MATH 7311 - Homework 0 - Motivation (Due Aug 29, 2017)

Review Questions.

(a) Prove that every Cauchy sequence in $\mathbb{R}^n$ converges to a point in $\mathbb{R}^n$.

(b) What is an open cover?

(c) What definition of compactness did we start with in class?

(d) Give a procedure to enumerate the rational numbers.

(e) Give an example of an uncountable set.

Prove:

Proposition 1. *A compact set $E \subset \mathbb{R}^n$ is bounded.*

Proof by contrapositive. Suppose E is not bounded. This means that E is not covered by any ball of finite radius. However, all of $\mathbb{R}^n$, and thus E, will be covered by the sequence of open balls

$$B_n := \{x : |x| < n\}.$$

Any union of a finite subset of these open balls will be an open ball of finite radius, and thus will not cover E. Therefore E is not compact.$\square$

Prove:

Proposition 2. *A compact set $E \subset \mathbb{R}^n$ is closed.*

Hint: Show that $\setminus E$ (i.e. the complement of E) is open: Let $\mathbf{x} \in \setminus E$. Let

$$G_k := \left\{\mathbf{y} : |\mathbf{y} - \mathbf{x}| > \frac{1}{k}\right\} \equiv \setminus\overline{B\left(\mathbf{x}, \frac{1}{k}\right)}.$$

Show that $\{G_k\}$ cover E, then conclude.

We will prove that the complement of E is open. Consider a point $\mathbf{x} \in \setminus E$ and a collection of sets

$$G_k := \{\mathbf{y} : |\mathbf{y} - \mathbf{x}| > \frac{1}{k}\}.$$

We can see that $\cup_{k=1}^{\infty} G_k = \mathbb{R}^n / \mathbf{x}$. Since $\mathbf{x} \notin E$, $E \subset \cup G_k$, which means $\{G_k\}$ is an open cover E. We know that E is compact so it can be covered by finitely many of the G_k. The G_k are ordered by containment ($G_k \subset G_{k+1}$) so E can be covered by exactly one of the G_k, call this set G_l. Now, all of the G_k with indices strictly greater than l have complements that are disjoint from E ($\setminus G_{k+1} \subset \setminus G_k$). The interiors of the complements of the G_k are the open balls $B(\mathbf{x}, \frac{1}{k})$. This sequence of open balls for all $k > l$ contains the point $\mathbf{x}$ and does not intersect E. The point $\mathbf{x}$ was arbitrarily chosen in $\setminus E$. Therefore $\setminus E$ is open, and E by definition is closed. $\square$

Prove:

Lemma 3. *Let $[a_j, b_j]$ be a nested sequence of nonempty closed intervals in $\mathbb{R}$: $[a_j, b_j] \supset [a_{j+1}, b_{j+1}]$ for all j. Then $\cap_{j=1}^{\infty}[a_j, b_j]$ is not empty.*

Definition. *Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and define*

$$[\mathbf{a}, \mathbf{b}] := \{\mathbf{x} \in \mathbb{R}^n : a_j \leq x_j \leq b_j,\ j = 1, ..., n\},$$

*where we call $[\mathbf{a}, \mathbf{b}]$ a closed n-*cell *or closed n-*interval.

Prove:

Lemma 4. *Let $[\mathbf{a}^j, \mathbf{b}^j]$ be a nested sequence of nonempty closed n-cells (n-intervals) in $\mathbb{R}^n$: $[\mathbf{a}_j, \mathbf{b}_j] \supset [\mathbf{a}_{j+1}, \mathbf{b}_{j+1}]$ for all j. Then $\cap_{j=1}^{\infty}[\mathbf{a}_j, \mathbf{b}_j]$ is not empty.*

Prove:

Lemma 5. *Any closed n-cell*

$$[\mathbf{a}^0, \mathbf{b}^0] \equiv \{\mathbf{x} \in \mathbb{R}^n : a_j^0 \leq x_j \leq b_j^0,\ j = 1, ..., n\}$$

in $\mathbb{R}^n$ is compact.

Hint: Suppose for contradiction that $[\mathbf{a}^0, \mathbf{b}^0]$ has an open cover $\{G_\alpha\}$ not containing a finite subcover. Set $c_j^0 := \frac{1}{2}(a_j^0 + b_j^0)$. Then the intervals $[a_j^0, c_j^0]$, $[c_j^0, b_j^0]$ determine 2^n closed n-cells whose union is $[\mathbf{a}^0, \mathbf{b}^0]$. At least one of these n-cells (having sides half the lengths of the sides of $[\mathbf{a}^0, \mathbf{b}^0]$) cannot be covered by a finite sub-cover of $\{G_\alpha\}$. Denote this cell by $[\mathbf{a}^1, \mathbf{b}^1]$. Continue this process ad infinitum and use Lemma 4 to extract a contradiction.

Prove:

Proposition 6. *A closed subset F of a compact set K in $\mathbb{R}^n$ is compact.*

Hint: if $\{G_\alpha\}$ is an open cover of F, then $\{G_\alpha, \backslash F\}$ is an open cover of K.

Prove:

Corollary 7. *The intersection of a closed set and a compact set in $\mathbb{R}^n$ is compact.*

Prove:

Proposition 8. *If E is closed and bounded in $\mathbb{R}^n$, then E is compact.*

Note: Propositions 1, 2, 8 constitute the Heine-Borel Theorem.

Prove:

Theorem 9 (Bolzano-Weierstrass Theorem)**.** *Every bounded infinite set E of points in $\mathbb{R}^n$ has a limit point (i.e. a point of accumulation).*

Hint: first by enclosing E in an n-cell, then using a sequence of bisections, as in the hint for Lemma 5, to obtain a nested family of cells each containing an infinite number of points.

Prove it again by enclosing the bounded infinite set in a compact n-cell K and assuming for contradiction that no point of K is a limit point of E. Then each $\mathbf{x} \in K$ would have a neighborhood $B(\mathbf{x}, \delta(\mathbf{x}))$ containing at most one point of E, namely $\mathbf{x}$ if $\mathbf{x} \in E$. Then conclude.

Prove:

Theorem 10. *If $E \subset \mathbb{R}^n$ is compact, then every sequence in E has a subsequence converging to a limit in E.*

The converse of Theorem 10 is a consequence of the next two lemmas. Prove them.

Lemma 11. *If every sequence in $E \subset \mathbb{R}^n$ has a subsequence converging to a limit in E, and if $\{G_\alpha\}$ is an open cover of E, then there is an $r > 0$ with the property that for each $\mathbf{y} \in E$ there is an $\alpha(\mathbf{y})$ such that $B(\mathbf{y}, r) \subset G_{\alpha(\mathbf{y})}$.*

Hint: if not, then for each $k \in \mathbb{N}$, there would be a $\mathbf{y}_k$ such that $B\left(\mathbf{y}_k, \frac{1}{k}\right)$ belongs to no G_α.

Lemma 12. *If every sequence in $E \subset \mathbb{R}^n$ has a subsequence converging to a limit in E, then E is totally bounded, i.e. for arbitrary $\epsilon > 0$, there is a finite number of points $\mathbf{x}_1, ..., \mathbf{x}_J$ such that $E \subset \cup_{j=1}^{J} B(\mathbf{x}_j, \epsilon)$.*

Hint: if not, there would be an $\epsilon > 0$ such that E could not be covered by a finite number of balls of radius ϵ. Choose $\mathbf{y}_1 \in E$, $\mathbf{y}_2 \in E \setminus B(\mathbf{y}_1, \epsilon)$, $\mathbf{y}_3 \in E \setminus B(\mathbf{y}_1, \epsilon) \setminus B(\mathbf{y}_2, \epsilon)$, ...

Prove:

Theorem 13. *If every sequence in $E \subset \mathbb{R}^n$ has a subsequence converging to a limit in E, then E is compact.*

Show that a number of our results in $\mathbb{R}^n$ are not readily exported to infinite-dimensional spaces by proving

Proposition 14. *The sequence of functions $\{f_k\}$, where $f_k(t) = \sqrt{2/\pi}\sin kt$, $k \in \mathbb{N}$, $0 \le t \le \pi$, in the space $C^0([0,\pi])$ of continuous real-valued functions on the interval $[0,\pi]$ endowed with the norm*

$$\|f\| = \sqrt{\int_0^\pi |f(t)|^2\,dt},$$

is bounded but has no convergent subsequence.

Proof for boundedness:

$$\begin{aligned}
\|f_k(t)\| &= \sqrt{\int_0^\pi |\sqrt{2/\pi}\sin kt|^2 dt} \\
&= \sqrt{2/\pi}\sqrt{\int_0^\pi |\sin kt|^2 dt} \\
&\le \sqrt{2/\pi}\sqrt{\int_0^\pi 1 dt} \\
&= \sqrt{2\pi}
\end{aligned}$$

So the sequence is bounded for all k. For the proof for no convergent subsequence, we will use the triangle inquality. Consider $\|f_n(t) - f_m(t)\|$ for natural numbers with $n \neq m$

$$\begin{aligned}
\|f_n(t) - f_m(t)\| &= \sqrt{\int_0^\pi |\sqrt{2/\pi}\sin nt - \sqrt{2/\pi}\sin mt|^2 dt} \\
&= \sqrt{2/\pi}\sqrt{\int_0^\pi |\sin nt - \sin mt|^2 dt} \\
\text{The triangle inequality: } &\ge \sqrt{2/\pi}\sqrt{|\int_0^\pi (\sin nt - \sin mt)^2 dt|} \\
&= \sqrt{2/\pi}\sqrt{|\int_0^\pi \sin^2 nt + \sin^2 mt - 2\sin mt \sin nt dt|} \\
\text{Since } n \neq m &= \sqrt{2/\pi}\sqrt{|\pi/2 + \pi/2 - 0|} \\
&= \sqrt{2}
\end{aligned}$$

This shows that as long as $m \neq n$ the norm of the difference between any two functions in the sequence will be greater than $\sqrt{2}$. Which proves that are no convergent subsequences. Prove:

Proposition 15. *Let E be compact in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be continuous on E relative to E. Then f is bounded on E.*

Hint: first by noting that for each $\epsilon > 0$ and each $\mathbf{x} \in E$, there is a $\delta(\epsilon, \mathbf{x}) > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ when $|\mathbf{x} - \mathbf{y}| < \delta(\epsilon, \mathbf{x})$ and $\mathbf{y} \in E$. Cover E with $B(\mathbf{x}, \delta(\epsilon, \mathbf{x}))$, then conclude.

Prove it again by assuming for contradiction that f is not bounded on E, in which case there would be a sequence $\mathbf{x}_k$ in E with $|f(\mathbf{x}_k)| \to \infty$.

Prove:

Theorem 16. *Let E be compact in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be continuous on E relative to E. Then f attains its infimum on E, i.e. f has a minimum on E.*

Prove:

Theorem 17. *Let E be compact in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be continuous on E relative to E. Then f is uniformly continuous on E.*

Prove:

Theorem 18. *The distance between two nonempty compact disjoint sets X and Y in $\mathbb{R}^n$ is positive.*

Hint: first by regarding $X \cup Y$ as a single compact set and covering it with sets of the form

$$B\left(\mathbf{x}, \frac{1}{3}|\mathbf{x} - \mathbf{y}|\right) \cup B\left(\mathbf{y}, \frac{1}{3}|\mathbf{x} - \mathbf{y}|\right)$$

for each $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Prove it again by using the definition of infimum to show that there are sequences $\mathbf{x}_k \in X$ and $\mathbf{y}_k \in Y$ such that

$$\text{dist}\,(X, Y) \equiv \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in X, \mathbf{y} \in Y\} = \lim_{k \to \infty} |\mathbf{x}_k - \mathbf{y}_k|.$$

Prove it yet again by defining $\text{dist}\,(\mathbf{x}, Y) := \inf_{\mathbf{y} \in Y} |\mathbf{x} - \mathbf{y}|$ and showing that $g(\mathbf{x}) := \text{dist}\,(\mathbf{x}, Y)$ is continuous on the compact set X. (This proof works if Y is merely closed.)