Efficient Trading in Taxable Portfolios [1]

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Abstract

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We determine optimal life-cycle trading strategies for portfolios subject to the American tax system. Our method employs Monte Carlo optimization, which accommodates long horizons (e.g., 40 to 60 years) and large numbers of trading periods (e.g., 480), while accounting for the full cost basis history for the portfolio's stock holdings. We develop many new results regarding questions about taxable portfolio investing that were previously unexplorable. Some of our results challenge current conventional wisdom, including establishing circumstances where raising the stock allocation is optimal though counterintuitive, and demonstrating the suboptimality of the 5/25 rebalancing rule, even as a rule of thumb.

Keywords: Taxes; Capital Losses; Portfolio Optimization; Expected Utility; Passive Investing.

JEL Classification codes: C6; G11.

"Death and taxes and childbirth! There's never any convenient time for any of them."
– Margaret Mitchell, Gone with the Wind

1 Introduction

Taxes make financial problems both more complicated and more interesting. The optimal portfolio problem is no exception to this general statement. The US tax code injects two options into portfolio analysis, due to the fact that capital losses can be deducted from an investor's ordinary income, which is taxed at higher rates than (long term) capital gains are taxed. First, Constantinides (1983) argues that when tax rates on losses are the same as (or greater than) the rate on gains, ceteris paribus, portfolio policies are tilted towards immediately opting to realize any losses (i.e., exercising a "tax put" option. This immediate exercise rule for the tax put option is easy to implement. Second, Constantinides (1984) points out a more subtle tax-timing (or "restart") option, where taking gains resets the cost basis and makes the tax put at-the-money, which allows an investor to better exploit the higher rate on losses. Both of these tax options are important, but it is far more challenging to determine how to optimally exercise the second option.

We consider a dynamic portfolio problem, where, at each time t, the investor chooses the fraction, $f \in [0, 1]$, of their wealth placed into a risky asset, subject to capital gains taxation and capital losses tax deductions, versus the remaining fraction, $(1-f)$, which is placed into a risk free asset. (For simplicity, we will refer to the risky asset as "stock" and the risk free asset as "cash" for the remainder of this paper.) Our choice of f is made in a tax-optimized manner, where the complexity of the

American tax code is considered over a large number of potential trading periods (~ 500). This optimal choice for f depends on the cost basis of the stock holdings in the portfolio, which, in turn, depends on the path of the stock's price history and on the investor's purchasing history, so the problem is heavily path-dependent and high-dimensional. In particular, the full cost basis of all the stock in the portfolio can be described by a vector of random, evolving length, where each stock purchase still held in the portfolio corresponds to an entry in the vector. Attempting to keep track of all of the possibilities for the basis can quickly make the equations for determining the optimal f intractable, as discussed, for example, in Ostrov and Wong (2011), therefore, many papers have chosen to assume that it is approximately acceptable to use the weighted-average cost basis for all the stock in the portfolio, instead of using the actual cost basis for each stock purchase. This assumption allows the vector describing the full basis to be replaced by a scalar representing the average cost basis, which can make solving a (dynamic programming) Bellman equation for the optimal f tractable.[1] Examples of papers that have used the average cost basis include Dammon, Spatt, and Zhang (2001), Dammon, Spatt, and Zhang (2004), Gallmeyer, Kaniel, and Tompaidis (2006), Tahar, Soner, and Touzi (2010), and Dai, Liu, Yang, and Zhong (2015).

Dybvig and Koo (1996) implemented a model with the full cost basis, but were limited to four periods before the problem became intractable. DeMiguel and Uppal (2005) subsequently used nonlinear programming (the SNOPT algorithm of Gill,

[1]The state space for the dynamic programming problem comprises the current wealth of the investor and the tax basis. If the weighted-average tax basis is used, then the state space is two-dimensional. If the complete tax basis is used, then the dimension of the state space has no definite upper bound.

Murray, and Saunders (2002)) with linear constraints to extend the model to ten periods. For their ten period model, they found that using the full cost basis provided a 1% certainty equivalent advantage over using the average cost basis, which is robust to various parameterizations.

In contrast to these previous models, we use a simulation algorithm with optimization that runs very fast. Our algorithm uses the nonlinear optimizer in the R programming language, which calls a simulation run in compiled C for 50,000 paths of portfolio wealth and policy. We generally run our algorithm for 40 year portfolio horizons with quarterly trading, i.e., 160 potential trading periods. However, we will also implement the model for longer horizons, e.g., 60 years, or for monthly trading, i.e., 480 periods, which takes marginally longer to run. Our approach has the advantage of being both simple and flexible, which yields a number of benefits, such as allowing us to 1) determine solutions using the full cost basis, even if the basis is large and complicated; 2) address a number of complex features of the American tax code that are generally difficult to accommodate; and 3) work with any desired stochastic processes to simulate the stock movement, although, for simplicity, we use geometric Brownian motion in this paper.

Our model keeps track of the entire tax basis, however, we can also implement a simplification to accommodate the average basis, which will allow us to compare the effect of using the average cost basis versus the full cost basis. This is key for understanding whether or not it is justified to use the average cost basis approximation that current dynamic programming (Bellman equation) approaches implement. DeMiguel and Uppal (2005) argue that using the average tax basis is reasonable, since, in their simulations, most stock holdings comprise a single basis, and only 4% of the holdings comprise additional bases. However, their conclusion is based on a ten-period model, where there are few periods for new purchases. With many more periods and more complex tax code features, as in our model, there is more potential for the difference in optimality between the full tax basis problem and the average tax basis problem to become large. Having a large number of periods is more like a Bellman model, where potential sales and purchases occur in continuous time or over multiple periods in discrete time.

A comprehensive paper by Dammon, Dunn, and Spatt (1989) examines both the tax put option and the restart option.[2] Their paper shows that the value of these options depends on the pattern of gains, the length of the portfolio's time horizon, T, and whether, at this horizon, the investor is alive and liquidates their portfolio or is deceased. (In the latter case, taxes are lower, since all capital gains are forgiven at death and the basis is reset.) We will also explore the effect of the investor being alive or deceased at the horizon T and show that it leads to substantially different optimal strategies, even when T is large. Unlike our paper, Dammon, Dunn, and Spatt (1989) do not determine an optimal trading policy. Instead they compare three trading rules suggested by Constantinides (1984) and compare the performance of these rules to a buy and hold strategy. They find that the restart option value is lower than that suggested by Constantinides (1984) for high volatility stocks, because such stocks also tend to have higher capital gains. They conclude that offset rules matter, hence in our model, we account for tax loss offsets and carry-forwards exactly as per the US

[2] See also the review by Dammon and Spatt (2012).

tax code.

It is well established (see, for example, Davis and Norman (1990), Dai, Liu, and Zhong (2011), Dai, Liu, Yang, and Zhong (2015), or, in the asymptotic case, Shreve and Soner (1994), Whaley and Wilmott (1997), Atkinson and Mokkhavesa (2002), Janacek and Shreve (2004), Rogers (2004), and Goodman and Ostrov (2010)) that, in the absence of taxes, the optimal trading strategy is to maintain the stock fraction within a specific "no-transaction region" (also referred to as a "hold region"). When the portfolio's stock fraction is in the interior of this no-transaction region, there is no trading. Trading only occurs to rebalance the portfolio so that it does not escape the no-transaction region. The same logic applies to the optimal strategy for taxable portfolios, but with one tweak: the optimal strategy, as per Constantinides (1983) and Ostrov and Wong (2011), requires realizing any capital losses as soon as possible (that is, exercising the "tax-put" option) — no matter where we are in the region — and then rebuying an essentially equivalent stock or index to avoid wash sale restrictions, as discussed further in Subsection 2.1. This process collects the tax advantages of realized losses without rebalancing the portfolio. We only rebalance, as before, to prevent the portfolio from escaping our region. Because we transact to realize losses in the interior of our region, we refer to any region using this optimal strategy in the presence of taxes as a "no-rebalancing region" instead of a "no-transaction region." In the case of a single stock, as in this paper, the region is an interval.

Because our simulation approach works forwards in time, whereas Bellman equation approaches either work backwards in time or employ specific models for which the value function's dependence on time is removed, we are able to obtain new results that complement the previous results obtained by Bellman equation approaches. By working forwards in time, we are not restricted to using a single cost basis or an average cost basis. We are not restricted to approximating the tax code for losses by either requiring all losses to be immediately claimed or only allowing losses to cancel gains. We are not encumbered by the "curse of dimensionality," which can impose less realistic model assumptions, such as an exponential distribution for the time of death of an investor or a restricted class of investor utility functions. On the other hand, our approach also has limitations for which the Bellman approach appears better suited. For example, in Dai, Liu, Yang, and Zhong (2015), the Bellman approach explores cases where it is better to sell stock even if it triggers short term capital gains taxes, which our approach cannot be generalized to determine. However, our algorithm can be generalized, as we will later show, to allow the values of the lower and upper bounds for the no-rebalancing interval to depend on the ratio of the stock basis in the portfolio to the current stock price, as is done in Dai, Liu, Yang, and Zhong (2015).

Working forward in time allows us to find an optimization algorithm that reveals many new results about the optimal no-rebalancing interval for the stock fraction. Some of our results confirm statements in the current literature, while others contradict previous claims, but many of our results provide insight into questions hitherto uncovered in the extant literature. The main findings of this paper are as follows:

First, as noted, we provide a simulation based method that quickly generates an optimal portfolio trading strategy. Our method utilizes the full cost basis history, a complex, realistic tax model, and many more trading periods (we implemented 480) than has been considered in the literature so far when using the full cost basis history (20 periods in Haugh, Iyengar, and Wang (2014)). Our model can be extended in a number of directions, including accommodating just about any stochastic model for stock price evolution and any utility function applied to the portfolio worth at the horizon time, T. Our model can be applied to multiple stocks, however, assumptions would have to be made about the geometric nature of the no-rebalancing region. In contrast, the approach used in DeMiguel and Uppal (2005) for the multiple stock question required no a priori knowledge about the geometry of the no-rebalancing region, although it was computationally limited to seven trading periods for two stocks and four trading periods for four stocks.

Second, our portfolio rule comprises an optimal no-rebalancing interval for the stock fraction, which can be specified by the interval's center, denoted by f^*, and the interval's width, denoted by Δf. We show that optimality is reduced much more by movement away from the optimal center than movement away from the optimal width. When the optimal interval width is positive, as opposed to zero, the reduction in optimality due to perturbations of the interval width from its optimal value is particularly small.

Third, as in Dammon, Dunn, and Spatt (1989), we find that materially different optimal portfolio choices are made depending on whether the investor is assumed to be alive or to expire when the portfolio is liquidated at the horizon time, T. Moreover, our results indicate that this optimal strategy difference does not vanish as T increases, even though the tax treatment is only different at time T, suggesting that these strategies have long memory.

Fourth, counterintuitively, the optimal stock fraction in a taxable portfolio (or more specifically, f^*, the interval's center, if the interval width $\Delta f > 0$) is often higher, not lower, than the optimal stock fraction for a portfolio without taxes. This is the case whether or not the investor is assumed to be alive or dead when the portfolio is liquidated at time T. Because there are generally more capital gains than capital losses, it is intuitive to think that taxes should make the stock less desirable. However, the tax rate used to credit losses is generally significantly higher than the tax rate for gains, which can often be exploited to make the taxable stock more, not less, desirable. We analyze when this is the case.

Fifth, our analyses show that static band rules that are commonly used in practice to determine the no-rebalancing region are generally problematic. The "5/25" rule of thumb[3], for example, generally suggests for our stock-cash scenario that we should use a no-rebalancing interval with a width $\Delta f = 0.10$, since that corresponds to ± 5 percentage points for the portfolio's stock fraction. Our results, however, show that the optimal width of the no-rebalancing interval can vary dramatically (and is often zero), depending on a number of parameters. Specifically, the optimal width increases as we increase the stock's expected return, the capital gains tax rate, or the size of the portfolio, and it decreases as we increase the risk free return, the capital losses tax rate, or the time horizon, T, of the portfolio. The optimal width is also increased if the investor is assumed to expire at time T. (Surprisingly, in our results, the optimal width does not appear to be sensitive to stock's volatility, when intuitively

[3]This rule of thumb recommends that a portfolio be rebalanced if the actual fraction of the portfolio in an asset class deviates from the desired fraction by 5 percentage points (in the case where the desired fraction is 25% or higher), or if the actual fraction deviates by more than 25% from the desired fraction (in the case where the desired fraction is less than 25%).

one might expect it to be. See, for example, the trading range in Section 3.2.1.) Further, we explain why, as a rule of thumb when we roughly average the effect of all these parameters, it is better to choose $\Delta f = 0$ (that is, continually rebalance) than to choose $\Delta f = 0.10$ as suggested by the 5/25 rule.

Sixth, counterintuitively, as the tax rate on gains increases, generally more, not less, investment in the stock is optimal in the case when the investor is assumed to expire at the portfolio horizon. This is because the amount of rebalancing (along with the associated capital gains) needed to maintain a desired fraction, f, of the portfolio in stock decreases as f increases from $f = 0.5$ to $f = 1$. Indeed, to maintain $f = 1$, no rebalancing is needed, so no capital gains are generated.

Seventh, we assessed monthly, quarterly, and semi-annual trading schemes and found that the choice among these three trading frequencies has no material effect on the optimal allocation strategy. This assessment is novel in comparison to the extant literature, which was unable to assess this issue due to limitations in the number of periods it was possible to consider.

Eighth, using a large collection of scenarios, mostly with a 40 year time horizon, we find that, on average, using the full cost basis provides only a 0.65% certainty equivalent advantage over using the average cost basis if the investor is alive at the portfolio horizon time, T. This advantage reduces even further to 0.27% when the investor is assumed to expire at time T. These numbers are a bit lower than the 1% figure reported by DeMiguel and Uppal, giving more support to the validity of using the average cost basis approximation in optimization models like the Bellman models that require the approximation for tractability.

Ninth, as in Dai, Liu, Yang, and Zhong (2015), we allow the no-rebalancing interval's upper and lower bounds to each depend on the ratio of the basis price of the portfolio's stock to the current stock price. Dai, Liu, Yang, and Zhong (2015) employs the average cost basis price, whereas since we allow for the full cost basis, we select the highest cost basis price in the portfolio, since that corresponds to the first stock that is traded. We find that allowing the no-rebalancing region's bounds to depend on this ratio of the highest cost basis to the current price yields no discernible certainty equivalent advantage.

Tenth, for completeness and comparison with Leland (2000), we also re-optimize the taxable portfolio in the presence of proportional transactions costs. We confirm Leland's result that transactions costs reduce portfolio churn (i.e., they increase the optimal width Δf), but unlike Leland, we do not find that transaction costs materially reduce the optimal fraction of the portfolio in the stock position. (That is, they do not reduce the optimal f^*.)

In the following section (Section 2), we present our modeling approach, explicitly accounting for the features of the tax code and for the evolving cost basis of all stock in the portfolio. This is followed by numerical simulations of the model in Section 3, where we report and explain the effect on the optimal trading strategy caused by varying the values of the stock's expected return, the cash interest rate, the stock's volatility, the investor's risk aversion, the tax rate on losses, the tax rate on gains, the initial portfolio worth, the portfolio's time horizon, and the period between trading opportunities. We then report and explain the effect on the optimal trading strategy of letting the strategy change halfway to liquidation (at time $\frac{T}{2}$), of using the average cost basis in place of the full cost basis, of allowing the optimal strategy to depend on the ratio of the portfolio's highest cost basis to the current stock price, and of incorporating transaction costs into the model. Section 4 concludes.

2 Model

In this section we present the assumptions behind our model and the details of our simulation of the model.

2.1 Assumptions and Notation

We have two assets in our portfolio model, stock and cash. The return on the stock is risky; the return on the cash is certain. Our model applies two sets of assumptions: those for the stock and cash positions and those for the tax model.

For the stock and cash positions, we make the following assumptions:

1. We assume the stock evolves by geometric Brownian motion with a constant expected return, μ, and a constant volatility, σ.

2. The tax-free continuously compounded interest rate for the cash position, r, is assumed to be constant. We note that cash positions with constant interest taxed at a constant rate can be converted to an equivalent tax-free rate, r, and, of course, periodically compounded interest at a constant rate can also be converted to a continuously compounded rate, r.

3. Except where otherwise specified, we assume that stock and cash can be bought

and sold in any quantity, including non-integer amounts, with negligible transaction costs.

In portfolios subject to tax, the impact of taxes on the optimal strategy is generally much stronger than the impact of transactions costs. At the same time, tax codes are generally more complex than models for transactions costs. For tax-free portfolios like 401(k)s and Roth IRAs where transaction costs take center stage, there is an extensive literature on portfolio optimization, such as Akian, Menaldi, and Sulem (1996); Atkinson and Ingpochai (2006); Bichuch (2012); Goodman and Ostrov (2010); Leland (2000); Liu (2004); and Muthuraman and Kumar (2006).

4. For simplicity, we do not consider dividends for the stock, although the model can easily be altered to approximate the effect of dividends by adjusting μ, the growth rate of the stock.

For the tax model, we make the following assumptions:

1. As stipulated by the tax code in the United States, we assume a limit of no more than \$3000 in net losses can be claimed at the end of each year. Net losses in excess of this amount are carried over to subsequent years. Should the investor expire when the portfolio is liquidated at time T, all remaining carried over capital losses are lost. We encode all these features into our portfolio simulation program.

2. For simplicity, we assume that for all times prior to the portfolio being liquidated, the capital gains tax rate, τ_g, is constant and applies to both long term

and short term gains. When the portfolio is liquidated, we use the capital gains rate $\tau^{liq} = \tau_g$ if the investor is alive, while $\tau^{liq} = 0$ if the investor is dead to reflect the fact that capital gains are forgiven in the U.S. tax system when an investor expires. Similarly, for both short term and long term capital losses that are claimed prior to the portfolio being liquidated, we assume a constant rate, τ_l, which corresponds to the marginal income tax rate of the investor.

Our model can be altered to accommodate different rates for short term and long term gains and losses, but optimizing with this short term/long term model poses difficulties, unless one restricts the strategy to something reasonable, but possibly suboptimal, such as assuming that short term gains will never be realized.

3. We allow for wash sale rules, but make an additional assumption, so that they have no effect. Specifically, we assume the presence of other stocks or stock indexes in our market with essentially the same value of μ and σ. For a loss to qualify for tax credit, wash sale rules in the U.S. require that the investor wait at least 31 days before repurchasing a stock that was sold at a loss. But if the investor sells all of a stock that is at a loss and then immediately buys a different stock with the same μ and σ, the investor will avoid triggering a wash sale, while still taking full advantage of the loss for tax purposes. This sell/repurchase strategy for any stock with losses lowers the cost basis, but because it allows for earlier use of the losses for taxes, it is always superior to the strategy of buying and holding when transaction costs are negligible (see, for example, Constantinides (1983) or Ostrov and Wong (2011)).

2.2 Trading Strategy

We implement the following trading strategy:

For simplicity, we start with a strictly cash portfolio, although initial portfolios containing stock positions with various cost bases could just as easily be accommodated. We then immediately buy stock so that the portfolio's stock allocation attains a selected fraction, f^{init}, of the total portfolio's worth.

After each time period of length h years passes in the simulation, we consider three types of trades. We first sell and repurchase any stock with a loss, while avoiding wash sale rstrictions as described above, to generate money from these capital losses.[4] Next, if f, the fraction of the portfolio's value in the stock position, is below a selected lower threshold, f^l (which is constant over time), we purchase stock until $f = f^l$. Finally, if f is above a selected upper threshold, f^u (which is also constant over time), then stock with the highest cost basis is sold until $f = f^u$. Choosing to sell the highest cost basis stock is optimal, since it minimizes capital gains.

At the end of each year, taxes on any net capital gains are paid from the cash position, and any net losses that can be realized are used to purchase additional stock.

We keep track of the cost basis of all stock purchases, which makes the problem and portfolio value path dependent, ruling out any easy dynamic programming approaches to determining the optimal stock proportions at each point in time for the portfolio, as explained in the introduction.

[4]This assumes that the investor has non-investment income, so the losses can be used to lower the tax on the income, which is equivalent to generating money from tax shields.

Our goal is to determine the three values for f^{init}, f^l, and f^u that optimize the expected utility of the portfolio at a specified final portfolio liquidation time, T. Our model can easily accommodate any utility function, however, in our simulations, we have chosen power law utility functions (i.e., constant relative risk aversion). Normally, power law utilities lead to results that are independent of the initial portfolio worth, however this will not be the case here due to the $3000 limit on losses that can be claimed per year.

2.3 The Algorithm for Simulating a Single Run over T Years

For a given f^{init}, f^l, and f^u, our algorithm works by Monte Carlo simulation over a large number of runs. For each run, we proceed with the following algorithm:

We start by purchasing stock so that our initially all-cash portfolio attains the given initial stock fraction, f^{init}, at $t = 0$.

We simulate the market over each time period h. From time t to time $t + h$, the stock price, S, advances by geometric Brownian motion, so

$$S_{t+h} = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) h + \sigma\sqrt{h}Z\right],$$

where Z is a standard normal random variable. The cash position, C, advances by $C_{t+h} = C_t \exp(rh)$.

As we buy and sell stocks over time, we develop a portfolio with stock positions purchased at various prices. We keep track of each of these separately to determine the correct basis for computing gains or losses when we sell the stock. We let J be the number of different purchase prices (corresponding to various purchase times) for the stock. Define B_j, where $j = 1, 2, ..., J$, be these J purchase prices in order of purchase times, and let N_j be the number of stocks purchased at each of these prices.[5] Since we sell and repurchase any stock that has a loss, we are guaranteed that $B_1 \leq B_2 \leq ... \leq B_J$. In the initial ($t = 0$) stock purchase, for example, we have that $J = 1$, since there is only one stock position; $B_1 = S_0$, the stock price at $t = 0$; and the number of shares bought is $N_1 = \frac{f^{init} W_0}{S_0}$, where W_0 is the initial worth of the portfolio.

After each time period h elapses, we consider our three possible trades in the following order:

1. Collect any losses: Define k such that positions $k, k+1, ..., J$ are the only positions with losses. That is, $B_k, B_{k+1}, ..., B_J$ are the only values of B_j that are greater than the current stock price, S. We sell all of these positions and then immediately buy them back, recalling that we are actually buying back other stocks or stock indexes that have the same μ and σ to avoid wash sale restrictions. We subtract these losses from our current gains, so the new value for the gains, denoted G, is

$$G = G^{old} - \sum_{j=k}^{J} N_j (B_j - S).$$

   Since the positions k through J have all been repurchased at the current stock price, we now have that $N_k = N_k^{old} + N_{k+1}^{old} + ... + N_J^{old}$, where $B_k = S$, and we

[5]Should our model be altered to distinguish between short term and long term gains and losses, the time of purchase, t_j, must also be recorded.

reduce J to equal k, since positions $k+1$ through J^{old} no longer exist.

2. Buy stock if the current stock fraction, f, is below f^l: We buy stock until $f = f^l$. Since we buy stock, we must create a new position at the current purchase price, so we increase J by one. For this new J position, we have $B_J = S$ and N_J equals the number of stocks needed to make $f = f^l$.

3. Sell stock if the current stock fraction, f, is above f^u: We begin selling position J, then position $J-1$, etc., until we have $f = f^u$. We add these gains to G. They must be gains, because all losses were already collected. If any positions are liquidated by this procedure, we reduce the value of J to reflect this.

This trading strategy of buying and selling stock so as to stay just within the interval $[f^l, f^u]$ is a standard optimizing strategy when there are either no transaction costs or proportional transaction costs, as is the case here. (See Davis and Norman (1990), for example.)

At the end of each year, we pay taxes or collect tax credits, depending on the sign of G. If $G > 0$, we have gains and so we remove $G \cdot \tau_g$ from the cash account and then set $G = 0$. If $G < 0$, we have losses, which we use to buy stock. Since we are buying stock, we increase J by one and then set $B_J = S$, the current stock price. If the losses, $-G$, are less than the annual limit of \$3000, they generate $-G \cdot \tau_l$ dollars, which purchases $N_J = \frac{-G \cdot \tau_l}{S}$ shares of stock, and we then set $G = 0$. If the losses are more than \$3000, we purchase $N_J = \frac{3000 \cdot \tau_l}{S}$ shares of stock, and then we set $G = G^{old} + 3000$, so the excess losses are carried over to the next year.

At the end of T years, we liquidate all the stock positions in the portfolio. If we assume the portfolio's owner is alive, capital gains from this liquidation are paid. If we assume the owner has expired, no capital gains are paid. Any carried over losses are lost.

As detailed in the next subsection, we will look to average the utility of the final portfolio worth over all of our Monte Carlo runs to determine an approximation for the expected utility. We use the same simulated stock runs for comparing the expected utilities for different f^{init}, f^l, and f^u combinations. This allows us to converge to the values of f^{init}, f^l, and f^u that optimize the expected utility for this fixed set of simulations. We then use different sets of simulations to check for consistency in the optimal values determined for f^{init}, f^l, and f^u.

2.4 The Expected Utility Estimator and The Optimization Program

Working with our model is quite computationally intensive. We generally chose a base case liquidation time, T, of forty years. Each forty year run is computed using the complex algorithm from the previous section, and we needed to average the utility at liquidation over a high number (eventually 50,000) of these runs to estimate the expected utility of the terminal wealth, $E[U(W_T)]$, corresponding to any specific values of f^{init}, f^l, and f^u. On top of this, the optimization algorithm requires several calls to this expected utility estimator for various values of f^{init}, f^l, and f^u.

Given this approach, we required both very fast computation for the expected utility estimator and an efficient optimizer. We therefore programmed the expected utility estimator in the C programming language, and compiled it and linked it to be callable from the R programming language so as to use the optimizer in R, which is stable, fast, and accurate. The optimization function in R is `constrOptim`, which is a constrained optimizer, as we need to restrict the values of f^{init}, f^l, f^u so that $0 \leq f^l \leq f^{init} \leq f^u \leq 1$.

We applied a power utility function,

$$U(W) = \frac{W^{1-\alpha}}{1-\alpha}, \tag{1}$$

to the terminal wealth, although we can easily accommodate any other utility function in our simulation program. The power law utility, however, is particularly suited to our model where f^l and f^u are constant with respect to time because, in the absence of taxes, Merton (1992) showed that, for the power law utility, the optimal fraction of the portfolio in stock is

$$f_{\text{Merton}} = \frac{\mu - r}{\alpha \cdot \sigma^2},$$

which is also constant with respect to time.

Given our choice of the power law utility, our estimate of the expected utility at time T becomes

$$E[U(W_T)] \approx \frac{1}{M} \sum_{m=1}^{M} \frac{1}{1-\alpha} \cdot (W_T(m))^{1-\alpha},$$

where α is the coefficient of relative risk aversion for the investor, m indexes each simulated run, M is the total number of runs, and $W_T(m)$ is the terminal wealth for run m generated in the simulation.

Optimization is run in two stages. In the first stage, we find the values of f^{init},

f^l, and f^u that are optimum over a case with only $M = 1000$ runs, which is fast, since M is small. We then use these three values as starting guesses for the optimum f^{init}, f^l, and f^u in the second stage, where we optimize over $M = 50,000$ runs. This two step process has the benefit of speed from the first stage and precision from the second stage. We used standard computing hardware for this simulated optimization procedure, and each optimization runs in under 5 minutes. We obtained similar results when using other sets of 50,000 sample paths generated using independent sets of random numbers.

3 Results and Analysis

In this section we present and analyze numerical results obtained from the simulation model described in the previous section. We begin in Subsection 3.1 with a discussion of the optimal stock fraction range, $[f^l, f^u]$, (i.e., the optimal no-rebalancing interval) for the "base case," where we assign the following values to the following nine parameters:

1. the stock growth rate, $\mu = 7\% = 0.07$ (per annum)

2. the risk free rate, $r = 3\% = 0.03$ (per annum)

3. the stock volatility, $\sigma = 20\% = 0.20$ (per annum)

4. the risk aversion parameter, $\alpha = 1.5$ in our utility function in equation (1)

5. the tax rate on losses, $\tau_l = 28\% = 0.28$

6. the tax rate on gains, $\tau_g = 15\% = 0.15$

7. the initial portfolio worth, $W_0 = \$100,000$

8. the time horizon before portfolio liquidation, $T = 40$ years

9. the period between potential trades, $h = 0.25$ years (i.e., quarterly trading and rebalancing).

We will consider the base case both where the investor expires at $T = 40$, so any remaining capital gains are forgiven, and where the investor is alive at $T = 40$, so taxes on any remaining net capital gains must be paid.

In Subsection 3.2, we show the sensitivity of this optimal stock fraction range to varying these nine parameter values, one at a time, from their base case values. Then, in Subsection 3.3, we consider the effect of changing the model in four ways:

1. Letting the stock fraction range, $[f^l, f^u]$, change values at time $\frac{T}{2} = 20$ years.

2. Using the average cost basis instead of the full cost basis.

3. Allowing the no-rebalancing interval's bounds to depend on the ratio of the highest cost basis to the current stock price.

4. Incorporating proportional transaction costs when we buy and sell stock.

A no-rebalancing interval can be defined by specifying f^l and f^u or by specifying the components

$$\begin{aligned} f^* &= \frac{f^l + f^u}{2} && \text{(the center (or midpoint) of the interval) and} \\ \Delta f &= f^u - f^l && \text{(the width of the interval).} \end{aligned}$$

We will often prefer to use f^* and Δf, since it makes more sense to analyze how changes affect the center and the width of the trading strategy, instead of how they affect f^l and f^u. In our simulations, we found the effect of varying f^{init} to be quite small. Therefore, we do not discuss or present the optimal values of f^{init}, even though our algorithm determines and uses them.

3.1 Optimal Strategy for the Base Case

For the base case specified above, if the investor expires at $T = 40$, our analysis shows that the optimal trading strategy is to set $f^* = 0.764$ and $\Delta f = 0.168$. For the base case if the investor is alive at $T = 40$, the optimal trading strategy is to set $f^* = 0.711$ and $\Delta f = 0$. Note that $\Delta f = 0$ means the investor is best off continually rebalancing.

The effect of dying versus living at time T on the optimal trading strategy that we observe here will hold in general, as we will see throughout this section. When the investor is alive at time T and must pay capital gains, there are two key effects: (1) Having to pay capital gains taxes at time T makes the stock less desirable, so f^* gets smaller when the investor is alive at liquidation. (2) Having to pay capital gains taxes at time T makes having capital gains at time T less desirable, so Δf also gets smaller. Further, even though dying or living at time T only affects the tax treatment at time T, it has a considerable effect on our optimal long term investing strategy, especially on the optimal Δf, as we see here and will see throughout this section.

As stated previously, in the absence of taxes, Merton (1992) shows that it is optimal to keep the stock fraction equal at all times to

$$f_{\text{Merton}} = \frac{\mu - r}{\alpha \cdot \sigma^2}, \tag{2}$$

which, for our base case, corresponds to $f_{\text{Merton}} = \frac{2}{3}$. But notice that this Merton stock fraction is smaller than either $f^* = 0.764$ or $f^* = 0.711$. That is, although it is counterintuitive, the optimal stock fraction in the taxable accounts is higher here than the optimal stock fraction in the account with no taxes.

Why? The answer is because τ_l, the refund tax rate for capital losses, is higher than τ_g, the tax rate for capital gains. Therefore, the advantage of culling capital losses from stock in a taxable account can, on average, outweigh the disadvantage of paying capital gains taxes, as is the case here. We emphasize that only the stock positions have different tax treatments here. The cash positions, both in our algorithm and in the Merton expression (2), use the same tax-free rate, $r = 0.03$. That is, we are not choosing a higher stock fraction in the taxable portfolio because it is better to keep the cash position shielded from taxes, as is often advised when considering a taxable account versus a 401(k) or Roth account.

Although our algorithm determines the optimal f^* and Δf, it can easily be simplified to determine the loss to an investor should they employ a suboptimal f^* and/or Δf. We quantify this loss using the certainty equivalent, C, which is defined by the equation

$$U(C) = E[U(W_T)],$$

where W_T is the portfolio worth at time T. That is, the investor has no preference between starting at $t = 0$ with Ce^{-rT} dollars that must be invested as cash at the risk free rate r until time T (leading to C dollars at $t = T$) or starting at $t = 0$ with W_0 dollars invested in our stock and cash portfolio until time T (leading to a random variable for the value of W_T).

To quantify the disadvantage of using a suboptimal strategy versus the optimal strategy, we use $\frac{C_{subopt}}{C_{opt}} < 1$, the ratio of the certainty equivalents under these two strategies, to define c, the certainty equivalent difference, by the following equation:

$$1 + c = \frac{C_{subopt}}{C_{opt}} = \frac{U^{-1}\left(E_{subopt}[U(W)]\right)}{U^{-1}\left(E_{opt}[U(W)]\right)},$$

Since $U(W) = \frac{W^{1-\alpha}}{1-\alpha}$, this can be re-expressed as

$$c = \left(\frac{E_{subopt}[U(W)]}{E_{opt}[U(W)]}\right)^{\frac{1}{1-\alpha}} - 1 \leq 0. \tag{3}$$

Note that the optimal strategy corresponds to $c = 0$, and c becomes progressively negative as the strategy becomes more suboptimal.

In Figure 1, we see the effect on c if we deviate from the optimal strategy. For the two panels on the left, we fix Δf at its optimal value and let f^* vary. For the two panels on the right, we fix f^* at its optimal value and let Δf vary. Noting the scale on the vertical axis, we see that the sensitivity of c to suboptimal values of f^* is far larger than the sensitivity to suboptimal values of Δf. That is, it is much more important for investors to determine their optimal portfolio stock fraction (or, more precisely, the center of their optimal no-rebalancing interval) than it is to determine the precise optimal width of this interval.

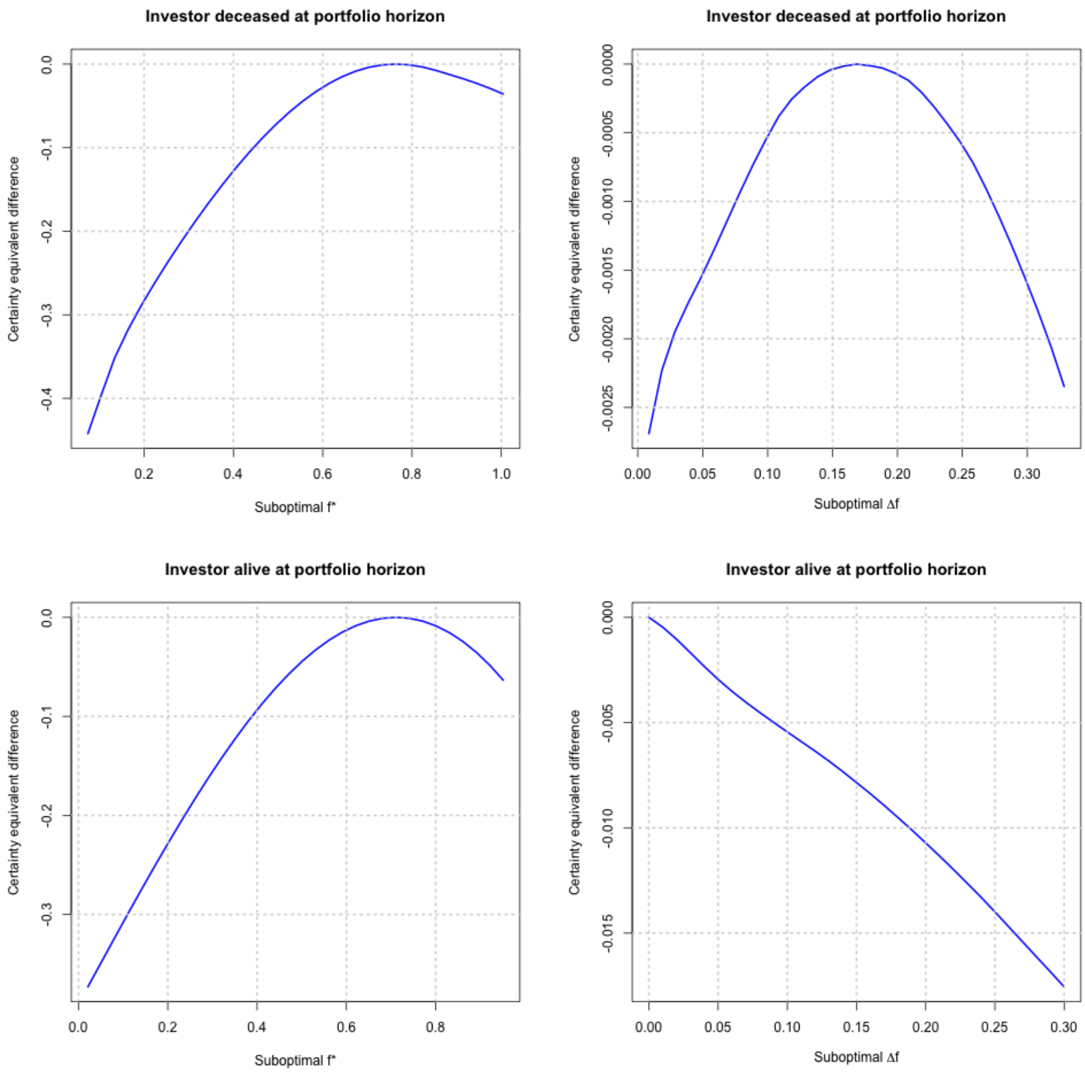


Figure 1: The loss to the investor, measured as the certainty equivalent difference, due to employing a suboptimal strategy to the base case. For the top two panels, the investor expires when the portfolio is liquidated, whereas the investor is alive during the portfolio liquidation in the bottom two panels. The two panels on the left examine the effect of using a suboptimal f^*, which is stronger than the effect of using a suboptimal Δf, examined in the two panels on the right.

But there's also considerable forgiveness should the investor get the precise value of the optimal f^* wrong due to the fact that, at least in the base case both for the investor dying (top left panel of Figure 1) or living (bottom left panel), the optimal f^* is on the interior of the domain $f \in [0, 1]$. Because of this, and the smoothness of the graphs, the derivative of c is zero at the optimum values of f^*. This leads to a small loss in the certainty equivalent difference for even moderate deviations from the optimal f^*. In the two base cases shown in Figure 1, for example, we see no more than a 1% loss in the certainty equivalent difference over 40 years, until the value chosen for f^* differs from the optimal f^* value by around 10 percentage points.

The same effect occurs for Δf in the base case where the investor expires (top right panel of Figure 1). That is, because the optimal Δf is an interior value in the domain of possible Δf values, the derivative of c must be zero at the optimal Δf, and so we see little sensitivity to moderate deviations from this value. This is not the case, however, in the base case where the investor expires (bottom right panel). Since the optimal Δf is zero, which is an endpoint of the domain of possible Δf values, the derivative of c need not be zero at the optimal Δf, which means the sensitivity of c to deviations from $\Delta f = 0$ will, in general, be greater, as is the case here.

There is considerable debate about whether, as a rule of thumb, a taxable portfolio should be rebalanced continually (but selling stock only with long term capital gains, of course) or rebalanced only when the portfolio deviates too much from its optimal stock fraction, as in the "5/25" rule discussed in the introduction. In this paper we will have numerous examples of where it is optimal to rebalance continually (i.e., the optimal $\Delta f = 0$) and almost equally numerous examples where it is optimal to rebalance only when the portfolio deviates too much (i.e., the optimal $\Delta f > 0$). It might therefore be assumed that this paper suggests that either rule of thumb is equally valid. However, this is not the case. If the optimal Δf isn't too large, the observation in the previous paragraph tells us that, as a rule of thumb, it is better to continually rebalance, because the loss due to being wrong by choosing $\Delta f = 0$ when the optimal Δf is a small positive value will generally be less than the loss due to being wrong by choosing a small positive value for Δf when the optimal $\Delta f = 0$. That said, using the wrong rule of thumb is unlikely to have significant consequences. After all, as we can see in the two right panels of Figure 1, there is a wide range of suboptimal Δf that can be used in the base case where the loss over 40 years in the certainly equivalent difference remains under 1%.

3.2 The Effect on Varying Parameters on the Optimal Strategy

In the remainder of this section, f^* and Δf will denote the f^* and Δf of the *optimal* no-rebalancing interval.

In the figures for this subsection, the parameters being varied are displayed on the horizontal axis and stock fractions, f, are displayed on the vertical axis. Recall that stock positions are constrained to be long only, so $f \in [0, 1]$. There are three curves on the graphs, which represent the upper boundary, f^u (red line), the lower boundary, f^l (blue line), and the midpoint, f^* (green line), between f^u and f^l. It is reasonable to think that f^* and the average value of f over time are almost equal, since, for example, they are equal to leading order as the interval width $\Delta f = f^u - f^l$ gets small in the related continuous time scenario considered in Goodman and Ostrov (2010). We will sometimes refer to f^* as the average stock fraction for this reason and, of course, because f^* is the average of the two values f^u and f^l.

In many of the figures, we will see that changes to our parameter values will cause

Δf to become bigger or smaller, often to the point of causing a transition between Δf being positive and being zero. These changes in the size of Δf are determined by shifting balances among a number of opposing factors.

There are two factors that push Δf to be bigger: (1) The bigger Δf is, the more useful deferring capital gains becomes, even when tax on the gains must be paid at time T due to the investor being alive. (2) If the investor is deceased at time T, then the bigger Δf is, the more likely that larger amounts of gains will be forgiven at time T.

In opposition to these effects are the factors that push Δf to be smaller: (1) The smaller Δf is, the more useful claiming capital losses becomes, since $\tau_l > \tau_g$ in current American tax law. That is, the smaller Δf is, the more we can take advantage of the two tax options discussed in Constantinides (1983) and Constantinides (1984). (2) The smaller Δf is, the more control we have over keeping the portfolio near or at the stock fraction that optimizes the investor's expected utility.

In this subsection, we demonstrate and then explain how f^* and Δf are affected by altering our model parameters, one at a time, from their base case values. We present the results for each parameter being varied using two graphs in each figure: the graph on the left will correspond to the case where the investor is assumed to expire at the portfolio horizon time T, and the graph on the right will correspond to the case where the investor is assumed to be alive at time T.

To begin we first keep track of how many levels there are in the tax basis on simulated path. Earlier work by DeMiguel and Uppal (2005) and Dai, Liu, Yang, and Zhong (2015) argued that there is rarely more than one level in the basis. We find otherwise, with an average of 9-10 levels in the basis over the life cycle. The distribution of basis levels across the 50,000 simulated paths is shown in Figure 2. This is for the optimal strategy under base case parameters.

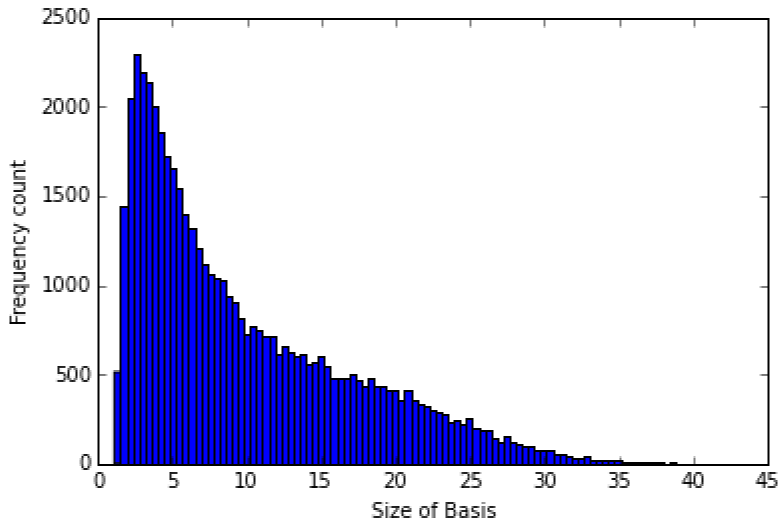


Figure 2: Distribution of the number of levels in the tax basis on average in one lifecycle. We show the histogram of the average basis levels for each path, across 50,000 paths. We assume the standard \$3000 annual limit for claiming losses and that all parameter values, except the varying parameter, take their base case values.

3.2.1 Varying the Stock and Cash Growth Rates, μ and r

The stock growth rate, μ: Critical to any long-run portfolio strategy is the assumed average growth rates of the portfolio's financial instruments. Financial planning tools make assumptions about this as a critical part of their process. Here, we first examine the effect of changing the expected stock growth rate, μ, over the values 0.06, 0.07, 0.08, ..., 0.12 per annum, while holding all the other parameters at their base case values given above.

In Figure 3, we see, as expected, that the optimal average stock fraction, f^* (represented by the green line), increases as μ increases, until the investor is best off placing the entire portfolio in stocks, so $f^* = 1$. And, of course, once this happens, the optimal stock fraction interval, $[f^l, f^u]$, represented by the blue and red lines, collapses to the point $f^l = f^u = 1$, so that there is no cash. Also we note, as expected from our discussion near the beginning of Subsection 3.1, that both f^* and Δf are smaller in the graph on the right of Figure 3, where the investor is alive at the portfolio's liquidation time $T = 40$. Finally, we note that the values seen in Figure 3 for the base case, $\mu = 0.07$, correspond, of course, to the numerical values given in Subsection 3.1.

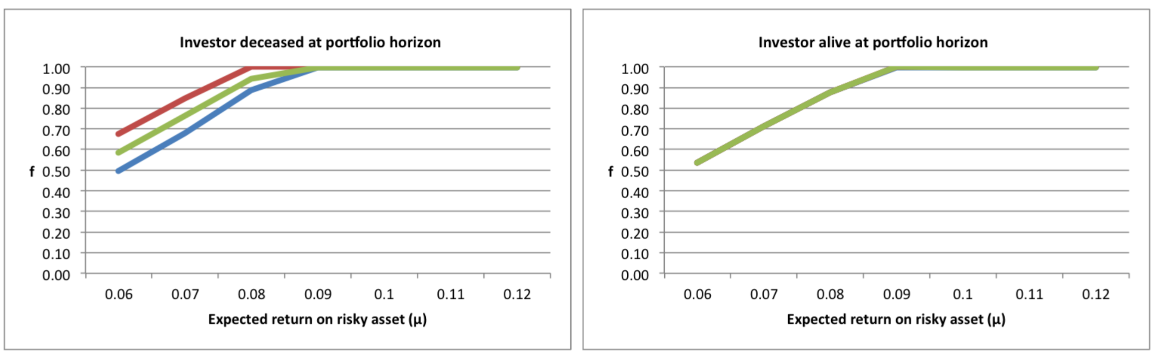


Figure 3: The optimal stock fraction range as the expected stock growth rate, μ, varies. In this figure, and those below, it is optimal to rebalance only enough so that the portfolio stock fraction stays between the blue curve, f^l, and the red curve, f^u. The green curve, f^*, represents the center (i.e., the midpoint) of this interval. We assume the standard \$3000 annual limit for claiming losses and that all parameter values, except the varying parameter, take their base case values.

The risk free rate, r: Increasing r, the cash interest rate (i.e., the risk free growth rate), essentially has the opposite effect on the optimal policy compared to increasing μ. We examine this effect in Figure 4 as we change the interest rate, r, over the values 0.01, 0.02, . . . , 0.06 per annum. Again, note that the base case value, $r = 0.03$, in Figure 4 corresponds, as it must, to the same values in Figure 3 at the base case value, $\mu = 0.07$.

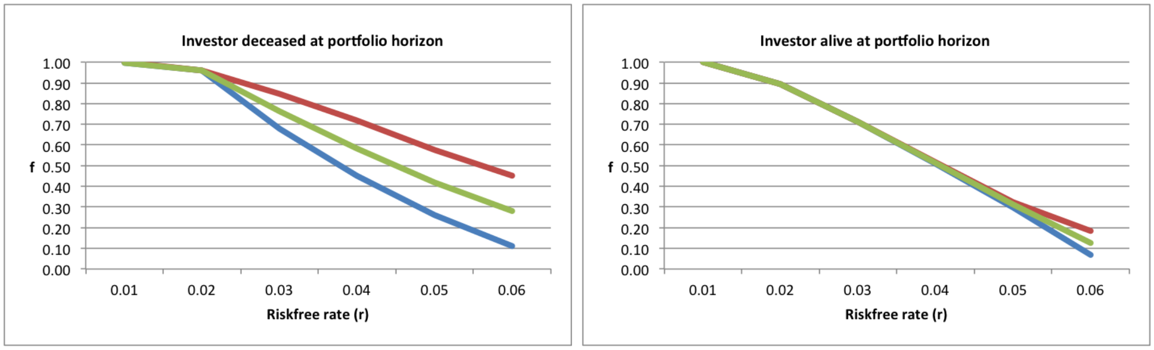


Figure 4: Optimal stock fraction range when varying the risk free rate, r.

The stock growth rate, μ, and the risk free rate, r, while holding $\mu - r$ constant: As a related analysis, we also examine the effect of increasing the stock growth rate, μ, to the values 0.04, 0.05, 0.06, ..., 0.16 per annum while equally increasing the risk free rate so as to hold the equity risk premium constant at $\mu - r = 0.04$.

Recall that in the absence of taxes, it is optimal to keep the stock fraction equal to $f_{\text{Merton}} = \frac{\mu - r}{\alpha \sigma^2}$ at all times. If we change μ while keeping the equity risk premium, $\mu - r$, constant, f_{Merton} doesn't change. That is, as long as $\mu - r = 0.04$, then the optimal Merton strategy stays unchanged from the base case, namely $f^* = \frac{2}{3}$ and $\Delta f = 0$.

But what happens to the optimal strategy in the presence of taxes? In Figure 5, we let μ vary while holding the equity risk premium constant at $\mu - r = 0.04$. We see that as μ increases, f^* decreases some and Δf increases.

Because $\mu - r$ is held constant, both μ and r increase equally in value, but the tax law's effect on the stock is quite different from its effect on the cash. As μ increases, there are fewer stock losses and more stock gains, which means fewer opportunities to

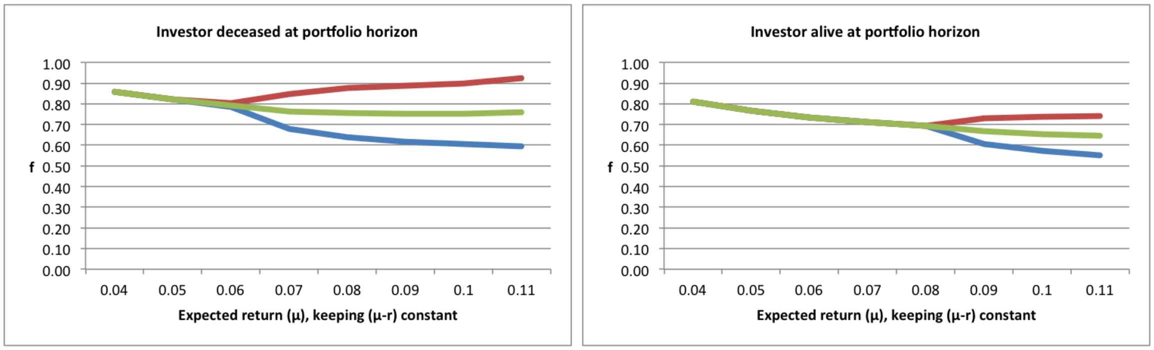


Figure 5: Optimal stock fraction range when varying the stock growth rate, μ, while keeping the excess return, $\mu - r$, constant.

take advantage of the tax break created for culling losses due to τ_l being greater than τ_g. Because culling losses is less advantageous, the stock becomes less useful relative to the cash, and so f^* decreases as μ (and r) increase.

As the probability of losses decreases, the ability of a small Δf to reap losses diminishes. At the same time, since there are more gains, the advantage of deferring paying taxes on these gains increases. Both of these effects push Δf to increase as μ (and r) increase.

3.2.2 Varying Risk and Risk Aversion, σ and α

The stock volatility, σ: Portfolio risk is a function of the volatility of the stock, σ. We examine the effect of changing the stock risk, σ, over the values 0.15, 0.20, 0.25, and 0.30. Not surprisingly, we see in Figure 6 that the optimal average stock fraction, f^*, decreases as risk increases, starting from an all stock position when σ is very low. The effect of σ on Δf, on the other hand, is surprisingly small. As is always the case when the investor is alive at time T, we have a smaller value for f^* and a smaller value for Δf. In fact, Δf shrinks to zero when the investor is alive here.

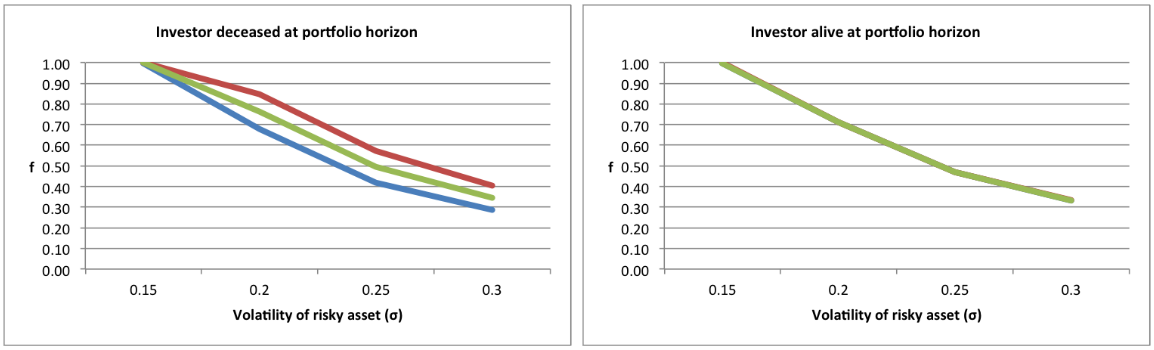


Figure 6: Optimal stock fraction range when varying the stock volatility, σ.

The risk aversion parameter, α: We also looked at changing the risk aversion parameter, α, over the values 0.7, 1.1, 1.5,..., 3.9. The results for changing α, as seen in Figure 7, are similar to the results for increasing σ, with similar reasoning.

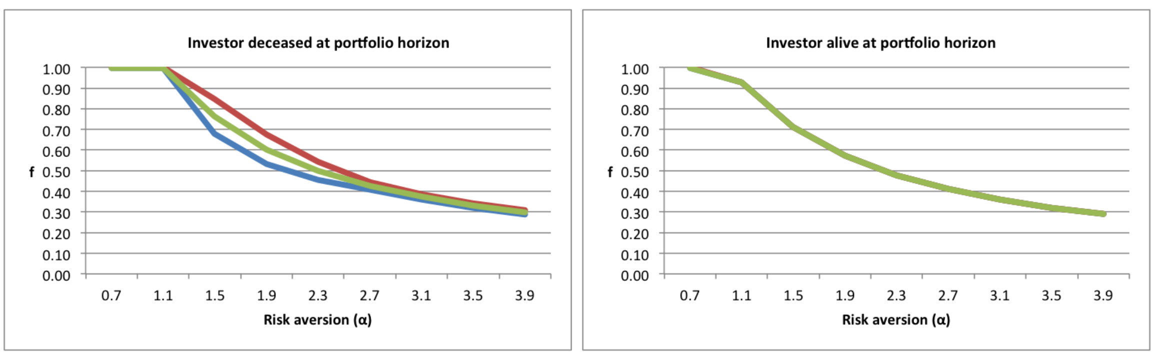


Figure 7: Optimal stock fraction range when varying the risk aversion coefficient α.

3.2.3 Varying Tax Rates, τ_l and τ_g

Since the U.S. tax code has different rates for capital losses (τ_l) versus capital gains (τ_g), we examine the impact of varying these tax rates separately.

The tax rate on losses, τ_l: We study the effect of changing the tax rates on capital losses, τ_l, over the values of the marginal tax rates in the current U.S. tax code: 0.1, 0.15, 0.25, 0.28, 0.33, 0.35, and 0.396. From a tax point of view, losses are a benefit, and the higher τ_l is, the greater the tax shielding experienced by the investor. As a consequence, in Figure 8, we see that the optimal average stock fraction, f^*, increases with τ_l, since the additional tax shielding mitigates the downside risk of holding stocks, thereby making the stock more desirable.

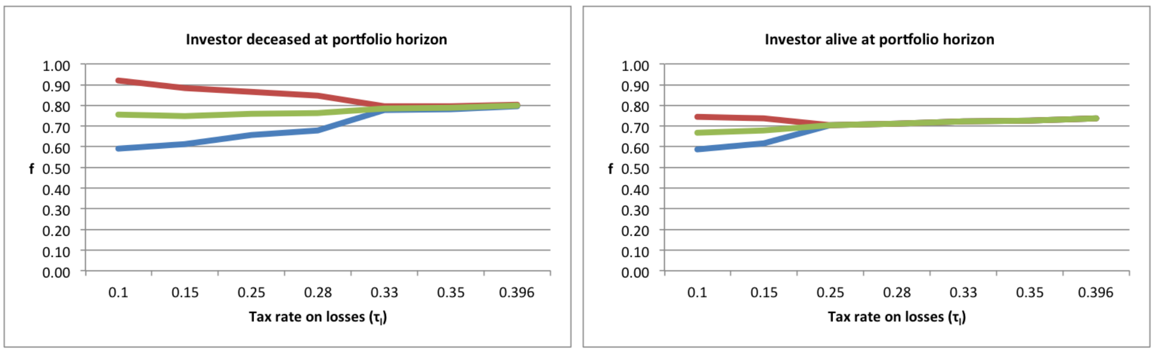
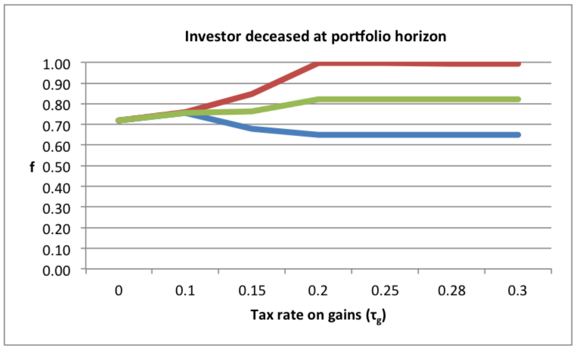


Figure 8: Optimal stock fraction range when varying the tax rate on losses, τ_l. Note that since the distances between our experimental values of τ_l are not uniform, the grid on the horizontal axis is not uniform.

Also, as expected, Δf decreases as τ_l increases, since the investor optimally rebalances more often as the benefits from taking losses increase.

The tax rate on gains, τ_g: We examined the effect of changing the capital gains tax rate, τ_g, over the values 0, 0.1, 0.15, 0.20, 0.25, 0.28, and 0.30. Given our analysis for τ_l, it is intuitive to think that as the capital gains tax rate, τ_g, increases, stock becomes less desirable, and so our desired stock fraction, f^*, should decrease. But a quick glance at the plot on the left in Figure 9 shows that this intuition is wrong!

Why should f^* increase? To provide some simple intuition, think about the case where the stock fraction, $f = 1$. Whether the stock goes up or down in worth, f continues to equal 1, so there is never a need to realize capital gains with their higher



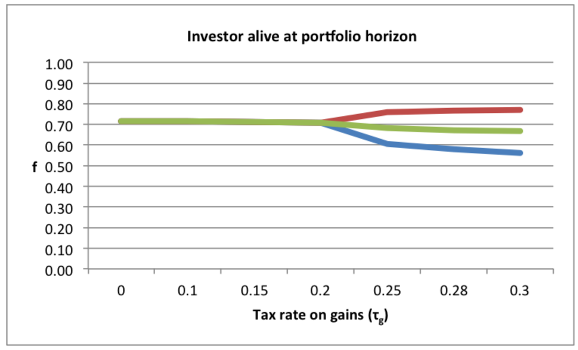


Figure 9: Optimal stock fraction range when varying the tax rate on gains, τ_g. Note that since the distances between our experimental values of τ_g are not uniform, the grid on the horizontal axis is not uniform.

tax rate, τ_g, before the liquidation time T. Therefore, in the case where capital gains are forgiven when the investor expires at time T, this highest possible value of f is clearly more desirable than, say, $f = .9$ or $f = .8$ etc., where capital gains will occur every time period h. And the desirability of this higher f increases as τ_g increases.

We can quantify and expand this intuition with a simple calculation over a year's time horizon. Let f be the fraction of the portfolio we want in stock, μ be the annual return for the stock, and r be the annual return for cash. Assume we have a portfolio worth \$1 at $t = 0$, so we have f dollars of stock and $(1 - f)$ dollars in cash. By $t = 1$, we have $(1 + \mu)f$ dollars of stock and $(1 + r)(1 - f)$ dollars of cash. Adding these gives a total portfolio worth of $(1 + r) + (\mu - r)f$ dollars at $t = 1$. Assume we choose $\Delta f = 0$. We then need to rebalance so that the stock fraction is f again. This means we want to have $((1 + r) + (\mu - r)f)f$ dollars of stock, and so we must sell

$$[(1 + \mu)f] - [((1 + r) + (\mu - r)f)f] = (\mu - r)f(1 - f)$$

dollars of stock as capital gains. This capital gains function, $(\mu - r)f(1 - f)$, is parabolic in f. It equals 0 at $f = 0$, increases to its maximum value at $f = \frac{1}{2}$ and then decreases back to 0 at $f = 1$. This establishes that if we are in the region where $f > \frac{1}{2}$, then *increasing* f actually *lowers* capital gains.

This calculation corresponds to the case where the investor expires at the portfolio liquidation time, T, and so capital gains are not paid at liquidation. When the investor is alive and capital gains taxes must be paid at liquidation, there are two opposing effects to balance: the higher deferral of capital gains that using a higher f provides (as explained in the previous paragraph) and the higher capital gains taxes that must be paid at liquidation when a higher f is used. We see in the plot on the right in Figure 9 that f^* now slightly decreases as τ_g increases, so this latter effect clearly outweighs the former effect in this case.

We also see in the plot on the right that Δf increases as τ_g increases. This is expected, since the higher τ_g is, the less advantageous it is to realize gains in order to reset the cost basis and, thereby, increase the likelihood of reaping the losses, which initially have a higher rate than the gains. Once τ_g surpasses $\tau_l = 0.28$, the situation is flipped, and capital gains become more destructive than capital losses are advantageous. This makes rebalancing progressively less desirable as τ_g increases, causing Δf to increase further.

3.2.4 Varying the Initial Portfolio Worth, W_0

Our power law utility function from equation (1) has the property of constant relative risk aversion. In the absence of taxes, this implies that the optimal stock fraction is independent of the portfolio size, as reflected by the absence of W_0 in the Merton expression given in equation (2).

If tax policy were strictly dictated by proportional factors, such as τ_g and τ_l, then we would also expect the optimal policy with taxes to be independent of W_0 (see, for example, Dammon, Spatt, and Zhang (2001)). However, the \$3000 limit on annual claimed losses is a constant, not proportional, factor, and therefore, the optimal strategy will be affected by W_0.

We see this effect in Figure 10. We study the effect of changing the initial size of the portfolio, W_0, over the values \$10,000, \$20,000, \$50,000, \$100,000, \$200,000, \$500,000, \$1,000,000, \$2,000,000 and \$5,000,000. As W_0 increases, there is a mild decline in f^* due to the fact that as W_0 increases, more and more losses must be carried over to subsequent years, making the stock less valuable. With a small portfolio, keeping Δf smaller corresponds, in general, to more collectable losses, which is desirable since the \$3000 limit rarely interferes. As the portfolio becomes larger, however, the \$3000 limit on losses is more easily reached, and the advantage of keeping Δf small is diminished. When this happens, the tax deferral provided by a larger Δf becomes a more dominant factor, and therefore Δf grows as W_0 increases in Figure 10.

3.2.5 Varying the Time Horizon before Portfolio Liquidation, T

We next look at the effect of the portfolio horizon, T, on the optimal strategy by changing T over the values 5, 10, 15, $\ldots$, 60 years.

We first consider the graph on the left in Figure 11 for the case where the investor expires at time T. As T first begins to increase, we see that f^l and f^u (and therefore

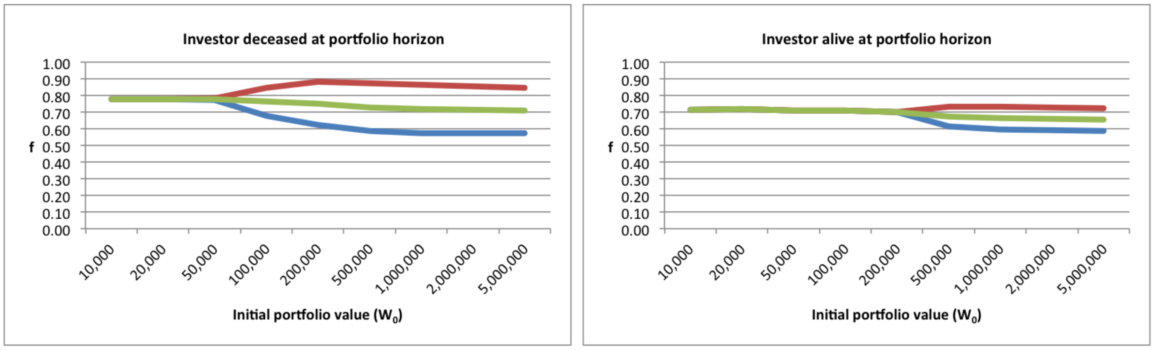


Figure 10: Optimal stock fraction range when varying the initial wealth, W_0. Note that since the distances between our experimental values of W_0 are not uniform, the grid on the horizontal axis is not uniform. In fact, the values of W_0 have been chosen so that the horizontal axis is close to logarithmic.

f^*) increase. This is no surprise since both the advantage of deferring capital gains taxes and the advantage of having capital gains forgiven at liquidation grow as T grows, which makes the stock's worth relative to cash increase. By $T = 15$, $f^u = 1$, so the advantages of having capital gains forgiven at liquidation now completely outweigh both the desire to position the portfolio at its optimal stock fraction and the desire to sell stock so that more losses can be generated for tax credits.

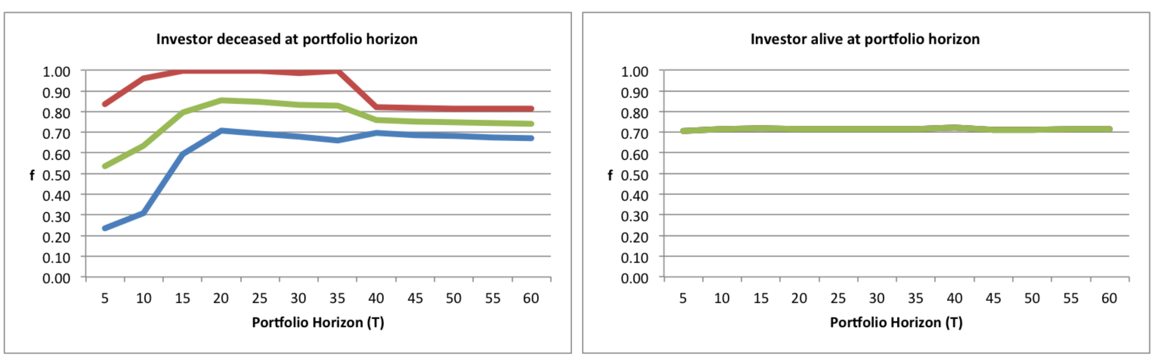


Figure 11: Optimal stock fraction range when varying the portfolio horizon, T (in years).

But by $T = 40$, this no longer holds, and we see f^u decrease. There are two effects behind this. Effect #1: When we use an interval of the form $[f^l, 1]$, we don't sell stock. In this case, because $\mu > r$, the longer the time horizon is, the more likely the stock fraction f is to drift into the high end of the $[f^l, 1]$ interval. The closer the stock fraction drifts towards 1, the farther it drifts from its optimal fraction, and so the portfolio gains too much risk unless we reduce f^u. Effect #2: If $f^u = 1$, so we don't sell stock over long time horizons, then, in the cases where the stock does well, the portfolio does particularly well, due, in part, to the significant forgiveness of particularly big gains at time T when the investor expires. On the other hand, in the cases where the stock, overall, does not do well, the portfolio will do even worse, since stock is not sold when it reaches an $f^u < 1$, which would have helped to generate losses over the long time horizon that would have cushioned the damage caused by bad returns. This greater "wealth disparity" over longer time horizons is penalized by the concavity of the utility function, eventually forcing f^u to be reduced as T increases.

In the case where the investor does not expire at the liquidation time T, we see in the graph on the right in Figure 11 that f^* is a constant just above 0.7 and $\Delta f = 0$. The fact that Δf is reduced when the investor does not expire at time T is, of course, expected. Because it is reduced to $\Delta f = 0$, the advantages of deferring taxes are not used and so f^* becomes independent of T, just as in the Merton case where $\Delta f = 0$, we have that $f_{\text{Merton}} = \frac{2}{3}$ regardless of the value of T.

3.2.6 Varying the Period between Potential Trading, h

We study the effect of changing the time period, h, between potential trades within the portfolio (i.e., culling losses and/or rebalancing) over the values $1/12 = 0.8333$

(monthly), $1/4 = 0.25$ (quarterly), and $1/2 = 0.5$ (semi-annually).

In Figure 12, we see that increasing h appears to have no real effect on f^* and only slightly increases Δf in the case on the left of Figure 12 when the investor expires at time $T = 40$.

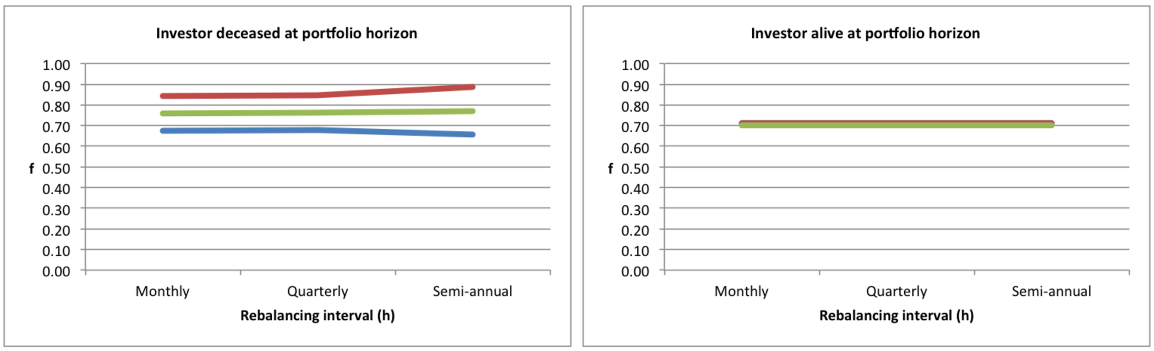


Figure 12: Optimal stock fraction range when varying the potential trading interval h. Note that since the two distances between the three values of h are not uniform, the grid on the horizontal axis is not uniform.

Why is there a slight increase in Δf in this case? One of the advantages of a smaller Δf is that it increases the likelihood of capital gains and capital losses, which is, overall, advantageous, since $\tau_l > \tau_g$. When we increase h, however, this advantage is reduced, since it becomes more likely that the losses will be cancelled by gains before they can be realized. With the advantage of a smaller Δf reduced, the factors that push Δf to expand become more dominant, and so Δf increases a little as h increases.

From a practical standpoint, however, it is more important to note that this increase in Δf is small. That is, the frequency of potential trading is not a particularly important factor on the optimal strategy, over the reasonable range of h values considered here.

3.3 The Effect of Changing the Model on the Optimal Strategy

In this subsection, we consider the effect of changing the model in four ways:

- Letting the optimal stock fraction range, $[f^l, f^u]$, change values when the portfolio is halfway to liquidation (i.e., at $\frac{T}{2} = 20$ years), instead of remaining constant.
- Using the average cost basis instead of the full cost basis to provide a quantitative measure of the suboptimality generated by using the average cost basis.
- Letting f^l and f^u each depend on the ratio of the highest cost basis in the portfolio to the current stock price to quantify any advantage this generates.
- Incorporating transaction costs when we buy and sell stock to understand their effect on the optimal stock fraction range, $[f^l, f^u]$, in the presence of taxes.

3.3.1 Effect of a Time Dependent No-Rebalancing Region

We study the effect of allowing f^l and f^u to change values when we transition from the initial 20 years, when there is a long time until liquidation, to the final 20 years, when there is a short time until liquidation.

We must now optimize over five variables instead of three: f^{init} (the initial stock fraction), $f^{l,0-20}$ and $f^{u,0-20}$ (the values of f^l and f^u in the initial 20 years), and $f^{l,20-40}$ and $f^{u,20-40}$ (the values of f^l and f^u in the final 20 years). As a typical example of our results, in Figure 13 we show the values of $f^{l,0-20}$, $f^{u,0-20}$, $f^{l,20-40}$ and

$f^{u,20-40}$ in the context of changing τ_g.

Unsurprisingly, when the investor expires at the liquidation time T, the ability to change stock proportions makes a significant difference, as shown in the left side plot in Figure 13. Specifically, since capital gains are forgiven at time T, the stock is more valuable in the final 20 years, so $f^{*,20-40} > f^{*,0-20}$, and to create more capital gains to be forgiven, we also see that $\Delta f^{20-40} > \Delta f^{0-20}$.

Equally unsurprisingly, we find that when the investor is alive at time T, the ability to change stock proportions makes little difference, so $f^{l,0-20} \approx f^{l,20-40}$ and $f^{u,0-20} \approx f^{u,20-40}$, as shown in the right side plot in Figure 13. The only difference over time is at the portfolio horizon, when there is a required liquidation and associated capital gains tax hit. This required liquidation diminishes the advantage of having stock, because it restricts the ability to defer capital gains taxes, and so $f^{*,20-40}$, the optimal stock proportion held when we are closer to liquidation, dips a little from $f^{*,0-20}$, the optimal proportion when we are farther from liquidation.

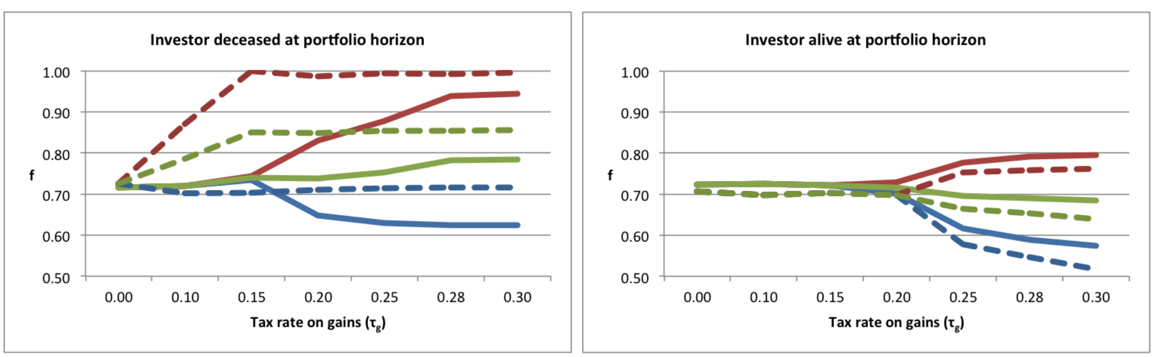


Figure 13: Optimal stock fraction range for the first and second half of the portfolio horizon when varying the tax rate on gains, τ_g. The entire horizon of the portfolio is 40 years. For the first 20 years stock fractions are shown in bold lines, and for the last 20 years they are shown in dashed lines. As before, the upper bound on the stock fraction is shown by a red line, the lower bound is shown by a blue line, and the average of these two bounds, f^*, is given by a green line.

3.3.2 Effect of the Average Cost Basis Versus the Full Cost Basis

As discussed in the introduction, this paper's simulation approach allows us to use the entire cost basis, resulting in a comprehensive solution to the portfolio problem with taxes. Preceding work that accounted for the full tax basis has been limited to very few allowable trading times and therefore was not suitable for modeling the entire long-horizon life cycle of an investor. Instead, for computational convenience in multiperiod optimization, most past work uses the average tax basis instead of the full tax basis. Using the average tax basis keeps the state space small, allowing a numerical solution of the dynamic programming problem. Of course, using the average costs basis, which is allowed in U.S. tax law for mutual funds, is guaranteed to be suboptimal compared to using the full cost basis.

But how sub-optimal is using the average cost basis?

Our large-scale algorithm enables us to directly compare the effect of using the full cost basis to the average cost basis for the long-horizon problem. To make this comparison, we reran each of the scenarios used to generate Figures 3–10 (except for Figure 5), using the average cost basis instead of the full cost basis. This allows us to compare, for example, the graphs in Figure 10, which use the full cost basis, to the graphs in Figure 14, which use the average cost basis. The observed changes between these figures where W_0 is varied, are quite similar to the observed changes when other parameters are varied.

Specifically, across our scenarios we found that using the average cost basis had very little effect on f^*, but it tended to increase Δf. The average increase in Δf

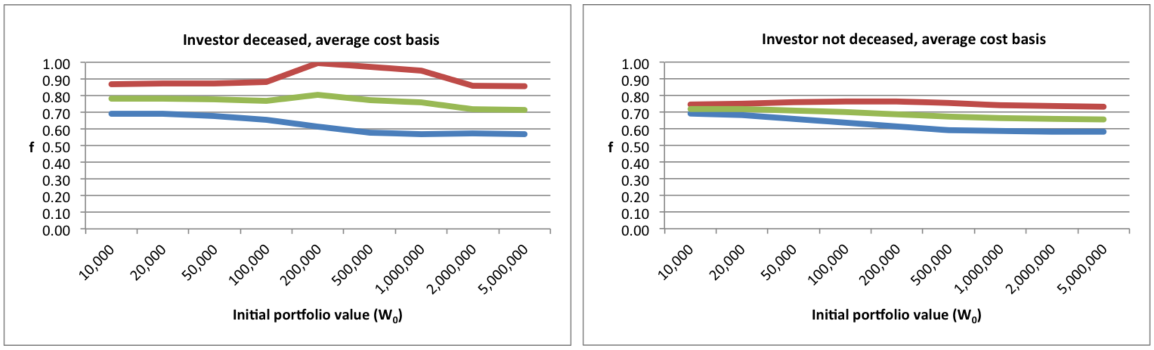


Figure 14: Optimal stock fraction range when the algorithm uses the average cost basis rather than the full cost basis, when varying the initial wealth, W_0. Note that since the distances between our experimental values of W_0 are not uniform, the grid on the horizontal axis is not uniform. The values of W_0 have been chosen so that the horizontal axis is close to logarithmic.

across our scenarios was 6.0 percentage points, both in the case where the investor is alive or expires at time T; the maximum increase in Δf was 16.8 percentage points when the investor was alive at time T and 18.6 percentage points when the investor expires at time T. The increase in Δf is explained by the fact that using the average cost basis makes losses in the portfolio less likely to occur, thereby reducing the advantage of keeping Δf small.

But the main concern when we switch to the average cost basis is not the change in the optimal policy. It is the loss to the investor created by this switch. To quantify this loss, we again use the certainty equivalent difference, which for our current circumstance, is

$$c = \left(\frac{E_{avg}[U(W)]}{E_{full}[U(W)]} \right)^{\frac{1}{1-\alpha}} - 1. \tag{4}$$

Applying equation (4) across each of our scenarios, we found that when the investor expires at time T, the average certainty equivalent difference, c, was only -0.27%, with a maximum certainty equivalent difference of -1.12%. When the investor is alive at time T, the average difference increases to -0.65%, with a maximum difference of -1.73%. For comparison, if we quantify the loss due to living at time T (no gains forgiven) versus expiring at time T (gains forgiven) over the same set of scenarios (using the full cost basis for both), the cost equivalent difference

$$\hat{c} = \left(\frac{E_{alive}[U(W)]}{E_{dead}[U(W)]} \right)^{\frac{1}{1-\alpha}} - 1.$$

averages -9.97% with a maximum difference of -23.6%.

The data in the above paragraph, by itself, gives considerable justification for past work that employs the average cost basis to determine trading strategies in taxable portfolios, since the loss generated by using the average cost basis in place of the full cost basis is clearly not that great. Yet the case for justifying these average cost basis models is even stronger: These models will generate an optimal f^* and Δf for the average cost basis, but, of course, these values for f^* and Δf would always be used with the full cost basis strategy in practice. So how much of the value of c in equation (4) is due to the f^* and Δf from the average cost basis model being suboptimal when the investor uses the full cost basis, as they would in practice, and how much is due to the effect of an investor actually using the average cost basis method versus using the full cost basis?

Consider the base case again. If the investor expires at $T = 40$, then, for the full cost basis optimization, we have that $f^* = 0.764$ and $\Delta f = 0.168$, while for the average cost basis optimization, we have that $f^* = 0.770$ and $\Delta f = 0.228$, which, using equation (4), corresponds to $c = -0.0019$. But recall from Subsection 3.1 that small changes to f^* and large changes to Δf, as we have here, usually have a small impact on the certainty equivalent difference. In fact if we compute

$$\tilde{c} = \left(\frac{E^a_{full}[U(W)]}{E_{full}[U(W)]} \right)^{\frac{1}{1-\alpha}} - 1,$$

where "E^a_{full}" means using the <u>full</u> cost basis with the values $f^* = 0.770$ and $\Delta f = 0.228$ that came from the <u>average</u> cost basis optimization, we get $\tilde{c} = -0.0003$. That is, the actual loss due to using the optimal strategy for the average cost basis model is much smaller than indicated above, specifically, it is only $\frac{\tilde{c}}{c} = \frac{3}{19}$ of the loss indicated above, as long as the full cost basis is actually employed for trading, as it would be by any investor interested in minimizing taxes.

The base case where the investor is alive at $T = 40$ shows less dramatic results. In this case, for the full cost basis optimization, we have that $f^* = 0.711$ and $\Delta f = 0$, while for the average cost basis optimization, we have that $f^* = 0.701$ and $\Delta f = 0.127$, which corresponds to $c = -0.0090$ and $\tilde{c} = -0.0060$. Therefore, $\frac{\tilde{c}}{c} = \frac{2}{3}$, where before it was only $\frac{3}{19}$. The higher value for $\frac{\tilde{c}}{c}$ is due to the optimal Δf now being zero for the full cost basis optimization, which, as seen in the right panels in Figure 1 and explained in Subsection 3.1, leads to a higher sensitivity to changes in Δf.

3.3.3 Effect of Allowing the No-Rebalancing Region to Vary with the Cost Basis to Current Stock Price Ratio

Papers such as Dai, Liu, Yang, and Zhong (2015) allow f^l and f^u to depend on b_{avg}, the ratio of the average cost basis to the current stock price, $\frac{B_{avg}}{S}$. Dai, Liu, Yang, and Zhong (2015) give examples in which optimizing the no-rebalancing interval, including optimizing its dependence on b_{avg}, leads to strategies that produce certainty equivalent advantages of between 0.84% and 5.20% over the strategy of deferring all short term gains and realizing all losses and long term gains. They do not consider the question we explore here: the certainty equivalent advantage created by allowing the no-rebalancing region to depend on the aforementioned ratio of the cost basis to the stock price versus not allowing such dependence.

Since our algorithm allows for the full cost basis, it makes little sense for us to employ b_{avg}. Instead, we define $b = \frac{B_J}{S}$, where, as before, B_J corresponds to the highest cost basis of the stock in the portfolio, which is the first stock that should be traded. We then define

$$\begin{aligned} f^l &= \max[0, c_1 - c_2 * \max[0, 1-b]] \\ f^u &= \min[1, c_3 + c_4 * \max[0, 1-b]], \end{aligned}$$

where we restrict all four c_i to be positive values and we restrict $0 \leq c_1 \leq c_3 \leq 1$. This model keeps $0 \leq f^l \leq f^u \leq 1$, and it lets Δf grow as b grows, creating the potentially desirable effect of discouraging selling stock as the capital gains implications of selling stock increase. Unlike Dai, Liu, Yang, and Zhong (2015), our model is restricted to f^l and f^u being linear functions of b, so the optimal dependence of f^l and f^u on b is only approximated here. In this new model, we optimize over five variables, f_{init}, c_1, c_2, c_3, and c_4, in place of optimizing over our normal three variables, f_{init}, f^l, and f^u. Since the subcase $c_2 = c_4 = 0$ in our five variable optimization yields our normal three variable optimization, we are guaranteed that the five variable optimization will be superior to our normal three variable optimization.

We considered the base case, both when the investor is deceased and alive at the portfolio horizon T. In either case, we found that using the five variable optimization versus the three variable optimization made no discernible difference. Specifically, in both cases the certainty equivalent difference between allowing (linear) dependence of f^l and f^u on b versus not allowing such dependence on b was less than a basis point over the course of $T = 40$ years, which is the limit of our model's ability to discern certainty equivalent differences. This evidence suggests that it is not necessary to use models where the boundaries of the no-rebalancing region, f^l and f^u, depend on b, the ratio of the (highest) cost basis to the current stock price.

3.3.4 Effect of Incorporating Transaction Costs

Finally, we investigate the effect of incorporating transactions costs into our model. This issue was also explored in Leland (2000), who found that transaction costs reduce optimal portfolio churn by 50%. He also found that capital gains taxes lead to lower investment in stock, and we find that this is not always the case.

Assume that for the current time t, the current number of shares of stock in the portfolio is N_t, the current stock price is S_t, the current worth of the portfolio's cash position is C_t, and the number of shares to be transacted at time t is n. Let e denote the proportion of the worth of a trade lost to transaction costs. Based on Domowitz, Glen, and Madhavan (2001) and Pollin and Heintz (2011), we assume transaction costs range from 0 to 50 basis points of the value of the transaction, i.e., we consider the e values 0 (our base case), $0.0005, 0.0010, \ldots, 0.0050$. These proportional transaction costs can be incurred at four places in the simulation:

1. When any stock has a capital loss, we sell it and buy back the same value of an equivalent stock, so our cash balance must be reduced by $2(nS_te)$, which corresponds to the transaction costs incurred both for selling and for repurchasing this stock. We note that while it is always optimal to sell and buy back stock with a capital loss when there are no transaction costs, this is no longer guaranteed to be the optimal strategy when we have transaction costs. However, for the small transaction costs we consider here, the investor will, generally, still be better off selling and buying back stock with a capital loss, so we continue to implement this strategy in our transaction cost model.

2. When the stock fraction falls below f_l, we buy stock so that the new stock fraction equals f_l *after* transaction costs. This must satisfy the following equation:

$$f_l = \frac{(N_t + n)S_t}{(N_t + n)S_t + C_t - nS_t - nS_te},$$

which, after rearrangement, leads to the following expression for the number of shares that will be bought:

$$n = \frac{N_tS_t(f_l - 1) + C_tf_l}{S_t(f_le + 1)}.$$

Using the subscript $(t+)$ to denote values just after rebalancing at time t, we then update the total number of shares of stock

$$N_{(t^+)} = N_t + n$$

and the cash balance

$$C_{(t^+)} = C_t - nS_t - nS_te.$$

3. When the stock fraction rises above f_u, we sell stock so that the new stock fraction equals f_u *after* transaction costs. This must satisfy the following equation:

$$f_u = \frac{(N_t - n)S_t}{(N_t - n)S_t + C_t + nS_t - nS_te},$$

which, after rearrangement, leads to the following expression for the number of shares that will be sold:

$$n = \frac{N_tS_t(f_u - 1) + C_tf_u}{S_t(f_ue - 1)}.$$

We then update the total number of shares of stock

$$N_{(t^+)} = N_t - n$$

and the cash balance

$$C_{(t^+)} = C_t + nS_t - nS_te.$$

4. Finally, at the end of the year, if there are capital losses, then the tax break generated by these losses is used to buy additional shares. This additional number of shares (after transaction costs) will be

$$n = \frac{\tau_l \cdot \min[3000, \max(0, -G)]}{S_t(1 + e)},$$

since $\$3,000$ is the annual limit allowed for taking tax losses and G, as before, represents the realized gains in the portfolio. Therefore, shares are only bought when $G < 0$; i.e., there are losses.

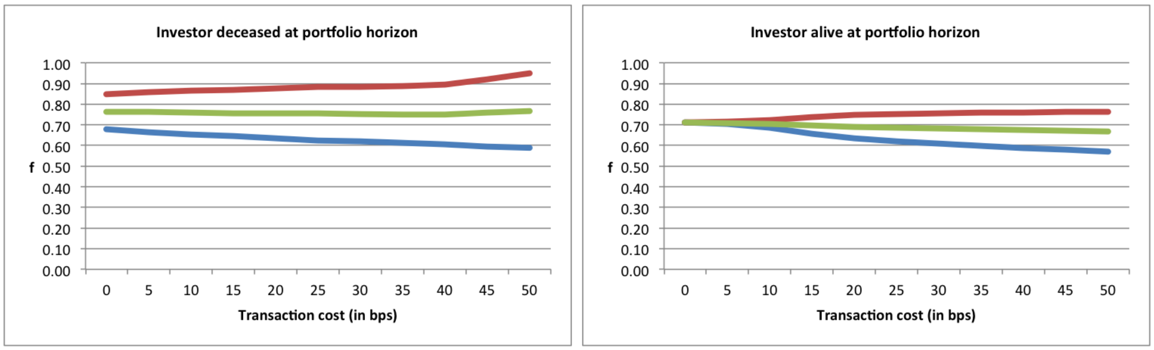


Figure 15: Optimal stock fraction range when varying transactions costs (in bps). These costs are stated in terms of the percentage of the value of each transaction lost to costs.

As expected, we see from Figure 15 that rebalancing occurs less often as transaction costs increase. That is, Δf increases as e increases. What is more surprising is that the optimal average stock fraction, f^*, is essentially unaffected by e, at least over the range of values for e considered here.

Considerable attention has been devoted to understanding the asymptotic effect of small proportional transaction costs in portfolios that are not subject to taxes. See for example, Atkinson and Mokkhavesa (2002), Goodman and Ostrov (2010), Janacek and Shreve (2004), Rogers (2004), Shreve and Soner (1994), and Whaley and Wilmott (1997), who all conclude that the order of growth of Δf is given by $\Delta f \sim O(e^{\frac{1}{3}})$ when e is small. What happens to this asymptotic expression in portfolios subject to taxes? The expression has no relevance to the case in the left panel of Figure 15 where the investor expires at time T, since $\Delta f \neq 0$ when $e = 0$ in this case. The case in the right panel of Figure 15, where the investor is alive at time T, has more potential to be connected to the asymptotic expression, since $\Delta f = 0$ when $e = 0$, and Δf grows as e grows. However, even in this case, it is clear from Figure 15 that Δf does not grow by order $e^{\frac{1}{3}}$, so the asymptotic formula for cases without taxes is not relevant to cases with taxes.

4 Conclusions

In this paper we have modeled the optimal trading strategy of a taxable portfolio with a stock position and a cash position in which we consider the effects of the American tax system, including keeping track of the cost basis of stocks whenever they are bought and sold, utilizing the tax benefits from stocks with losses, and using the annual \$3000 limit on claimable losses along with carrying over losses above this limit. We develop a large-scale simulated optimization program that offers facile computation on standard hardware, extending the number of trading periods into the hundreds (∼500) from earlier models (∼20 periods).

For our model, we have determined the optimal static interval $[f^l, f^u]$ within which to maintain f, the fraction of the portfolio invested in stock. When the portfolio is inside this interval, it is optimal to trade only to reap capital losses, not to rebalance. When the portfolio strays outside this interval, it is optimal to rebalance the portfolio back to the nearest endpoint, f^l or f^u. We note that a generalization of the static interval is feasible by making f^l and f^u functions of a state variable such as the tax basis.

Our experiments to determine this optimal interval provide a number of insights concerning the best strategies for investors to pursue in their taxable portfolios. A number of our conclusions differ from conventional wisdom and many investors' intuitions:

- The optimal value of f, the fraction of the portfolio in stock, is often *higher* for taxable accounts than for tax-free accounts like the Roth IRA (as also noted in Dammon, Spatt, and Zhang (2004)).

  This is the case even if the cash position is not taxed and if we assume that the investor is alive at liquidation, meaning that taxes on all capital gains are paid. This is explained by the fact that τ_l, the refund tax rate for capital losses, which is equal to the marginal income tax rate, is higher than τ_g, the (long term) tax rate for capital gains. Therefore, the benefit of culling capital losses from stock can, on average, outweigh the disadvantage of paying capital gains taxes, making the stock more useful in the taxable portfolio than in the tax-free portfolio. For example, in Subsection 3.1, we considered a base case where the investor is alive at the time of liquidation, and we found that the optimal value of $f = f^l = f^u$ was about four percentage points higher than the value of f_{Merton}, the optimal constant stock fraction in a tax-free portfolio.

- If the capital gains tax rate increases, then f, the fraction of the portfolio in stock, should be *raised*, not lowered.

  In the previous section, we establish this both experimentally and with an intuitive explanation. This conclusion assumes that the portfolio is designed to be given to a beneficiary, so that gains are forgiven at death, and it assumes that the stock position is larger than the cash position.

- The 5/25 rule for rebalancing taxable portfolios is less than ideal, even as a rule of thumb.

  The "5" part of the 5/25 rule corresponds to an interval width $\Delta f = f^u - f^l = 2 \times 0.05 = 0.10$. That is, it recommends using an interval for the stock fraction with a width of 10 percentage points. (The "25" part of the rule is irrelevant for our stock-cash model if f^*, the center of the optimal interval, is between 0.20 and 0.80.)

  But we find that there are common circumstances where the optimal $\Delta f = 0$ and almost equally common circumstances where the optimal $\Delta f > 0$. Moreover, because the effect of using a suboptimal Δf will generally be more harmful when the optimal $\Delta f = 0$ than when the optimal $\Delta f > 0$ (for reasons explained in Subsection 3.1), our analysis suggests that, for a rule of thumb, it makes more sense to adopt $\Delta f = 0$, in other words, adopt a strategy of continual rebalancing, if transaction costs are small.

- The optimal interval in which the stock fraction, f, should not be rebalanced is not improved by allowing it to get bigger as the stock price increases further from its purchase price.

  It is intuitive to think that since the capital gains taxes from selling stock get larger as the difference between the stock price and the cost basis grows, it should be optimal to let Δf increase as this difference grows so as to avoid these progressively costly sales. However, as we saw in the previous section, allowing f^l and f^u to linearly depend on b, the ratio of the highest basis price to the current stock price, leads to no discernible certainty equivalent advantage, suggesting that the attention to the dependence of the optimal f^l and f^u on b in previous papers, such as Dai, Liu, Yang, and Zhong (2015), is likely unnecessary.

Overall, our model shows that the optimal width, $\Delta f = f^u - f^l$, of the stock fraction interval...

- ...significantly increases when the stock's expected return, μ, increases or the risk free interest rate, r, decreases; when the tax rate for capital gains, τ_g, increases or the tax rate for capital losses, τ_l, decreases; when the initial portfolio size, W_0, increases; or when the lifetime of the portfolio, T, decreases.

- ...slightly increases when the time interval between potential transactions, h, increases or when the utility function risk aversion parameter, α, in equation (1) decreases.

- ...and is essentially unaffected when the stock's volatility, σ, changes.

and that the optimal portfolio stock fraction — or, more specifically, the optimal center, $f^* = \frac{f_l+f_u}{2}$, of the stock fraction interval...

- ...significantly increases when μ increases or when r, σ, or α decrease.

- ...slightly increases when τ_l increases; when W_0 decreases; when τ_g increases, assuming the investor expires at time T; or when τ_g decreases, assuming the investor is alive at time T.

- ...is essentially unaffected when h changes or, in the case where the investor is alive at time T, when T changes.

- ...and, in the case where the investor expires at time T, increases when T initially increases, but as T increases further, it levels off, then slowly decreases, and finally levels off again.

Finally, our model provides a number of other insights about optimal investing in taxable portfolios:

- Using a suboptimal value for f^* is generally far more detrimental to the investor than using a suboptimal value for Δf.

- The optimal strategy is almost completely unaffected by considering potential trading on a monthly versus quarterly versus semi-annual basis.

- When the optimal stock fraction interval, $[f^l, f^u]$, is allowed to depend on time, we see far more dynamic behavior for this interval when the investor expires at time T than when the investor is alive at time T. Unsurprisingly, when the investor expires at time T, both f^* and Δf increase over time. When the investor is alive at time T, f^* decreases slightly over time, while Δf stays essentially constant.

- While using the full cost basis history versus the average cost basis has a significant effect on the optimal stock fraction interval, $[f^l, f^u]$, it has a surprisingly small effect on the investor's outcome, generally leading to a certainty equivalent difference of less than 1 percent over 40 years. This justifies the use of the average cost basis approximation, which had to be employed by many previous papers using Bellman equation approaches to investigate optimal taxable portfolio strategies.

- As the magnitude of proportional transaction costs increases, the width, Δf, of the optimal stock fraction interval also, unsurprisingly, increases, however, the center, f^*, for this interval remains surprisingly constant.

There are a number of ways in which this research may be extended to generalize the setting in this paper, by capitalizing on the fact that we can solve an optimization problem over simulated portfolios with one risk free and one risky asset while tracking the full tax basis. First, we may extend this simulation approach to the case of multiple risky assets, where it may be able to clarify the geometry of the optimal no-rebalancing region. Second, the model may be extended to account for the difference in tax treatment between short term and long term capital gains. Third, the optimal evolution of the shape of the rebalancing region over time can be further explored. We leave these interesting avenues open for further research.

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