Algebra 1 Homework 5
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3.1 #3: Proof. Let a and b be elements of G. Then

$$\begin{aligned}\text{Inn}(ab) &= \phi_(ab) \\ &= ab[.](ab)^{-1} \\ &= ab[.]b^{-1}a^{-1} \\ &= \text{Inn}(a) \circ \text{Inn}(b)\end{aligned}$$

Since $Z(G)$ is the set $\{a \in A | gag^{-1} = a\}$, we have that $a \in Z(G)$ if and only if $\text{Inn}(a) = Id_G$. This is because $gag^{-1} = a$ for all g if and only if $g = aga^{-1}$ for all g.

Part 2. Any automorphism of D_8 must respect the group operation. Thus any automorphism is defined by its actions on the generators. Let $\phi \in \text{Aut}(D_8)$, then $\phi(r)$ and $\phi(s)$ completely define ϕ. We know ϕ sends elements to elements of the same order. Thus $\phi(r)$ can be r or r^3, and $\phi(s)$ can be one of s, r^2, sr, sr^2 or sr^3. Now we know $|\text{Aut}(D_8)| \leq 10$.

To compute $\text{Aut}(D_8)$ we can use the two isomorphisms: ρ and ϕ. The are defined with $\rho(r) = r$ and $\rho(s) = sr$; and $\phi(r) = r^3$ and $\phi(s) = s$. We can see that $|\rho| = 4$ and $|\phi| = 2$ this tells us that $|\text{Aut}(D_8)|$ is either 8 or 4, and because no composition of ρ creates the permutation ϕ we know that $|\text{Aut}(D_8)| = 8$. A quick computation shows that $\rho \circ \phi(r) = \phi \circ \rho^{-1}(r) = r^3$ and that $\rho \circ \phi(s) = \phi \circ \rho^{-1}(s) = sr$. So $\text{Aut}(D_8) \cong D_8$ and $\text{Aut}(D_8)$ is generated by ϕ and ρ.

Now let's compute $\text{Inn}(D_8)$. We know $Z(D_8) = \langle r^2 \rangle$ so if we consider the homomorphism $\phi : D_8 \to \text{Inn}(D_8)$. We have by the first isomorphism theorem that $\text{Inn}(D_8) \cong D_8/\langle r^2 \rangle$. This group is isomorphic to the Klein 4-Group.

3.4. Let $\phi : G \to \bar{G}$ be an epimorphism, let $N = ker(\phi)$ and let H be a subgroup of G containing N.

$\to$ Suppose H is normal in G. We know for all g, $gHg^{-1} = H$. Also, since ϕ is an epimorphism any element $\bar{g} \in \bar{G}$ satisfies $\bar{g} = \phi(g)$ for at least one $g \in G$. Consider $\bar{g}\phi(H)\bar{g}^{-1}$ there is a g for which this equals $\phi(g)\phi(H)\phi(g)^{-1} = \phi(gHg^{-1}) = \phi(H)$. So we know that if H is normal then $\phi(H)$ must be normal.

$\leftarrow$ Suppose $\phi(H)$ is normal in $\bar{G}$. Since ϕ is an epimorphism we can write $\phi(H) = \bar{g}\phi(H)\bar{g}^{-1} = \phi(gHg^{-1})$ This implies that $gHg^{-1} = nH$, that the preimages are equal up to an element from N. However, $N \leq H$ so $nH = H$, and $gHg^{-1} = H$.Thus H is normal and we're done.

For the next part: we need to show that that the lattice subgroups of G that contain N is in bijection with the subgroup lattice of $\bar{G}$. We know from the first isomorphism theorem that $G/N \cong \bar{G}$.

Let π be the natural projection from G to G/N. If H is a subgroup such that $N \leq H \leq G$ then $\pi(H)$ will be a subgroup of G/N because π is a homomorphism. Likewise $\phi(H)$ will be a subgroup of $\bar{G}$ because the groups are isomorphic. Now suppose $\bar{S} \leq \bar{G}$ then $\bar{S}$ is isomorphic to S a subgroup of G/N. Consider $\pi^{-1}(S)$, this is the group SN which is a subgroup of G that contains N.

Last part: Let $B \leq A$ be subgroups of G containing N. Let aB be an element of the set A/B. We know that each element in A/B maps to a coset of $\phi(A)/\phi(B)$, because ϕ is surjective. However the preimage, $\phi^{-1}(\bar{a}\phi(B))$, is a coset in A/B since A and B contain N. So $|A : B| = |\phi(A) : \phi(B)|$ and we're done.

3.5.

First part: Suppose H is a normal subgroup of G with prime index p and K is a subgroup of G but is not a subgroup of H. Let π be the natural map from G to G/H. We know that $H \leq HK \leq G$ This tells us that

$$\pi(HK) \cong HK/H \leq G/H.$$

Then by the second isomorphism theorem we get that $K/(H \cap K) \leq G/H$. Since K is not a subgroup of H and G/H has prime order. This extends to:

$$KH/H \cong K/(H \cap K) \cong G/H.$$

From this relation we have $|K : H \cap K| = p$ immediately. To see that $G \cong HK$ is not so hard. This follows from the fourth isomorphism theorem, because π is an epimorphism and G and HK are both subgroups of G that contain H. Since their projections are isomorphic, they themselves must be isomorphic.

Second part: Suppose H is a normal subgroup of G with prime index p and K is a subgroup of G with $|K : K \cap H| \neq p$. The same relations as before hold from the second isomorphism theorem.

$$KH/H \cong K/(H \cap K) \leq G/H$$

But this time since $|K : K \cap H| \neq p$ we know that $K/(H \cap K)$ must be trivial. So this means $H \cap K = K$, or $K \leq H$.

3.6.

1. Suppose G and G' are groups, let multiplication in $G \times G'$ be defined as $(g, g')(h, h') = (gh, g'h')$. This makes $G \times G'$ a group.

- Closure: Let $(g, g')(h, h') = (gh, g'h')$ for any $g, h \in G$ and $g', h' \in G'$. Since G and G' are both closed, $(gh, g'h') \in G \times G'$.
- Associativity: $(g, g')((h, h')(k, k')) = (g, g')(hk, h'k') = (ghk, g'h'k')$ by G and G' being associative. Also $((g, g')(h, h'))(k, k') = (gh, g'h')(k, k') = (ghk, g'h'k')$ again because G and G' are associative. Thus $G \times G'$ is associative.
- If e is the identity in G and e' is the identity in G' then for all (g, g') in $G \times G'$ we have $(e, e')(g, g') = (eg, e'g') = (g, g')$ and also $(g, g')(e, e') = (ge, g'e') = (g, g')$. So $G \times G'$ has an identity element.
- We have $(g, g')^{-1} = (g^{-1}, g'^{-1})$. So inverses exist.

Now we know that $G \times G'$ is a group.

2. Let M and N be normal subgroups of G such that $G = MN$. We define a homomorphism $\phi : G \to G/M \times G/N$ by $\phi(g) = (gM, gN)$. The kernel of this homomorphism is $M \cap N$. if $x \in M \cap N$ then $\phi(x) = (M, N)$ which is the identity element of $G/M \times G/N$.

Pg 87 #17.

- (a) The order of D_16 is 16 and the order of $\langle r^4 \rangle$ is 2. So $|\bar{G}| = |D_16/\langle r^4 \rangle| = 16/2 = 8$.

- (b) Any element in $\bar{G}$ is of the form $\bar{s}^a\bar{r}^b$ where a is either 1 or 0 and b is one of $0, 1, 2, 3$. The order of the element s does not change in $\bar{G}$. The order of r is now 4 instead of 8 since r^4 has been modded out.

- (c) (I will omit the bars, these are all elements in $\bar{G}$ though) $|s| = 2, |sr| = 2, |sr^2| = 2, |sr^4| = 2, |r| = 4, |r^2| =$

- (d) $\bar{r}\bar{s} = \bar{s}\bar{r}^3$, $sr^{\bar{-}2}s = \bar{r}^2$, and $sr^{\bar{-}1}sr = \bar{r}^2$

- (e) Omitting the bars: $rH = \{rs, r^3, rsr^2, r\} = \{sr^3, r^3, sr, r\} = Hr$. And for conjugation by s, $sH = \{1, sr^2, r^2, s\} = \{1, r^2s, r^2, s\} = Hs$. So H is fixed under conjugation by the generators, so H is normal. Each non-identity element of H has order 2, and H has order 4, thus H is the Klein-4 group.

  The preimage of H in G is $\{1, r^2, r^4, r^6, s, sr^2, sr^4, sr^6\}$. This is isomorphic to D_8. with isomorphism $\phi(s) = s$ and $\phi(r^2) = r$.

- The center of $\bar{G}$ is $\langle \bar{r}^2 \rangle$. Also $\bar{G}/Z(\bar{G})$ is isomorphic to the Klein 4-Group. This is because $\bar{G}$ is isomorphic to D_4 with isomorphism $\phi(\bar{r}) = r$ and $\phi(\bar{s}) = s$. In one of the previous problems we showed that $D_8/Z(D_8) \cong K_4$.

Pg 88 #32. The subgroups Q_8 and $\langle 1 \rangle$ are trivially normal. The subgroups $\langle i \rangle$, $\langle j \rangle$ $\langle k \rangle$ all have index 2 and thus are normal. Q_8 mod each one is isomorphic to Z_2 the only group of order 2. The subgroup $\{1, -1\}$ is normal as well because it is the center of Q_8. $Q_8/\langle -1 \rangle \cong Z_2 \times Z_2$. This is $i^2, j^2, k^2 = -1$, so i, j and k all have order 2.