- Risk-sharing issues and the design of optimal derivative securities,
- Risk measurement, management, and control,
- Valuation and analysis of the options embedded in capital projects,
- Valuation and hedging of exotic options, and
- New areas for further development (such as natural resources and environmental economics).

We invite contributions in these and related areas.

The first issue of the *Review* reflects the broad orientation of the journal by presenting four papers at the frontiers of derivatives research. The first paper, by Sanjiv Das and Silverio Foresi, presents two models of interest-rate dynamics that combine the jump and diffusion components of interest-rate processes. They derive closed-form solutions for pricing bonds and tractable schemes for pricing contingent claims. The second paper, by Don Rich, examines the effect of default risk on the pricing of European options. The model he proposes yields some insights into the setting of margin requirements for exchange-traded options. The next paper by In-Joon Kim and George Yu derives valuation models for American options for a range of stochastic processes for the underlying asset. These results provide a useful analytical framework for modeling American-style options in a wide range of applications. The last paper in the issue, by George Pennacchi, Peter Ritchken, and L. Sankarasubramanian, links finite state models of the term structure with models that specify the pricing kernel. They identify a restriction in terms of investors' forecasts of the drift of the pricing kernel.

We would like to dedicate this first issue to the memory of Fischer Black, whose pioneering work laid the foundations of this field and has influenced the thinking of many researchers in financial economics, in general, and derivatives, in particular.

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Exact Solutions for Bond and Option Prices with Systematic Jump Risk

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Abstract. A variety of realistic economic considerations make jump-diffusion models of interest rate dynamics an appealing modeling choice to price interest-rate contingent claims. However, exact closed-form solutions for bond prices when interest rates follow a mixed jump-diffusion process have proved very hard to derive. This paper puts forward two new models of interest-rate dynamics that combine infrequent, discrete changes in the interest-rate level, modeled as a jump process, with short-lived, mean reverting shocks, modeled as a diffusion process. The two models differ in the way jumps affect the central tendency of interest rates; in one case shocks are temporary, in the other shocks are permanent. We derive exact closed-form solutions for the price of a discount bond and computationally tractable schemes to price bond options.

Keywords: Jump-diffusions, Bonds, Options

1. Introduction

One of the most interesting anomalies in option pricing is that bond options with different strike prices trade at prices that imply different volatilities of the interest-rate process. While this finding is puzzling in light of the diffusion models of interest-rate dynamics of Vasicek (1977) or Cox, Ingersoll, and Ross (1985b), this effect is to be expected if the there are "jumps" in interest rates. With jumps, not only the volatility but also higher order aspects of the distribution, such as the fact that interest rates are leptokurtic, become important in the determination of option values.

This observation, coupled with the realistic appeal of jumps, due for example to discrete changes of the central bank's interest-rate targets, often linked to currency markets' instability, suggest that modeling jumps may be quite important in pricing interest-rate contingent claims. To date, however, exact closed-form solutions for bond prices when interest rates follow a mixed jump-diffusion process have proved hard to derive. The contribution of this paper is to offer two new models of interest-rate dynamics that combine infrequent, discrete changes in the interest-rate level, modeled as a jump process, with short-lived, mean reverting shocks, modeled as a diffusion process, and based on these models, to derive exact closed-form solutions for discount bonds and computationally tractable expressions for other contingent claims, such as options on discount bonds.

Obviously, the leptokurtosis of interest rates and the "smile" effect in option prices could be due to variations in volatility, which is not accounted for by simpler diffusion models.

Table 1. The short rate: descriptive statistics

Data frequency	Mean	Variance	Skewness	Excess kurtosis	Number of observations
U.S.$ daily:					1264
r	6.3615	1.5886	0.03474	−0.94064	
dr	−0.00057	0.072291	0.57903	17.3382	
					396
	5.9153	7.5614	1.0614	1.3207	
	−0.00169	0.6530	−0.8620	8.64524	
					72
	5.0167	4.4914	0.0950	−1.3985	
	0.0291	0.0792	−0.0193	0.3209	
					72
	10.0984	14.0039	−0.0980	−1.5600	
	−0.1007	0.1252	−0.6953	4.4023	

Note: This table provides descriptive statistics at different frequencies. The short rate used is the one-month T-bill rate. U.S. dollar monthly data spans the period January 1960 to December 1992. Weekly and monthly data span the period January 1987 to December 1992. Weekly data is as of the last trading day of the week. Data have been obtained from Salomon Brothers and CRSP. Yen and Pound rates were obtained from Reuters and are for a period of six years from 1989 through 1994. Statistics are provided on the interest rates (r) as well as the changes in interest rates (dr). Raw data is in percentage terms.

Consider however how the kurtosis of interest rates changes as the sampling frequency changes. Table 1 shows that the distribution of daily changes in interest rates displays more kurtosis than the distribution of monthly changes in interest rates. This is a natural implication of a process with jumps, since the instantaneous probability of jumps does not depend on the sampling frequency. A stochastic volatility process, in contrast, has exactly the opposite implication, as the effects of stochastic volatility on higher moments of the distribution are inversely proportional to the sampling frequency: in a stochastic volatility model, kurtosis tends to disappear as the sampling interval shrinks.

A jump-diffusion model may capture this and other, higher-order, aspects of the distribution of the underlying asset that are difficult to accommodate in a diffusion setting. The focus of this paper is on pricing of interest-rate sensitive securities, with an eye to maintain tractability in pricing options on bonds. This interest in options is easy to understand. While disregarding the effect of jumps on the distribution of interest rates may result in comparatively small mispricing of discount bonds, mispricing is likely to become severe with options and other levered contingent claims.

We develop two models to price discount bonds and bond options in a jump-diffusion setting. The two models are different in the way jumps affect the conditional distribution of interest rates. In the first model, interest rates revert toward a constant central tendency, and their diffusion component displays constant conditional variance. This model for interest rates is much in the spirit of Vasicek (1977), and it reduces to the Ornstein-Uhlenbeck model considered by Vasicek (1977) when there are no jumps. Jumps are infrequent events that displace interest rates by discrete amounts but do not change their central tendency. The model of the jump component is rich enough to capture possible asymmetries in the size

and the probability of positive and negative shocks: this makes it particularly suitable to fit higher-order aspects of the distribution of interest rates, such as skewness and kurtosis. Within this paradigm, we derive an exact closed form solution for discount bond prices. We also derive an expression for the transition probability density function of interest rates. We then combine the expressions for the bond price and the probability density and develop a scheme for pricing options on bonds that is computationally attractive.

In the second model, jumps change the conditional central tendency of interest rates. Interest rates have two components—a high-frequency component, modeled as a diffusion, which represents short-lived deviations from the central tendency of interest rates, and a low-frequency component, due to jumps that change the central tendency. This model captures the notion that interest rates oscillate smoothly around a central tendency that changes infrequently—for example, because of the central bank intervention in setting new interest rate targets (Balduzzi, Bertola, and Foresi, 1993) or because of changes in the interest rate regime (Naik and Lee, 1994). While in the first model the interest rate is the only one, albeit complicated, factor, here the shock to the central tendency introduces a second factor in the term structure (Balduzzi, Das, and Foresi, 1995). To derive bond and option prices, we exploit a decomposition for the underlying process similar to the decomposition used by Merton (1976) to price options on stocks. In Merton (1976) the log stock price relative is the product of two independent random variables, one driven by a diffusion process and the other driven by a compound Poisson process. We posit a similar process for the logarithm of the pricing kernel, which is essentially the "underlying" in bond pricing, and derive exact closed-form expressions for discount bonds as well as a formula for bond options which is straightforward to implement numerically.

An important feature of our pricing results is that they reflect a remuneration for jump risk, which in our setting is systematic, as it affects both interest rates and the pricing kernel.

2. An interest-rate model with asymmetric jumps

In this section we consider a model in which there are discrete, infrequent changes in interest rates, which, absent these jumps, would be fluctuating smoothly around a fixed long-term central tendency. The interest rates process that we consider generalizes the Ornstein-Uhlenbeck process of Vasicek (1977), as we add a jump shock component to the standard diffusion shock. Jump-diffusion models of interest rates considered to date do not admit exact solutions for the price of discount bonds even in relatively simple cases, such as when jumps are of constant size or drawn from a normal distribution. Approximate solutions to the valuation problem are usually derived solving a linearized version of the fundamental valuation equation (see Ahn and Thompson, 1988). An interesting feature of the model we consider here is that it allows for realistic asymmetries in the probability of positive and negative shocks. We show that in this somewhat richer setting one can derive an exact closed form solution for the price of discount bonds.

As in Constantinides (1992) and Turnbull and Milne (1991), we start from a *pricing kernel*, m, a stochastic discount factor that regulates prices of contingent claims. The existence of such a pricing kernel is equivalent to the absence of pure arbitrage opportunities (see, e.g., Duffie, 1992). The time t price P of a contingent claim that entitles the owner to receive a cash flow H at time $t+\tau$ satisfies the pricing relation

$$P(t, H(t+\tau)) = E_t\left(\frac{m(t+\tau)}{m(t)} H(t+\tau)\right) \tag{1}$$

In the case of a discount bond paying off one dollar with certainty at maturity $t+\tau$, the price, $P(t,\tau)$, is

$$P(t,\tau) = E_t\left[\frac{m(t+\tau)}{m(t)}\right]$$

Assumption 1. *The pricing kernel m is given by*

$$\frac{dm}{m} = r\,dt + \lambda dz - \lambda_J h dt + \lambda_J\, dN_h, \tag{2}$$

with interest rate r

$$dr = a(b-r)\,dt + \sigma\, dz + J(\alpha, \psi)\, dN_h, \tag{3}$$

where $a > 0$ is the parameter that governs mean-reversion in interest rates, $b > 0$ determines the central tendency of the interest rate, σ is the standard deviation of changes of the interest rate, which is constant. λ determines the impact of a diffusion shock to the interest rate on the pricing kernel, and $\lambda_J < 1$ determines the impact of a jump on the pricing kernel. $J(\alpha,\psi)$ denotes the jump. The distribution of the jump size, in absolute value, is exponential (mean jump size $= 1/\alpha$), with $a\alpha > 1$; the sign of the jump is positive with probability ψ; dz is a standard Wiener process increment, and dN_h denotes Poisson arrivals independently and identically distributed, with constant arrival rate h. The components of the jump are distributed independent of each other and of the diffusion component.

The main feature of the process of interest-rate jumps summarized in equation (3) is that we model separately the uncertainty about the occurrence of a jump in the interest-rate level, and the direction and size of the jump. A second important feature of our modeling approach is that both jump and diffusion risk are systematic, in the sense that these shocks affect both the interest rate r and the pricing kernel m.

The process for r in Assumption 1 extends the Ornstein-Uhlenbeck model of interest-rate fluctuations used by Vasicek (1977). Absent jumps ($h = 0$), the interest-rate process reduces to the Ornstein-Uhlenbeck one, and in fact, in this model, as in the Ornstein-Uhlenbeck model, interest rates may become negative.

While it is natural to model the jump arrival as a Poisson-distributed random variable and the sign of the jump as binomial, the assumption that the jump size follows an exponential distribution plays a crucial role in making this model of jumps tractable.

It is well known that the expected rate of change of the pricing kernel is equal to minus the instantaneous interest rate, r (see, e.g., Cox, Ingersoll, and Ross, 1995a, Theorem 1, in a diffusion setting), $E\,(dm/m) = -r dt$, a property that the process for the pricing kernel in Assumption 1 satisfies by construction. It would be straightforward to obtain this particular

pricing kernel in general equilibrium, as the marginal utility of a representative investor in a production economy a la Cox, Ingersoll, and Ross (1985b)—where the representative investor has a time-separable logarithmic utility function and the production technology has constant returns to scale—by suitably choosing the stochastic process governing the mean and volatility of production (see, e.g., Constantinides, 1991; Sun, 1992).

2.1. *Bond pricing*

To derive bond prices, we calculate the following expectation:

$$P(t,\tau) = E_t\left[\frac{m(t+\tau)}{m(t)}\right]$$

Cox, Ingersoll, and Ross (1985a) show that, in a pure diffusion setting, this expectation could be calculated solving an appropriate "fundamental" partial differential equation. In the presence of jumps, the fundamental valuation equation becomes a partial difference-differential equation (Ahn and Thompson, 1988). We derive the fundamental valuation appropriate for our economy in three steps.

The first step involves using Ito's Lemma and the analogous lemma for the jump component (see, e.g., Kushner, 1967, p. 15) to calculate the instantaneous return on a bond:

$$dP = \left[a(b-r)]P_r + P_t + \frac{1}{2}\sigma^2 P_{rr}\right] dt + \sigma P_r\, dz + [P(r+J) - P(r)]\, dN, \tag{4}$$

where the subscripts denote derivatives with respect to the subscript variable. The instantaneous expected return on a bond $E(dP)$ is then

$$E(dP) = \left[a(b-r)]P_r + P_t + \frac{1}{2}\sigma^2 P_{rr}\right] dt + E([P(r+J) - P(r)]\, dN). \tag{5}$$

The last term can be rewritten as follows:

$$\begin{aligned} E([P(r+J,\tau) - P(r,\tau)]\; dN) &= E\,[P(r+J,\tau) - P(r,\tau) | \, dN = 1]\, E(dN) \\ &= E\,[P(r+J,\tau) - P(r,\tau) | \, dN = 1]\, h\, dt \\ &= qhP(r,\tau)dt, \end{aligned} \tag{6}$$

where q is the instantaneous expected rate of change of the bond price if a jump occurs,

$$\left[\frac{P(r+J,\tau) - P(r,\tau)}{P(r,\tau)}\middle| dN = 1\right]$$

Second, by equation (1), $E_t[\frac{m(t+\Delta t)P(t+\Delta t,\tau-\Delta t)}{m(t)P(t,\tau)}] = 1$, and taking the limit for $\Delta t \to 0$,

$$= \lim_{\Delta t\to 0} E_t\left[\frac{m(t+\Delta t)P(t+\Delta t,\tau-\Delta t)}{m(t)P(t,\tau)}\right]$$

we derive

$$0 = E\left(\frac{dP}{P}\right) + E\left(\frac{dm}{m}\right) + E\left[\frac{dm}{m}\frac{dP}{P}\right]$$

since $E\left(\frac{dm}{m}\right) = -rdt$, this gives the standard capital asset pricing relation

$$E\left(\frac{dP}{P}\right) - rdt = -E\left[\frac{dm}{m}\frac{dP}{P}\right] \tag{7}$$

The third step involves using equation (4) to calculate the $E[\frac{dm}{m}\frac{dP}{P}]$ and $E(\frac{dP}{P})$ to substitute into equation (7). Consider the premium, $-E[\frac{dm}{m}\frac{dP}{P}]$. Using equation (4), the independence of dN and the other jump components, and equation (6), we obtain

$$\left[\frac{dm}{m}\frac{dP}{P}\right] = \left(\lambda\sigma\frac{P_r}{P} + h\lambda_J q\right) dt. \tag{8}$$

In this economy there are two types of risk and, correspondingly, two risk premiums that explain the instantaneous excess returns, $E(\frac{dP}{P}) - rdt$. The first risk premium is due to diffusion risk, $\lambda\sigma\frac{P_r}{P}$, and is given by minus the covariance of the rate of change of the pricing kernel and the change of the diffusion component of the interest rate. The second risk premium is due to jump risk, $h\lambda_J q$, and is given by minus the expectation of the product of the rate of change of the pricing kernel due exclusively to the jump component, and the jump in the interest rate. Finally, by equations (5) and (6), the expected rate of change of the price is

$$E(dP) = \left[a(b-r)]P_r + P_t + \frac{1}{2}\sigma^2 P_{rr}\right] dt + qhP(r,\tau)dt. \tag{9}$$

Substituting equations (9) and (8) into (7) we derive our fundamental valuation equation for interest-rate sensitive securities:

$$[a(b-r) - \lambda\sigma]P_r - P_\tau + \frac{1}{2}\sigma^2 P_{rr} + (qh(1-\lambda_J) - r)P = 0. \tag{10}$$

The price of a discount bond solves the valuation equation (10) subject to the boundary condition $P(r, 0) = 1$.

Proposition 1 *When the interest rate and the pricing kernel follow the joint jump-diffusion process described in Assumption 1, the time t price of a discount bond that promises to pay one dollar at maturity $t + \tau$ is*

$$P(r,\tau) = e^{A(\tau) - rB(\tau)},$$

where

$$B(\tau) = \frac{1 - e^{-a\tau}}{a}$$

and

$$\begin{aligned} A(\tau) = {} & \frac{\sigma^2}{4a^3} - \frac{\sigma^2}{4e^{2a\tau}a^3} - \frac{a(\lambda\sigma - ab) + \sigma^2}{a^3} + \frac{a(\lambda\sigma - ab) + \sigma^2}{e^{a\tau}a^3} \\ & + \frac{[2h(1-\lambda_J)a^2 + 2\alpha h(1-\lambda_J)a^3 - 2a(\lambda\sigma - ab)]\tau}{2a^2(-1+\alpha a)(1+\alpha a)} \\ & + \frac{2\alpha^2 a^3(\lambda\sigma - ab) - 4\alpha h(1-\lambda_J)a^3\psi - \sigma^2 + \alpha^2 a^2\sigma^2]\tau}{2a^2(-1+\alpha a)(1+\alpha a)} \\ & - \frac{\alpha h(1-\lambda_J)\psi \ln(\alpha a)}{1+\alpha a} - \frac{[\alpha h(1-\lambda_J) - \alpha h(1-\lambda_J)\psi]\ln(\alpha a)}{\alpha a - 1} \\ & + \frac{\alpha h(1-\lambda_J)\psi \ln(1 - e^{-a\tau} + \alpha a)}{1+\alpha a} \\ & \frac{[\alpha h(1-\lambda_J) - \alpha h(1-\lambda_J)\psi]\ln(-1 + e^{-a\tau} + \alpha a)}{\alpha a - 1} \end{aligned}$$

Proof: We conjecture and verify that the bond price is of the form $P(t,\tau) = \exp(A(\tau) - r(t)B(\tau))$. Using the distributional assumptions on the sign and size of the jump in Assumption 1, we can write

$$q = \frac{\int_0^\infty [\psi\,(P[r+J,\tau] - P[r,\tau]) + (1-\psi)\,(P[r-J,\tau] - P[r,\tau])]\,\alpha\exp(-\alpha J)\,dJ}{P(r,\tau)}.$$

Using the conjecture that $P(r,\tau) = \exp(A(\tau)\quad rB(\tau))$, we can calculate the integral in the previous expression explicitly,

$$q(\tau) = \frac{B(\tau)(1-\psi)}{\alpha - B(\tau)} - \frac{B(\tau)\psi}{\alpha + B(\tau)}.$$

Take the appropriate derivatives, $P_r = -PB$, $P_{rr} = PB^2$, and $P_\tau = P(A_\tau - rB_\tau)$, substitute them into equation (10), and rearrange:

$$r[aB\quad B_\tau - 1] + [(\lambda\sigma - ab)B - A_\tau + \frac{1}{2}\sigma^2 B^2 + qh(1-\lambda_J)] = 0.$$

The problem is reduced to find a solution to the two ordinary differential equations obtained by equating each of the terms in square brackets to zero. The solution to the first ODE, subject to the appropriate boundary condition, gives $B(\tau)$,

$$B(\tau) = \frac{1 - e^{-a\tau}}{a},$$

which is identical to that in Vasicek's (1977) Ornstein-Uhlenbeck diffusion model. Inspection of the second ODE, shows that, since $B(\tau)$ and $q(B(\tau))$ are both functions of τ, the original conjecture for the bond price function is verified. Substituting explicit expressions for $B(\tau)$ and $q(\tau)$ in the second ODE, integrating, and using the appropriate boundary condition yields $A(\tau)$.

It is possible for the prices of pure discount bonds to be greater than 1. This is because, as in Vasicek, the interest rate can become negative. The most troubling case occurs as $\tau \to \infty$ and the bond price also goes to ∞. One way to prevent this from happening is to place restrictions on the parameter space; for example one could assume that the coefficient of τ in the solution of A above to be negative, i.e., the coefficients of τ in lines 2 and 3 of $A(\tau)$ are negative.

Our solution gives reasonable long bond prices only when mean reversion is strong enough relative to the variance of the process. To simplify matters, consider the case in which $\lambda = \lambda_J = 0$. To guarantee that the price does not diverge to infinity with the maturity of the bond, we need $ab > \sigma^2/2$ for the diffusion component. This condition is necessary to prevent infinite and positive bond prices even in the Vasicek model. The corresponding restriction for the jump component is $a\alpha > 1$. To see the relation to the condition on the diffusion component, recall that the mean of the jump is $1/\alpha$, and its variance is $1/\alpha^2$, so that we could rewrite the condition as $a(1/\alpha) > 1/\alpha^2$. When this inequality is violated, in a sense, the problem is worse than for Vasicek. For $a\alpha < 1$, there is some finite maturity $\overline{\tau}$ such that $(-1 + e^{-a\overline{\tau}} + \alpha a) = 0$, and inspection of the last term of the solution for $A(\tau)$ shows that our solution for the bond price becomes meaningless for $\tau \geq \overline{\tau}$.

To see what happens to long bonds when a and α become too small, consider the expected rate of return due to exclusively to jumps:

$$q(\tau) = \frac{B(\tau)(1-\psi)}{\alpha - B(\tau)} - \frac{B(\tau)\psi}{\alpha + B(\tau)} = (1 - e^{-a\tau})\left(\frac{1-\psi}{\alpha a - 1 + e^{-a\tau}} - \frac{\psi}{\alpha a + 1 - e^{-a\tau}}\right)$$

The sensitivity to interest rate changes becomes higher the smaller a, that is the weaker the mean reversion. The jump becomes larger the smaller α. A positive jump in rates causes only a small drop in price, since the price of long bonds is already low and it cannot drop below zero. A negative jump in rates causes a strong increase in price. This tends to make the expectation of returns due to jumps positive and large. This effect becomes stronger the longer the maturity:

$$\lim_{\alpha a \downarrow 1} \lim_{\tau \uparrow \infty} q(\tau) = \lim_{\alpha a \downarrow 1} \left(\frac{(1-\psi)}{\alpha a - 1} \quad \frac{\psi}{\alpha a +}\right) = \infty.$$

The solution is implemented, and Figure 1 displays a graph of the results for an indicative set of parameters. Inspection of the solution shows that $\lim_{r\to\infty} P(r, \tau) = 0, \ \ \forall \tau$. One can verify that the solution reduces to that of Vasicek (1977) when $h = 0$ and the probability of a jump arrival is zero. When $h > 0$, jumps and jump risk affect prices and yields of bonds of different maturity, but not the semielasticity of the bond price with respect to the level of the interest rate, providing a justification for immunization strategies based on a simple Vasicek (1977) model even in the presence of jumps.

To derive the bond price, we have calculated the expectation $P(t, \tau) = E_t\,[m(t+\tau)/m(t)]$ by solving the fundamental valuation equation (10); Cox, Ingersoll, and Ross (1985a) show

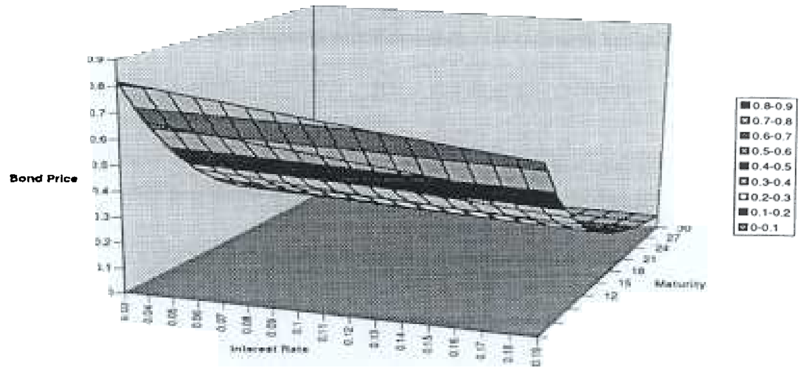

Figure 1. Jump-diffusion bond prices using the closed-form solution. This figure presents a plot of zero-coupon bond prices using the closed-form solution in Proposition 1. The prices of risk are assumed to be zero. The other parameters used are $a = 0.5$, $b = 0.13$, $\sigma = 0.14$; $\alpha = 200$, $\psi = 0.5$, $h = 15$.

that this is equivalent to calculate the expectation of the final payoff, discounted at the risk free rate r,

$$P(t, \tau) = E_t^*\left[\exp\left(-\int_t^{t+\tau} r(s)\,ds\right)\right]$$

where the expectation E^* is taken with respect to a risk-neutral probability measure, which in our case corresponds to the process

$$dr = [a(b - r) - \lambda\sigma]\,dt + \sigma\,dz + J(\alpha, \psi)\,dN_{h(1-\lambda_J)}, \tag{11}$$

where the diffusion risk premium can be viewed as changing the drift of the process, and the jump risk premium can be viewed as changing the arrival rate of the jump. The two procedures are equivalent because they give rise to the same partial differential equation. In the following section, we propose a scheme to price bond options that is based on this second procedure and that exploits the exact solution for bond price function and our knowledge of the distribution of interest rates. ■

2.2. *The distribution of interest rates with asymmetric jumps*

In this section we provide an expression for the characteristic function and probability density function of interest rates, conditional on the current interest-rate level. The procedure followed here is similar to that of Heston (1993) and Bates (1993). We concentrate on the distribution of the *risk-neutral* interest-rate process. By setting $\lambda = \lambda_J = 0$, our expressions reduce to those appropriate for the objective probability distribution of interest rates.

Proposition 2. *Under the risk-neutral process (11), the characteristic function of the probability distribution of the interest rate of* $r(t+\tau)$ *conditional on* $r(t)$ *is*

$$\phi(r(t),\tau;s)=\exp[\hat{A}(\tau;s)-\hat{B}(\tau;s)r(t)],$$

where

$$\begin{aligned}\hat{A}(\tau;s) &= is(ab-\lambda\sigma)\left(\frac{1-e^{-a\tau}}{a}\right)-s^2\sigma^2\left(\frac{1-e^{-2a\tau}}{4a}\right)\\ &\quad+\frac{ih(1-\lambda_J)(1-2\psi)}{a}\left[Arctan\left(\frac{s}{\alpha}e^{-a\tau}\right)-Arctan\left(\frac{s}{\alpha}\right)\right]\\ &\quad+\frac{h(1-\lambda_J)}{2a}\ln\left(\frac{\alpha^2+s^2e^{-2a\tau}}{\alpha^2+s^2}\right)\\ \hat{B}(\tau;s) &= -is\exp(-a\tau).\end{aligned}$$

Proof: Consider a function $\phi(r,\tau;s)$ that is twice-differentiable in the interest rate r and once-differentiable in τ; s is a constant. The Kolmogorov backward equation based on the risk-neutral process (11) is given by

$$\begin{aligned}&[a(b-r)-\lambda\sigma]\frac{\partial\phi}{\partial r}-\frac{\partial\phi}{\partial\tau}+\frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial r^2}\\ &\quad+h(1-\lambda_J)E[\phi(r+J,\tau;s)-\phi(r,\tau;s)|dN=1]=0.\end{aligned}$$

To obtain the characteristic function of the probability distribution of the interest rate, we solve this equation subject to the initial condition $\phi(r,0;s)=\exp(isr)$, for some constant s. We conjecture and verify a solution for the characteristic function of the form $\phi(r,\tau;s)=\exp[\hat{A}(\tau;s)-r\hat{B}(\tau;s)]$. Taking the appropriate derivatives and substituting them into the Kolmogorov backward equation above yields

$$[a\hat{B}+\hat{B}_\tau]+r[ab-\lambda\sigma](-\hat{B})-\hat{A}_\tau+\frac{1}{2}\sigma^2\hat{B}^2+\hat{q}(\hat{B})h(1-\lambda_J)=0,$$

where

$$\hat{q}(\hat{B}(\tau))=\frac{\hat{B}(\tau)(1-\psi)}{\alpha-\hat{B}(\tau)}-\frac{\hat{B}(\tau)\psi}{\alpha+\hat{B}(\tau)},$$

and the problem is reduced to find a solution to the two ordinary differential equations obtained by equating each of the terms in square brackets to zero, (1) $a\hat{B}+\hat{B}_\tau=0$, subject to $\hat{B}(0)=-is$, and (2) $\hat{A}_\tau=[ab-\lambda\sigma](-\hat{B})+\frac{1}{2}\sigma^2\hat{B}^2+\hat{q}(\hat{B})h(1-\lambda_J)$, subject to $\hat{A}(0)=0$. The solution to the first ODE gives $\hat{B}(\tau)$. Inspection of the second ODE confirms that the original conjecture is verified, since $\hat{B}$ and $\hat{q}$ are functions of τ only: substituting the solution for $\hat{B}$ into the second ODE, integrating, and using the appropriate boundary condition, provides the solution for $\hat{A}(\tau)$.

The conditional probability density function $f(r(t+\tau)\mid r(t))$ is obtained from the characteristic function using the inversion formula:

$$f(r(t+\tau)\mid r(t))=\frac{1}{\pi}\int_0^\infty \mathrm{Re}[\exp(-isr(t+\tau))\phi(r(t),\tau;s)]\,ds.$$

Once the probability density function has been calculated, it is possible to implement a scheme for pricing options. This is taken up in the following section. ■

2.3. *Option pricing*

The price of a derivative security $C(r(t),\tau)$ with payoff function $H(r(t+\tau))$ is the expected discounted value of the payoff under the risk-neutral measure,

$$C(r(t),\tau)=E_t^*\left[\exp\left(-\int_t^{t+\tau}r(s)\,ds\right)H(r(t+\tau))\right],$$

for a European-type option, and

$$C(r(t),\tau)=\sup_{t\le\hat{t}\le t+\tau}E_t^*\left[\exp\left(-\int_t^{\hat{t}}r(s)\,ds\right)H(r(\hat{t}))\right],$$

for an American-type option, where $\hat{t}$ is a stopping time defined on the interval $[t,t+\tau]$. In order to implement a scheme to price American-type securities, we utilize a discrete-time version of the above equation:

$$C(r(t),\tau)=\sup_{t\le\hat{t}\le t+\tau}E_t^*\left[\exp\left(-\sum_{s=1}^{(\hat{t}-t)/h}r(s)\,\Delta t\right)H(r(\hat{t}))\right].$$

This equation is implemented on a discrete-time, discrete-space, two-dimensional grid, using a discrete state space for the interest rate r, over a finite support, *i.e.*, $r\in[r_{min},r_{max}]$. The probability density function under the risk-neutral measure has been derived in the previous section. The probability of moving from $r(t)$ to $r(t+\Delta t)\in[r_{min},r_{max}]$ is then apportioned over the finite space grid using the approximation $[\Pr[r(t+\Delta t)\mid r_t]\approx f(r(t+\Delta t)\mid r_t)\,\Delta r]$, where Δr is the constant interval between points on the grid on the interest-rate axis. Once the probabilities have been established, payoffs can be computed and then backward recursion is used to evaluate the price of any American-type interest-rate derivative security. Boundary conditions are checked at each node on the grid during the process of backward recursion.

In Table 2, sample results of the scheme above are displayed. We use the model to price one-year call options on two-year maturity coupon bonds. The options are priced at a strike price of par, and are of the American type. The exercise of the option is undertaken cum accrued interest. Prices are reported for varying values of skewness, which is regulated by the probability of positive jumps ψ, jump size, whose average is inversely related to α, and jump frequency h.

Table 2. Option prices for the binomial-exponential jump-diffusion model

a. $h = 10$

	Jump Sign (ψ) (skewness)		
Jump Size (α)	0.2	0.5	0.8
	4.2584	1.8508	0.4893
	3.3181	1.7708	0.7518

b. $h = 20$

	Jump Sign (ψ) (skewness)		
Jump Size (α)	0.2	0.5	0.8
	6.8840	1.9211	0.0999
	5.0263	1.8197	0.2703

Note: The table reports call option prices on a \$100 coupon bond. The maturity of the options is one year and that of the bond is two years. Options are American and, if exercised, are paid for with accrued interest. The option is struck at par. The coupon on the bond is 10 percent. Option values are reported for a range of jump signs (skewness), jump size (kurtosis), and jump frequencies. The first panel uses low frequency of jumps $h = 10$ times per year, and the second panel reports values at a higher frequency of jumps, $h = 20$ times per year. α is the parameter for the mean jump size and the mean jump size is given by $\frac{1}{\alpha}$. The other parameters are $a = 0.5$, $b = 0.1$, $\sigma = 0.05$. The prices of risk are assumed to be zero.

The main features of our solution for option values can be summarized as follows. When jumps are asymmetric, the distribution of interest rates becomes skewed: with negative skewness ($\psi < 0.5$) call options are more valuable than when there is positive skewness ($\psi > 0.5$); in general, the option price is decreasing in skewness. Quite intuitively, option values increase with the jump frequency when there is negative skewness. When there is positive skewness, option values decrease as the jump frequency increases. For symmetric jumps (zero skewness), the higher the frequency of jumps, the greater the option value. Analogous effects occur when considering the jump parameter α (remember that $\frac{1}{\alpha^2}$ is the jump variance) which drives the variance of the jumps. As α increases, the jump variance decreases and for positive skewness, the effect of the jump decreases, raising the option price. The opposite effect occurs for negative skewness.

3. Infrequent changes in the central tendency

In this section we consider an economy in which interest rates fluctuate smoothly around a central tendency, which changes infrequently by discrete amounts. This provides an alternative yet analogous modeling approach to the one developed in the previous sections. The

model complements and extends the category of kernel pricing models (Constantinides, 1992).

Assumption 2. *The pricing kernel m is given by*

$$m(t) = \exp[-y(t) - X(t)],$$

with

$$\begin{aligned} dX &= (x + \lambda^2/2)\,dt + \lambda\,dz, \\ dx &= a(b - x)\,dt + \sigma dz, \\ y(t) &= y(0)\left[1 - h\mu t + \sum_{j=1}^{N(t)} J_j\right] \end{aligned}$$

where x is the drift of X, λ determines the impact of a diffusion shock to interest rate on the pricing kernel through X, $a > 0$ is the parameter that governs mean-reversion in x, $b > 0$ determines the central tendency of x, σ is the standard deviation of changes of x, and dz is the increment to a standard Wiener process. N is a Poisson counter, with arrival rate h. The random variables J_j are independently and identically distributed, with characteristic function $\phi(J)$. The rate of change of y is the sum of N jumps J, compensated by $h\mu t$. Further, we assume that the diffusion and the jump components are independent.

Absent jumps ($h = 0$), $-E(dm/m) = x\,dt = r\,dt$, and, given the dynamics for x, this model reduces to the Ornstein-Uhlenbeck model of Vasicek (1977); as in that model, interest rates may become negative.

3.1. *Bond pricing*

Proposition 3. *When the pricing kernel follows the process described in Assumption 2, the time t price of a discount bond that promises to pay one dollar at maturity $t + \tau$ is*

$$P(t, \tau) = \exp[A(\tau) - B(\tau)\,x(t)]\exp[-h\tau\,(1 - \phi(t) - y(t)\mu)],$$

where

$$\begin{aligned} A(\tau) &= \frac{(B(\tau) - \tau)(a(ab - \lambda\sigma) - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(\tau)^2}{\text{[illegible]}} \\ B(\tau) &= \frac{1 - e^{-a\tau}}{a}, \end{aligned}$$

and where

$$\phi(t) = \phi(J; -y(t)/i)$$

denotes the characteristic function of the jump J evaluated at $y(t)/i$, with i the imaginary number $i = \sqrt{-1}$.

Proof: Because of the independence of the jump and diffusion components,

$$P(t,\tau) = \frac{E_t[m(t+\tau)]}{m(t)}$$
$$= e^{y(t)h\mu\tau} E_t(\exp[X(t) - X(t+\tau)]) E_t\left(\exp\left[-y(t)\sum_{j=1}^{N(\tau)} J_j\right]\right),$$

and the problem of calculating the bond price is reduced to the problem of evaluating two expectations, $E_t(\exp[X(t) - X(t+\tau)])$ and $E_t(\exp[-y(t)\sum_{j=1}^{N(\tau)} J_j])$. We first compute $V[x(t),\tau] = E_t(\exp[X(t) - X(t+\tau)])$. Conjecture that $V[x(t),\tau] = \exp(A(\tau) - x(t)B(\tau))$, take the appropriate derivatives, $V_x = -VB$ $V_{xx} = VB^2$, and $V_\tau = V(A_\tau - xB_\tau)$, and substitute them into the appropriate partial differential equation. This yields

$$x[aB + B_\tau - 1] + \left[(\lambda\sigma - ab)B - A_\tau + \frac{1}{2}\sigma^2 B^2\right] = 0.$$

The problem is reduced to find a solution to the two ordinary differential equations obtained by equating each of the terms in square brackets to zero. The solution to the first ODE, subject to the appropriate boundary condition, gives B. Substituting the expression for B in the second ODE, integrating, and using to the appropriate boundary condition, we obtain A; obviously, $V[x(t),\tau]$ is the same as the bond price in the Vasicek (1977, p. 182, eq. (18) model, since our process for interest rates reduces to the Ornstein-Uhlenbeck one when the jumps are zero. For the jump-related expectation, we proceed as follows. Observe that $W(y(t),\tau) = y(t)\sum_{j=1}^{N(\tau)} J_j$ is a compound-Poisson process. Let $\phi(W(y(t),\tau); s) = E_t(e^{isW(y(t),\tau)})$ be the characteristic function of this compound-Poisson process. Since N follows a Poisson distribution with parameter $h\tau$, the moment generating function of $[N(t+\tau) - N(t)]$ is

$$g_N(\tau; s) = e^{-h\tau(1-s)}$$

Using the results in Karlin and Taylor (1981, p. 428, theorem 9.1) on the relation between characteristic functions and moment generating functions of compound processes, we can write $\phi(W; s) = g_N[\phi(J; s)]$, which is the moment generating function of the Poisson process evaluated at $\phi(J; s)$, the characteristic function of the jump,

$$E_t(e^{isW(y(t),\tau)}) = \exp[-h\tau(1 - \phi(J; s))].$$

To find the second expectation, $E_t(e^{-y(t)W})$, we need to evaluate this moment generating function at $s = -y(t)/i$, hence the result.

The yield to maturity on the pure discount bond is therefore affine in $x(t)$ and $\phi(t) + y(t)\mu$

$$\frac{\ln P(t,\tau)}{} = -\frac{A(\tau)}{} + \frac{B(\tau)}{\tau} x(t) + [1 - \phi(t) - y(t)\mu]\, h$$

(see Duffie and Kan, 1993, for a discussion of affine models of yields in a diffusion setting).

In our model, the instantaneous interest rate is

$$r(t) = x(t) + [1 - \phi(t) - y(t)\mu]\, h,$$

where x imparts short-lived fluctuations to interest rates, while changes in $[1 - \phi(t) - y(t)\mu]\, h$ can be viewed as a permanent shift of the central tendency of interest rates. Also note that, while the impact of x on yields becomes less important with maturity, the impact of $y(t)$ and $\phi(t)$ is the same for all maturities, which is to be expected, as changes in the interest rates due to $[1 - \phi(t) - y(t)\mu]\, h$ are permanent changes in the level of interest rates.

To make this model of bond pricing operational, we need to provide a reasonable description of the distribution of jumps. In the following, we look at an example based on the assumption of normally distributed jumps. ■

Proposition 4. *Assume that the pricing kernel follows the process described in Assumptions 2 and that the jump J is distributed normal with mean θ and variance γ^2. Then the time t price of a discount bond that promises to pay one dollar at maturity $t + \tau$ is*

$$P(t,\tau) = \exp[A(\tau) - B(\tau)x(t)] \exp\left[-h\tau\left(1 - y(t)\mu - e^{-\theta y(t) + \gamma^2 y(t)^2/2}\right)\right],$$

with $A(\tau)$ and $B(\tau)$ as given in Proposition 3.

Proof: The characteristic function of a normal distribution with mean θ and variance γ^2 evaluated at $s = -y(t)/i$ is

$$\phi(J; s) = e^{-\theta y(t) + \gamma^2 y(t)^2/2};$$

then the result follows from Proposition 3.

We can assess the effect of the uncertainty about the magnitude and direction of jumps on yields using the approximation $1 - y\mu - e^{-\mu y + \gamma^2 y^2/2} \approx (\theta - \mu)y - \gamma^2 y^2/2$. When $\gamma \to 0$, the jump tends to a constant. Since $1 - y\mu - e^{-\theta y} \approx (\theta - \mu)y$, jumps increase (decrease) the central tendency of interest rates if $\theta - \mu > 0$ (< 0), assuming that $y(0) > 0$. When $\theta = \mu$, y is a martingale, and the effect of jumps on yields is through uncertainty only: the variance of the jump tends to increase the price of bonds and to drive yields down. ■

3.2. *Option pricing*

In what follows, we exploit the bond pricing solution of Proposition 4 to develop an option pricing formula that is easy to implement numerically.

We define $C_0(t,\tau)$ to be the time 0 price of a European call option with maturity t, on a bond with maturity $t+\tau$—that is, a bond that will have maturity τ when the option expires. The exercise price is X. The price of the option,

$$C_0(t,\tau) = E_0\left[\frac{m(t)}{m(0)} \max[0, P(t,\tau) - X]\right]$$

can be found as follows. Define $y_n(t)$ to be the value of $y(t)$ when exactly n jumps occur in the interval $[0, t]$, with $Prob(N = n) = \frac{e^{-ht}(ht)^n}{n!}$. Note that, conditional on $y_0(t)$, $y_n(t)$ is normally distributed.

Employing this structure we now present the option valuation formula:

Proposition 5. *Assume that the pricing kernel follows the process described in Assumption 2 and that the jump J is distributed normal with mean θ and variance γ^2. Then the time 0 price of a European call option with a strike price X, which expires in t periods and which is written on a bond which will pay off one dollar in $t + \tau$ periods, is*

$$\begin{aligned}
C_0(t,\tau) &= \sum_{n=0}^{\infty} \frac{e^{-ht}(ht)^n}{n!} \int_0^{\infty} F[y_n(t)]D[y_n(t),\tau]C_n[t,\tau,K_n]d\mathcal{N}[y_n(t)],\\
F[y_n(t)] &= \exp[-y_n(t) + y(0)],\\
D[y_n(t),\tau] &= \exp[-h\tau(1 - e^{-\theta y_n(t) + \gamma^2 y_n(t)^2/2} - y_n(t)\mu)],\\
C_n[t,\tau,K_n] &= V(x(0), t+\tau)N(d_1) - K_n V(x(0), t)N(d_2),\\
K_n &= \frac{1}{D[y_n(t),\tau]}X,\\
V(x(t),\tau) &= A(\tau)\exp[-x(t)B(\tau)],\\
d_1 &= \frac{1}{\sigma_V}\ln\left[\frac{V(0,t+\tau)}{V(0,t)K_n}\right] + \frac{\sigma_V}{2} \qquad d_2 = d_1 - \sigma_V,\\
\sigma_V^2 &= \frac{\sigma^2}{2a}(1 - e^{-2a\tau})B(\tau)^2,
\end{aligned}$$

$\mathcal{N}[y_n(t)]$ denotes the cumulative distribution function of $y_n(t)$, which, conditional on $y_0(t)$, is a normal random variable with mean and variance

$$E_0[y_n(t)] = y(0)[1 - h\mu t + n\theta], \quad Var_0[y_n(t)] = n\gamma^2 y(0)^2,$$

and $N(d)$ denotes the cumulative distribution function of a standard normal variable.

Proof: The call option price is

$$C_0(t,\tau) = \sum_{n=0}^{\infty} \frac{e^{-ht}(ht)^n}{n!} E_0^{x,y_n}\left(\frac{m_n(t)}{m(0)}\max[0, P_n(t,\tau) - X]\right), \tag{12}$$

where $m_n(t)$ is the pricing kernel when $N(t) = n$, *i.e.*, when exactly n jumps occur in the interval $[0, t]$

$$\frac{m_n(t)}{m(0)} = \exp[-X(t) + X(0) - y_n(t) + y(0)],$$

and $P_n(t,\tau)$ is the discount bond price at time t, when exactly n jumps occur

$$P_n(t,\tau) = V(x(t),\tau)D[y_n(t),\tau].$$

Since we can write the pricing kernel as

$$\frac{m_n(t)}{m(0)} = F[y_n(t)]\exp[-X(t) + X(0)],$$

the typical expectation in equation (12), $E_0^{x,y_n}(\frac{m_n(t)}{m(0)}\max[0, P_n(t,\tau) - X])$, is

$$\begin{aligned}
&E_0^{x,y_n}\left(\frac{m_n(t)}{m(0)}\max[0, P_n(t,\tau) \quad X]\right)\\
&= E_0^{x,y_n}\left(\frac{m_n(t)}{m(0)}\max[0, V(x(t),\tau)D_n[y_n(t),\tau] - X]\right)\\
&= E_0^{x,y_n}(F[y_n(t)]\exp[-X(t) + X(0)]\max[0, V(x(t),\tau)D_n[y_n(t),\tau] - X])\\
&= E_0^{x,y_n}\Bigg(F[y_n(t)]\exp[-X(t) + X(0)]D_n[y_n(t),\tau]\max\\
&\qquad \times \left[0, V(x(t),\tau) - \frac{X}{D_n[y_n(t),\tau]}\right]\Bigg)\\
&= E_0^{y_n}\Bigg(F[y_n(t)]D_n[y_n(t),\tau]E_0^x\Bigg[\exp[-X(t) + X(0)]\max\\
&\qquad \times \left[0, V(x(t),\tau) - \frac{X}{D_n[y_n(t),\tau]}\right]\Bigg]\Bigg)\\
&= E_0^{y_n}(F[y_n(t)]D_n[y_n(t),\tau]C_n[t,\tau,K_n]),
\end{aligned}$$

where the last step uses the exact bond option price solution due to Jamshidian (1989), but with the exercise price $(\frac{X}{D[y_n(t),\tau]})$, which varies with the number and size of jumps. Substituting this expectation in equation (12) gives the result.

This expression for the option price is computable numerically from the initial values of the state variables, $x(0)$ and $y(0)$. ■

4. Conclusions

In this paper we develop two models to price interest-rate derivative securities when the underlying interest rate may be subject to jumps. We derive exact solutions for the price of pure discount bonds and tractable expressions for bond option prices. Exact expressions obtain because while jumps affect interest rates and state prices, we need not know the exact time of occurrence of the jump to calculate the bond price: unlike other models considered to date, the "location" of the jumps is not essential. In the first model, the diffusion component of interest rates displays constant conditional variance and jumps are infrequent events that displace interest rates by discrete amounts but do not change their central tendency. In the second model, jumps change the conditional central tendency of interest rates. In this second model, interest rate have two factors or components, a high-frequency component, modeled as a diffusion, which represents short-lived deviations from the central tendency of interest rates, and a low-frequency component, due to jumps that change the central tendency. This model captures the notion that interest rates oscillate smoothly around a central tendency that may change infrequently. An important feature of our solutions for bond and option prices is that they reflect a remuneration for systematic diffusion and jump risk.

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The Valuation and Behavior of Black-Scholes Options Subject to Intertemporal Default Risk

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Abstract. This paper addresses the valuation and behavior of European options subject to intertemporal writer default risk. The framework allows the timing of default and recovery value to be uncertain. Default is said to occur if the writer's creditworthiness violates a specified critical level—both stochastic. Various recovery scenarios are considered including linking recovery to the moneyness of the option at the time of default. In an application of the model, it is estimated that current customer margin requirements for exchange-traded options are set far in excess of the fair market value.

Keywords: Default, Creditworthiness, Options, Margin Requirements, Risk Management, Default Premium, Hedging, Derivatives, Forwards

> *If the final product of the efforts of the financial theorists was only an assemblage of abstractions, those abstractions are the essential insights into how people do act and how people should act as they engage in the competitive battle. Mere abstractions cannot tell investors whether to buy or sell—in the end, that secret remains hidden from us—but they can tell us how to manage our affairs so that the uncertainties of human existence do not defeat us.* —*Bernstein (1992, p. 306)*

1. Introduction

Assuming away default risk is certainly tenable for exchange-traded options because the back-up system used by the Options Clearing Corporation virtually insures that contracts will be honored.[1] But what about customized or over-the-counter (OTC) options, which are widely used today and are not guaranteed against writer default?[2] For firms using these instruments, accurate assessment of writer creditworthiness is key to maintaining a successful hedging position; if the writer defaults, the hedge is worthless. Herein lies the practical need for a theory that draws on the brilliant Black and Scholes (1973) formulation but incorporates the important feature of credit risk. As one prominent practitioner put it: "Although not yet the case, it seems likely that in the future the credit rating of a writer will be an explicit determinant of an OTC stock index option's price" (Hudson, 1991). The purpose of this paper is to value and analyze European options written by economic agents subject to intertemporal default risk.

Formulating a theory for risky options is controversial because the terms of, and conditions triggering, default have to be known and agreed upon by all parties. Two approaches have been proposed: (1) allowing the writer to default at any time (Hull and White, 1995) and (2) allowing default to occur only at specific points in time (Johnson and Stulz, 1987;