Topology Homework 2
Lee Fisher
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1. Show that a space is simply connected if and only if all paths having the same endpoints are fixed endpoint homotopic.

   $\rightarrow$ Suppose X is simply connected. Consider two points x_0 and x_1, and two paths α and β connecting the points. We say $\bar{\beta}$ is $\beta(1-t)$. We can say that $[\alpha] \cong [\alpha] * [\bar{\beta} * \beta]$ because the path $[\bar{\beta} * \beta]$ is homotopic to a constant path at x_1. Then by associativity we have $[\alpha] \cong [\alpha * \bar{\beta}] * [\beta]$. Next, because $[\alpha * \bar{\beta}]$ is a loop based at x_0 we have $[\alpha] \cong [1] * [\beta]$. Finally we get $[\alpha] \cong [\beta]$ because $[1]$ is the identity. Thus α and β are fixed endpoint homotopic.

   $\leftarrow$ Suppose any two paths in X are fixed end point homotopic. Consider a loop that begins and ends at the point x_0 and a path that is constant at x_0. These paths have the same endpoints so they are homotopic. Thus any loop based at x_0 will be homotopic to the path that is constant at x_0, and X is by definition simply connected.

2. We need to show that $(g \circ f)_* = g_* \circ f_*$. This is not so hard. Let $[\gamma] \in \pi_1(X, x_0)$. Then we have $(g \circ f)_* \circ [\gamma] = [g \circ f \circ \gamma]$ by definition. On the other hand, lets look at $g_* \circ f_* \circ [\gamma]$, this is $g_*(f_*([\gamma]))$ which is $g_*([f \circ \gamma])$, and finally $[g \circ f \circ \gamma]$. So the maps are equal.

3.

4. Let $p : E \rightarrow B$ be a covering map where B is connected and there is some point $b \in B$ for which $|p^{-1}(b)| = k$. Consider two subsets of B where $U = \{x \in B : |p^{-1}(x)| = k\}$ and $V = \{y \in B : |p^{-1}(y)| \neq k\}$. We can say that if $x \in U$, because p is a covering map, there will be some open set containing x that is in U; therefore U is open. As well if $y \in V$ then there will be an open set containing y that is in V; so V also is open. Since p is an onto function we have that $U \cap V = \emptyset$ and $U \cup V = B$. If V is nonempty this forms a separation of B. So since B is connected we have $V = \emptyset$ and $U = B$.

5.

6.