Liability Directed Investing in a Behavioral Portfolio Theory Framework [2]

Sanjiv Das
Santa Clara University

Seoyoung Kim
Santa Clara University

Meir Statman
Santa Clara University

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Abstract

Liability Directed Investing (LDI), like behavioral portfolio theory (BPT), is centered on investors directed to optimal portfolios by their liabilities. The liabilities of pension funds are to their beneficiaries. The liabilities of individuals are their goals - liabilities to themselves.

Risk in BPT-LDI is measured by shortfalls from target terminal wealth whereas risk in mean-variance portfolio theory (MVPT) is measured by variance of returns or terminal wealth. BPT-LDI investors, unlike MVPT investors, are not always averse to variance. Lottery tickets with negative expected returns and high variance are never optimal for MVPT investors but are optimal for BPT-LDI investors with little current wealth relative to target terminal wealth. This is because a \$1 portfolio containing stocks, bonds, and options is less likely to bring investors to their \$1 million target terminal wealth than a \$1 lottery ticket. The same is true in less extreme cases such as those facing pension funds that are underfunded relative to their liabilities and individuals who quit secure jobs and take second mortgages to start high technology companies that might carry them to high target terminal wealth.

Optimal BPT-LDI portfolios maximize expected terminal wealth subject to the condition that expected shortfalls from their liability, a target terminal wealth, not exceed expected shortfall allowance. We explore initial optimal portfolios and subsequent portfolios as they are rebalanced on the way to the terminal date, highlighting differences between the BPT-LDI rebalancing method and the fixed-proportion and market-proportion rebalancing methods.

Introduction

Behavioral portfolio theory (BPT) developed by Shefrin and Statman (2000) describes liability directed investing (LDI). Investors in the BPT-LDI framework are similar to investors in the mean-variance (MVPT) framework in their pursuit of high expected wealth and dislike of risk, but whereas mean-variance investors measure risk by the standard deviation of terminal wealth, BPT-LDI investors measure it by the expected shortfall from their liabilities, namely target terminal wealth. The target terminal wealth of a pension fund is its liability to its beneficiaries. The target terminal wealth of an individual is her 'liability' to herself, such as the amount that would fund her target standard of living during retirement.

Consider an investor striving to maximize her expected wealth at her terminal date. She is willing to accept the risk reflected in a particular expected shortfall from her target wealth at that terminal date. We find the optimal initial portfolio for this investor and the optimal subsequent portfolios as she rebalances them each period on her way to the terminal date. We also identify the composition of optimal portfolios as underlying parameters vary, such as the expected returns of stocks and bonds, the shortfall allowance, and the target terminal date.

Mean-variance and behavioral portfolio theories

The prescriptions of mean-variance portfolio theory are premised on the assumption that investors are always risk-averse, where risk is defined as the variance of portfolio returns. For clarity, we refer to the risk-aversion of mean-variance investors as variance-aversion. One implication of variance-aversion is that mean-variance investors never buy lottery tickets, such as one that costs \$1 and offers a 0.00005 percent chance for a \$1 million prize. This is because this lottery ticket offers a 50 percent expected loss, coupled with a very high variance, making it inferior to a sure \$1 with a zero expected return and zero variance.

Behavioral investors, in contrast, might find it optimal to buy this and similar lottery tickets if all they have is $1 of current wealth and their target terminal wealth is $1 million at their target date, when the lottery prize is paid to the winner. The behavior of behavioral investors is quite rational as a lottery ticket offers them some chance, however small, of reaching their $1 million target terminal wealth whereas a diversified portfolio of stocks and bonds offers an even smaller chance of reaching their target terminal wealth. In that, behavioral investors are similar to the Dubins and Savage (1976) investors in a casino at night with $1,000 whose target wealth is $10,000 by morning. The optimal portfolio for these investors consists of a single bold bet, wagering their entire $1,000 for a small chance of winning $10,000. Investors who make timid bets do worse since they have even smaller chances of winning $10,000.

Behavioral investors perceive risk as shortfall from target wealth. They do not always prefer high-variance investments but they prefer high-variance investments over low-variance investments when high-variance investments provide a greater chance of avoiding shortfalls from target terminal wealth. Thus, we refer to behavioral investors as shortfall-averse but not variance-averse.

Friedman and Savage (1948) emphasized the effect of target wealth on choices. “Men will and do take great risks to distinguish themselves even when they know what the risks are,” they wrote, adding: "An unskilled worker…may jump at an actuarially fair gamble that offers a small chance of lifting him out of the class of unskilled workers and into the "middle" or "upper" class, even though it is far more likely… to make him one of the least prosperous unskilled workers." Behavioral investors do not seek variance. Rather, variance is payment for a chance to reach their target wealth.

The road to behavioral portfolio theory started with this Friedman and Savage article. Four years later, Markowitz wrote two papers reflecting two very different views of behavior. In one (1952a) he created mean-variance portfolio theory and in the other (1952b) he extended Friedman and Savage's framework, noting that people aspire to move up from their *current* social class or "customary wealth." So, people with $1 might accept lottery-like odds in hope of winning $1 million whereas people with $1 million might accept lottery-like odds in hope of winning $100 million. Kahneman and Tversky (1979) extended the work of Markowitz (1952b) into prospect theory. Prospect theory describes the behavior of people who accept lottery-like odds when they are in the domain of losses, such as when low-variance investments would leave them below their target wealth, but reject such odds when low-variance investments would take them to their target wealth or above it.

BPT-LDI investors choose portfolios on an efficient frontier, as MVPT investors do. MVPT investors choose portfolios on the mean-variance efficient frontier with combinations of expected terminal wealth and variance of terminal wealth that fit them best. BPT-LDI investors choose portfolios on the BPT-LDI efficient frontier with combinations of expected terminal wealth and expected shortfalls from target terminal wealth that fit them best. The BPT-LDI framework also offers investors a portfolio rebalancing method that is distinct from the fixed-proportions and market-proportions portfolio rebalancing methods in the MVPT framework.

Shefrin and Statman's BPT model is in a single-period setting, describing the construction of optimal portfolios that are not rebalanced over time as they depart from optimality. We explore the construction of BPT-LDI portfolios and their rebalancing over time, in a multi-period setting.

We find, in general, that relatively high portfolio wealth induces relatively high optimal allocations to risky assets, such as stocks, but there are circumstances where relatively *low* portfolio wealth induces relatively *high* optimal allocations to stocks. For example, an investor with little current wealth relative to her $1 million target terminal wealth might find no feasible portfolio that will take her to her target terminal wealth if she sets her shortfall allowance to $150,000. She can choose to temper her aspirations, reducing her target terminal wealth to a level that allows a feasible portfolio. Or she can choose to maintain her target terminal wealth and accept high variance, as lottery buyers do, by increasing her shortfall allowance to a level consistent with a feasible portfolio that provides a chance of reaching her target terminal wealth if stock prices zoom, even as it also increases the chance of financial ruin if stock prices plunge.

In general, optimal allocations to stocks are relatively low when investment horizons are relatively short, since short horizons provide little time to recoup losses and reach target terminal wealth. But there are circumstances where optimal stock allocations are relatively *high* when investment horizons are relatively short. For example, an investor standing one year before her terminal date with much wealth relative to her target terminal wealth might find that she can maximize her expected wealth with an all-stock portfolio, subjecting herself to only a small chance of failing to reach her target terminal wealth even if stock prices tumble that year. Yet she subjects herself to a substantial chance of failing to reach her target terminal wealth when standing five years before her terminal date if stock prices tumble year after year during these five years, leaving her far below her target terminal wealth.

Increases in expected stock returns induce increases in optimal stock allocations when unaccompanied by increases in the volatility of stock returns. This is because increases in expected stock returns increase expected returns of portfolios containing stocks and expected terminal wealth without increasing expected shortfalls. Conversely, increases in the volatility of stock returns unaccompanied by increases in their expected returns induce reductions in optimal stock allocation as higher volatility increases the likelihood of portfolio shortfalls without increasing expected terminal wealth.

Increases in the returns of risk-free bonds induce high *stock* allocations, since relatively low allocations to risk-free bonds when risk-free returns are high can provide more money than higher allocations to risk-free bonds when risk-free returns are low. The increase in money provided by higher risk-free returns facilitates increases in stock allocations, allowing higher expected terminal wealth without higher risk of shortfalls.

Related work includes an early paper on portfolio construction for liability directed investors by Sharpe and Tint (1990). Martellini and Milhau (2009) and Ang, Chen, and Sundaresan (2012) extend that work. Binsbergen and Brandt (2009) show that riskier portfolios with ex-post mandatory contributions by plan sponsors provide higher expected utility than ex-ante lower risk VaR-constrained portfolios. Basak, Shapiro, and Tepla (2006), and Browne (2000) explore the optimal way to reach target terminal wealth or minimize the time to reach target terminal wealth. Amenc, Martellini, Goltz, and Milhau (2010) analyze the management of portfolios where risk is measured by shortfalls, and Das and Statman (2013) show that shortfall risk can be managed with options and structured products. Other explorations of portfolios in shortfall or value-at-risk (VaR) settings include Roy (1952), Levy and Sarnat (1972), Basak and Shapiro (2001), Basak, Shapiro, and Tepla (2006), and Akcay and Yalcin (2010).

BPT-LDI portfolios

Several parameters play roles in BPT-LDI portfolios and, for concreteness, we assign values to them. The return of risk-free bonds is $r_f = 3\%$. Stock returns follow a normal distribution with an expected return $\mu = 7\%$ and standard deviation $\sigma = 20\%$; i.e., $N(\mu, \sigma^2)$. Later in the paper we consider stock returns that follow a non-normal distribution, reflecting an actual stock return distribution.

We denote the interval of each period as h = one year, and the terminal date as T = 20 years from today. The number of periods is then $N = T/h = 20$. The beginning of each period is indexed by j, where $j \in \{1,2,...,N\}$. Wealth at the beginning of first period is W_0 = \$500,000, target terminal wealth is H = \$1 million, and the expected shortfall allowance from that target wealth is K = \$150,000.

At the beginning of each period j an investor allocates a proportion $w_j \in [0,1]$ to stocks, and $(1 - w_j)$ to bonds. The portfolio return, r_j , during period j is normally distributed with mean and variance:

$$r_j \sim N\left[w_j \mu h + \left(1 - w_j\right) r_f h, w_j^2 \sigma^2 h\right] \tag{1}$$

We can express the portfolio return during period j as:

$$r_j = w_j\left(\mu h + \sigma\sqrt{h} \cdot z\right) + \left(1 - w_j\right) r_f h, \quad z \sim N(0,1) \tag{2}$$

The multi-period return distribution, under the assumption that returns are independent and identically distributed is:

$$R \sim N\left[\left(w_j \mu h + \left(1 - w_j\right) r_f h\right)(N - j + 1), w_j^2 \sigma^2 h(N - j + 1)\right] \equiv N\left[\mu_j, \sigma_j^2\right] \tag{3}$$

where $(N$-j+1) is the number of periods remaining before the terminal date. We write the mean and variance of R_j compactly as μ_j and σ_j^2, respectively.

$$W_N = W_{j-1} e^{R_j} \tag{4}$$

Where W_N is lognormally distributed, since $R_j \sim N[\mu_j, \sigma_j^2]$. Under the lognormal distribution, expected wealth is $E[W_N] = exp[\mu_j + 0.5\, \sigma_j^2]$.

The BPT-LDI efficient frontier

Consider an investor striving to maximize her expected wealth at her terminal date and is willing to accept the risk reflected in a particular expected shortfall from her target wealth at that terminal date. For example, consider an investor with $500,000 today who sets her target wealth at $1 million by the terminal date, 20 years from now. She is willing to allow a $150,000 expected shortfall from her target wealth at the terminal date.

The BPT-LDI efficient frontier is depicted in Figure 1. Each point on the frontier is associated with a portfolio where the allocation to stocks is some percentage ranging from 0% to 100% and the remainder is allocated to risk-free bonds. Each point on the frontier and the corresponding portfolio is also associated with a pair of expected terminal wealth and expected shortfall from a $1 million target terminal wealth that we derive from equations (1) – (4), and Appendix A.

A 100% bond portfolio provides a terminal wealth of $911,100, consisting of the initial $500,000 compounded continuously at an annual return of 3% for 20 years. The corresponding expected shortfall at the terminal date is $88,900. The 100% bond portfolio is sub-optimal, falling below the BPT-LDI efficient frontier. It is dominated, for example, by a portfolio with 85.43% in bonds and 14.57% to stocks. That portfolio has an expected shortfall of $88,900, identical to the expected shortfall of a 100% bond portfolio, yet a higher expected terminal wealth of $1,032,400.

The optimal portfolio where the expected shortfall allowance from the $1 million target terminal wealth is $150,000 consists of 30.76% in stocks and 69.24% in bonds. The probability that this portfolio would result in a shortfall from the $1 million target terminal wealth, computed from the current distribution of returns at the portfolio horizon with the chosen weights, is 28.93%. The standard deviation of shortfall around its $150,000 expected shortfall, conditioned on falling short of the $1 million target, is $167,897. This standard deviation implies that, conditioned on falling short of the $1 million target, our investor has a 50% chance of a shortfall from the $1 million target ranging from $60,320 to $220,011. The portfolio provides an expected terminal wealth of $1,210,200, calculated by compounding continuously the initial $500,000 portfolio wealth at the annual expected return of the portfolio for 20 years.

Investors who conclude that a 50% chance of a shortfall ranging from $60,320 to $220,011 is more than they are willing to bear can choose a shortfall allowance smaller than $150,000, such as $100,000. The corresponding optimal portfolio consists of 17.44% in stocks and 72.56% in bonds. The higher allocation to bonds whose return is lower than the expected return of stocks increases the probability of shortfall but reduces its standard deviation. The probability that this portfolio would result in a shortfall from the $1 million target terminal wealth is 38.30%.

The portfolio with the lowest expected shortfall on the BPT-LDI efficient frontier consists of 5.13% in stocks and 94.87% in bonds. That portfolio combines an expected shortfall of $60,500 with an expected terminal wealth of $950,200. An all-stock portfolio is also on the BPT-LDI efficient frontier, providing a relatively high expected terminal wealth $3,024,800, but also a relatively high $342,500 expected shortfall.

Varying portfolio parameters

Allocations in optimal BPT-LDI portfolios depend on their underlying parameters. We vary the portfolio parameters, one at a time, and consider their effect on optimal portfolio allocations. We begin with the previously stated set of parameters: $500,000 in initial wealth, a $1 million target terminal wealth, a $150,000 expected shortfall allowance, 20 years to the target terminal date, a 3% risk-free annual bond return, a 7% annual stock expected return and a 20% standard deviation of the annual return of stocks.

Varying initial wealth

There is no feasible portfolio when initial wealth is short of $428,700. An investor with less than that amount, whether an individual or a pension fund, can construct a feasible portfolio in a number of ways we detail later. A pension fund can increase its initial wealth to $428,700, such as with a cash contribution from its sponsoring corporation or public entity. Alternatively, it can extend its target terminal date beyond 20 years, take more risk by increasing its shortfall allowance beyond $150,000, or decrease its $1 million target terminal wealth, such as by reducing payments to be made to the pension fund's beneficiaries.

The optimal portfolio when the initial wealth is $428,700 consists of 14.55% in stocks and a corresponding 85.45% in bonds. Figure 2 shows that the allocation to stocks is higher, gradually and monotonically, when initial wealth is higher, implying that investors can 'afford' to take more risk as the ratio of initial wealth to target terminal wealth increases. The optimal allocation to stocks is 35.39% when the initial wealth is $548,800, even though that amount would have enabled our investor to reach her $1 million target terminal wealth with no shortfall in a 100% bond portfolio. She prefer to allocate 35.39% of her portfolio to stocks, however, because such allocation yields a higher expected terminal wealth, $3,320,000, while conforming to her $150,000 expected shortfall allowance.

People with high wealth relative to their target wealth can 'afford' to take investments with high variance of returns. People with low wealth relative to their target wealth can feel 'compelled' to accept investments with high variance of returns. People with low wealth who find no feasible portfolio that will take them to their target terminal wealth can choose to temper their aspirations, lowering their target terminal wealth. Or they can choose to accept investments with high variance of returns as they strive to reach their target terminal wealth.

Think of an investor with $311,600 who sets his target terminal wealth at $1 million, but is relatively shortfall-averse, willing to accept no more than $150,000 as his expected shortfall allowance. That investor finds no feasible portfolio that will take him to his $1 million target terminal wealth with only a $150,000 expected shortfall allowance. Yet he can find a feasible portfolio if he tempers his aspirations, lowering his target terminal wealth from $1 million to $786,000. The feasible portfolio consists of 19.13% in stocks and 80.87% in bonds. Alternatively, he can accept a higher expected shortfall allowance, such as $300,000.

Figure 3 presents the minimum initial wealth and optimal stock allocation necessary for a feasible portfolio for investors who are willing to modify their expected shortfall allowances. For example, the minimum feasible wealth is $312,000 and the optimal allocation to stocks is 31.2% when the expected shortfall allowance is $300,000. The minimum feasible wealth is $429,000 and the optimal allocation to stocks is 15.2% when the expected shortfall allowance is $150,000.

Combining the lessons of Figures 2 and 3 we can see a U-shaped relation between initial wealth and optimal stock allocation. Optimal stock allocation is relatively high when initial wealth is either very high or very low relative to target terminal wealth. Stock allocation is intermediate between high and low when initial wealth is intermediate relative to target terminal wealth.

Varying the investment horizon

Panel A of Figure 4 shows that the optimal allocation to stocks increases monotonically as the number of periods remaining before the terminal date increases. For example, the optimal allocation to stocks is 29.37% when the number of periods remaining before the terminal date is 19, but it is 30.76% when the number is 20. This conforms to the common advice to reduce portfolio risk as the investment horizon becomes shorter and investors come closer to the terminal dates of their portfolios. There is no feasible portfolio when the investment horizon is shorter than 16 years.

Panel A of Figure 4 also shows that the optimal stock allocation is relatively high when the expected shortfall allowance is relatively high. For example, the optimal stock allocation is 30.76% when the number of periods remaining is 20 and the expected shortfall allowance is $150,000, but it is 81.00% when the expected shortfall allowance is $300,000. Moreover, feasible portfolios are available when the expected shortfall allowance is $300,000 as long as the investment horizon is no shorter than 8 years.

Panel B of Figure 4 shows that circumstances exist where the optimal stock allocation is relatively *high* when the investment horizon is relatively short. As noted earlier, an investor with high wealth relative to target terminal wealth when standing one year before her terminal date might find that she can maximize her expected wealth with an all-stock portfolio, bearing only a small chance of failing to reach her target terminal wealth even if stock prices tumble that year. Yet she bears a substantial chance of failing to reach her target terminal wealth when standing five years before her terminal date if stock prices tumble year after year during these five years, increasing her shortfall much above her allowed shortfall and leaving her far below her target terminal wealth.

Think of an investor whose initial wealth is $850,000 and whose expected shortfall allowance is $150,000. The optimal stock allocation for this investor is 57.45% when the investment horizon is 5 years, yet it is *higher*, 66.82%, when the investment horizon is *shorter*, 2 years. The relatively high $850,000 wealth when the investor is 2 years away from her target terminal date implies that the likelihood that shortfall would exceed $150,000 is relatively small, allowing for a relatively high stock allocation. The likelihood that shortfall would exceed $150,000 is higher when the investor is 5 years away from her target terminal date, as there is a greater likelihood that a string of poor returns during 5 years would drive the shortfall above $150,000.

Varying the expected return of stocks

Figure 5 shows that the optimal initial stock allocation is monotonically higher when the expected return of stocks is higher. A monotone pattern is also evident when the expected shortfall allowance is $300,000 rather than $150,000, but the optimal stock allocation is now higher. For example, the optimal portfolio is a 100% stock portfolio when the expected return of stocks is 10% and the expected shortfall allowance is $300,000, but the optimal stock allocation is only 41.57% when the expected shortfall allowance is $150,000.

Increases in the 'equity premium' in the form of high expected stock returns relative to risk-free bond returns increase the attractiveness of high allocations to stocks when the volatility of stock returns is not increased. High expected stock returns increase the advantage of stocks over risk-free bonds in yielding high expected terminal wealth. Increases in the volatility of the returns of stocks would have muted the advantage of stocks over risk-free bonds, but keeping the volatility of stock returns constant implies that the advantage of higher stock allocations in yielding high terminal wealth is not muted by higher volatility of stock returns, as the likelihood of shortfall is not increased. Indeed, higher expected stock returns diminish the likelihood of shortfall as they increase expected terminal wealth to levels higher than target terminal wealth.

Varying the volatility of stocks

Figure 6 shows that the optimal stock allocation declines monotonically as volatility increases. The optimal portfolio is a 100% stock portfolio when the standard deviation is 9%, but it is 30.76% when the standard deviation is 20%. Optimal stock allocation is higher when the expected shortfall allowance is higher. For example, the optimal stock allocation is 47.28% when the expected shortfall allowance is $150,000 and the standard deviation is 15%, but the optimal stock allocation is 100% when the expected shortfall allowance is $300,000.

As noted is the discussion about varying the expected returns of stocks, higher stock volatility without accompanying higher expected stock returns increases the risk of shortfall without increasing expected terminal wealth. Low volatility of stock returns, however, increases their attractiveness. Indeed, when volatility declines to zero the optimal portfolios is composed entirely of stocks, as it leaves an investor with a simple choice between stocks as a risk-free asset with a 7% return and risk-free bonds with a 3% return.

Varying the risk free return of bonds

Figure 7 shows that there is no feasible portfolio when the risk-free return falls below 2.10%, given a $150,000 expected shortfall allowance. The optimal stock allocation when the risk-free return is 2.10% is 16.42%.The optimal stock allocation is monotonically higher as the risk-free return is higher, once it exceeds 2.10%. For example, the optimal stock allocation is 30.76% when the risk-free return is 3%. The optimal stock allocation is also higher when the expected shortfall allowance is higher. We see that the optimal stock allocation increases from 30.76% to 81.00% when the risk free return is 3% and the expected shortfall allowance increases from $150,000 to $300,000.

The *increase* in optimal stock allocation and the corresponding decrease in the allocation to risk-free bonds as the risk free return increases might seem counterintuitive, but it is an example of the familiar interplay between 'income effect' and 'substitution effect.' The substitution effect favors an increase in the allocation to risk-free bonds as their return increases relative to the expected return of stocks. But the income effect favors an increase in the allocation to stocks as the higher return of bonds acts as the equivalent of an increase in income.

The BPT-LDI rebalancing method and other rebalancing methods

The optimal portfolio at the beginning of the initial period is not likely to be optimal at the beginning of the second period, because parameters determining the optimal portfolio likely change during the period. Rebalancing at the beginning of the second period consists of changes in portfolio allocations from the optimal allocations at the beginning of the initial period to the optimal allocations at the beginning of the second period. We refer to this rebalancing method as the BPT-LDI rebalancing method and note its difference from the fixed-proportions and market-proportion rebalancing methods prevalent in the mean-variance framework.

Portfolios in the fixed-proportions rebalancing method are rebalanced to proportions in the optimal initial mean-variance portfolio, such as 60-40 proportions, where 60% is allocated to stocks and 40% to bonds. High stock returns relative to bond returns increase the relative allocation to stocks, say to 70-30. Fixed-proportions rebalancing consist of selling stocks and buying bonds in amounts necessary to restore the 60-40 proportions.

Allocations in the market-proportions rebalancing method are buy-and-hold allocations. Allocations begin with the proportions in the optimal initial portfolio, such as 60-40, and are rebalanced automatically by the market. High stock returns relative to bond returns increase the allocation to stocks, say to 70-30, which is accepted as optimal since it reflects the change in the relative overall values of stocks and bonds in the market.

Two rationales are offered for fixed-proportions rebalancing. First, investors who have chosen 60-40 portfolios have declared, in effect, that their optimal portfolio on the mean-variance efficient frontier is a 60-40 portfolio. This portfolio balances optimally their desire for high expected returns against their aversion to variance. Portfolios with other proportions are sub-optimal for 60-40 investors who respond by rebalancing their portfolios to the 60-40 proportions.

The second rationale for fixed-proportions rebalancing is built on the claim that returns are mean-reverting. Bernstein (2010) argued for that rationale, noting that the "fairly convincing evidence for mean reversion in the long run…and the everyday experience of disciplined practitioners strongly suggest that rebalancing usually, but not always, pays off well before the 'Keynesian long run' expires."

Sharpe (2010a) offered "adaptive asset allocation" portfolio rebalancing, which corresponds to market-proportions rebalancing. Sharpe's rationale for market-proportions rebalancing is that it is "macro-consistent in the sense that all investors can follow such strategies." (p. 54). He noted that fixed-proportion portfolio rebalancing requires that some investors trade with other investors whenever securities prices change. In contrast, rebalancing to market-proportions requires no trading. Responding to Bernstein's arguments in favor of fixed-proportions rebalancing Sharpe (2010b) noted that while mean-reversion might exist, he is "reluctant to second-guess the collective views of investors around the globe."

BPT-LDI portfolio rebalancing along portfolio paths

We turn now to paths of optimal portfolios from the initial date to the terminal date. We optimize portfolios at the beginning of each period, ensuring that portfolios maximize expected terminal wealth subject to expected shortfall allowance.

As noted earlier, our investor begins with a $500,000 initial wealth and a $1 million target terminal wealth. She sets her expected shortfall allowance to $150,000 at the terminal date, which is 20 years away. The optimal initial portfolio consists of 30.76% in stocks and 69.24% in bonds.

Suppose that the realized return of the portfolio during the first period is 0%, such that her wealth at the beginning of the second period remains at $500,000. But now she has only 19 periods before the terminal date. The optimal allocation to stocks at the beginning of the second period, presented in Table 1, is 29.37%. Portfolio rebalancing consists of a decrease in stock allocation from 30.76% to 29.37% and a corresponding increase in bond allocation.

Now suppose that the realized return of the portfolio during the second period is 10%. The optimal stock allocation at the beginning of the third period is then 33.69%, and our investor rebalances her portfolio by an increase in stock allocation from 29.37% to 33.69% and a corresponding decrease in bond allocation.

There will be times when our investor finds that her wealth is too low for any feasible portfolio that conforms to her target terminal wealth and expected shortfall allowance. Pension funds describe such times in the language of underfunding. Suppose that the realized return of the portfolio during the third period is a 20% loss, such that her wealth at the beginning of the fourth period is $440,000. This level of wealth places our investor in an infeasible portfolio position; there is no portfolio such that she can reach her $1 million target terminal wealth with a $150,000 expected shortfall allowance. Our investor can cure her infeasible portfolio by one of four remedies.

First, she can increase current wealth with a cash contribution. Table 1 shows that a contribution of $33,100, increasing current wealth from $440,000 to $473,100, is the minimal contribution necessary for a feasible portfolio. The corresponding optimal stock allocation at the beginning of the fourth period is 14.65%. Second, she can reduce her target terminal wealth by $55,200, from $1 million to $944,800, which is the minimal decrease necessary for a feasible portfolio. The corresponding optimal stock allocation is 16.19%. Third, she can increase her shortfall allowance by $39,300, from $150,000 to $189,300, which is the minimal increase necessary for a feasible portfolio. The corresponding optimal stock allocation is 19.58%. Fourth, she can postpone her terminal date by 3 years such that it extends to 23 years from the initial date, the minimal postponement necessary for a feasible portfolio. The corresponding optimal stock allocation is 20.76%.[1]

Optimal portfolios when return distributions are non-normal

Our discussion so far centered on optimal portfolios where return distributions are normal. Normal distributions are easily tractable, simplifying analysis, but they do not account well for 'fat tails,' namely high probabilities of large gains or large losses. High probabilities of large losses are especially worrisome.

We construct the distribution of stock returns from actual weekly CRSP 1-10 stock index returns during the period from July 1926 through January 2013.[2] The fat tails in the distributions are evident in Figure 8 which also shows a normal distribution with the same mean and variance. The moments of the actual return distribution are: mean = 0.0017, standard deviation = 0.0216, skewness = -0.3969, and kurtosis = 10.7762.

Figure 9 compares the BPT-LDI efficient frontier when the distribution of stock returns is the actual non-normal distribution to the efficient frontier when the distribution is normal. We construct the efficient frontier using equations (1)--(4) and Appendix A when the distribution of returns is normal, and by simulation when the distribution is the actual non-normal one.

The allocation to stocks is lower under the actual non-normal distribution than under the normal distribution for any given expected shortfall allowance. As a consequence, the expected

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[1] Our approach is similar to that of DFA's SmartNest. See "DFA Gives Managed Accounts a New Dimension," in *Retirement Income Journal*, by Kerry Pechter, October 24, 2012. DFA calls this contribution dependent portfolio targeting "Managed DC".

[2] From Ken French's web site. http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

terminal wealth under the actual non-normal distribution is lower. For instance, the optimal stock allocation is 77.99% when the distribution is the actual non-normal distribution and the expected shortfall allowance is $200,000. The expected terminal wealth is $2,557,869. The optimal stock allocation is higher, at 79.35% when the distribution is normal, and the expected terminal wealth is also higher, at $2,608,947.

We explore further the effect of fat tails on the BPT-LDI efficient frontier by 'fattening' the left tail of the actual weekly return distribution. We begin by identifying the 100 most negative weekly returns and multiplying them by a factor of 1.2, making them even more negative. Next, we adjust all returns such that we retain the mean return of the original actual distribution. The mean remains at 0.0017 but the standard deviation is now higher, at 0.0227, than the original 0.0216, the skewness is more negative, at -0.9837, than the original -0.3969, and the kurtosis is 13.8268, higher than the original 10.7762.

The optimal stock allocation when the distribution is normal and the expected shortfall allowance is $200,000 is 73.23% and expected terminal wealth is $2,612,260. The optimal stock allocation when the left tail is fattened by a factor of 1.2 is lower, at 70.22%, and the expected terminal wealth is also lower, at $2,283,258.

Conclusion

Liability Directed Investing (LDI), like behavioral portfolio theory (BPT), is centered on institutional investors who are directed to optimal portfolios by their liabilities to their beneficiaries or individual investors who are directed to optimal portfolios by their goals – i.e., liabilities to themselves. Risk in BPT-LDI is measured by expected shortfalls from target terminal wealth whereas risk in mean-variance portfolio theory (MVPT) is measured by variance of returns or terminal wealth. Investors in MVPT are variance-averse, whereas investors in BPT-LDI are shortfall-averse. We discuss circumstances where shortfall-aversion makes variance-*seeking* optimal.

Optimal BPT-LDI portfolios maximize expected terminal wealth subject to the condition that the expected shortfall from a target terminal wealth not exceed expected shortfall allowance. We explore initial optimal portfolios and subsequent portfolios as they are rebalanced on the way to the terminal date, highlighting differences between the BPT-LDI rebalancing method and the fixed-proprtion and market-proportion rebalancing methods.

The investors we portray in this paper begin with some initial wealth and make no contributions to their portfolios beyond that date, unless they choose to cure infeasible portfolios with contributions. Future work would incorporate contributions, such as employer and employee contributions into pension funds or savings by individuals. Target terminal wealth can be regarded as the present value of an annuity paying annual pensions in the future. Future work would explore the effect of changes in interest rates that affect the present values of such annuities. Future work would also extend the range of assets available to investors beyond a stock index and risk-free bonds.

The investors we portray also have only one goal, such as retirement. Future work would allow multiple goals in multiple mental accounts, such as retirement, education, and bequest. The multiple mental account framework adds a method for curing infeasible portfolios beyond cash contributions, reduction of target terminal wealth, increase in shortfall allowance, and postponement of terminal date. Specifically, it allows transfers of wealth from lower ranking mental accounts to higher ranking ones.

APPENDIX

LDI-BPT portfolio constructions and rebalancing

Consider a portfolio with wealth W_{j-1} at the beginning of period *j*. Portfolio wealth at the end of the terminal period *N* is then described by:

$$W_N = W_{j-1}e^{R_j}$$

where W_N is lognormally distributed since $R_j \sim N(\mu_j, \sigma_j^2)$.

Based on this, an LDI-BPT investor can proceed to find the optimal allocation to the risky asset w_j given his target wealth and shortfall tolerance. Specifically, this entails finding the highest value of w_j to maximize the expected value of the portfolio at the horizon, i.e., $E_j[W_N]$, subject to the constraint that the expected shortfall below a set target, *H*, is less than *K*. The expected shortfall constraint of the portfolio for period *j* is expressed as:

$$ES_j = E_j[H - W_N | W_N < H] \leq K$$

Or, since *H* is exogenous, we have

$$ES_j = H - E_j[W_N | W_N < H]$$

which leaves us to compute $E_j[W_N | W_N < H]$, the expectation of a truncated lognormal variable W_N. The solution is as follows:

$$E_j[W_N | W_N < H] = W_j \cdot exp\left\{\mu_j + \sigma_j\lambda(\alpha) + \frac{1}{2}\sigma_j^2[1 - \delta(\alpha)]\right\}$$

where

$$\alpha = \frac{H_j - \mu_j}{\sigma_j}$$

$$H_j = ln\left(\frac{H}{W_j}\right)$$

$$\lambda(\alpha) = \frac{-\phi(\alpha)}{\Phi(\alpha)}$$

$$\delta(\alpha) = \lambda(\alpha)[\lambda(\alpha) - \alpha]$$

where $\phi(\cdot)$ is the standard normal density function, and $\Phi(\cdot)$ is the standard normal distribution function. The proof of this result is as follows. Because W_N is lognormal, the expectation is

$$E_j[W_N|W_N < H] = W_j \cdot exp\left[E_j\left(R_j\middle|R_j < H_j\right) + \frac{1}{2}Var_j\left(R_j\middle|R_j < H_j\right)\right]$$

Since R_j is normally distributed, standard calculations for truncated normal random variables give us that $E_j\left(R_j\middle|R_J < H_j\right) = \mu_j + \sigma_j\lambda(\alpha)$, and $Var_j\left(R_j\middle|R_j < H\right) = \sigma_j[1 - \delta(\alpha)]$, where the expressions for $\lambda(\alpha)$ and $\delta(\alpha)$ are given above.

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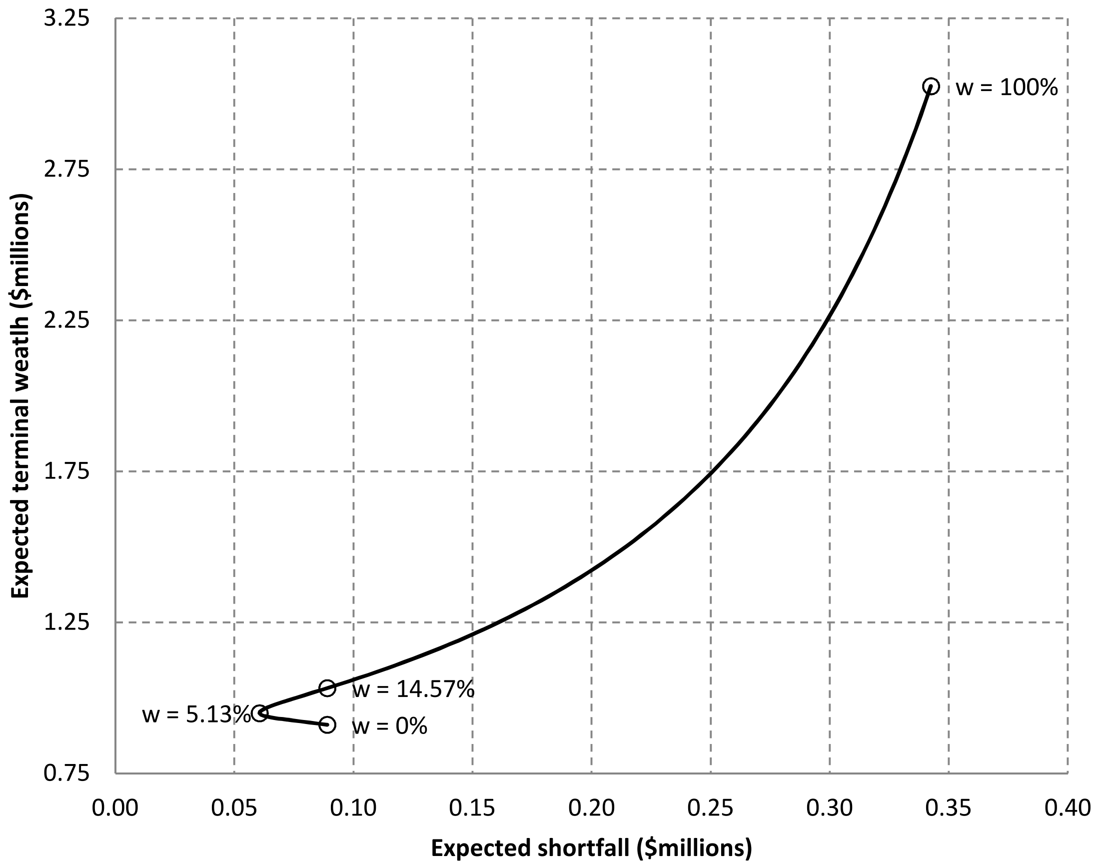


Figure 1: The BPT-LDI efficient frontier: Efficient pairs of expected terminal wealth and expected shortfall. The target terminal wealth is H = \$1 million and the initial wealth is W_0 = \$500,000. The investment horizon is N = 20-years, the bond risk-free return is r_f = 3%, the stocks' expected return is μ = 7%, with standard deviation of σ = 20%. An allocation to stocks, w, and bonds, 1-w, corresponds to each pair of expected terminal wealth and expected shortfall.

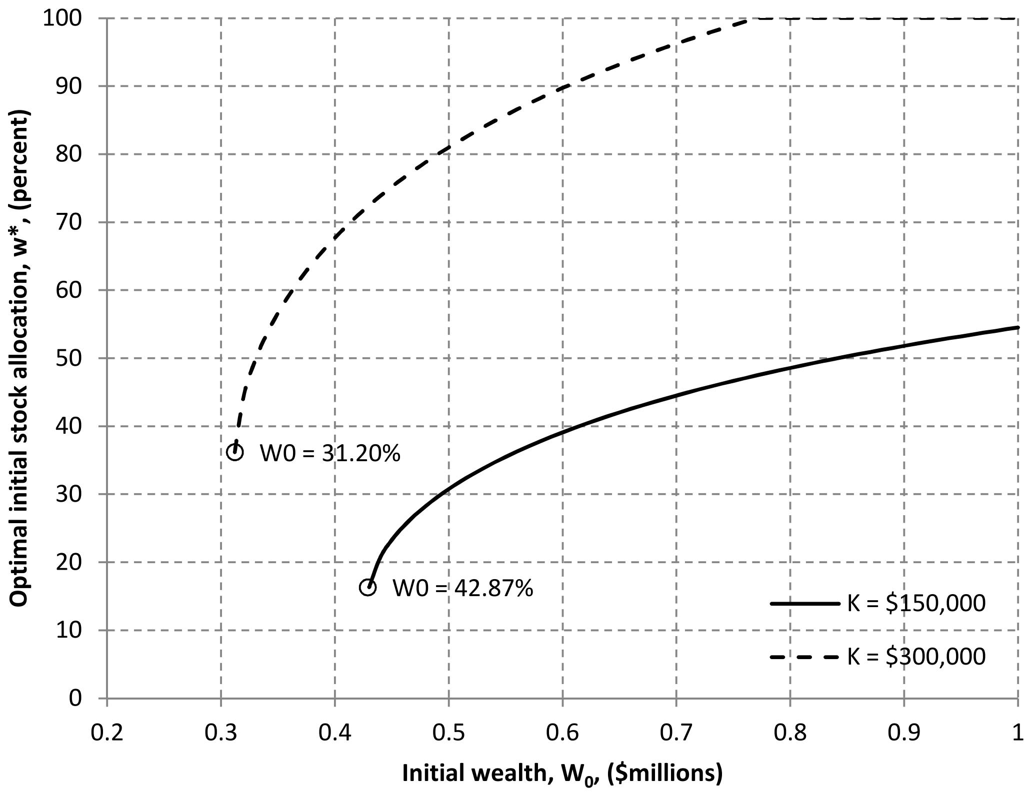


Figure 2: The relation between portfolio wealth and optimal stock allocation. The solid line presents the case where the shortfall allowance, K, is $150,000. The broken line presents the case where the shortfall allowance, K, is $300,000.

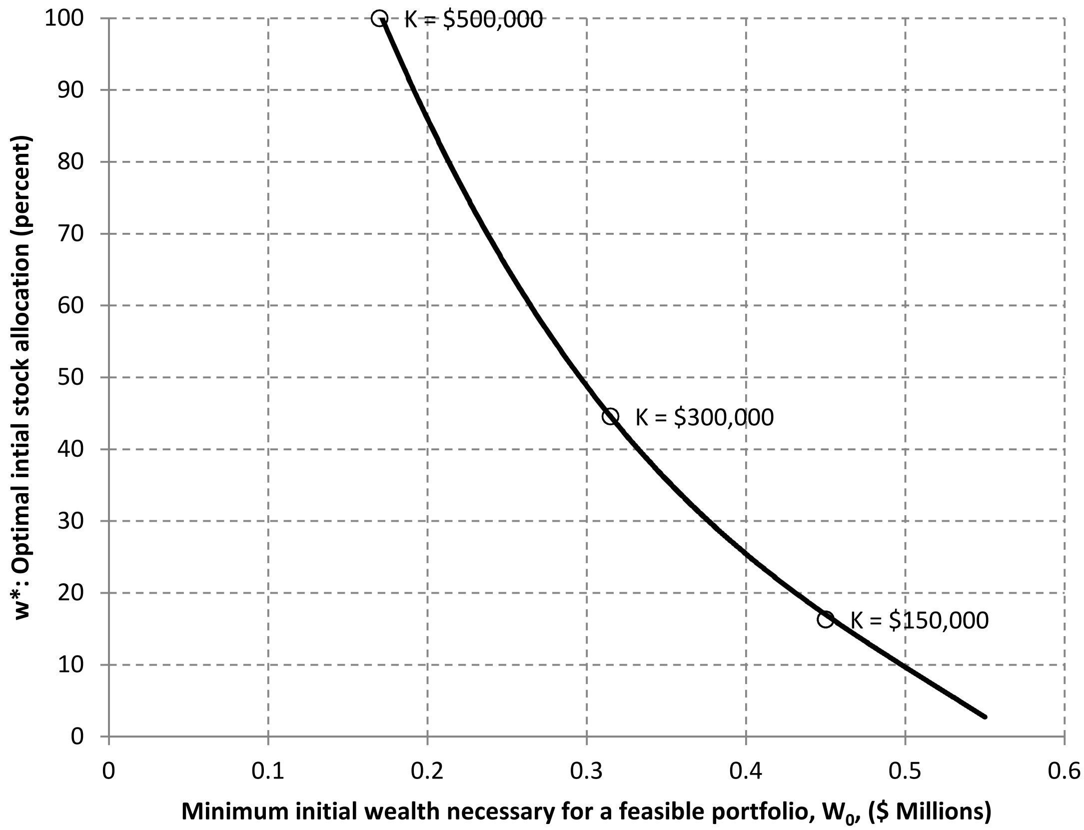
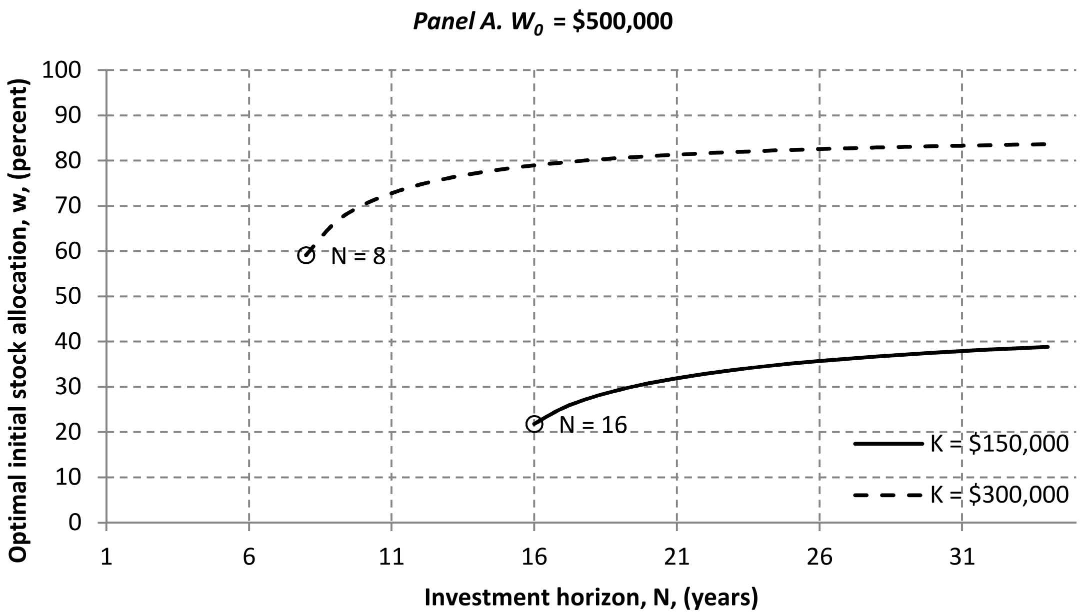


Figure 3: Minimum wealth necessary for a feasible portfolio and corresponding optimal stock allocation under varying shortfall allowances. For example, the minimum initial wealth necessary for a feasible initial portfolio is $170,000 when the shortfall allowance is $500,000. The corresponding optimal initial stock allocation is w^* = 100%. For an expected shortfall allowance of $300,000, the minimum initial wealth necessary is $312,000, and the optimal stock allocation is w^* = 36.2%.



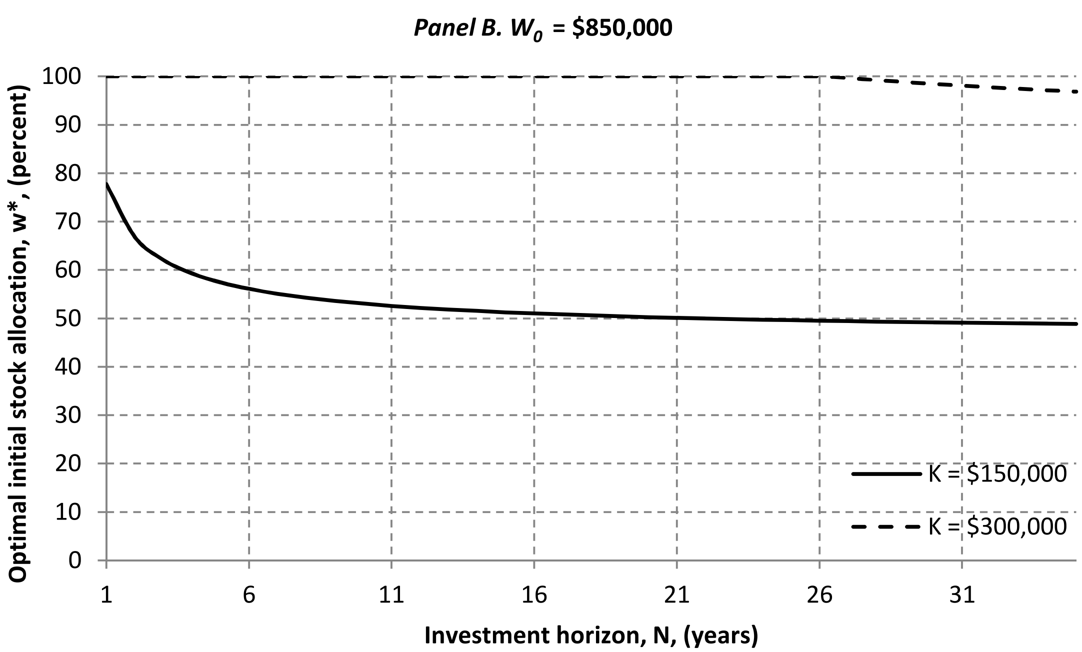


Figure 4: The relation between investment horizon and optimal stock allocation. Wealth at the beginning of the horizon is $500,000 in Panel A, and $850,000 in Panel B. The solid line presents the case where the shortfall allowance, K, is $150,000. The broken line presents the case where the shortfall allowance, K, is $300,000.

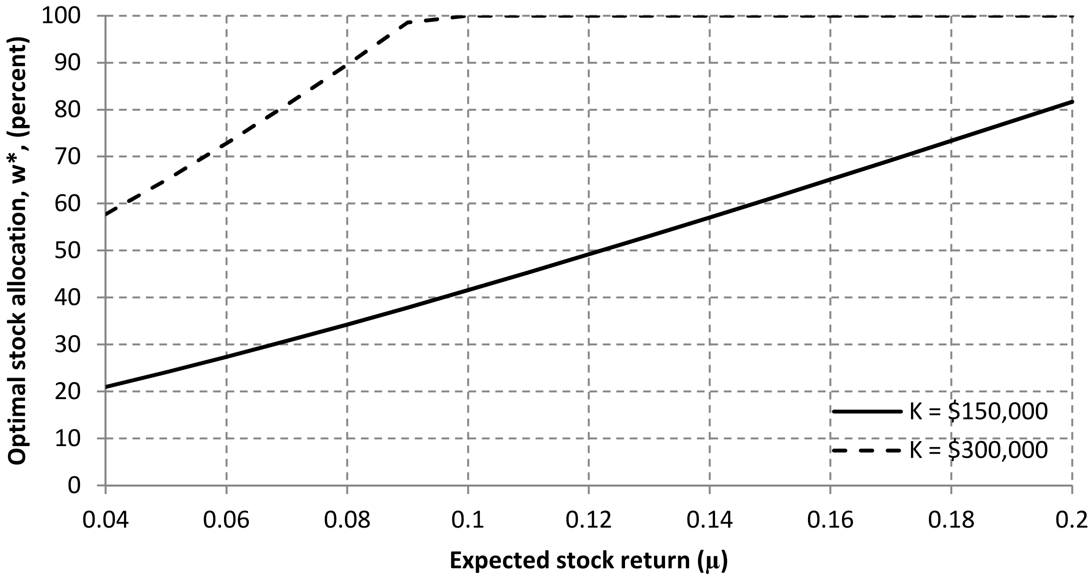


Figure 5: The relation between expected stock return and optimal stock allocation. The solid line presents the case where the shortfall allowance, K, is $150,000. The broken line presents the case where the shortfall allowance, K, is $300,000.

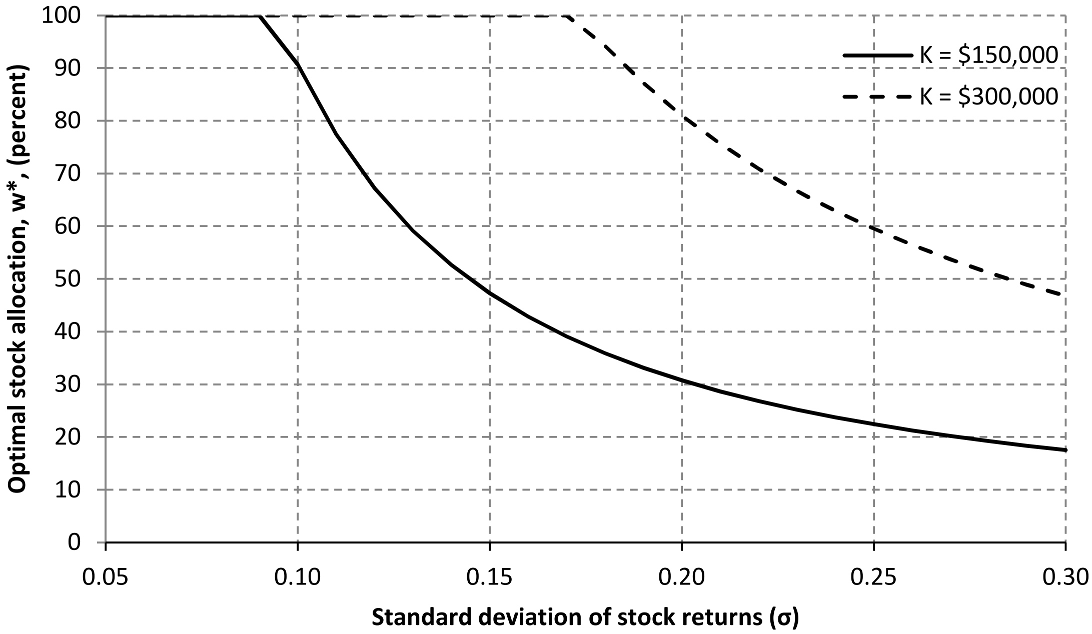


Figure 6: The relation between standard deviation of stock returns and optimal stock allocation. The solid line presents the case where the shortfall allowance, K, is $150,000. The broken line presents the case where the shortfall allowance, K, is $300,000.

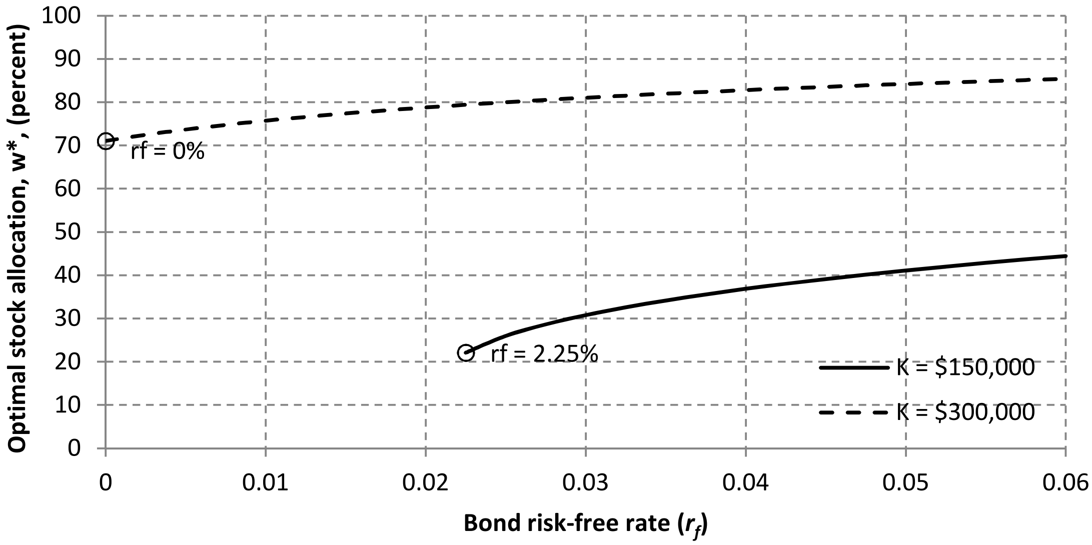
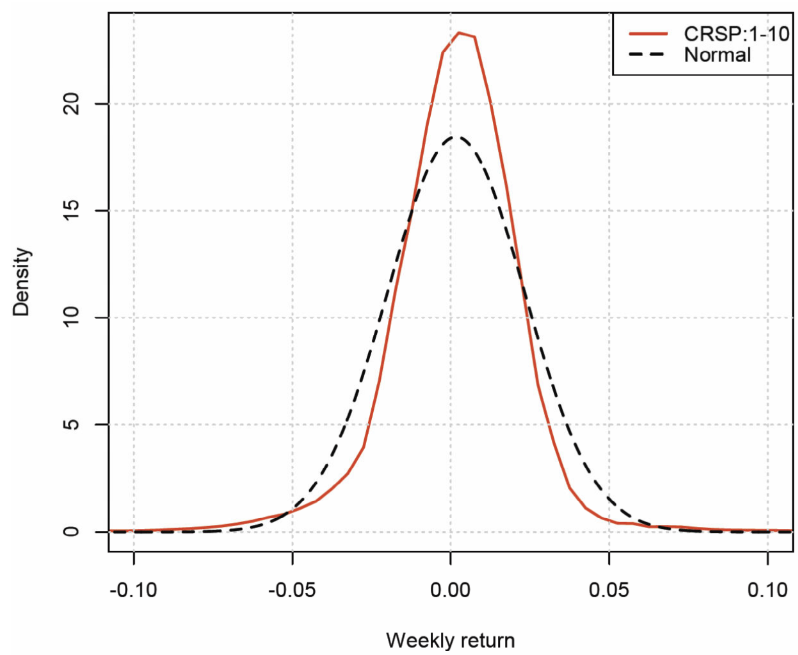


Figure 7: The relation between bond risk-free return and optimal stock allocation. The solid line presents the case where the shortfall allowance, K, is $150,000. The broken line presents the case where the shortfall allowance, K, is $300,000.



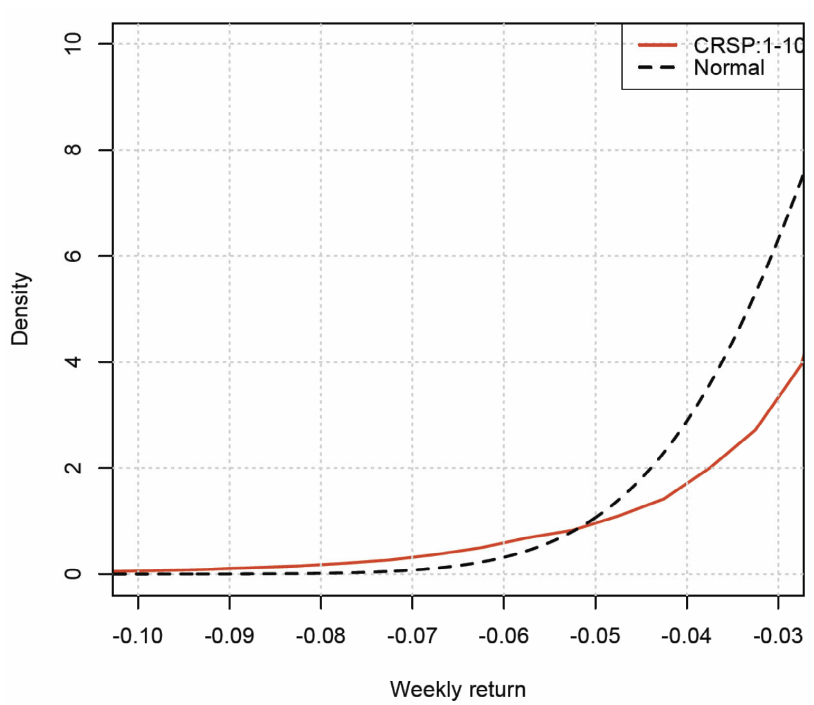


Figure 8: The distribution of actual weekly returns of the CRSP 1-10 stock market index from July 1926 through January 2013, and a normal distribution with the same mean and variance. The parameters of the CRSP 1-10 distribution are: mean = 0.0017, standard deviation = 0.0216, skewness = -0.3969, and kurtosis = 10.7762. The upper panel depicts full distributions, and the lower panel zooms in on the left tails.

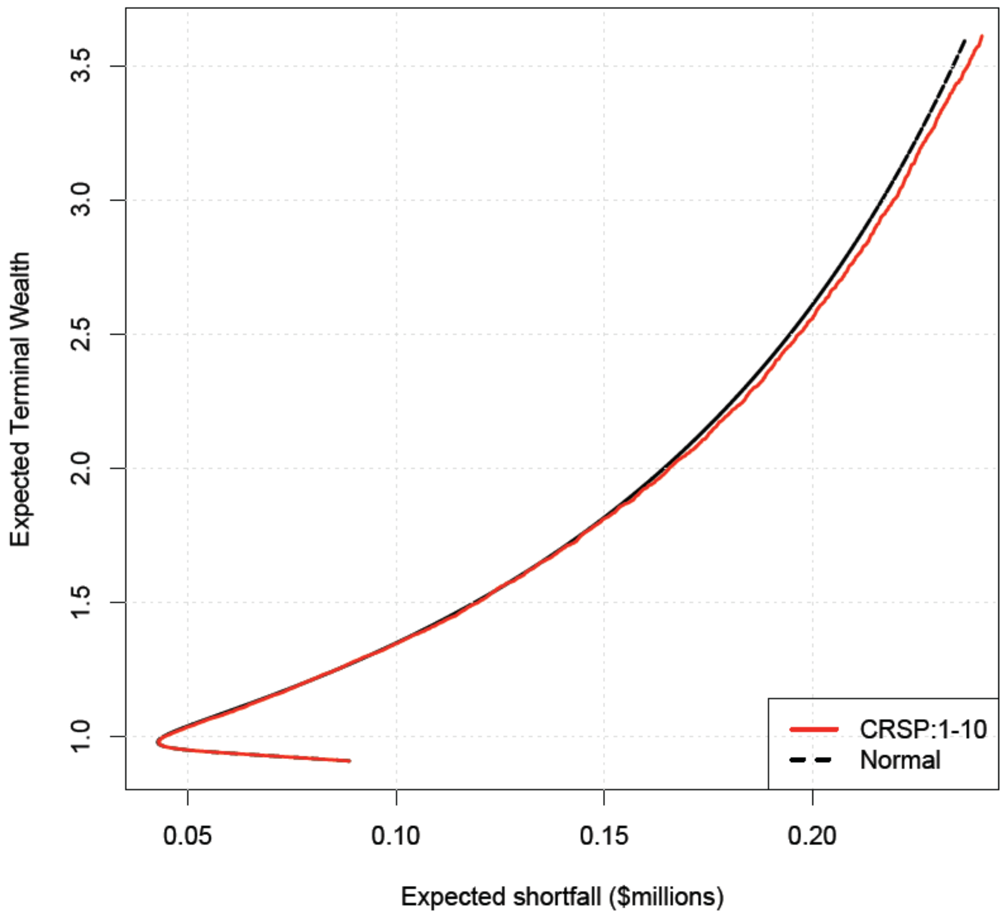


Figure 9: The BPT-LDI efficient frontier when stock returns follow the CRSP 1-10 distribution and when they follow a normal distribution with the same mean and variance.

Table 1: Initial portfolio allocation and subsequent rebalancing based on a sample realized return path (Panel A) and remedies when a portfolio is infeasible (Panel B) for an LDI-BPT investor with an initial shortfall allowance of K = \$150,000. Allocations to stocks and bonds are between zero and 100%, inclusive. Here, we consider an investor with an initial wealth of \$500,000 and a target terminal wealth of \$1,000,000. The initial investment horizon is N = 20 years. The expected annual stock return is $\mu_j = 7\%$, with a standard deviation of $\sigma_j = 20\%$. The annual bond risk-free return is $r_f = 3\%$. We also report the standard deviation of shortfall, σ_{ES}.

Panel A. Sample realized return path

j	Realized return	W_{j-1}	w_j^*	σ_{ES}
1	N/A	\$500,000	30.76%	\$167,897
2	0%	\$500,000	29.37%	\$162,650
3	+10%	\$550,000	33.69%	\$170,582
4	-20%	\$440,000	infeasible	
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Panel B. Remedies when portfolio becomes infeasible at j=4

Remedy	Change	K	H	N	W_{j-1}	w_j^*
Do nothing	---	\$150,000	\$1,000,000	20	\$440,000	infeasible
Increase W_j	\$33,100	\$150,000	\$1,000,000	20	\$473,100	14.65%
Increase N	3	\$150,000	\$1,000,000	23	\$440,000	20.76%
Increase K	\$39,300	\$189,300	\$1,000,000	20	\$440,000	19.58%
Decrease H	\$55,200	\$150,000	\$994,800	20	\$440,000	16.19%