+m_{i,j}-(1-rs)d_{i,j}+1)}} \\
&\qquad = \frac{(1-q^{-1})^n}{q^{m+(k-1)(\sum(m_{i,j}-d_{i,j}+1))}} \cdot \frac{t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a-rm+k\sum(a_{i,j}-rd_{i,j}))}}{\prod_j \left(q^{m_{i,j}-d_{i,j}+1} - t^{r_{\mathfrak{D}}(a_{i,j}-rd_{i,j})}\right)}.
\end{aligned}$$

Summing over i, we see that $\|\alpha\|_{s,\mathfrak{D}}$ is a rational function in $t_{\mathfrak{D}}$ of the form

$$\frac{P_\alpha(t_{\mathfrak{D}})}{t_{\mathfrak{D}}^e \prod_E (q^{m_E-d_E+1} - t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a_E-rd_E)})},$$

where P_α is a polynomial with coefficients lie in $\mathbf{Z}[q^{-1}]$ and e is a nonnegative integer (coming from the term $t_{\mathfrak{D}}^{r_{\mathfrak{D}}(a-rm+k\sum(a_{i,j}-rd_{i,j}))}$), as desired. $\square$

This proposition allows us to view $t_{\mathfrak{D}} = q^{-s/r_{\mathfrak{D}}}$ as a formal variable, and view the total norm $\|\alpha\|_{\mathfrak{D}} := (\|\alpha\|_{s,\mathfrak{D}})_s$ of a form α as an element in the function field $\mathbf{Q}(t_{\mathfrak{D}})$. Hence, there is a $\mathbf{Q}(t_{\mathfrak{D}})$-valued norm

$$\|{-}\|_{\mathfrak{D}} \colon \varinjlim_L H^0(X, \lfloor r(K_X+D+L)\rfloor) \longrightarrow \mathbf{Q}(t_{\mathfrak{D}}).$$

As a consequence, when $t_{\mathfrak{D},0} = q^{-s_0/r_{\mathfrak{D}}}$ is a transcendental number for some s_0, we can determine the rational function $\|\alpha\|_{\mathfrak{D}}$ by the value $\|\alpha\|_{s_0,\mathfrak{D}}$. So the norms $\|{-}\|_{s,\mathfrak{D}}$ on the space $H^0(X, \lfloor r(K_X+D+L)\rfloor)$, $0 \le \mathrm{Re}(s) < s_r(\mathfrak{X},\mathfrak{D},L)$, are in fact determined by $\|{-}\|_{s_0,\mathfrak{D}}$.

2. Characterizing birational models. Suppose that $f \colon X \dashrightarrow X'$ is a birational map between smooth projective varieties over $\mathbf{K}$. Then the properness of X' shows that there is an open set $U \subset X$ on which $f|_U \colon U \to X'$ is a birational morphism with $\operatorname{codim}_X(X \setminus U) \ge 2$. It follows that

$$f^* \colon \left(H^0(X', rK_{X'}), \|{-}\|_{1/r}\right) \longrightarrow \left(H^0(U, rK_X|_U), \|{-}\|_{1/r}\right) \cong \left(H^0(X, rK_X), \|{-}\|_{1/r}\right)$$

is an isometry by change of variables. This means that the normed space

$$\left(H^0(X, rK_X), \|{-}\|_{1/r}\right)$$

only reflects the birational class of X.

For simplicity, let $V_{r,X} = H^0(X, rK_X)$ and denote by $\Phi_{r,X}$ the r-canonical map

$$\Phi_{|rK_X|} \colon X \dashrightarrow \mathbf{P}(V_{r,X})^\vee.$$

For a klt pair (X, D) and an $\mathbf{R}$-divisor L on X, let $V_{r,(X,D),L} = H^0(X, \lfloor r(K_X + D + L)\rfloor)$ and denote by $\Phi_{r,(X,D),L}$ the map

$$\Phi_{|\lfloor r(K_X+D+L)\rfloor|} \colon X \dashrightarrow \mathbf{P}(V_{r,(X,D),L})^\vee.$$

For a linear subspace of V of $V_{r,(X,D),L}$, let Φ_V denote the map determined by the linear system $|V|$.

Using the p-adic analogue of the trick in [Ant14, Theorem 5.2], we can prove that:

THEOREM 2.1. *Let X and X' be smooth projective varieties over $\mathbf{K}$ of dimension n with nonempty $\mathbf{K}$-points, r a positive integer. Suppose there is a ($\mathbf{K}$-linear) isometry*

$$T \colon \big(V_{r,X'}, \|{-}\|_{1/r}\big) \longrightarrow \big(V_{r,X}, \|{-}\|_{1/r}\big).$$

Then the images of the r-canonical maps $\Phi_{r,X}$ and $\Phi_{r,X'}$ are birational to each other under the identification

$$\mathbf{P}(T)^\vee \colon \mathbf{P}(V_{r,X})^\vee \xrightarrow{\sim} \mathbf{P}(V_{r,X'})^\vee.$$

When $\mathfrak{X}$ and $\mathfrak{X}'$ are both smooth over $\mathcal{O}$, the statement also holds when the norm $\|{-}\|_{1/r}$ is replaced by $\|{-}\|_s$ for any positive number s.

Proof. Let $\alpha_0, \alpha_1, \dots, \alpha_N$ be a basis of $V_{r,X}$, and let $\alpha'_i = T^{-1}(\alpha_i)$, which forms a basis of $V_{r,X'}$. Then the r-canonical maps $\Phi_{r,X}$ and $\Phi_{r,X'}$ can be realized as

$$[\alpha_0 : \cdots : \alpha_N] \colon X \dashrightarrow \mathbf{P}^N \quad \text{and} \quad [\alpha'_0 : \cdots : \alpha'_N] \colon X' \dashrightarrow \mathbf{P}^N,$$

respectively.

In the following, let $s = 1/r$ if X and X' are just smooth projective varieties over $\mathbf{K}$.

Let $d\nu = |\alpha_0|^s \, d\mu_{\mathfrak{X}}^{1-rs}$ and $g_i = \frac{\alpha_i}{\alpha_0}$ on $X^\circ = X \setminus \{\alpha_0 = 0\}$, $d\nu' = |\alpha'_0|^s \, d\mu_{\mathfrak{X}'}^{1-rs}$ and $g'_i = \frac{\alpha'_i}{\alpha'_0}$ on $X'^\circ = X' \setminus \{\alpha'_0 = 0\}$. Using the isometry T, we get

$$\begin{aligned} \int_{X^\circ(\mathbf{K})} \Big|1 + \sum_i \lambda_i g_i\Big|^s d\nu &= \Big\|\alpha_0 + \sum_i \lambda_i \alpha_i\Big\|_s \\ &= \Big\|\alpha'_0 + \sum_i \lambda_i \alpha'_i\Big\|_s = \int_{X'^\circ(\mathbf{K})} \Big|1 + \sum_i \lambda_i g'_i\Big|^s d\nu' \end{aligned} \tag{1}$$

for all $(\lambda_1, \dots, \lambda_N) \in \mathbf{K}^N$.

Now, we need a p-adic analogue of Rudin's result [Rud08, 7.5.2]:

CLAIM. We have

$$\int_{X^\circ} h \circ (g_1, \dots, g_N) \, d\nu = \int_{X'^\circ} h \circ (g'_1, \dots, g'_N) \, d\nu' \tag{2}$$

for all nonnegative Borel function $h \colon \mathbf{K}^N \to \mathbf{R}$.

Proof of Claim. Let W be the set of all Borel function h such that (2) holds. It follows that W is invariant under translations, dilations, and scaling. So it suffices to prove that $\mathbf{1}_{\mathcal{O}^N} \in W$.

Let $G = (g_1, \dots, g_N)\colon X^\circ \to \mathbf{A}^N$ and $G' = (g'_1, \dots, g'_N)\colon X'^\circ \to \mathbf{A}^N$. Define

$$B_s(y) = B_s(y_1, \dots, y_N) = \int_{\mathcal{O}^N} \Big|1 + \sum_i \lambda_i y_i\Big|^s d\lambda,$$

where $y = (y_1, \dots, y_N) \in \mathbf{K}^N$ and $d\lambda = d\lambda_1 \cdots d\lambda_N$. By (1) and Fubini's theorem,

$$\begin{aligned}\int_{X(\mathbf{K})} B_s \circ G\, d\nu &= \int_{\mathcal{O}^N} \int_{X^\circ(\mathbf{K})} \Big|1 + \sum_i \lambda_i g_i\Big|^s d\nu d\lambda \\ &= \int_{\mathcal{O}^N} \int_{X'^\circ(\mathbf{K})} \Big|1 + \sum_i \lambda_i g'_i\Big|^s d\nu' d\lambda = \int_{X'(\mathbf{K})} B_s \circ G'\, d\nu'.\end{aligned}$$

By change of variables, we see that

$$B_s(y) = \begin{cases} 1 & \text{if } \max|y_i| \le 1, \\ b(s) \cdot \max|y_i|^s & \text{if } \max|y_i| \ge 1, \end{cases} \tag{3}$$

where $b(s) = \int_{\mathcal{O}} |y|^s\, dy = \frac{q-1}{q-t} \in (0,1)$ and $t = q^{-s}$. So, if we define the "cut-off" function

$$B_{s,0}(y) = t^{-1} B_s(\pi y) - B_s(y) = \begin{cases} t^{-1} - 1 & \text{if } \max|y_i| < 1, \\ t^{-1} - b(s) & \text{if } \max|y_i| = 1, \\ 0 & \text{if } \max|y_i| > 1, \end{cases} \tag{4}$$

then $B_{s,0}$ also satisfies (2), and hence lies in W. Since $B_{s,0}$ is a Schwartz–Bruhat function that is supported in $\mathcal{O}^N$ with $\int_{\mathcal{O}^N} B_{s,0}(y)\, dy \neq 0$ and is well-defined on $\mathcal{O}^N/\pi\mathcal{O}^N$, we see that

$$\mathbf{1}_{\mathcal{O}^N}(y) = \left(\frac{1}{\mu_{\mathcal{O}}(\pi\mathcal{O}^N)} \int_{\mathcal{O}^N} B_{s,0}(u)\, du\right)^{-1} \sum_{\bar{z} \in \mathcal{O}^N/\pi\mathcal{O}^N} B_{s,0}(y+z) \tag{5}$$

also lies in W, as desired. □

Taking $h = \mathbf{1}_{G(X^\circ)}$ in the claim, we get

$$\|\alpha_0\|_s = \int_{G^{-1}(G(X^\circ))} |\alpha_0|^s\, d\mu_X^{1-rs} = \int_{G'^{-1}(G(X^\circ))} |\alpha'_0|^s\, d\mu_{X'}^{1-rs} \le \|\alpha'_0\|_s = \|\alpha_0\|_s.$$

So the inequality above has to be an equality. Let U be the intersection of $G(X^\circ)$ and $G'(X'^\circ)$. The equality implies that the $\mathbf{K}$-points $U(\mathbf{K})$ of U has full measure in $G'(X'^\circ)(\mathbf{K})$ with respect to $G'_*\nu'$.

Let $\bar{X} \subseteq \mathbf{P}^N$ (resp. $\bar{X}' \subseteq \mathbf{P}^N$) be the image of $\Phi_{r,X}$ (resp. $\Phi_{r,X'}$), and let $\bar{n}$ (resp. $\bar{n}'$) be the dimension of $\bar{X}$ (resp. $\bar{X}'$). Since the general fiber of $X \dashrightarrow \bar{X}$ has dimension $n - \bar{n}$, the image of an n-dimensional polydisc Δ in $X(\mathbf{K})$ under $X \dashrightarrow \bar{X}$ has dimension $\bar{n}$ (at a generic point). The same argument applies to $X' \dashrightarrow \bar{X}'$. Thus, we conclude that $\bar{n} = \bar{n}'$ and that $U(\mathbf{K})$ contains an $\bar{n}$-dimensional polydisc.

It follows that U is a quasi-projective $\mathbf{K}$-variety of dimension at least $\bar{n}$. Since both $\bar{X}$ and $\bar{X}'$ have dimension $\bar{n}$, and both contain U, we conclude that $\bar{X}$ and $\bar{X}'$ are birational to each other under $\mathbf{P}(T)^\vee$. □

COROLLARY 2.2. *Let $\mathfrak{X}$ and $\mathfrak{X}'$ be smooth projective varieties over $\mathcal{O}$ of relative dimension n, r a positive integer. Suppose that* $q \ge \max\{q_0(\mathfrak{X}), q_0(\mathfrak{X}')\}$, *the* *birationality assumption* (♠) *holds for* $V = V_{r,X}$ *and* $V' = V_{r,X'}$*, and there is an isometry*

$$T\colon \big(V_{r,X'}, \|{-}\|_s\big) \longrightarrow \big(V_{r,X}, \|{-}\|_s\big)$$

for some positive number s*. Then there is a birational map* $f\colon X \dashrightarrow X'$ *such that* $T = u \cdot f^*$ *for some* $u \in \mathbf{K}^\times$*.*

Proof. The condition $q \geq \max\{q_0(\mathfrak{X}), q_0(\mathfrak{X}')\}$ ensures that both $X(\mathbf{K})$ and $X'(\mathbf{K})$ are nonempty. Since the images $\bar{X}$ and $\bar{X}'$ are birational to each other and both $\Phi_{r,X}$ and $\Phi_{r,X'}$ are birational maps, X and X' are birational to each other. The birational map $f\colon \mathfrak{X} \dashrightarrow \mathfrak{X}'$ comes from the identification

$$\mathbf{P}(T)^\vee\colon \mathbf{P}(V_{r,X})^\vee \xrightarrow{\sim} \mathbf{P}(V_{r,X'})^\vee.$$

Therefore, $T = u \cdot f^*$ for some $u \in \mathbf{K}^\times$. ◻

Imitating the proof, Theorem 2.1 and Corollary 2.2 are also generalized to the pair case (with only difference in notation):

Theorem 2.3. *Let* (X, D) *and* (X', D') *be projective klt pairs over* $\mathbf{K}$ *of dimension* n *with nonempty* $\mathbf{K}$*-points,* r *a positive integer. Let* L *(resp.* L'*) be an* $\mathbf{R}$*-divisor on* X *(resp.* X'*), and let* V *(resp.* V'*) be a linear subspace of* $V_{r,(X,D),L}$ *(resp.* $V_{r,(X',D'),L'}$*). Suppose* $1/r < \min\{s_r(X, D, V), s_r(X', D', V')\}$ *and there is an isometry*

$$T\colon \big(V', \|{-}\|_{1/r,D'}\big) \longrightarrow \big(V, \|{-}\|_{1/r,D}\big).$$

Then the images of the maps Φ_V *and* $\Phi_{V'}$ *are birational to each other.*

When $(\mathfrak{X}, \mathfrak{D})$ *and* $(\mathfrak{X}', \mathfrak{D}')$ *are both projective klt pairs, the statement also holds when* $1/r$ *is replaced by any positive number* $s < \min\{s_r(\mathfrak{X}, \mathfrak{D}, V), s_r(\mathfrak{X}', \mathfrak{D}', V')\}$*.*

Corollary 2.4. *Let* $(\mathfrak{X}, \mathfrak{D})$ *and* $(\mathfrak{X}', \mathfrak{D}')$ *be projective klt pairs over* $\mathcal{O}$ *of relative dimension* n*,* r *a positive integer. Let* L *(resp.* L'*) be an* $\mathbf{R}$*-divisor on* X *(resp.* X'*), and let* V *(resp.* V'*) be a linear subspace of* $V_{r,(X,D),L}$ *(resp.* $V_{r,(X',D'),L'}$*).*

Suppose that both $\mathfrak{X}(\mathbf{K})$ *and* $\mathfrak{X}'(\mathbf{K})$ *contains polydiscs, the birationality assumption* (♠) *holds, and there is an isometry*

$$T\colon \big(V', \|{-}\|_{s,\mathfrak{D}'}\big) \longrightarrow \big(V, \|{-}\|_{s,\mathfrak{D}}\big)$$

for some positive number $s < \min\{s_r(\mathfrak{X}, \mathfrak{D}, V), s_r(\mathfrak{X}', \mathfrak{D}', V')\}$*. Then there is a birational map* $f\colon \mathfrak{X} \dashrightarrow \mathfrak{X}'$ *such that* $T = u \cdot f^*$ *for some* $u \in \mathbf{K}^\times$*.*

3. Characterizing the K-equivalence. We say two projective varieties $\mathfrak{X}$ and $\mathfrak{X}'$ over $\mathcal{O}$ (resp. X and X' over $\mathbf{K}$) are *K-equivalent on the $\mathbf{K}$-points* if there is a projective smooth variety $\mathfrak{Y}$ over $\mathcal{O}$ (resp. Y over $\mathbf{K}$) with birational morphisms $\phi\colon \mathfrak{Y} \to \mathfrak{X}$ and $\phi'\colon \mathfrak{Y} \to \mathfrak{X}'$ (resp. $\phi\colon Y \to X$ and $\phi'\colon Y \to X'$) such that $\phi^* K_X = \phi'^* K_{X'}$ on the $\mathbf{K}$-points $Y(\mathbf{K})$. Equivalently,

$$K_Y = \phi^* K_X + \sum m_E E = \phi'^* K_{X'} + \sum m_E E$$

on $Y(\mathbf{K})$. For K-equivalence between klt pairs $(\mathfrak{X}, \mathfrak{D})$ and $(\mathfrak{X}', \mathfrak{D}')$, simply replace the relation $\phi^* K_X = \phi'^* K_{X'}$ by $\phi^*(K_X + D) = \phi'^*(K_{X'} + D')$.

REMARK 3.1. Note that this is a slightly weaker definition of the original K-equivalence, as it only captures the exceptional divisors that contain K-points. This means that we are working in the category of K-analytic spaces (coming from algebraic varieties). To include all exceptional divisors (as K-schemes), one may take a field extension.

When $\mathfrak{X}$ and $\mathfrak{X}'$ are K-equivalent on the $\mathbf{K}$-points over $\mathcal{O}$, the pullbacks of the canonical measures $\phi^* d\mu_{\mathfrak{X}}$ and $\phi'^* d\mu_{\mathfrak{X}'}$ are equal to each other. Consequently, the natural linear transformation

$$\big(V_{r,X'}, \|{-}\|_s\big) \longrightarrow \big(V_{r,X}, \|{-}\|_s\big) \tag{6}$$

is an isometry:

$$\|\alpha'\|_s = \int_{Y(\mathbf{K})} |\phi'^*\alpha'|^s \cdot \phi'^* d\mu_{\mathfrak{X}'}^{1-rs} = \int_{Y(\mathbf{K})} |\phi^*\alpha|^s \cdot \phi^* d\mu_{\mathfrak{X}}^{1-rs} = \|\alpha\|_s.$$

The main result of this section is to show that the isometry (6) gives us K-equivalence on the $\mathbf{K}$-points between X and X' when X and X' satisfy (♠).

In order to prove this, let us consider a proper birational morphism $\phi\colon Y \to X$ over $\mathbf{K}$ such that

$$K_Y = \phi^* K_X + \sum_{E\in\mathcal{E}} m_E E$$

on $Y(\mathbf{K})$. Suppose that the r-canonical maps

$$\begin{array}{ccc} Y & \overset{\Phi_{r,Y}}{\dashrightarrow} & \mathbf{P}\big(V_{r,Y}\big)^\vee \\ \Big\downarrow{\scriptstyle \phi} & & \wr\Big\downarrow{\scriptstyle \mathbf{P}(\phi^*)^\vee} \\ X & \overset{\Phi_{r,X}}{\dashrightarrow} & \mathbf{P}\big(V_{r,X}\big)^\vee . \end{array}$$

map X and Y birationally to their images.

For $\alpha \in V_{r,X}$, its pullback $\phi^*\alpha$ lies in $V_{r,Y}$. It is then natural to compare the difference between $\|\alpha\|_s$ and $\|\phi^*\alpha\|_s$. But there might be a problem: Y may not be defined over $\mathcal{O}$, so there may not be a canonical measure $\mu_{\mathfrak{Y}}$ on $Y(\mathbf{K})$. Instead, we construct a measure μ_Y temporarily as follows:

(†) decompose $Y(\mathbf{K})$ into finitely many polydiscs $Y_j \cong \pi^{k_j}\mathcal{O}^n$, and take $\mu_Y|_{Y_j}$ to be the canonical measure $\mu_{\mathcal{O}}$ on $\pi^{k_j}\mathcal{O}^n$. Using this measure μ_Y, we define the s-norm $\|{-}\|_s$ on $V_{r,Y}$ similarly:

$$\|\beta\|_s = \int_{Y(\mathbf{K})} |\beta|^s \, d\mu_Y^{1-rs}.$$

THEOREM 3.2. *For a fixed positive number $s \neq \frac{1}{r}$, the Jacobian $J_\phi\colon Y(\mathbf{K}) \to \mathbf{R}_{\geq 0}$ defined by the formula $\phi^* d\mu_{\mathfrak{X}} = J_\phi \, d\mu_Y$ is determined by the data*

$$\big(V_{r,X}, \|{-}\|_s\big) \quad \text{and} \quad \big(V_{r,Y}, \|{-}\|_s\big). \tag{7}$$

In particular, the set of divisors $\mathcal{E} = \{E\}$ and the positive integers m_E, $E \in \mathcal{E}$, are also determined by (7).

Proof. It follows from the construction (†) of μ_Y that $J_\phi\colon Y(\mathbf{K}) \to \mathbf{R}_{\geq 0}$ is continuous. Let $\alpha_0, \ldots, \alpha_N$ be a basis of $V_{r,X}$, $g_i = \frac{\alpha_i}{\alpha_0}$ a rational function on Y for each i, and $d\nu = |\phi^*\alpha_0|^s\, d\mu_Y^{1-rs}$. Then for a form $\alpha = \alpha_0 + \sum_i \lambda_i\alpha_i$, we have

$$\|\alpha\|_s = \int_{Y^\circ(\mathbf{K})} J_\phi^{1-rs}\left|1+\sum\lambda_i g_i\right|^s d\nu, \quad \|\phi^*\alpha\|_s = \int_{Y^\circ(\mathbf{K})}\left|1+\sum\lambda_i g_i\right|^s d\nu,$$

where $Y^\circ = Y \setminus \{\phi^*\alpha_0 = 0\}$.

For each Borel function $h\colon \mathbf{K}^N \to \mathbf{R}$, denote by $I(h) = (I_X(h), I_Y(h))$ the integrals given by

$$I_X(h) = \int_{Y^\circ(\mathbf{K})} J_\phi^{1-rs}\cdot (h\circ G)\, d\nu, \quad I_Y(h) = \int_{Y^\circ(\mathbf{K})} (h\circ G)\, d\nu,$$

where $G = (g_1, \ldots, g_N)\colon Y^\circ \to \mathbf{A}^N$.

As in the proof of Theorem 2.1, consider the functions $B_s(y)$ and $B_{s,0}(y)$ (defined in (3) and (4)). Let $A = \alpha_0 + \sum \mathcal{O}\alpha_i$. It follows that

$$I(B_s) = \big(I_X(B_s), I_Y(B_s)\big) = \left(\int_A \|\alpha\|_s\, d\alpha, \int_A \|\phi^*\alpha\|_s\, d\alpha\right),$$

and hence $I(B_{s,0})$, is determined by (7). Using (5), we see that $I(\mathbf{1}_{\mathcal{O}^N})$, and hence $I(h)$ for each nonnegative Borel function $h\colon \mathbf{K}^N \to \mathbf{R}$, is determined by (7).

Let U be a Zariski open dense subset of Y° on which $G|_U\colon U \to \mathbf{A}^N$ is an immersion and $G^{-1}(G(U)) = U$. Then for each $y \in U(\mathbf{K})$,

$$J_\phi(y) = \left(\lim_{\varepsilon\to 0^+} \frac{I_X\big(\mathbf{1}_{B_\varepsilon(G(y))}\big)}{I_Y\big(\mathbf{1}_{B_\varepsilon(G(y))}\big)}\right)^{\frac{1}{1-rs}}$$

is determined by (7) (note that $s \neq 1/r$). By the continuity of J_ϕ, we can then determine the function $J_\phi\colon Y \to \mathbf{R}_{\geq 0}$.

Thus, the union of the divisors (that contain $\mathbf{K}$-points)

$$\bigcup_{E\in\mathcal{E}} E = \big\{y \in Y(\mathbf{K}) \,\big|\, J_\phi(y) = 0\big\}$$

is determined by (7).

In order to find the order m_E of E, we pick a generic point $y \in E$ that does not lies in any other $E' \in \mathcal{E}$. Say y lies in the polydisc $Y_j \cong \pi^{k_j}\mathcal{O}^n$. Then under some coordinate $u = (u_1, \ldots, u_n)$,

$$\phi^* d\mu_X = J_\phi(u)\, d\mu_Y = m(u)\cdot |u_1|^{m_E}\, du,$$

for some nonvanishing continuous function $m(u)$. We may assume that $m(u) \equiv q^{-m}$ is a constant on some polydisc $\Delta^n \subseteq Y_j$ that contains y. Then for j large,

$$\begin{aligned}\big\{u \in \Delta^n \,\big|\, J_\phi(u) = q^{-j}\big\} \neq \varnothing \quad &\Longleftrightarrow \quad \{u_1 \in \Delta \mid |u_1|^{m_E} q^{-m} = q^{-j}\} \neq \varnothing \\ &\Longleftrightarrow \quad j \in m_E\mathbf{Z} + m.\end{aligned}$$

Therefore, m_E can be determined by J_ϕ, and hence, by (7). □

Corollary 3.3. *Let $\mathfrak{X}$ and $\mathfrak{X}'$ be smooth projective varieties over $\mathcal{O}$ of relative dimension n such that*

$$T\colon \big(V_{r,X'}, \|{-}\|_s\big) \longrightarrow \big(V_{r,X}, \|{-}\|_s\big)$$

is an isometry for some positive number $s \neq 1/r$. Suppose that the r-canonical maps $\Phi_{r,X}$ and $\Phi_{r,X'}$ maps X and X' birationally to their images. Then there exists a projective smooth variety Y over $\mathbf{K}$ with birational morphisms $\phi\colon Y \to X$ and $\phi'\colon Y \to X'$ such that $\phi^* d\mu_X$ is proportional to ${\phi'}^* d\mu_{X'}$. In particular, X and X' are K-equivalent on the $\mathbf{K}$-points.

Proof. It follows from Theorem 2.1 that X and X' are birational. So there is a resolution $\phi\colon Y \to X$, such that the birational map $f\colon X \dashrightarrow X'$ factors through some birational morphism $\phi'\colon Y \to X'$:

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X \arrow[rr, dashed, "f"] & & X' \end{tikzcd}$$

Note that the r-canonical map $\Phi_{r,Y}$ also maps Y birationally to its image.

Write

$$\phi^* K_X = K_Y + \sum m_E E, \quad {\phi'}^* K_{X'} = K_Y + \sum m'_E E,$$

where m_E, m'_E are nonnegative integers (viz. could be zero). Let μ_Y be the measure on $Y(\mathbf{K})$ we constructed in (†), and let $\|{-}\|_s$ be the s-norm induced by μ_Y. By Theorem 3.2, the Jacobian $J_\phi = \phi^* d\mu_{\mathfrak{X}}/d\mu_Y$ and $\{(E, m_E)\}_{E\in\mathcal{E}}$ are determined by

$$\big(V_{r,X}, \|{-}\|_s\big) \quad \text{and} \quad \big(V_{r,Y}, \|{-}\|_s\big),$$

while $J_{\phi'} = {\phi'}^* d\mu_{\mathfrak{X}'}/d\mu_Y$ and $\{(E, m'_E)\}_{E\in\mathcal{E}}$ are determined by

$$\big(V_{r,X'}, \|{-}\|_s\big) \quad \text{and} \quad \big(V_{r,Y}, \|{-}\|_s\big).$$

Since $T = u\cdot f^*$ for some $u \in \mathbf{K}^\times$, we see that

$${\phi'}^* d\mu_{\mathfrak{X}'} = J_{\phi'}\, d\mu_Y = |u|^{\frac{1}{1-rs}} J_\phi\, d\mu_Y = |u|^{\frac{1}{1-rs}}\, \phi^* d\mu_{\mathfrak{X}}$$

and $\{(E, m_E)\}_{E\in\mathcal{E}} = \{(E, m'_E)\}_{E\in\mathcal{E}}$, which shows that X is K-equivalent to X' on the $\mathbf{K}$-points. □

Remark 3.4.

(i) In the proof above, if we assume further more that there are resolutions $\phi\colon \mathfrak{Y} \to \mathfrak{X}$ and $\phi'\colon \mathfrak{Y} \to \mathfrak{X}'$ over $\mathcal{O}$, then this shows that $\mathfrak{X}$ and $\mathfrak{X}'$ are K-equivalent on the $\mathbf{K}$-points (over $\mathcal{O}$).

(ii) For other positive number s', using ${\phi'}^* d\mu_{\mathfrak{X}'} = \upsilon\, \phi^* d\mu_{\mathfrak{X}}$ for some $\upsilon \in \mathbf{R}_{>0}$, we see that

$$\begin{aligned} \|f^*\alpha\|_{s'} = \int \|\phi^*\alpha\|^{s'} J_{\phi'}^{1-rs'}\, d\mu_Y^{1-rs'} &= \upsilon^{1-rs'} \int \|\phi^*\alpha\|^{s'} J_{\phi}^{1-rs'}\, d\mu_Y^{1-rs'} \\ &= \upsilon^{1-rs'} \|\alpha\|_{s'}, \end{aligned}$$

i.e., under f^*, $\|{-}\|_{s'}$ on $V_{r,X'}$ is proportional to $\|{-}\|_{s'}$ on $V_{r,X}$.

(iii) When X and X' are already birational to each other, say $f\colon X \dashrightarrow X'$ is a birational map, the order on the s-norms also detects the K-partial ordering (defined in [Wan98]). More precisely, if $s < 1/r$ and

$$\|f^*\alpha'\|_s \le \|\alpha'\|_s, \quad \forall \alpha' \in H^0(X', rK_{X'}), \tag{8}$$

then $X \le_K X'$, i.e., there exists a birational correspondence (over $\mathbf{K}$)

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

with $\phi^* K_X \le \phi'^* K_{X'}$ on $Y(\mathbf{K})$. If $s > 1/r$, then the condition (8) implies $X \ge_K X'$.

(iv) The result also applies to the case in which r is a negative integer, for example, when both X and X' are Fano with $-r$ sufficiently large.

Again, the theorem and the corollary above are also generalized to the log pair case.

Corollary 3.5. *Let $(\mathfrak{X}, \mathfrak{D})$ and $(\mathfrak{X}', \mathfrak{D}')$ be projective klt pairs over $\mathcal{O}$ of relative dimension n. Let L (resp. L') be an $\mathbf{R}$-divisor on X (resp. X'), and let V (resp. V') be a linear subspace of $V_{r,(X,D),L}$ (resp. $V_{r,(X',D'),L'}$).*

Suppose that the maps Φ_V and $\Phi_{V'}$ maps X and X' birationally to their images, and there is an isometry

$$T\colon \big(V', \|{-}\|_{s,\mathfrak{D}'}\big) \longrightarrow \big(V, \|{-}\|_{s,\mathfrak{D}}\big)$$

for some positive number $s < \min\{s_r(X, D, V), s_r(X', D', V')\}$ with $s \neq 1/r$. Then (X, D) and (X', D') are K-equivalent on the $\mathbf{K}$-points.

Remark 3.6. Let $P(\partial_s) = \sum c_k \partial_s^k$ be a linear differential operator of degree $d \ge 1$ where $\partial_s = \frac{d}{ds}$. Then

$$P(\partial_s)\|\alpha\|_{s,D} = \int_{X(\mathbf{K})} P(a) t^a \, d\mu_{\mathfrak{X},\mathfrak{D}},$$

where $a = v(|\alpha|/d\mu^r_{\mathfrak{X},\mathfrak{D}})$. An isometry

$$\big(V', P(\partial_s)\|{-}\|_{s,\mathfrak{D}'}\big) \longrightarrow \big(V, P(\partial_s)\|{-}\|_{s,\mathfrak{D}}\big)$$

also gives us K-equivalence between $(\mathfrak{X}, \mathfrak{D})$ and $(\mathfrak{X}', \mathfrak{D}')$ when they both satisfy ($\spadesuit$) (even when $s = 0,\ 1/r$).

Indeed, replace $B_s(y)$ in the proof of Theorem 3.2 by

$$B_s^{P,a}(y) = \int_{\mathcal{O}^N} P\big(v\big(1 + \textstyle\sum \lambda_i y_i\big) + a\big) \cdot \big|1 + \textstyle\sum \lambda_i y_i\big|^s \, d\lambda.$$

Then the function

$$B_{s,0}^{P,a}(y) := (-1)^{d+1} \sum_{j=0}^{d+1} \tbinom{d+1}{j} (-t)^{-j} B_s^{P,a}(\pi^j y)$$

is supported in $\pi^{-d}\mathcal{O}^N$, and one can prove that the integral

$$Q^P(a) := \int_{\pi^{-d}\mathcal{O}^N} B^{P,a}_{s,0}(y)\,dy$$

is a polynomial in a with $\deg Q^P = d - \delta_{t1}$. Since $Q^P \neq 0$, we can then determine the Jacobian function J_ϕ following the proof above.

Applying the same method in Theorem 3.2, we can prove that:

PROPOSITION 3.7. *Let $(\mathfrak{X}, \mathfrak{D})$ be a projective klt pair over $\mathcal{O}$, L be an $\mathbf{R}$-divisor on X, V be a linear subspace of $V_{r,(X,D),L}$ such that Φ_V maps X birationally to its image. Then the $\mathbf{Q}(t_{\mathfrak{D}})$-valued total norm $\|-\|_{\mathfrak{D}} = (\|-\|_{s,\mathfrak{D}})_s$ on V can be determined by any two norms $\|-\|_{s_1,\mathfrak{D}}$ and $\|-\|_{s_2,\mathfrak{D}}$ with $s_1 \neq s_2$.*

Proof. Suppose that the data $\|-\|_{s_1,D}$ and $\|-\|_{s_2,D}$ with $s_1 \neq s_2$ are given. Then for each $\alpha \in V$ we can determine

$$I_i(h) = \int_{U(\mathbf{K})} (h \circ G) \cdot |\alpha|^{s_i}\, d\mu^{1-rs_i}_{\mathfrak{X},\mathfrak{D}}$$

for all nonnegative Borel function $h\colon \mathbf{K}^N \to \mathbf{R}$, where U is a Zariski dense open dense subset of X and $G\colon U \to \mathbf{A}^N$ is the immersion defined by Φ_V (up to a choice of basis).

Thus, we can determine

$$\frac{|\alpha|}{d\mu^r_{\mathfrak{X},\mathfrak{D}}}(x) = \lim_{\varepsilon\to 0^+} \left(\frac{I_2\big(B_\varepsilon(G(x))\big)}{I_1\big(B_\varepsilon(G(x))\big)} \right)^{\frac{1}{s_2-s_1}}$$

for each $x \in U(\mathbf{K})$. Let

$$Z = \left\{ u \in \mathbf{K}^N \,\middle|\, I_1\big(B_\varepsilon(u)\big) \neq 0,\ \forall \varepsilon > 0 \right\}.$$

It is clear that Z contains $G(U)(\mathbf{K})$ and is contained in the closure $\overline{G(U)(\mathbf{K})}$ of $G(U)(\mathbf{K})$.

Take

$$h(u) = \mathbf{1}_Z(u) \cdot \lim_{\varepsilon\to 0^+} \left(\frac{I_2\big(B_\varepsilon(u)\big)}{I_1\big(B_\varepsilon(u)\big)} \right)^{\frac{s-s_1}{s_2-s_1}},$$

so $h(G(x)) = (|\alpha|/d\mu^r_{\mathfrak{X},\mathfrak{D}})^{s-s_1}(x)$ for each $x \in U(\mathbf{K})$. We see that

$$\|\alpha\|_{s,\mathfrak{D}} = \int_{U(\mathbf{K})} \left(\frac{|\alpha|}{d\mu^r_{\mathfrak{X},\mathfrak{D}}} \right)^{s-s_1} \cdot |\alpha|^{s_1}\, d\mu^{1-rs_1}_{\mathfrak{X},\mathfrak{D}} = I_1(h)$$

is determined by the norms $\|-\|_{s_1,\mathfrak{D}}$ and $\|-\|_{s_2,\mathfrak{D}}$. ◻

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