Fractals, Compression and Contraction Mapping

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One of the most often cited applications of the study of fractals is that of their use in image compression. Such an application is not surprising, since seemingly complicated and intricate fractal images have relatively simple mathematical descriptions in terms of iterated mappings. Given also that fractals have been found to model well a wide variety of natural forms, it seems natural that we should try to exploit their self-similar properties to encode images of such forms.

We examine a simple example of a fractal, the Koch curve, to illustrate the principle of encoding a fractal image. Referring to Figure 2, we construct the Koch curve by first taking a line segment of length 1, K_0. We then construct K_1 by combining the images of this segment under four transformations, each involving a dilation of factor 1/3 composed with either or both a rotation and translation. Combining the images of K_1 under the same four transformations yields K_2, and the Koch curve itself (K_∞) is the limit of this process as it is iterated. (Peitgen, in [4], calls this method of drawing fractals the "Multiple Reduction Copy Machine" or MRCM.)

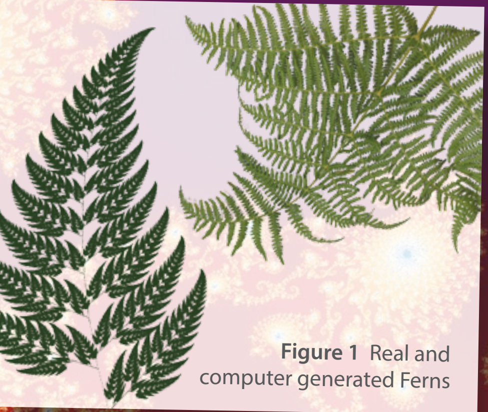


Figure 1 Real and computer generated Ferns

We can see that this object is self-similar, in the sense that we can find arbitrarily small portions of the curve that are related to the whole by a similarity transformation.

The fractal fern of Figure 1 is also the limit of four affine transformations iterated in the same manner. Since each affine transformation may be represented by a 2×2 matrix giving the homogenous part of the transformation and a 2-component vector giving the inhomogeneous (translation) part of the transformation, a figure that is the limit of n iterated affine transformations can be encoded as a collection of $6n$ real numbers – a much more efficient encoding than a pixel-by-pixel representation. These ideas also generalise in an obvious manner to subset of higher dimensional Euclidean space.

We might then ask whether we can measure how 'close' a perfectly self-similar or self-affine fractal is to a given 'imperfect' real life image that we are trying to approximate. We might also ask how many iterations of the kind described above we need to carry out to get a reasonable approximation of the limiting set. More theoretically, we might question whether we can be sure that such iterations will indeed tend to a definite limit, and, given that any such limit will be invariant under the iteration, whether it matters with what set we start. Could we, for example, have begun our construction of the Koch curve with a circle rather than a line segment?

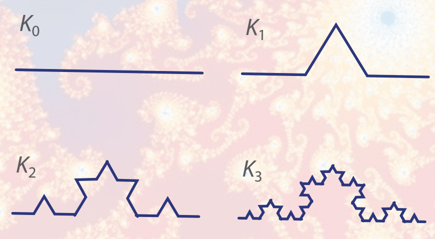


Figure 2 Construction of the Koch curve

In this article, we shall see that, by considering subsets of Euclidean space as points in a metric space, we can measure how different two images are, and by applying the contraction mapping theorem, we can see that limit sets of the sort described above do exist, that our starting point in their construction does not matter, and we can also obtain an estimate for how rapid the convergence is.

Definitions

For reference, we enumerate here a few standard definitions and theorems that we shall use later.

Definition 1 A **metric space** is an ordered pair $(\mathcal{X}, d)$, where $\mathcal{X}$ is a set and $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a function with the following properties:

(i) $d(x,y) \geq 0 \;\; \forall\, x,y \in \mathcal{X}$, with $d(x,y) = 0$ if and only if x = y;

(ii) $d(x,y) = d(x,y) \;\; \forall\, x,y \in \mathcal{X}$;

(iii) $d(x,z) \leq d(x,y) + d(y,z) \;\; \forall\, x,y,z \in \mathcal{X}$.

The notion of convergence of a sequence to a limit carries over to metric spaces in an obvious way, as does the following related notion:

Definition 2 Let (x_n) be a sequence of points in a metric space $(\mathcal{X}, d)$. We say that (x_n) is **Cauchy** if, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$.

Clearly every convergent sequence is a Cauchy sequence. The converse is also true for an important class of metric spaces:

Definition 3 A metric space $(\mathcal{X}, d)$ is **complete** if every Cauchy sequence in $\mathcal{X}$ converges.

We remark that the metric space formed by $\mathbb{R}^n$ with the usual Euclidean metric is complete.

Definition 4 Let $(\mathcal{X}, d)$ be a metric space. Then $f : \mathcal{X} \to \mathcal{X}$ is a **contraction** if there exists a non-negative real number $c < 1$ such that $d(f(x), f(y)) \leq c \times d(x,y)$ for all $x, y \in \mathcal{X}$.

Our central theorem tells us about the behaviour of contractions under iteration (for a proof, see, for example, [3]).

Theorem 5: Contraction Mapping Let $(\mathcal{X}, d)$ be a non-empty complete metric space and $f : \mathcal{X} \to \mathcal{X}$ a contraction. Then there exists a unique $x_0 \in \mathcal{X}$ such that $f(x_0) = x_0$, and furthermore, $\lim_{n\to\infty} f^n(x) = x_0$ for all $x \in \mathcal{X}$.

In the final section we will refer to a corollary:

Corollary 6 Let $(\mathcal{X}, d)$ be a non-empty complete metric space and $f : \mathcal{X} \to \mathcal{X}$ such that f^n is a contraction. Then the same conclusions hold as for Theorem 5.

For most of the time, we shall restrict our attention to compact subsets of metric spaces.

Definition 7 Let $(\mathcal{X}, d)$ be a metric space. Then we say $A \subseteq \mathcal{X}$ is compact if every covering of A by open sets has a finite subcovering.

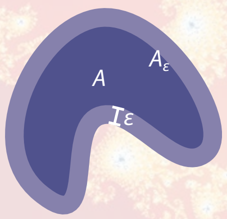
The important properties of compact sets which we need are that they are closed and bounded.

Hausdorff Distance

Our starting point is a way of turning a collection of subsets of Euclidean space into a complete metric space, so that we can talk about limits and convergence, and make use of the considerable information provided by the contraction mapping theorem. The concept we require is due to Hausdorff, who formulated a notion of 'distance' between compact subsets of a metric space which makes the set of compact subsets of a given metric space into a metric space itself. Furthermore, if our initial metric space is complete, then so is the space of compact subsets with the Hausdorff metric.

We require a further concept before introducing the Hausdorff distance itself:

Definition 8 Let A be a subset of a metric space $(\mathcal{X}, d)$. The **ε-collar** of A, denoted A_ε, is the set $\{x \in \mathcal{X} : \exists\, a \in A \text{ with } d(a,x) \leq \varepsilon\}$, i.e. the set of all points at a distance at most ε from the set A.



Definition 9 Let A and B be compact subsets of a metric space $(\mathcal{X}, d)$. If we write $\rho'(A,B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon\}$ then the **Hausdorff distance** $\rho(A,B)$ between A and B, is defined by $\rho(A,B) = \max\{\rho'(A,B), \rho'(B,A)\}$.

It follows straightforwardly from the definition that ρ' satisfies all the axioms for a metric space in definition 1 except (ii), so the final part of the definition is essentially a symmetrisation. An alternative definition sometimes used (for example in [3]) but which does the same job is $\rho(A,B) = \rho'(A,B) + \rho'(B,A)$. The proof that the resulting metric space inherits completeness is given in [2] and as an exercise in [3].

The Hutchinson Operator

Now that we have some way of measuring 'closeness' of compact subsets of metric spaces, our next task is to show that the iterated transformation applied in Figure 2 to construct the Koch curve is indeed a contraction, so that we may apply Theorem 5. The following treatment follows quite closely that of [4]. We work in $\mathbb{R}^m$.

We have a collection of affine transformations, $T_1, T_2, \ldots, T_n$, and at each iteration we apply the transformation

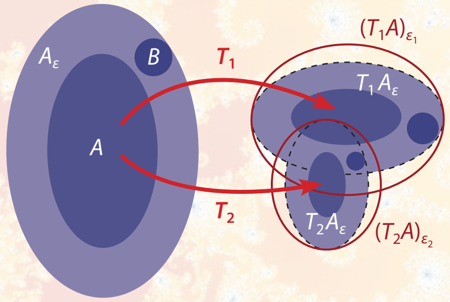
$$T : A \mapsto \bigcup_{i=1}^{n} T_i A.$$

(This is known as the **Hutchinson operator**, after Hutchinson who first analysed its properties.) We impose the condition that each T_i should itself be a contraction with respect to the Euclidean metric, with constant $c_i < 1$.

We now show that T is a contraction with constant $c = \max\{c_1, c_2, \ldots, c_n\}$ on the metric space of compact subsets of $\mathbb{R}^m$ equipped with the Hausdorff metric. (See diagram below for the following.) Let A and B be compact subsets of $\mathbb{R}^m$ with $\rho'(B,A) = \delta$. Then for any $\varepsilon > \delta$ we have $B \subseteq A_\varepsilon$. Clearly then $T_i B \subseteq T_i A_\varepsilon$, for each i, but since T_i is contractive on $\mathbb{R}^m$, $T_i A_\varepsilon \subseteq (T_1 A)_{\varepsilon_i}$, where $\varepsilon_i = c_i \varepsilon < c\varepsilon$. Hence $T_i B \subseteq (T_i A)_{\varepsilon_i} \subseteq (T_1 A)_{c\varepsilon}$, yielding

$$\bigcup_{i=1}^{n} T_i B \subseteq \bigcup_{i=1}^{n} (T_i A)_{c\varepsilon} = \left(\bigcup_{i=1}^{n} T_i A\right)_{c\varepsilon}.$$

So $TB \subseteq (TA)_{c\varepsilon}$ for all $\varepsilon > \delta$, and hence $\rho'(TB, TA) \leq c\delta$. Therefore $\rho(TB, TA) \leq c \times \rho(A,B)$ and so T is indeed a contraction.



An important practical observation which can be made from the above proof is that the contraction constant calculated for T is equal to the largest of the individual contraction constants of the transformation T_i. It is clear from the proof that, in general, we can do no better than this. In the usual proof of the contraction mapping theorem, it is shown that, for a contraction f with constant c,

$$d\left(f^n(x), f^{n+k}(x)\right) \leq d(x, f(x)) \frac{c^n}{1-c}.$$

Since this inequality holds for all k, the expression $c^n/(1-c)$ provides an estimate for how quickly the iterations converge to the unique fixed point. As might be expected, we see that the larger the constant c, the slower the convergence. Hence the MRCM method of drawing fractals is only as rapid as is allowed by the 'least contractive' contraction. It is, however, worth remarking that a given transformation may or may not be contractive, depending on the choice of metric, and that the contraction constants will vary according to the metric used. Since the notion of Hausdorff distance works for any metric space, not just $\mathbb{R}^m$ with the Euclidean metric, we may certainly replace the Euclidean metric in the above analysis with any other making $\mathbb{R}^m$ into a complete metric space, to be able to draw conclusions about the convergence properties of a wider variety of Hutchinson operators.

Julia Sets

We conclude with some brief, informal remarks about how these ideas may be applied to producing images of another rather famous class of fractals. For a given polynomial $f : \mathbb{C} \to \mathbb{C}$, the **Julia Set** of f, $J(f)$, is the closure of the set of repelling (unstable) fixed and periodic points of f. This is non-trivially equivalent to the definition as the boundary of the basins of attraction of the attractive fixed points of f (for details see [1]), and the set $J(f)$ has the property that $f(J) = f^{-1}(J) = J$. The most famous example of these objects are those associated with the mapping $f : z \mapsto z^2 + c$ for various $c \in \mathbb{C}$ (like the example shown in Figure 3). In this case we notice that the inverse mapping

$f^{-1}: z \mapsto \{\pm\sqrt{z-c}\}$ seems to play the role of a non-linear Hutchinson operator, in that each point (other than c itself) has two images, and the fractal of interest is invariant under the transformation.

We might well then ask whether the mapping is contractive. Here a partial answer is suggested by the theory of conformal mappings, which tells us that for a conformal mapping $g: \mathbb{C} \to \mathbb{C}$, the approximate scaling in length near a point z_0 in $\mathbb{C}$ is $|g'(z_0)|$. The criterion for a fixed point z_0 of a mapping g to be attractive, viz. $|g'(z_0)| < 1$, is therefore the same as the criterion for the mapping to be locally contractive. Any point close to $J(f)$ is, by definition, close to some repelling periodic point of f (whose period we shall denote by p), which in turn will be an attractive periodic point of $f_1^{-1}: z \mapsto +\sqrt{z-c}$ and $f_2^{-1}: z \mapsto -\sqrt{z-c}$. Hence the iterate T^p of the Hutchinson operator T defined by these two mappings will be a local contraction, and so Corollary 6 suggests that, at least if we consider sets not 'too far' in terms of Hausdorff distance from $J(f)$, the iteration will converge in the same manner as for the self-affine fractals discussed above. In fact the convergence is very good, and although after a finite time the iterates do not in general approximate all parts of the Julia set evenly, this is how many fractal packages produce their images of Julia Sets.

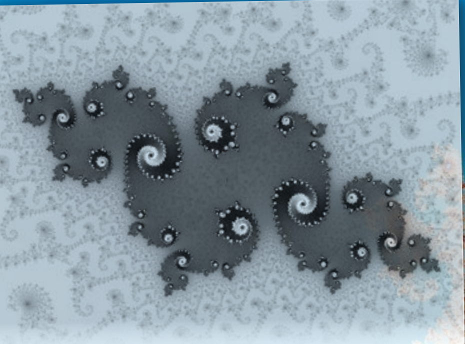


Figure 3 A Julia Set

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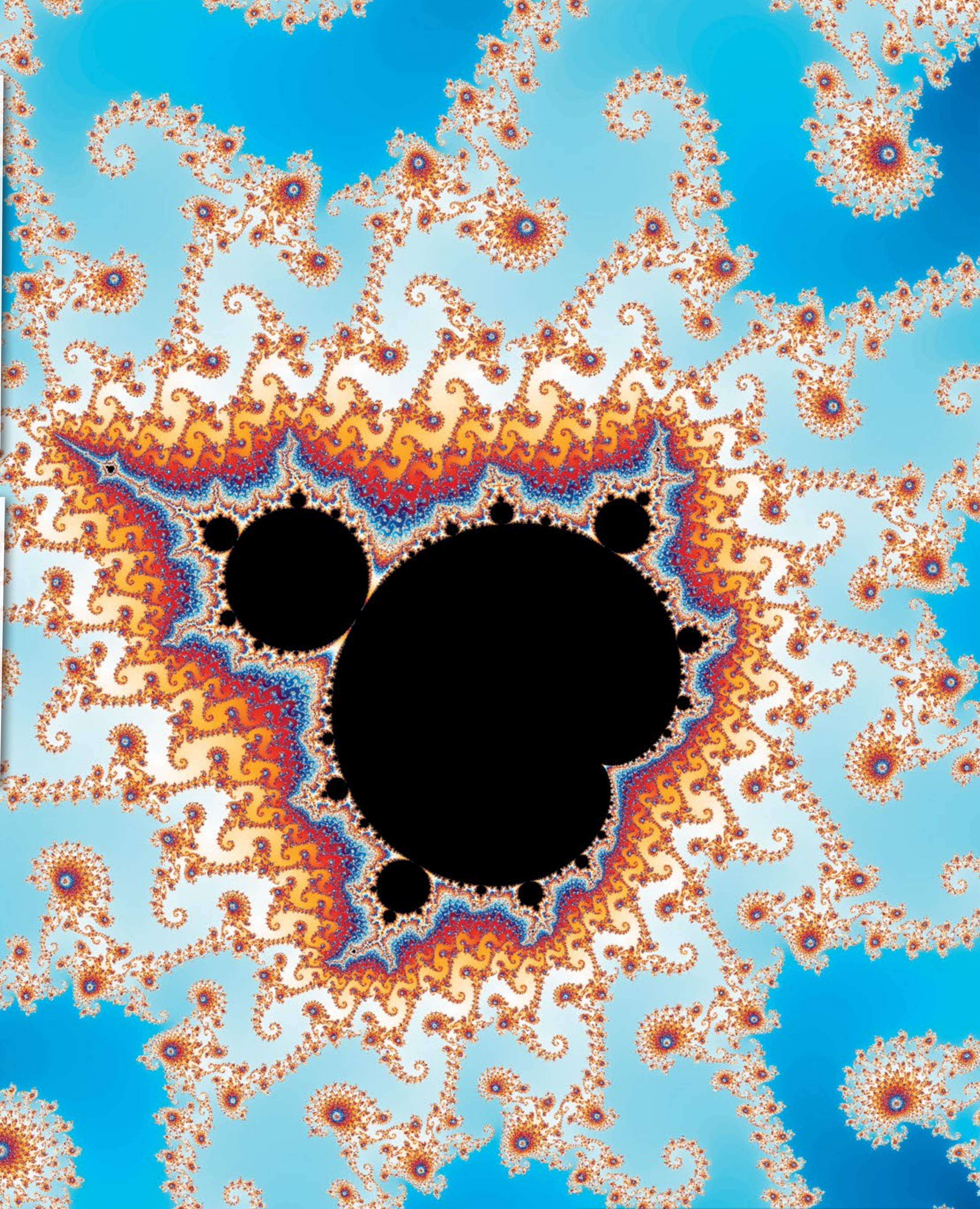

Images of the Mandelbrot Set
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