**Local Volatility and the Recovery Rate of Credit Default Swaps**

by

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Abstract

Credit default swap (CDS) spreads can only be decomposed into the probability of default and the loss-given-default by imposing some structure. Das and Hanouna (2009) impose structure by employing a hybrid binomial tree for equities and a recovery function. They obtain accurate estimates for CDS spreads by fitting the model to historical equity volatilities. We extend their approach by including the full implied volatility surface, developing an implied binomial tree with a jump to default based on the Derman and Kani (1994) tree. We then evaluate the effect of including the full volatility surface on CDS recovery rate.

*JEL Classification:* C02; C13; G12; G13

*Keywords:* credit default swap, recovery rates, implied tree models; implied volatility; local volatility; option pricing

## 1. Introduction

The yield on a corporate bond can be disentangled into two components: (1) the yield of a risk-free asset with the same maturity and (2) the spread over this risk-free asset based on the issuer's credit rating. If the risk-free rate is the London interbank offered rate (LIBOR), then the credit spread is the asset swap spread (ASW) or the credit default swap (CDS). This credit spread is the compensation an investor wants for bearing the default exposure associated with the issuer. In a risk-neutral world, this credit spread should be equal to the product of the probability that the issuer defaults (PD) and the loss the investor incurs in the case of a default (loss given default or LGD). This pivotal relationship is referred to as the "credit triangle." The problem with this relationship is that although the credit spread is known in the form of a market price (ASW or CDS), the PD and LGD are not separately identified since it is the result of a product. Additionally, the credit spread might include other components that are not related to default risk such as liquidity risk and any embedded options. If we abstract from this non-default component and want to decompose the credit triangle, we need some structure on either the PD or the LGD to identify the other component. Most research has been done on the identification of either the PD or the LGD alone. The modelling of default probabilities alone for example in a structural model is done in Merton's structural model (Merton, 1974), where credit spreads and equity are linked via option payoffs. The modelling of the LGD alone is focuses on the determinants of actual defaults (see, e.g., Altman et al., 2005). The simplest solution, of course, is to assume a constant recovery rate, as practitioners often do by using 40% as an estimate for the recovery rate on senior loans; however, this approach fails to take into account that LGD can be time varying: when a company defaults in a good economic environment, the recovery rate may be high, while if the company defaults in a recession, the assets might be sold for fire sale prices (see, e,g., Altman and Kuehne, 2012). Another drawback for all of the above approaches is that they are based on the physical world PD's and LGD's while the credit triangle relationship is based on risk-neutral PD's and LGD's. Bakshi et al. (2006) show that the risk-neutral PD's are higher than their physical counterpart, but also (not surprisingly given the credit triangle) that the risk-neutral LGD's are lower than their physical counterpart. Therefore, mixing physical world and risk-neutral variables can be complicated.

There is research attempting to estimate the joint identification of the LGD and PD for sovereign and corporate CDS spreads. Merrick (2001) assumes a linear relationship for default to estimate the recovery rate for Argentina. Using a reduced form model, Zhang (2003) tests the model on Argentina as well. Pan and Singleton (2008) estimate recovery rates for sovereign countries using time-series data assuming a constant recovery rate. Doshi (2011) uses bonds from the same issuer with different seniority (i.e. senior versus subordinated). Using a similar approach, Schläfer and Uhrig-Homburg (2010)investigate different seniority of CDS premia (CDS versus LCDS) to examine the recovery rates assuming the same default likelihood.

Berd (2005) demonstrates that it is possible to disentangle the credit triangle using CDS contracts. Das and Hanouna (2009) develop a model that identifies the PD and LGD jointly with information from the CDS and equity markets. Augmenting the binomial tree of Cox, Ross, and Rubinstein (1979) with a jump to default, they calibrated their model with CDS prices in order to estimate the term structure of PD and LGD. The advantage of the Das-Hanouna (DH hereafter) model is that it is a parsimonious model that assumes no explicit correlation between the default likelihood and recovery rates. However, as DH note, one deficiency of their model is that it does not use the full option volatility surface. The model only uses the at-the-money volatility and therefore ignores the fact that out-of-the money put options trade at a higher implied volatility. To include the full volatility surface, the Cox-Rubenstein-Ross (CRR hereafter) tree needs to be replaced by an implied tree (Derman and Kani, 1994 and Barle and Cakici, 1998) and then augmented with a jump to default. DH state that they will leave this extension for further research because it will enhance complexity but note that it is an important extension as it will use the entire volatility surface and raise the model's information content.

In this paper, we extend the DH model to include the full volatility surface and then look at the effect on the recovery rate. To our knowledge, the augmentation of an implied tree with a jump to default has not been done. Derman and Kani (1994), DK hereafter, develop a binomial tree that can exactly price options: each node in the tree is implied from the prices of traded options. However, this binomial tree does not contain a jump to default. We are only aware of an extension to a trinomial tree

(Derman and Chriss, 1996) where the stock price can either go up, down or stay at the same level. In continuous-time framework, Carr and Madan (2010) develop a model for local volatility that is enhanced by a jump to default. However, because their model is in a continuous–time framework, it more challenging to implement in contrast to a binomial tree structure that is easy to calibrate.

This paper contributes to the literature in three important areas. First our research contributes to the literature on the development of models for PD and LGD for asset and risk management applications, as well as regulatory requirements. Basel II requires banks not only to estimate the risk of default but also to estimate the recovery rate in case of a default. Second, we provide evidence as to whether market participants would be able to better explain CDS spreads by utilizing options' information provided by the market. That is, we look at whether both volatility smile and term structure are priced into the CDS market. Finally, we provide empirical evidence as to whether the model might help in pricing longer-term options. The most liquid CDS have five-year tenors, while the option market only has maturities up to two years. So the question our empirical evidence helps answer is whether market participants might benefit by using CDS contracts to price longer-term options.

There are four sections that follow. In Section 2 we outline our methodology, first describing the DH model and then developing an implied binomial tree with a jump to default. After describing our dataset in Section 3, we present our empirical results in Section 4. Section 5 concludes.

## 2. Methodology

In this section, we discuss our methodology, starting with a description of the DH model followed by how we can use option information obtained from the market to construct implied trees. We then develop our model that adds a jump to default tree to the implied tree.

### 2.1 Das-Hanouna model

DH developed a model that uses information from equity and credit markets to extract implied recovery rates. The model is based on three building blocks, (1) a binomial tree enhanced by a jump to default, (2) a CDS pricing model and (3) a recovery function. We describe each building block below.

*2.1.1 First building block: Binomial tree enhanced by a jump to default*

The first building block is based on a standard CRR binomial tree. In this binomial tree, the initial stock price $S_j^i$ in state *i* can either go up to state *i+1* in the next period *j+1* ($S_{j+1}^{i+1}$) or down to state *i* ($S_{j+1}^{i}$). The stock price changes in the tree with a constant volatility: it goes up to $S_j^i e^{\sigma\sqrt{h}}$ or goes down to $S_j^i e^{-\sigma\sqrt{h}}$ in the next period. The movements are thus governed by two factors, the volatility $\sigma$ and the time step *h.* Note also that the binomial tree is a recombining tree, since as the stock price goes up in period *j+1* and then down in period *j+2* we arrive at the same price as at time *j.* The probability that the stock price goes up is *q* and therefore the probability that the stock price goes down is $(1 - q)$.

Then DH enhance the CRR tree with a jump to default. More specifically, in addition to an up and down movement as just described, the stock price *S* can also jump to default (jumps to a price of zero) with probability $\lambda$. The company will survive if it does not default, therefore the probability of survival is $1 - \lambda$. Since we will work more with the survivorship probability than with the default probability, we adopt the letter *z* for the survivorship probability and a $(1 - z)$ for the default probability. Below we schematically show the three movements as follows

$$S_j^i \quad \longrightarrow \begin{cases} S_{j+1}^{i+1} = S_j^i e^{\sigma\sqrt{h}} & \text{Up state with probability } q_j^i \, z_j^i \\ S_{j+1}^{i} = S_j^i e^{-\sigma\sqrt{h}} & \text{Down state with probability } (1\text{-}q_j^i)\, z_j^i \\ 0 & \text{Default with probability } 1 - z_j^i \end{cases} \qquad (1)$$

A requirement in a binomial tree is that the branching process is risk-neutral, meaning that the forward price of the stock should be equal to the product of the probabilities of the states and the stock prices of the different states. If $f_{j,j+1}$ and $d_{j,j+1}$ are respectively the one-period risk-free rate and dividend rate (for period *j* to period *j*+1), then the forward price $F_{j,j+1}^i$ of the stock at time *j* and state *i* is equal to $S_j^i \exp\left((f_{j,j+1} - d_{j,j+1})\, h\right)$.

Given equation (1) and the requirement of the risk-neutral branching process, the stock prices and probabilities should satisfy:

$$F^i_{j,j+1} = q^i_j z^i_j S^{j+1}_{j+1} + (1 - q^i_j) z^i_j S^j_{j+1} \quad (2)$$

with the additional requirement being that the transition probabilities should lie on the interval [0,1].

The unknown parameters in equation (2) are the probabilities $z^i_j$ and $q^i_j$. Note also that $z^i_j$ is state and time dependent, therefore if we have five layers in the binomial tree, we need for the first four layers an estimate of $z^i_j$ and therefore we need 10 survival probabilities to construct the tree.[1] To identify equation (2), DH assumed a functional form for $z^i_j$: the survival probability is a function of the stock price *S*. The structure is setup in two stages.

In the first stage, the default intensity $\xi^i_j$ is modeled as a function of the stock price as follows:

$$\xi^i_j = \frac{1}{(S^i_j)^b} \quad (3)$$

where *b* is assumed to be greater than 0 so that default intensity decreases if the stock price rises. The stock price should act as a buffer as it does in structural models. The more buffer there is, the less is the default intensity.

In the second stage, the probability that the firm will survive between period *j* and *j+1*, $z^i_j$, will be an exponential function of the default intensity and time step *t* as follows:

$$z^i{}_j = e^{-\xi^i_j h} \quad (4)$$

Since $\xi^i_j$ and *t* are always positive, the survival probability $z^i_j$ will be between 0 and 1. In the limit, this means that the firm always survives ($z$ = 1) or always defaults ($z$ = 0).

The probability of default will be 1 minus the survival probability

$$\delta^i_j = 1 - z^i_j \quad (5)$$

The subscript indicates that the default probability is known at time *j* for the next period (*j*+1). The superscript *i* denotes the state.

[1] If *n* is the number of layers, we need $\frac{n(n+1)}{2}$ estimates.

The relation between $b$ and $z$ is positive: if $b$ decreases, the survival probability decreases. The relationship between S and $z$ is positive; the higher the stock price, the higher the survival probability.

*2.1.2 Second building block: CDS pricing model*

A CDS is an insurance contract, where the seller of the contract promises to pay out the credit loss on a bond when there is a credit event and the buyer of the contract promises to pay the periodic premium up to the time of any credit event (should one occur). The most common credit events as defined by the International Swaps and Derivatives Association (ISDA) are bankruptcy, failure to pay and restructuring.[2]

DH use a standard CDS pricing model as described in, for example, Duffie (1999). The two legs of a CDS are priced as follows:

1. The seller of protection promises to pay the recovery rate when a credit event occurs, so the value of this leg of the contract is:

$$B_n = \sum_{j=1}^{N} Q(T_{j-1})(1 - e^{-\xi h})\, D(T_j)(1 - R) \tag{6}$$

where $Q(T_{j-1})(1 - e^{-\xi h})$ is the probability of surviving until period ($j$-1) and then a credit event (a default for example) in period $j$, $D(T_j)$ is the discount function and R is the recovery rate.

2. For the buyer of the contract this is:

$$A_n = C_n h \sum_{j=1}^{N} Q(T_{j-1})\; D(T_j) \tag{7}$$

where C is the premium of the contract.

Then both legs should equal each other $A_n = B_n$, allowing us to solve for the coupon $C_n$ of the $n$-period CDS contract.

*2.1.3 Third building block: Flexible recovery function*

The third building block is the bond's recovery rate and is modelled as follows

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[2] Next to these three credit events there are also Obligation Default, Obligation Acceleration and Repudiation/Moratorium (Markit, 2008, p. 27).

$$R(i,j) = \Phi(a_o + a_1 \delta_j^i) \tag{8}$$

where $\Phi(.)$is the cumulative normal distribution, $\delta_j^i$ is the default probability already defined in first building block and $a_0$ and $a_1$ are parameters to be estimated. This functional form has the advantage that it is always positive (i.e., one cannot lose more than one owns). Then we can solve the model for $a_0$, $a_1$ and $b$ given that all the other variables are market observables.

### 2.2 Extension of the model

We would like to know how volatility is incorporated in the DH model and therefore we inspect the three parameters ($a_0$, $a_1$ and $b$). It is parameter $b$ that determines local volatility in the DH model as the parameters $a_0$ and $a_1$ only affect the recovery rate via expression (8) . To analyze how the $b$ parameter influences the volatility surface, we could theoretically calculate the local volatility at each node, but we choose to calculate implied volatilities directly from option prices because it is simpler and in line with the way we calculate implied volatilities later.

The CRR binomial tree was originally developed as an easy alternative to understand the Black-Scholes formula (Black and Scholes, 1973). The method starts with creating a binomial tree for a call option with a given volatility and initial stock price. The last layer in the binomial tree gives the stock prices at option's expiration date. Then the call option price is calculated by subtracting the exercise price from each terminal state price and multiplying it by the Arrow-Debreu state price ($\lambda$). The price of a call option with exercise price $K$ and time to expiration $j$ is:

$$C[K,j] = \sum_{i=0}^{j} \lambda_j^i \max(S_j^i - K, 0) \tag{9}$$

The price of a put option is:

$$P[K,j] = \sum_{i=0}^{j} \lambda_j^i \max(K - S_j^i, 0) \tag{10}$$

The Arrow-Debreu price ($\lambda$) is the price of a security that has a one unit payoff if the tree arrives at that state and a 0 unit payoff otherwise. If there is no discounting, the Arrow-Debreu price is equal to the probability of reaching that state. Otherwise, the probability needs to be discounted using the risk-free rate of interest. The Arrow-Debreu price is thus in fact the discounted state probability.

When we have calculated the price of an option as explained above, we can plug the price and all the parameters (time to expiration, interest rate, stock price and strike) back into the Black-Scholes formula to solve for the implied volatility.[3] The implied volatility is just another way of quoting the option's price. We can construct an implied volatility surface by varying (1) the exercise price (e.g., the log moneyness) and (2) the time to expiration.

The relationship between the moneyness and the implied volatility is flat in the CRR binomial tree, just as in the Black-Scholes formula. However, the relationship that has been observed in the options market is not flat, but has a skewed structure with a negative slope. Out-of-the-money put options (negative log moneyness) have higher implied volatilities than at-the-money options (zero log moneyness). This phenomenon is referred to as the "volatility skew".[4]

The relationship between time and implied volatility is flat in the CRR binomial tree, just as in the Black-Scholes formula. However, the relationship that has been observed in the options market is not flat, but either upward or downward sloping and this phenomenon is called "the term structure of volatility"

DH augmented the CRR binomial tree with an additional node so that the stock price can also jump to default. The risk of a default increases if either *b* becomes lower or the stock price becomes lower as can be seen in equation (3). As we know from the above, the CRR binomial tree will produce a flat volatility curve and the question we would like to answer is if the addition of the *b* parameter would change this. Therefore, we run a few scenarios, where we vary several parameters; that is, we investigate the impact of the parameter *b*, the moneyness (*M*), initial stock price (*S*), and the number of steps in the binomial tree on the implied volatility. In our experiment, we analyze the impact on a two-year option, setting interest rates and dividends equal to zero.[5]

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[3] It may be a surprise, but the implied volatility is not exactly equal to the volatility we have used to construct the CRR binomial tree. One of the reasons is that the CRR method is only approximating the Black-Scholes formula asymptotically (e.g., the time step in the binomial tree should be very small).

[4] Three patterns that are observed in the options market: (1) volatility smile (curvature around the at-the-money point-and prevalent in currency markets), (2) volatility skew (slope of the curve-and prevalent in equity markets), and (3) volatility smirk (superposition of the smile and the smirk-prevalent). See Zhang and Xian (2008).

[5] We first calculate option prices for each exercise price by first calculating the payoff in the binomial tree and second multiplying this payoff by the Arrow-Debreu security price. Then we derive the implied volatility by the obtained from the Black-Scholes model.

The results of the experiment are shown in Figure 1. In panel a of the figure, we have set the time step in the binomial tree to 0.5 year (6 months)[6] and volatility to 25%. Then we vary the *b* from 0.5 to 2 with steps of 0.5 and include a very high *b* of 100. If we set *b* to 100, equation (3) will be approximately zero and therefore the survival probability *z* is approximately one. Therefore, this case can be seen as the standard CRR binomial tree where there is no jump to default.

Two things are quite apparent from panel a in Figure 1: (1) when we increase *b* from the initial value of 0.5, the curve rapidly converges from a skewed curve to a kind of hump shaped curve where there is no visible difference between a curve with a *b* value of 1.5 and a *b* value of 100 and (2) the curve converges not to the straight line where $\sigma = 0.25$ but to a rather hump shaped curve.

Considering this last observation, we know that the binomial model for pricing options is only asymptotically an approximation for the Black-Scholes formula. Therefore, we increase the step size in the tree from 0.5 years to 0.05 years in panel b in Figure 1 and see that the implied volatility converges to a straight line as there is hardly any difference between the highest values of *b* and the straight line $\sigma = 0.25$. In panel c in Figure 1 we repeat the exercise of the top left only with a volatility of 50% instead of 25%; panels d and e in Figure 1 show the results from repeating the exercise but with an initial stock price of 10 instead of 100.

Based on the graphs in Figure 1, we can make two observations on the volatility structure in the DH model. First, the skew is only apparent for stocks with low values for *b* and/or low stock prices. As is shown in in the study by Das and Hanouna (2009),[7] the best three quintiles in terms of expected default frequency have a *b* value of around 1.25 and a stock price of around 20 should therefore based on our analyses have a rather flat volatility surface. Second, when there is a skew, the at-the-money volatility (moneyness is zero) is much higher than the option market indicates. Das and Hanouna (2009) report that the worst quintile in terms of expected default frequency has a *b* value of -0.35. However we see in Figure 1 that a higher value of *b* not only skews the implied volatility curve,

[6] This means we will have five layers in the tree. The initial stock price at time $t$=0 and then then the layers at $t$=0.5, $t$=1, $t$=1.5 and $t$=2.

[7] See Table 2 in their study. Note, however, the there is a printing error in the published article as the *b* values are omitted. They are available in the authors' working paper.

but also shifts the implied volatility upwards. This indicates that even the at-the-money volatility is not similar to the at-the-money option implied volatility.

For these reasons, we explore the extension of the model wherein we calibrate the binomial model based on information from options. These models, referred to as implied binomial trees, do not allow for a jump to default. Therefore we have to enhance these models with an additional jump to default as is done in DH.

The original idea of a binomial implied tree came from DK, who constructed a tree that is recombining and can exactly price the options with different strike prices and thus different implied volatilities. Barle and Cakici (1998), BC hereafter, observed that the DK methodology can sometimes lead to negative transition probabilities and non-risk-neutrality at the branch level, especially when interest rates are high. They proposed two modifications: (1) setting the option's exercise price equal to the stock's forward price instead of the stock price and (2) centering the tree on the stock's forward rate instead of the initial stock price. We included both modifications in our implied tree with a jump to default.

### 2.3 Implied tree with a jump to default

The world of implied trees is not as simple as the world of the CRR tree. In the CRR world, if we know the volatility at time $t$ = 0, the next nodes in the tree are given by the exponential of the volatility times the time step (see equation (1)). In the world of implied trees, we need to construct the tree node by node in each layer calibrating it to option prices. Therefore, we start with a cryptic description, stating that the stock price $S$ can go in the next period to one of the following three states:

$$S_j^i \longrightarrow \begin{cases} S_{j+1}^{i+1} & \text{with probability } q_j^i \ z_j^i \\ S_{j+1}^{i} & \text{with probability } (1\text{-}q_j^i)\ z_j^i \\ 0 & \text{with probability } 1 - z_j^i \end{cases} \tag{11}$$

where the subscript $j$ indicates time and subscript $i$ denotes the state. Both $i$ and $j$ start at 0, so the initial state of the world is $S_0^0$. From time $j$=0 to time $j$=1, the stock can either go down to state $i$=0 or up to state $i$=1.

#### 2.3.1 Risk Neutrality

The binomial trees of DK, BC, and CRR are based on the no-arbitrage condition. Therefore, the branching process in our implied tree with a jump to default must satisfy the risk-neutral condition based on (11):

$$F_{j,j+1}^{i} = z_j^i q_j^i S_{j+1}^{i+1} + z_j^i (1 - q_j^i) S_{j+1}^i \tag{12}$$

where $F_{j,j+1}^{i}$ is the stock's forward rate for the next period and is equal to $S_j^i exp^{(r_{j,j+1} - qd_{j,j+1})}$, e.g. the stock price *S* plus the interest *r* minus the dividend *d*.

We can write equation (12) in terms of probability $q_j^i$:

$$q_j^i = \frac{\frac{F_{j,j+1}^{i}}{z_j^i} - S_{j+1}^i}{S_{j+1}^{i+1} - S_{j+1}^i} \tag{13}$$

In the BC implied tree there is no default probability and therefore $z_j^i$ equals one. If we do this in expression (13), then we arrive at no-arbitrage condition in the BC model. Based on that equation, BC proposed two modifications to the original DK tree. We will apply the same two modifications to our implied tree with a jump to default model.

First we set the exercise price $K_j^i$ equal to the forward rate $F_{j,j+1}^{i}$ divided by the survival probability $z_j^i$:

$$K_j^i = \frac{F_{j,j+1}^{i}}{z_j^i} \tag{14}$$

Second, we let the tree grow with $K_j^i$. Therefore, the BC tree will be a nested solution of our model when the survival probability $z_j^i$ is equal to one.

### 2.3.2 Arrow-Debreu Pricing

In the previous section we introduced the Arrow-Debreu price ($\lambda$) as the discounted probability of reaching a certain node in a tree. The most upper and lower nodes in a binomial tree are the most easy to describe in terms of Arrow-Debreu pricing since they can only descend from one previous node.

We can describe the Arrow-Debreu price for the upper most branch as:

$$\lambda_j^i = e^{-\mathrm{rh}} \lambda_{j-1}^{i-1} q_{j-1}^{i-1} * z_{j-1}^{i-1} \tag{15}$$

The Arrow-Debreu price $\lambda_j^i$ is defined as the previous Arrow-Debreu price $\lambda_{j-1}^{i-1}$ times the probability of the stock going up in the next period $q_{j-1}^{i-1}$ times the probability that it does not default $z_{j-1}^{i-1}$ and then we discount the value using $e^{-r\mathrm{h}}$ .

In a similar way we can define the Arrow-Debreu prices for the lowest nodes in the tree as:

$$\lambda_j^i = e^{-\mathrm{rh}} \lambda_{j-1}^i \left(1 - q_{j-1}^i\right) z_{j-1}^i \tag{16}$$

where now $\left(1 - q_{j-1}^i\right)$ is the probability that the stock goes down.

For the nodes in between the upper and lower nodes (if any), we need to calculate the Arrow-Debreu prices from the two directions it can come from:

$$\lambda_j^i = e^{-\mathrm{rh}} \lambda_{j-1}^{i-1} * q_{j-1}^{i-1} z_{j-1}^{i-1} + \lambda_{j-1}^i \left(1 - q_{j-1}^i\right) z_{j-1}^i \tag{17}$$

### 2.3.4 Binomial Option Pricing

We can value an option with the Black-Scholes formula, but since we need to know the stock prices in the binomial tree, we apply binomial option pricing where the value of the option is equal to the value of the payoff of that option times the Arrow-Debreu security price in that state. So the price of a call option $C[K,j]$ with exercise price *K* and time to maturity *j* is equal to the payoff in layer *j* times the corresponding Arrow-Debreu state price $\lambda_j^i$:

$$C[K,j] = \sum_{i=0}^{j} \lambda_j^i \max\left(S_j^i - K, 0\right) \tag{18}$$

and the expression for the put option:

$$P[K,j] = \sum_{i=0}^{j} \lambda_j^i \max\left(K - S_j^i, 0\right) \tag{19}$$

#### 2.3.5 The upper nodes in the implied tree with a jump to default

Now that we have all the ingredients (risk-neutrality, Arrow-Debreu and binomial option pricing) we can start building the implied tree with a jump to default.

We know from expression (14) that the exercise price is equal to $K_j^i$. So therefore expression (18) becomes::

$$C\left[K_{j-1}^k, j\right] = \sum_{i=0}^{j} \lambda_j^i \max\left(S_j^i - K_{j-1}^k, 0\right) \tag{20}$$

The exercise price $K_{j-1}^k$ is also chosen because it falls between the stock price when it goes up ($S_j^{i+1}$) and when it goes down ($S_j^i$) as probability $q_j^i$ must be between zero and one. Therefore we will separate expression (20) in the counters that sum up to *k* and the ones above:

$$C\left[K_{j-1}^k, j\right] = \sum_{i=0}^{k} \lambda_j^i \max\left(S_j^i - K_{j-1}^k, 0\right) + \sum_{i=k+1}^{j} \lambda_j^i \max\left(S_j^i - K_{j-1}^k, 0\right) \tag{21}$$

Then we know that the payoff of the first summation part in expression (21) is always zero (as the exercise price is above the stock price) and the payoff in the second summation part is always positive (as the stock price is always above the exercise price). Therefore, expression (21) reduces to:

$$C\left[K_{j-1}^k, j\right] = \sum_{i=k+1}^{j} \lambda_j^i \left(S_j^i - K_{j-1}^k\right) \tag{22}$$

Then we separate both *i=j* and *i=k+1* from the summation term

$$C\left[K_{j-1}^k, j\right] = \lambda_j^j\left(S_j^j - K_{j-1}^k\right) + \lambda_j^{k+1}\left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j-1} \lambda_j^i \left(S_j^i - K_{j-1}^k\right) \tag{23}$$

Remember from the Section 2.3.2 on Arrow-Debreu pricing that the most upper and most lower branch are the simplest to describe as they can only descend from one previous node. We can use this to replace the $\lambda$'s (Arrow-Debreu prices) in equation (23) with the $\lambda$'s from equations (15) to (17). We know that the $\lambda$'s for node *i=j* are simpler as expressed in equation (15) than the other $\lambda$'s as expressed in equation (17).

$$C\left[K_{j-1}^{k},j\right]=e^{-r\mathrm{h}}\Big\{\lambda_{j-1}^{j-1}q_{j-1}^{j-1}z_{j-1}^{j-1}*\left(S_j^j-K_{j-1}^k\right)+(\lambda_{j-1}^k q_{j-1}^k z_{j-1}^k \\ +\lambda_{j-1}^{k+1}\left(1-q_{j-1}^{k+1}\right)z_{j-1}^{k+1}\,(S_j^{k+1}-K_{j-1}^k) \\ +\sum_{i=k+2}^{j-1}(\lambda_{j-1}^{i-1}q_{j-1}^{i-1}z_{j-1}^{i-1}+\lambda_{j-1}^{i}z_{j-1}^{i}(1-q_{j-1}^{i}))\,(S_j^i-K_{j-1}^k)\Big\} \tag{24}$$

Then with some mathematical manipulation that we show stepwise in Appendix A, we arrive at:

$$S_j^{k+1}=\frac{S_j^k\left(C\left[K_{j-1}^k,j\right]e^{r\mathrm{h}}-\rho_{up}\right)-\lambda_{j-1}^k z_{j-1}^k K_{j-1}^k\left(K_{j-1}^k-S_j^k\right)}{C\left[K_{j-1}^k,j\right]e^{r\mathrm{h}}-\rho_{up}-\lambda_{j-1}^k z_{j-1}^k\left(K_{j-1}^k-S_j^k\right)} \tag{25}$$

where $\rho_{up}=\sum_{i=k+2}^{j}\lambda_{j-1}^{i-1}z_{j-1}^{i-1}\left(\frac{F_{j-1,j}^{i-1}}{z_{j-1}^{i-1}}-K_{j-1}^k\right)$

Equation (25) parametrizes $S_j^{k+1}$ in terms of $S_j^k$ and reduces to the solution of BC's equation (12) when we set $z$ equal to one.

#### 2.3.5 The central nodes in the implied tree with a jump to default

When we start growing the tree, we do not know $S_j^k$ and as a result cannot solve equation (25). Consequently, we have to make an assumption about the central nodes as was done by DK and BC. Following BC, we center the tree on the expected path. BC centered the tree on the stock's forward price using the risk-neutrality condition in expression (13) when $z$ is equal to one; therefore we center the tree on the forward rate divided by the survival probability as in our model.

So when $j$ is even[8]

$$S_j^{j/2}=\frac{F_{j-2,j}^{(j-2)/2}}{\left(z_{j-2}^{\frac{j-2}{2}}\right)^2} \tag{26}$$

and we can use expression (26) in expression (25), so that the node above the central node is given by:

$$S_j^{k+1}=\frac{\frac{F_{j-2,j}^{(j-2)/2}}{\left(z_{j-2}^{(j-2)/2}\right)^2}\left(C\left[K_{j-1}^k,j\right]*e^{r\mathrm{h}}-\rho_{up}\right)-\left(\lambda_{j-1}^k*z_{j-1}^k*K_{j-1}^k\left(K_{j-1}^k-\frac{F_{j-2,j}^{(j-2)/2}}{\left(z_{j-2}^{(j-2)/2}\right)^2}\right)\right)}{C\left[K_{j-1}^k,j\right]*e^{r\mathrm{h}}-\rho_{up}-\lambda_{j-1}^k z_{j-1}^k*\left(K_{j-1}^k-S_j^k\right)}\ \forall\, j=even \tag{27}$$

When $j$ is odd, we choose the following centering condition:

[8] We will make a small approximation here as we take $\left(\boldsymbol{z}_{j-2}^{((j-2)/2)}\right)^2$ instead of $\boldsymbol{z}_{j-1}^{((j-2)/2)}*\boldsymbol{z}_{j-1}^{((j-2)/2-1)}$. We take the survival probability at $j$-2 as this node has one central node instead of $j$-1 that has two central nodes. This this is mathematically easier to implement.

$$S_j^{(j+1)/2} = \frac{\left(\frac{F_{j-1,j}^{(j-1)/2}}{z_{j-1}^{(j-1)/2}}\right)^2}{(S_j^{(j-1)/2}} \tag{28}$$

This is similar to the approach used by CRR and BC where they center the tree on the exercise price. In CRR, the centering condition is $S_j^{(j+1)/2} = \frac{\left(S_{j-1}^{(j-1/2)}\right)^2}{(S_j^{(j-1)/2}}$ and in the BC model the centering condition is $S_j^{(j+1)/2} = \frac{\left(F_{j-1,j}^{(j-1)/2}\right)^2}{(S_j^{(j-1)/2}}$.

We can then replace the lower central node prices $S_j^k$ in expression (25) with expression (28) which takes the form of (e.g.: $k$=($j$-1)/2 and $k$+1=($j$+1)/2)

$$S_j^k = \frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}} \tag{29}$$

which results in:

$$\left(S_j^{k+1}\right) = \frac{\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\left(C\left(K_{j-1}^k, j\right) * e^{\mathrm{rh}} - \rho_{up}\right) - \lambda_{j-1}^k * z_{j-1}^{k,k+1} * K_{j-1}^k\left(K_{j-1}^k - \frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right)}{C\left[K_{j-1}^k, j\right] * e^{\mathrm{rh}} - \rho_{up} - \lambda_{j-1}^k z_{j-1}^k * \left(K_{j-1}^k - \frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right)} \tag{30}$$

Then after some mathematical manipulations, that we show stepwise in Appendix B, this becomes:

$$S_j^{k+1} = K_{j-1}^k \frac{\lambda_{j-1}^k * F_{j-1,j}^k + \left(C\left[K_{j-1}^k, j\right] * e^{\mathrm{rh}} - \rho_{up}\right)}{\lambda_{j-1}^k * F_{j-1,j}^k - \left(C\left[K_{j-1}^k, j\right] * e^{\mathrm{rh}} - \rho_{up}\right)} \quad \forall\, j = odd \;\; \& \, k = (j+1)/2 \tag{31}$$

which is our solution for the upper central node ($k$=($j$+1)/2) and $j$ is not even.

We can use the centralizing condition (29) to find the lower central node:

$$S_j^k = K_{j-1}^k \frac{\lambda_{j-1}^k * F_{j-1,j}^k - \left(C\left[K_{j-1}^k, j\right] * e^{\mathrm{rh}} - \rho_{up}\right)}{\lambda_{j-1}^k * F_{j-1,j}^k + \left(C\left[K_{j-1}^k, j\right] * e^{\mathrm{rh}} - \rho_{up}\right)} \quad \forall\, j = odd \;\; \& \, k = (j-1)/2 \tag{32}$$

Equation (32) reduces to the solution of BC's equation (13) when we set $z$ equal to one.

### 2.3.6 The lower nodes in the implied tree with a jump to default

In BC and DK the solution for the lower part of the tree is much simpler since we already know the upper nodes. In our case the price of the put option has another component that represents the price of a jump to default:

$$P\left(K_{j-1}^{k}, j\right) = \sum_{i=0}^{j} \lambda_{j}^{i} \max\left(K_{j-1}^{k} - S_{j}^{i}, 0\right) + \lambda_{j}^{jtd} max\left(K_{j-1}^{k} - 0{,}0\right) \tag{33}$$

Since we have assumed that the price of the stock jumps to zero in the case of a default, the last part can be simplified to just $\lambda_{j}^{jtd} K_{j-1}^{k}$ . This last part is the price of a jump to default option that only pays off in case of a default and therefore has very similar characteristics as a CDS. We can express this jump to default option as:

$$P_{jtd}\left(K_{j-1}^{k}, j\right) = \lambda_{j}^{jtd} K_{j-1}^{k} \tag{34}$$

We do not know the price of $\lambda_{j}^{jtd}$ in expression (33) at time $j$ . Consequently, similar to what we have done with the Arrow-Debreu prices in equations (15) through (17), we express $\lambda_{j}^{jtd}$ in terms of $j$-1. Therefore:

$$\lambda_{j}^{jtd} = e^{-r\mathrm{h}} * \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * \left(1 - z_{j-1}^{i}\right)\right) \tag{35}$$

The Arrow-Debreu price of a jump to default in period *j,* $\lambda_{j}^{jtd}$, is equal to the discounted value of the Arrow Debreu price of a jump to default in period *j-1,* $\lambda_{j}^{jtd}$, plus the marginal probability of default coming from the individual nodes in the layer $j$-1. The good thing about expression (34) is that it contains information that we already know, so that we can easily solve the expression for the lower nodes

Now we can use expressions (33) through (35) and find the solution for the lower nodes in the tree and the derivation of the solution is much simpler, w as shown in appendix C:

$$S_{j}^{k} = \frac{S_{j}^{k+1} * \left(e^{r\mathrm{h}}\left(P_{df}\left[K_{j-1}^{k}, j\right]\right) - \rho_{down}\right) - \lambda_{j-1}^{k} * F_{j-1,j}^{k} * \left(S_{j}^{k+1} - K_{j-1}^{k}\right)}{e^{r\mathrm{h}}\left(P_{df}\left[K_{j-1}^{k}, j\right]\right) - \rho_{down} - \lambda_{j-1}^{k} * z_{j-1}^{k} * \left(S_{j}^{k+1} - K_{j-1}^{k}\right)} \tag{36}$$

with $\rho_{down} = \sum_{i=0}^{k-1} \lambda_{j-1}^{i} * z_{j-1}^{i} * \left(K_{j-1}^{k} - \frac{F_{j-1,j}^{i}}{z_{j-1}^{i,}}\right)$

and $P_{df}\left[K_{j-1}^{k}, j\right] = P\left[K_{j-1}^{k}, j\right] - P_{jtd}\left[K_{j-1}^{k}, j\right]$

and $P_{jtd}\left[K_{j-1}^{k}, j\right] = e^{-r\mathrm{h}}\left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * \left(1 - z_{j-1}^{i}\right)\right) K_{j-1}^{k}$

Expression (36) reduces to the of BC's equation (15) when we set $z_{j-1}^{k}$ equal to one. Expression (36) above reveals the price of a default-free put option $P_{df}$. Carr and Madan (2010) also use default-free stock processes in their (continuous-time) version of local volatility enhanced with a jump to default. Here the price of the default-free put option is equal to the price of a put option minus the price of the jump to default put option. There is an interesting relation between the price of a jump to default put option and the price of a CDS as indicated by Carr and Wu (2011)[9], but we leave that empirical test to further research.

**2.4 Requirements for the Implied Tree**

As documented by BC, an implied tree should satisfy the following criteria:

1. Correct reproduction of the volatility smile
2. Node transition probabilities (*q*) lying in the [0,1] interval only
3. Risk-neutral branching process (forward price of the underlying asset equals the conditional expected value of itself) at each step.

The tree is calibrated with option prices to fulfill requirement one. As documented by BC, implied trees sometimes violate requirement 2.

From the no-arbitrage condition in expression (13), we can deduce that $q_j^i$ will be in the limit [0,1] only when the ratio $\frac{F_{j,j+1}^{i}}{z_j^i}$ satisfies:

$$S_{j+1}^{i+1} > \frac{F_{j,j+1}^{i}}{z_j^i} > S_{j+1}^{i} \tag{37}$$

So that the stock price $S_{j+1}^{i+1}$ is limited to the following inequality:

$$\frac{F_{j,j+1}^{i}}{z_j^i} < S_{j+1}^{i+1} < \frac{F_{j,j+1}^{i+1}}{z_j^{i+1}} \tag{38}$$

[9] The price of a digital default swap (a CDS that pays one dollar at default) is equal to the price of the jump to default put option divided by the strike price.

The stock price in period $j+1$ at node level $i+1$ should be in between the forward rate divided by the survivorship probability at $j$ and node level $i$ and the forward rate divided by the survivorship probability at $j$ and node level $i+1$. If there is no default probability, then the survivor probability will always be equal to one and inequality (38) is equal to that derived by BC.[10]

If the stock price does not satisfy inequality (38), we follow the override procedure suggested by Härdle and Mysickova (2008), replacing it by:

$$\ln S_{j+1}^{i+1} = \ln \frac{S_j^i}{S_{j-1}^{i-1}} + \ln S_j^i \tag{39}$$

The logarithmic difference between adjacent stock prices in the tree is the same

When the resulting stock price still does not satisfy (38), we apply the second override procedure suggested by Härdle and Mysickova( 2008):[11]

$$S_{j+1}^{i+1} = \frac{1}{2}\left(\frac{F_{j,j+1}^{i+1}}{z_j^{i+1}} + \frac{F_{j,j+1}^{i}}{z_j^{i}}\right) \tag{40}$$

The stock price is equal to the average of the two forward rates.

### 2.5 Volatility Surface

Although in the Black-Scholes model the implied volatility is constant, in practice the implied volatility of an option depends on the strike price and the time to expiration. The implied volatility for at-the-money put options is usually lower than for (far) out-of-the-money put options. This effect is referred to as the "skew effect." Also options with the same strike price, but different expiration dates trade on different implied volatilities. This is referred to as the term structure of volatility. Together the skew and term structure effects are usually referred to as the volatility surface. This volatility surface is a dynamic relationship: if markets are panicking, the skew might be steeper and when the company is reporting earnings on a short horizon, the term structure of volatility might be inverted.

[10] See equation (5) in their paper.

[11] For the DH model we also set a minimum value of *b* based on the expression (39). The stock goes either up to $e^{\sigma\sqrt{h}}$ or down to $e^{-\sigma\sqrt{h}}$. Then we can determine the minimum value for *b* so that expression (39) is satisfied. Setting this minimum value for *b* also for the implied models helps in finding the initial solution. For the other models, in a limited number of cases, it was also helpful to set a minimum value for *b*.

The implied volatility surface is derived from quoted option prices that have different strikes and maturities. If we construct the implied tree, we might need strike prices or time to maturities that are not available directly from quoted option prices. Therefore we would like to have a continuous functional form for the implied volatility. Derman and Kani (1994, p. 10) provide an example of how to model the effect of the strike in the form of a deterministic function: the implied volatility increases linearly by 0.5 percentage points with every 10 point drop in the strike price. While several deterministic functions have been documented,[12] we choose to follow Dumas et al. (1998) who run a polynomial regression on quoted option prices. The implied volatility $\sigma$ is a function of the moneyness *M* and time to expiration *T*:

$$\sigma(M,T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT \tag{41}$$

where *MT* is the cross-product of *M* and *T* and *T* is defined as the log of the time to expiration and we define moneyness *M* as:

$$M = \log\left(\frac{K}{F}\right) \tag{42}$$

where *K* is the strike price and *F* is the stock's forward price. Since *T* and *M* are both in log terms, the intercept can be interpreted as the implied volatility of a one year at-the-money option.

We estimate the regression given by (41) in Section 4 and then use the discrete volatility function when constructing the binomial tree.

## 3. Data

In this section we describe the data needed to estimate the model.

### 3.1 Choice of universe

To estimate impact of the implied tree on the recovery rates of corporate bonds, we need data from companies that are listed on a stock exchange, have equity options, and have public debt outstanding so that there is an active CDS market. Additionally, we would like to have companies

[12] See also Ait-Sahalia and Lo (1998) where they use a kernel regression.

with a long track record and prefer to have a predefined list of stocks rather than select stocks for this study ourselves.

Consequently, we would like to have a universe of companies satisfying the above restrictions that is compiled by some external provider. However, we have found that indices are segmented in asset class as well as credit quality. For example, there is no fixed-income benchmark that includes only listed companies. Fixed-income benchmark constructers typically only include assets based on the fact that they have debt outstanding. The same is true for equity indices. For CDS indices, the main relevant reason for a constructer such as Markit is trading volume. Additionally, we observe that many fixed-income benchmarks are segmented by either investment grade or high yield corporate issues.

We considered using all available CDS quotes from the Markit CDS database and filter on the listed companies; however, this information is not available from Markit. Moreover, we would also have many companies included only for a few months. While we considered using the Markit CDX index, this index also separates credit quality and not all constituents of the index have publicly traded stocks. Another alternative that we considered was the NASDAQ 100 index, especially because the index is diverse in credit quality. Unfortunately, for this index not many stocks have CDS quotes. Therefore we choose the 30 companies that represented the Dow Jones Industrial Index (DJIA) as of September 24, 2012. We acknowledge that this universe is tilted towards the higher graded companies in the universe.

### 3.2 CDS data

We use the Markit Corporate Debt CDS database. This database contains the daily pricing history for single name CDS spreads from January 2001 to June 2014. The CDS spreads in the database are composite spreads based on contributions from different brokers. Each reference name has different entries per date because currency, clause and seniority can be different. We restrict our sample to the 30 stocks in the DJIA and take the corresponding spreads that (1) are denominated in

U.S. dollars (USD), (2) are senior in tier, and (3) have a modified restructuring document clause.[13] Although the database contains information on the full term structure of CDS spreads,[14] we are only interested in tenors up to two years because listed stock options typically have a maximum time to expiration of two years. Three tenors are available for this part of the term structure: six months, one year, and two years.

Five-year tenors are seen to be the most liquid contracts; we compare in Figure 2 the availability of this contract with the three tenors that we want to use. At the start of the Markit database in January 2001 not all 30 companies are represented or have a no full CDS term structure. The Markit CDS spread is a composite of quotes from individual contributors (brokers) and so if there is no CDS spread available either there are no contributors or the Markit composite rules are not fulfilled.[15] At the end of January 2001, only 13 stocks had a CDS spread for the five-year tenor and for the six-month tenor there was no CDS pricing at all. In subsequent years more and more contracts were quoted in the market and appearing in CDS database. At the end of July 2003 all 30 stocks had a CDS quote for the five-year tenor. It took however until the end of August 2005 before the CDS spreads of all 30 stocks were consistently available in the database.[16] The one- and two-year tenors followed the five-year tenor closely in availability, with only the six-month tenor lagging. When all five-year tenors had CDS spreads available at the end of July 2003, only 12 six-month tenors had a CDS price. The availability of the five-year tenors stabilized after August 2005; however, the availability of six-month tenor quotes varied widely after this date. In August 2005 there were 26 CDS spreads, but only 20 in January 2007.

If we want to start our sample when all prices are available, we can only start in April 2013. However, since prices of CDS with six-month tenors are not that different from one-year tenors and

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[13] This document clause is the most prevalent for US Investment Grades (Markit, 2008, p. 28)

[14] The database contains tenors (in years) of 0.5, 1, 2, 3, 4, 5, 7, 10, 15, 20 and 30. Not all those tenors are always filled.

[15] The composite price is the average of the prices provided to Markit by its contributors once those prices failing any one of the data quality tests have been excluded. In order to form a composite, Markit requires at least three distinct contributors submitting curves of which at least two pass all data cleaning tests Markit (2011). Microsoft (MSFT) had no public debt outstanding in the period 2001-2009 (source Bloomberg).

[16] There is one exception in that MSFT did not have a price for the five-year tenor at the end of February 2009 (seen as the height of the credit crisis)

since most of the time only a small part of the six month tenors are not available, we decided to start the sample in August 2005, the date from when the five-year tenor is consistently available. When a tenor is not available, we use the average of the longer and shorter tenor and when no shorter tenor is available, we use the higher tenor. We also incorporate the 18-month tenor as done in DH by interpolating the quoted prices. For each company we therefore have four points on the CDS curve.

<<Insert Figure 2 about here>>

### 3.3 Equities, options and yield curves

We collect all other data that are needed to estimate the model from the Ivy DB Option Metrics[17] database that is available on Wharton Research Database. The data in Optionmetrics is available for a much longer history[18] than is available for the CDS data and therefore we match it to the length of the CDS data: August, 31 2005 till June 30, 2014. We obtained four different files for all 30 stocks in our universe: (1) the volatility surface, (2) standardized options, (3) security prices, and (4) zero-coupon yield curves.

From the volatility surface file we take the implied volatility, implied strike, and the time to expiration at each month end for all 30 companies. The forward price in the standardized file includes the dividend and the interest rate. However, we prefer to calculate the forward price ourselves[19] for two reasons. First, not all forward prices are available at all times.[20] Second, we may need forward prices longer than two years (up to five years) and the option market and consequently Optionmetrics only has information up to two years.

[17] We downloaded for each of the 30 companies all the information in the menu (1) standardized options, (2) volatility surface, and (3) securities prices.
[18] Data are available from January, 1 1996 to August 31, 2014.
[19] In cases where we have a forward price, we calculate the dividend rate as follows: $\mathrm{dr}_t = \log\left(\frac{P\_forward_t}{r_t}\right)/t$ . In cases where we do not have a forward rate, we calculate it as $P\_forward_t = P_{Spot}\exp\left((r_t - dr_{t-1})\ \mathrm{TTM}\right)$. So for the longer-term option, we take the same dividend rate as in the prior period.
[20] There seems to be a discrepancy between the volatility surface file and the standardized files at some points. For example, for the ticker BAC on 07/31/2013, the volatility surface goes up to 730 days while the standardized file only goes up to 365 days.

We only include options that are out-of-the-money since in the money options trade less and are more expensive. Therefore, we only include call options where the moneyness is greater than zero ($M > 0$) and put options where the moneyness is smaller than zero ($M<0$).

The zero-coupon yield curve is only available in certain standard maturities[21]. We simply interpolate the zero-coupon curve to match the maturities with the tenors of the CDS.

## 4 Empirical Results

In this section we report the results of the tests of our model.

### 4.1 Discrete Volatility Function

We run the cross-sectional regression in equation (40) for each of the 30 companies at each month end for the period August 2005 to June 2014. We only included the implied volatilities with a time to expiration of 6, 12, 18, and 24 months to match the tenors from the CDS available quotes. In Table 1, we report that the regressions explain most of the variation as can be seen in the last two columns. All R-squares exceed 0.9 (the average is 0.97). In the preceding columns, we report the average of the coefficients over the 107 regressions. The second column (C) shows the constant in equation (40) and represents the volatility of an at-the-money option with a one-year time to expiration. For example, Alcoa (AA) has an average volatility of 38.7% over this period. This volatility is not constant, as the standard deviation of the monthly coefficients is also large (13.9%).

The fourth column ($M$) shows the results for the (log) moneyness. All stocks have a negative sign for the moneyness, which clearly confirms that there is volatility skew. The average of the moneyness coefficients varies from -10 to -22. For AA this indicates that the implied volatility of a put option that has a strike price that is approximately 10% below the forward price, the implied volatility is 2% higher.

The sixth column ($M^2$) shows the results for the squared moneyness variables. This variable represents the smile effect of the implied volatility. All average coefficients are positive and range from 6 to 22.6 and we see that the standard deviations are sometimes high. If we take the same

[21] Maturities are 9, 15, 50, 78, 169, 260, 351, 442, 559, 624, 715, and 806 days.

example as above, the effect of this variable on the implied volatility for AA is only an increase of 0.07%.

The eighth column ($T$) shows the coefficients from the time to expiration. All but one (i.e. AA) coefficients are positive and are around one. So if we extend the time to expiration from one year to two years, we increase the implied volatility on average by 0.7%. When we shorten the time to maturity, we decrease the implied volatility.

The tenth column ($T^2$) shows the coefficient from the squared time to expiration. The results are more mixed, most are negative but there are also some positive average coefficients. The effect from extending the time to expiration from one year to two years on the implied volatility when we have a coefficients of -0.5, as for example for IBM, will be a decrease in implied volatility of -0.25%. If we sum the two effects for the time to expiration for IBM, the impact of an extension of the time to expiration from one year to two years will be +0.5%.

The twelfth column ($MT$) shows the interaction effect. All average coefficients are positive, ranging from 3.4 to 9.6. This implies that the skew declines as maturity increases.

<Insert Table I about here>

### 4.2 Fitting the model

We can now calibrate the three parameters of the model ($a_0$, $a_1$, and $b$) using a least squares fit of the model CDS spreads to the market CDS spreads. The function that we minimize over is the mean squared error (MSE):

$$MSE = \frac{1}{4}\sum_{k=1}^{4}(CDS_{model} - CDS_{market})^2 \tag{43}$$

We fit the DH model and then the implied model with a jump to default where the local volatility is defined by the deterministic volatility function in equation (41). We fit the implied model with a jump to default based on different specifications for the implied volatility function[22] in equation (41). We specify:

(1) The constant volatility (CV) model where we include only the $\beta_0$ coefficient and set all other $\beta$ coefficients to zero. This model differs from the DH model in that it has constant volatility at each node, where the local volatility in the DH model increases as the stock price becomes lower.

(2) The term structure (TS) model where we include $\beta_0$, $\beta_3$, and $\beta_4$ coefficients and set the $\beta_1$, $\beta_2$, and $\beta_5$ coefficients to zero. Here the local volatility is constant in a certain layer in the tree, but differs between layers.

(3) The skew (SK) model where we include the $\beta_0$, $\beta_1$, and $\beta_2$ coefficients and set the $\beta_3$, $\beta_4$, and $\beta_5$ to zero.

(4) The volatility surface (VS) model that includes all $\beta$'s.

We calibrate the three parameters for each month for the period August 2005 to June 2014. The averages over those months for the four specifications above are reported in Table 2. Instead of reporting the MSE from expression (43), we report the error relative to the level of the CDS spread. Therefore, we use the same measure as is used in DH, the relative root mean square error (RRMSE). This measure is defined as the squared root of the MSE divided by the average of the four spreads in that month. For example for Boeing Company (BA), the average RRMSE for the full period in the DH model is 1.7%, that is, the standard deviation of the error is 1.7% around the mean of the four CDS spreads (six months, one year, 18 months and one year). In the last row we report the average RRMSE over the individual companies which is equal to 2% for the DH model. DH (2009, p. 1845) report a RRMSE of 4.05% for the companies with the lowest default probability over the period January 2000 to July 2002. We also have separated the sample into three equal subperiods. The first subperiod (August 2005 to July 2008) reflects the 36 months till the beginning of the credit crisis (Lehman collapse was in September 2008) with relatively low CDS spreads. The second subperiod

[22] In theory we could run a separate regression for each model, but we choose for simplicity to use the regression with all variables included and then select the $\beta$'s from that model for the submodel.

(August 2008 to June 2011) contains the 36 months that include the credit crisis with very high CDS spreads and the third subperiod (July 2011 to June 2014) covers the 35 months in the recovery period where CDS spreads decreased from their high levels. Although, the subperiods have different levels of credit spreads, we see that the average RRMSE over all companies for the DH model decreases in time from 2.3% in the first subperiod to 1.6% in the last subperiod.

<Insert Table 2 about here>

The second specification that we consider, the constant volatility model, has a RRMSE that is slightly above that of the DH model for the full period (2.2% versus 2%). Also we see that the RRMSE decreases with time as the error in the last subperiod is lower than the first subperiod.

The TS model has the lowest RRMSE of all models. The average RRMSE is slightly better than the DH model both for the full sample as well as all subperiods.[23] The TS model has a flat volatility structure, but the volatilities differ by tenor. On an individual company level, the TS model does not always have a lower RRMSE than the DH model. For the full period, the TS model has a lower RRMSE in 18 of the 30 cases. In the first subperiod the TS model has a lower error in 9 out of the 30 companies, in the second subperiod in 20 out of the 30 companies and in the last subperiod in 21 out of the 30 companies. Also for the TS specification, the errors seem to decrease in time as the average RRMSE for the first subperiod is 2.5%, the second subperiod is 1.9% and the last subperiod is 1.4%.

The results for the specifications where the skew parameters ($\beta_3$, $\beta_4$ and $\beta_5$) are included (SK and VS), the mean RRMSE are all well above the DH model and the errors do not have a specific trend in time. The error for the VS model is surprisingly stable at 2.3% both for the full period as for all subperiods. The error for the SK specification is a bit lower than for the VS specification, but for the subperiods there is no clear pattern.

[23] Although the table shows for formatting reasons an average MSE of 2% for both models, the average MSE for the TS model is 1.98% in the subsample August 2008- June 2011, while average MSE for the DH model is 2.01%.

Based solely on the perspective of the RRMSE metric, it seems that the CDS spreads are priced only in the dimension of time and not in the dimension of moneyness (i.e., skew). The fact that the RRMSE increases if we add the volatility skew suggests that the CDS market prices a different skew. Potentially we can imply the "CDS implied volatility" as a counterpart of the "option implied volatility" with our model if we have enough points on the CDS curve. Then we can minimize the RRMSE by calibrating not only the parameters $a_0$, $a_1$ and $b$, but also calibrate the β's of the deterministic volatility function. However, in our model we only have four prices on the CDS curve and therefore have an over-identification problem if we fit the eight parameters of the joint model. Consequently, we need more points on the CDS curve, for example, a curve that is specified per quarter or even per month instead of half a year. We leave this exercise for further research. Another interesting area for further research is to see if the mispricing with the skew is exploitable; that is, can we construct a trading strategy that buys the underpriced options/CDS and sell the overpriced options/CDS? Carr and Wu (2010, p. 488) mention that market participants see the arbitrage strategy of selling credit insurance through CDS and hedging the exposure by buying (deep) out of the money put options on the same company as a profitable strategy.

Although the differences between the RRMSE of the different models are close to each other, the estimated parameters are different. In Table 3, we report the average values of the different estimated parameters for the full sample and also the average of the stock price for the full period. If we compare the estimated values for the DH model with the results in the DH paper (p.1845) we see that the value of our parameter $b$ (1.24) isclose to the one in their paper (1.34) but that the values for $a_0$ (3.9 versus 2.1) and especially $a_1$ (-454 versus -4.1) are very different. The difference can be due to the estimation period or to the fact that $a_0$ and $a_1$ are highly correlated and therefore setting some restrictions on the value of $a_1$ alters the values of the estimated parameters but not so much of the RRMSE.

<Insert Table 3 about here>

Also as we know from DH (p. 1844), parameter $a_0$ has no economically determined sign in the model, but the other two parameters do. Parameter $a_1$ is negative in the full sample for all companies and model specifications and from equation (8) we know that this results in an inverse relation between the default probability and the recovery rate: if the default probability goes up, the recovery rate goes down. Parameter $b$ is positive (as required) for all companies and models and we know from equations (3) and (4) that as stock prices fall, the probability of default rises.

If we compare the $b$ parameters in the DH model with the ones from the implied models we see that they are lower for the constant volatility models but more or less the same for the specifications that include the skew. What we have seen in section 2.2 and also in Figure 1 is that a decrease in the value of the $b$ parameter will increase the ATM volatility and create a skew. So therefore, allowing a skew (either in the DH models or the SK and VS) will reduce the value of the $b$ and therefore by expressions (3) through (5) increase the default probability. However, although the DH, SK and VS models have the same value for the $b$, it does not mean that the default probabilities are equal, as the underlying binomial tree can be completely different. We will look at this in the Section 4.4.

The value of $b$ is more or less the same between the DH model and the SK and VS specifications, but the $a_0$ and $a_1$ parameters are different. The fact that these parameters are different gives us some indication that the recovery rates are different; however this is also dependent on the underlying binomial tree as we will see in Section 4.4.

### 4.3 Reverse engineering the implied volatilities from the tree

Although we have set the DH model as the benchmark for the different specifications of our implied model, we have to say that this is not a truly fair benchmark. It is not fair because the DH is less restrictive in volatility. As we have shown in Section 2.2, a lower value of $b$ in the DH model will create a volatility skew, but also increase the "at-the-money" implied volatility of the model to potentially unrealistic high levels. The implied models are by construction mimicking the implied volatility of the option market. So even if the DH model produces a lower error than an implied model, the implied model is more realistic in volatility terms.

<Insert Figure 3 about here>

Given that we have estimated the different models by calibrating the three parameters, we can also inspect the produced implied trees and reengineer the implied volatility from each tree to check if the developed model works and to evaluate if the model has used overrides to be risk neutral. Therefore, although we could look at an individual node level and calculate the local volatility of that node, it is complicated to calculate the local volatility over several nodes. Instead, we choose to calculate the binomial option price with Arrow-Debreu prices for the call option via expression (9) and for the put option via expression (35) and then calculate the implied volatilities using the Black-Scholes formula.

We also know that the binomial model is only an approximation for the Black-Scholes formula when the time step goes to zero. In our case, we have a time step of 0.5 and therefore we introduce an error in the binomial option price and thus ultimately in the implied volatility that is calculated via the Black-Scholes formula. However, as suggested by BC, we used Black-Scholes option prices to estimate our models, so for specific exercise prices in the tree, there should be no error. Therefore, if we want to look at the implied volatility for the two-year options, we can use the four exercise prices that we have used to estimate the model and compare those with the discrete volatility function. For other exercise prices, we will have an error in the option prices and therefore an error in the implied volatility. Ideally, we would like to choose the exercise price ourselves, since we would like to compare how well the implied model is in pricing the volatility tree over time. Since we cannot do that without introducing a large error, we choose the exercise prices that we have used to optimize the model and then interpolate these exercise prices to set levels of moneyness (-0.3,-0.15, 0.15 and 0.3). We have chosen this small band of moneyness to ensure that most trees fit. In such cases where the moneyness does not fit, we set the value of the implied volatility equal to the value of the discrete volatility tree.

We compute the implied volatilities for all months in our database for each company and then first average for each company and then average over all 30 Dow Jones companies. In Figure 3 we plot the results for the DH, TS and the VS models. Panel a shows the result for the full period for the TS model and as we know, this model has a flat volatility. First we plot the line that represents the Discrete Volatility Function (DVF) and then we mark the points that represent the re-engineered implied volatility for the TS model for the four exercise prices. The four implied volatilities from the TS model correctly reproduce the volatility as they are all on the DVF. The DH model is as we have seen earlier in Figure 1 well above this curve for the ATM options and is also skewed. In panel c,e and f we have done the same exercise, but for the three subperiods that we have defined earlier. The TS specification correctly represents the DVF for the first subperiod, while for the two other subperiods correctly represents the implied volatilities for the positive moneyness options and is marginally off for the negative moneyness options. It seems that it is more difficult to keep constant volatility in the lower part of the tree than in the upper part of the tree. That can, for example, also be seen if we go to the VS specification, where the DVF has a skew and the VS model is able to correctly reproduce the volatility both with positive moneyness as well as negative moneyness. The DH model also lies well above the implied model indicating that the DH model has a higher volatility than the option market. However, both in the first subperiod as in the last subperiod, the lowest moneyness of the DH model and the VS specification are almost equal.

What is clear from the above exercise is that the implied model with a jump to default correctly reproduces the DVF in general and if we specify a DVF with a skew the algorithm doesn't need many overrides as the implied volatility lies exactly on the DVF.

### 4.4 Default probabilities

The default probability at each node in the tree can be easily calculated by expression (5). To calculate the default probability for over one layer of nodes we need to do backward recursion with the probabilities. Das and Hanouna (2009, equation (15)) give the forward probability of default as:

$$\delta_{j,j+1}^{forward} = \sum_{i=0}^{j} p_j^i \delta_j^i \quad (44)$$

The forward default probability is given by the default probabilities $(\delta_j^i)$ discounted by the cumulative probability of reaching that state ($p_j^i$). The cumulative default probability is calculated both with node probabilities $q_j^i$ as with the survival probabilities $z_j^i$. The forward probability gives the probability that the company defaults between time *j* and *j+1*

We can also write the cumulative probability:

$$\delta_{j+1}^{cum} = \sum_{i=0}^{j} p_j^i \delta_j^i + \sum_{i=0}^{j} (1 - p_j^i) \quad (45)$$

The cumulative probability is equal to the forward probability and the probability that the company already defaulted in an earlier layer.

Now we can use expression (45) to calculate the two-year default probability for the companies in the DJIA for the different model specifications and subperiods. The first column in Table 4 shows the results for the DH model for the full period. Note that in contrast to the convention in the interest rate and spread markets, these are not annualized probabilities but cumulative probabilities. The average cumulative default probability for the DH model for the full period is 4.1%, thus approximately 2% per year. Two companies have rather high default probabilities − Alcoa (AA) and American Express (AXP) −while the rest of the companies are substantially lower. We see also that the default probability in the first subperiod was the lowest of all subperiods, while the second subperiod (that included the credit crisis) has the highest default probability.

<Insert Table 4 about here>

Focusing on the different specifications for the implied models, we see that the flat volatility models (CV and TS) have substantially lower default volatilities than the ones that include the skew. We also see that the specification that includes the full volatility surface (VS) has the highest default probability of all specifications as well as the DH model in all periods. The effect of the skew is thus that the default probability increases (at the expense of the recovery rate as we will see later). On an individual basis, the VS specification is almost always higher than the TS specification, except in the first subperiod. This suggests that the skew was much less apparent in the first subperiod than in the other periods as we have seen in Figure 3.

**4.5 Recovery rates**

Expression (14) in Das and Hanouna (2009) states that the forward recovery rate can be expressed as:

$$R_{j,j+1}^{forw} = \sum_{i=0}^{j} p_j^i R_j^i \tag{46}$$

The forward recovery rate is equal to the recovery rate discounted by the probability of reaching that state.

We have a problem with this expression as the probabilities $p_j^i$ in expression (46) do not sum to one. The reason is they only sum over the states when the company has not defaulted yet. Therefore we propose an improvement of the expression in Das and Hanouna (2009):

$$R_{j,j+1}^{forw} = \frac{\sum_{i=0}^{j} p_j^i R_j^i}{\sum_{i=0}^{j} p_j^i} \tag{47}$$

where we divided the expression (48) by the sum of the probabilities: $\sum_{i=0}^{j} p_j^i$.

The problem in expression (46) can be easily seen in Figure 4 where we simulated two simple cases with no discounting (interest rates are zero). In the first case, we assume that the recovery rate is always 100% when $a_0$ equals 100, $a_1$ equals zero, and $b$ is set to one so there is some default probability. When we run expression (46) in this case, the term structure of forward rates is downward sloping, purely because there is some default probability. When we adjust expression (47), we see that the term structure is stable at 100%, which should be the correct forward recovery rate.

<Insert Figure 4 about here>

In the second case we let the term structure of recovery rates vary by setting $a_0$ to 3 and $a_1$ to -200. In this simulation, the term structure of recovery rates is downward sloping for both expressions but for different reasons. For expression (46) it is downward sloping as a result of the default probability and $a_1$<0, while for expression (47) it is only as a result of $a_1$<0. The $a_1$ parameter thus determines the slope of the recovery rate term structure.

In Figure 5 we have applied expression (47) to the estimated DH model and some specifications of the implied model (TS and VS). We can use expression (47) to calculate the marginal (or forward) recovery rates. We start with the forward rate at the initial node for time period 0.5. We calculate expression (47) for all dates for each company and then average over the months and companies. Then we continue for the 0.5-year forward rate at point 0.5 and repeat the same procedure for all months and all companies. The results in figure 5 show that the implied model with the VS specification has the highest recovery rate while the implied model with the term structure specifications has the lowest term structure of recovery. The 0.5-year forward rate at 1.5 year in panel a of Figure 5 is 72.1% for the VS specification and 64.9% for the TS specification, while the DH model shows a recovery rate of 69.9%. From panels b, c, and d we also see that these recovery rates change over time and for the different models. The recovery rate of the DH model is the most volatile. In all panels we see that the forward curve is downward sloping, which is something that we expected given that the average $a_1$ parameter is less than zero.

Instead of calculating the forward recovery rate we can also calculate the j-period (cumulative) recovery rate, that is defined as the implied recovery rate for a firm between time is null and time is *j:*

$$R_{0,j}^{cum} = \sum_{i=0}^{j} p_j^i R_j^i + \sum_{i=0}^{j} (1 - p_j^i) R_{j-1}^{cum} \tag{48}$$

The cumulative recovery rate is equal to the recovery rates of the nodes that have not defaulted yet plus the cumulative recovery rate in the last period weighted by the (cumulative) default probability.

<Insert Table 5 about here>

The effect on the recovery rates can be seen in Table 5, where we depict the two-year recovery rate calculated with expression (48). For the full period, on average the two-year recovery rate is around 75%, the constant volatility specifications (CV and TS) are a bit lower at 70%. DH report a comparable average recovery rate of 73.55%. We have in our sample probably better quality companies and analyze a different period. In the subperiod that is closest to the period in DH study, we report a recovery rate of 78.55%. Another reason could be that we only include tenors up to two years while the DH study includes tenors up to five years. With a downward sloping recovery rate curve that results in a higher recovery rate. As we might have expected from the idea that in bad times recovery rates are lower because companies have to sell the assets at fire sale prices, we see that in the second subperiod the recovery rates are much lower than in the other subperiods. The recovery rate for the DH model drops from 78.8% in the first subperiod to 68.0% in the second subperiod and for the CV specification the drop is even more dramatic, from 75.2% in the first subperiod to 61% in the second subperiod. What can also be observed from Table 5 is that the recovery rates for models that include the skew are more stable. The recovery rate drops only from 77.3% in the first subperiod to 75.6% in the second subperiod, while the last subperiod is even higher at 81.4%.

We can also inspect the time-series behavior of the average default probabilities and recovery rates. For each month end, we take the average for all firms of the two-year default probabilities and two-year recovery rates and we plot them in Figure 6. In panel a we see that the average recovery rates for the 30 companies vary from 48% to 86%. When we compare the DH model with the VS specification, we see that the VS model is more stable: for example it varies from 60% to 86% while the DH model varies from 51% to 86%. In panel b we see that the recovery rate is initially stable and then from May 2007 it declined, while default probabilities were unchanged. They continued to go down for the TS and DH model until October 2008 and then bounced after they had their lowest point in January 2010. The VS specification had its lowest value much earlier, in July 2008, and was much more stable in the period thereafter. This stability of the VS specification is clearest in panel d, where the recovery stays almost all of the last subperiod at around 80%.

The default probabilities are also not equal in the different models. In general, the VS specification shows the highest default probability.

Including the full information from the option market with the VS model increases both the default probability and recovery rate in comparison to the DH model.

<Insert Figure 6 about here>

**5. Conclusion**

The CDS spread is the product of the PD and the LGD. Since only the CDS spreads are known from market prices, the LGD and PD are not separately identified unless some structure is imposed. Das and Hanouna (2009) developed a model ("the hybrid tree") that identifies PD and LGD jointly by combining information from the CDS and equity markets. Their model is easy to calibrate and shows good performance. One deficiency of their model is that it fails to include information from the options market. The model only includes the ATM volatility while the implied volatility in the option market shows a skewed pattern in relation to the moneyness and a term structure pattern in relation to maturity. Also, as we show in this paper, their model produces even for ATM options unrealistic high volatility levels when compared to option implied volatility.

To overcome the deficiency in the Das-Hanouna model, we develop a new type of binomial tree, an implied tree with a jump to default, that exactly prices the option volatility surface. Our framework allows us to estimate five different model specifications over different volatility specifications, while nesting the Barle and Cakici (1998) model.

In the empirical part of the paper we find that our model mimics the different specifications of the implied volatility curve well. Surprisingly, the specification with the lowest error was not the specification that includes the full volatility specification (e.g., both the term structure and skew), but the specification that includes only the term structure. One explanation for this could be that the option and CDS markets insure different elements: a CDS is triggered when there is a credit event while the payoff of an option is dependent on the stock price. Another explanation can be that one of the two markets is more efficient (includes more information) and therefore is leading the other market. Therefore, an interesting area for future research is to compute a CDS implied volatility surface with the model we developed. The dynamics of the CDS implied volatility and option implied volatility surfaces could help us in answering questions about efficiency and integration.

Our main theoretical contribution to the literature is that we have found an elegant extension to the implied binomial trees models by adding a jump to default. In addition to that theoretical contribution, market participants can use the model to gain more insight into the decomposition of the credit spread into LGD and PD. Structurers that focus purely on recovery rates alone can gain important insights since for example the model with term structure specification produces lower LGD's than the Das and Hanouna (2009) model as well as a LGD that is based on more realistic levels of volatility. Additionally, structurers (or traders) can use the model to price longer term options. The CDS market is much richer than the option market because there are actively traded vehicles up to 30 years in tenor while time to maturities in the option market typically only goes out to two years. We can use these longer tenors in our model to price longer-term options. Finally, regulators and bank risk managers can use the model for more accurate LGD parameters in Basel 2.

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**Table** 1

**Discrete Volatility Function**

For each stock we run the cross-sectional regression $\sigma(M,T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 XT$ at each month-end for the period August, 31 2005 to June, 30 2006. In total, 107 regressions are run for each stock and we report the mean of each coefficient and the standard deviations around this mean. *M* is defined as log (Strike Price/Forward Price) and *T* is defined as the log of the time to expiration. All coefficients are in percentages.

| | Mean coefficients (standard deviations) | | | | | | | | | | | | Goodness of fit | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ticker | C | | M | | $M^2$ | | T | | $T^2$ | | MT | | $R^2$ | |
| AA | 38.7 | (13.9) | -10.1 | (5.6) | 6.0 | (5.9) | -0.5 | (2.6) | 0.5 | (1.6) | 3.4 | (3.5) | 0.90 | (0.13) |
| AXP | 32.5 | (14.9) | -19.3 | (5.6) | 10.2 | (6.2) | 0.1 | (2.7) | -0.4 | (1.3) | 7.9 | (3.1) | 0.98 | (0.01) |
| BA | 29.6 | (7.2) | -16.3 | (4.5) | 10.6 | (4.2) | 0.7 | (1.5) | -0.1 | (1.0) | 6.5 | (2.5) | 0.99 | (0.01) |
| BAC | 38.8 | (22.0) | -17.3 | (7.2) | 10.9 | (9.9) | 0.2 | (4.5) | 0.4 | (1.8) | 6.8 | (5.6) | 0.96 | (0.06) |
| CAT | 32.4 | (8.9) | -15.9 | (5.9) | 8.0 | (3.9) | 0.4 | (1.6) | 0.0 | (1.1) | 6.3 | (2.4) | 0.98 | (0.02) |
| CSCO | 30.2 | (6.8) | -14.6 | (5.3) | 12.0 | (4.8) | 0.7 | (1.3) | -0.3 | (0.8) | 5.0 | (2.9) | 0.96 | (0.03) |
| CVX | 25.5 | (6.3) | -19.2 | (6.8) | 14.7 | (7.0) | 0.6 | (1.5) | -0.6 | (1.4) | 7.7 | (3.1) | 0.98 | (0.01) |
| DD | 27.0 | (8.0) | -17.8 | (5.6) | 14.4 | (7.0) | 0.5 | (1.7) | -0.5 | (0.9) | 7.2 | (2.8) | 0.98 | (0.02) |
| DIS | 27.6 | (7.3) | -18.1 | (5.3) | 13.9 | (6.5) | 0.6 | (1.7) | -0.6 | (0.9) | 6.6 | (2.6) | 0.97 | (0.02) |
| GE | 28.2 | (13.1) | -17.0 | (8.8) | 16.5 | (12.9) | 1.0 | (1.6) | -0.1 | (1.4) | 8.9 | (6.2) | 0.93 | (0.13) |
| HD | 29.0 | (8.8) | -19.0 | (5.4) | 13.5 | (7.0) | 0.8 | (1.6) | -0.6 | (1.1) | 6.7 | (2.8) | 0.98 | (0.02) |
| HPQ | 32.7 | (6.4) | -14.0 | (4.4) | 9.2 | (3.6) | 0.5 | (1.4) | -0.2 | (1.0) | 5.1 | (2.1) | 0.98 | (0.04) |
| IBM | 23.2 | (6.0) | -18.4 | (5.6) | 12.5 | (4.9) | 0.7 | (1.4) | -0.5 | (1.0) | 7.5 | (2.9) | 0.98 | (0.03) |
| INTC | 29.9 | (7.3) | -13.8 | (5.5) | 13.1 | (6.4) | 0.8 | (1.3) | 0.0 | (0.8) | 4.9 | (2.9) | 0.95 | (0.08) |
| JNJ | 18.4 | (4.2) | -19.3 | (5.6) | 19.6 | (7.6) | 1.1 | (1.3) | -0.6 | (0.9) | 8.1 | (3.0) | 0.98 | (0.02) |
| JPM | 32.2 | (13.7) | -22.2 | (5.8) | 14.1 | (5.9) | 0.2 | (2.4) | 0.1 | (1.5) | 8.0 | (3.2) | 0.98 | (0.02) |
| KO | 19.7 | (5.3) | -18.1 | (5.3) | 17.8 | (7.7) | 1.2 | (1.1) | -0.5 | (0.9) | 7.4 | (3.0) | 0.97 | (0.02) |
| MCD | 22.3 | (5.5) | -18.9 | (5.6) | 17.7 | (7.6) | 0.9 | (1.2) | -0.5 | (1.0) | 6.7 | (2.6) | 0.98 | (0.02) |
| MMM | 23.5 | (5.6) | -18.5 | (6.4) | 11.6 | (7.5) | 0.8 | (1.4) | -0.4 | (1.0) | 8.5 | (2.9) | 0.98 | (0.02) |
| MRK | 26.2 | (6.9) | -17.6 | (6.0) | 17.0 | (8.9) | 0.8 | (1.8) | -0.3 | (1.4) | 7.3 | (3.1) | 0.97 | (0.02) |
| MSFT | 26.5 | (7.7) | -16.5 | (4.3) | 15.1 | (6.4) | 1.0 | (1.3) | 0.1 | (0.8) | 5.3 | (2.5) | 0.97 | (0.02) |
| PFE | 25.6 | (6.6) | -15.8 | (7.0) | 17.3 | (11.9) | 1.1 | (1.1) | -0.3 | (1.2) | 5.7 | (3.1) | 0.93 | (0.13) |
| PG | 19.9 | (4.9) | -21.1 | (6.4) | 17.8 | (10.0) | 1.1 | (1.1) | -0.8 | (0.9) | 9.6 | (3.7) | 0.97 | (0.05) |
| T | 23.4 | (7.3) | -19.4 | (6.7) | 25.1 | (15.8) | 1.1 | (1.5) | -0.4 | (1.3) | 8.3 | (3.6) | 0.95 | (0.10) |
| TRV | 25.9 | (9.0) | -20.3 | (5.1) | 13.5 | (6.8) | 0.2 | (2.4) | -0.7 | (1.3) | 8.2 | (2.2) | 0.98 | (0.02) |
| UNH | 32.1 | (10.0) | -14.7 | (4.9) | 9.1 | (6.1) | 0.5 | (1.7) | -0.3 | (1.4) | 5.8 | (2.9) | 0.98 | (0.02) |
| UTX | 25.4 | (6.2) | -20.6 | (5.5) | 13.3 | (6.0) | 1.1 | (1.5) | -0.4 | (1.2) | 8.8 | (2.8) | 0.98 | (0.01) |
| VZ | 23.5 | (6.4) | -17.7 | (5.2) | 22.6 | (9.2) | 1.3 | (1.4) | -0.6 | (1.1) | 7.5 | (3.5) | 0.97 | (0.02) |
| WMT | 21.5 | (5.6) | -18.0 | (5.3) | 16.8 | (8.2) | 1.2 | (0.8) | -0.6 | (1.0) | 7.1 | (2.9) | 0.98 | (0.01) |
| XOM | 24.0 | (6.1) | -20.2 | (6.5) | 15.9 | (7.3) | 0.8 | (1.6) | -0.6 | (1.0) | 8.2 | (3.5) | 0.97 | (0.02) |

# Table 2

## Estimation results: Relative Root Mean Squared Error

We calibrate the 3 parameters {$a_0$,$a_1$ and b} so that the mean squared error (MSE) is minimized. The MSE is defined as: $MSE = \frac{1}{4}\sum_{k=1}^{4}(CDS_{model} - CDS_{market})^2$ . We start with the calibration for the Das and Hanouna model (DH) for the full period (August 2005-June 2014). Then we calibrate for the implied model where we gradually increase the number of variables from the discrete volatility function: $\sigma(M,T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT$. In the constant volatility model (CV) we only include $\beta_0$, in the term structure model we include $\beta_0$, $\beta_3$ and $\beta_4$, in the skew (SK) we include $\beta_0$, $\beta_1$ and $\beta_2$ and finally in then volatility surface mode (VS) we include all 5 β's. The full period has 107 monthly observations, and to evaluate the performance in different subperiod, we split the sample in three, where the first two subsamples have 35 observations and the last subsample has 36 observations. We report the relative root mean square error where take the standard deviation of the MSE and divided it with the average of the CDS spreads (similar to DH)

| | Full period | | | | | August 2005 - July 2008 | | | | | August 2008- June 2011 | | | | | July 2011- June 2014 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ticker | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS |
| AA | 2.1 | 3.3 | 2.4 | 2.8 | 2.4 | 1.3 | 1.8 | 1.9 | 1.8 | 1.8 | 1.9 | 2.4 | 1.9 | 1.9 | 1.9 | 3.2 | 5.7 | 3.4 | 4.7 | 3.4 |
| AXP | 1.5 | 2.4 | 1.5 | 2.4 | 2.1 | 2.6 | 3.5 | 2.7 | 3.1 | 2.4 | 0.9 | 2.1 | 1 | 2.5 | 2 | 0.9 | 1.5 | 0.8 | 1.6 | 1.7 |
| BA | 1.7 | 2 | 1.6 | 1.7 | 1.6 | 1.4 | 1.5 | 1.3 | 1.3 | 1.1 | 1.8 | 2.1 | 1.7 | 2.5 | 1.9 | 2 | 2.3 | 1.7 | 1.2 | 1.9 |
| BAC | 1.3 | 1.6 | 1.6 | 1.5 | 1.9 | 2.2 | 2.4 | 2.7 | 1.9 | 2.8 | 1.1 | 1.4 | 1.2 | 1.4 | 1.1 | 0.6 | 0.9 | 1 | 1.1 | 1.9 |
| CAT | 1.8 | 2.1 | 1.7 | 2.1 | 2.2 | 2.1 | 2 | 1.9 | 1.8 | 1.7 | 1.1 | 1.3 | 1.2 | 2.3 | 1.9 | 2.3 | 2.9 | 2.1 | 2.2 | 2.9 |
| CSCO | 3.2 | 2.9 | 2.7 | 3.2 | 3 | 5.9 | 5.2 | 5.1 | 4.3 | 4.1 | 2 | 1.9 | 1.7 | 2.7 | 2.1 | 1.6 | 1.5 | 1.2 | 2.5 | 2.8 |
| CVX | 2.7 | 2.5 | 2.4 | 2.4 | 2.7 | 3.5 | 3.5 | 3.5 | 3.2 | 3.1 | 2.2 | 2.3 | 1.9 | 2.7 | 2.6 | 2.2 | 1.8 | 1.9 | 1.5 | 2.3 |
| DD | 1.8 | 2 | 1.8 | 2.4 | 2.3 | 1.2 | 1.6 | 1.7 | 1.6 | 1.7 | 2.4 | 2.5 | 2.4 | 3 | 3.1 | 1.8 | 2 | 1.4 | 2.4 | 2.3 |
| DIS | 1.9 | 2 | 2 | 3 | 2.7 | 2.9 | 3.2 | 3 | 2.3 | 2.1 | 1.9 | 2 | 2 | 3.1 | 3.1 | 0.7 | 0.7 | 0.9 | 3.7 | 2.7 |
| GE | 3.1 | 4 | 3.4 | 3.3 | 3.3 | 4.8 | 6.8 | 5.8 | 5.3 | 5.8 | 2.4 | 2.5 | 2.2 | 2.3 | 1.9 | 1.9 | 2.7 | 2.2 | 2.2 | 2.3 |
| HD | 1.9 | 2 | 1.7 | 2 | 1.8 | 1.4 | 1.8 | 1.6 | 1.7 | 1.5 | 1.4 | 1.5 | 1.3 | 2.5 | 1.9 | 2.9 | 2.7 | 2.1 | 1.9 | 1.9 |
| HPQ | 1.5 | 2 | 1.7 | 1.9 | 1.7 | 1.5 | 2 | 1.8 | 1.5 | 1.4 | 1.8 | 2 | 1.8 | 2.2 | 2.1 | 1.2 | 2 | 1.6 | 2 | 1.7 |
| IBM | 2.2 | 2.5 | 2.2 | 2.1 | 2.2 | 1.8 | 2.5 | 2.3 | 2.3 | 2 | 2.8 | 2.8 | 2.6 | 3 | 2.9 | 2 | 2.3 | 1.6 | 1.2 | 1.7 |
| INTC | 1.8 | 1.8 | 1.7 | 2 | 1.8 | 2.3 | 2.5 | 2.3 | 2.4 | 1.9 | 1.8 | 1.7 | 1.5 | 1.7 | 1.2 | 1.4 | 1.3 | 1.3 | 2 | 2.4 |
| JNJ | 2.4 | 2.5 | 2.3 | 1.8 | 2.2 | 2.8 | 2.9 | 3 | 2.5 | 2.4 | 2.6 | 2.7 | 2.6 | 2.2 | 2.6 | 2 | 2 | 1.3 | 0.8 | 1.5 |
| JPM | 1.3 | 1.5 | 1.4 | 2 | 1.9 | 1.7 | 1.9 | 1.8 | 1.9 | 1.6 | 1.6 | 1.8 | 1.4 | 2.4 | 1.9 | 0.7 | 1 | 0.9 | 1.7 | 2.2 |
| KO | 2 | 2.4 | 2.3 | 1.8 | 2.2 | 2.4 | 2.9 | 3.1 | 2.3 | 2.9 | 2.1 | 2.6 | 2.3 | 2.2 | 2.5 | 1.6 | 1.6 | 1.3 | 1 | 1.2 |
| MCD | 2.3 | 2.1 | 2 | 2 | 2.4 | 3.2 | 3.1 | 3 | 2.5 | 2.2 | 1.3 | 1.2 | 1.2 | 2 | 2.5 | 2.2 | 2 | 1.7 | 1.4 | 2.5 |
| MMM | 1.6 | 1.6 | 1.7 | 2.1 | 2.7 | 1.9 | 2.1 | 2.1 | 1.8 | 1.7 | 2 | 2 | 1.8 | 2.2 | 2.3 | 0.8 | 0.6 | 1.2 | 2.3 | 3.9 |
| MRK | 1.6 | 1.6 | 1.5 | 2.5 | 2.1 | 1.5 | 1.7 | 1.6 | 2.4 | 2 | 2.2 | 2.4 | 1.9 | 3.5 | 2.5 | 0.9 | 0.8 | 0.8 | 1.8 | 1.7 |
| MSFT | 2.5 | 2.5 | 2.2 | 2 | 1.9 | 1.8 | 1.5 | 1.5 | 1.8 | 1 | 4.1 | 4.1 | 3.7 | 2.4 | 3.1 | 1.5 | 1.9 | 1.6 | 1.9 | 1.7 |
| PFE | 1.7 | 1.6 | 1.6 | 1.8 | 2.1 | 1.4 | 1.4 | 1.5 | 1.6 | 2.3 | 1.7 | 1.7 | 1.8 | 2.2 | 2.1 | 2.1 | 1.7 | 1.5 | 1.7 | 2 |
| PG | 2 | 2.2 | 1.7 | 1.8 | 2.1 | 1.8 | 2.1 | 1.9 | 2 | 1.7 | 2.1 | 2.1 | 1.8 | 2 | 2.1 | 2.1 | 2.2 | 1.4 | 1.4 | 2.4 |
| T | 1.8 | 1.8 | 1.6 | 2.4 | 2.7 | 1.3 | 1.9 | 1.7 | 2.8 | 3.4 | 3.1 | 3 | 2.7 | 2.6 | 3 | 0.9 | 0.6 | 0.5 | 1.9 | 1.6 |
| TRV | 1.7 | 2 | 1.6 | 1.9 | 1.9 | 2.2 | 2.5 | 2.6 | 2.6 | 2.4 | 1.8 | 2 | 1.4 | 1.9 | 1.4 | 1.1 | 1.5 | 0.9 | 1.2 | 1.8 |
| UNH | 2 | 2.6 | 2.1 | 1.5 | 1.9 | 2.2 | 2.9 | 2.8 | 2.4 | 2.6 | 0.8 | 1.2 | 1 | 1.2 | 1 | 3.1 | 3.6 | 2.5 | 1 | 2.2 |
| UTX | 1.5 | 1.7 | 1.6 | 2.3 | 3 | 1.6 | 2.2 | 2.1 | 2.1 | 2 | 1.8 | 1.8 | 1.7 | 2.9 | 2.6 | 1.1 | 1.1 | 1 | 2 | 4.3 |
| VZ | 2.4 | 2.6 | 2.2 | 2.8 | 2.4 | 4.4 | 4.6 | 3.8 | 3.5 | 3.1 | 2.2 | 2.5 | 2.3 | 3.8 | 2.6 | 0.6 | 0.7 | 0.4 | 1.2 | 1.5 |
| WMT | 1.6 | 1.6 | 1.5 | 1.7 | 2.2 | 1.5 | 1.8 | 1.7 | 1.7 | 1.8 | 1.8 | 1.9 | 1.6 | 2 | 2 | 1.5 | 1.1 | 1.2 | 1.4 | 2.7 |
| XOM | 2.9 | 2.6 | 2.5 | 3.1 | 3.5 | 3.2 | 2.7 | 2.4 | 1.8 | 1.6 | 3.9 | 3.7 | 3.3 | 4 | 3.7 | 1.5 | 1.3 | 1.7 | 3.5 | 5.1 |
| **Mean** | 2.0 | 2.2 | 1.9 | 2.2 | 2.3 | 2.3 | 2.6 | 2.5 | 2.3 | 2.3 | 2.0 | 2.2 | 1.9 | 2.4 | 2.3 | 1.6 | 1.8 | 1.4 | 1.9 | 2.3 |

## Table 3
## Estimation results: Mean Model Parameters

We report the mean estimated model parameters {$a_0$, $a_1$ and b} for the full period August 2005 till June 2014 for the 5 different model specifications. The first specification is the DH model and the four subsequent models are based on different specifications of the discrete volatility function: $\sigma(M,T) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 T + \beta_4 T^2 + \beta_5 MT$. In the constant volatility model (CV) we only include $\beta_0$, in the term structure model we include $\beta_0$, $\beta_3$ and $\beta_4$, in the skew (SK) we include $\beta_0$, $\beta_1$ and $\beta_2$ and finally in then volatility surface mode (VS) we include all 5 β's. The full period has 107 monthly observations and we also report the mean of the stock price over this period

| | DH Model | | | Constant Volatility (CV) | | | Term Structure (TS) | | | Skew (SK) | | | Volatility Surface (VS) | | | Stock Price |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ticker | $a_0$ | $a_1$ | B | $a_0$ | $a_1$ | b | $a_0$ | $a_1$ | b | $a_0$ | $a_1$ | b | $a_0$ | $a_1$ | b | S |
| AA | 2.7 | -92.2 | 1.40 | 3.6 | -245.8 | 1.53 | 3.7 | -221 | 1.46 | 3.5 | -210.9 | 1.52 | 3.6 | -214.9 | 1.53 | 18.9 |
| AXP | 4.8 | -180.5 | 0.90 | 5.3 | -626.1 | 1.09 | 5.6 | -751.6 | 1.11 | 5.8 | -338.3 | 1.00 | 5.7 | -529.9 | 1.09 | 52.2 |
| BA | 4.2 | -370.8 | 1.01 | 5.1 | -762.6 | 1.11 | 5.1 | -868 | 1.11 | 5.4 | -422.5 | 1.04 | 5 | -781.4 | 1.12 | 77.7 |
| BAC | 2.2 | -139.5 | 1.49 | 2.6 | -256.1 | 1.60 | 2.8 | -365.4 | 1.57 | 3.2 | -93.8 | 1.40 | 3 | -77.5 | 1.33 | 24.1 |
| CAT | 4.2 | -328 | 1.03 | 4.7 | -798.3 | 1.08 | 4.9 | -821.8 | 1.06 | 5.5 | -577.5 | 1.02 | 5.2 | -674.5 | 1.05 | 75.2 |
| CSCO | 2.5 | -261.1 | 1.63 | 2.7 | -584.1 | 1.69 | 2.9 | -668.6 | 1.73 | 3 | -228.9 | 1.62 | 3 | -207.5 | 1.59 | 21.6 |
| CVX | 3.8 | -571.3 | 1.10 | 4.4 | -1022.4 | 1.15 | 4.4 | -1351.9 | 1.19 | 5.7 | -366.2 | 1.03 | 4.9 | -347.3 | 1.21 | 89.3 |
| DD | 4.1 | -669.5 | 1.29 | 5.2 | -1165.2 | 1.34 | 5.3 | -1346.1 | 1.36 | 6 | -333.9 | 1.06 | 5.7 | -312.9 | 1.13 | 45.6 |
| DIS | 3.5 | -560.1 | 1.41 | 4.2 | -1157.6 | 1.51 | 4.3 | -1352.4 | 1.52 | 4.6 | -621.8 | 1.41 | 4.5 | -252 | 1.35 | 39.4 |
| GE | 4.2 | -216.6 | 1.34 | 5.5 | -622.1 | 1.40 | 5.8 | -430.7 | 1.39 | 5 | -312.8 | 1.31 | 5.5 | -336.3 | 1.23 | 24.6 |
| HD | 3.6 | -282.2 | 1.11 | 4.1 | -595.9 | 1.24 | 3.8 | -743.4 | 1.29 | 4.9 | -231.1 | 1.06 | 4 | -244.7 | 1.10 | 42.5 |
| HPQ | 3 | -259.9 | 1.37 | 3.6 | -524.1 | 1.39 | 3.7 | -553 | 1.42 | 4 | -327.6 | 1.38 | 4 | -632.6 | 1.44 | 35.4 |
| IBM | 5.7 | -446.8 | 0.84 | 6.6 | -1186.9 | 0.97 | 6.4 | -1329.2 | 1.00 | 6.9 | -619.5 | 0.89 | 6.2 | -1240.9 | 0.98 | 138.5 |
| INTC | 2.1 | -206.9 | 1.68 | 2.4 | -503.8 | 1.80 | 2.5 | -471.7 | 1.80 | 2.6 | -226.2 | 1.77 | 2.8 | -162.2 | 1.46 | 21.9 |
| JNJ | 4.8 | -546.3 | 1.15 | 5.8 | -1332 | 1.21 | 5.9 | -1648 | 1.29 | 6.2 | -592.5 | 1.12 | 5.8 | -777.1 | 1.21 | 67.5 |
| JPM | 3.9 | -415.3 | 1.23 | 4.8 | -894.3 | 1.24 | 4.9 | -755 | 1.22 | 5.6 | -175.4 | 1.12 | 5.3 | -169 | 1.06 | 43 |
| KO | 5.2 | -568.4 | 1.24 | 6.1 | -999.7 | 1.30 | 6.1 | -1203.4 | 1.35 | 6.2 | -601.6 | 1.27 | 6.1 | -1493.9 | 1.36 | 51.6 |
| MCD | 4.4 | -504.1 | 1.11 | 4.9 | -1200 | 1.22 | 4.9 | -1252 | 1.26 | 5.6 | -395.1 | 1.05 | 4.8 | -341.5 | 1.07 | 68.6 |
| MMM | 4.3 | -890.5 | 1.18 | 5.1 | -1780.8 | 1.24 | 5.1 | -2060.3 | 1.25 | 5.7 | -1686.5 | 1.24 | 5.2 | -1732.1 | 1.26 | 86.6 |
| MRK | 3.3 | -774.4 | 1.48 | 4.1 | -1302.4 | 1.49 | 4.3 | -1468.7 | 1.45 | 4.7 | -479.4 | 1.44 | 5 | -448 | 1.19 | 39.7 |
| MSFT | 2.2 | -190 | 1.49 | 2.5 | -434.9 | 1.59 | 2.5 | -366.1 | 1.56 | 5.3 | -242.2 | 1.28 | 2.4 | -138.9 | 1.41 | 28.4 |
| PFE | 2.6 | -560.6 | 1.59 | 3.5 | -837.3 | 1.72 | 3.5 | -855 | 1.71 | 5.2 | -261 | 1.42 | 4.6 | -208.2 | 1.19 | 22.4 |
| PG | 5.5 | -664.7 | 1.08 | 6.8 | -1455.2 | 1.14 | 7.1 | -1380.9 | 1.16 | 7.1 | -824.5 | 1.08 | 7.5 | -991 | 1.12 | 65 |
| T | 3.2 | -875.5 | 1.56 | 3.8 | -1283 | 1.60 | 4 | -1529.8 | 1.60 | 5.4 | -236.2 | 1.22 | 5.7 | -281 | 1.10 | 31.6 |
| TRV | 4.8 | -342.2 | 1.01 | 6.1 | -869.5 | 1.10 | 6.7 | -992.1 | 1.12 | 6.4 | -556.1 | 1.02 | 6.8 | -694.7 | 1.05 | 57.1 |
| UNH | 4.2 | -222 | 0.93 | 4.6 | -552 | 1.10 | 4.5 | -624.6 | 1.13 | 4.8 | -363.7 | 1.04 | 4.3 | -502 | 1.15 | 47.6 |
| UTX | 4.9 | -608.2 | 1.08 | 5.5 | -1271 | 1.18 | 5.9 | -1736.2 | 1.19 | 6.4 | -849.9 | 1.12 | 6.1 | -1112 | 1.18 | 74.4 |
| VZ | 4.3 | -364.1 | 1.18 | 4.7 | -704.8 | 1.30 | 4.8 | -884 | 1.33 | 6 | -485.2 | 1.21 | 6.4 | -412.6 | 1.20 | 37.8 |
| WMT | 5.1 | -1028 | 1.22 | 6.6 | -1941 | 1.26 | 6.8 | -2133.2 | 1.25 | 6.4 | -937.6 | 1.18 | 6.4 | -1488.1 | 1.24 | 56.9 |
| XOM | 3.8 | -483.5 | 1.18 | 4.1 | -832.1 | 1.24 | 3.9 | -795.5 | 1.24 | 4.2 | -461.6 | 1.18 | 3.6 | -409.9 | 1.18 | 78.5 |
| Mean | 3.9 | -454.1 | 1.24 | 4.6 | -924.7 | 1.33 | 4.7 | -1032 | 1.34 | 5.2 | -468.7 | 1.22 | 5 | -573.8 | 1.22 | 52.1 |

## Table 4

**Estimation results: Two-year (cumulative) default probabilities**

We report the two-year cumulative probability ($\delta_j^{cum}$) as calculated by the expression (47): $\delta_j^{cum} = \sum_{i=0}^{j} p_j^i * \delta_j^i + \sum_{i=0}^{j}(1 - p_j^i)$. The probability $p_j^i$ indicates the probability of reaching a state from all possible paths in the binomial tree that is governed both by the node probability $p_j^i$ as by the survival probability $z_j^i$.

| | Full period | | | | | August 2005 - July 2008 | | | | | August 2008- June 2011 | | | | | July 2011- June 2014 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ticker | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS |
| AA | 9.4 | 8.3 | 8.9 | 8 | 8.3 | 3.3 | 1.3 | 1.8 | 1.1 | 1.3 | 16.6 | 12.6 | 12.5 | 12.1 | 11.4 | 8.4 | 11.1 | 12.3 | 11 | 12.3 |
| AXP | 10.7 | 5.4 | 4.3 | 8.3 | 6.3 | 9.2 | 3.9 | 3.5 | 5.2 | 3.6 | 16.4 | 10.3 | 7.4 | 15.1 | 10.4 | 6.4 | 2 | 1.8 | 4.6 | 4.8 |
| BA | 4.5 | 2.3 | 2.1 | 3.4 | 3.5 | 1.1 | 0.8 | 0.8 | 1.2 | 0.9 | 6.9 | 4 | 3.6 | 4.2 | 5.8 | 5.6 | 2.1 | 1.9 | 4.7 | 3.8 |
| BAC | 7.1 | 4.7 | 5.8 | 6.4 | 8.3 | 1.1 | 0.8 | 1.3 | 2.9 | 2.2 | 14.9 | 8.6 | 10.4 | 10.9 | 14.8 | 5.3 | 4.8 | 5.5 | 5.5 | 8 |
| CAT | 5.5 | 2.4 | 2.5 | 4.9 | 5.1 | 1 | 0.9 | 0.8 | 1 | 0.9 | 9.5 | 4.1 | 4.8 | 9.6 | 10.5 | 6 | 2.2 | 2 | 4.2 | 3.8 |
| CSCO | 2.8 | 1.5 | 1.4 | 3.6 | 4.4 | 1.3 | 0.8 | 0.8 | 1 | 0.8 | 3.7 | 2.5 | 2.4 | 6.2 | 8.2 | 3.5 | 1.2 | 1 | 3.7 | 4.2 |
| CVX | 2.1 | 1.2 | 1.5 | 3.4 | 3.7 | 0.6 | 0.4 | 0.4 | 0.7 | 0.7 | 3.7 | 1.9 | 2.9 | 4.5 | 6.5 | 1.8 | 1.2 | 1.1 | 5 | 3.8 |
| DD | 2.9 | 2.6 | 2.5 | 4.4 | 4.3 | 1 | 0.7 | 0.9 | 1 | 1.2 | 4 | 5.7 | 5.3 | 7.4 | 8.1 | 3.8 | 1.3 | 1.2 | 4.7 | 3.7 |
| DIS | 2.5 | 1.1 | 1.1 | 3.1 | 4 | 1.7 | 0.9 | 0.8 | 1.2 | 0.8 | 3.5 | 1.8 | 1.7 | 5.1 | 7.3 | 2.3 | 0.5 | 0.7 | 2.9 | 3.8 |
| GE | 6.6 | 4.4 | 4.8 | 5.7 | 6.9 | 2.1 | 2 | 1.8 | 1.7 | 2.6 | 13.6 | 8.6 | 9.1 | 11.2 | 12.6 | 4.2 | 2.5 | 3.4 | 4 | 5.6 |
| HD | 6.2 | 3.2 | 3 | 4.8 | 4.7 | 3.2 | 2 | 1.5 | 2 | 2.8 | 12.5 | 6 | 6.1 | 8.8 | 8.4 | 2.8 | 1.4 | 1.2 | 3.5 | 2.8 |
| HPQ | 4.8 | 2.2 | 2 | 3.3 | 3.5 | 0.9 | 0.6 | 0.7 | 0.9 | 0.7 | 8.2 | 3.2 | 2.9 | 5.4 | 6.1 | 5.4 | 2.8 | 2.6 | 3.8 | 3.7 |
| IBM | 2.6 | 1.3 | 1.2 | 2.4 | 1.6 | 2.1 | 1.1 | 1 | 1.1 | 1 | 2.6 | 1.4 | 1.3 | 2.5 | 1.8 | 3.2 | 1.5 | 1.3 | 3.7 | 1.9 |
| INTC | 3.5 | 2 | 2.2 | 3.6 | 4.5 | 0.7 | 0.5 | 0.9 | 0.6 | 1.4 | 6.6 | 4.6 | 4.1 | 6 | 6.9 | 3.2 | 0.8 | 1.5 | 4.2 | 5.3 |
| JNJ | 1.8 | 1.1 | 1 | 2.4 | 2.7 | 0.8 | 0.5 | 0.5 | 0.7 | 0.8 | 1.8 | 1.5 | 1.4 | 3.4 | 4.5 | 2.7 | 1.2 | 1.2 | 3 | 3 |
| JPM | 5.3 | 3.5 | 3.7 | 5.3 | 5.9 | 1.2 | 0.9 | 1.2 | 2 | 2.7 | 11 | 7.5 | 7.8 | 9.7 | 9 | 3.8 | 2.1 | 2.1 | 4.1 | 5.9 |
| KO | 3.6 | 2.2 | 2.2 | 2.6 | 2.7 | 1 | 0.9 | 0.8 | 0.7 | 0.8 | 7.6 | 4.3 | 4.3 | 4.6 | 5.1 | 2.3 | 1.5 | 1.7 | 2.5 | 2.2 |
| MCD | 2.7 | 1.2 | 1.3 | 3.3 | 3.5 | 1.4 | 0.8 | 0.9 | 1.9 | 1.7 | 4.9 | 2 | 2.1 | 5.4 | 6.6 | 1.8 | 0.9 | 0.9 | 2.7 | 2.2 |
| MMM | 2.2 | 1.1 | 1.2 | 2.1 | 2.5 | 1.1 | 0.7 | 0.6 | 0.9 | 0.6 | 3 | 1.9 | 2.2 | 3.5 | 4.1 | 2.6 | 0.7 | 0.7 | 2 | 2.9 |
| MRK | 2.1 | 1.1 | 1.6 | 3.4 | 5.2 | 0.5 | 0.4 | 0.5 | 0.9 | 0.7 | 2.1 | 1.5 | 2.4 | 6 | 10 | 3.6 | 1.4 | 1.9 | 3.4 | 5 |
| MSFT | 3.4 | 1.6 | 2.1 | 4.2 | 4 | 1.5 | 0.6 | 0.9 | 1.5 | 0.8 | 4 | 3 | 3 | 6.3 | 6.6 | 4.6 | 1.2 | 2.4 | 4.8 | 4.5 |
| PFE | 2.8 | 1.5 | 1.9 | 5.1 | 4.6 | 0.3 | 0.3 | 0.5 | 1.9 | 1 | 4.9 | 2.8 | 3.7 | 8 | 7.5 | 3.1 | 1.4 | 1.6 | 5.3 | 5.2 |
| PG | 2.8 | 1.7 | 1.9 | 3 | 3.8 | 1.1 | 0.7 | 0.7 | 1.3 | 0.9 | 4.2 | 2.6 | 2.9 | 5 | 5.6 | 3.1 | 1.7 | 2.3 | 2.7 | 4.8 |
| T | 2.1 | 1 | 1.4 | 3.9 | 4.7 | 1.9 | 0.7 | 0.8 | 1.4 | 0.7 | 3.2 | 1.5 | 2.5 | 5.9 | 6.7 | 1.3 | 0.7 | 1 | 4.5 | 6.6 |
| TRV | 4.3 | 2.4 | 3.2 | 3.1 | 4.1 | 1.3 | 0.9 | 1.1 | 1.6 | 1.4 | 6.2 | 3.9 | 5.7 | 3.9 | 6 | 5.3 | 2.3 | 2.7 | 3.9 | 5 |
| UNH | 6.6 | 2.8 | 2.9 | 4.1 | 3.6 | 2.1 | 1.3 | 1.2 | 1.8 | 2.1 | 13.2 | 5.1 | 5.2 | 6.9 | 5.5 | 4.4 | 2.2 | 2.3 | 3.7 | 3.1 |
| UTX | 3.4 | 1.7 | 1.8 | 2.6 | 4.3 | 1.8 | 0.9 | 1.1 | 1.4 | 0.6 | 4.6 | 2.5 | 2.7 | 4.2 | 7.2 | 3.9 | 1.5 | 1.5 | 2.3 | 5.2 |
| VZ | 4.7 | 2 | 3.2 | 3.7 | 5.2 | 3 | 1.6 | 1.5 | 1.8 | 1.7 | 5.5 | 2.2 | 5.6 | 5.1 | 10.3 | 5.6 | 2.1 | 2.4 | 4.1 | 3.5 |
| WMT | 2.2 | 1.2 | 1.7 | 2.1 | 2.4 | 1.2 | 0.6 | 0.8 | 1.5 | 0.6 | 3.8 | 1.9 | 2.9 | 2.7 | 4.1 | 1.5 | 1 | 1.5 | 2.2 | 2.6 |
| XOM | 2 | 1.3 | 1.8 | 2.2 | 2.6 | 1.6 | 0.7 | 1.7 | 1.6 | 1.7 | 3.6 | 2.4 | 3 | 3.2 | 4.1 | 0.8 | 0.7 | 0.5 | 1.9 | 1.9 |
| **Mean** | 4.1 | 2.3 | 2.5 | 3.9 | 4.4 | 1.7 | 1 | 1.1 | 1.5 | 1.3 | 6.9 | 4.1 | 4.4 | 6.4 | 7.4 | 3.7 | 1.9 | 2.1 | 4 | 4.4 |

## Table 5

### Estimation results: Two-year recovery rates

We estimate the two-year recovery rates based on the expression: $R_j^{cum} = \sum_{i=0}^{j} p_j^i * R_j^i + \sum_{i=0}^{j}(1 - p_j^i) * R_{j-1}^{cum}$ . The implied recovery rate for period j at time zero is equal to the recovery rates in each node times the probability of reaching that node (from all paths) plus the recovery rate from the previous layer in the three times the probability of default.

| | Full period | | | | | August 2005 - July 2008 | | | | | August 2008- June 2011 | | | | | July 2011- June 2014 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ticker | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS | DH | CV | TS | SK | VS |
| AA | 73.4 | 67.9 | 70.6 | 70.3 | 70.7 | 74.6 | 72.1 | 75.4 | 73.6 | 73.4 | 68.6 | 57.9 | 60.3 | 60.2 | 59.6 | 77.1 | 73.9 | 76.2 | 77.4 | 79.4 |
| AXP | 81.7 | 70.9 | 69.8 | 79.2 | 75.9 | 87.8 | 75.7 | 77.9 | 79.6 | 77.6 | 71.9 | 62.2 | 58.8 | 73 | 65.7 | 85.4 | 75 | 72.7 | 85.2 | 84.5 |
| BA | 78 | 71.9 | 71.3 | 76.6 | 77.5 | 80 | 77.2 | 76.8 | 80.8 | 78.8 | 64.9 | 57.8 | 58.6 | 61.2 | 68.8 | 89.4 | 80.9 | 78.8 | 88.1 | 85.3 |
| BAC | 64.9 | 60.4 | 64.7 | 70.1 | 75.4 | 69.4 | 66.5 | 68.1 | 76.4 | 74 | 68.9 | 58.5 | 65.7 | 70 | 79 | 56.2 | 55.9 | 60.1 | 63.8 | 73 |
| CAT | 76.7 | 70.6 | 70.9 | 77.6 | 78.2 | 78.6 | 78.5 | 77.1 | 79 | 78.6 | 65.2 | 56 | 60.2 | 71 | 76.6 | 86.5 | 77.4 | 75.5 | 83.1 | 79.3 |
| CSCO | 71.3 | 65.9 | 65.3 | 72.6 | 76.1 | 78.7 | 69.8 | 69.6 | 75.2 | 72.6 | 60.2 | 57.4 | 56.9 | 66.9 | 78.2 | 75.2 | 70.7 | 69.5 | 75.6 | 77.5 |
| CVX | 73.4 | 70.1 | 69.4 | 77 | 77.5 | 72.4 | 70.6 | 70.2 | 71.6 | 70.7 | 63.1 | 58.4 | 60.7 | 68.3 | 76.5 | 85.2 | 81.7 | 77.6 | 91.3 | 85.6 |
| DD | 73.8 | 71.2 | 71.8 | 80.6 | 78.9 | 75.5 | 72.8 | 74.2 | 76.2 | 77 | 62.1 | 63.4 | 65.6 | 83.4 | 83 | 84.1 | 77.5 | 75.7 | 82.2 | 76.6 |
| DIS | 76.3 | 71.8 | 71.7 | 76.3 | 84.6 | 87.5 | 82.9 | 81.5 | 84 | 82.4 | 68.9 | 64.4 | 64.4 | 72.5 | 86.7 | 72.4 | 68.1 | 69.3 | 72.3 | 84.6 |
| GE | 66.4 | 59.7 | 62.8 | 67.6 | 71.8 | 69.7 | 70.9 | 71 | 72.5 | 75 | 63.7 | 48.6 | 52.6 | 60.6 | 66.7 | 65.9 | 59.6 | 64.9 | 69.8 | 73.9 |
| HD | 79.1 | 73.1 | 71.8 | 77.8 | 79.3 | 77.3 | 73.4 | 71 | 75.7 | 78.8 | 69.8 | 62.6 | 64.8 | 75.9 | 79.7 | 90.4 | 83.7 | 79.9 | 81.9 | 79.5 |
| HPQ | 76.6 | 72.1 | 70.7 | 75.9 | 74.7 | 79.2 | 76.4 | 77.3 | 78 | 78 | 76.9 | 70.1 | 66.1 | 79.2 | 77.3 | 73.7 | 69.7 | 68.8 | 70.3 | 68.7 |
| IBM | 84 | 77.8 | 76.6 | 82.5 | 80 | 87.4 | 81.8 | 81.5 | 84.1 | 84.1 | 73.7 | 68.1 | 67.1 | 71.6 | 71.3 | 91.1 | 83.7 | 81.3 | 92.1 | 84.7 |
| INTC | 74.4 | 66.6 | 70 | 75.9 | 80.5 | 72.9 | 71.1 | 73.1 | 70.4 | 72.9 | 79.2 | 62.9 | 69.5 | 77 | 82.9 | 71 | 65.8 | 67.4 | 80.4 | 85.9 |
| JNJ | 79.5 | 74.7 | 72.5 | 82.4 | 80.8 | 82.5 | 82.4 | 82.6 | 83.8 | 84.3 | 64.9 | 58.5 | 53.9 | 72.1 | 69.9 | 91.3 | 83.6 | 81.3 | 91.5 | 88.2 |
| JPM | 68.8 | 65.3 | 68.2 | 74.5 | 79.6 | 71.6 | 70.2 | 72 | 75.1 | 83.2 | 68.4 | 63.5 | 69.7 | 75.5 | 77.1 | 66.3 | 62.1 | 62.7 | 72.7 | 78.4 |
| KO | 80.7 | 76.4 | 75.9 | 79.7 | 79.4 | 77.4 | 77.8 | 77.8 | 79.5 | 82.4 | 76.8 | 68.8 | 68.9 | 73.1 | 72 | 87.9 | 82.9 | 81.1 | 86.7 | 83.9 |
| MCD | 82.6 | 76.2 | 74.6 | 85.6 | 85.3 | 85.3 | 81.4 | 81.4 | 85.3 | 83.4 | 73.5 | 63.9 | 61.8 | 81.1 | 84.4 | 89.3 | 83.6 | 80.8 | 90.6 | 88.1 |
| MMM | 78.2 | 72 | 70.5 | 76 | 74.4 | 86.6 | 82 | 81.8 | 82.9 | 80.5 | 69.2 | 60.3 | 58.4 | 70.2 | 64 | 78.7 | 73.7 | 71.3 | 75 | 78.7 |
| MRK | 68.6 | 65.3 | 66.9 | 72.1 | 77.3 | 70.8 | 69.8 | 70.5 | 72.1 | 71.9 | 56.4 | 53 | 56 | 67.6 | 82.6 | 79 | 73.5 | 74.4 | 76.7 | 77.5 |
| MSFT | 75.4 | 70.6 | 72.7 | 66.7 | 79.5 | 80.9 | 70.8 | 72.9 | 55.6 | 72.4 | 73.8 | 74.6 | 74.5 | 72.3 | 83.5 | 71.6 | 66.3 | 70.8 | 72.5 | 82.7 |
| PFE | 70.7 | 67.2 | 69 | 52.1 | 80.5 | 66.3 | 67 | 68.5 | 41.5 | 68.3 | 60.5 | 54.8 | 61 | 54.1 | 84 | 85.6 | 80.2 | 77.7 | 60.8 | 89.4 |
| PG | 79.8 | 74.2 | 74.8 | 80.6 | 80.6 | 82.9 | 77.2 | 77.3 | 81.9 | 78.3 | 69.1 | 63.5 | 63.4 | 74.5 | 74.1 | 87.5 | 82 | 83.9 | 85.6 | 89.6 |
| T | 69.4 | 63.7 | 64.8 | 74.7 | 78.6 | 77.2 | 72.6 | 73.3 | 68.8 | 64.3 | 64.2 | 55.4 | 59.4 | 78.5 | 83.3 | 66.7 | 63.1 | 61.5 | 77 | 88.4 |
| TRV | 76 | 68.4 | 69.7 | 75.4 | 76.9 | 76 | 72 | 73.5 | 78.4 | 78 | 65 | 55.2 | 58.7 | 63.1 | 67.4 | 87.3 | 78.2 | 77.3 | 85.1 | 85.7 |
| UNH | 82.5 | 71.9 | 72.1 | 78.6 | 75.7 | 82.3 | 76.8 | 76.7 | 80.3 | 79.9 | 74.9 | 58.2 | 59.4 | 68.2 | 62.7 | 90.5 | 81 | 80.4 | 87.5 | 84.7 |
| UTX | 77 | 71.6 | 71 | 75.5 | 79.2 | 83.9 | 77.9 | 77 | 82.7 | 76.5 | 65 | 61.9 | 62.8 | 69 | 74.4 | 82.1 | 75.2 | 73.4 | 74.9 | 86.8 |
| VZ | 78.3 | 71.6 | 75.1 | 77.7 | 79.4 | 85.8 | 80.2 | 79 | 81.3 | 80.5 | 65.5 | 56.2 | 70 | 69.5 | 86.4 | 83.7 | 78.7 | 76.3 | 82.4 | 71 |
| WMT | 75.8 | 72.1 | 73.2 | 79.7 | 77.1 | 80.2 | 78 | 77.8 | 84.3 | 76.9 | 67.2 | 62 | 63.8 | 69.6 | 70 | 80.2 | 76.5 | 78.1 | 85.3 | 84.7 |
| XOM | 77.5 | 74.9 | 75.3 | 75.5 | 76.5 | 84.4 | 81.3 | 87 | 83.9 | 83.6 | 75.2 | 70.9 | 69.6 | 72.1 | 79.8 | 72.9 | 72.3 | 69.2 | 70.5 | 66 |
| **Mean** | 75.7 | 70.2 | 70.8 | 75.6 | 78.1 | 78.8 | 75.2 | 75.8 | 76.5 | 77.3 | 68.2 | 61 | 62.8 | 70.7 | 75.6 | 80.1 | 74.6 | 73.9 | 79.6 | 81.4 |

**Figure 1** Volatility skew in the Das-Hanouna model

We start a binomial tree either with a stock price *S* of 100 or 10. Then we let the tree grow either with a volatility 25% or 50% and with a timestep of either 0,5 or 0.05. The tree will have a length of 2 year. Then we vary the default probability in the tree by letting *b* vary from 0.5 to 100. In the resulting trees we calculate the value of the option at different moneyness points (-0.2, -0.1, 0, 0.1 and 0.2). The implied volatility of is then calculated by using the Black-Scholes formula

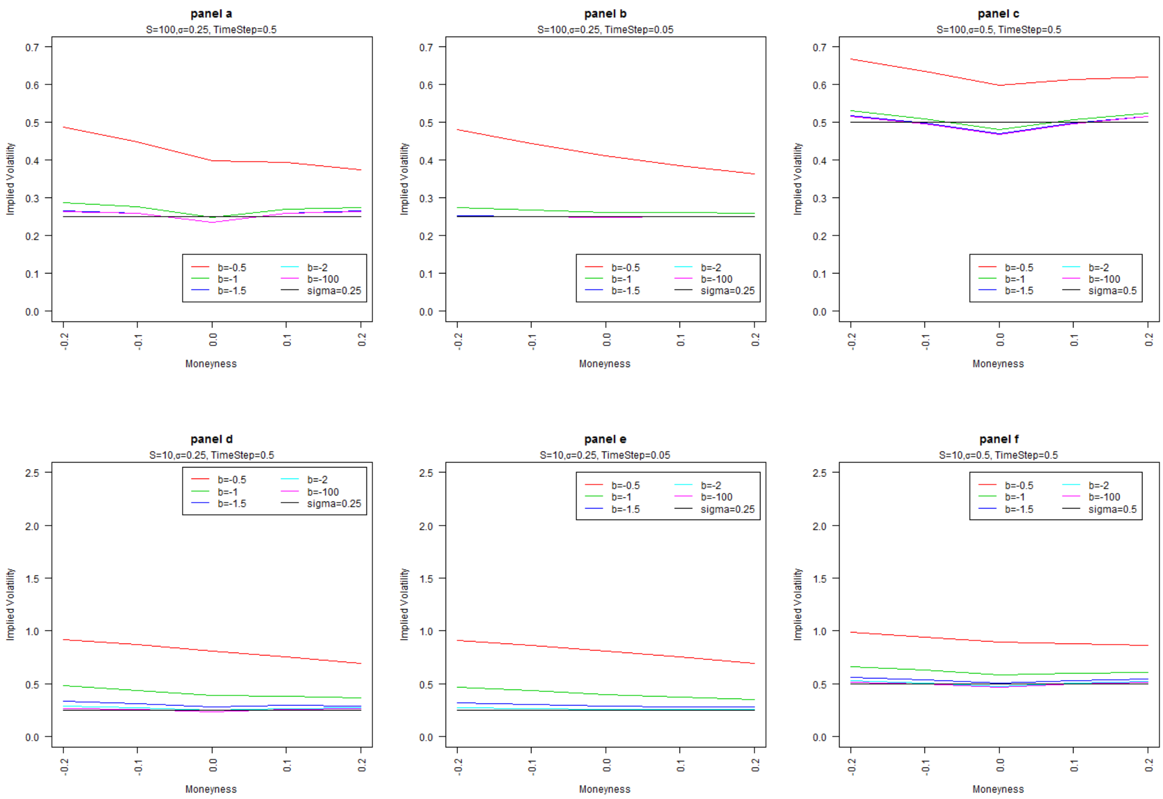
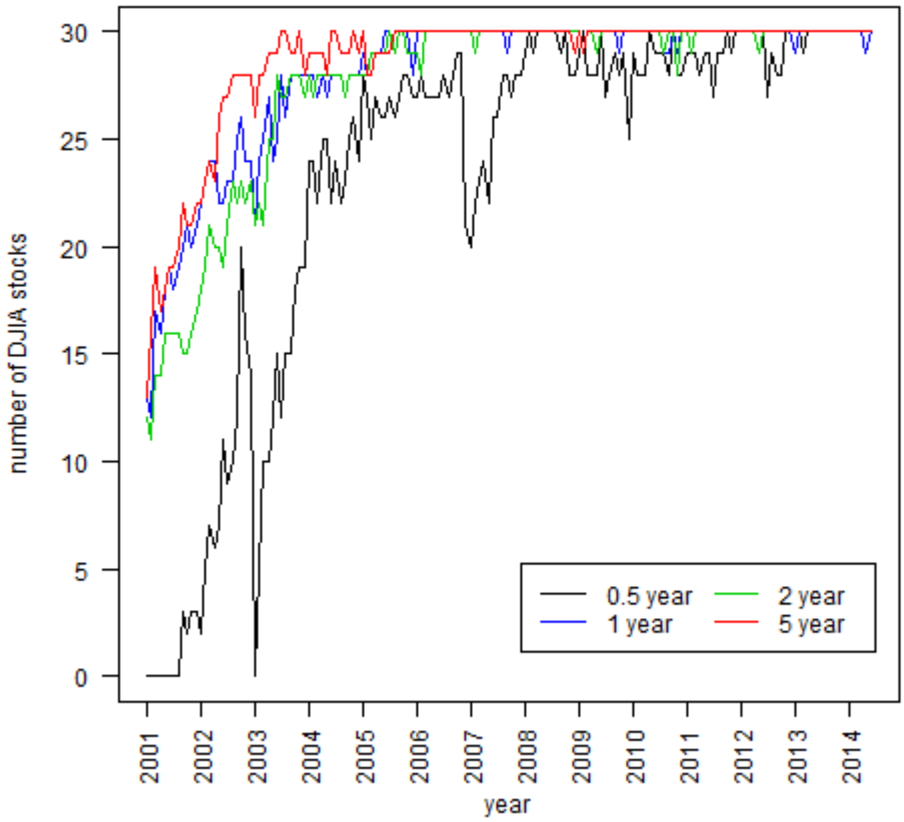


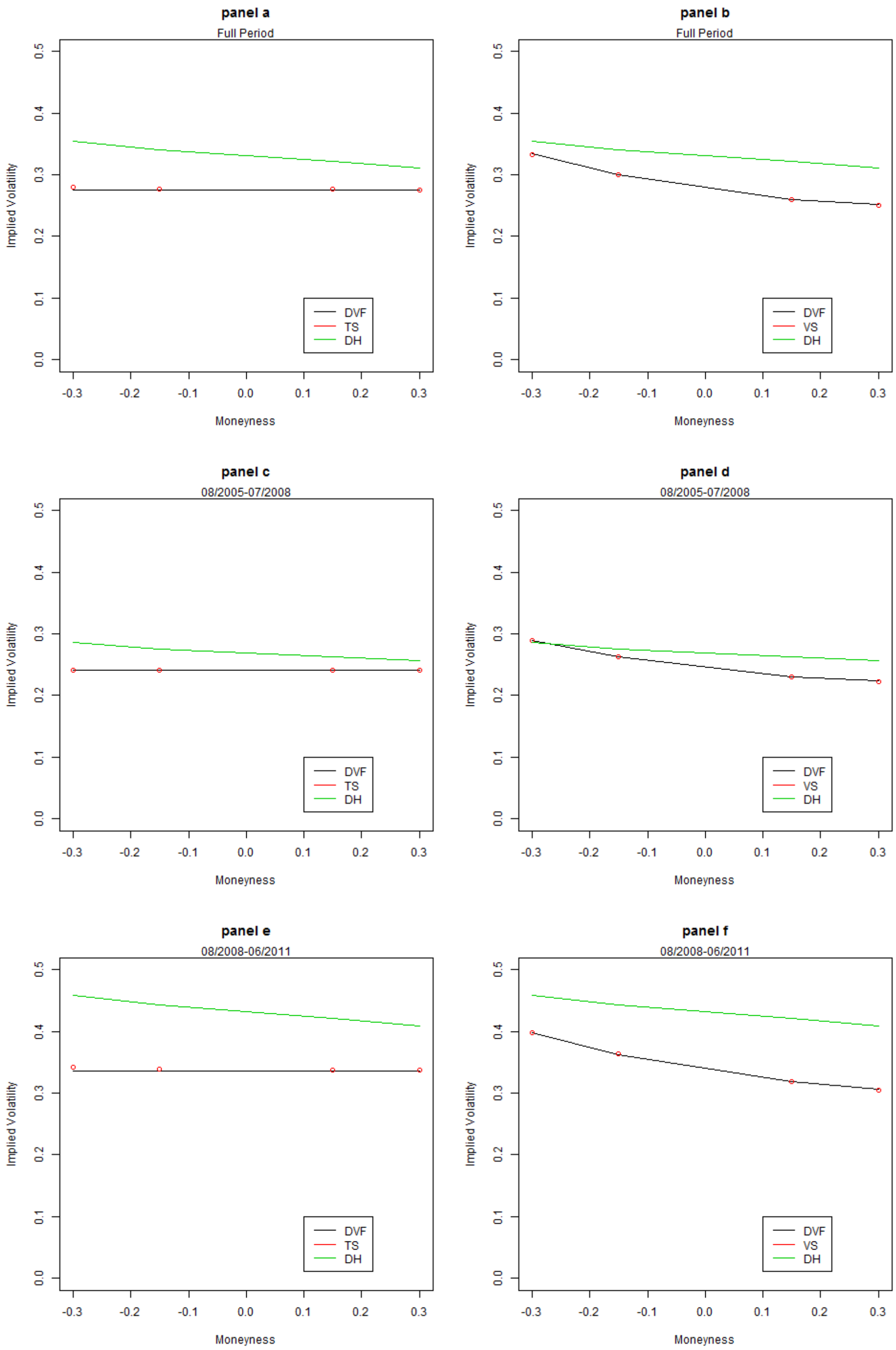
Figure 2 Representation of DJIA companies in the Markit CDS database

The Dow Jones Industrial Index consists of 30 companies and we check how many of these companies were priced in the Markit CDS database for the period January 2001 till June 2014. We checked for the three tenors we are interested in:  half year, one year and two years plus we added a five-year tenor that is viewed by market participants  as the most liquid CDS spread.

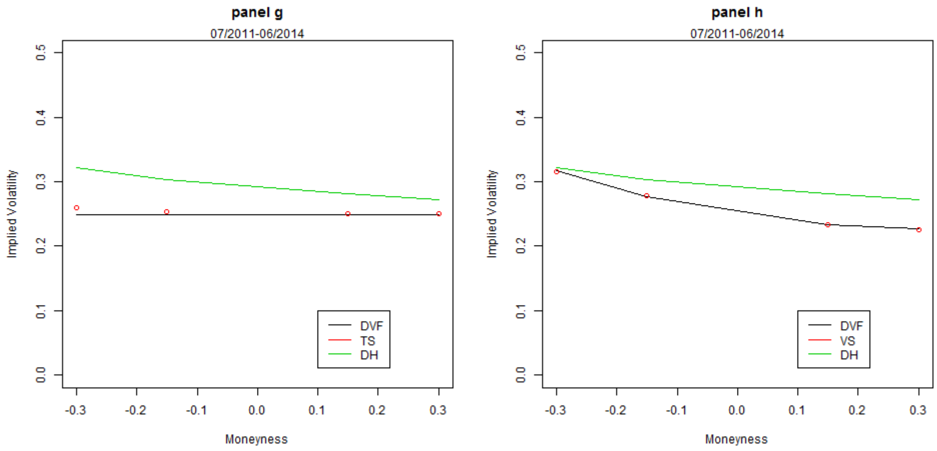


## Figure 3 Re-engineered implied volatilities

We calculate the two year implied volatility for a specific date with the Black-Scholes formula with as input the 2-year binomial option prices from the DH, TS and VS models for the four exercise prices (moneyness) that we have in the fourth layer of the binomial tree. Then we interpolate this moneyness to -0.3, -0.1, 0.1 and 0.3. Then we run this for all dates for a company and take the average for the period. We do this for all 30 companies and report the average for these companies.



## Figure 3 Re-engineered implied volatilities (concluded)

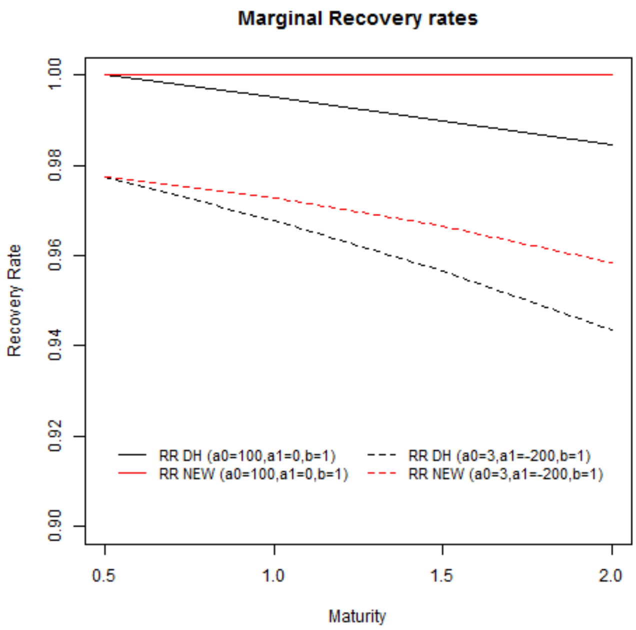


## Figure 4 Marginal recovery rates

We simulate the recovery rates for two expressions for the recovery rate. The first expression is as in Das and Hanouna (2009 expression (14)):

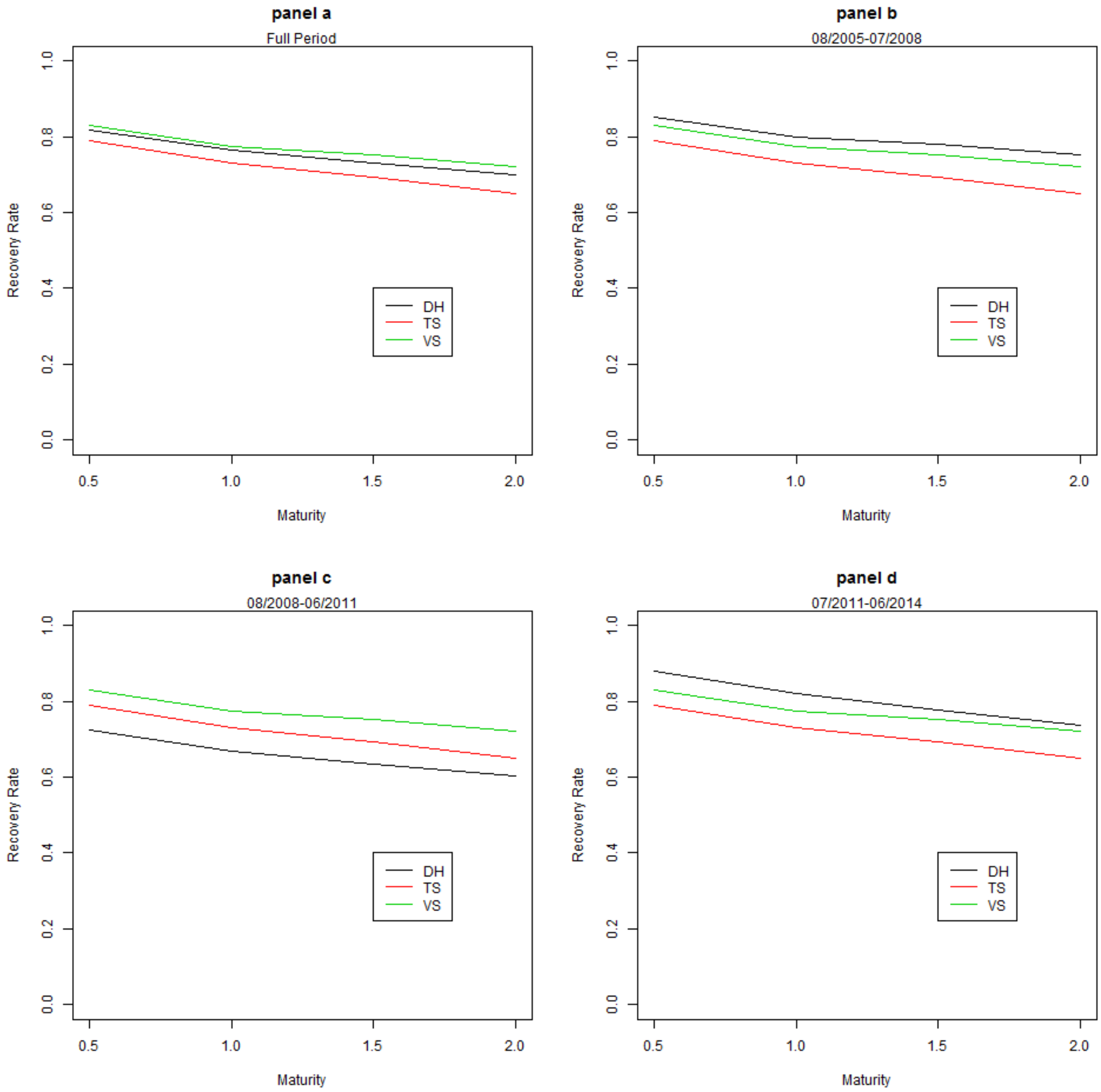
$R_j^{marg} = \sum_{i=0}^{j} p_j^i * R_j^i$ and the second is our expression: $R_j^{marg} = \frac{\sum_{i=0}^{j} p_j^i * R_j^i}{\sum_{i=0}^{j} p_j^i}$.

Then we simulate the recovery rates in two cases: (1) the recovery rate is set to a constant value of one with $a_0$ is 100 and $a_1$ set to zero and (2) the recovery rate is non-constant and dependent on the development of the stock price ($a_1$=-200). Additionally we have set interest rate to zero and $b$ to one.



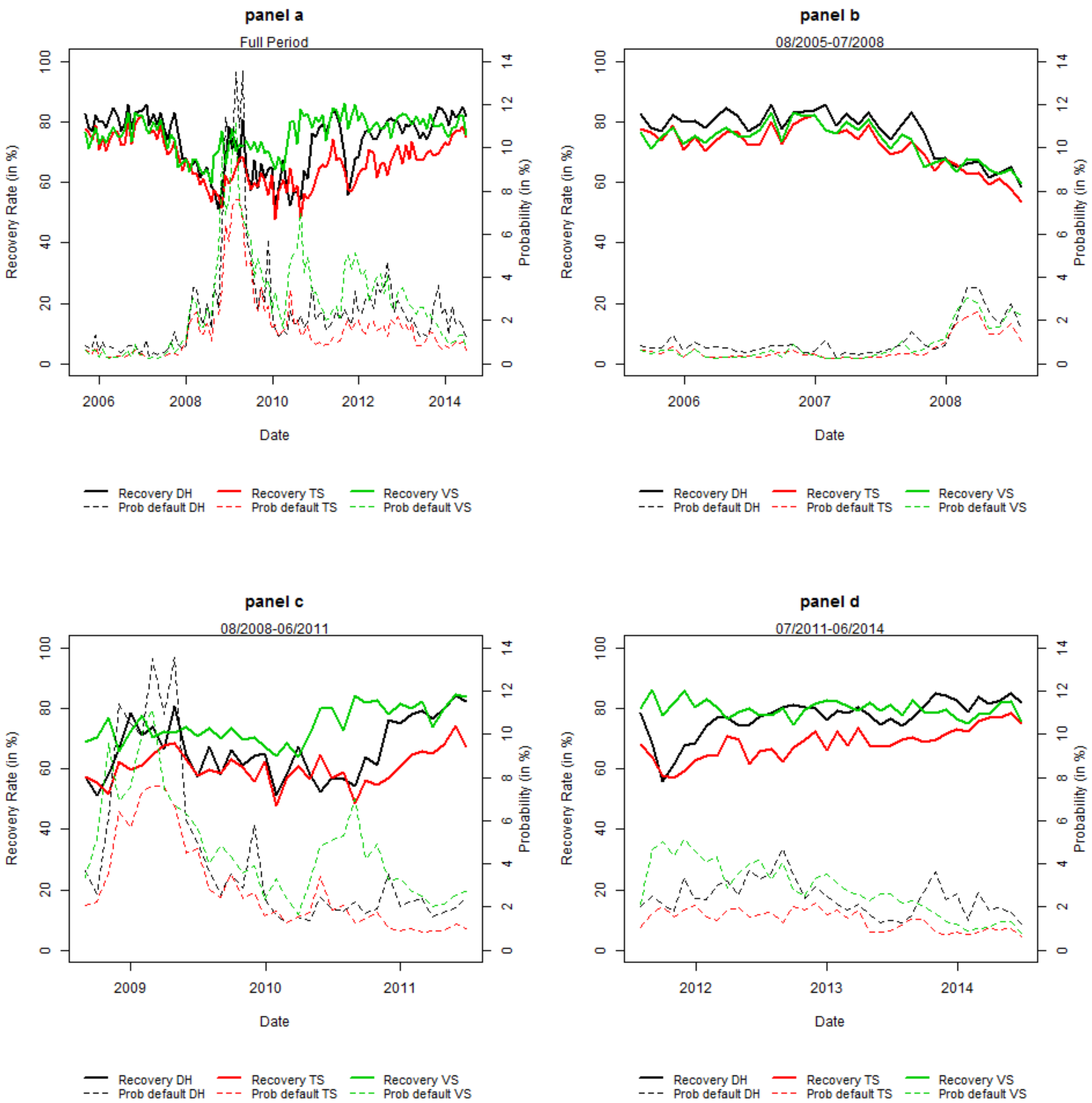
## Figure 5 Term structure of forward recovery rates

For each company we run for the full period (August 2005 till June 2014) expression (48), $R_{j,j+1}^{forw} = \frac{\sum_{i=0}^{j} p_j^i * R_j^i}{\sum_{i=0}^{j} p_j^i}$, that gives the forward (or marginal) recovery rate for the four tenors (0.5, 1, 1.5 and 2 years). Then we average over all companies and dates to come up with the recovery rate for that specific tenor for of either the Das Hanouna (DH) model or the implied models with term structure (TS) and volatility surface (VS) specification.



## Figure 6 2-year default probabilities and recovery rates

The time-series of recovery rates and default probabilities are shown for two different model specifications of the implied model and the Das and Hanouna model (DH). The Term Structure (TS) specification includes the term structure parameters while the Volatility Surface (VS) specification includes all parameters. Panels b to d show the different sub-periods.



## Appendix A Intermediate steps to go from equation (24) to equation (25)

We start with the equation (24):

$$C\left[K_{j-1}^{k}, j\right] = e^{-r\mathrm{h}} * (\lambda_{j-1}^{j-1} * q_{j-1}^{j-1} * z_{j-1}^{j-1} * \left(S_j^j - K_{j-1}^k\right) + (\lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k} + \lambda_{j-1}^{k+1} * \left(1 - q_{j-1}^{k+1}\right) * z_{j-1}^{k+1} * (S_j^{k+1} - K_{j-1}^k) + \sum_{i=k+2}^{j-1} (\lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1} + \lambda_{j-1}^{i} * z_{j-1}^{i} * (1 - q_{j-1}^{i}))\,(S_j^i - K_{j-1}^k)) \tag{A.1}$$

then we bring $e^{-r\Delta t}$ to the left side of expression A.1 and we separate the summation terms in two individual terms and separate the summation terms in two individual terms to obtain:

$$e^{r\mathrm{h}} * C\left[K_{j-1}^{k}, j\right] = \left(\lambda_{j-1}^{j-1} * q_{j-1}^{j-1} * z_{j-1}^{j-1} * \left(S_j^j - K_{j-1}^k\right) + \left(\lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k} + \lambda_{j-1}^{k+1} * \left(1 - q_{j-1}^{k+1}\right) * z_{j-1}^{k+1} * \left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j-1} \lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1}\,(S_j^i - K_{j-1}^k)\right)\right) + \sum_{i=k+2}^{j-1} \lambda_{j-1}^{i} * z_{j-1}^{i} * (1 - q_{j-1}^{i})\,(S_j^i - K_{j-1}^k)) \tag{A.2}$$

Now we change the counters of the last (summing) term from $\sum_{i=k+2}^{j-1}$ to $\sum_{i=k+3}^{j}$:

$$e^{r\mathrm{h}} * C\left[K_{j-1}^{k}, j\right] = \left(\lambda_{j-1}^{j-1} * q_{j-1}^{j-1} * z_{j-1}^{j-1} * \left(S_j^j - K_{j-1}^k\right) + \left(\lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k}\right)\left(S_j^{k+1} - K_{j-1}^k\right) + \lambda_{j-1}^{k+1} * \left(1 - q_{j-1}^{k+1}\right) * z_{j-1}^{k+1} * \left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j-1} \lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1}\,(S_j^i - K_{j-1}^k)\right)\right) + \sum_{i=k+3}^{j} \lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * (1 - q_{j-1}^{i-1})\,(S_j^{i-1} - K_{j-1}^k)) \tag{A.3}$$

And then we can add the first and third term to the summation so that both summation terms run from *i=k+2* to *j* and our last expression reduces to:

$$e^{r\mathrm{h}} * C\left[K_{j-1}^{k}, j\right] = \lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k})\left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j} \left(\lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1}\right)(S_j^i - K_{j-1}^k) + \sum_{i=k+2}^{j} (\lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * (1 - q_{j-1}^{i-1}))\,(S_j^{i-1} - K_{j-1}^k)) \tag{A.4}$$

and we can combine the summation terms again and expand them:

$$e^{r\mathrm{h}} * C\left[K_{j-1}^{k}, j\right] = \lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k})\left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j} \lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1} * S_j^i - \lambda_{j-1}^{i-1} * q_{j-1}^{i-1} * z_{j-1}^{i-1} K_{j-1}^k + \lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * (1 - q_{j-1}^{i-1})\; S_j^{i-1} - \lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * (1 - q_{j-1}^{i-1}) K_{j-1}^k \tag{A.5}$$

so that we use the no-arbitrage condition from equation (12) and use the fact that the probabilities sum to one to arrive at:

$$e^{r\mathrm{h}} * C\left[K_{j-1}^{k}, j\right] = \lambda_{j-1}^{k} * q_{j-1}^{k} * z_{j-1}^{k}\left(S_j^{k+1} - K_{j-1}^k\right) + \sum_{i=k+2}^{j} \lambda_{j-1}^{i-1} * (F_{j-1,j}^{i-1} - z_{j-1}^{i-1} * K_{j-1}^k) \tag{A.6}$$

The two variables that are still not known are $q_{j-1}^{k}$ and $S_j^{k+1}$. Therefore bring these two variables to the left side and set $\rho_{up} = \sum_{i=k+2}^{j} \lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * (\frac{F_{j-1,j}^{i-1}}{z_{j-1}^{i-1}} - \frac{F_{j-1,j}^{k}}{z_{j-1}^{k}})$ as this summation term now consist of forward prices that are all known at $j$-1 and bring $\lambda_{j-1}^{i-1} * z_{j-1}^{i=0}$ to the other side:

$$q_{j-1}^{k}\left(S_j^{k+1} - K_{j-1}^{k}\right) = \frac{e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}}{\lambda_{j-1}^{k} z_{j-1}^{k}} \tag{A.7}$$

Then we can use equation (13) and (14) where we change the $j$ counter in (13) to $j$-1 and replace $q_{j-1}^{k}$ in A.11:

$$\left(\frac{K_{j-1}^{k} - S_j^{k}}{S_j^{k+1} - S_j^{k}}\right) * \left(S_j^{k+1} - K_{j-1}^{k}\right) = \frac{e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}}{\lambda_{j-1}^{k} z_{j-1}^{k}} \tag{A.8}$$

And move everything but $S_j^{k+1}$ to the right side and expand the right side by setting them all in the same denominator:

$$S_j^{k+1} = \left(\frac{e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}}{\lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right)}\right) * S_j^{k+1} - \frac{\left(e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}\right) * S_j^{k}}{\lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right)} + \frac{\lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right) * K_{j-1}^{k}}{\lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right)} \tag{A.9}$$

Bring the $S_j^{k+1}$ term to the left side and set the right side under the same denominator:

$$\left(1 - \frac{e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}}{\lambda_{j-1}^{k} z_{j-1}^{k} * (\frac{F_{j-1,j}^{k}}{z_{j-1}^{k}} - S_j^{k})}\right) * S_j^{k+1} = -\frac{\left(e^{rh} * C\left[K_{j-1}^{k}, j\right] - \rho_{up}\right) * S_j^{k} - \lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right) * K_{j-1}^{k}}{\lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - S_j^{k}\right)} \tag{A.10}$$

Expand the left side and everything but $S_j^{i}$ to the right side and rearrange:

$$\boldsymbol{S_j^{k+1}} = \frac{\boldsymbol{S_j^{k}}\left(\boldsymbol{C}\left[K_{j-1}^{k}, \boldsymbol{j}\right] * \boldsymbol{e^{rh}} - \boldsymbol{\rho_{up}}\right) - \boldsymbol{\lambda_{j-1}^{k}} * \boldsymbol{z_{j-1}^{k}} * K_{j-1}^{k}\left(K_{j-1}^{k} - \boldsymbol{S_j^{k}}\right)}{\boldsymbol{C}\left[K_{j-1}^{k}, \boldsymbol{j}\right] * \boldsymbol{e^{rh}} - \boldsymbol{\rho_{up}} - \boldsymbol{\lambda_{j-1}^{k}} \boldsymbol{z_{j-1}^{k}} * \left(K_{j-1}^{k} - \boldsymbol{S_j^{k}}\right)} \tag{A.11}$$

which is equation (25)

## Appendix B Intermediate steps to go from equation (30) to equation (31).

We start with equation (30)

$$(S_j^{k+1}) = \frac{\frac{\left(K_{j-1}^{k}\right)^2}{S_j^{k+1}}\left(C\left(K_{j-1}^{k}, j\right) * e^{rh} - \rho_{up}\right) - \lambda_{j-1}^{k} * z_{j-1}^{k,k+1} * K_{j-1}^{k}\left(K_{j-1}^{k} - \frac{\left(K_{j-1}^{k}\right)^2}{S_j^{k+1}}\right)}{C\left[K_{j-1}^{k}, j\right] * e^{rh} - \rho_{up} - \lambda_{j-1}^{k} z_{j-1}^{k} * \left(K_{j-1}^{k} - \frac{\left(K_{j-1}^{k}\right)^2}{S_j^{k+1}}\right)} \tag{B.1}$$

First we multiply both sides with the denominator on the right:

$$S_j^{k+1}\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}-\lambda_{j-1}^k z_{j-1}^k*\left(K_{j-1}^k-\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right)\right)$$
$$=\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)-\lambda_{j-1}^k*z_{j-1}^k*K_{j-1}^k\left(K_{j-1}^k-\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right) \tag{B.2}$$

Bring the right side together using the same denominator:

$$S_j^{k+1}\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}-\lambda_{j-1}^k z_{j-1}^k*\left(K_{j-1}^k-\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right)\right)$$
$$=\frac{\left(K_{j-1}^k\right)^2*\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)-\left(\lambda_{j-1}^k*z_{j-1}^k*K_{j-1}^k\left(S_j^{k+1}K_{j-1}^k-\left(K_{j-1}^k\right)^2\right)\right)}{S_j^{k+1}} \tag{B.3}$$

Multiply both sides with *S* and expand the left side:

$$\left(S_j^{k+1}\right)^2*\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}-\lambda_{j-1}^k F_{j-1,j}^k+\lambda_{j-1}^k z_{j-1}^k\frac{\left(K_{j-1}^k\right)^2}{S_j^{k+1}}\right)$$
$$=\left(K_{j-1}^k\right)^2\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)-\left(\lambda_{j-1}^k*z_{j-1}^k*K_{j-1}^k\left(S_j^{k+1}K_{j-1}^k-\left(K_{j-1}^k\right)^2\right)\right) \tag{B.4}$$

Expand the right side and bring everything to the left side:

$$\left(S_j^{k+1}\right)^2*\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}-\lambda_{j-1}^k F_{j-1,j}^k\right)+\lambda_{j-1}^k z_{j-1}^k S_j^{k+1}\left(K_{j-1}^k\right)^2-\left(K_{j-1}^k\right)^2\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)+\lambda_{j-1}^k$$
$$*z_{j-1}^k*S_j^{k+1}\left(K_{j-1}^k\right)^2-\lambda_{j-1}^k*z_{j-1}^k*K_{j-1}^k\left(K_{j-1}^k\right)^2 \tag{B.5}$$

Collect in terms of $\boldsymbol{\left(S_j^{k+1}\right)^2}$, $\left(\frac{F_{j-1,j}^k}{z_{j-1}^k}\right)^2$ and $\boldsymbol{S_j^{k+1}}$

$$\left(\boldsymbol{S_j^{k+1}}\right)^2*\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}-\lambda_{j-1}^k F_{j-1,j}^k\right)-\left(K_{j-1}^k\right)^2\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}+\lambda_{j-1}^k*F_{j-1,j}^k\right)+2*\boldsymbol{S_j^{k+1}}$$
$$*\lambda_{j-1}^k*z_{j-1}^k\left(K_{j-1}^k\right)^2 \tag{B.6}$$

Then factoring:

$$-\left(F_{j-1,j}^k-\boldsymbol{S_j^{k+1}}*z_{j-1}^k\right)*\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*F_{j-1,j}^k+\lambda_{j-1}^k*\left(F_{j-1,j}^k\right)^2-\rho_{up}*F_{j-1,j}^k+C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*\boldsymbol{S_j^{k+1}}*z_{j-1}^k\right.$$
$$\left.-F_{j-1,j}^k*\lambda_{j-1}^k*\boldsymbol{S_j^{k+1}}*z_{j-1}^k-\rho_{up}*\boldsymbol{S_j^{k+1}}*z_{j-1}^k\right)=0 \tag{B.7}$$

We can split the above into two equations

$$-\left(F_{j-1,j}^k-\boldsymbol{S_j^{k+1}}*z_{j-1}^k\right)=0 \tag{B.8}$$

and

$$\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*F_{j-1,j}^k+\lambda_{j-1}^k*\left(F_{j-1,j}^k\right)^2-\rho_{up}*F_{j-1,j}^k+C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*\boldsymbol{S_j^{k+1}}*z_{j-1}^k-F_{j-1,j}^k*\lambda_{j-1}^k\right.$$
$$\left.*\boldsymbol{S_j^{k+1}}*z_{j-1}^k-\rho_{up}*\boldsymbol{S_j^{k+1}}*z_{j-1}^k\right)=0 \tag{B.9}$$

This last expression can be easily solved. Collecting the *S* and bringing all rest to the right:

$$\boldsymbol{S_j^{k+1}}\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*z_{j-1}^k-F_{j-1,j}^k*\lambda_{j-1}^k*z_{j-1}^k-\rho_{up}*z_{j-1}^k\right)$$
$$=-C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}*F_{j-1,j}^k-\lambda_{j-1}^k*\left(F_{j-1,j}^k\right)^2+\rho_{up}*F_{j-1,j}^k \tag{B.10}$$

Dividing the right side with everything on the left side except *S* and factoring by $K_{j-1}^k$:

$$\boldsymbol{S_j^{k+1}}=K_{j-1}^k\frac{\lambda_{j-1}^k*F_{j-1,j}^k+\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)}{\lambda_{j-1}^k*F_{j-1,j}^k-\left(C\left[K_{j-1}^k,j\right]*e^{\mathrm{rh}}-\rho_{up}\right)}\quad\forall\, j=odd\quad \&\, k=(j+1)/2 \tag{B.11}$$

which is equation (31).

## Appendix C Intermediate steps to go from equation (33) to equation (36)

Equation (33) states that the value of a put option is equal to:

$$P\left(K_{j-1}^{k},j\right)=\sum_{i=0}^{j}\lambda_j^i \max\left(K_{j-1}^{k}-S_j^i,0\right)+\lambda_j^{jtd}max\left(K_{j-1}^{k}-0,0\right) \tag{C.1}$$

We know that $S_j^{i+1} > K_{j-1}^k$ and that $S_j^i < K_{j-1}^k$ and since we look at a put option we are only interested in stock prices that are below the strike price and therefore we can reduce equation C.1:

$$P\left(K_{j-1}^{k},j\right)=\sum_{i=0}^{k}\lambda_j^i * \left(K_{j-1}^{k}-S_j^i\right)+\lambda_j^{jtd}K_{j-1}^{k} \tag{C.2}$$

Then we take out *i*=0 and *i*=*k* from the summation:

$$P\left(K_{j-1}^{k},j\right)=\lambda_j^0 *\left(K_{j-1}^{k}-S_j^0\right)+\lambda_j^k *\left(K_{j-1}^{k}-S_j^k\right)+\sum_{i=1}^{k-1}\lambda_j^i * \left(K_{j-1}^{k}-S_j^i\right)\ +\ \lambda_j^{jtd}K_{j-1}^{k} \tag{C.3}$$

Now, we can replace the lambdas in equation C.3 with the lambdas from expressions (15) to (17)

$$\begin{aligned}P\left(K_{j-1}^{k},j\right)=&\left(e^{-\mathrm{rh}} * \lambda_{j-1}^0 * z_{j-1}^0(1-q_{j-1}^0)\right)*\left(K_{j-1}^{k}-S_j^0\right)+\left(e^{-r\Delta t} * (\lambda_{j-1}^{k-1} * z_{j-1}^{k-1} * q_{j-1}^{k-1}\right.\\&\left.+\lambda_{j-1}^{k} * z_{j-1}^{k} * (1-q_{j-1}^{k}))\right)*\left(K_{j-1}^{k}-S_j^k\right)\\&+\sum_{i=1}^{k-1}\left(e^{-r\Delta t} * (\lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * q_{j-1}^{i-1}+\lambda_{j-1}^{i} * z_{j-1}^{i} * (1-q_{j-1}^{i}))\right)\\&* \left(K_{j-1}^{k}-S_j^i\right)+\lambda_j^{jtd} * K_{j-1}^{k}\end{aligned} \tag{C.4}$$

Additionally we can express the jump to default Arrow-Debreu price as expressed in (35) and replace it in expression (C.4) and reorder:

$$\begin{aligned}e^{\mathrm{rh}}P\left[K_{j-1}^{k},j\right]=&\left(\lambda_{j-1}^0 * z_{j-1}^0 * \left(1-q_{j-1}^0\right)\right)*\left(K_{j-1}^{k}-S_j^0\right)+\left(\lambda_{j-1}^{k-1} * z_{j-1}^{k-1} * q_{j-1}^{k-1}\right)\left(K_{j-1}^{k}-S_j^k\right)\\&+\left(\lambda_{j-1}^{k} * z_{j-1}^{k} * (1-q_{j-1}^{k})\right)*\left(K_{j-1}^{k}-S_j^k\right)\\&+\sum_{i=1}^{k-1}\left((\lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * q_{j-1}^{i-1}+\lambda_{j-1}^{i} * z_{j-1}^{i} * (1-q_{j-1}^{i}))\right) * \left(K_{j-1}^{k}-S_j^i\right)\\&+\left(\lambda_{j-1}^{jtd}+\sum_{i=0}^{K}\lambda_{j-1}^{i} * (1-z_{j-1}^{i})\right) * K_{j-1}^{k}\end{aligned} \tag{C.5}$$

Separating the summations term in two terms: t

$$\begin{aligned}e^{r\Delta t}P\left[K_{j-1}^{k},j\right]=&\left(\lambda_{j-1}^0 * z_{j-1}^0 * \left(1-q_{j-1}^0\right)\right)*\left(K_{j-1}^{k}-S_j^0\right)\\&+\left(\lambda_{j-1}^{k-1} * z_{j-1}^{k-1} * q_{j-1}^{k-1}\right)\left(K_{j-1}^{k}-S_j^k\right)+\left(\lambda_{j-1}^{k} * z_{j-1}^{k} * (1\right.\\&\left.-q_{j-1}^{k})\right)*\left(K_{j-1}^{k}-S_j^k\right)+\sum_{i=1}^{k-1}\left(\lambda_{j-1}^{i-1} * z_{j-1}^{i-1} * q_{j-1}^{i-1}\right)\left(K_{j-1}^{k}-S_j^i\right)\\&+\sum_{i=1}^{k-1}\left(\lambda_{j-1}^{i} * z_{j-1}^{i} * (1-q_{j-1}^{i})\right)\left(K_{j-1}^{k}-S_j^i\right)\\&+\left(\lambda_{j-1}^{jtd}+\sum_{i=0}^{K}\lambda_{j-1}^{i} * (1-z_{j-1}^{i})\right)K_{j-1}^{k}\end{aligned} \tag{C.6}$$

Now we change the counters in the first (summing) term from $\sum_{i=1}^{k-1}$ to $\sum_{i=0}^{k-2}$:

$$e^{rh}P[K_{j-1}^{k},j] = \left(\lambda_{j-1}^{0} * z_{j-1}^{0} * \left(1-q_{j-1}^{0}\right)\right) * \left(K_{j-1}^{k} - S_{j}^{0}\right) + \left(\lambda_{j-1}^{k-1} * z_{j-1}^{k-1} * q_{j-1}^{k-1}\right)\left(F_{j-1,j}^{k} - S_{j}^{k}\right) + (\lambda_{j-1}^{k} * z_{j-1}^{k} * (1 - q_{j-1}^{k})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) + \sum_{i=0}^{k-2} (\lambda_{j-1}^{i} * z_{j-1}^{i} * q_{j-1}^{i})(K_{j-1}^{k} - S_{j}^{i+1}) + \sum_{i=1}^{k-1} (\lambda_{j-1}^{i} * z_{j-1}^{i} * (1 - q_{j-1}^{i}))(K_{j-1}^{k} - S_{j}^{i}) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) K_{j-1}^{k} \tag{C.7}$$

And then we can add the first and second term to the summing so that both summing terms run from *i=0* to *k-1* and our expression reduces to:

$$e^{rh}P[K_{j-1}^{k},j] = (\lambda_{j-1}^{k} * z_{j-1}^{k-1}(1 - q_{j-1}^{k-1})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) + \sum_{i=0}^{k-1} (\lambda_{j-1}^{i} * z_{j-1}^{i} * q_{j-1}^{i})(K_{j-1}^{k} - S_{j}^{i+1}) + \sum_{i=0}^{k-1} (\lambda_{j-1}^{i} * z_{j-1}^{i} * (1 - q_{j-1}^{i}))(K_{j-1}^{k} - S_{j}^{i}) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) K_{j-1}^{k} \tag{C.8}$$

And then we can combine the two summation terms again:

$$e^{rh}P[K_{j-1}^{k},j] = (\lambda_{j-1}^{k} * z_{j-1}^{k}(1 - q_{j-1}^{k})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) + \sum_{i=0}^{k-1} (\lambda_{j-1}^{i} * z_{j-1}^{i} * q_{j-1}^{i})(K_{j-1}^{k} - S_{j}^{i+1}) + (\lambda_{j-1}^{i} z_{j-1}^{i}(1 - q_{j-1}^{i})(K_{j-1}^{k} - S_{j}^{i})) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) * K_{j-1}^{k} \tag{C.9}$$

Now we can expand the summation part:

$$e^{rh}P[K_{j-1}^{k},j] = (\lambda_{j-1}^{k} * z_{j-1}^{k} * (1 - q_{j-1}^{k})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) + \sum_{i=0}^{k-1} ((\lambda_{j-1}^{i} * z_{j-1}^{i} * q_{j-1}^{i} K_{j-1}^{k} - \lambda_{j-1}^{i} * z_{j-1}^{i} * q_{j-1}^{i} S_{j}^{i+1} + (\lambda_{j-1}^{i} z_{j-1}^{i}(1 - q_{j-1}^{i})(K_{j-1}^{k}) - (\lambda_{j-1}^{i} z_{j-1}^{i}(1 - q_{j-1}^{i})) S_{j}^{i})) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) * K_{j-1}^{k} \tag{C.10}$$

We can use the no-arbitrage condition from equation (12) and sum the exercise prices to arrive at:

$$e^{rh}P[K_{j-1}^{k},j] = (\lambda_{j-1}^{k} * z_{j-1}^{k} * (1 - q_{j-1}^{k})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) + \sum_{i=0}^{k-1} (\lambda_{j-1}^{i} * (z_{j-1}^{i} K_{j-1}^{k} - F_{j-1,j}^{i})) + \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) * K_{j-1}^{k} \tag{C.11}$$

The two variables that are still not known are $q_{j-1}^{k}$ and $S_{j}^{k}$. Therefore, bringing these two variables to the left side:

$$(\lambda_{j-1}^{k} * z_{j-1}^{k} * (1 - q_{j-1}^{k})) * \left(K_{j-1}^{k} - S_{j}^{k}\right) = e^{rh}P[K_{j-1}^{k},j] - \sum_{i=0}^{k-1} (\lambda_{j-1}^{i} * (z_{j-1}^{i} K_{j-1}^{k} - F_{j-1,j}^{i})) - \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) * K_{j-1}^{k} \tag{C.12}$$

The term $\sum_{i=0}^{k-1}(\lambda_{j-1}^{i} * (z_{j-1}^{i} K_{j-1}^{k} - F_{j-1,j}^{i}))$ on the right hand side is the sum of all the forwards that lie below the current node and we set that equal to $\rho_{down}$. The term $\left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) K_{j-1}^{k}$ on the right hand side is all known in period *j* and if we use expression (34) and (35) we can state that:

$$P_{jtd}[K_{j-1}^{k}, j] * e^{r\Delta t} = \left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right) K_{j-1}^{k} \tag{C.13}$$

Therefore, when we bring $\lambda_{j-1}^{k} * z_{j-1}^{k}$ to the other side, C.12 becomes:

$$(1 - q_{j-1}^{k})) * (K_{j-1}^{k} - S_{j}^{k}) = \frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}}{\lambda_{j-1}^{k} * z_{j-1}^{k}} \tag{C.14}$$

Now we can replace $q_{j-1}^{k}$ with the no arbitrage condition equation (13): $q_{j}^{i} = \frac{K_{j-1}^{k} - S_{j+1}^{i}}{S_{j+1}^{i+1} - S_{j+1}^{i}}$. This is the same as replacing $1 - q_{j-1}^{k}$ with $\frac{S_{j}^{k+1} - K_{j-1}^{k}}{S_{j}^{k+1} - S_{j}^{k}}$ [24]

$$\frac{S_{j}^{k+1} - K_{j-1}^{k}}{S_{j}^{k+1} - S_{j}^{k}} * (K_{j-1}^{k} - S_{j}^{k}) = \frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}}{\lambda_{j-1}^{k} * z_{j-1}^{k}} \tag{C.15}$$

Bring $S_{j}^{k+1} - K_{j-1}^{k}$ to the right side, multiply both sides by $S_{j}^{k+1} - S_{j}^{k}$ and subtract $K_{j-1}^{k}$ from both sides:

$$-S_{j}^{k} = \frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}}{(\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})} * (S_{j}^{k+1} - S_{j}^{k}) - K_{j-1}^{k} \tag{C.16}$$

Separating both terms on the right side and bring $S_{j}^{k}$ to the left

$$\frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}}{(\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})} * S_{j}^{k} - S_{j}^{k} = \frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}}{(\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})} * S_{j}^{k+1} - K_{j-1}^{k} \tag{C.17}$$

Give both sides the same denominator and bringing it under the same denominator:

$$\frac{e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}} - (\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})}{(\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})} S_{j}^{k} = \frac{\left(e^{r\mathrm{h}} P[K_{j-1}^{k}, j] - \rho_{down} - P_{jtd}[K_{j-1}^{k}, j] * e^{r\mathrm{h}}\right) * S_{j}^{k+1} - (\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k}) K_{j-1}^{k}}{(\lambda_{j-1}^{k} * z_{j-1}^{k})(S_{j}^{k+1} - K_{j-1}^{k})} \tag{C.18}$$

Rearranging we obtain:

$$S_{j}^{k} = \frac{S_{j}^{k+1} * \left(e^{r\mathrm{h}}(P_{df}[K_{j-1}^{k}, j]) - \rho_{down}\right) - \lambda_{j-1}^{k} * F_{j-1,j}^{k} * \left(S_{j}^{k+1} - K_{j-1}^{k}\right)}{e^{r\mathrm{h}}(P_{df}[K_{j-1}^{k}, j]) - \rho_{down} - \lambda_{j-1}^{k} * z_{j-1}^{k} * \left(S_{j}^{k+1} - K_{j-1}^{k}\right)} \tag{C.19}$$

[24] We also change the subscripting to *j-1*

which is equation (36) with: $\rho_{down} = \sum_{i=0}^{k-1} \lambda_{j-1}^{i} * z_{j-1}^{i} * \left(K_{j-1}^{k} - \frac{F_{j-1,j}^{i}}{z_{j-1}^{i,}}\right)$, $P_{jtd}[K,j] = e^{-r\mathrm{h}}\left(\lambda_{j-1}^{jtd} + \sum_{i=0}^{K} \lambda_{j-1}^{i} * (1 - z_{j-1}^{i})\right)K_{j-1}^{k}$ and $P_{df}\left[K_{j-1}^{k}, j\right] = P\left[K_{j-1}^{k}, j\right] - P_{jtd}\left[K_{j-1}^{k}, j\right]$.