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Of Smiles and Smirks: A Term Structure Perspective

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Abstract

An extensive empirical literature in finance has documented not only the presence of anomalies in the Black-Scholes model, but also the term structures of these anomalies (for instance, the behavior of the volatility smile or of unconditional returns at different maturities). Theoretical efforts in the literature at addressing these anomalies have largely focused on two extensions of the Black-Scholes model: introducing jumps into the return process, and allowing volatility to be stochastic. We employ commonly used versions of these two classes of models to examine the extent to which the models are theoretically capable of resolving the observed anomalies. We find that each model exhibits some term structure patterns that are fundamentally inconsistent with those observed in the data. As a consequence, neither class of models constitutes an adequate explanation of the empirical evidence, although stochastic volatility models fare somewhat better than jumps.

I. Introduction and Summary

It is widely acknowledged today that financial data from currency and equity markets differ in systematic ways from the model of Black and Scholes (1973). Two anomalies have been particularly well documented: i) the presence of a greater degree of excess kurtosis in *unconditional* returns distributions than is consistent with normality; and ii) the presence of an implied volatility smile or skew in options data that indicates the existence of excess kurtosis (and possibly skewness) in the *conditional* returns distribution. An extensive empirical literature has also documented the term structures of these anomalies (i.e., the manner in which they change with maturity), and has identified some striking common patterns that appear to hold across markets; we review these findings in Section I.C.

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Attempts in the finance literature at reconciling the theory with the data have mostly centered around two approaches, both of which involve generalizations of the basic Black-Scholes framework. One, the class of *jump-diffusion* models, augments the Black-Scholes returns distribution with a Poisson-driven jump process. The other, the class of *stochastic volatility* models, extends the Black-Scholes model by allowing the volatility of the return process to itself evolve randomly over time.[1]

It is easy to see intuitively how jump-diffusions and stochastic volatility models could each lead to returns distributions that exhibit both skewness and excess kurtosis. It appears plausible, therefore, that for a *given maturity*, each class of models could be made consistent with observed degrees of deviation from the Black-Scholes model. It is less obvious whether the theoretical predictions of either class of models are—or can be made—consistent with the observed *term structures* of these deviations. This question is of obvious importance in judging how well a model fits data from a given market and especially in pricing options in that market.[2] Yet, despite the vast empirical work on this topic, it has not been investigated in the literature. It forms the subject matter of this paper.

Identifying the term structures of skewness, kurtosis, and the implied volatility smile that can result in either class of models also serves two other important purposes. First, it provides a simple method for distinguishing between the two classes of models on the basis of their empirical implications and, ipso facto, for judging the better model in a given setting. Second, it has been pointed out (see, e.g., Duffie and Pan (1997)) that knowledge of the term structures of conditional skewness and kurtosis is central to the construction of models for measuring such quantities as value-at-risk; our results should help in this direction.

Our results and their implications are described in Sections I.A and I.B that follow. Section I.C then relates our paper to others in the literature.

A. Main Results: A Synopsis

We employ commonly used versions of either class of models to answer the questions raised above. In a nutshell, our central finding is that while each class of models does well at addressing some of the documented anomalies, there are also basic and important inconsistencies between the implications of either model and empirical patterns in the data. As such, neither model is capable of providing an adequate description of the data. On the whole, however, we find that stochastic volatility models fare better than jump-diffusions.

The key to analyzing the consistency of either class of models with observed patterns in the data lies in the term structures of skewness and kurtosis in each model. For *unconditional* returns, this is obvious, since empirical departures from Black-Scholes in this case are stated precisely in these terms. Characterizing *con-*

[1]For work on jump-diffusions, see, e.g., Ahn (1992), Amin (1993), Ball and Torous (1985), Bates (1991), (1996), Das and Foresi (1996), Jarrow and Rosenfeld (1984), or Merton (1976). On stochastic volatility, see Amin and Ng (1993), Heston (1993), Hull and White (1987), Melino and Turnbull (1990), Nandi (1998), Stein and Stein (1991), and Wiggins (1987).

[2]If a model's term structure implications differ qualitatively from observed patterns in a given market, the model cannot reasonably be used to price options in that market. Otherwise, parameter estimates made at one frequency will lead to systematic errors in pricing at other maturities.

ditional skewness and kurtosis analytically will enable us to identify the factors that drive the way implied volatilities change with maturity in either model. This is especially important since closed-form solutions are not often available for option prices in either class of models.

As the first step in our analysis, therefore, we derive closed-form solutions for skewness and kurtosis in both classes of models, conditionally and unconditionally. These closed-forms are used to provide a detailed analytical characterization of the dependence of these quantities on the models' parameters, including, especially, the length of the horizon over which they are calculated. Using numerical techniques, we then derive implied volatilities in either class of models for a range of maturities and parameterizations, and relate the behavior of these implied volatilities to the behavior of skewness and kurtosis in the model.

Concerning jump-diffusions, we find that such models are capable of reproducing observed patterns of skewness and kurtosis at short maturities under reasonable parameterizations. Consistent with empirical patterns, moreover, the degree of skewness and excess kurtosis in a jump-diffusion declines as maturity increases. However, this dissipation is far more rapid than would be suggested by the data. As a result, jump-diffusions can generate realistic implied volatility smiles at short maturities, but not at long maturities: the implied volatility smile flattens out too quickly. Finally, we find that the term structure of implied volatilities of at-the-money forward options in a jump-diffusion is always an *increasing* function of the time-to-maturity. This puts the model at odds with the data, which suggest that decreasing or non-monotone term structure patterns frequently arise in practice (e.g., the term structure for S&P 500 index options that prevailed in October/November 1997).

Unlike jump-diffusions, we find that stochastic volatility models are *not* capable of generating high levels of skewness and kurtosis at short maturities under reasonable parameterizations. As a consequence, stochastic volatility models cannot generate implied volatility smiles as sharp as those typically observed empirically. Moreover, conditional skewness and kurtosis in stochastic volatility models are always hump-shaped in the length of the horizon;[3] indeed, for plausible parameter values, both quantities must be *increasing* over short to moderate maturities. This implies, contrary to the data, that the smile does not flatten out appreciably as maturity increases. On the question of the term structure of implied volatilities of at-the-money forward options, however, stochastic volatility models do well. We find that a variety of patterns are possible in this model: increasing, decreasing, or even non-monotone (for example, U-shaped). Thus, qualitatively speaking, these models are able to capture an important aspect of financial market data better than jump-diffusions.

One additional finding bears mention. Unlike the substantial differences in the behavior of conditional kurtosis in the two models, our closed-form solutions show that *unconditional* kurtosis in both models could be a monotone decreasing

[3]This result corrects a misstatement in Bates (1996) and Das and Foresi (1996) that conditional excess kurtosis in stochastic volatility models is an increasing function of maturity. The difference is non-trivial. We show that there are parameter configurations for which the hump occurs sufficiently early that the curve becomes empirically indistinguishable from an everywhere *decreasing* function. However, these values are implausible.

function of maturity. This result is surprising and unintuitive, especially in light of the fact that conditional kurtosis in stochastic volatility models can never be a monotone function of maturity. Moreover, since decreasing unconditional kurtosis is the pattern most commonly observed in the data (see Section I.C), both models could be consistent with the unconditional data. Thus, our results also suggest that conditional (e.g., options) rather than unconditional data should be used to judge the better model in a given setting.

Finally, we should qualify our results in two directions. First, through a large part of the paper, we work with specific commonly employed versions of the respective classes of models (viz., lognormally distributed jumps and a square-root process for volatility). However, our results have strong intuitive underpinnings and we do not believe they depend unduly on these distributional choices. Second, in either class of models, the derivation of option prices and implied volatilities requires invoking the respective risk-neutral distributions. The question of the appropriate transformation from the original to the risk-neutral world is a non-trivial one in each case; we take as our guide in this process the equilibrium model of Bates (1996), and adopt the risk-neutralized distributions derived there. Once again, we do not believe this affects the nature of our results in any significant manner; indeed, the behavior of implied volatilities in both models accords extremely closely with the analytical properties we derive for the term structures of conditional skewness and kurtosis.

B. Implications

Empirical examination has established very strong evidence in favor of time-varying volatility (see, for example, the survey of Bollerslev, Chou, and Kroner (1992)). On the other hand, a number of studies have also found that markets are characterized by the presence of jumps, especially when data at short frequencies are used (e.g., Ball and Torous (1985), Drost, Nijman, and Werker (1995), Jarrow and Rosenfeld (1984)). Moreover, in a direct comparison of the alternatives, Jorion (1988) finds that the jump sub-model does better than its stochastic volatility counterpart, though the combined model does better still. Partially echoing these results, Bates (1996) finds that the stochastic volatility sub-model can explain the volatility smile only under parameter values that are implausible given the time-series properties of implied volatilities, and that the jump sub-model does better in this regard. Our results are consistent with all of these studies. Indeed, since a primary message of the paper is that the two models speak to different aspects of the data, our results even offer a post-hoc rationalization of these findings.[4]

Taking all this into account, the best solution appears to be the use of a model that combines time-varying volatility with jumps. Such a model would, for ex-

[4]An alternative, stronger interpretation of our results is also possible. Our entire analysis in this paper concerns models that directly extend the "normal returns" model of Black and Scholes (1973). This was a deliberate choice on our part, since such models are, far and away, the most commonly employed ones in the literature. However, our results are also qualified by this modeling strategy. Thus, one could view our results as saying that returns processes in financial markets are fat-tailed and skewed in a more fundamental sense than can be captured by simply introducing these features into a model that has neither of these effects. We would not disagree with this interpretation, but would not like to push it without further research into either class of models.

ample, be able to generate adequate kurtosis at short maturities (through the jump component) and at moderate maturities (through the stochastic volatility component). It would also be able to account for a wide range of shapes for the term structure of implied volatilities of at-the-money forward options. Unfortunately, such a choice would not be a parsimonious one. In this context, our results offer at least two possible ways to form a preliminary inference about the suitability of either class of models.

First, it is possible to extract information on *conditional* skewness and kurtosis from option prices using a variety of techniques, and then to use our results on the sharply differing possible patterns of conditional skewness and kurtosis in the two models. One such technique is the use of Gram-Charlier approximations as described in Jarrow and Rudd (1982). Backus, et al. (1997) carry out this procedure using data on $/DM option prices. They conclude that the resulting downward sloping kurtosis curve in their paper is consistent with a jump-diffusion, but not with a stochastic volatility model. Campa, Chang, and Rieder (1997) describe and implement a number of alternative techniques in this regard; see also Jackwerth and Rubinstein (1996), (1998) and Rosenberg (1996).

Second, we find that certain term structure patterns (decreasing or U-shaped) for implied volatilities of at-the-money forward options cannot be consistent with a jump-diffusion model. This offers a direct route for judging the appropriateness of jump-diffusions in a specific context, since data on term structures of implied volatilities for nearest-the-money options are relatively easy to get.

Two additional consequences of this paper may be of some importance, both related to the closed-form expressions we provide for the higher-order moments in the two classes of models. First, these closed-forms provide identification conditions that should enable efficient econometric testing using method-of-moments models. Second, these closed-forms should also find use in the construction of models for estimating measures of risk (such as value-at-risk) in a portfolio of derivatives (see Duffie and Pan (1997)).

C. The Related Literature

An extensive literature in finance has documented that excess kurtosis in unconditional returns declines with an increase in the length of the holding period. Early references to this phenomenon may be found in Kon (1984) and elsewhere; for instance, while Fama (1965) identified the presence of fat tails in stock returns using daily data, Blattberg and Gonedes (1974) found that the normality assumption fits stock returns well if monthly data are used. More recently, Jorion (1988) reports that excess kurtosis in the $/DM exchange rate and in the value-weighted CRSP index are, respectively, 3.29 and 2.92 under weekly data, but fall to 1.56 and 0.89 under monthly data. Bates (1996) contains references to other studies that have reported similar results.

The behavior of implied volatilities at different maturities has similarly received extensive attention.[5] Two important findings stem from this literature.

[5]In an early reference to this issue, Black (1975) discusses the tendency of the Black-Scholes model to overprice short maturity at-the-money options. More recent work includes Backus, et al. (1997), Bodurtha and Courtadon (1987), Campa and Chang (1995), Derman and Kani (1994), Heynen,

First, it appears indisputable that the implied volatility smile in most markets is deepest at short maturities and flattens out monotonically as maturity increases.[6] Second, a substantial literature has focused on an examination of the term structure of implied volatilities of at-the-money forward options. A variety of shapes (increasing, decreasing, and even sometimes non-monotone) for this curve have been documented in virtually all equity and currency markets.

Although the patterns of these anomalies have been known for some time now, relatively few papers have attempted to derive theoretical implications of asset price processes at varying time frequencies, especially using higher-order moments. Two recent such papers are Backus, et al. (1997), and Drost, Nijman, and Werker (1995). The former uses Gram-Charlier approximations to examine the impact of skewness and kurtosis on option prices, volatility smiles, and the term structure of implied volatilities. The latter models security prices as evolving according to continuous-time GARCH diffusions (cf., Drost and Nijman (1993)), and derives an overidentifying relationship between the variance and kurtosis that must be satisfied at any frequency.

It is also necessary to distinguish our approach from the literature that has studied the term structure of implied volatilities for at-the-money forward options (see, e.g., Stein (1989), Heynen, Kemna, and Vorst (1994), Campa and Chang (1995), and Xu and Taylor (1994)). One important difference is that each of these papers takes as given an underlying model of time-varying volatility, and is concerned with an empirical aspect of the term structure; we are, in contrast, interested in the theoretical implications of different models for the shape of the term structure. A second, and vital difference, is that many of these papers use an approximation argument that leads them to replace implied volatility with average expected volatility over the given horizon. This has one important consequence: the resulting shape of the term structure of implied volatilities is entirely decided by the mean reversion factor, i.e., it is upward sloping if current volatility is below its long-term mean, flat if it is at the long-term mean, and downward sloping otherwise. Our results suggest that a much richer variety of shapes is possible than is implied by this approximation argument.

Finally, as discussed above, direct econometric tests for choosing between jump-diffusions and stochastic volatility models have also been attempted (Bates (1996), or Jorion (1988)). Interestingly, given our emphasis on different frequencies, Jorion (1998) finds in his study that based on monthly data, there is not much

et al. (1994), Rosenberg (1996), Rubinstein (1994), Skiadopolous, Hodges, and Clewlow (1998), Stein (1989), Xu and Taylor (1994), and Zhu (1997). The Equity Derivatives Research publications of Goldman Sachs also carry detailed monthly information on this subject covering several equity markets.

[6]At first blush, this appears to suggest that excess kurtosis of the *conditional* returns also declines monotonically with maturity, but, as this paper's referee pointed out, this need not necessarily be the case. Implied volatility smiles are invariably plotted with a fixed set of strike prices applying to all maturities. Since the volatility of returns is greater at longer maturities, this means that, in standard deviation terms, the range of strike prices is smaller at longer maturities. Thus, it is possible that the implied volatility smile becomes flatter at longer maturities even though conditional kurtosis increases with maturity.

difference between the models he considers, but that this is not true under weekly data.[7]

II. Geometric Brownian Motion: A Brief Review

Let S_0 denote the initial (time-0) price of the asset. Under Geometric Brownian Motion (GBM), the time-t price S_t evolves according to

$$S_t = S_0 \exp\{\alpha t + \sigma W_t\}, \tag{1}$$

where α and σ are given constants, and W_t is a standard Brownian motion process.

Let $h > 0$ denote the length of time between observations of the price, and let $Z_t(h) = \ln(S_{t+h}/S_t)$ denote the continuously compounded return from holding the asset over the time interval $[t, t+h]$. From (1), $Z_t(h)$ is given by

$$Z_t(h) = \alpha h + \sigma(W_{t+h} - W_t). \tag{2}$$

Since (W_τ) is a Wiener process, we have $(W_{t+h} - W_t) \sim N(0, h)$. It follows from (2), therefore, that the returns $Z_t(h)$ are independent of t, and are themselves normally distributed with a mean of αh and a variance of $\sigma^2 h$,

$$Z_t(h) \sim N\left(\alpha h, \sigma^2 h\right). \tag{3}$$

Now recall that, if X is any random variable with mean m and variance $s^2 > 0$, then the skewness and kurtosis of X are defined as

$$\text{Skewness}(X) = \frac{E[(X-m)^3]}{s^3} \quad \text{and} \quad \text{Kurtosis}(X) = \frac{E[(X-m)^4]}{s^4}, \tag{4}$$

and that the *excess kurtosis* of X is given by Excess Kurtosis $(X) = [\text{Kurtosis}(X) - 3]$. It is well known that any normal distribution has zero skewness and a kurtosis of three. Therefore, expression (3) implies that the skewness and excess kurtosis of returns in a GBM are zero, both conditionally and unconditionally. The systematic violation of these conditions in practice has led to the development of the two classes of models we examine over the remainder of this paper.

III. Jump-Diffusion Models

A jump-diffusion model is obtained by augmenting the return process in a GBM with a Poisson jump process. Given any date $t \geq 0$, and a holding period of length $h > 0$, the returns $Z_t(h)$ over the period $[t, t+h]$ in such a model are given by

$$Z_t(h) = \begin{cases} x, & \text{if } K = 0 \\ x + y_1 + \cdots + y_K, & \text{if } K \geq 1 \end{cases}, \tag{5}$$

[7] On a broader note, there is a literature in finance that looks at the estimation of continuous-time models with discrete-time data (e.g., Melino (1994)). Especially relevant in this context is the recent work of Ait-Sahalia (1996).

where i) $x \sim N(\alpha h, \sigma^2 h)$, ii) $y_1, y_2, \ldots,$ is an i.i.d. sequence with common distribution G, and iii) K is distributed Poisson with parameter λh, where $\lambda > 0$ (for $k = 0, 1, 2, \ldots,$ we have $\text{Prob}(K = k) = e^{-\lambda h}(\lambda h)^k / k!$).

Since $Z_t(h)$ does not depend on t in any way, conditional and unconditional returns distributions coincide in a jump-diffusion model. To simplify exposition in the sequel, we shall, therefore, write $Z(h)$ for $Z_t(h)$.

Throughout, we will require that the first four moments of the jump size G exist. That is, letting $\nu_n = E(y^n)$ denote the nth moment of G, we will assume that ν_n is finite for $n = 1, 2, 3, 4$. This condition is a mild one that is met under virtually all the distributions used in practice; it is also a necessary requirement in the context of any sensible discussion of higher-order moments. No other restrictions are placed on G.

A. Moments of the Jump-Diffusion Model

The following proposition is the central result of this section.

Proposition 1. The variance, skewness, and kurtosis of the process (5) are given by

$$\text{(6)} \qquad \text{Variance}[Z(h)] \;=\; E\left[(Z(h) - E[Z(h)])^2\right] \;=\; h(\sigma^2 + \lambda\nu_2).$$

$$\text{(7)} \qquad \text{Skewness}[Z(h)] \;=\; \frac{E[(Z(h) - E[Z(h)])^3]}{[\text{var}(Z(h))]^{3/2}} \;=\; \frac{1}{\sqrt{h}}\left[\frac{\lambda\nu_3}{(\sigma^2 + \lambda\nu_2)^{3/2}}\right].$$

$$\text{(8)} \qquad \text{Kurtosis}[Z(h)] \;=\; \frac{E[(Z(h) - E[Z(h)])^4]}{[\text{var}(Z(h))]^2} \;=\; 3 + \frac{1}{h}\left[\frac{\lambda\nu_4}{(\sigma^2 + \lambda\nu_2)^2}\right].$$

Therefore,

1. Concerning Skewness:
   a) If $\nu_3 = 0$, then the skewness of the jump-diffusion is also zero.
   b) If $\nu_3 > 0$, the skewness is positive and inversely proportional to $\sqrt{h}$.
   c) If $\nu_3 < 0$, the skewness is negative and inversely proportional to $\sqrt{h}$.
2. The excess kurtosis of the jump-diffusion process (5) is always strictly positive and is inversely proportional to h.

Proof. See Appendix A. □

We can illustrate the content of Proposition 1 by using a simple example of the most commonly employed distribution for the jump size, the normal distribution. Let G be a normal distribution with mean μ and variance γ^2. (This is the case studied in Merton (1976) and Jorion (1988).) Then, from (7) and (8), we have

$$\text{(9)} \qquad \text{Skewness}(Z(h)) \;=\; \frac{1}{\sqrt{h}}\left[\frac{\lambda(\mu^3 + 3\mu\gamma^2)}{(\sigma^2 + \lambda\gamma^2 + \lambda\mu^2)^{3/2}}\right],$$

$$\text{(10)} \qquad \text{Kurtosis}(Z(h)) = 3 + \frac{1}{h}\left[\frac{\lambda(\mu^4 + 6\mu^2\gamma^2 + 3\gamma^4)}{(\sigma^2 + \lambda\gamma^2 + \lambda\mu^2)^2}\right].$$

Note that skewness in this case is positive if $\mu > 0$, and negative if $\mu < 0$.

B. Skewness and Kurtosis in the Jump-Diffusion Model

The closed-form expressions for skewness and kurtosis provided in Proposition 1 make it easy to verify that the model can match observed levels of skewness and kurtosis at specified intervals. Table 1 illustrates this point by presenting skewness and kurtosis values for the jump-diffusion for a range of parameter values, assuming $\lambda = 5$ and $G \sim N(\mu, \gamma^2)$. (The choice of parameter values is explained in the table.) As the numbers reveal, even for the intermediate and reasonable values of μ and γ, the model is capable of generating a considerable amount of skewness and excess kurtosis at weekly intervals. Note also that, as γ increases, excess kurtosis increases very rapidly (for example, compare kurtosis at $\gamma = 0.02$ and $\gamma = 0.03$).

TABLE 1

Skewness and Kurtosis in the Jump-Diffusion Model

Parameters			Skewness			Excess Kurtosis		
σ	μ	γ	1 Week	1 Month	3 Months	1 Week	1 Month	3 Months
0.134	0	0.02	0	0	0	0.312	0.072	0.024
0.134	−0.001	0.02	−0.015	−0.007	−0.004	0.313	0.072	0.024
0.134	−0.01	0.02	−0.160	−0.077	−0.044	0.452	0.104	0.035
0.125	0	0.03	0	0	0	1.580	0.365	0.122
0.125	−0.001	0.03	−0.034	−0.016	−0.009	1.582	0.365	0.122
0.125	−0.01	0.03	−0.341	−0.164	−0.095	1.844	0.425	0.142
0.087	0	0.05	0	0	0	12.188	2.813	0.938
0.087	−0.001	0.05	−0.095	−0.046	−0.026	12.190	2.813	0.938
0.087	−0.01	0.05	−0.929	−0.446	−0.258	12.534	2.892	0.964

This table presents the values of skewness and excess kurtosis in the jump-diffusion model when $\lambda = 5$ and $G \sim N(\mu, \gamma^2)$, for a range of values of μ and γ. To facilitate comparison across these values, the total variance of annual returns is fixed at approximately 0.02 in all cases. When $\gamma = 0.02$, this means about 10% of the annual variance comes from the jump-component; the figure rises to about 23% when $\gamma = 0.03$, and to more than 60% when $\gamma = 0.05$.

Proposition 1, however, also indicates that the skewness and kurtosis of a jump-diffusion decrease very rapidly as the horizon increases; for instance, excess kurtosis at weekly intervals ($h = 1/52$) should be more than four times that at monthly intervals ($h = 1/12$). Reflecting this rate of decline, Table 1 shows that, for reasonable parameter configurations, excess kurtosis becomes virtually negligible over a three-month horizon. This aspect of the model is troubling; certainly, the rate of decline appears much higher than would be suggested by empirical estimates of unconditional kurtosis at different frequencies. It also implies that implied volatility smiles should be nearly non-existent at three months, which is not typically the case in practice. We will return to this issue in Section VI.A.

IV. Conditional Returns under Stochastic Volatility

Stochastic volatility models generalize Geometric Brownian Motion by allowing the volatility of the return process itself to evolve stochastically over time. From (1), the cumulative returns $x_t = \ln(S_t/S_0)$ under a GBM obey $x_t = \alpha t + \sigma W_t$; in stochastic differential form,

$$dx_t = \alpha dt + \sigma dW_t. \tag{11}$$

In a stochastic volatility model, the instantaneous variance $V = \sigma^2$ is not required to be constant, but is allowed to change with time. That is, (11) is replaced by the joint process,

$$dx_t = \alpha dt + \sqrt{V_t} dW_t, \tag{12}$$

$$dV_t = \xi(t, V_t)dt + \beta(t, V_t)dB_t, \tag{13}$$

where W_t and B_t are standard Brownian motion processes. For the purposes of analysis, we must choose a functional form for (13). A popular specification, and one that we adopt, is that V_t evolves according to a mean-reverting square-root process,

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dB_t, \tag{14}$$

where κ, θ, and η are all strictly positive constants. This specification has been widely used in the literature (see, e.g., Bates (1996), Heston (1993), Heynen, Kemna, and Vorst (1994)), and, as such, is a natural choice. It also possesses significant analytical advantages over alternatives, particularly for the pricing of options (see Bates (1996)). Nonetheless, it seems important to emphasize that the qualitative aspects of our results do not appear to depend too much on this assumption; we obtained similar conclusions, for instance, when the square-root process was replaced by an Ornstein-Uhlenbeck process. Finally, we adopt a general specification for the relationship between W_t and B_t,

$$dB_t \, dW_t = \rho dt. \tag{15}$$

Expression (15) captures the possibility that increases in volatility could be related to the level of asset prices. Expressions (12), (14), and (15) complete the model description.

As earlier, let $Z_t(h) = x_{t+h} - x_t$ denote the returns from holding the asset between times t and $t+h$. It is immediate from (12) that these returns will depend on the path taken by the volatility process V_t over this interval. Moreover, it is also immediate from (14) that this path will depend on the initial value V_t of the volatility at time t. Thus, the returns $Z_t(h)$ depend on time-t information through V_t (but not, of course, through x_t). It follows that conditional and unconditional returns distributions will not necessarily coincide in a stochastic volatility model. In this section, we examine the properties of the conditional returns distribution; the unconditional distribution is the subject of the following section. The proofs of all results obtained in this section are in Appendix B.

For notational ease, let $V_t = v$. Denote by $F(v, h, s) = E[\exp\{isZ_t(h)\}|x, v]$, the characteristic function of $Z_t(h)$ conditional on time-t information. In Appendix B, we derive a closed-form expression for this characteristic function. Differentiation of this characteristic function enables us to recover all the moments of $Z_t(h)$ in the usual way, and enables us to obtain the following:

Proposition 2. Conditional on time-t values, the skewness and kurtosis of the stochastic volatility model are given by

$$\text{(16)} \qquad \text{Skewness}(Z_t(h)) = \left(\frac{3\eta\rho e^{\frac{1}{2}\kappa h}}{\sqrt{\kappa}}\right) \times \left[\frac{\theta(2 - 2e^{\kappa h} + \kappa h + \kappa h e^{\kappa h}) - v(1 + \kappa h - e^{\kappa h})}{(\theta[1 - e^{\kappa h} + \kappa h e^{\kappa h}] + v[e^{\kappa h} - 1])^{3/2}}\right],$$

$$\text{(17)} \qquad \text{Kurtosis}(Z_t(h)) = 3\left[1 + \eta^2\left(\frac{\theta A_1 - v A_2}{B}\right)\right],$$

where, letting $y = \kappa h$,

$$\text{(18)} \qquad A_1 = [1 + 4e^y - 5e^{2y} + 4ye^y + 2ye^{2y}] + 4\rho^2[6e^y - 6e^{2y} + 4ye^y + 2ye^{2y} + y^2e^y],$$

$$\text{(19)} \qquad A_2 = 2[1 - e^{2y} + 2ye^y] + 4\rho^2[2e^y - 2e^{2y} + 2ye^y + y^2e^y],$$

$$\text{(20)} \qquad B = 2\kappa[\theta(1 - e^y + ye^y) + v(e^y - 1)]^2.$$

Proof. See Appendix B.1. □

Expressions (16)–(20) look menacing, but they turn out to be surprisingly tractable from an analytical standpoint. We use them in the next two subsections to identify properties of the skewness and kurtosis of the conditional returns $Z_t(h)$.

A. Skewness of the Conditional Returns

For notational simplicity, let $S_t(h)$ denote the skewness of $Z_t(h)$. When we wish to emphasize the dependence of $S_t(h)$ on the correlation ρ, we will write $S_t(h; \rho)$. We show in this subsection that, for $\rho \neq 0$, the absolute skewness $|S_t(h)|$ must necessarily be a hump-shaped function of h. The following result establishes several preliminary properties of $S_t(\cdot)$ that are also of independent interest.

Proposition 3. $S_t(h)$ has the following properties:

1. $S_t(h)$ is positive if $\rho > 0$, zero if $\rho = 0$, and negative if $\rho < 0$.
2. $S_t(h; \rho) = -S_t(h; -\rho)$ for all ρ.
3. $\lim_{h\downarrow 0} S_t(h) = \lim_{h\uparrow\infty} S_t(h) = 0$.

Proof. See Appendix B.2. □

Properties 1 and 3 of Proposition 3 imply that when $\rho \neq 0$, skewness *cannot* be a monotone function of h as it was in the case of jump-diffusions. Indeed, in Appendix B.2, we provide an analytical proof that a very simple pattern must obtain, namely:

> If $\rho \neq 0$, absolute skewness $|S_t(h)|$ increases from zero to a maximum, and then decreases asymptotically back to zero.

For empirical relevance, this result raises an obvious question: for a given $\rho \neq 0$, how large is the value h at which absolute skewness is maximized? In Appendix B.2, we show that this maximizing value of h depends on only two parameters of the problem: the rate of mean reversion κ and the ratio $a = v/\theta$ of current volatility to its long-term mean.

Plausible values for κ (i.e., ones typically found empirically) are in a neighborhood of unity. If the current value of the volatility v is not excessively off its long-term mean level θ (i.e., if a and a^{-1} are both not very large), then it can be shown that the maximizing h is of the order of several months, and even years (see Table 2 of Das and Sundaram (1998)). From an empirical standpoint, therefore, the increasing part of $S_t(h)$ is very relevant, creating a sharp contrast with conditional skewness in jump-diffusion models.

B. Kurtosis of the Conditional Returns

For notational simplicity, let $K_t(h)$ denote the kurtosis of $Z_t(h)$. As with skewness, when we wish to emphasize the dependence of kurtosis on the correlation, we will write $K_t(h\,;\rho)$. The following result establishes important properties of $K_t(\cdot)$ including symmetry in ρ.

Proposition 4. $K_t(h)$ has the following properties:

1. Excess kurtosis is strictly positive everywhere: $K_t(h) > 3$ for all $h > 0$.
2. The degree of kurtosis only depends on the absolute value of ρ: $K_t(h\,;\rho) = K_t(h\,;-\rho)$.
3. $\lim_{h\downarrow 0} K_t(h) = \lim_{h\uparrow\infty} K_t(h) = 3$.

Proof. See Appendix B.3. □

Properties 1 and 3 imply, once again, that kurtosis cannot be a monotone function of h as it was for jump-diffusions. Indeed, the same procedure we used earlier establishes that:

> Kurtosis is hump-shaped as a function of h, increasing from zero to a maximum, and then decreasing asymptotically back to zero again.

We omit the details here, since the expressions are significantly more complicated than in the earlier case. However, carrying out the relevant computations shows that behavior of kurtosis in h depends on three factors: κ, a, and the degree of correlation ρ. It can be shown that the maximizing value of h increases with ρ and a, and decreases with κ. More importantly, as with skewness, it is again the case that, for plausible parameter values, this maximizing value of h is quite large (of the order of several months, and even years; see Table 3 in Das and Sundaram (1998)), making the increasing portion of the kurtosis curve significant from an empirical standpoint.

C. Values of Skewness and Kurtosis

We have shown that the stochastic volatility model has very different patterns of skewness and kurtosis from jump-diffusions. It is also possible to see, using the closed-form expressions for these quantities, that the degree of skewness and kurtosis the stochastic volatility model can generate for common parameterizations is less than that of jump-diffusions.

Table 2 summarizes the degree of conditional skewness and excess kurtosis for a range of possible values of the relevant parameters: $\rho \in \{-0.25, 0\}$, $\kappa \in \{1, 5\}$, $a \in \{0.75, 1, 1.25\}$, and $\eta \in \{0.1, 0.4\}$. The table reveals two important points. First, even for low mean reversion ($\kappa = 1$), and implausibly high volatility (η=0.4), the model does not generate a high degree of kurtosis at weekly intervals. (For the more reasonable value of $\eta = 0.1$, the model fails to generate a substantial degree of excess kurtosis at any interval.) Second, excess kurtosis is—as we have already seen—initially an increasing function of the horizon h. This suggests, contrary to observations, that the implied volatility smile may remain pronounced even at relatively long maturities (say, one year). We return to this point in Section VI.B when discussing implied volatility smiles in the stochastic volatility model.

V. Unconditional Returns under Stochastic Volatility

In Appendix C.1, we describe the derivation of the characteristic function of unconditional returns in the stochastic volatility model using the characteristic function of conditional returns. Successive differentiation of this function delivers all the moments of the unconditional returns. Letting $Z(h)$ denote unconditional returns over period of length h, and $y = \kappa h$, this results in the following simple expressions for unconditional skewness and kurtosis,

$$\text{(21)} \qquad \text{Skewness } (Z(h)) = 3\left(\frac{\rho\eta}{\sqrt{\kappa\theta}}\right)\left[\frac{1 - e^y + ye^y}{y^{3/2}e^y}\right],$$

$$\text{(22)} \qquad \text{Kurtosis } (Z(h)) = 3\left[1 + \frac{\eta^2}{\kappa\theta y^2 e^y}\left(1 - e^y + ye^y + 4\rho^2[2 - 2e^y + y + ye^y]\right)\right].$$

Expressions (21) and (22) may be used to derive properties of unconditional skewness and kurtosis in the stochastic volatility model. We begin with unconditional skewness.

A. Skewness of the Unconditional Returns

Let $S(h)$ denote the skewness of the unconditional returns $Z(h)$. As earlier, when we wish to emphasize the dependence of $S(h)$ on ρ, we will write $S(h\,;\rho)$. The skewness of unconditional returns behaves in qualitatively the same way as skewness of the conditional returns. The following preliminary result is important.

TABLE 2

Conditional Skewness and Kurtosis in the Stochastic Volatility Model

Parameters				Skewness			Excess Kurtosis		
ρ	κ	a	η	1 Week	1 Month	3 Months	1 Week	1 Month	3 Months
0	1	0.75	0.1	0.00	0.00	0.00	0.03	0.10	0.26
0	1	1.00	0.1	0.00	0.00	0.00	0.02	0.08	0.21
0	1	1.25	0.1	0.00	0.00	0.00	0.02	0.06	0.17
0	5	0.75	0.1	0.00	0.00	0.00	0.02	0.08	0.12
0	5	1.00	0.1	0.00	0.00	0.00	0.02	0.06	0.11
0	5	1.25	0.1	0.00	0.00	0.00	0.01	0.05	0.10
−0.25	1	0.75	0.1	−0.06	−0.12	−0.19	0.03	0.12	0.30
−0.25	1	1.00	0.1	−0.05	−0.11	−0.17	0.02	0.09	0.24
−0.25	1	1.25	0.1	−0.05	−0.09	−0.16	0.02	0.07	0.20
−0.25	5	0.75	0.1	−0.06	−0.10	−0.14	0.03	0.09	0.14
−0.25	5	1.00	0.1	−0.05	−0.09	−0.13	0.02	0.07	0.13
−0.25	5	1.25	0.1	−0.05	−0.09	−0.12	0.02	0.06	0.11
0	1	0.75	0.4	0.00	0.00	0.00	0.40	1.64	4.20
0	1	1.00	0.4	0.00	0.00	0.00	0.30	1.25	3.33
0	1	1.25	0.4	0.00	0.00	0.00	0.24	1.01	2.76
0	5	0.75	0.4	0.00	0.00	0.00	0.37	1.21	1.94
0	5	1.00	0.4	0.00	0.00	0.00	0.29	0.99	1.73
0	5	1.25	0.4	0.00	0.00	0.00	0.23	0.83	1.56
−0.25	1	0.75	0.4	−0.24	−0.48	−0.77	0.45	1.85	4.76
−0.25	1	1.00	0.4	−0.21	−0.42	−0.69	0.34	1.41	3.77
−0.25	1	1.25	0.4	−0.19	−0.38	−0.63	0.27	1.14	3.12
−0.25	5	0.75	0.4	−0.23	−0.42	−0.54	0.42	1.38	2.25
−0.25	5	1.00	0.4	−0.20	−0.38	−0.52	0.32	1.12	2.01
−0.25	5	1.25	0.4	−0.18	−0.35	−0.49	0.26	0.95	1.82

This table presents the values of conditional skewness and excess kurtosis at various horizons in the stochastic volatility model for a range of values of the four relevant parameters: the correlation ρ between the returns and volatility processes, the coefficient κ of mean reversion in the volatility process, the ratio a of current volatility to its long-term mean, and the volatility of volatility η.

TABLE 3

Unconditional Skewness and Kurtosis in the Stochastic Volatility Model

Parameters			Skewness			Excess Kurtosis		
ρ	k	η	1 Week	1 Month	3 Months	1 Week	1 Month	3 Months
0	1	0.1	0.00	0.00	0.00	1.49	1.46	1.38
0	5	0.1	0.00	0.00	0.00	0.29	0.26	0.21
−0.25	1	0.1	−0.05	−0.11	−0.17	1.49	1.47	1.41
−0.25	5	0.1	−0.05	−0.09	−0.13	0.29	0.27	0.22
0	1	0.4	0.00	0.00	0.00	23.85	23.35	22.12
0	5	0.4	0.00	0.00	0.00	4.65	4.20	3.30
−0.25	1	0.4	−0.21	−0.42	−0.69	23.89	23.51	22.56
−0.25	5	0.4	−0.20	−0.38	−0.52	4.69	4.33	3.57

This table presents the values of unconditional skewness and excess kurtosis at various horizons in the stochastic volatility model for a range of values of the four relevant parameters: the correlation ρ between the returns and volatility processes, the coefficient κ of mean reversion in the volatility process, and the volatility of volatility η.

Proposition 5. $S(h)$ has the following properties:

1. $S(h)$ is positive if $\rho > 0$, zero if $\rho = 0$, and negative if $\rho < 0$.
2. $S(h\,;\rho) = -S(h\,;-\rho)$.
3. $\lim_{h\downarrow 0} S(h) = \lim_{h\uparrow 0} S(h) = 0$.

Proof. See Appendix C.1. □

Properties 1 and 3 of Proposition 5 show that unconditional skewness cannot be monotone in h if $\rho \neq 0$. The natural question, therefore, is: how does skewness behave for $h > 0$? Analysis of the derivative $S'(h)$ provides the answer: there is one and only one value of h (given approximately by $h = 2.15/\kappa$) at which $S'(h) = 0$. Combined with Proposition 5, this means:

> If $\rho \neq 0$, then absolute skewness $|S(h)|$ increases from zero to a maximum and then decreases asymptotically to zero.

A remarkable feature of this result is that the point where skewness reaches its extremum values depends on only a *single* parameter of the model, namely κ. The only part played by ρ is in determining whether this extremum is a maximum (if $\rho > 0$) or a minimum (if $\rho < 0$). Moreover, even for a high value of κ such as $\kappa = 5$, the value of h that maximizes unconditional skewness is of the order of $2.15/5 = 0.43$ years, or almost five months. Thus, for an empirically relevant interval of values of h, unconditional skewness in the stochastic volatility model is *increasing* in h.

B. Kurtosis of the Unconditional Returns

Let $K(h)$ denote the kurtosis of the conditional returns given h; the term $K(h;\rho)$ will have the obvious interpretation. The behavior of unconditional kurtosis is different from (and much more complex than) that of conditional kurtosis. We begin with some important properties of $K(h)$.

Proposition 6. $K(h)$ has the following properties:

1. Excess kurtosis is strictly positive everywhere: $K(h) > 3$ at all $h > 0$.
2. Excess kurtosis only depends on the absolute value of ρ: $K(h\,;\rho) = K(h\,;-\rho)$.
3. $\lim_{h\downarrow 0} K(h) = 3(1+b)$ where $b = \eta^2/2\kappa\theta$, and $\lim_{h\uparrow\infty} K(h) = 3$.

Proof. See Appendix C.2. □

Two aspects of this result deserve emphasis. First, in contrast to conditional excess kurtosis, excess kurtosis of the unconditional distribution does *not* approach zero as $h \to 0$. We find this behavior puzzling, and are unable to see an intuitive explanation.

Second, since excess kurtosis has a strictly positive limit as $h \to 0$, but goes to zero as $h \to \infty$, Proposition 6 does *not* rule out the possibility that excess kurtosis could be monotone in h. Indeed, it can be shown analytically (see Das and Sundaram (1998)) that for $|\rho| < ½$, excess kurtosis is, in fact, strictly decreasing on $h \geq 0$. On the other hand, at $\rho = \pm 1$, it can also be shown (again, see Das and Sundaram (1998)) that there is a unique value of y, denoted y^*, for instance, (and given approximately by $y^* = 1.83$), such that kurtosis is increasing in h for $h < y^*/\kappa$, and decreasing in h for $h > y^*/\kappa$. Combining these results with the continuity of excess kurtosis in ρ, we have the following summary description of the behavior of excess kurtosis:

1. When $|\rho| < ½$, excess kurtosis is a strictly decreasing function of h.
2. When $|\rho|$ is sufficiently large, excess kurtosis is a hump-shaped function of h, increasing to a maximum and then decreasing asymptotically to zero.

C. Values of Unconditional Skewness and Kurtosis

Analysis of the closed-form expressions provided above reveals that for moderate values of η, the stochastic volatility model generates non-negligible excess kurtosis at weekly intervals only if mean reversion is also low. For example, suppose $\rho = -0.25$ and $\eta = 0.1$. Then, at $\kappa = 1$, weekly excess kurtosis is 1.49; at $\kappa = 5$, it falls to just 0.29. The degree of skewness is also small at all maturities. In all cases, moreover, excess kurtosis dissipates very slowly, significantly slower than empirical estimates appear to suggest.

VI. The Behavior of Implied Volatilities

To price options under jumps or stochastic volatility, certain of the models' primitive variables must be replaced with their risk-adjusted counterparts (see Bates (1996) for the arguments). In the case of the jump-diffusion, assuming that the jump size distribution is $N(\mu, \gamma^2)$, the parameters that need adjustment are the jump intensity λ and the mean of the jump size μ; the variance of the jump size γ^2 is not affected in this process. In the case of stochastic volatility, the rate of mean reversion κ and the long-run mean level θ must both be adjusted; the volatility of volatility η remains unchanged. Lending analytical tractability to this process is the fact that the risk-adjusted parameters are, like their counterparts in the original model, also constants (again, see Bates (1996)).

Rather than complicate the exposition by introducing new terms, we will continue to use the notation introduced earlier. It is to be kept in mind, however, that some of the parameters (namely, the ones identified in the last paragraph) are risk-neutralized versions.

A. Implied Volatilities under a Jump-Diffusion

Once the appropriate variables have been adjusted for risk, options under a jump-diffusion may be priced using Merton's (1976) formula. Implied volatility estimates can then be backed out using the Black-Scholes model as the benchmark.

We carried out this procedure for a large number of parameter configurations. Typical results from these computations are reported in Table 4. We present implied volatilities for five maturities and seven strike prices when $\mu < 0$, so there is negative skewness. (For the corresponding results when $\mu = 0$, see Table 6 of Das and Sundaram (1998).) The choice of parameter configurations is the same as in Table 1, Section III. As there, the intermediate value of γ is to be considered the most reasonable choice.

TABLE 4

Implied Volatilities in the Jump-Diffusion Model

Negative Skewness: $\lambda = 5, S = 100, r = 0$

μ	γ	σ	Months	85	90	95	100	105	110	115
−0.01	0.02	0.1342	1	0.1581	0.1505	0.1453	0.1425	0.1413	0.1410	0.1417
−0.01	0.02	0.1342	2	0.1484	0.1459	0.1440	0.1428	0.1420	0.1415	0.1412
−0.01	0.02	0.1342	3	0.1461	0.1447	0.1437	0.1429	0.1423	0.1419	0.1416
−0.01	0.02	0.1342	6	0.1444	0.1438	0.1434	0.1430	0.1426	0.1424	0.1421
−0.01	0.02	0.1342	12	0.1437	0.1434	0.1432	0.1430	0.1429	0.1427	0.1426
−0.01	0.03	0.1245	1	0.1749	0.1604	0.1477	0.1409	0.1397	0.1430	0.1499
−0.01	0.03	0.1245	2	0.1559	0.1495	0.1448	0.1418	0.1405	0.1405	0.1416
−0.01	0.03	0.1245	3	0.1503	0.1467	0.1440	0.1422	0.1411	0.1407	0.1408
−0.01	0.03	0.1245	6	0.1459	0.1445	0.1434	0.1426	0.1419	0.1415	0.1412
−0.01	0.03	0.1245	12	0.1442	0.1436	0.1432	0.1427	0.1424	0.1421	0.1419
−0.01	0.05	0.0866	1	0.2112	0.1877	0.1566	0.1290	0.1389	0.1645	0.1830
−0.01	0.05	0.0866	2	0.1777	0.1629	0.1466	0.1347	0.1354	0.1451	0.1557
−0.01	0.05	0.0866	3	0.1648	0.1543	0.1442	0.1372	0.1361	0.1403	0.1464
−0.01	0.05	0.0866	6	0.1515	0.1467	0.1426	0.1398	0.1385	0.1387	0.1400
−0.01	0.05	0.0866	12	0.1459	0.1440	0.1424	0.1412	0.1404	0.1400	0.1398

This table presents implied volatilities in the jump-diffusion model when there is negative skewness. The jump intensity is $\lambda = 5$ and the jump size G is distributed $N(\mu, \gamma^2)$, where μ=-0.01. Results are presented for the same range of values of γ and σ considered in Table 1. To facilitate comparison across these values, the total variance of annual returns is fixed at approximately 0.02 in all cases. When γ=0.02, this means about 10% of the annual variance comes from the jump component; the figure rises to about 23% when $\gamma = 0.03$, and to more than 60% when $\gamma = 0.05$. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is taken to be 100, and the interest rate to be zero.

Implied volatility smiles at any maturity may be read off Table 4 by considering the row corresponding to the maturity. Since we assume an interest rate of zero, the term structure of implied volatilities for the at-the-money options is simply the column corresponding to the strike price of 100. Examination of these tables reveals several interesting points about the properties of the volatility smiles.

Concerning the structure of the smile at a fixed maturity (for instance, one month), we have seen that at low levels of γ, the jump-diffusion model does not produce much excess kurtosis even at monthly intervals (Table 1). Reflecting this, the implied volatility smile is shallow even at short maturities for $\gamma = 0.02$. As γ increases from 0.02 to 0.03, however, the degree of excess kurtosis increases rapidly (see Table 1) and, as a consequence, the smile becomes deeper, comparable to empirically observed levels at short maturities.

Second, to move from a two-sided smile to a one-sided smirk requires a substantial amount of skewness in the distribution of the jump size. Table 4 describes the case $\mu = -0.01$, which corresponds to an average annual return from the jump component of about -5%.[8] At this value, the smirk at short maturities begins to resemble observed levels. At smaller values of μ (corresponding to, e.g., an annual expected return from the jump component of -3%), the skew was barely noticeable, even at short maturities.

Third, echoing the rapid decline of skewness and excess kurtosis in the model at longer horizons, the smile flattens out extremely quickly, much more rapidly, indeed, than one would expect in practice. At three months, for example, the implied volatility curve is almost flat unless γ is implausibly high.

The last, and perhaps most important point, concerns the term structure of at-the-money implied volatilities. Reflecting the monotone decreasing nature of excess kurtosis in this model, this term structure is always a monotone *increasing* function of maturity. In particular, the model is unable to generate *decreasing* term structures of the sort that have been observed in equity markets in recent years. Equally importantly, the slope of the term structure implied by the model is tiny: even for unreasonably high parameter values, the difference between the implied volatilities at one year and one month is only about one percentage point, compared to the two to three percentage points documented in practice (see, e.g., Campa and Chang (1995) or Derman and Kani (1994)). The jump-diffusion model appears, on both counts, to be an inadequate description of reality.

B. Implied Volatilities in the Stochastic Volatility Model

Option prices in the stochastic volatility model may be computed using the method introduced in Heston (1993); this involves a Fourier inversion of the characteristic function of the conditional returns to obtain the risk-neutral density.

This procedure was carried out for a large number of parameter configurations. Typical results are reported in Tables 5–8. (Tables 12 and 13 of Das and Sundaram (1998) provide further information.) Two values each are considered for the rate of mean reversion κ, the volatility of volatility η, and the correlation ρ; and three values for the initial volatility v_0 (corresponding, respectively, to v_0 being below, at, and above its long-term mean level θ). The parameter θ is itself fixed at 10%. The values of other parameters are provided in the tables.

First, as shown earlier in Table 2, the stochastic volatility model has only a limited ability to generate excess kurtosis unless the value of η is very high. Reflecting this, Tables 5–8 show that, at the realistic value of $\eta = 0.1$, the depth of the volatility smile (the highest minus the lowest implied volatility in any row) is insubstantial, rarely exceeding 1–2%. At the implausibly high value of $\eta = 0.4$, however, the smile becomes much deeper, comparable to empirically observed levels.

Second, conditional excess kurtosis in the stochastic volatility model can be substantially higher at moderate maturities than at short ones. Reflecting this, in almost all cases, the depth of the smile reduces only very slowly as the horizon increases. Indeed, the smile remains quite pronounced even at 12 months in some cases (e.g., when $\kappa = 1$ and $\eta = 0.4$ in Table 5).

[8]Care has to be taken in interpreting these numbers. Since the value of λ has been risk-neutralized, all contributions of the jump component to total returns or total variance are in the risk-neutral world.

TABLE 5

Implied Volatilities in the Stochastic Volatility Model

$S = 100, r = 0, \theta = 0.01$
Correlation $\rho = 0$, Initial Volatility $v_0 = 0.0075$

Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.0979	0.0914	0.0888	0.0874	0.0884	0.0905	0.0953
1	0.1	2	0.0971	0.0917	0.0887	0.0873	0.0885	0.0909	0.0951
1	0.1	3	0.0968	0.0918	0.0888	0.0875	0.0886	0.0910	0.0949
1	0.1	6	0.0963	0.0919	0.0892	0.0882	0.0891	0.0913	0.0946
1	0.1	12	0.0956	0.0924	0.0904	0.0897	0.0903	0.0919	0.0943
1	0.4	1	0.1378	0.1168	0.0984	0.0823	0.0973	0.1140	0.1287
1	0.4	2	0.1325	0.1148	0.0950	0.0786	0.0940	0.1113	0.1259
1	0.4	3	0.1298	0.1119	0.0920	0.0764	0.0910	0.1085	0.1231
1	0.4	6	0.1214	0.1041	0.0856	0.0734	0.0847	0.1008	0.1149
1	0.4	12	0.1061	0.0928	0.0794	0.0720	0.0790	0.0908	0.1022
5	0.1	1	0.0979	0.0922	0.0905	0.0895	0.0902	0.0915	0.0956
5	0.1	2	0.0972	0.0933	0.0917	0.0910	0.0916	0.0928	0.0958
5	0.1	3	0.0972	0.0941	0.0929	0.0923	0.0928	0.0937	0.0960
5	0.1	6	0.0974	0.0958	0.0952	0.0949	0.0951	0.0956	0.0968
5	0.1	12	0.0981	0.0975	0.0973	0.0972	0.0973	0.0974	0.0979
5	0.4	1	0.1314	0.1152	0.0982	0.0856	0.0972	0.1121	0.1258
5	0.4	2	0.1267	0.1112	0.0956	0.0857	0.0948	0.1083	0.1209
5	0.4	3	0.1221	0.1077	0.0941	0.0866	0.0934	0.1051	0.1166
5	0.4	6	0.1123	0.1016	0.0931	0.0894	0.0927	0.0998	0.1081
5	0.4	12	0.1037	0.0980	0.0942	0.0929	0.0941	0.0971	0.1014

This table presents implied volatilities in the stochastic volatility model when there is no skewness and the initial volatility is below its long-term mean level. Two levels each are considered for the mean reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

Third, the introduction of skewness into the model by considering $\rho = -0.25$ results in a significant asymmetry in the volatility smile, and also increases the depth of the smile. (The latter effect arises from the fact that kurtosis also increases in this case as ρ changes; see Table 2.) This is immediate from a comparison of the otherwise identical Tables 5 and 8.

Fourth, Table 4 also showed that the degree of excess kurtosis in the stochastic volatility model decreases as the ratio $a = v_0/\theta$ of initial volatility to its long-term mean increases. As a consequence, the smiles in Tables 5–7 exhibit a monotonic pattern, decreasing in depth (ceteris paribus) as v_0 increases.

Finally, the term structure of at-the-money implied volatilities displays a variety of possible shapes in this model. When volatility starts out below its long-term mean, mean reversion pulls it upward. This exerts an upward pull on the term structure. However, since kurtosis is higher at longer maturities, a downward pull is also exerted on the term structure. The final shape of the term structure reflects these effects. When mean reversion is weak (κ is small) and the kurtosis impact is large (η is high), the latter dominates initially, and a U-shaped term structure

TABLE 6

Implied Volatilities in the Stochastic Volatility Model

$S = 100, r = 0, \theta = 0.01$
Correlation $\rho = 0$, Initial Volatility $v_0 = 0.0100$

Parameters k	Parameters η	Maturity Months	Strike Prices 85	90	95	100	105	110	115
1	0.1	1	0.1079	0.1028	0.1012	0.1002	0.1009	0.1021	0.1060
1	0.1	2	0.1071	0.1027	0.1006	0.0996	0.1004	0.1021	0.1055
1	0.1	3	0.1066	0.1024	0.1002	0.0993	0.1001	0.1018	0.1050
1	0.1	6	0.1051	0.1015	0.0994	0.0986	0.0993	0.1010	0.1037
1	0.1	12	0.1030	0.1002	0.0986	0.0980	0.0985	0.0998	0.1018
1	0.4	1	0.1422	0.1255	0.1083	0.0955	0.1073	0.1223	0.1362
1	0.4	2	0.1397	0.1227	0.1047	0.0914	0.1037	0.1194	0.1333
1	0.4	3	0.1368	0.1197	0.1014	0.0886	0.1005	0.1165	0.1304
1	0.4	6	0.1281	0.1114	0.0942	0.0838	0.0934	0.1083	0.1219
1	0.4	12	0.1112	0.0984	0.0860	0.0795	0.0857	0.0967	0.1077
5	0.1	1	0.1067	0.1021	0.1010	0.1003	0.1008	0.1016	0.1050
5	0.1	2	0.1050	0.1016	0.1005	0.0999	0.1003	0.1012	0.1038
5	0.1	3	0.1039	0.1012	0.1002	0.0997	0.1001	0.1008	0.1029
5	0.1	6	0.1019	0.1004	0.0999	0.0996	0.0998	0.1002	0.1013
5	0.1	12	0.1006	0.1000	0.0998	0.0997	0.0998	0.0999	0.1004
5	0.4	1	0.1385	0.1224	0.1069	0.0966	0.1060	0.1194	0.1325
5	0.4	2	0.1324	0.1175	0.1031	0.0947	0.1024	0.1147	0.1268
5	0.4	3	0.1271	0.1133	0.1006	0.0940	0.1000	0.1108	0.1219
5	0.4	6	0.1159	0.1055	0.0975	0.0941	0.0971	0.1038	0.1118
5	0.4	12	0.1058	0.1003	0.0967	0.0954	0.0965	0.0994	0.1035

This table presents implied volatilities in the stochastic volatility model when there is no skewness and the initial volatility is at its long-term mean level. Two levels each are considered for the mean reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

results.[9] However, when mean reversion is strong and the kurtosis effect is small, a monotone upward sloping structure obtains (see, for example, the case $\kappa = 5$ and $\eta = 0.1$ in Table 5).[10] Thus, stochastic volatility models are able to generate many of the patterns observed in reality. There is one problem, however. In all cases, the difference between 12-month implied volatilities and one-month implied volatilities is around just one percentage point, compared to the two to three percentage points that have been documented empirically (see, e.g., Campa and Chang (1995), Derman and Kani (1994), or the Equity Derivatives Research publications of Goldman Sachs).

[9] In some cases, the term structure continues to decline even at 12 months. See Table 5 for the case $\kappa = 1$ and $\eta = 0.4$.

[10] Analogously, when volatility starts out above its long-term mean, the term structure slopes downward, since both the kurtosis effect and mean reversion work to pull it in the same direction; see Table 7.

TABLE 7

Implied Volatilities in the Stochastic Volatility Model

$S = 100, r = 0, \theta = 0.01$
Correlation $\rho = 0$, Initial Volatility $v_0 = 0.0125$

Parameters k	Parameters η	Maturity Months	Strike Prices 85	90	95	100	105	110	115
1	0.1	1	0.1176	0.1133	0.1122	0.1115	0.1120	0.1128	0.1161
1	0.1	2	0.1165	0.1128	0.1113	0.1106	0.1112	0.1123	0.1152
1	0.1	3	0.1157	0.1122	0.1106	0.1099	0.1105	0.1117	0.1143
1	0.1	6	0.1134	0.1104	0.1087	0.1081	0.1087	0.1100	0.1122
1	0.1	12	0.1099	0.1075	0.1062	0.1057	0.1061	0.1072	0.1089
1	0.4	1	0.1494	0.1330	0.1175	0.1072	0.1166	0.1301	0.1433
1	0.4	2	0.1463	0.1301	0.1137	0.1029	0.1129	0.1271	0.1402
1	0.4	3	0.1433	0.1270	0.1103	0.0996	0.1095	0.1240	0.1372
1	0.4	6	0.1342	0.1183	0.1023	0.0933	0.1016	0.1153	0.1282
1	0.4	12	0.1158	0.1038	0.0922	0.0865	0.0920	0.1023	0.1128
5	0.1	1	0.1152	0.1112	0.1106	0.1100	0.1103	0.1108	0.1138
5	0.1	2	0.1123	0.1094	0.1085	0.1080	0.1084	0.1090	0.1113
5	0.1	3	0.1102	0.1078	0.1070	0.1066	0.1069	0.1075	0.1093
5	0.1	6	0.1062	0.1048	0.1043	0.1041	0.1043	0.1046	0.1057
5	0.1	12	0.1030	0.1024	0.1022	0.1022	0.1022	0.1023	0.1028
5	0.4	1	0.1446	0.1292	0.1151	0.1066	0.1143	0.1264	0.1388
5	0.4	2	0.1377	0.1234	0.1102	0.1030	0.1096	0.1209	0.1323
5	0.4	3	0.1319	0.1186	0.1068	0.1010	0.1063	0.1163	0.1268
5	0.4	6	0.1193	0.1093	0.1017	0.0986	0.1014	0.1077	0.1153
5	0.4	12	0.1078	0.1025	0.0991	0.0978	0.0989	0.1017	0.1057

This table presents implied volatilities in the stochastic volatility model when there is no skewness and the initial volatility is above its long-term mean level. Two levels each are considered for the mean reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

VII. Conclusion

This paper has emphasized the importance of looking at the theoretical implications of pricing models at different frequencies. We carried out this task for two of the most common approaches used in the finance literature to generate skewness and kurtosis in returns distributions—jump-diffusions and stochastic or time-varying volatility. We found that the models had dramatically different implications for the term structures of skewness and kurtosis, both conditionally and unconditionally. Reflecting the differences in conditional returns, implied volatility estimates in the two models behave very differently.

In sum, neither model is able to capture all the aspects of the data adequately. The presence of jumps creates considerable kurtosis at short maturities, but not always at moderate or long maturities; this picture is reversed with stochastic volatility models. Reflecting this, implied volatility smiles in jump-diffusions can be quite sharp at short maturities, but tend to die out much quicker than empirical observations would appear to suggest. On the other hand, under reasonable parameterizations, implied volatility smiles in stochastic volatility models are quite

TABLE 8

Implied Volatilities in the Stochastic Volatility Model

$S = 100, r = 0, \theta = 0.01$
Correlation $\rho = -0.25$, Initial Volatility $v_0 = 0.0100$

Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.1145	0.1079	0.1040	0.1002	0.0979	0.0965	0.0988
1	0.1	2	0.1137	0.1077	0.1033	0.0996	0.0975	0.0966	0.0983
1	0.1	3	0.1130	0.1073	0.1029	0.0992	0.0972	0.0964	0.0979
1	0.1	6	0.1113	0.1061	0.1019	0.0985	0.0965	0.0959	0.0969
1	0.1	12	0.1086	0.1042	0.1006	0.0978	0.0960	0.0952	0.0956
1	0.4	1	0.1545	0.1356	0.1157	0.0954	0.0984	0.1105	0.1229
1	0.4	2	0.1513	0.1325	0.1118	0.0911	0.0950	0.1080	0.1204
1	0.4	3	0.1488	0.1294	0.1083	0.0882	0.0922	0.1055	0.1184
1	0.4	6	0.1426	0.1220	0.1011	0.0839	0.0869	0.0998	0.1131
1	0.4	12	0.1304	0.1119	0.0947	0.0819	0.0827	0.0923	0.1033
5	0.1	1	0.1129	0.1069	0.1036	0.1002	0.0980	0.0964	0.0982
5	0.1	2	0.1109	0.1060	0.1027	0.0998	0.0979	0.0966	0.0975
5	0.1	3	0.1094	0.1051	0.1022	0.0997	0.0979	0.0967	0.0971
5	0.1	6	0.1063	0.1034	0.1013	0.0995	0.0982	0.0971	0.0969
5	0.1	12	0.1035	0.1019	0.1007	0.0996	0.0987	0.0979	0.0974
5	0.4	1	0.1502	0.1324	0.1141	0.0965	0.0976	0.1079	0.1194
5	0.4	2	0.1436	0.1269	0.1097	0.0945	0.0945	0.1036	0.1139
5	0.4	3	0.1379	0.1223	0.1066	0.0937	0.0927	0.1002	0.1094
5	0.4	6	0.1254	0.1132	0.1019	0.0936	0.0914	0.0946	0.1005
5	0.4	12	0.1134	0.1059	0.0995	0.0948	0.0925	0.0925	0.0945

This table presents implied volatilities in the stochastic volatility model when there is negative skewness and the initial volatility is at its long-term mean level. Two levels each are considered for the mean reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

shallow, but they may not flatten out appreciably as maturity increases. Last, although the data present a great variety of patterns of the term structure of implied volatilities for at-the-money options, jump-diffusions are only able to produce a single shape (monotone increasing); stochastic volatility models fare much better in this regard.

Finally, we provide closed-form solutions in this paper for the higher-order moments of jump-diffusions and stochastic volatility models. These closed-forms played a central role in our analysis, but they may also find use in areas beyond this immediate paper. They may, for example, be used in the construction of models to estimate value-at-risk as explained in Duffie and Pan (1997). They may also be used in method-of-moments estimation procedures where the moments may exploit data at different intervals. To the best of our knowledge, this has not been undertaken so far in the literature.

Appendices

Appendix A. Proof of Proposition 1

In stochastic differential form, the returns process under the jump-diffusion process of Section III may be represented as $dx = \alpha dt + \sigma dZ + J d\pi(\lambda)$, where α is the drift and σ the standard deviation of the diffusion component, J is the jump size, and π is a Poisson process with intensity parameter λ. In the notation of Section III, J has distribution G; its first four moments are finite, and are denoted by ν_m, $m = 1,2,3,4$.

We first identify the characteristic function of the distribution of x_{t+h}, given $x_t = x$. Denoting this characteristic function by $F(x,s)$, a standard argument shows that F is the solution to the following Kolmogorov Backward Equation, subject to the boundary condition that $F(x,0) = e^{isx}$,

$$\alpha \frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial h} + \lambda E[F(x+J) - F(x)] \quad = \quad 0.$$

It is readily checked that the solution to this differential equation is

$$F(x,s) \quad = \quad \exp\left\{ isx + \alpha ish - \frac{1}{2}\sigma^2 s^2 h + \lambda h E(e^{isJ} - 1)\right\}.$$

Since $Z_t(h) = x_{t+h} - x_t$, it is immediate that the characteristic function of $Z_t(h)$ is simply $F^* = e^{-isx}F$. The moments of $Z_t(h)$ may now be computed from the characteristic function F^* by successive differentiation; i.e., if ζ_n represents the nth moment of $Z_t(h)$, then

$$\zeta_n \quad = \quad \frac{1}{i^n}\left[\frac{\partial^n F^*}{\partial s^n}\right]\Bigg|_{s=0}.$$

Using these computed moments, it is not hard to compute the variance, skewness, and kurtosis of $Z_t(h)$ and to verify that they are as provided in Proposition 1. □

Appendix B. Proofs for Section IV

1. Deriving the Skewness and Kurtosis

Let x denote the time-t value of the returns process. Let $\bar{F}(x,v,h,s)$ be the characteristic function of x_{t+h} given time-t information. A standard argument (see, e.g., Duffie (1996)) establishes that $\bar{F}$ may be obtained as the solution to the Kolmogorov Backward Equation,

$$\text{(B-1)} \quad \alpha\frac{\partial \bar{F}}{\partial x} + \frac{1}{2}v\frac{\partial^2 \bar{F}}{\partial x^2} + \kappa(\theta - v)\frac{\partial \bar{F}}{\partial v} + \frac{1}{2}\eta^2 v\frac{\partial^2 \bar{F}}{\partial v^2} - \frac{\partial \bar{F}}{\partial h} + \rho\eta v\frac{\partial^2 \bar{F}}{\partial x \partial v} \quad = \quad 0,$$

subject to the initial condition,

$$\text{(B-2)} \qquad \bar{F}(x,v,0\,;s) \quad = \quad e^{isx}, \qquad \text{for all } x, v, s.$$

Proposition B.1. The solution of (B-1)–(B-2) has the form,

$$\text{(B-3)} \qquad \bar{F}(x, v, h\,; s) \;=\; C(h)\exp[isx + A(h) + vB(h)].$$

Letting $\gamma = \kappa - \rho\eta is$ and $\phi = \sqrt{\gamma^2 + \eta^2 s^2}$, we have $A(h) = i\alpha sh$, and

$$\text{(B-4)} \qquad B(h) \;=\; \frac{-s^2\left[e^{\phi h} - 1\right]}{(\phi + \gamma)\left[e^{\phi h} - 1\right] + 2\phi},$$

$$C(h) \;=\; \left[\frac{2\phi\left(e^{(\phi+\gamma)\frac{h}{2}}\right)}{(\phi + \gamma)\left[e^{\phi h} - 1\right] + 2\phi}\right]^{\frac{2\kappa\theta}{\eta^2}}.$$

Proof. This may be verified by direct substitution. □

It is an easy matter now to obtain the characteristic function $F(h, v, s)$ of $Z_t(h)$. Indeed, since $Z_t(h) = x_{t+h} - x$, F is related to the function $\bar{F}$ of Proposition B.1 through $F(v, h, s) = \bar{F}(x, v, h, s)e^{-isx}$. Substituting for $\bar{F}(x, v, h, s)$ from (B-3), we obtain

$$\text{(B-5)} \qquad F^*(v, h, s) \;=\; C(h)\exp\{A(h) + vB(h)\},$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are as given in Proposition B.1. Expression (B-5) may now be used to obtain the moments of $Z_t(h)$ in the usual way: if we let $\xi_m = E[(Z_t(h))^m]$, then we have

$$\xi_m \;=\; \frac{1}{i^m} \cdot \frac{\partial^m F^*}{\partial s^m}(v, h\,; s)\bigg|_{s=0}.$$

Grinding through the relevant computations yields the expressions for skewness and kurtosis presented in Section IV.

2. Proof of Proposition 3

Property 1 of Proposition 3 will be established if we can show that the last term in parentheses on the RHS of (16) is positive. We will first show that the denominator is positive, and then that the numerator is positive. For the first part, it suffices to show that

$$\text{(B-6)} \qquad \theta(1 - e^y + ye^y) + v(e^y - 1)] \;>\; 0 \qquad \text{for all } y \;>\; 0.$$

Consider first the term $1 - e^y + ye^y$. At $y=0$, this term is equal to zero. Its derivative at any $y > 0$ is given by the strictly positive quantity ye^y. Therefore, this term is strictly positive at all $y > 0$. The term $(e^y - 1)$ is obviously also strictly positive at all $y > 0$. Since θ and v are positive, (B-6) is established.

Turning to the numerator, it evidently suffices to show that for all $y > 0$, we have

$$\text{(B-7)} \qquad 2 - 2e^y + y + ye^y \;>\; 0 \qquad \text{and} \qquad 1 + y - e^y \;<\; 0.$$

For notational ease, let $h(y) = 2 - 2e^y + y + ye^y$. It is easily checked that $h(0) = h'(0) = 0$, and that $h''(y) = ye^y > 0$ for all $y > 0$. It follows immediately that

$h(y)$ is positive for all $y > 0$, establishing the first part of (B-7). The other part is trivial since, by definition, $e^y = \sum_{n=0}^{\infty}(y^n/n!) > 1 + y$ if $y > 0$. This completes the proof of Property 1.

Property 2 is immediate from (16). Finally, to see Property 3, let

$$g(y) \quad = \quad e^{\frac{1}{2}y}\left[\frac{\theta(2 - 2e^y + y + ye^y) - v(1 + y - e^y)}{(\theta(1 - e^y + ye^y) + v(e^y - 1))^{3/2}}\right].$$

The expression on the right-hand side of this equation has the form $0/0$ at $y = 0$, and ∞/∞ as $y \uparrow \infty$. However, using L'Hopital's rule repeatedly, it can be seen that $\lim_{y \downarrow 0} g(y) = 0$ and that $\lim_{y \uparrow \infty} g(y) = 0$. From (16), it is immediate that

$$\lim_{h \downarrow 0} S(h) \quad = \quad \frac{3\eta\rho}{\sqrt{k}} \cdot \lim_{y \downarrow 0} g(y) \quad \text{and} \quad \lim_{h \uparrow \infty} S(h) \quad = \quad \frac{3\eta\rho}{\sqrt{k}} \cdot \lim_{y \uparrow \infty} g(y).$$

It follows that $S(h)$ goes to zero as $h \downarrow 0$ and $h \uparrow \infty$. □

In the remainder of this subsection, we prove that conditional (absolute) skewness is a hump-shaped function of h. To see this, observe first from (16) that, although other parameters affect the *level* of conditional skewness, the pattern of dependence on h only relies on *two* variables: the rate of mean reversion κ, and the ratio $a = v/\theta$ of current volatility to its long-term mean. Now consider the derivative $S_t'(h)$. It can be seen by direct calculation that h satisfies $S_t'(h) = 0$ if and only if $y = \kappa h$ satisfies

$$\text{(B-8)} \qquad (1 - a)(4 - 3a) + (1 - a)^2 y - (8 - 8a + 3a^2)e^y$$
$$- 2a(2 - a)ye^y + (4 - a)e^{2y} - ye^{2y} - 2(1 - a)y^2 e^y \quad = \quad 0.$$

Equation (B-8) has a unique solution for each value of a. Combined with Properties 1 and 3 of Proposition 2, this uniqueness implies that if $\rho \neq 0$, absolute skewness increases from zero to a maximum, then decreases asymptotically back to zero.

Finally, note that the value of h that maximizes $S_t(h)$ is simply the value of h for which (B-8) holds. Thus, the maximizing value of h only depends on two numbers: κ and $a = v/\theta$. Table 2 in Das and Sundaram (1998) describes the way the maximizing value of h changes as these parameters change.

3. Proof of Proposition 4

To prove Property 1, it suffices to show that $A_1 > 0$ and $A_2 < 0$. Now, A_1 is of the form $f(y) + 4\rho^2 g(y)$, where

$$f(y) \quad = \quad 1 + 4e^y - 5e^{2y} + 4ye^y + 2ye^{2y},$$

$$g(y) \quad = \quad 6e^y - 6e^{2y} + 4ye^y + 2ye^{2y} + y^2 e^y.$$

A simple computation shows that $f(0) = 0$ and $f'(y) = 4e^y(2 - 2e^y + y + ye^y)$. The term in parentheses is strictly positive for $y > 0$, as we showed in the proof of Proposition 3 (see equation (B-7). Therefore, $f'(y) > 0$ for all $y > 0$, and so $f(y) > 0$ for all $y > 0$. Similarly, it can be checked that $g(0) = g'(0) = g''(0) =$

$g'''(0) = 0$, and that $g^{(4)}(y) > 0$ for all $y > 0$. Therefore, $g(y) > 0$ for all $y > 0$. Since f and g are both strictly positive functions, A_1 is strictly positive. $A_2 < 0$ is checked similarly. The details are omitted. This proves Property 1.

Property 2 is immediate from (17). This leaves Property 3. Let $h(y) = [\theta(1 - e^y + ye^y) + v(e^y - 1)]^2$. Then, for $y = \kappa h$, we have

$$K(h) = 3\left[1 + \frac{\eta^2}{2k}\left(\frac{\theta A_1 - vA_2}{h(y)}\right)\right].$$

Now, $A_1/h(y) = [f(y)/h(y)] + 4\rho^2[g(y)/h(y)]$, where f and g were defined above. Repeated use of L'Hopital's rule shows that the limits as $y \downarrow 0$ and $y \uparrow \infty$ of $[f(y)/h(y)]$ and $[g(y)/h(y)]$ are all zero. Thus, $A_1/h(y)$ tends to zero as y tends to zero or becomes unbounded. A similar argument shows that $A_2/h(y)$ also tends to zero as y tends to zero or to infinity. Since κ is fixed, $y \downarrow 0$ if and only if $h \downarrow 0$, and $y \uparrow \infty$ only if $h \uparrow \infty$, so we finally obtain $\lim_{h\downarrow 0} K(h) = \lim_{h\uparrow\infty} K(h) = 3$. This completes the proof of the proposition. □

Appendix C. Proofs for Section V

We begin with a derivation of the characteristic function of unconditional returns. In Appendix B.1, we showed that *conditional* on $V_t = v$, the characteristic function $F(v, h, s)$ of $Z_t(h)$ has the form,

$$D(h, s) \exp\{vB(h, s)\}. \tag{C-1}$$

Now, it is well known that the square-root process (14) has a stationary density that is γ, given by

$$g(v) = \frac{1}{\Gamma(\nu)}\omega^\nu v^{\nu-1} e^{-\omega v}, \tag{C-2}$$

where $\omega = 2\kappa/\eta^2$ and $\nu = 2\kappa\theta/\eta^2$. Combining (C-1) and (C-2), the characteristic function of the *unconditional* returns $Z(h)$, denoted, for example, $P(h, s)$, is given by

$$P(h, s) = \int F^*(v, h, s) g(v) dv = D \int \exp\{vB\} g(v) dv. \tag{C-3}$$

But this last integral on the right-hand side is simply the moment generating function $E(e^{tv})$ of the γ distribution evaluated at $t = B$. It is well known that this function has the form $[\omega/(\omega - B)]^\nu$. Therefore, P has the form,

$$P(h, s) = D(h, s)\left(\frac{\omega}{\omega - B(h, s)}\right)^\nu. \tag{C-4}$$

Successive differentiation of the function P delivers all the moments of the unconditional returns, and leads, in particular, to the forms of skewness and kurtosis provided in Section V.

4. Proof of Proposition 5

As usual, let $y=\kappa h$. Property 1 of Proposition 5 will be proved if we can show that the term $(1 - e^y + ye^y)$ is strictly positive for all $y > 0$. But we have already done this in the proof of Proposition 3 above (see equation (B-6)). Property 2 of Proposition 5 is immediate from the form of skewness (21). This leaves Property 3. Now, the limits as $h \downarrow 0$ and $h \uparrow \infty$ of $S(h)$ are clearly determined entirely by the limits as $y \downarrow 0$ and $y \uparrow \infty$ of the function $h(y)$ defined by

$$\text{(C-5)} \qquad h(y) = \frac{1 - e^y + ye^y}{y^{3/2}e^y}.$$

Indeed, we need only show that $h(y)$ tends to zero as y tends to 0 or $+\infty$. As $y \downarrow 0$, both the numerator and the denominator of the right-hand side of (C-5) approach zero. However, repeated application of L'Hopital's rule shows that $\lim_{y \downarrow 0} h(y)=0$, establishing one part of the desired result. To see the other part, note that $h(y)$ can be rewritten as

$$\text{(C-6)} \qquad h(y) = \frac{1}{y^{3/2}e^y} - \frac{1}{y^{3/2}} + \frac{1}{y^{1/2}}.$$

Since each term on the right-hand side of (C-6) tends to zero as $y \uparrow \infty$, it is clearly the case that $h(y)$ also goes to zero as $y \uparrow \infty$. This completes the proof of Proposition 5. □

5. Proof of Proposition 6

For notational ease, define $g(y) = [(1 + 8\rho^2) - (1 + 8\rho^2)e^y + (1 + 4\rho^2)ye^y + 4\rho^2 y]/[y^2e^y]$. Then, we have $K(h) = 3[1 + (\eta^2 g(\kappa h))/(\kappa\theta)]$. To prove Property 1 now, it suffices to show that $g(y) > 0$ at all $y > 0$. A simple calculation shows that we have $g(0) = g'(0) = 0$, and that $g''(y) = e^y + (1 + 4\rho^2)ye^y > 0$, which is strictly positive at all $y > 0$. Property 1 follows immediately. Property 2 is an immediate consequence of the fact that $K(h)$ only depends on ρ^2. Finally, to see Property 3, note that

$$\text{(C-7)} \qquad \lim_{h \downarrow 0} K(h) = 3\left[1 + \frac{\eta^2}{\kappa\theta}(\lim_{y \downarrow 0} g(y))\right],$$

$$\text{(C-8)} \qquad \lim_{h \uparrow \infty} K(h) = 3\left[1 + \frac{\eta^2}{\kappa\theta}(\lim_{y \uparrow \infty} g(y))\right].$$

Repeated applications of L'Hopital's rule to g establish that

$$\text{(C-9)} \qquad \lim_{y \downarrow 0} g(y) = \frac{1}{2} \quad \text{and} \quad \lim_{y \uparrow \infty} g(y) = 0.$$

Substituting (C-9) into (C-7) and (C-8) completes the proof of Property 3. □

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