Algebra 1 Homework 6
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4.2 #1: Let ϕ be map from G to the group of permutations of G/H. We define ϕ by $\phi(g) = gxH$, for any xH in G/H. That is ϕ sends an element in G to the left multiplication bijection of G/H. The kernel of this homomorphism is the set

$$\begin{aligned}\mathrm{Ker}(\phi) &= \{g \in G | gxH = xH \ \forall x \in G\}\\ &= \{g \in G | x^{-1}gxH = H \ \forall x \in G\}\\ &= \{g \in G | x^{-1}gx \in H \ \forall x \in G\}\\ &= \{g \in G | g \in xHx^{-1} \ \forall x \in G\}\\ &= \cap_{x \in G} xHx^{-1}\end{aligned}$$

So we know four things:

- $\mathrm{Ker}(\phi)$ is a normal subgroup of G because the kernel of a homomorphism is a normal subgroup.
- H contains $\mathrm{Ker}(\phi)$ because $eHe^{-1} = H$ is in the intersection that defines $\mathrm{Ker}(\phi)$.
- $G/\mathrm{Ker}(\phi)$ is isomorphic to a subgroup of the bijections of G/H by the first isomorphism theorem.
- The index of $\mathrm{Ker}(\phi)$ is a divisor of $n!$ because of Cayley's theorem and because $|S_{G/H}| = n!$.

That ends part 1.

4.2 #2: Let p be the smallest prime factor of $|G|$ and let H have index p. We start with the subgroup K of G contained in H where:

$$K = \bigcap_{x \in G} xHx^{-1}$$

.

Since G is finite we have that:

$$|G : K| = |G : H||H : K|$$

Lets say $|H : K| = k$, so $|G : K| = pk$. We know from the last part that $pk|p!$, so $k|(p-1)!$. We'll look at the prime factorization $|G| = pp_1 \dots p_n$ where each $p_i \leq p_{i+1}$ and p is the smallest prime in the factorization. As well, the prime factorization of $|H|$ is $p_1p_2 \dots p_n$. So $k = |H : K|$ is either 1 or a product of some subset of $p_1, p_2, \dots p_n$. All of these primes are at least size p, so any product of them cannot divide into $(p-1)!$. Therefore $k = 1$ and $H = K$.

Since $H = K = \cap xHx^{-1}$, this tells us that H is fixed under conjugation by G, and thus that H is normal. This ends part 2.

4.2 #3: This is a simple consequence of the previous part. 2 is the smallest prime number so for any subgroup of index 2, the previous part will apply and that subgroup will be normal. This ends part 3.

4.2 #4: Let N be a normal subgroup and K be a conjugacy class of G. Suppose $K \cap N \neq \emptyset$ this means there is an $a \in K \cap N$. Consider $b \in K$, we know $b = g_1 k g_1^{-1}$ for a fixed $k \in G$ and some $g_1 \in G$, and we know that $a = g_2 k g_2^{-1}$ for another $g_2 \in G$. After some algebra we have that $b = g_1 g_2^{-1} a (g_1 g_2^{-1})^{-1}$, so $b \in N$. Now we know that $K \subseteq N$ and this ends part 4.

4.3 #1: For Q_8: First off we know that for any element g in any group G, $\langle g \rangle \leq C_G(g)$. Cyclic groups are always abelian so any element in the cyclic subgroup will belong to the centralizer.

Here is what this means for Q_8 in particular. The center of Q_8 is $\{1, -1\}$. The element i is not in the center, but $|\langle i \rangle| = 4$ and $|Q_8 : \langle i \rangle| = 2$, so $C_{Q_8}(i) = \langle i \rangle$. So we have $Q_8 / C_{Q_8}(i) = \{i, -i\}$ and the same reasoning holds for j and k.

Thus the conjugacy classes of Q_8 are: $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$, and $\{k, -k\}$.

For D_8, there are three abelian subgroups with index 2. These three normal subgroups: $\langle s, r^2 \rangle$, $\langle r \rangle$, and $\langle rs, r^2 \rangle$, union to be the entirety of D_8 and all contain r^2 in their intersection. From the previous problem this tells us that the conjugacy classes of D_8 are properly contained in these groups.

On the next tier of the subgroup lattice we have $\langle s \rangle$, $\langle r^2 s \rangle$, $\langle rs \rangle$, $\langle r^3 s \rangle$, and the center: $\langle r^2 \rangle$. If x is not in the center then $\langle x \rangle$ is contained in one abelian subgroup of order 4, so this means $|C_{D_8}(x)| \geq 4$. The order of the centralizers must divide 8, but they cannot be equal to 8, because D_8 is not Abelian. So each centralizer has order 4. Now we can compute the congugacy classes by finding $D_8 / C_{D_8}(x)$, they are: $\{1\}$, $\{r^2\}$, $\{r, r^3\}$, $\{s, sr^2\}$, and $\{sr, sr^3\}$.

4.3 #2 a): Consider a t-cycle, $c = (c_1, \ldots, c_t)$ and a random permutation σ. If $c(c_i) = c_j$ then consider $\sigma c \sigma^{-1}(\sigma(c_i))$. By composition this is $\sigma c(c_i) = \sigma(c_j)$. So, we know that if $c = (c_1, \ldots, c_t)$ then $\sigma c \sigma^{-1} = (\sigma(c_1), \ldots, \sigma(c_t))$.

So since any conjugate of a t-cycle is a t-cycle then the conjugacy class of a t-cycle must contain only t-cycles. Consider the previous t-cycle c, and another one, $d = (d_1, \ldots d_t)$. Now we construct the permutation σ such that $\sigma(t_i) = d_i$ for all $i \leq t$, then we have $\sigma c \sigma^{-1} = d$. So any t-cycle is the conjugate of any other t-cycle.

4.3 #2 b): To construct a t-cycle from S_n, we have n choices for the first spot, $n-1$ choices for the second entry, and so on all the way down to $n - t + 1$ choices for the t^{th} entry. Also t-cycles are the same up to cyclicly reordering the elements, there are t such shifts. Altogether we know there are $\frac{n!}{(n-t)!t}$ unique t-cycles.

We know that the order of the conjugacy class of an element is $|G|/|C_G(x)|$. So we have $|S_n|/|C_{S_n}(c)| = \frac{n!}{(n-t)!t}$ and we know $|S_n| = n!$, so we have $|C_{S_n}(c)| = t(n-t)!$.

4.3 #2 c): Consider the alternating group A_5. From the previous question we can see that it contains 24 5-cycles, 20 3-cycles, 15 pairs of 2-cycles, and the identity element.

We need to construct the conjugacy classes of each of the cycle groups. First we need to look at how conjugacy classes in A_n are related to the corresponding classes in S_n. We have either that $C_{S_n}(x) \subset A_n$ or the opposite.

In the first case we see that $C_{S_n}(x) \cap A_n = C_{S_n}(x)$, and since A_n has exactly half as many elements as S_n we have that the conjugacy class of x in A_n has exactly half as many elements as the same conjugacy class in S_n.

In the second case, if $C_{S_n}(x)$ is not a subset of A_n then there is an odd element τ in $C_{S_n(x)}$. Consider for any other odd element σ we have $\sigma x \sigma^{-1} = \sigma\tau x \tau^{-1}\sigma^{-1} = \sigma\tau x(\sigma\tau)^{-1}$. This is implies the conjugacy classes are the same in A_n and S_n.

We can start constructing the conjugacy classes with the 5-cycles. We see that if σ conjugates (12345) to (13524), then σ can be written as (2354) which is odd. So these elements are not conjugates in A_5. Thus from the previous part there are 2 5-cycle conjugacy classes, each having size 12.

Now we look at the three cycles. The centralizer of (123) contains the odd element (45), so the conjugacy class of (123) in A_5 is the same as in S_5. So there is only one conjugacy class of three cycles.

Finally we consider pairs of two cycles. Again the centralizer of (13)(24) contains the odd element (13). So there is one conjugacy class for pairs of 2-cycles.

So we know that the class equation for A_5 is $60 = 12 + 12 + 20 + 15 + 1$. Any normal subgroup of A_5 is a union of conjugacy classes and it must contain the element 1. It also must be a divisor of 60. This reveals the only possible normal subgroups have size 60 or size 1, and we're done.

4.4 #1: Let G be a group of order $11^2 * 13^2$. From Sylow's theorem we know that $n_{11} = 1 \bmod 11$ and that $n_{11}|13^2$, this means $n_{11} = 1$. Also we know that $n_{13} = 1 \bmod 13$ and that $n_{13}|11^2$, again tells us that $n_{13} = 1$. This tells us that the groups $P \in Syl_{11}(G)$ and $Q \in Syl_{13}(G)$ subgroups are unique. Since they are unique, $\forall g \in G$ we have that $gPg^{-1} = P$ and $gQg^{-1} = Q$. So the Sylow p-subgroups are both normal. From the section on normal subgroups we know that PQ is a subgroup of G, and since $|PQ| = |G|$, we have $G = PQ$. The subgroups P and Q themselves both have order a prime squared, so, from exercise 4.1, we can say that P and Q are both Abelian, and thus $G = PQ$ is Abelian.

4.4 #2: Consider a group G of order $77 = 7*11$. Let's look at n_7 and n_11 the number of Sylow 7-subgroups and Sylow 11-subgroups. From Sylow's theorem we have $n_7 = 1 \bmod 7$ and $n_7|11$, this tells $n_7 = 1$. Also we have that $n_{11} = 1 \bmod 11$ and $n_{11}|7$, again we know $n_{11} = 1$. So from the same reasoning as the last problem we see that G is the product of subgroups of order 11 and 7. These subgroups are both prime order, and thus cyclic. So $G = \mathbb{Z}_{11} \times \mathbb{Z}_7$. This group has an element of order 77, namely $(1, 1)$. Thus $G = \mathbb{Z}_{77}$.

4.4 #3: Let G be a group of order 30. Consider the collections of Sylow 3-subgroups and Sylow 5-Subgroups. We have $n_5 = 1 \bmod 5$ and $n_5|6$, and that $n_3 = 1 \bmod 3$ and $n_3|10$.

If either n_5 or n_3 is one then we are done, this means G will contain a normal subgroup P of order 3, and at least one subgroup Q of order 5. This means PQ will be a subgroup of order 15, and since P and Q both have prime order, they are cyclic, and PQ will also be the cyclic group $\mathbb{Z}_{15}$.

The only other options for n_3 and n_5 are if $n_3 = 10$ and $n_5 = 6$. This would mean that the G would contain $6 * 4 = 24$ elements of order 5 and $10 * 2 = 20$ elements of order 3. This is impossible though, because a group of order 30 cannot contain more than 44 elements.

So to conclude, we know that at least one of n_3 or n_5 will be 1. This guarantees the existence of a cyclic subgroup of order 15 in G.