Topology Homework 2

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Problem 1: Show that a space is simply connected if and only if all paths having the same endpoints are fixed endpoint homotopic.

$\rightarrow$ Suppose X is simply connected. Consider two points x_0 and x_1, and two paths α and β connecting the points. We say $\bar{\beta}$ is $\beta(1-t)$. We can say that $[\alpha] \cong [\alpha] * [\bar{\beta} * \beta]$ because the path $[\bar{\beta} * \beta]$ is homotopic to a constant path at x_1. Then by associativity we have $[\alpha] \cong [\alpha * \bar{\beta}] * [\beta]$. Next, because $[\alpha * \bar{\beta}]$ is a loop based at x_0 we have $[\alpha] \cong [1] * [\beta]$. Finally we get $[\alpha] \cong [\beta]$ because $[1]$ is the identity. Thus α and β are fixed endpoint homotopic.

$\leftarrow$ Suppose any two paths in X are fixed end point homotopic. Consider a loop that begins and ends at the point x_0 and a path that is constant at x_0. These paths have the same endpoints so they are homotopic. Thus any loop based at x_0 will be homotopic to the path that is constant at x_0, and X is by definition simply connected.

Problem 2: We need to show that $(g \circ f)_* = g_* \circ f_*$.

This is not so hard. Let $[\gamma] \in \pi_1(X, x_0)$. Then we have $(g \circ f)_* \circ [\gamma] = [g \circ f \circ \gamma]$ by definition. On the other hand, lets look at $g_* \circ f_* \circ [\gamma]$, this is $g_*(f_*([\gamma]))$ which is $g_*([f \circ \gamma])$, and finally $[g \circ f \circ \gamma]$. So the maps are equal.

Problem 3: Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $F : [0,1] \times [0,1] \to B$ be continuous with $F(0,0) = b_0$. We want to lift the entire function.

From the uniqueness of path lifting we can lift the edges of the domain, the paths $F(t,0)$ and $F(0,s)$, to unique paths in E. Now we procede by cutting the domain of F into small closed cells. We start with two lists $0 = t_1 < t_2 \cdots < t_n = 1$ and $0 = s_1 < s_2 \cdots < s_m = 1$, where for each $I_a \times J_b = [t_a, t_{a+1}] \times [s_b, s_{b+1}]$ we have an open set $U_{a,b}$ containing $F(I_a \times J_b)$ where $p^{-1}(U_{a,b})$ is a disjoint union of open sets in E.

Next we will lift one square at a time. We'll first lift all of the squares in row one $I_a \times J_1$, and then procede to lift the next row, squares like $I_a \times J_2$, and so on. Now let's lift the first square. We know that the image of the bottom and left edges of this square lift to unique paths in E. We also know that the image of the entire square will lift to a collection of disjoint subsets of E. Since the unique paths $\tilde{F}(t,0)$ and $\tilde{F}(0,s)$ agree on the point $(0,0)$, there will be a unique subset of E for which $\tilde{F}|_{I_1 \times J_1}$ will match $\tilde{F}$ on the boundaries. In this way we extend the lift to the first square.

As we procede to all other squares we can see that each subsequent square will share two adjacent sides with either some previous squares or the left or bottom edges of the domain. In this way we can lift the entire function on $[0,1] \times [0,1]$ to a unique $\tilde{F}$.

For the last part we suppose F is a path homotopy. This means $F(t,0)$ and $F(t,1)$ are fixed. When we specify that $\tilde{F}(0,0) = e_0$ we make a unique lift to an $\tilde{F}$. This one must also have $\tilde{F}(t,0)$ and $\tilde{F}(t,1)$ fixed. Then by continuity we have $\tilde{F}(0,s)$ and $\tilde{F}(1,s)$ are two homotopic paths in E.

Problem 4: Let $p : E \to B$ be a covering map where B is connected and there is some point $b \in B$ for which $|p^{-1}(b)| = k$.

Consider two subsets of B, where $U = \{x \in B : |p^{-1}(x)| = k\}$ and $V = \{y \in B : |p^{-1}(y)| \neq k\}$. We can say that because p is a covering map if $x \in U$ there will be some open set containing x that is a subset of U; therefore U is open. As well if $y \in V$ then there will be an open set containing y that is a subset of V; so V also is open.

Since p is an onto function we have that $U \cap V = \emptyset$ and $U \cup V = B$. If V is nonempty this forms a separation of B. So since B is connected we have $V = \emptyset$ and $U = B$. We conclude, every point has k preimages.

Problem 5: Suppose B is simply connected, E is path connected, and $p : E \to B$ is a covering map with $p(e_0) = b_0$.

We can define a map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$; we say if $[f] \in \pi_1(B, b_0)$ then $[f]$ lifts to a unique class of paths beginning at e_0, $[\tilde{f}]$, we say $\phi([f]) = [\tilde{f}](1)$. The map ϕ sends every loop in $\pi_1(B, b_0)$ to the endpoint of its lift in E.

Since E is path connected, the map ϕ will be surjective. Every path in E connecting e_0 to another point, say e_1, in $p^{-1}(b_0)$ will be projected to a loop in B by p. Therefore there will be at least one loop for which $\phi([f]) = e_1$.

We know that $|\pi_1(B, b_0)| \geq |p^{-1}(b_0)|$ since ϕ is onto. As well $\pi_1(B, b_0)$ is trivial, because B is simply connected. This means $|p^{-1}(b_0)| = 1$. So, p covers the point b_0 by exactly one point. Now by the previous we know that p is a 1-fold covering, also known as an injective map. Since covering maps are surjective and continuous by supposition we have that p is a homeomorphism.

Problem 6: Let $h : (X, x_0) \to (Y, y_0)$ be an inessential map. We want to show that h_* is trivial.

We know that there is a homotopy η that deforms h to a constant map on Y. Consider the induced map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. If $[x] \in \pi_1(X, x_0)$ we can consider $h_* \circ [x] = [h \circ x]$ and since η is a homotopy we have $\eta \circ [h \circ x] \cong [h \circ x]$, but also $\eta \circ [h \circ x] \cong [(\eta \circ h) \circ x] \cong [1_{y_0} \circ x] \cong [1]$. This tells us that $h_*([x]) = 1$. So the map induced by an inessential map is trivial.