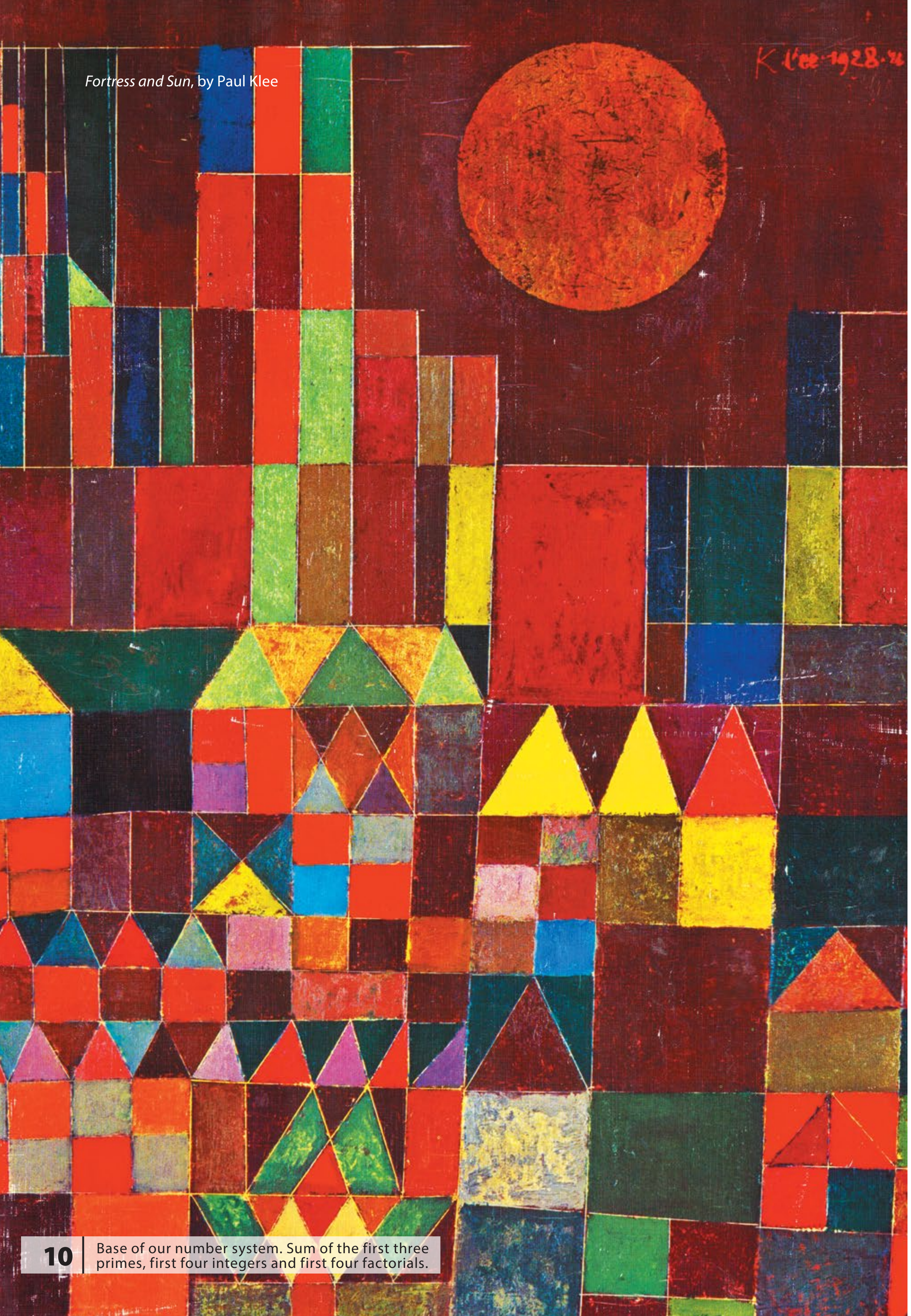


Fortress and Sun, by Paul Klee

Pentaplexity

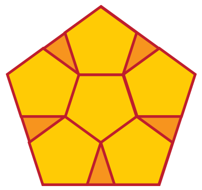
Sir Roger Penrose

First published in issue 39, 1978

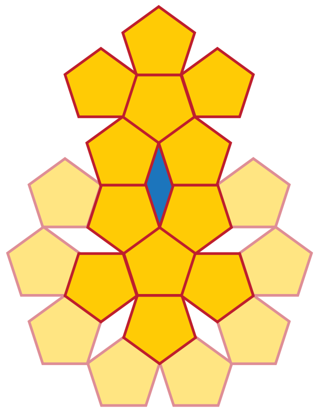
Certain shapes, when matched correctly, can form a tiling of the entire plane but only in a non-periodic way. These tilings have a number of remarkable properties, and I shall give here a brief account explaining how these tiles came about and indicating some of their properties.

The starting point was the observation that a regular pentagon can be subdivided into six smaller ones, leaving only five slim triangular gaps. This is familiar as part of the usual "net" which folds into a regular dodecahedron, as shown in Figure 1. Imagine now, that this process is repeated a large number of times, where at each stage the pentagons of the figure are subdivided according to the scheme of Figure 1. There will be gaps appearing of varying shapes and we wish to see how best to fill these. At the second stage of subdivision, diamond-shaped gaps appear between the pentagons (Figure 2). At the third, these diamonds grow "spikes", but it is possible to find room, within each such "spiky diamond", for another pentagon, so that the gap separates into a star (pentagram) and a "paper boat" (or Jester's cap?) as shown in Figure 3. At the next stage, the star and the boat also grow spikes, and, likewise, we can find room for new pentagons within them, the remaining gaps being new stars and boats (as before). These subdivisions are shown in Figure 4.

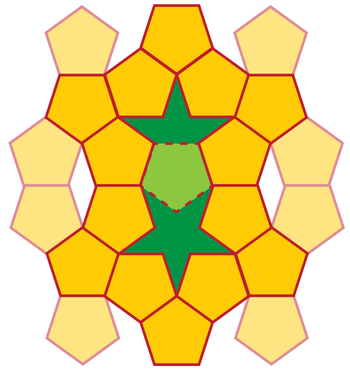
Since no new shapes are now introduced at subsequent stages, we can envisage this subdivision process proceeding indefinitely. At each stage, the scale of the shapes can be expanded outwards so that the new pentagons that arise become the



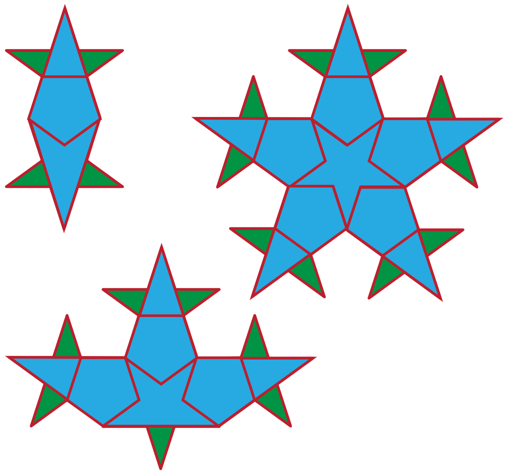
▲ Figure 1



▲ Figure 2



▲ Figure 3



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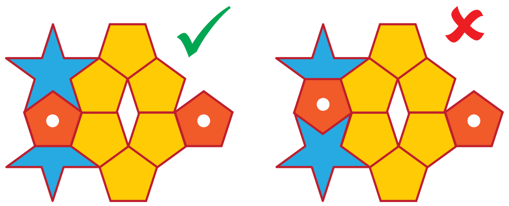
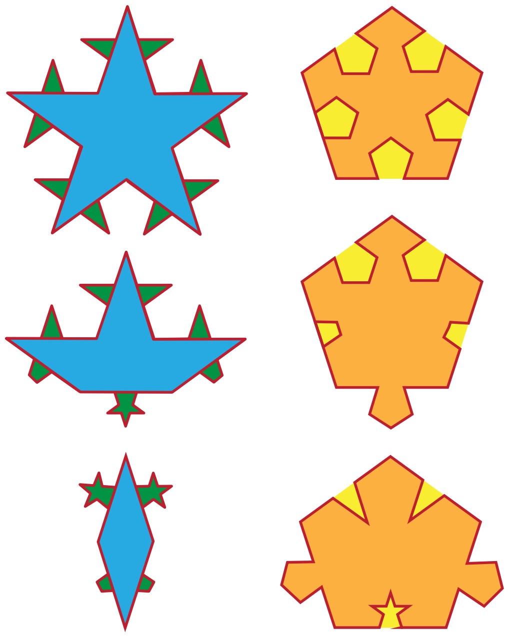


▲ Figure 5



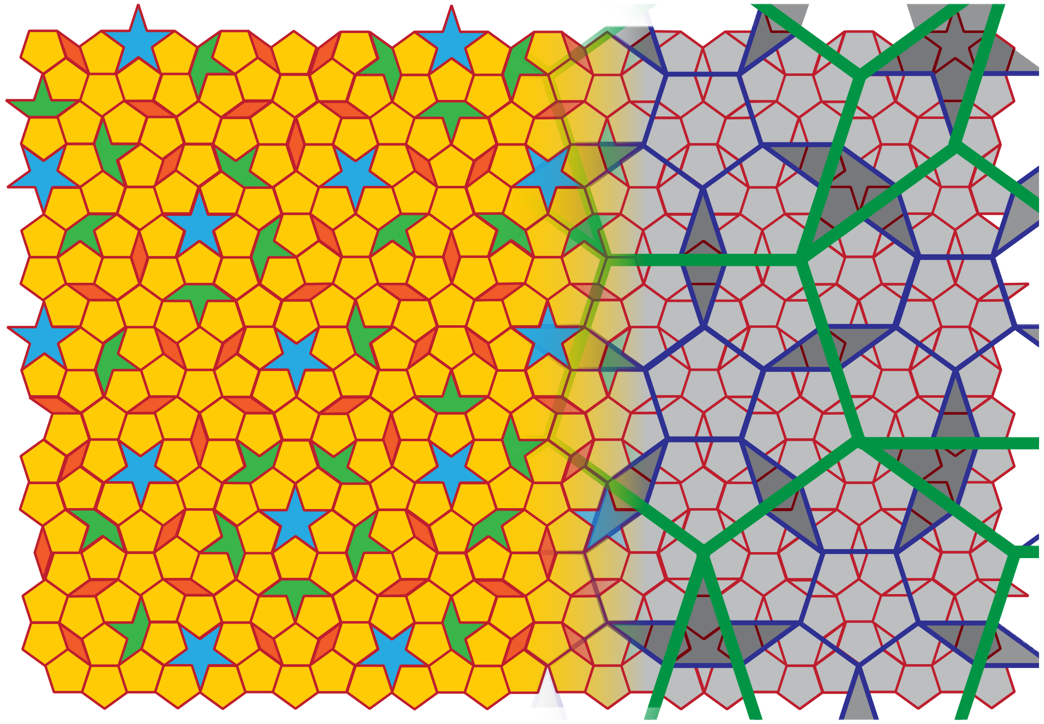
▲ Figure 6

same size as those at the previous stage. As things stand, however, this procedure allows ambiguity that we would like to remove. The subdivisions of a "spiky diamond" can be achieved in two ways, since there are two alternate positions for the pentagon. Let us insist on just *one* of these, the rule being that given in Figure 5. (When we examine the pattern of surrounding pentagons we necessarily find that they are arranged in the type of configuration shown in Figure 5.) It may be mentioned that had the opposite rule been adapted for subdividing a "spiky diamond", then a contradiction would appear at the next stage of subdivision, but this never happens with the rule of Figure 5.

This procedure, when continued to the limit, leads to a tiling of entire plane with pentagons, diamonds, boats and stars. But there are many "incorrect" tilings with the same shapes, being not constructed according to the above prescription. In fact, "correctness" can be *forced* by adopting suitable matching rules. The clearest way to depict these rules is to modify the shapes to make a kind of infinite jigsaw puzzle, where a suggested such modification is given in Figure 6. It is not hard to show that any tiling with these six shapes is forced to have a hierarchical structure of the type just described.

Properties of these Tilings

Furthermore, the forced hierarchical nature of this pattern has a number of very remarkable properties. In the first place, it is necessarily non-periodic (i.e. without any period parallelogram). More about this later. Secondly, though the completed pattern is not uniquely determined – for there are $2^{\aleph_0}$ different arrangements – these different arrangements are, in a certain "finite" sense, all indistinguishable from one another! Thus, no matter how large a finite portion is selected in one such pattern, this finite portion will appear somewhere in *every* other completed pattern (infinitely many times, in fact). Thirdly, there are many unexpected and aesthetically pleasing features that these patterns exhibit (see Figure 7). For example, there are many regular decagons appearing, which tend to overlap in places. Each decagon is surrounded by a ring of twelve pentagons, and there are larger rings of various kinds also. Note that every straight line segment of the pattern extends outwards to infinity, to contain an infinite number of line segments of the figure. The hierarchical arrangement of Figure 7 is brought out in Figure 8.

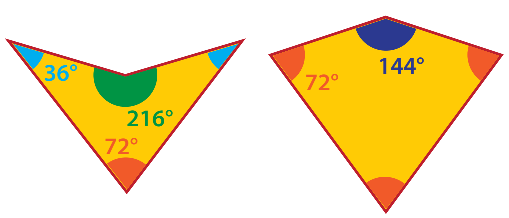


▲ Figure 7 ▲ Figure 8

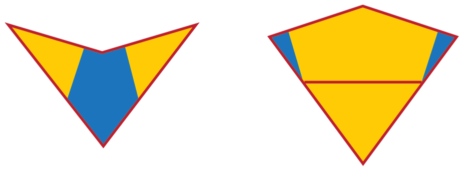
After I had found this set of six tiles that forces non-periodicity, it was pointed out to me (by Simon Kochen) that Raphael Robinson had, a number of years earlier, also found a (quite different) set of six tiles that forces non-periodicity. But it occurred to me that with my tiles one can do better. If, for example, the third "pentagon" shape is eliminated by being joined at two places to the "diamond" and at one place to the bottom of the "boat", then a set of *five* tiles is obtained that forces non-periodicity. It was not hard to reduce this number still further to four. And then, with a little slicing and rejoining, to *two*!

The two tiles so obtained are called "kites" and "darts", names suggested by John Conway. The precise shapes are illustrated in Figure 9. The matching rules are also shown, where vertices of the same colour must be placed against one another. There are many alternative ways to colour these tiles to force the correct arrangement. One way brings out the relation to the pentagon-diamond-boat-star tilings shown in Figure 10. A patch of assembled tiles (partly coloured in this way) is shown in Figure 11. The hierarchical nature of the kite-dart tilings can be seen directly, and is illustrated in Figure 12. Take any such tiling and bisect each dart symmetrically with a straight line segment. The resulting half-darts and kites can then be collected together to make darts and kites on a slightly larger scale: two half-darts and one kite make a large dart; two half-darts and two kites make a large kite. It is not hard to convince oneself that every correctly matched kite-dart tiling is assembled in this way. This "inflation" property also serves to prove non-periodicity. For suppose there were a period parallelogram. The corresponding inflated kites and darts would also have to have the same period parallelogram. Repeat the inflation process many times, until the size of the resulting inflated kites and darts is greater than that of the supposed period parallelogram. This gives a contradiction.

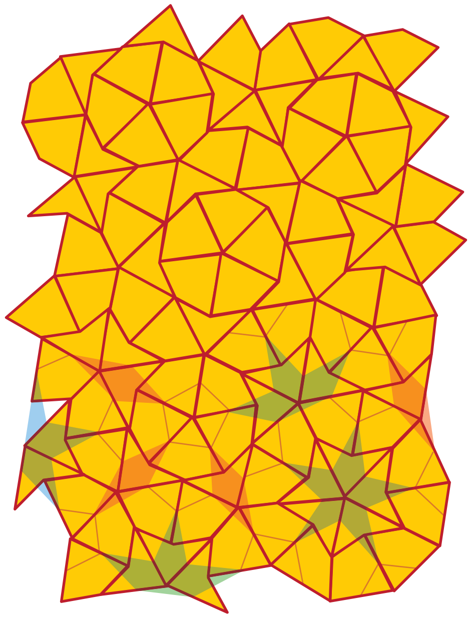
The contradiction with periodicity shows up in another striking way. Consider a very large area containing d darts and k kites, which is obtained referring to the inflation process a large number of times. The larger the area, the closer the ratio $x = k/d$ of kites to darts will be to satisfying the recurrence relation $x = (1 + 2x)/(1 + x)$, since, on inflation, a dart and two kites make a larger kite, while a dart and a kite make larger dart. This gives, in the limit of an infinitely large pattern,



▲ Figure 9



▲ Figure 10



▲ Figure 11

$x = \frac{1}{2}\left(1 + \sqrt{5}\right) = \varphi$, the golden ratio! Thus we get an *irrational* relative density of kites to darts – which is impossible for a periodic tiling. (This is the *numerical* density. The kite has φ times the area of the dart, so the total area covered by kites is φ^2 (= $1 + \varphi$) times that covered by darts.)

Jigsaws and beyond

There is another pair of quadrilaterals which, with suitable matching rules, tiles the plane only non-periodically: a pair of rhombuses as shown in Figure 13. A suitable shading is suggested in Figure 14, where similarly shaded edges are to be matched against each other. In Figure 15, the hierarchical relation to the kites and darts is illustrated. The rhombuses appear mid-way between one kite-dart level and the next inflated kite-dart level.

Many different jigsaw puzzle versions of the kite-dart pair or the rhombus pair can evidently be given. One suggestion for modified kites and darts, in the shape of two birds, is illustrated in Figure 16.

Other modifications are also possible, such as alternative matching rules, suggested by Robert Ammann (see Figure 17) which force half the tiles to be inverted.

Many intriguing features of these tilings have not been mentioned here, such as the pentagonally-symmetric rings that the stripes of Figure 14 produce, Conway's classification of "holes" in kite-dart patterns (i.e. regions surrounded by "legal" tilings but which cannot themselves be legally filled), Ammann's three-dimensional version of the rhombuses (four solids that apparently fill space only non-periodically), Ammann's and Conway's analysis of "empires" (the infinite system of partly disconnected tiles whose positions are forced by a given set of tiles). It is not known whether there is a *single* shape that can tile the Euclidean plane non-periodically. For the hyperbolic (Lobachevski) plane a single shape can be provided which, in a certain sense, tiles only non-periodically (see Figure 18) – but in another sense a periodicity (in one direction only) can occur. (This remark is partly based on suggestions of John Moussouris.)

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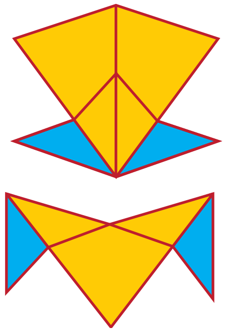

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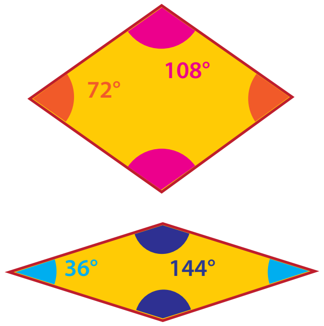

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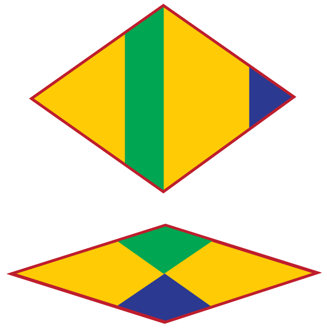

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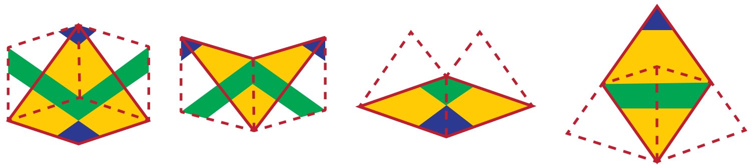
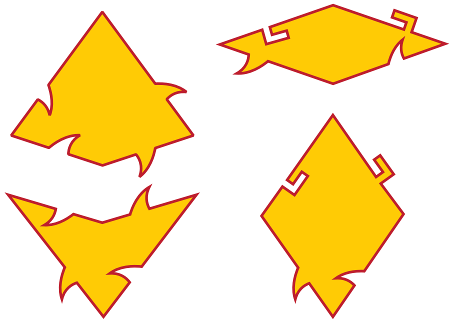


▲ Figure 15

▲ Figure 16



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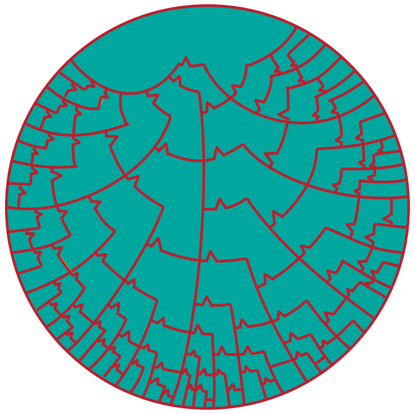

▲ Figure 18