MATH 7311: HW 2

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Problem 1.

Proof. □

Problem 2 (3.3)**.**

Will be cleaned up later. But it should be mostly correct.

Lemma 2.1. *The product of two perfect sets are perfect.*

Proof. Let X, Y be perfect sets. Consider $(x, y) \in X \bar{\times} Y$. Let U be an open neighborhood of x. Then $U \times Y$ is an open neighborhood of (x, y). However since (x, y) is a limit point of $X \times Y$ it follows that $U \times Y \cap X \times Y \neq \phi$. This implies that $U \cap X \neq \phi$. Since this holds for any open neighborhood x is a limit point of X and as such $x \in X$. By the same reasoning $y \in Y$ which implies that $(x, y) \in X \times Y$. Therefore $X \times Y$ contains all of its limit points.

Next suppose that $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$ and so it follows that x and y are limit points of X and Y respectively as they are perfect. This in turn implies that (x, y) is a limit point of $X \times Y$.

Therefore the product of two perfect sets is perfect. □

Proof. Define $f([a, b]) = [a, \frac{a+b}{3}] \cup [\frac{2(a+b)}{3}, b]$ and $f(\bigcup_1^n [a_k, b_k]) = \bigcup_1^n f([a_k, b_k])$. Then we can define $C_k = f^k([0, 1])$ and the Cantor set is defined as $C = \bigcap_1^\infty C_k$.

Similarly we can define

$$g([a,b]\times[c,d]) = \left[a, \frac{a+b}{3}\right] \times \left[c, \frac{c+d}{3}\right] \cup \left[a, \frac{a+b}{3}\right] \times \left[\frac{2(c+d)}{3}, d\right] \cup \left[\frac{2(a+b)}{3}, b\right] \times \left[c, \frac{c+d}{3}\right] \cup \left[\frac{2(a+b)}{3}, b\right] \times \left[\frac{2(c+d)}{3}\right.$$

and $g(\bigcup_1^n [a_k, b_k] \times [c_k, d_k]) = \bigcup_1^n g([a_k, b_k], [c_k, d_k])$. Then let $D_k = g^k([0, 1]^2)$ and define $D = \bigcap_1^\infty D_k$.

Note that $g([a, b] \times [c, d]) = f([a, b]) \times f([c, d])$.

Now we will show that $D = C^2$, that $|D| = 0$, and that D is perfect.

- First we'll show that $D_k = C_k^2$. For $k = 0$ we have $g([0, 1]^2) = [0, 1]^2 = [0, 1] \times [0, 1] = f([0, 1]) \times f([0, 1])$. Assume that $D_n = C_n^2$ for $n < k$.Then for k we have $D_k = g^k([0, 1]^2) = g(g^{k-1}([0, 1]^2)) = g(f^{k-1}([0, 1]) \times f^{k-1}) = f^k([0, 1]) \times f^k([0, 1]) = C_k^2$ Therefore by induction $D_k = C_k^2$ for all k.

  Let $(x, y) \in D$. Then $(x, y) \in D_k$ for all k. However $D_k = C_k^2$ which implies that $x, y \in C_k$ for all k and as such $x, y \in C$. Therefore $D \subset C$.

  Next let $(x, y) \in C^2$. Then $x, y \in C_k$ for all k. However this implies that $(x, y) \in C_k^2 = D_k$ for all k and as such $(x, y) \in D$ and $D \subset C$.

  Therefore $C^2 = D$.

- First let us look at how g affects the measure of a set. Let $I = [a, b] \times [a, a + b]$ be a square interval. Then $|I| = b^2$ and if we take $g(I)$ we get four smaller squares with measure $\left(\frac{b}{3}\right)^2 = |I|/9$. However since there are four we have $|g(I)| = \frac{4|I|}{9}$ when I is a square.

  Then if we look at D_k the measure of D_k is

$$|D_k| = g^k([0, 1]^2) = \left(\frac{4}{9}\right)^k$$

We know that D is measurable since it is the countable intersection of measurable sets. Since $D \subset D_k$ for all k it follows that $|D| < \left(\frac{4}{9}\right)^k$ for all k and as such $|D| = 0$.

Therefore the measure of D is 0.

- From our lemma the product of two perfect sets is perfect. This implies that $C^2 = D$ is also perfect.

  Therefore D is a perfect set.

$\square$

Problem 3 (3.7)**.**

Proof. $\square$

Problem 4 (3.9)**.**

Proof. $\square$

Problem 5 (3.10)**.**

Proof. Let E_1, E_2 be measurable sets. Then by Carathéodory's Theorem we have that

$$|E_1| = |E_1 \cap E_2| + |E_1 \setminus E_2$$

and

$$|E_2| = |E_1 \cap E_2| + |E_2 \setminus E_1$$

If we add the two equations together we get

$$|E_1| + |E_2| = 2|E_1 \cap E_2| + |E_1 \setminus E_2| + |E_2 \setminus E_1|$$

Since $E_1 \setminus E_2, E_2 \setminus E_1, E_1 \cap E_2$ are disjoint it follows from two applications of Lemma 4.7 of the notes that

$$|E_1 \cup E_2| = |E_1 \Delta E_2| + |E_1 \cap E_2| = |E_1 \setminus E_2| + |E_2 \setminus E_1| + |E_1 \cap E_2|$$

$\square$

Problem 6 (3.12)**.**

Proof. $\square$

Problem 7 (3.13)**.**

- *Proof.* By definition $|E|_i = \sup\{|F|\}$ where F is a closed subset of E. Likewise, $|E|_e = \inf\{|G|\}$ where G is an open set containing E. Consider particular satisfying setss, F and G. We know $F \subset G$, so $|F| \leq |G|$. This holds for any pair of sets F and G satisfying the restraint, so we can conclude $|E|_i \leq |E|_e$. $\square$

- *Proof.* $\rightarrow$ Suppose E is measurable. By Lemma 3.22 we know that for any $\epsilon > 0$ there is a closed F, a subset of E for which $|E/F|_e < \epsilon$. Consider F, we know from the earlier part that $|F| \leq |E|_i \leq |E|_e$. However, $E = F \cup (E/F)$, so $|E|_e \leq |F| + |E/F|_e < |F| + \epsilon$. Finally we have $|F| \leq |E|_i \leq |E|_e < |F| + \epsilon$. Let ϵ tend to zero and we have it. $|E|_i = |E|_e$.

  $\leftarrow$ Suppose $|E|_i = |E|_e$. Since these measures are equal, we can say that for any $\epsilon > 0$ there will be F a closed subset of E and G an open superset of E such that, $|G| - |F| < \epsilon$. Consider two such satisfying sets. Then we have $G/E \subset G/F$. Which implies that $|G/E|_e \leq |G/F|_e$. Since G and F are both measurable this simplifies to: $|G/E|_e \leq |G| - |F| < \epsilon$. This is precisely the definition of measurability, so E is measurable. $\square$