Algebra 1 Homework 3
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1. Page 40 #2. Consider $\phi : G \to H$ an isomorphism. Let $x \in G$ with $|x| = n$. This means $x^n = 1_G$. Therefore $\phi(x^n) = \phi(1_G)$, and since ϕ is an isomorphism we have $\phi(x)^n = 1_H$. So the order of $\phi(x)$ is at most n.

   To prove the order is equal to n suppose there is a $k < n$ for which $\phi(x)^k = 1_H$, this means $\phi(x^k) = 1_H$ and that $x^k = \phi^{-1}(1_H)$. Finally $x^k = 1$ which contradicts $|x| = n$. So $|\phi(x)| = |x|$.

   If ϕ is only a homomorphism we don't get this result. Take $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_{\neq}, +)$ by $\phi(x) = x\text{mod}2$. ϕ is a homomorphism but $|1| = \infty$ and $|\phi(1)| = 2$.

2. Page 40 #3. Let $\phi : G \to H$ be an isomorphism. Consider $a, b \in G$. We have $\phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a)$. So $\text{Im}(\phi)$ is commutative. Because ϕ is a bijection we have $\text{Im}(\phi) = H$, so H is abelian.

   For the other direction note that ϕ^{-1} is also an isomorphism, and therefore if H is abelian then G will be abelian by the same argument.

   More generally, if $\phi : G \to H$ is a homomorphism and G is abelian then H will be abelian provided $\text{Im}(\phi) = H$. In other words, so ϕ must be onto to ensure that if G is abelian, then so is H.

3. Page 40 #4. Consider the multiplicative groups $\mathbb{R} - 0$ and $\mathbb{C} - 0$. Consider, for sake of contradiction, an isomorpism $\phi : \mathbb{C} - 0 \to \mathbb{R} - 0$. Now, $\phi(i) \in \mathbb{R} - 0$. Say $\phi(i) = x$, then we have $\phi(i)^2 = x^2$, which means $x^2 = \phi(-1)$.

   We will now prove $\phi(-1) = -1$. We have $\phi(-1)^2 = \phi(1) = 1$. This means $\phi(-1)$ equals either 1 or -1. If $\phi(-1) = 1$ then $\phi(-1) = \phi(1)$ which contradicts ϕ being one to one. So, since $\phi(-1) = -1$ we have $x^2 = -1$, where x is real. This equation has no solutions over the real numbers, this contradicts ϕ being well defined. Thus the multiplicative groups $\mathbb{C} - 0$ and $\mathbb{R} - 0$ are not isomorphic.

4. Page 40 #7. D_8 and Q_8 are not isomorphic. D_8 has 4 elements of order 2 (V,H,D,D') while Q_8 has only one element of order 2 (-1).

5. Page 40 #17. Let G be a map and consider the map $\phi : G \to G$ by $\phi(g) = g^{-1}$.

   $\to$ Suppose ϕ is a homomorphism. Consider $a, b \in G$ then we have $b^{-1}a^{-1} = (ab)^{-1} = \phi(ab) = \phi(a)\phi(b) = a^{-1}b^{-1}$. So $b^{-1}a^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Thus G is abelian.

   $\leftarrow$ Suppose G is abelian. Consider $a, b \in G$. We have $\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$. Thus ϕ is a homomorphism.

6. Page 41 #25.

(a) Consider a vector in polar coordinates, and the product:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r\cos(\phi) \\ r\sin(\phi) \end{bmatrix} \tag{1}$$

(b)

(c)

7. Page 41 #26.

8. Page 48 #3

9. Page 48 #10(a).

10. Page 60 #1.

11. Page 60 #3.

12. Page 60 #12(a).