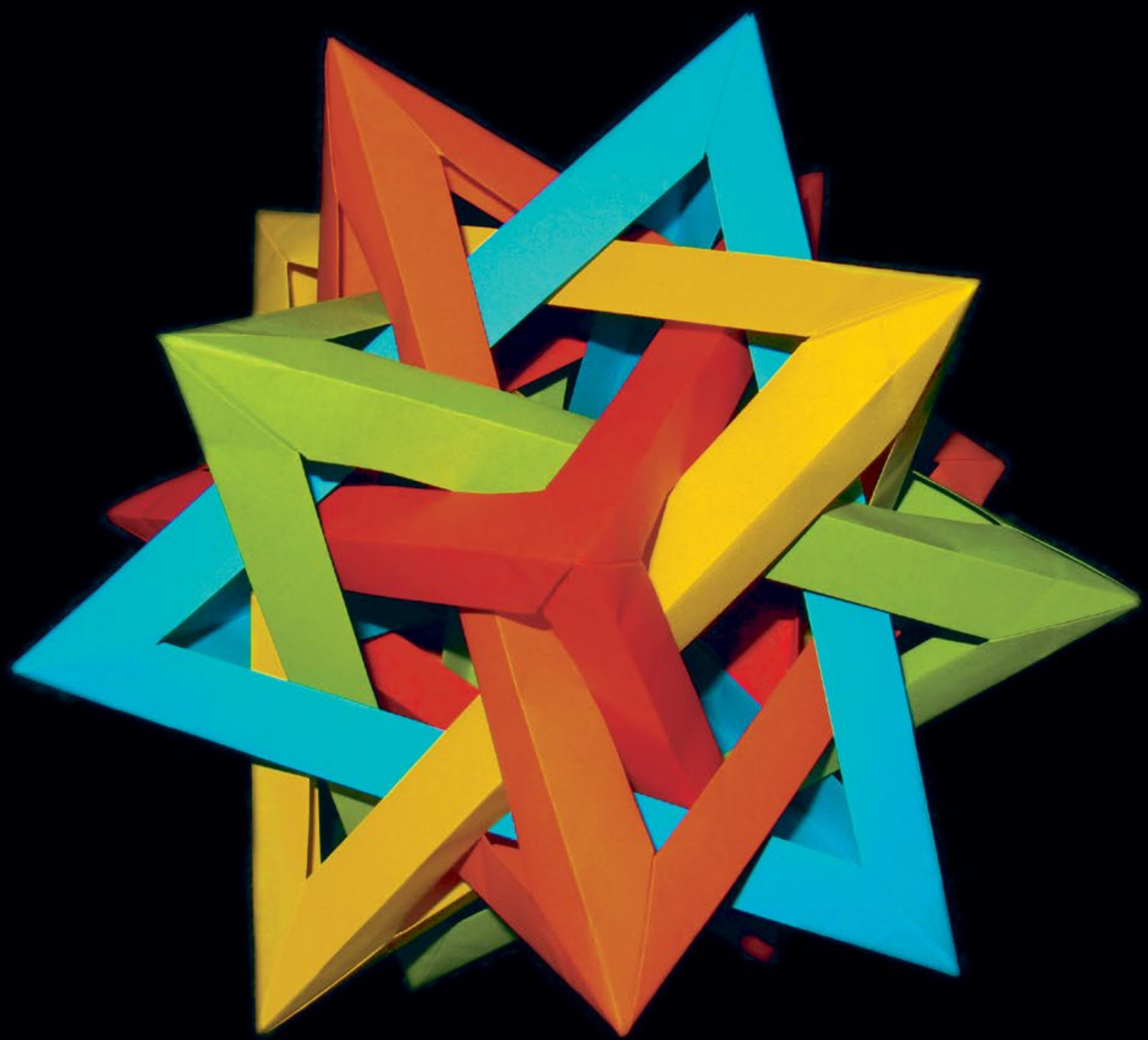
These five interlocking tetrahedra are made from 30 individual pieces, using no scissors or glue.
Philipp Legner



Mathematical Origami

Philipp Legner, St John's College

Origami shapes and paper models of Archimedean solids are not only nice to look at, they give rise to an interesting area of mathematics. Similarly to the idea of *constructing* polygons using nothing but a straight edge and a compass, you can think about which shapes and solids you can *fold* using a sheet of paper and no other tools. The results are surprisingly different from ruler and compass geometry!

Even more beautiful objects can be created if you *are* allowed to use scissors and glue. I have included photos and folding patterns of braided platonic solids, knotted pentagons and interlocking polyhedra.

The Axioms of Origami

In 1992, the Italian-Japanese mathematician Humiaki Huzita published a list of all possible operations that are possible when folding paper.

O1 We can fold a line connecting any two points P and Q.

O2 We can fold any two points onto each other.

O3 We can fold any two lines onto each other.

O4 Given a point P and a line L, we can make a fold perpendicular to L passing through P.

O5 Given two points P and Q and a line L, we can make a fold that passes through P and places Q onto L.

O6 Given two points P and Q and two lines K and L, we can make a fold that places P onto line K and places Q onto line L.

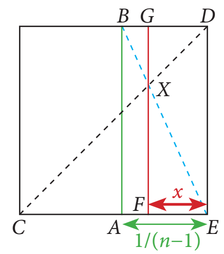
A seventh one was discovered by Koshiro Hatori:

O7 Given a point P and two lines K and L, we can fold a line perpendicular to K placing P onto L.

This set of axioms is much more powerful than the one corresponding to straight edge and compass: connecting any two points with a straight line and drawing a circle of radius r around any point. There are many interesting consequences: you can trisect angles, double cubes and even construct regular heptagons and 19-gons.

Even more surprisingly, we can use Origami to fold *any* rational number. Consider a square piece of paper of side length 1 and suppose, for induction, that we can fold one side into $(n-1)$th, as shown. Then we can also fold it into nth the by

- folding the along **CD**;
- folding the line **EB**;
- folding the line **FG** through X, perpendicular to the edge of the paper.



Now observe that

$$\frac{\frac{1}{n-1}}{x} = \frac{|AE|}{|FE|} = \frac{|AB|}{|FX|} = \frac{1}{1-x}$$

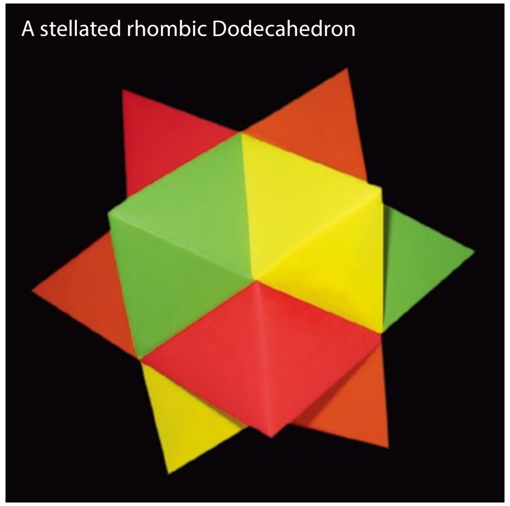
$$1 - x = x(n-1)$$

$$x = 1/n,$$

so we have divided the side of the square into nth. We can easily divide the side of the square into halves. Thus, by induction, we can use origami to fold any ratio, as required.

One can make a perfect regular pentagon simply by knotting a strip of paper.

All by Philipp Legner



A stellated rhombic Dodecahedron

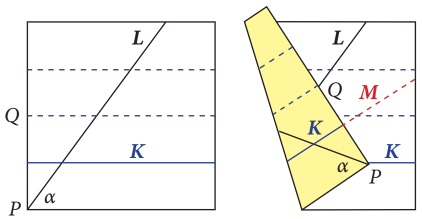
Cunning Constructions

The proofs of the following constructions are left to the reader. They are based on simple geometric relations and can also be found in [3].

Trisecting the Angel

We start with a square piece of paper and fold a crease L, as shown below, to create any angle α at P. To trisect α, we have to

- fold the paper into **quarters** from top to bottom and define K and Q as shown;
- simultaneously fold P onto K and Q onto L using axiom 6, and don't reopen;
- extend K by a new crease M.

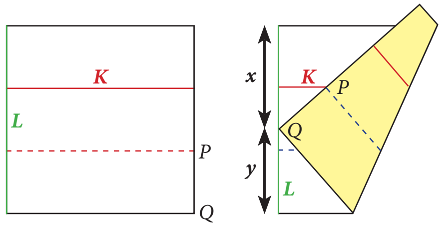


If we now open the paper and extend M to its full length, it will divide α in the ration 1:2. Halving the larger part of the angle then splits α into three equal parts.

Doubling the Cube

Even the ancient Greeks knew that it is impossible to double a cube, i.e. construct $\sqrt[3]{2}$ using nothing but ruler and compass. It was rather discouraging when the oracle in Delphi prophesied that a plague could be defeated by doubling the size of the altar to Apollo – if only they had known ancient Japanese Origami artists…

Again let us start with a square sheet of paper. We first fold the paper into thirds (since we can fold any ratios), and define K, L, P and Q as shown below. We now fold P onto K and Q onto L using axiom 6, and the ratio of the lengths x and y in the diagram is precisely $\sqrt[3]{2}$.

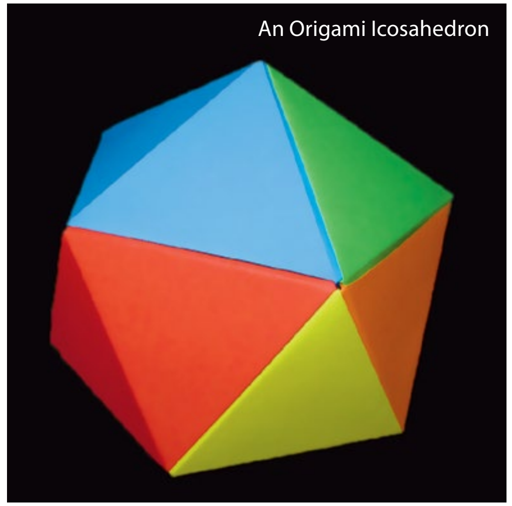


Incidentally, the third "classical" problem that is impossible with straight edge and compass, squaring the circle, is impossible even using Origami, since it involved constructing the transcendental ratio $\sqrt{\pi}$.

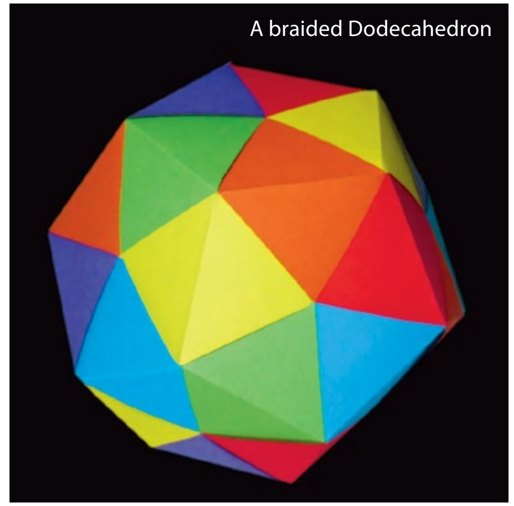
Solving Cubic Equations

It is known that quadratic equations can be solved with straight edge and compass. With Origami, we can also solve *cubic* equations.

Suppose have an equation $x^3 + ax^2 + bx + c = 0$. Let $P = (a,1)$ and $Q = (c,b)$ in a coordinate system. Furthermore, let K be the line $y = -1$ and L be the line $x = -c$ as shown below. Using axiom 6, we can simultaneously place P onto K (at P') and Q onto L (at Q') to create a new line M. Suppose that M has equation $y = \alpha x + \beta$, for some $\alpha, \beta \neq 0$.



An Origami Icosahedron

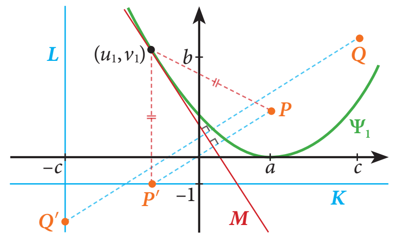


A braided Dodecahedron

Let Ψ_1 be the parabola $4y = (x - a)^2$ with focus P and directrix K. Then M is a tangent to Ψ_1 at a point (u_1, v_1) – this is illustrated by the dotted red lines in the diagram below. Differentiating gives $\alpha = \frac{1}{2}(u_1 - a)$ and we can deduce $\beta = -\alpha^2 - a\alpha$.

Let Ψ_2 be the parabola $4cx = (y - b)^2$ with focus Q and directrix L. Again M is a tangent to Ψ_2 at a point (u_2, v_2) and we can find $\beta = b + c/\alpha$.

Setting these two results equal shows that $\alpha = \frac{1}{2}(u_1 - a)$ satisfies $x^3 + ax^2 + bx + c = 0$, i.e. is the solution we are looking for.



Doubling the cube is equivalent to solving the cubic $x^3 - 2 = 0$, while trisecting the angle is equivalent to solving $x^3 + 3tx^2 - 3x - t = 0$ with $t = 1/\tan\theta$ and $x = \tan(\theta/3 - \pi/2)$.

We can define the set $\mathcal{O}$ of *Origami Numbers*, numbers that can be constructed using origami. It *includes* the corresponding set for straight edge and compass constructions, and is *the same* as for constructions using a marked rule and compass.

We can also construct many regular polygons using Origami: precisely those with $2^a 3^b \rho$ sides, where ρ is a product of distinct Pierpont primes, that is, primes of the form $2^u 3^v + 1$.

The Art of Folding Paper

Finally let us look at some of the artistic aspects of paper folding. The word *Origami* (折り紙) originates from the Japanese *oru* (to fold) and *kami* (paper). Japanese monks were among the first to turn the amusement into a sophisticated art. On the next page you can see some great examples.

Especially useful for creating mathematical solids is *modular origami*: you fold many individual pieces (such as faces) separately and then stack them together.

An ingenious method for creating Platonic solids is "braiding" particularly shaped strips of paper. When using differently coloured paper, this produces some of the most beautiful and decorative objects. The patterns can be found on the following page. Start with the tetrahedron before attempting the larger solids – you may need lots of paper clips, or another pair of hands!

References, Further Reading

Robert Lang's website (see below) is a great place to start both for building origami and reading about the mathematical background.

1. R. Lang, P. Wang-Iverson, and M. Yim, *Origami^5*, CRC Press (2011)
2. R. Lang, *www.langorigami.com*
3. J. Krier, *math.uttyler.edu/nathan/classes/senior-seminar/JaemaKrier.pdf*
4. Cut-The-Knot, *www.cut-the-knot.org/pythagoras/PaperFolding/index.shtml*

Upupa Epops, Fugu and Crayfish

Sipho Mabona

For best effect, enlarge this page to A3 and copy it on heavy, coloured paper. Start by cutting out the required number of strips for each solid and carefully creasing all lines in the same direction. The stars should be visible on the inside of the bottom face; only for the Dodecahedron and Icosahedron some strips are added later. It will be helpful to use paperclips to hold the finished faces in place.

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
3 needed	4 needed	4 needed	6 needed	6 needed