Algebra 1 Homework 3
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1. Page 40 #2. Consider $\phi : G \to H$ an isomorphism. Let $x \in G$ with $|x| = n$. This means $x^n = 1_G$. Therefore $\phi(x^n) = \phi(1_G)$, and since ϕ is an isomorphism we have $\phi(x)^n = 1_H$. So the order of $\phi(x)$ is at most n.

   To prove the order is equal to n suppose there is a $k < n$ for which $\phi(x)^k = 1_H$, this means $\phi(x^k) = 1_H$ and that $x^k = \phi^{-1}(1_H)$. Finally $x^k = 1_G$ which contradicts $|x| = n$. So $|\phi(x)| = |x|$.

   If ϕ is only a homomorphism we don't get this result. Take $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_2, +)$ by $\phi(x) = x \bmod 2$. ϕ is a homomorphism but $|1| = \infty$ and $|\phi(1)| = 2$.

2. Page 40 #3. Let $\phi : G \to H$ be an isomorphism. Consider $a, b \in G$. We have $\phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a)$. So $\mathrm{Im}(\phi)$ is commutative. Because ϕ is a bijection we have $\mathrm{Im}(\phi) = H$, so H is abelian.

   For the other direction note that ϕ^{-1} is also an isomorphism, and therefore if H is abelian then G will be abelian by the same argument.

   More generally, if $\phi : G \to H$ is a homomorphism and G is abelian then H will be abelian provided $\mathrm{Im}(\phi) = H$. In other words, ϕ must be onto to ensure that if G is abelian, then so is H.

3. Page 40 #4. Consider the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$. As well, for sake of contradiction, an isomorpism $\phi : \mathbb{C} - \{0\} \to \mathbb{R} - \{0\}$. Now, $\phi(i) \in \mathbb{R} - \{0\}$. Say $\phi(i) = x$, then we have $\phi(i)^2 = x^2$, which means $x^2 = \phi(-1)$.

   We will now prove $\phi(-1) = -1$. We have $\phi(-1)^2 = \phi(1) = 1$. This means $\phi(-1)$ equals either 1 or -1. If $\phi(-1) = 1$ then $\phi(-1) = \phi(1)$ which contradicts ϕ being one to one. Since $\phi(-1) = -1$ we have $x^2 = -1$, where x is real. This equation has no solutions over the real numbers, this contradicts ϕ being well defined. Thus the multiplicative groups $\mathbb{C} - \{0\}$ and $\mathbb{R} - \{0\}$ are not isomorphic.

4. Page 40 #7. D_8 and Q_8 are not isomorphic. D_8 has 4 elements of order 2 (s, sr, sr^2, sr^3) while Q_8 has only one element of order 2 (-1).

5. Page 40 #17. Let G be a map and consider the map $\phi : G \to G$ by $\phi(g) = g^{-1}$.

   $\to$ Suppose ϕ is a homomorphism. Consider $a, b \in G$ then we have $b^{-1}a^{-1} = (ab)^{-1} = \phi(ab) = \phi(a)\phi(b) = a^{-1}b^{-1}$. So $b^{-1}a^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Thus G is abelian.

   $\leftarrow$ Suppose G is abelian. Consider $a, b \in G$. We have $\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$. Thus ϕ is a homomorphism.

6. Page 41 #25.

- Take a vector in polar coordinates, and the product:
$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix} r\cos(\phi)\\ r\sin(\phi)\end{bmatrix} = \begin{bmatrix} r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)\\ r\cos(\phi)\sin(\theta) + r\sin(\phi)\cos(\theta)\end{bmatrix} \tag{1}$$
$$= \begin{bmatrix} r\cos(\phi+\theta)\\ r\sin(\phi+\theta)\end{bmatrix} \tag{2}$$
$$\tag{3}$$
Multiplication by this matrix will rotate a vector in $\mathbb{R}^2$ through an angle of θ.
- We need to show that ϕ respects the group structure of D_{2n} to prove that ϕ is a homomorphism. We need that $\phi(r)^n = \phi(s)^2 = I$ and that $\phi(s)\phi(r) = \phi(r)^{-1}\phi(s)$.
We know that $\theta = 2\pi/n$ and that $\phi(r)$ is a rotation matrix through and angle of θ. If we multiply two rotation matrices together we will add their angles. So from this, we have $\phi(r)^n = I$.
To show $\phi(s)^2 = I$ is a simple calculation:
$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
Done. For the last part we need $\phi(s)\phi(r) = \phi(r)^{-1}\phi(s)$. So, for the left hand side we have:
$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} = \begin{bmatrix}\sin(\theta) & \cos(\theta)\\ \cos(\theta) & -\sin(\theta)\end{bmatrix}$$
And for the right hand side we have:
$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^{-1}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}\sin(\theta) & \cos(\theta)\\ \cos(\theta) & -\sin(\theta)\end{bmatrix}$$
So $\phi(r)$ and $\phi(s)$ satsify all the relations that generate D_{2n}. This means ϕ will be a homomorphism from D_{2n} to $GL_2(\mathbb{R})$.
- In the previous part I showed that $\phi(r)$ and $\phi(s)$ satisfy all the relations that the generators for D_{2n} satisfy. This means the image of ϕ will be isomorphic to D_{2n} (with isomorphism ϕ). So we know ϕ must be injective, otherwise $|Im(\phi)| < |D_{2n}|$ which we know is impossible.

7. Page 41 #26. In the same way as the last problem we will show that $\phi(i)$ and $\phi(j)$ satisfy all the same relations as i and j satisfy as generators of Q_8. That is: $\phi(i)^4 = I$, $\phi(i)^2 = \phi(j)^2$, and $\phi(j)^{-1}\phi(i)\phi(j) = \phi(i)^{-1}$.
The first one is easy; since $\phi(i)$ is diagonal we have $\phi(i)^2 = -I$ and $(\phi(i)^2)^2 = \phi(i)^4 = I$. We get the second one almost as easily:
$$\phi(j)^2 = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = \phi(i)^2$$
Then we have to prove the third relation:
$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$
$$= \begin{bmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{bmatrix}$$

This proves it. We know that $\phi(i)$ and $\phi(j)$ as elements of $GL_2(\mathbb{C})$ satisfy all the relations that generate Q_8. So just like before, ϕ will be an injective homomorphism, and the image of ϕ will be isomorphic to Q_8.

8. Page 48 #3 We'll do both parts by constructing the Cayley tables.

Table 1: The Other Subset

$\circ$	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

Table 2: One Subset

$\circ$	1	r^2	sr	sr^3
1	1	r^2	sr	sr^3
r^2	r^2	1	sr^3	sr
sr	sr	sr^3	1	r^2
sr^3	sr^3	sr	r^2	1

Because their Cayley Tables are both tables of groups of order 4, these sets must be subgroups.

9. Page 48 #10(a). Let H and K be subgroups of G. Consider $H \cap K$. To prove $H \cap K$ is a subgroup we will show it is closed under multiplication and inverses. Let $x \in H \cap K$, because $x \in H$ and H is a subgroup $x^{-1} \in H$; likewise $x^{-1} \in K$. Therefore $x^{-1} \in H \cap K$.

   Consider $x, y \in H \cap K$. Well, $x, y \in H$ so $xy \in H$, likewise $xy \in K$. So we have $xy \in H \cap K$. We conclude $H \cap K$ is a subgroup.

10. Page 60 #1. Find all the subgroups of $\mathbb{Z}_{45} = \langle x \rangle$, giving a generator for each. $\mathbb{Z}_{45}$ has subgroups of order 45, 15, 9, 5, 3, and 1. To generate these subgroups we can do it in this order: $|\langle 1 \rangle| = 45$, $|\langle 3 \rangle| = 15$, $|\langle 5 \rangle| = 9$, $|\langle 15 \rangle| = 3$, $|\langle 0 \rangle| = 1$. The picture of subgroup containment looks like this.

$$\begin{tikzcd}
 & & \langle 1 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle 3 \rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 5 \rangle \arrow[dl, dash] \\
\langle 9 \rangle \arrow[dr, dash] & & \langle 15 \rangle \arrow[dl, dash] & \\
 & \langle 0 \rangle & &
\end{tikzcd}$$

Figure 1: Subgroups of $\mathbb{Z}_{45}$

11. Page 60 #3. The generators of $\mathbb{Z}/48\mathbb{Z}$ will be numbers less than 48 and relatively prime with 48. These are 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, and 47.

12. Page 60 #12(a). $Z_2 \times Z_2$ is not cyclic. The group has order 4 but the orders of its elements are: $|(0,0)| = 1$, $|(1,0)| = 2$, $|(1,1)| = 2$, and $|(0,1)| = 2$. $Z_2 \times Z_2$ has no elements of order 4 so it is not cyclic.