**Algebra 1 Homework 4**
**Lee Fisher**
**2017-09-17**

**2.7 #1:**

First we need to show that $N$ is a normal subgroup of $G$. For a fixed $g \in G$, consider the coset $gN$. Elements in $gN$ are of the form $gn$. Where $n$ has the property that for any $x \in G$ there is an $h \in H$ with $n = xhx^{-1}$. Let $x = g^{-1}$, then we have $n = g^{-1}h_*g$, and also that $gn = h_*g$ for some $h_* \in H$.

To conclude we need to prove that $h_* \in \cap xHx^{-1} = N$ then we'll have $gn = h_*g = eh_*e^{-1}g \in Ng$. We know $h_* = gng^{-1}$ where $n \in N$. Consider a $y \in G$ then $yh_*y^{-1} = ygng^{-1}y^{-1} = (yg)n(yg)^{-1}$. However, we know $yg$ is an arbitrary element in $G$ and $n \in eHe^{-1} = H$. So we conclude, $h_* \in N$ and $gn = h_*g \in Ng$. So $gN = Ng$ and $N$ is normal in $G$.

Now to prove that $N$ is the largest normal subgroup contained in $H$. Let $M$ be a normal subgroup of $G$ contained in $H$. Consider $m \in M$ we know that $m \in H$ by supposition and that because $M$ is normal, for all $g \in G$, $gmg^{-1} \in M$. So $m \in N$. Any normal subgroup contained in $H$ must be a subset of $N$.

**2.7 #2.**

1. - Reflexivity: $a \sim a$ because $a \in HaK$. Because $H$ and $K$ are both subgroups and $eae \in HaK$.
   - Symmetry: Suppose $a \sim b$. This means $a \in HbK$. For some $h \in H$ and $k \in K$, $a = hbk$. With some algebra: $h^{-1}ak^{-1} = b$, so $b \in HaK$.
   - Transitivity: Suppose $a \sim b$ and $b \sim c$. So $a \in HbK$ and $b \in HcK$, so $a = h_1bk_1$ and $b = h_2ck_2$, therefore $a = (h_1h_2)c(k_1k_2)$, and $a \in HcK$. So $a \sim c$ and we're done.
2. Suppose $f(aW) = f(bW)$

$$\begin{aligned}
f(aW) &= f(bW) \\
aWxK &= bWxK \\
\text{This implies } \exists e, k_1 \in K \text{ such that:} & \\
aWx &= bWxk_1 \\
\text{As well } \exists e, w \in W \text{ such that:} & \\
awx &= bxk_1 \\
a(xk_2x^{-1})x &= bxk_1 \\
axk_2 &= bxk_1 \\
b^{-1}axk_2 &= xk_1 \\
b^{-1}ax &= xk_1k_2^{-1} \\
b^{-1}a &= xk_1k_2^{-1}x^{-1} \\
&\to b^{-1}a \in xKx^{-1}
\end{aligned}$$

We also know by supposition that $a$ and $b$ are in $H$, and that since $H$ is a subgroup $b^{-1}a$ is also in $H$. So $b^{-1}a \in W \to aW = bW$. So we know $f$ is injective.

Next part: Let $L$ be the set of representatives of left cosets of $W$ in $H$. Consider: $L_h = \{hxK | h \in L\}$. We need $L_{h_1} \cap L_{h_2} = \emptyset$ when $h_1 \neq h_2$ and $\cup L_h = HxK$.

First, suppose $a \in L_{h_1}$ and $a \in L_{h_2}$. This means that $a = h_1xk_1$ and $a = h_2xk_2$. So $h_1xk_1 = h_2xk_2$ implies $h_2^{-1}h_1 = xk_2k_1^{-1}x^{-1}$. This means $h_1$ and $h_2$ lie in the same left coset of $W$. So we know the $L_h$ are disjoint.

Secondly, suppose $a \in HxK$. This means $a = hxk$, thus $a \in L_h$. So for every $a \in HxK$ there is an $L_h$ that contains $a$. So $\cup L_h = HxK$. Also $\cup L_h$ cannot exceed $HxK$ because each $L_h$ contains only elements from $HxK$.

Last part (this was mostly copy + pasted): Let $R$ be the set of representatives of right cosets of $M = x^{-1}Hx \cap K$ in $K$. Consider: $R_k = \{Hxk | k \in R\}$. We need $R_{k_1} \cap R_{k_2} = \emptyset$ when $k_1 \neq k_2$ and $\cup R_k = HxK$.

First, suppose $a \in R_{k_1}$ and $a \in L_{k_2}$. This means that $a = h_1xk_1$ and $a = h_2xk_2$. So $h_1xk_1 = h_2xk_2$ implies $k_1k_2^{-1} = x^{-1}h_1^{-1}x$. This means $k_1$ and $k_2$ lie in the same right coset of $M$. So we know the $R_k$ are disjoint.

Secondly, suppose $a \in HxK$. This means $a = hxk$, thus $a \in R_k$. So for every $a \in HxK$ there is an $R_l$ that contains $a$. So $\cup R_k = HxK$. Also $\cup R_k$ cannot exceed $HxK$ because each $R_k$ contains only elements from $HxK$.

3. Suppose $H$ and $K$ are finite. First we will prove that $|W| = |M|$. Let $f : W \to M$ by $f(w) = x^{-1}wx$. If $w \in W$ then we know $w \in H$ and $\exists k \in K$ where $w = xkx^{-1}$. Now, $f(w) = x^{-1}wx = x^{-1}xkx^{-1}x = k \in K$. So $f(w) \in K$ and also $f(w) \in x^{-1}Hx$. The function is a bijection with inverse given by $f^{-1}(m) = xmx^{-1}$. So $|W| = |M|$.

   We can show that $|HxK| = |H||K|/|W|$ relatively easily. If $h_1xk_1 = h_2xk_2 \in HxK$ then that means $h_1W = h_2W$. So this tells us the number of elements in $HxK$ is the product of the number of elements in $H$ and $K$, not counting elements congruent mod $W$. Thus:

$$|HxK| = \frac{|H||K|}{|W|} = \frac{|H||K|}{|M|}.$$

**2.8.**

1. 
   - Let $x, y \in C_G(A)$, consider $xy^{-1}$. We know $ax = xa$ for all $a \in A$ and $ay = ya \to a = yay^{-1} \to y^{-1}a = ay^{-1}$. Therefore $axy^{-1} = (ax)y^{-1} = (xa)y^{-1} = x(ay^{-1}) = xy^{-1}a$ and $xy^{-1} \in C_G(A)$. So $C$ is a subgroup of $G$.
   - Let $x, y \in N_G(A)$, consider $xy^{-1}$. We know $xAx^{-1} = A$ and $yAy^{-1} = A \to A = y^{-1}Ay$. Therefore $xy^{-1}A(xy^{-1})^{-1} = x(y^{-1}Ay)x^{-1} = xAx^{-1} = A$. We're done, $N_G(A)$ is a subgroup of G.
   - Let $x \in C_G(A)$, so for all $a \in A$ $xa = ax$. This implies that $xA = Ax$. This means $xAx^{-1} = A$, so $x \in N_G(A)$.
2. Suppose that $A$ is a subgroup of $G$. Let $n \in N_G(A)$. We have $nAn^{-1} = A$ by definition of $N_G(A)$. This means $nA = An$ for all $n \in N_G(A)$ and thus $A \trianglelefteq N_G(A)$ and we're done.
3. Consider $g \in G$. We know that for all $z \in Z(G)$, $gz = zg$. So $gZ(G) = Z(G)g$ and thus $Z(G) \trianglelefteq G$.

4. Let $[G : H] = 2$ Consider $g \in G$ either $g \in G - H$ or $g \in H$, if $g \in H$ then clearly $gH = Hg$. If $g \in G - H$ then $gH = G - H$ because $H$ has index 2 and there are only 2 cosets. From the same logic we have $Hg = G - H$, so $gH = Hg$ and $H$ is normal.

   For an example, consider $D_6$. $[D_6 : \langle s \rangle] = 3$ but $sr \neq rs$ so $\langle s \rangle$ is not a normal subgroup of $D_6$.

5. If $n$ is even, $Z(D_{2n}) = \langle r^{n/2} \rangle$. If $n$ is odd $Z(D_{2n}) = e$. We know the rotations will always commute, and that $sr = r^{-1}s$. This means only elements with $r = r^{-1}$ will be in $Z(D_{2n})$, that is $r^2 = 1$. So the center of the odd groups is just the identity, and for the even ones it is the identity and $r^{n/2}$.

6. Let $N$ be a subgroup of $Q_8$. We know from the structure of $Q_8$ that for every $q \in Q$ and every $n \in N$ either $qn = nq$ or $qn = -nq$. If $N$ is just the group containing 1 then we're done. If $N$ contains an element $n \neq 1$, then $N$ must contain $-n$ because $i * (-i) = j * (-j) = k * (-k) = 1$ and $-(-1) = 1$. So $N = -N$ as long as $N \neq \{1\}$. Then we have $qN = Nq$ for all $q \in Q$. Done.

**2.9 #1.** Let $G$ have prime order. We know that if $H$ is a subgroup of $G$ then $|H|$ divides $|G|$. However, since $|G|$ is prime, we have $|H| = 1$ or $|H| = |G|$. Thus $H = G$ or $H = 1$ and $G$ has no nontrivial subgroups. This proves $G$ has no non-trivial subgroups.

Also, $\forall g \in G$ we know $g^{|}G| = 1$, we also know that the order of each element must divide evenly into $|G|$, therefore every element has order either $|G|$ or $|1|$, we conclude there is at least one generator for $G$, so $G$ is cyclic.

**2.9 #2.**

Let $G$ be a group and $H$ be a subgroup of $Z(G)$, also suppose $G/H$ is cyclic. Let $G/H = \langle g \rangle$ and let $a, b \in G$. We can write $a = g^n h_1$ and $b = g^m h_2$. Then we use group properties:

$$\begin{aligned} ab &= g^n h_1 g^m h_2 \\ \text{Since } h_1 \in Z(G) &= g^n g^m h_1 h_2 \\ &= g^{n+m} h_1 h_2 \\ &= g^m g^n h_2 h_1 \\ \text{Since } h_2 \in Z(G) &= g^m h_2 g^n h_1 \\ &= ba \end{aligned}$$

So we have $G$ is Abelian.

**2.10.**

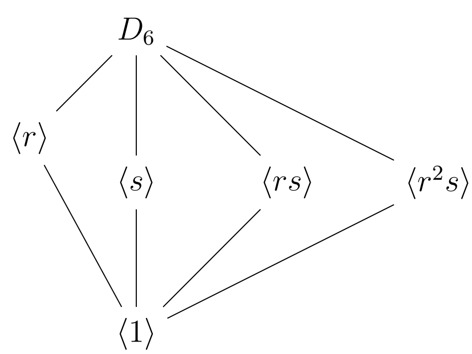


Figure 1: Subgroups of $\mathbb{D}_6$

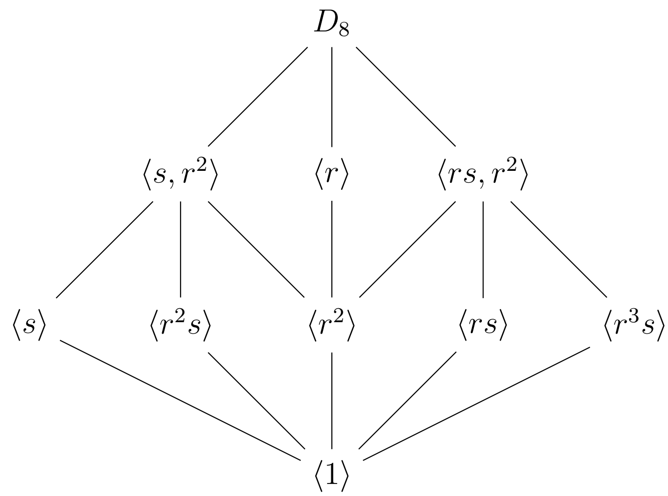


Figure 2: Subgroups of $\mathbb{D}_8$

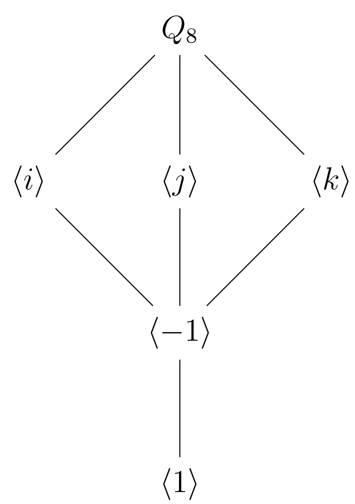


Figure 3: Subgroups of $\mathbb{Q}_8$