Topology Homework 1
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1. Construct several examples of homotopic and non-homotopic maps. No proofs required. Here are some examples of homotopic maps:

    - The straight lines $\vec{r_1}(t) = (2t+1, t-3)$ and $\vec{r_2}(t) = (4t+5, -6t+2)$ will be homotopic in $\mathbb{R}^2$.
    - Any two paths on a Torus will be homotopic because the Torus is path connected.
    - Pick two paths in the upper half plane of $\mathbb{R}^2$ without the x-axis.

    Here are some examples of non-homotopic maps:

    - In $\mathbb{R}^2$ without the x-axis, take one path in the upper half plane and another path in the lower half plane. These paths won't be homotopic in this space.
    - Our space is the image of the sine curve $f(x) = sin(1/x)$, for all $x > 0$ union the closed interval of the y-axis, $[-1, 1]$. Take one path to be the closed interval $[-1, 1]$ and the other to be a connected piece of the sine curve. This space isn't path connected, so the two paths won't be homotopic.
    - Take the space to be R^3 without the xy-plane. Pick one path to have only positive z-coordinates and the other to have only negative z-coordinates. These paths won't be homotopic.

2. To prove that fixed endpoint homotopy is an equivalence relation we need to prove three things.

    - Reflexivity. Consider a path f, a homotopy from f to f would be the constant map $\phi(x,t) = f(x)$.
    - Symmetry. Suppose $\phi(x,t)$ is a homotopy from f_1 to f_2. Then $\phi(x, 1-t)$ will be a homotopy from f_2 to f_1.
    - Transitivity. Suppose $\phi_1(x,t)$ is a homotopy from f_1 to f_2 and $\phi_2(x,t)$ is a homotopy from f_2 to f_3. Then we a define a new homotopy:

    $$\phi_3(x,t) = \begin{cases} \phi_1(x, 2t) & t \in [0, 1/2] \\ \phi_2(x, 2t-1) & t \in (1/2, 1] \end{cases}$$

    $\phi_3(x,t)$ satisfies $\phi_3(x,0) = f_1$ and $\phi_3(x,1) = f_3$. As well ϕ_3 is continuous at the point $t = 1/2$ because $\phi_1(x, 2*1/2) = \phi_2(x, 2*1/2-1) = f_2$. This means homotopy is transitive.

3. Here are some examplse of fixed endpoint homotopic maps:

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    Here are some examples of non fixed-endpoint homotopic maps:

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4. A space X is contractible if the identity map $\mathrm{Id}_X : X \to X$ is homotopic to a constant map.

- Convex open subsets of $\mathbb{R}^n$ are contractible. A convex subset, X is by definition one in which the straight line between any two points in X inside of X. Specifically, there exists a point $\mathbf{x}_0$ for which every point in the set X is connected to that point by a straight line. If $\mathbf{x} \in X$ then define the homotopy ϕ, by $\phi(\mathbf{x}, t) = t(\mathbf{x}_0 - \mathbf{x}) + \mathbf{x}$. This is a continuous map and contracts every point in the convex set X to a single point.
- Consider a contractible space X, and two points a and b in X. We know there is a map $\phi(x, t)$ which contracts every point in X to a single point x_0. To make a path $\gamma(t)$ from a to b take:

$$\gamma(t) = \begin{cases} \phi(a, 2t) & t \in [0, 1/2] \\ \phi(b, -2t + 2) & t \in (1/2, 1] \end{cases}$$

  This is a continuous path from a to b because $\gamma(\frac{1}{2}) = x_0$. This proves contractible spaces are path connected.
- Suppose Y is a contractible space. Consider two maps $f_1 : X \to Y$ and $f_2 : X \to Y$. Since Y is contractible, there is a contraction map, ϕ which sends every point to element of Y, x_0. We can define a homeomorphism, say η from f_1 to f_2 in a similar way as before.

$$\eta(x, t) = \begin{cases} \phi(f_1(x), 2t) & t \in [0, 1/2] \\ \phi(f_2(x), -2t + 2) & t \in (1/2, 1] \end{cases}$$

  This is a continuous map because both cases match at the point $t = \frac{1}{2}$, and $\eta(x, 0) = f_1$ and $\eta(x, 1) = f_2$. Thus, any two maps from $X \to Y$ will be homeomorphic.
- Suppose X is contractible and Y is path-connected. Consider two maps $f_1 : X \to Y$ and $f_2 : X \to Y$. To make a homeomorphism from f_1 to f_2 we need a contraction, ϕ of X to x_0 and a path p from $f_1(x_0)$ to $f_2(x_0)$. Then we define a homeomorphism:

$$\eta(x, t) = \begin{cases} f_1(\phi(x, 3t)) & t \in [0, 1/3] \\ p(3t - 1) & t \in (1/3, 2/3] \\ f_2(\phi(x, -3t + 3)) & t \in (2/3, 1) \end{cases}$$

  This is a homeomorphism from f_1 to f_2. η first contracts the domain of the map f_1 to a point, then η moves the image $f_1(x_0)$ to the point $f_2(x_0)$ along a path in Y, and then lastly decontracts the domain back to X from x_0. $\eta(x, 0) = f_1(x)$ and $\eta(x, 1) = f_2(x)$. η is made of continuous pieces that agree at the transition points $t = 1/3$ and $t = 2/3$.

  If Y is not path connected, then the maps f_1 and f_2 could fail to be homeomorphic if they don't satisfy $f_1(x_0) = f_2(x_0)$, for some contraction point x_0.

5. The fundamental group $\pi_1(X, x_0)$ is a group.

   - Closure: Consider ϕ_1 and ϕ_2 equivalence class representatives of loops in $\pi_1(X, x_0)$. The composition

$$\phi_1 * \phi_2 = \begin{cases} \phi_1(2t) & t \in [0, 1/2] \\ \phi_2(2t - 1) & t \in (1/2, 1] \end{cases}$$

     Is a continuous map that takes on the value x_0 at the points $t = 0$ and $t = 1$, and thus is a loop and is an element of $\pi_1(X, x_0)$.

- Associativity: Consider three maps ϕ_1, ϕ_2, and ϕ_3. Now compare the compositions $\phi_1 * (\phi_2 * \phi_3)$ and $(\phi_1 * \phi_2) * \phi_3$. By definition:

$$\phi_1 * (\phi_2 * \phi_3) = \begin{cases} \phi_1(2t) & t \in [0, 1/2] \\ \phi_2(4t-2) & t \in (1/2, 3/4] \\ \phi_3(4t-3) & t \in (3/4, 1] \end{cases}$$

and

$$(\phi_1 * \phi_2) * \phi_3 = \begin{cases} \phi_1(4t) & t \in [0, 1/4] \\ \phi_2(4t-1) & t \in (1/4, 1/2] \\ \phi_3(2t-1) & t \in (1/2, 1] \end{cases}$$

These two loops are homeomorphic. The following homeomorphism was constructed from pages of algebra and guesswork. It is not pretty but it works:

$$\eta(t,s) = \begin{cases} \phi_1\left(2t(1+s)\right) & t \in \left[0, \frac{1}{2s+2}\right] \\ \phi_2\left(\frac{(s+2)(t(2s+2)-1)}{2s+1}\right) & t \in \left(\frac{1}{2s+2}, \frac{3}{2s+4}\right] \\ \phi_3\left(\frac{t(2s+4)-3}{2s+1}\right) & t \in \left(\frac{3}{2s+4}, 1\right] \end{cases}$$

It's easy enough to verify that $\eta(t,0) = \phi_1 * (\phi_2 * \phi_3)$ and $\eta(t,1) = (\phi_1 * \phi_2) * \phi_3$. As well, it's not difficult to see that $\eta(0,s) = \phi_1(0) = x_0$ and $\eta(1,s) = \phi_3(1) = x_0$. To show that the functions match at the other endpoints is more annoying, but... here we go!

$$\begin{aligned} \phi_1\left(2\left(\frac{1}{2s+2}\right)(1+s)\right) &= \phi_1(1) \\ &= x_0 \\ \phi_2\left(\frac{(s+2)((\frac{1}{2s+2})(2s+2)-1)}{2s+1}\right) &= \phi_2(0) \\ &= x_0 \end{aligned}$$

So far so good, here's the other one:

$$
\begin{aligned}
\phi_2\left(\frac{(s+2)((\frac{3}{2s+4})(2s+2)-1)}{2s+1}\right) &= \phi_2\left(\frac{(\frac{3(s+2)}{2s+4})(2s+2)-(s+2)}{2s+1}\right)\\
&= \phi_2\left(\frac{\frac{3}{2}(2s+2)-(s+2)}{2s+1}\right)\\
&= \phi_2\left(\frac{3(s+1)-(s+2)}{2s+1}\right)\\
&= \phi_2\left(\frac{3s+3-s-2}{2s+1}\right)\\
&= \phi_2\left(\frac{2s+1}{2s+1}\right)\\
&= \phi_2(1)\\
&= x_0\\
\phi_3\left(\frac{\frac{3}{2s+4}(2s+4)-3}{2s+1}\right) &= \phi_3\left(\frac{3-3}{2s+1}\right)\\
&= \phi_3(0)\\
&= x_0.
\end{aligned}
$$

Okay, so the functions used for the construction of this map are all continuous when $s, t \in [0,1]$, so this will be a homeomorphism from $\phi_1 * (\phi_2 * \phi_3)$ and $(\phi_1 * \phi_2) * \phi_3$. Therefore, we arrive at the conclusion of associativity.

- Identity Element: Anything in the homotopy class of the map constant at x_0 will be an indentity element of this group. If ϕ is a loop in X then $x_0 * \phi$

$$
\mathrm{Id}_{x_0} * \phi = \begin{cases} x_0 & t \in [0, 1/2] \\ \phi(2t-1) & t \in (1/2, 1] \end{cases}
$$

This is homeomorphic to ϕ with homeomorphism

$$
\eta(t,s) = \begin{cases} x_0 & t \in [0, 1/2(1-s)] \\ \phi(t\frac{2}{s+1} + \frac{s-1}{s+1}) & t \in (1/2(1-s), 1] \end{cases}
$$

This will satisfy $\eta(t,0) = x_0 * \phi$, $\eta(t,1) = \phi$, and be continuous for all $s, t \in [0,1]$. Likewise $\phi * x_0$ will be homeomorphic to ϕ with homeomorphism:

$$
\eta(t,s) = \begin{cases} \phi(\frac{2t}{1+s}) & t \in [0, 1/2(1+s)] \\ x_0 & t \in (1/2(1+s), 1] \end{cases}
$$

This will also be continuous and satisfy the similar corresponding constraints. This proves the identity element exists in the fundamental group.

- Inverse Elements: Consider a path $\phi(t)$ in $\pi_1(X, x_0)$. We denote $\phi^{-1}(t)$ as $\phi(1-t)$. To check that this is homeomorphic to the constant map we can compute $\phi * \phi^{-1}$

$$
\phi * \phi^{-1} = \begin{cases} \phi(2t) & t \in [0, 1/2] \\ \phi(2-2t) & t \in (1/2, 1] \end{cases}
$$

This is homeomorphic to the identity map with homeomorphism:

$$\eta(t,s) = \begin{cases} \phi(2t(1-s)) & t \in [0,1/2] \\ \phi((2-2t)(1-s)) & t \in (1/2,1] \end{cases}$$

This will satisfy $\eta(t,0) = \phi * \phi^{-1}$, $\eta(t,1) = x_0$, and be continuous for all $s,t \in [0,1]$. Done. $\pi_1(X,x_0)$ is a group.

6. Consider two groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ where x_0 and x_1 lie in the same path connected component. We will show that these two groups are isomorphic. Consider a path p that goes from x_0 to x_1, and a path p^{-1}, the same path backwards, that goes from x_1 to x_0. Let ϕ_0 be a loop based at x_0, it can be associated with a loop based at x_1 by the function, $f(\phi_0) = p^{-1} * \phi_0 * p$. This function is an isomorphism because it is continuous and invertible. The inverse is given by: $f^{-1}(\phi_1) = p * \phi_1 * p^{-1}$. Then to show that it works $f^{-1}(f(\phi_0)) = p * p^{-1} * \phi_0 * p * p^{-1}$, from associativity and the fact that $p * p^{-1} = x_0 * \phi_0 * x_0$, and then again from associativity and x_0 being the identity element, we arrive at $x_0 * \phi_0 * x_0 = \phi_0$. Likewise $f(f^{-1}(\phi_0)) = p^{-1} * p * \phi_0 * p^{-1} * p = x_0 * \phi_0 * x_0 = \phi_0$. Now we're done, we've showed that f is invertible and continuous, and thus an isomorphism from $\pi_1(X,x_0)$ to $\pi_1(X,x_1)$.