1. Verify that the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} f_1\left(4^{n-1}x\right),$$

is not differentiable at any point $x \in \mathbb{R}$. Recall that

$$f_1(x) = |x|, \text{ for } |x| \leq \frac{1}{2}, \text{ and } f_1(x+n) = f_1(x),$$

for all integers n and real numbers x.

Proof. Suppose for sake of contradiction that $f(x)$ had a derivative at some point x_0. Since, by the Weierstrass M-Test, $f(x)$ converges uniformly, we can say that if $f(x)$ is differentiable anywhere, it's derivative must be:

$$f'(x_0) = \sum_{n=1}^{\infty} f_1'\left(4^{n-1}x_0\right).$$

The function $f_1'(x)$ has period one and is piecewise defined.

$$f_1'(x) = \begin{cases} undefined & x - \lfloor x \rfloor = 0, \frac{1}{2} \\ 1 & 0 < x - \lfloor x \rfloor < \frac{1}{2} \\ -1 & \frac{1}{2} < x - \lfloor x \rfloor \end{cases}$$

There are two cases for x_0. If $x_0 = \frac{m}{2^k}$ for some integers m and k then for $n = k$, $f_1'(4^{k-1}\frac{m}{2^k})$ will not be defined, and thus $f'(x_0)$ will not exist. For the other case, if $x_0 \neq \frac{m}{2^k}$, then $f(x_0)$ will be the sum of the sequence $f_1'(4^{n-1}x_0)_{n=1}^{\infty}$. This sequence is made up entirely of the value 1 or -1, which means the sequence cannot converge to zero, and thus that the derivative cannot exist here either.