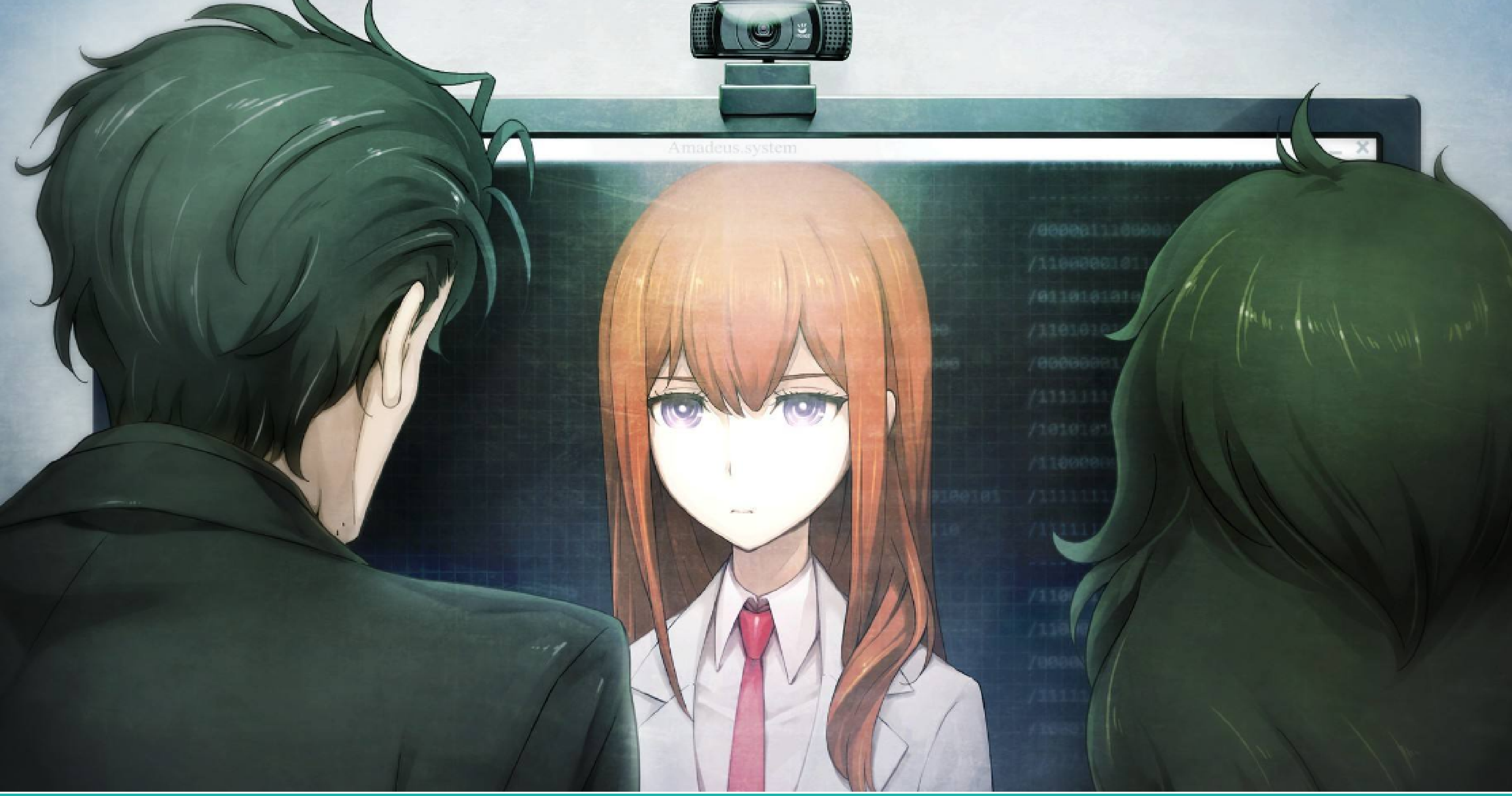

MAT3040 Advanced Linear Algebra NOTE

Date: autumn, 2022

El Psy Kongroo

Contents

Chapter difference between MATH2040 and MATH3040

we generalize something we learned in MAT2040:

1. vector space: $(x_1, ..., x_n)^t \in \mathbb{R}^n \Rightarrow v \in V$
2. linear transformation: $T : \mathbb{R}^n \to \mathbb{R}^m \Rightarrow T : V \to V$
3. eigenvector: $Av = \lambda v \Rightarrow Tv = \lambda v$
4. linear operator: $T : \mathbb{R}^n \to \mathbb{R}^n \Rightarrow T : V \to V$
5. inner product: $\langle v, w \rangle = \sum_{i=1}^n v_i w_i \to \langle v, w \rangle : V \times V \to \mathbb{C}$
6. symmetric matrix $A^t = A \Rightarrow$ self-adjoint linear operator $S : V \to V : \langle S(v), w \rangle = \langle v, S(w) \rangle$ we will prove that S also has an orthonormal basis of eigenvectors with real numbers.

Chapter why generalize from $\mathbb{R}^n$ to V

Chapter vector space

Chapter subspace

Chapter direct sum

Chapter spanning set

Chapter linear independence

Chapter dimension

Definition 8.1 (dimension)

let V be a vector space over $\mathbb{F}$, then dimension of V is defined by

$$dim_{\mathbb{F}(V)=}\begin{cases} m & \text{if V is finitely generated and m is the size of any basis of V} \\ \infty & \text{if V is not finitely generated} \end{cases}$$

♣

Example 8.1

1. $dim_{\mathbb{F}(\mathbb{F}^n)=n}$
2. $dim_{\mathbb{F}}(M_{m\times n}(\mathbb{F})) = m \times n$ since $B = \{e_{11}, e_{12}, ..., e_{mn}\}$ is a basis of $M_{m\times n}(\mathbb{F})$
3. $dim_{\mathbb{F}}(\mathbb{F}[x]) = \infty$
4. $W = \{A \in M_{3\times 3}(\mathbb{F}) | A^t = -A\} = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{F} \right\}$

   a basis of W is $\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$

so the dimension $dim_{\mathbb{F}}(W) = 3$

✍ **Exercise 8.1** if $V = span(v_1, v_2, ..., v_n)$, we can remove some v_i to get a basis of V. how about if we have $\{w_1, ...w_k\}$ that is linear independent in V, can we extend it to $\{w_1, ..., w_k, v_1, ..., v_r\}$ to be a basis of V?

Theorem 8.1 (basis extension)

let V be a finite dimensional vector space. for any $\{w_1, ..., w_k\}$ that is linear independent set in V, $k \leqslant n$, there are vectors $\{v_{k+1}, ..., v_n\}$ s.t. $\{w_1, ..., w_k, v_{k+1}, ..., v_n\}$ is a basis of V.

♡

Proof take a basis $\{u_1, ..., u_n\}$ of V. we consider $S = \{w_1, ..., w_k, u_1, ..., u_n\}$ which spans V since $\{u_1, ..., u_n\}$ is a basis of V. if S is linear independent, we are done.
if S is linear dependent, we want to exclude several vectors in S: we have

$$\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_k w_k + \beta_1 u_1 + ... + \beta_n u_n = 0 \quad \alpha, \beta \text{ are not all } 0$$

suppose $\exists \beta_1 \neq 0$, we have

$$u_1 = -(\frac{\alpha_1}{\beta_1} w_1 + \frac{\alpha_2}{\beta_1} w_2 + ... + \frac{\alpha_k}{\beta_1} w_k + \frac{\beta_2}{\beta_1} u_2 + ... + \frac{\beta_n}{\beta_1} u_n) \in \{w_1, w_2, ..., w_k, u_2, ..., u_n\} := S_2$$

we have: $\{u_1, ..., u_n\} \subset \{w_1, ..., w_k, u_2, ..., u_n\}$ since $u_1 \in \{w_1, ..., w_k, u_2, ..., u_n\}$, so $V \subset \{w_1, ..., w_k, u_2, ..., u_n\} = S_2$. if S_2 is linear independent, we are done. otherwise, we repeat the argument with one less vector $u_i \notin S_2$ and so on until $S_r := \{w_1, ..., w_k, u_r, ..., u_n\}$ is linear independent. we notice that the process finishes in finitely many times. so we are done.
(if all β are zero, we have

$$\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_k w_k + \beta_1 u_1 + ... + \beta_n u_n = \alpha_1 w_1 + ... + \alpha_k w_k = 0 \rightarrow a_1 = a_2 = ... = a_k = 0$$

and then all α, β are zero, we reach a contradiction.)

Theorem 8.2 (complementation)

let $dim(V) = n$ and $W \leqslant V$, then $\exists W' \leqslant V s.t.\ W \oplus W' = V$

♡

Proof let $\{w_1, ..., w_k\}$ be a basis of W $(k \leqslant n)$ and $W \leqslant v$, then $dim(W) = k \leqslant n = dim(V)$. by the theorem of basis extension, we have $\{w_{k+1}, ..., w_n\}$ s.t. $B = \{w_1, ..., w_k, w_{k+1}, ..., w_n\}$ is a basis of V. let $W' = \{w_{k+1}, ..., w_n\} \leqslant V$, we want to show that $W \otimes W' = V$:

we first check that $W + W' = V$: take $v \in V$, since B is a basis of V, we have

$$v = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \in W + W' \to v \in W + W'$$

and thus $V \subset W + W'$

we check that $W \cap W' = \{0\}$: suppose $\exists u \in W \cap W'$, then

$$u = \beta_1 w_1 + ... + \beta_k w_k = \gamma_{k+1} w_{k+1} + ... + \gamma_n w_n$$

so we have

$$\beta_1 w_1 + ... + \beta_k w_k + (-\gamma_{k+1}) w_{k+1} + ... + (-\gamma_n) w_n = 0$$

since $\{w_1, ..., w_n\}$ is a basis and thus linear independent, we must have

$$\beta_1 = ... = \beta_k = \gamma_1 = ... = \gamma_n = 0 \to u = 0$$

so $W \cap W' = \{0\}$

Chapter linear transformation

Definition 9.1 (linear transformation)

let V,W be vector spaces over $\mathbb{F}$, a linear transformation $T : V \to W$ is a function from V to W satisfying $T(\alpha v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$ for all $\alpha_1, \alpha_2 \in \mathbb{F}$ and $v_1, v_2 \in V$

♣

Example 9.1

1. $T : \mathbb{R}^n \to \mathbb{R}^m : T(x) = Ax, A \in M_{m\times n}(\mathbb{F})$
2. $T : \mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R}), T(f) := f''$ is a linear transformation: we check two things:

   (a). $T(f + g) = T(f) + T(g)$

   (b). $T(\alpha f) = \alpha T(f)$
3. $T : \mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R}) : T(f) := \int_0^x f(t)dt$ is also a linear transformation.

Remark

1. if the target space $W = V$, then the linear transformation is called a linear operator.
2. $T(O_V) \to T(O_W)$ for all linear transformations: since $T(0 \cdot v) = 0 \cdot T(v) = 0_W$
3. if $T : V \to W$ and S$W \to U$ are linear transformations, then $S \circ T : V \to U$ is a linear transformation.

Remark

1. $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R} : T(A) := tr(A) = \sum_{i=1}^n A_{ii}$ is a linear transformation
2. $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R} : A \to det(A)$ is not a linear transformation if $n > 1$ since $det(A + B) \neq det(A) + det(B)$
3. $T : M_{m\times n}(\mathbb{R}) \to M_{k\times n}(\mathbb{R}) : A \to C \cdot A, \in M_{k\times m}(\mathbb{R})$ is a linear transformation.

Chapter kernel and image

Definition 10.1

$T : V \to W$ is a linear transformation, the kernel of T is defined by $ker(T) := \{v \in V|T(v) = 0_w\}$
the range or image of T is defined by $im(T) := T(V) \in W|v \in V$

Example 10.1 a. $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(V) = Av$, $ker(T) = \{v \in \mathbb{R}^n|Av = 0\} := Null(A)$; $im(T) = \{A(v) \in \mathbb{R}^m|v \in \mathbb{R}^m\} := Col(A)$, the space spanned by the columns of A
b. $T : c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) : R(f) = f''$, $ker(T) = \{f \in C^\infty(\mathbb{R})|f'' = 0\} = \{f(x) = ax + b|a, b \in \mathbb{R}\} = span_\mathbb{R}\{g(x) = x, h(x) = 1\}$, $im(T) = \{f''|f \in C^\infty(\mathbb{R})\} = C^\infty(\mathbb{R})$, which means that every infinitely differentiable function $g(x) \in C^\infty(\mathbb{R})$ is the second derivative of a function f, i.e. $f'' = g$

Remark in example b, T is not injective but is surjective, why?

Proposition 10.1

a. let $T : V \to W$ be linear transformations, then $ker(T) \le V$ and $im(T) \le W$;
b. T is injective $\Leftrightarrow ker(T) = \{0_V\}$;
c. T is surjective $\Leftrightarrow im(T) = W$

Proof

1. let $v_1, v_2 \in ker(T), a, b \in \mathbb{F}$, we show that $av_1 + bv_2 \in ket(T)$, then $ker(T) \leqslant V$: consider $T(av_1 + bv_2) = aT(v_1) + bT(v_2) = a \cdot 0_W + b \cdot 0_W = 0_W \in ker(T)$. so we have $av_1 + bv_2 \in ker(T)$
2. let $T(v_1), T(v_2) \in im(T), a, b \in \mathbb{F}$. we show that $aT(v_1) + bT(v_2) \in im(T)$: $T(v_1), T(v_2) \in im(T) \to v_1, v_2 \in V \to (av_1 + bv_2) \in V$ and $T(av_1 + bv_2) \in im(T) \to aT(v_1) + bT(v_2) \in im(T)$
3. we show that: T is injective $\Leftrightarrow ker(T) = \{0_v\}$.
   $(\Rightarrow)$: suppose T is injective. we have $T(v_1) = T(v_2) \to v_1 = v_2$, let $\forall v \in ker(T)$, then $T(v) = 0_W \to v = 0_V$ since T is an injective and $T(0_V) = 0_W \to 0_V \in ker(T)$. so we have proved that $ker(T) = \{0\}$
   $(\Leftarrow)$: suppose $ker(T) = \{0\}$, we prove that T is an injective. suppose $T(x) = T(y) \to T(x) - T(y) = T(x - y) = 0_W$, since $ker(T) = \{0_V\}$, we have $x - y = 0_V \to x = y$, so T is an injective.
4. we prove that T is a surjective $\Leftrightarrow im(T) = W$: recall the definition of T is a surjective: $\forall w \in W, \exists v \in V$ s.t. $T(v) = w$, by the definition we see the equivalence.

Remark if $\{v_1, v_2, ..., v_n\}$ is a basis of V, one can determine $T(v) = T(a_1v_1 + ... + a_nv_n)$ for all $v \in V$ by knowing the values of $\{T(v_1), ..., T(v_n)\}$

Definition 10.2 (isomorphism)

T is a linear transformation. if T is bijective, $T : V \to W$ is an isomorphism

Proposition 10.2

if $T : V \to W$ is an isomorphism, then $T^{-1} : W \to V$ exists, and is also a linear transformation

Proof we want to show that $T^{-1}(aw_1 + bw_2) = aT^{-1}(w_1) + bT^{-1}(w_2)$, $\forall w_1, w_2 \in T(V) = W$ and $a, b \in \mathbb{R}$: let $w_1 = T(v_1)$ and $w_2 = T(v_2) \in W$, then we have $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. since T is a linear transformation, we have $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$,

$$T^{-1}(T(av_1 + bv_2)) = T^{-1}(aT(v_1) + bT(v_2)) = T^{-1}(aw_1 + bw_2)$$

we have $av_1 + bv_2 = T^{-1}(aw_1 + bw_2) \to aT^{-1}(w_1) + bT^{-1}(w_2) = T{-}1(aw_1 + bw_2)$
(we have $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$). we are done.

Theorem 10.1 (rank-nullity)

let $T : V \to W$ be a linear transformation, $dim(V) < \infty$, then $dim(ker(T)) + dim(im(T)) = dim(V)$

Proof we have $ker(T) \subset V$, by complementation theorem, we have $\exists V_1 \leqslant V$ s.t. $ker(T) \oplus V_1 = V$, then $dim(ker(T)) + dim(V_1) = dim(V)$ proved in HW2 so we only need to show that $dim(V_1) = dim(im(T))$. we consider the restricted map: $T\big|_{V_1} : V_1 \to T\big|_{V_1}(V_1) := T(v_1), \forall v_1 \in V_1, T(v_1) \in W$

two claims are required in the proof of the rank-nullity theorem:

Claim 10.1

$T\big|_{V_1}$ is an isomorphism so $dim(V_1) = dim(T(V_1))$

Proof

1. $T\big|_{V_1}$ is a surjective: since all elements in the space $T(V_1)$ is of the form $T\big|_{V_1}(v_1)$ for $v_1 \in V_1$, we conclude that $T\big|_{V_1}$ is a surjective.
2. $T\big|_{V_1}$ is an injective. we check that $ker(T\big|_{V_1}) = \{0_{V_1}\}$: suppose $u_1 \in ker(T\big|_{V_1}) \leqslant V_1 \leqslant V$, we prove that $u_1 = 0$: $T\big|_{v_1}(u_1) = T(u_1) = 0_{T(V_1)}$ since $u_1 \in ker(T\big|_{V_1})$, and thus $u_1 \in ker(T)$, so we have $u_1 \in ker(T) \cap V_1$, recall we have $ker(T) \oplus V_1 = V \to V_1 \cap ker(T) = \{0\}$, so $u_1 = 0_{V_1}$, we are done.

Claim 10.2

$dim(T(V_1)) = dim(im(T))$

Proof $T(V_1) \leqslant T(V) = im(T)$ since $V_1 \leqslant V$. we check: $T(V) \leqslant T(V_1)$: consider $T(v) \in T(V)$, since $ker(T) \oplus V_1 = V$, $\exists v_k \in ker(T)$ and $\exists v_n \in V_1$ s.t. $v = v_k + v_n$, $T(v) = T(v_k + v_n) = T(v_k) + T(v_n)$, since $v_k \in ker(T) \to T(v_k) = 0$, $T(v) = 0 + T(v_n) = T(v_n) \in T(V_1)$, so we have $T(V) \leqslant T(V_1)$, and recall $T(V_1) \leqslant T(V)$, we have $T(V_1) = T(V) \to dim(T(V_1)) = dim(T(V)) = dim(im(T))$

Chapter coordinate vectors

our goal is to translate vector v ∈ V into $(x_1, ..., x_n)^T \in \mathbb{F}^n$

Definition 11.1 (coordinate vector)

let V be a vector space over $\mathbb{F}$ with $dim(V) = n < \infty$, consider $B = \{v_1, v_2, ..., v_n\}$, which is an ordered basis of V, then the coordinate vector $v \in V$ is given by $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$, then $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n$

Remark here the **ordered** basis means that $B = \{v_1, v_2, ..., v_n\}$ and $B' = \{v_2, v_1, ...v_n\}$ are different as ordered basis.

Example 11.1 $V = P_3(\mathbb{R}) = \{\text{polynomials of degree} \leqslant 3\}$, $B = \{1, x, x^2, x^3\}$ ordered basis of V. then $v = 15 + \pi x^2 + exercise^3$, $[v]_B = (15, 0, \pi, e)^T \in R^4$. let $B' = \{x^2, x^3, x, 1\}$, $[v]_{B'} = (\pi, e, 0, 15)^T \in \mathbb{R}^4$. we find the order of B matters.

Proposition 11.1

the function $\phi : V \longrightarrow \mathbb{F}^n$ gives an isomorphism of vector space.

Proof 1. show that ϕ is a well-defined function: $\forall v \in V$, suppose we have

$$v = \alpha_1 v_1 + ... + \alpha_n v_n = \beta_1 v_1 + ... + \beta_n v_n$$

we have $[v]_\alpha = \left\{ \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_n \end{matrix} \right\} = \left\{ \begin{matrix} \beta_1 \\ \vdots \\ \beta_n \end{matrix} \right\} \in F^n$, so we have $\alpha_i = \beta_i, \forall i = 1, 2, ..., n$ since B is a basis of V, thus we conclude that ϕ is a well-defined function. 2. show that ϕ is surjective
3. show that ϕ is injective

Example 11.2 based on the proposition, we can check the linear independence of $\begin{cases} x + 3x^2 + 4x^3 \\ 1 + 5x^2 \\ 20x + x^2 + 10x^3 \end{cases}$ by checking the linear independence of $\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 20 \\ 1 \\ 10 \end{pmatrix} \right\}$

Definition 11.2 (well-defined function)

a function $\phi : A \longrightarrow B$ is called well-defined if the following holds:
1. for a fixed $a \in A$, there is only one output $\phi(a)$ in the image
2. the only output $\phi(a)$ is in codomain B.

Chapter change of basis

given two bases B and B', what is the relation between $[v]_B$ and $[v]_{B'}$?

Definition 12.1

let $B = \{v_1, v_2, ..., v_n\}$, $B' = \{v'_1, v'_2, ..., v'_n\}$ be two bases of V, suppose: $v'_j = \sum_{i=1}^{n} \alpha_{ij} v_i$ for $j = 1,2,...,n$. the change of basis matrix is $C_{BB'} = (\alpha_{ij})_{n\times n}$, it is important to understand the meaning of i and j

Example 12.1 we continue the example in the last chapter:

let the basis B be the standard basis and let the basis B' be: $\begin{cases} x + 3x^2 + 4x^3 \\ 1 + 5x^2 \\ 20x + x^2 + 10x^3 \end{cases}$ the matrix representation is $C_{BB'}(\alpha_{ij}) =$

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ since we have $\begin{cases} x + x^2 = 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \\ 1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \end{cases}$

Example 12.2 let B be the standard basis and $B'' = \{2 + x, 1 + 3x + x^2, 5\}$, we have the matrix representation is $C_{BB'} = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Proposition 12.1

given two bases B and B', $\forall v \in V$, we have $C_{BB'}[v]_{B'} = [v]_B$

Proof recall we have $B = \{v_1, ...v_n\}$ and $B' = \{v'_1, ..., v'_n\}$ $v'_j = \sum_{i=1}^{n} \alpha_{ij} v_i$ where $(\alpha)_{ijnn}$ is the matrix of change of basis $C_{BB'}$.

1. for $v = v'_j$, LHS $= C_{BB'}[v_j]_{B'} = C_{BB'}[0v'_1 + ... + 1v'_j + ... + 0v'_n]_{B'} = C_{BB'}e_j =$ the j-th column of $C_{BB'} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{1j} \end{pmatrix}$

   RHS $= [v'_j]_B = [\alpha_{1j}v_1 + \alpha_{2j}v_2 + ... + \alpha_{nj}v_j]_B = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$
2. for general $v = k_1v'_1 + ... + k_nv'_n, k_1, ..., k_n \in \mathbb{F}$. then $C_{BB'}[v]_{B'} = C_{BB'}[k_1v'_1 + ... + k_nv'_n]_{B'} = C_{BB'}(k_1[v'_1]_{B'} + ... + k_n[v'_n]_{B'}) = k_1C_{BB'}[v'_1]_{B'} + ... + k_nC_{BB'}[v'_n]_{B'} = k_1[v'_1]_B + ... + k_n[v'_n]_B == [k_1v'_1 + ... + k_nv'_n]_B = [v]_B$. so we are done.

Example 12.3 we continue the example: $v = 3 + 2x + 4x^2$, $[v]_B = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $[v]_{B'} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ and we check that

$$(\alpha_{ij})[v]_{B'} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = [v]_B$$

Corollary 12.1

$C_{BB'}$ is invertible with $C_{BB'}^{-1} = C_{B'B}$

Proof from the proposition, we have $C_{BB'}[v]_{B'} = [v]_B$, we do twice change of basis and get $C_{B'B}C_{BB'}[v]_{B'} = C_{B'B}[v]_B = [v]_{B'}$, let $M = C_{B'B}C_{BB'}$, we have $Mx = x, \forall x \in F^n \to M = I_n$.

Chapter matrix representation of linear transformation

Definition 13.1 (matrix representation of linear transformation)

let

$$T : V \to W$$

be a linear transformation,

$$A = v_1, v_2, ..., v_n$$

is an ordered basis of V,

$$B = w_1, w_2, ..., w_m$$

is an ordered basis of W. the matrix representation of T with respect to A and B is given by the (n,m) matrix

$$(T_{BA}) = (\beta_{ij})$$

with entries given by:

$$T(v_1) = \beta_{11}w_1 + \beta_{21}w_2 + ... + \beta_{m1}w_m$$

$$T(v_2) = \beta_{12}w_1 + \beta_{22}w_2 + ... + \beta_{m2}w_m$$

$$...$$

$$T(v_n) = \beta_{1n}w_1 + \beta_{2n}w_2 + ... + \beta_{mn}w_m$$

♣

Example 13.1 let T: $P_3(\mathbb{R}) \to P_2(\mathbb{R}) : T(P(x)) = P'(x)$, let $A = \{x^2, 1, x^3, x\}$, $B = \{X, X^2, 1\}$. we have: $T(x^2) = 2 \cdot x + 0 \cdot x^2 + 0 \cdot 1$; $T(1) = 0 \cdot x + 0 \cdot x^2 + 0 \cdot 1$; $T(x^3) = 0 \cdot x + 3 \cdot x^2 + 0 \cdot 1$; $T(x) = 0 \cdot x + 0 \cdot x^2 + 1 \cdot 1$, so the (3,4) matrix is

$$T_{BA} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 13.1

$T_{BA}[v]_A = [T(v)]_B$*: it means that we can calculate* $T(v)$ *on RHS by pure matrix multiplication on LHS.*

♠

Proof

now we want to find the relations between $T_{BA}, T_{DB}, (S \circ T)_{DA}$, given $T : V_A^n \to W_B^m$ and $S : W_B^m \to U_D^p$. clearly that $T_{BA}, S_{DB}, (S \circ T)_{DA}$ are matrices.

Theorem 13.1 (functoriality)

$(S \circ T)_{DA} = S_{DB} \times T_{BA}$

composition of linear transformation↔matrix multiplication

♡

Proof we write $A = \{v_1, v_2, ..., v_n\}$, $B = \{w_1, w_2, ..., w_m\}$, $D = \{u_1, u_2, ..., u_p\}$. recall the definition of matrix representation, we have:

$$T(v_j) = \sum_{i=1}^{m} (T_{BA})_{ij} w_i$$

$$S(w_j) = \sum_{i=1}^{p} (S_{DB})_{ij} u_i$$

we notice that the j-th column is $(S\cdot T)_{DA}\times(e_j)$

$$(S\cdot T)_{DA}\times(e_j)=(S\cdot T)_{DA}[v_j]_A=[(S\cdot T)(v_j)]_D=[S(T(v_j))]_D=[S(\sum_{i=1}^{m}(T_{BA})_{ij}w_i)]_D=$$

$$[\sum_{i=1}^{m}(T_{BA})_{ij}S(w_i)]_D=[\sum_{i=1}^{m}(T_{BA})_{ij}(\sum_{k=1}^{p}(S_{DB})_{ki}u_k)]_D=[\sum_{i=1}^{p}((\sum_{i=1}^{m}(S_{DB})_{ki}(T_{BA})_{ij})u_k]_D$$

$$=[\sum_{k=1}^{p}((S_{DB})(T_{BA}))_{kj}u_k]_D=\begin{pmatrix}(S_{DB}\times T_{BA})_{1j}\\(S_{DB}\times T_{BA})_{2j}\\\cdots\\(S_{DB}\times T_{BA})_{nj}\end{pmatrix}$$

which is the j-th column of $S_{DB}\times T_{BA}$.
this is the definition of matrix multiplication: $(S_{DB})_{p\times m}\times(T_{BA})_{m\times n}$

✍ **Exercise 13.1** given $T:V\to V$, B, C are two bases of V, then we can write down matrix representations. what are the relations among $(T)_{BB},(T)_{BC},(T)_{CB},(T)_{CC}$?

Example 13.2 the relation between $(T)_{CC}$ and $(T)_{BB}$ is given by: $(C_{BC})\times(T_{CC})\times(C_{CB})=T_{BB}$, where C_{BC},C_{CB} are the change of basis matrices

Remark T_{BB},T_{CC} are "similar" matrices if the above formula holds. the two matrices have the same characteristic polynomials, eigenvalues, determinant, diagonalizability.
we have to give a definition for the determinant of a linear operator T:

Definition 13.2 (determinant of linear operator)

$T:V\to V$ is a linear operator, B is a basis of V, $det(T):=det(T_{BB})$, where T_{BB} is the matrix representation of the linear operator T. ♣

Chapter quotient space

Remark

the rough idea is to partition a big vector space V into union of smaller spaces.

Definition 14.1 (coset)

let V be a vector space and $W \leqslant V$. for any $v \in V$, define a coset (with representative v) by: $v + W := \{v + w | w \in W\} \subset V$

Example 14.1 there are two cosets:
a. $W = span\{e_1, e_2\} \leqslant V = \mathbb{R}^3$
b. $W = span(0, 0, 1)^T$

Remark for $v \neq v' \in V$, we can have $v + W = v' + W$

Proposition 14.1

two cosets $v + W = v' + W$ are equal (as subsets of V) $\Leftrightarrow v - v' \in W$

Proof $(\rightarrow)$: $v + W = v' + W \rightarrow v + 0 \in v + W = v' + W \rightarrow v = v' + w \rightarrow v - v' = w \in W$
$(\leftarrow)$: want to show that $v + W \subset v' + W$: suppose $v - v' = w \in W$, $\forall v + w' \in v + W$, $v + w' = v - v' + v' + w' = w + v' + w' \in v' + W \rightarrow v + W \subset v' + W$. and similarly we can prove the other inclusion.

Example 14.2 one application is to solve the system in linear algebra: $Ax = b$

Definition 14.2 (quotient space V/W)

let $W \leqslant V$, the quotient space V/W is defined as the collection of all W-cosets: $V/W := \{v + W | v \in V\}$

we can define a vector space structure on V/W, i.e. the addition and scalar multiplication.

Exercise 14.1 since we have defined addition on quotient space, we wonder that is it true that

$$(v + W) + (u + W) = (v' + W) + (u' + W)$$

given $v \neq v'$ and $u \neq u'$

Proof LHS = $(v + u) + W$, RHS = $(u' + v') + W$, note that $(v + u) - (v' - u') = (v - v') + (u - u')$, from the above remark we know that $(v - v') \in W$ and $(u - u') \in W$, so $(v + u) - (v' - u') \in W$, use the remark again, we have

$$(v + u) + W = (v' + u') + W$$

similar to the above question14.1, we also need to check scalar multiplication: if $v + W = v' + W$ given $v \neq v'$, is it true that

$$\alpha(v + W) = \alpha(v' + W) \Leftrightarrow (\alpha v) + W = (\alpha v') + W$$

Theorem 14.1

let V be a vector spce, $W \leqslant V$, define a map $\Pi : V \rightarrow V/W$ by $\Pi(v) = v + W$, here $v + W$ is an element in V/W. then Π is a surjective linear transformation, Π is called canonical projection map, with $ker(\Pi) = W$

Proof it is obvious that Π is surjective. we only need to show that Pi is linear transformation and $ker(\Pi) = W$:
to show that $ker(\Pi) = W$: $\forall x \in ker(\Pi), \Pi(x) = 0_{V/W} \Leftrightarrow \Pi(x) = 0_V + W \Leftrightarrow x + W = 0_V + W \Leftrightarrow x - 0_V \in W \Leftrightarrow x \in W$, since $\Pi(x) = x + W$

Corollary 14.1

if $dim(V) < \infty$, $W \leqslant V$, then $dim(V/W) = dim(V) - dim(W)$

Proof by rank-nullity theorem, we have: $\dim(V) = \dim(ker(\Pi)) + \dim(im(\Pi))$, from 14.1 we have $ker(\Pi) = W$ and thus we have $dim(V) = dim(W) + dim(im(\Pi))$ and since Π is surjective, we know that $im(\Pi) = V/W$, $dim(im(\Pi)) = dim(V/W) \rightarrow dim(V) = dim(ker(\Pi)) + dim(im(\Pi)) = dim(V/W) + dim(W)$

Example 14.3

Example 14.4

Theorem 14.2

let $T : V \longrightarrow W$ be a linear transformation, and V' is a vector subspace and $V' \subset ker(T)$, then we can define a linear transformation: $\bar{T} : V/V' \longrightarrow W$ by $\bar{T}(v + V') = T(v)$

Proof we first check that T is well-defined: suppose $v, u \in V$, $v \neq u$ such that $v + V' = u + V'$, we prove that $T(v) = T(u)$: since $v + V' = u + V' \Leftrightarrow u - v \in V' \rightarrow u - v \in ker(T) \rightarrow T(u - v) = 0_W \rightarrow T(u) = T(v)$ so we conclude that T is well-defined.

Chapter dual vector space

Definition 15.1 (dual(or dual space))

let V be a vector space over $\mathbb{F}$, the dual of V is $V^* := Hom_{\mathbb{F}}(V, \mathbb{F}) = \{$all linear transformations $\phi : V \to \mathbb{F}\}$, i.e. V^* contains all functions $\phi : V \to \mathbb{F}$ satisfying 1. $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$; 2. $\phi(\alpha v) = \alpha\phi(v)$

♣

Example 15.1

Remark a. sometimes the elements $\phi : V \to \mathbb{F}$ in v^* are called linear functionals.
b. there is a vector space structure on V^*: if $\phi, \psi \in V^*$, then $\phi + \psi \in V^*$ and satisfies that $(\phi + \psi)(v) = \phi(v) + \psi(v)$ and $(\alpha\phi)(v) = \alpha\phi(v)$
c. to describe a function $f : A \to B$, we need to specify $f(a), \forall a \in A$
d. if we further know that $f : V \to W$ is a linear transformation, we only need to specify $f(v_i)$, where $B = \{v_i | i \in I\}$(I may be infinite) is a basis of V

Definition 15.2 (basis of dual space)

let V be a vector space with basis $B = \{v_i | i \in I\}$. for each $i \in I$, we define $\phi_i : V \to \mathbb{F}$ by:

$$\phi_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

and for all $v = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + ... + \alpha_l v_{i_l} \in V$, $\phi_i(v) = \alpha_i\phi_i(v_{i_1}) + \alpha_1\phi_i(v_{i_2}) + \alpha_l\phi_i(v_{i_l})$, by construction, we have: $\phi_i \in V^*$ and we define $B^* := \{\phi_i \in V | i \in I\}$. the question is: whether B^* is a basis for V^*?

♣

Example 15.2

Proposition 15.1

B^* is linearly independent in V^*

♠

Proof take a finite subset $\{\phi_{i_1}, \phi_{i_2}, ..., \phi_{i_k} \subset B^*\}$ and consider

$$\alpha_1\phi_{i_1} + \alpha_2\phi_{i_2} + ... + \alpha_k\phi_{i_k} = 0_{V^*}$$

want to show that $\alpha_1 = \alpha_2 = ... = \alpha_k$:

we have: $(\alpha_1\phi_{i_1} + \alpha_2\phi_{i_2} + ... + \alpha_k\phi_{i_k})(v_{i_1}) = 0_{V^*}(v_{i_1}) \in \mathbb{F}$. recall $\phi_{i_k}(v_{i_n}) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$ we have $(\alpha_1\phi_{i_1} + \alpha_2\phi_{i_2} + ... + \alpha_k\phi_{i_k})(v_{i_1}) = \alpha_1 = 0$ and similarly we have $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$

Corollary 15.1

if $dim(V^*) < \infty$, then B^* is a basis of V^*

♡

Proof suppose $dim(V) = n$, then $dim(V^*) = dim(Hom_{\mathbb{F}}(V, \mathbb{F}))$=$dim(V)dim(\mathbb{F}) = dim(V) \times 1$. we have shown that $B^* = \{\phi_1, \phi_2, ..., \phi_n\}$ is linear independent in $V^* \to B^*$ is a basis of v^*.

Remark if $dim(V) = \infty$, B^* may not be a basis of V^*

Example 15.3 a. $V = \mathbb{R}[x]$, the polynomial in $\mathbb{R}$, $B = \{1, x, x^2, ...\}$, $B^* - \{\phi_0, \phi_1, ...\}$ with $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$ we know that B^* is linearly independent but it does not span V^*
(recall that $\forall x \in span(B^*), x = \alpha_1\phi_1 + \alpha_2\phi_2 + ... + \alpha_n\phi_n$ is finite)

Chapter annihilators

Definition 16.1

let V be a vector space, $S \subset V$, the annihilator of S is defined as $Ann(s) := \{\phi \in V^* | \phi(S) = 0, \forall s \in S\}$ ♣

Example 16.1

Proposition 16.1

1. *$Ann(s) \leqslant V^*$ is a vector space.*
2. *$S \subset S' \subset V$, then $Ann(S) \supset Ann(S')$* ♠

Proof 1.review how to show a set is a vector space?
2. suppose $\phi \in Ann(S')$, want to show that $\phi \in Ann(S)$. for all $s \in S$, by definition of $\phi \in Ann(S')$, we know that $\phi(s) = 0, \forall s \in S \subset S'$, which means that $\phi \in Ann(s) \to Ann(s) \supset Ann(S')$

Theorem 16.1

a. $Ann(S) = Ann(span(S))$;
b. if $dim(V) < \infty$ and $W \leqslant V$, then $dim(Ann(W)) = dim(V) - dim(W)$ ♡

Proof a. from the above proposition, we know that $Ann(S) \supset Ann(span(S))$ since $S \subset span(S)$, we only need to show the other inclusion. let $\phi \in Ann(S)$, suppose $S_1, S_2, ..., S_k \in S$, for any $\alpha_1 S_1 + \alpha_2 S_2 + ... + \alpha_k S_k \in span(S)$ we have $\phi(\alpha_1 S_1 + \alpha_2 S_2 + ... + \alpha_k S_k) = \alpha_1 \phi(S_1) + \alpha_2 \phi(S_2) + ... + \alpha_k \phi(S_k) = \alpha_1 \times 0 + \alpha_2 \times 0 + ... + \alpha_k \times 0 = 0 \to \phi \in Ann(span(S))$, so we have $Ann(S) \subset Ann(span(S))$

Proof b.

Example 16.2

Remark if $dim(V) < \infty$ and $W \leqslant V$, then $dim(V/W) = dim(V) - dim(W)$, $dim(Ann(w)) = dim(V) - dim(W)$, dim(Ann(W)) = dim(V/W). we can construct an isomorphism: $\Phi(Ann(W)) \to V \backslash W$ as follows: let $B := \{v_1, v_2, ..., v_k, v_{k+1}, ...v_n\}$ and $\{v_1, v_2, ..., v_k\}$ is a basis of W, then:
(a) $\{\phi_{k+1}, ..., \phi_n\}$ is a basis of Ann(W) by the above theorem.
(b) $\{v_{k+1} + W, ..., v_n + W\}$ is a basis of $V \setminus W$ by HW5., then we can define $\Phi(\phi_l) := v_l + W$, (l = k+1,...,n), and this is an isomorphism: $Ann(W) \to V \setminus W$. but this is not a good isomorphism since it involves a choice of basis of V

✍ **Exercise 16.1** is there a way to construct an isomorphism: $Ann(W)$ and $V \setminus W$ without using a choice of basis, i.e. construct a natural isomorphism?

the answer yes.

Definition 16.2 **($\Phi : Ann(W) \to (V/W)^*$)**

let $f \in Ann(W)$, then $(f : V \to \mathbb{F}) \in V^*$ is a linear transformation with $W \in ker(f)$, by the theorem 14.1, we can define a linear transformation $\tilde{f} : V/W \to \mathbb{F}$ by $\tilde{f}(v+W) = f(v)$. we define $\Phi : Ann(W) \to (V/W)^*$ by $\phi(f) = \tilde{f}$ ♣

Proposition 16.2

Φ is a linear transformation ♠

Proof prove that $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$:
a. prove that $\Phi(f+g) = \Phi(f) + \Phi(g)$: $LHS = \Phi(f+g) = f \tilde{+} g$, $RHS = \Phi(f) + \Phi(g) = \tilde{f} + \tilde{g}$, for any $v + W \in V/W$,

$LHS = (\tilde{f} + \tilde{g})(v + W) = (f + g)(v) = f(v) + g(v)$, $RHS = \tilde{f}(v + W) + \tilde{g}(v + W) = f(v) + g(v) \Rightarrow LHS = RHS$
b. prove that $\Phi(\alpha f) = \alpha\Phi(f)$

Proposition 16.3

Φ is an injective transformation.

Proof only need to show that $ker(\Phi) = \{0_{Ann(W)}\}$: for all $f \in ker(\Phi)$, $\Phi(f) = \tilde{f}(v + W) = f(v) = 0, \forall v \in V \to f = 0_{V^*} = 0_{Ann(W)}$, we conclude that $ker(\Phi) = \{0_{Ann(W)}\}$.

Corollary 16.1

if V is finite dimensional, then $\Phi : Ann(W) \to (V/W)^*$ is an isomorphism

Proof we know that $dim(Ann(W)) = dim(V/W) = dim(V/W)^*$ by 16.1 and we have shown that Φ is injective, then Φ is bijective and Φ is an isomorphism.

if V is infinite dimensional, we can still show that Φ is an isomorphism. the method is to construct another linear transformation Ψ s.t. $\Phi \circ \Psi$ is identity and $\Psi \circ \Phi$ is also identity. then we find the inverse of Φ is Ψ, Φ is invertible and must be isomorphic.
invertible $\Leftrightarrow$ isomorphic $\Leftrightarrow$ surjective and injective!

Chapter transpose of linear transformation

Definition 17.1

let $T: V \to W$ be a linear transformation, define: $T^t: W^* \to V^*$ by the following: let $f \in W^*$, then $(T^t(f): V \to \mathbb{F}) \in V^*$ is defined by $T^t(f)(v) := f(T(v)), \forall v \in V$, where $f: W \to \mathbb{F}$, $T(v) \in W$, $f(T(v) \in \mathbb{F})$

Remark by construction, $T^t(f)(v)$ is just a function. we want to check that $T^t(f): V \to \mathbb{F}$ is indeed an element in V^*, i.e. $T^t(f)$ is a linear transformation: $V \to \mathbb{F}$. to show that, we need to check the linearity of $T^t(f)$: $T^t(f)(v+v') = T^t(f)(v) + T^t(f)(v')$ and $T^t(f)(\alpha v) = \alpha T^t(f)(v)$

Proof to show that $T^t(f)(v+v') = T^t(f)(v) + T^t(f)(v')$: $T^t(f)(v+v') = f(T(v+v')) = f(T(v)+T(v')) = f(T(v)) + f(T(v')) = T^t(f)(v) + T^t(f)(v')$

after the remark, we can safely say that $T^t(f): W^* \to V^*$ is well-defined.

Proposition 17.1

T^t is a linear transformation: check the linearity of $T^t(f)$: $T^t(f+g) = T^t(f) + T^t(g)$ and $T^t(\alpha f) = \alpha T^t(f)$

Proof only show the first equation: $T^t(f+g) = (f+g)(T(v)) = f(T(v)) + g(T(v)) = T^t(f) + T^t(g)$. the proof of the other equations is similar.

we have checked two different things in the above two proofs. the first thing is the linearity of the function $T^t(f)(v): V \to \mathbb{F}$ and the second thing is the linearity of the function $T^t(f): W^* \to V^*$.

Theorem 17.1

let V, W be finite dimensional vector space. $T: V \to W$ is a linear transformation with transpose $T^t: W^* \to V^*$, consider $\mathcal{A}: \{v_1, v_2, ..., v_n\}$ is a basis of V, $\mathcal{B}: \{w_1, w_2, ..., w_m\}$ is a basis of W, $\mathcal{A}^* = \{\phi_1, \phi_2, ..., \phi_n\}$ is a basis of V^*, then the matrix representations

$$(T^t)_{A^*B^*} = (T_{BA})^t$$

where T_{BA} is the matrix representation of the linear transform $T: V \to W$, and $(T_{BA})^t$ is the transpose matrix of (T_{BA}). transpose of linear transformations $\leftrightarrow$ transpose of matrix.

Proof recall that $T: V \to W$ and $\mathcal{A} = \{v_1, v_2, ..., v_n\}$ is a basis of V, $\mathcal{B} = \{w_1, w_2, ..., w_m\}$ is a basis of W. we have $T(v_j) = \sum_{l=1}^m \alpha_{lj} w_l$
$T^t: W^* \to V^*$ and $\mathcal{A}^* = \{\phi_1, \phi_2, ..., \phi_n\}$ is a basis of V^*, $\mathcal{B}^* = \{\psi_1, \psi_2, ..., \psi_m\}$ is a basis of W^*. we have $T^t(\psi_i) = \sum_{k=1}^n \beta_{ki}\phi_k$.
notice that we can calculate $T^t(\psi_i)(v_j)$ in two ways: the first is

$$T^t(\psi_i)(v_j) = \psi_i(T(v_j)) = \psi_i(\sum_{l=1}^m \alpha_{lj} w_l) = \sum_{l=1}^m \alpha_{lj}\psi_i(w_l) = \alpha_{ij}$$

since $\psi_i(w_l) = 0$ if $l \neq i$
the second way is:

$$T^t(\psi_i)(v_j) = \psi(T(v_j)) = (\sum_{k=1}^n \beta_{ki}\phi_k)(T(v_j)) = \sum_{k=1}^n \beta_{ki} T(\phi_k(v_j)) = \beta_{ji}$$

since $\phi_k(v_j) = 0$ if $k \neq j$. so we have $\beta_{ji} = \alpha_{ij}$. this shows that $(T^t)_{A^*B^*} = (T_{BA})^t$

we finally come back to the origin why we introduce the transpose of linear transformation: the inverse of Φ in 16: consider the surjective linear transformation $\Pi: V \to V/W : \Pi(v) = v + W$, then we know that its transpose is $\Pi^t := (V/W)^* \to V^*$.

we claim that: for all $h \in (V/W)^*$, $\Pi^t(h) \in Ann(W)$, so Π^t actually maps $(V/W)^*$ onto $Ann(W)$.

Proof let $w \in W$, $\Pi^t(h)(w) = h(\Pi(w)) = h(w + W) = h(0_W + W) = h(0_{V/W})$. recall the definition of $h : (V/W \rightarrow \mathbb{F})$, so we have $h(0_{V/W}) = 0_{\mathbb{F}}$, which implies that $\Pi^t(h) \in Ann(W)$

finally we need to show that $\Pi^t \cdot \Phi$ and $\Phi \cdot \Pi$ are both identity maps and we conclude that Φ is an isomorphism.

Proof

Chapter preliminary for characteristic polynomial

Definition 18.1 (leading term)

♣

Definition 18.2 (monic)

♣

Definition 18.3 (greatest common divisor for polynomial)

♣

Remark using euclidean algoritheorem we can find the gcd of polynomials.

Theorem 18.1 (bezout's theorem)

♡

Chapter characteristic polynomials and minimal polynomials

Definition 19.1 (eigenvector; eigenvalue)

let $T : V \to T$ be a linear operator. a non-zero vector $v \in V$ is an eigenvector of T with eigenvalue λ if $T(v) = \lambda v$

Remark it is enough to find the eigenvalues of the matrix $T_{\mathcal{B}\mathcal{B}} \in M_{n\times n}(\mathbb{F})$

Definition 19.2 (characteristic polynomial)

let $dim(V) = n < \infty$, and $T : V \to V$ be a linear operator, then the characteristic polynomial of T is defined as

$$\chi_T(x) := \chi_{T_{\mathcal{B}\mathcal{B}}}(x) = det(xI - T_{\mathcal{B}\mathcal{B}})$$

Exercise 19.1 if using another basis $\mathcal{A}$ and get another matrix $T_{\mathcal{A}\mathcal{A}}$, is it true that $\chi_{T_{BB}}(x) = \chi_{T_{AA}}(x)$? the answer is yes.

Proof

Definition 19.3 (P(T))

let $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 \in \mathbb{F}(V)$ be a polynomial and $T : V \to V$ is a linear operator, we define $P(T) : V \to V$ by $P(T) = a_n(T \circ T \circ ... \circ T) + ... + a_2(T \circ T) + a_1 T + a_0 I$
note $P(T)$ is still a polynomial and later you will meet $m_T(T)$, which follows the definition of P(T)

Example 19.1

Definition 19.4 (minimal polynomial)

let $T : V \to V$ be a linear operator, then the minimal polynomial m_T of T is a polynomial satisfying:
1. $m_T(x)$ is monic, (monic means $a_n = 1$)
2. $(m_T(T) : V \to V)$ is the zero operator
3. $m_T(x)$ is the polynomial of smallest positive degree satisfying 1. and 2.

Example 19.2

Theorem 19.1 (Cayley-Hamilton)

$\chi_T(T) = 0$

Proposition 19.1

the properties of minimal polynomials are as follow:
1. if $dim(V) = n < \infty$, then $m_T(x)$ always exists.
2. $m_T(x)$ is unique, i.e. suppose p(x) and q(x) are polynomials satisfying the definition of minimal polynomial, then $p(x) \equiv q(x)$
3. for any polynomials $f(x) \in \mathbb{F}[x]$ s.t. $f(T) = 0_{V\to V}$ then $m_T(x)|f(x)$

Remark we introduce the strategy for finding $m_T(x)$:
1. by cayley-hamilton theorem, we have $\chi_T(T) = O_{V\to V}$
2. by property 3. we have $m_T(x)|\chi_T(x)$
3. run through all factors $f(x)$ of $\chi_T(x)$ and pick the one with smallest degree satisfying $f(T) = 0_{V\Rightarrow V}$

Example 19.3

Remark for a linear transformation T, we wonder what is the relation between minimal polynomial and eigenvalues?
a. if λ is an eigenvalue of T, we have $(x - \lambda)|m_T(x)$

Chapter triangularizable operator

Definition 20.1 (diagonalizable)

we say a linear transformation $T: V \to V$ is diagonalizable if: $\exists$ a basis $\mathcal{B}$ s.t. $T_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \lambda_1 & \dots & O \\ \vdots & \lambda_i & \vdots \\ O & \dots & \lambda_n \end{pmatrix}$ ♣

Remark a. T is diagonalizable if and only if B is a basis of eigenvectors of T. (T is diagonalizable $\Leftrightarrow \exists B = \{v_1, ..., v_n\}$ s.t. $T(v_1) = \lambda_1 v_1 + 0v_2 + ... + 0v_n$, and $T(v_2) = 0v_1 + \lambda_2 v_2 + 0v_3 + ... + 0v_n$,..., and $T(v_n) = 0v_1 + ... + 0vn - 1 + \lambda_n v_n$ and which leads to B being a basis of eigenvectors of T.)
b. the characteristic polynomial $\chi_T(x)$ is not enough to determine whether T is diagonalizable. we give a counter example: ???.
but $\chi_T(x)$ is enough to determine whether T is triangularizable:

Definition 20.2 (triangularizable)

$T: V \to V$ is said to be triangularizable if $\exists$ basis B of V s.t. $T_{BB} = \begin{pmatrix} \lambda_1 & * \\ O & \lambda_n \end{pmatrix}$ ♣

Theorem 20.1 (triangularizable)

let $T: V \to V$, then T is triangularizable if and only if $\chi_T(x) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_n)$, i.e. $\chi_T(x)$ can be factorized into linear factors. ♡

Proof

Example 20.1

Remark if V is a finite dimensional vector space over $\mathbb{C}$, then every $T: V \to V$ is triangularizable: recall the fundamental theorem of algebra that is: all polynomial in $\mathbb{C}$ including the characteristic polynomial $\chi_T(x)$ can be factorized into linear factors.

Example 20.2 a. if $v \in V$ is an eigenvector of $T: V \to V$, then $W = span\{v\}$ is T-invariant: for any $\alpha v \in W, T(\alpha v) = \alpha T(v) = \alpha(\lambda v) = (\alpha\lambda)v \in W$
b. the $\lambda - eigenspace: E_n := \{v = V | T(v) = \lambda v\}$ is also a T-invariant subspace of V.
c. more generally, for all polynomials $g(x) = \mathbb{F}[x]$, $ker(g(T)) \leqslant V$ is T-invariant.

in HW5, we have the following question:

✍ **Exercise 20.1**

(IMPORTANT EXERCISE TO BE USED LATER) Let $T: V \to V$ be a linear operator, and $W \leq V$ be a T-invariant subspace, i.e. $T(W) \leq W$. Let $A = \{\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{v_{k+1}}, \ldots, \mathbf{v_n}\}$ be an ordered basis of V, with $B = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ an ordered basis of W as in Question 2. Assume that

$$T_{AA} = \begin{pmatrix} P & Q \\ O & R \end{pmatrix}$$

where $P \in M_{k\times k}(\mathbf{R}), Q \in M_{k\times(n-k)}(\mathbf{R}), O \in M_{(n-k)\times k}(\mathbf{R}), R \in M_{(n-k)\times(n-k)}(\mathbf{R})$. (a) Show that the entries of $O \in M_{(n-k)\times k}(\mathbf{R})$ are all zeros.
(b) Find the matrix representation $(T|_W)_{BB}$ of $T|_W : W \to W$.

(c) Using the basis C of V/W given in Question 3, compute the matrix representation $(\tilde{T})_{CC}$ for the linear transformation $\tilde{T} : V/W \to V/W$ defined in Question 2.

Lemma 20.1

under the setting of this question, we have: $\chi_T(x) = (\chi_{T/W}(x)(\chi_{\tilde{T}}(x))$

Proof

Corollary 20.1

let V be a vector space over $\mathbb{C}$, then all linear operator $T : V \to V$ are triangularizable.

Proof

Theorem 20.2 (cayley-hamilton)

let $T : V \to V$ be linear operator on finite dimensional vector over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $\chi_T(T) = 0_{V\to V} \Leftrightarrow m_T(x)|\chi_T(x)$

Proof

Proposition 20.1

if $T : V \to V$ is triangularizable, then $\chi_T(T) = 0_{V\to V}$

Proof

Proposition 20.2

$T : V \to V$, V is a finite dimensional vector space over $\mathbb{R}$, then $\chi_T(T) = 0_{V\to V}$

Remark by cayley-hamilton theorem, if $m_T(x)|\chi_T(x)$, so we can find $m_T(x)$ by looking at factors of $\chi_T(x)$, more precisely, if $\chi_T(x) = p_1(x)^{e_1}p_2(x)^{e_2}..p_k(x)^{e_k}$, $p_1(x), ..., p_k(x)$ are distinct irreducible polynomials, then $m_T(x) = p_1(x)^{f_1}p_2(x)_{f_2}...p_k(x)^{f_k}$ with $f_k \leqslant e_k$

Proposition 20.3

let $T : V \to V$ be such that $\chi_T(x) = p_1(x)^{e_1}p_2(x)^{e_2}..p_k(x)^{e_k}$ for p_1, p_2, p_k irreducible polynomials $e_k \geqslant 1$ integers, then $m_T(x) = m_T(x) = p_1(x)^{f_1}p_2(x)_{f_2}...p_k(x)^{f_k}$ *with* $1 \leqslant f_i \leqslant e_i, \forall i = 1, 2, ..., k$

Proof

Chapter primary decomposition theorem

Theorem 21.1 (primary decomposition theorem)

let $T : V \to V$ be a linear operator, $dim(V) < \infty$, suppose $m_T(x) = p_1(x)^{e_1} p_2(x)^{e_2} ... p_k(x)^{e_k}$, $p_i(x)$ are distinct irreducible and monic, let $v_i := ker(p_i(T)^{e_i}) \leqslant V$, then:

1. $V = V_1 \oplus V_2 \oplus ... \oplus V_k$

2. v_i are T-invariant subspaces of V

3. the minimal polynomial of $T|_{v_i} : V_i \to V_i$ is $m_{T|_{v_i}}(x) = p_i(x)^{e_i}$

♡

to prove the theorem, we need the following three claims:

claim 1: $v_i \in ker(p_i(T)^{e_i}) := V_i$

claim 2: the sum $V = V_1 + V_2 + ... + V_k$ is a direct sum

claim 3: if $m_{T|_{v_i}}(x) = p_i(x)^{g_i}.1 \leqslant g_i \leqslant e_i$, then $g_i = e_i$

Corollary 21.1

$T : V \to V$ is a linear transform, V is a finite dimensional vector space, then T is diagonalizable $\Leftrightarrow$ $m_T(x) = (x - \mu_1)...(x - \mu_k)$ for distinct $\mu_1, ..., \mu_k$

♡

Proof ($\leftarrow$): by primary decomposition theorem, let $V = V_1 \oplus V_2 \oplus ... \oplus V_k$ where $V_i = ker(T - \mu_i I) = \{v|(T - \mu_i I)v = 0\} = \{v|Tv = \mu_i v\}$ is the $\mu_i - eigenspace$ of T. take B_i being the basis of V_i for each $i = 1, 2, ..., k$, then $B = B_1 \cup B_2 \cup ... \cup B_k$ is a basis of V, where each vector in B_i is an eigenvector of T with eigenvalue μ_i, so we conclude B is a basis of T. then we finish the direction (B is a basis with all vectors being eigenvectors, and T is diagonalizable by a remark above.)

Chapter jordan normal form

Theorem 22.1

suppose $T : V \to V$ *is s.t.* $m_T(x) = (x-u_1)^{e_1}(x-u_2)^{e_2}...(x-u_k)^{e_k}$*, there is a basis A of V s.t.* $T_{\mathcal{A}\mathcal{A}} = \begin{pmatrix} J_1 & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & J_k \end{pmatrix}$ *is block diagonal, where each* J_r *looks like* $J_r = \begin{pmatrix} u_m & 1 & \dots & O \\ \vdots & \ddots & \ddots & 1 \\ O & \dots & \dots & u_m \end{pmatrix}$ *for some* $1 \leqslant m \leqslant k$

♡

Remark a. the sizes of each J_r need not to be the same

b. there can be two blocks J_r and J_s having the same diagonal entries u_m

c. each $u_1, u_2, ...u_m$ must appear in at least one J_r

d. in the special case if all the jordan blocks are of 1×1 sizes i.e. $J_r = (u)$ for all r, then $T_r r$ is a diagonal matrix.

Example 22.1

Chapter inner product space

Definition 23.1 (bilinear form)

let V be a vector space over $\mathbb{R}$, a bilinear form on V is a function $B : V \times V \to \mathbb{R}$ satisfying:

a. $B(u+v,w) = B(u,w) + B(v+w)$

b. $B(u,v+w) = B(u,v) + B(u,w)$

c. $B(u,\lambda v) = B(\lambda v,u) = \lambda B(u,v), \quad \lambda \in \mathbb{R}$

♣

Example 23.1

Remark there are no positive definite bilinear form for complex vector space V for any $v \neq 0$: for any $v \neq 0$ and $B(v,v) > 0$, we have $B(iv,iv) = -B(v,v) < 0$. so we have trouble defining $||v|| := \sqrt{B(v,v)}$ for complex vector space V.

Definition 23.2 (sesquilinear)

let V be a complex vector space. a sesquilinear form on V is a function $B : V \times V \to \mathbb{C}$ satisfying a. and b. in the definition of bilinear form and c: $B(X,\lambda y) = \lambda B(x,y)$ and $B(\lambda x,y) = \overline{\lambda} B(x,y)$, $\lambda \in \mathbb{C}$

♣

Definition 23.3 (conjugate symmetric; positive definite)

a sesquilinear form is conjugate symmetric if $B(x,y) = \overline{B(y,x)}$

a sesquilinear form is positive definite if $B(v,v) \geqslant 0, \forall v \in V$ and equality holds iff $v = 0$

♣

Example 23.2

Definition 23.4 (inner product)

let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. an inner product of V is a bilinear (or sesquilinear) form $B : V \times V \to \mathbb{R}(\mathbb{C})$ on V s.t. B is (conjugate) symmetric and positive definite,

♣

Definition 23.5 (orthogonal;norm;unit vector)

a. let $(V, \langle\cdot\rangle)$ be inner product. we say $u,v \in V$ are orthogonal if $\langle u,v\rangle = 0$

b. the norm of $v \in V$ is $||v|| := \sqrt{\langle v,v\rangle}$

c. if $||v|| = 1$, v is a unit vector.

♣

Remark we have cauchy-schwarz inequality: $|\langle u,v\rangle| \leqslant ||u|| \cdot ||v||$ and it implies the triangle imequality: $||u+v|| \leqslant ||u|| + ||v||$

Remark[gram-schmidt process] $\{v_1,...,v_k\}$ is a basis of V, we can apply the gram-schmidt process to get $\{w_1,w_2,...,w_k\}$ orthonormal basis of V

a. $w_1 = \frac{v_1}{||v_1||}$

b. suppose we have $\{w_1,...,w_k\}$, then w_{k+1} is obtained by $w'_{k+1} := v_{k+1} - (\langle v_{k+1},w_1\rangle w_1 + ... + \langle v_{k+1},w_k\rangle w_k)$ and $w_{k+1} := \frac{w'_{k+1}}{||w'_{k+1}||}$

Theorem 23.1 (rietz representation theorem)

let V be inner product space, we define a function $\phi : V \to V^* : V \to \phi(v) := \phi_v : V \to \mathbb{F}$, where ϕ_v is defined by $\phi_v(w) = \langle v,w\rangle, \forall w \in V$. then:

a. $\phi_v \in V^*$

b. ϕ *is injective*

c. ϕ *is a $\mathbb{R}$-linear transformation i.e.* $for \gamma,\delta \in \mathbb{R}$, $\phi(\gamma v +') = \gamma\phi(v) + \phi(v')$

♡

Corollary 23.1

Remark we wonder: how about $\mathbb{C}$-inner product space V with $dim(V) = n < \infty$ in such cases, we treat V as a real vector space of dimension 2n.

Corollary 23.2

Corollary 23.3

Chapter adjoint operator

Theorem 24.1

let V be a finite dimensional inner product space, and $T : V \to V$ be a linear operator for any $v \in V$, there exists a unique $a_v \in V$ s.t. $\langle a_v, w\rangle = \langle v, T(w)\rangle$, $\forall w \in V$

Definition 24.1 (adjoint map)

let V be a finite dimensional inner product space and $T : V \to V$, then the adjoint map $T^* : V \to V$ of T is defined by $T^* : v \to V$ of T is defined by $T^*(v) := a_v$, where a_v appears in the above theorem.
Rietz's theorem guarantees the existence and uniqueness of a_v

Remark in short, the ajoint map $T^* : V \to V$ satisfies $\langle T^*(v), w\rangle = \langle v, T(w)\rangle$, $\forall v, w \in V$

Example 24.1

Proposition 24.1

the map $T^* : V \to V$ is a linear transformation.

Example 24.2

Proposition 24.2

let V be finite dimensional inner product space, suppose $\mathcal{B} = \{e_1, \ldots, e_n\}$ orthonormal basis of V, then the matrix representation of $T : V \to V$ and $T^* : V \to V$ are related by $(T^*)_{\mathcal{B}\mathcal{B}} = \overline{(T_{\mathcal{B}\mathcal{B}})^t}$

Chapter self-adjoint operators

Definition 25.1 (self-adjoint)

let V be inner product space, a linear operator $T : V \to V$ is self-adjoint if its adjoint operator $T^* : V \to V$ is equal to itself, i.e. $T^* = T$

Example 25.1

Remark 1. we will generalize all nice results on symmetric matrices $A = A^t$ to all self-adjoint operators $T = T^*$ for finite dimensional vector space V, namely if $T = T^*$ is self-adjoint, then T is diagonalizable, and it has an orthonormal basis of eigenvectors with real eigenvalues.
2. for $V = \mathbb{C}^n$, $T(v) = Av$ is self-adjoint $\iff A = \overline{A^t}$, therefore for complex $n \times n$ matrices, the conjugate symmetric matrices have the property in 1.

Proposition 25.1

let $T : V \to V$ be self-adjoint, then all eigenvalues λ of T are real

Proof suppose $T(v) = \lambda v$ ($v \neq 0$ is an eigenvector), $\lambda \in \mathbb{C}$, then $\overline{\lambda}\langle v, v\rangle = \langle \lambda v, v\rangle = \langle T(v), v\rangle = \langle T^*(v), v\rangle = \langle v, T(v)\rangle = \langle v, \lambda v\rangle = \lambda\langle v, v\rangle$, so we have

$$(\overline{\lambda} - \lambda)(\langle v, v\rangle) = 0$$

since $v \neq 0$, we have $\langle v, v\rangle \neq 0$ and $\lambda = \overline{\lambda}$ and thus λ is real.

Proposition 25.2

if T is self-adjoint, $T(v) = \lambda v$ and $T(w) = \mu w$, $v, w \neq 0$, λ, μ are distinct eigenvalues, then $v \perp w$, i.e. $\langle v, w\rangle = 0$

Proof

Proposition 25.3

let V be a finite dimensional inner product space, and $T : V \to V$ is self-adjoint, then T always has an eigenvector.

Proposition 25.4

let $T : V \to V$ be self-adjoint ($dim(V) < \infty$), if $U \leqslant V$ is T-invariant, then $U^\perp \leqslant V$ is also T-invariant, recall $U^\perp = \{w \in V | \langle w, u\rangle = 0, \forall u \in U\}$

Theorem 25.1

let V be a finite-dimensional inner product space, and $T : V \to V$ is self-adjoint, then V has an orthonormal basis of eigenvectors of T with real eigenvalues.

Chapter unitary operators

Definition 26.1 (unitary operator)

let V be $\mathbb{C}$-inner product space. an operator $T : V \to V$ is called unitary if $\langle T(x), T(y)\rangle = \langle x, y\rangle, \forall x, y \in V$

Remark if $T : V \to V$ is unitary, then $||T(x)|| = ||x||, \forall v \in V$, i.e. T preserves the norm of x, which is easily to be checked by taking $x = y$ in the definition.

Example 26.1

Remark what is meaning of $\overline{A^t}A = I_{n\times n}$ for $A \in M_{n\times n}(\mathbb{C})$? ...

Definition 26.2 (unitary matrix)

$A \in M_{n\times n}(\mathbb{C})$ is called unitary matrix if $\overline{A^t}A = I$ we have $T(x) = Ax$ is unitary $\leftrightarrow$ $A \in M_{n\times n}(\mathbb{C})$ is a unitary matrix

Theorem 26.1

let $T : V \to V$ be unitary, and $dim(V) < \infty$, then V has an orthonormal basis of eigenvector of T. moreover, the eigenvalues $\mu \in C$ of T must have $||\mu|| = 1$

Lemma 26.1

let V be $\mathbb{C}$-space, then $T : V \to V$ is unitary $\Leftrightarrow T^*T : V \to V$ is identity map.

Lemma 26.2

let v be an eigenvector of a unitary operator $T : V \to V$, then its eigenvalue $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$

Lemma 26.3

let $T : V \to V$ be unitary operator and $U \leqslant V$ is T-invariant, then $U \leqslant V$ is $T^* - invariant$.

Proposition 26.1

let $T : V \to V$ be unitary operator and $U \leqslant V$ is T-invariant, then $U^\perp \leqslant V$ is also $T-$invariant

Theorem 26.2

let V be a finite dimensional $\mathbb{C}$-inner product, and $T : V \to V$ is unitary, then there is an orthonormal basis of eigenvectors in V with eigenvalues $|\lambda| = 1$

Chapter normal operators

Definition 27.1 (normal operator)

let V be a $\mathbb{C}$-inner product space, we say $T : V \longrightarrow V$ is normal if $T^*T = TT^*$

♣

Example 27.1 all self-adjoint operators (over $\mathbb{C}$-inner product space) is normal since $TT^* = TT = T^2$ and $T^*T = TT = T^2$. notice that unitary operators are normal since $TT^* = T^*T = I$

Lemma 27.1

let T be a normal operator. we have 1. $||T(v)|| = ||T^*(v)||, \forall v \in V$;
2. $T - \lambda I$ is also normal operator, $\forall \lambda \in \mathbb{C}$;
3. $T(v) = \lambda v \longrightarrow T^*(v) = \overline{\lambda}(v)$;
4. if we have $T(v) = \lambda v$ and $T(w) = \mu w, \lambda \neq \mu$, then we have $\langle v, w\rangle = 0$

♡

Theorem 27.1

let V be a finite dimensional $\mathbb{C}$-inner product space, and $T : V \longrightarrow V$ is a normal operator. then there is an orthonormal basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ of V s.t. each v_i is an eigenvector of T.

♡

Corollary 27.1 (spectral theorem)

let $T : V \longrightarrow V$ be a normal space on a finite dimensional $\mathbb{C}$-inner product space V, then there are self-adjoint operators, $p_1, p_2, ..., p_k : V \longrightarrow V$ s.t.
a. $p_i^2 = p_i$
b. $p_i p_j = 0$ if $i \neq j$
c. $\sum_{i=1}^k p_i = Id$; and
d. $\sum_{i=1}^k \lambda_i p_i = T$ for some $\lambda_i \in \mathbb{C}$; we decompose T into a linear combination of projection operators p_i

♡

Remark we will prove that these $p_i = a_i(T)q_i(T) : V \longrightarrow V$ are self-adjoint in HW11.

Chapter tensor product

the motivation is that we try to understand the k-linear maps $f : V_1 \times V_2 \times ... \times V_k \rightarrow W$ $(V_1, V_2, ..., V_k, W)$ are vector spaces, i.e. $f(v_1, ..., a_i v_i + b v_i', v_k) = af(v_1, ..., v_i, ..., v_k) + bf(v_1, ..., v_i', ..., v_k)$

Example 28.1

1. the inner product with bilinear form: $V \times V \rightarrow \mathbb{R}$ (which is not a linear transformation)
2. the cross product: $\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $u \times v$ is 2-linear: $(au_1 + bu_2) \times v = a(u \times v) + b(u \times v)$
3. $f : M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \times ... \times M_{n\times n}(\mathbb{R}) : (A_1, A_2, ..., A_k) \rightarrow A_1 \cdot A_2 ... \cdot A_k$ is $k - linear$
4. $det : \mathbb{R}^(n \times 1) \times ... \times \mathbb{R}^(n \times 1) \rightarrow \mathbb{R} : det(v_1, ..., v_n) = det(v_1|...|v_n)$, we know that the function det is n-linear i.e. we have

$$det(v_1|...|v_i + v_i'|...|v_n) = det(v_1|...|v_i|...|v_n) + det(v_1|...|v_i'|...v_n)$$

Remark $V_1 \times V_2 \times ... \times V_k$ has a vector space structure i.e. $(v_1, ..., v_k) + (v_1', ..., v_k') = (v_1 + v_1', ..., v_k + v_k')$ and $\alpha(v_1, ..., v_k) = (\alpha v_1, ..., \alpha v_k)$. we wonder whether a k-linear map $f : V_1 \times V_2 \times ... \times V_k \rightarrow W$ be a linear transformation between vector spaces? if yes, we could apply many theorems on linear transformations to the map f. however, the answer is NO.

Example 28.2 one example is the bilinear map: we know that is not a linear transformation (and we define the sesquilinear transformation)

we are going to consstruct a tensor product space $V_1 \otimes V_2 \otimes ... \otimes V_k$ such that:

1. $\exists$ an injective map $i : V_1 \times ... \times V_k \rightarrow V_1 \otimes V_2 \otimes ... \otimes V_k : i(v_1, ..., v_k) = v_1 \otimes v_2 \otimes ... \otimes v_k$. the image of i, $V_1 \otimes V_2 \otimes ... \otimes V_k$ contains all $(v_1, ..., v_k) \in V_1 \times V_2 \times ... \times V_k$
2. for any k-linear $f : V_1 \times ... \times V_k \rightarrow W$, we have linear transformation $\Phi : V_1 \times ... \times V_k \rightarrow W$ s.t. $f = \Phi \circ i$, i.e. $f(v_1, ..., v_k) = \Phi(v_1 \otimes ... \otimes v_k)$ is completely determined by Φ, and thus rather than study f, which is not a linear transformation, we study Φ, which is a linear transformation and contains all information of f.

Definition 28.1 ($V \otimes W$; $V_1 \otimes V_2 \otimes ... \otimes V_k$)

we only define $V \otimes W$ (dimension = 2) and it can be easily generalized to any dimension k. let V, W be vector space over $\mathbb{F}$, we define a set $\mathscr{S} := \{(v, w)|v \in V, w \in W\}$ and a vector space $\mathscr{X} := (\mathscr{S})$, $\mathscr{X}$ is a very big uncountable dimensional vector space.

♣

Remark WARNING it is important to note that the elements of $\mathscr{X}$ are of the form $a_1(v_1, w_1) + ... + a_k(v_k, w_k)$, however, there are no relations among different $(v_i, w_i)'s$. for example, $(v, 0_W), (0_V, w)$ and (v, w) are 3 linearly independent vectors in $\mathscr{X}$ with no relationship among them: i.e. $(v, 0) + (0, w) \neq (v, w)$. another example is (v, w) and $(2v, 2w)$ are linearly independent in $\mathscr{X}$, i.e. $2(v, w) \neq (2v, 2w)$. the only possible relation on $\mathscr{X}$ is like: $2(v, w) + 5(v, w) = 7(v, w)$

Definition 28.2 (the set y)

let $y \leqslant \mathscr{X}$ be a vector space spanned by the following vectors:

$$y := \begin{cases} (v, w_1 + w_2) - 1(v, w_1) - 1(v, w_2) \\ 1(v_1 + v_2, w) - 1(v_1, w) - 1(v_2, w) \\ 1(kv, w) - k(v, w) \\ 1(v, lw) - l(v, w) \end{cases} \text{for all possible} \begin{cases} v, v_1, v_2 \in V \\ w, w_1, w_2 \in W \\ k, l \in \mathbb{F} \end{cases}$$

we observe that y is also a very big vector subspace of $\mathscr{X}$ with uncountable dimension.

♣

Definition 28.3 ($V \otimes W$ and $v \times w$)

we define $V \otimes W := \mathscr{X}/y$, for $v \in V$ and $w \in W$, we also define $v \otimes w := (v, w) + y \in \mathscr{X}/y =: V \otimes W$.

♣

Example 28.3

1. we study $(2v)\otimes w \in V\otimes W$: $(2v)\otimes w := (2v,w)+y = (2v,w)-[(2v,w)-2(v,w)]+y$ $((2v,w)-2(v,w)\in y)$ we have $(2v,w)+y = 2(v,w)+y = 2((v,w)+y) = 2(v\otimes w)$, similarly we have $v\otimes(lw) = l(v\otimes w)$
2. consider $(v_1+v_2,w)\otimes w \in V\otimes W$: $(v_1+v_2)\otimes w = (v_1+v_2,w)+y = (v_1+v_2,w)-[(v_1+v_2,w)-(v_1,w)-(v_2,w)]+y = (v_1,w)+y+(v_2,w)+y = v_1\otimes w + v_2\otimes w$, similarly we have $v\otimes(w_1+w_2) = v\otimes w_1 + v\otimes w_2$
3. we study $\begin{pmatrix}2\\3\end{pmatrix}\otimes\begin{pmatrix}4\\5\\6\end{pmatrix}\in\mathbb{R}^2\otimes\mathbb{R}^3$: let $\begin{pmatrix}2\\3\end{pmatrix} = 2e_1+3e_2$, and $\begin{pmatrix}4\\5\\6\end{pmatrix} = 4f_1+5f_2+6f_3$, we have

$$\begin{aligned}(2e_1+3e_2)\otimes(4f_1+5f_2+6f_3) &= (2e_1)\otimes(4f_1+5f_2+6f_3)+(3e_2)\otimes(4f_1+5f_2+6f_3)\\ &= 2\cdot4(e_1\otimes f_1)+2\cdot5(e_1\otimes f_2)+2\cdot6(e_1\otimes f_3)\\ &+3\cdot4(e_2\otimes f_1)+3\cdot5(e_2\otimes f_2)+3\cdot6(e_3\otimes f_3)\end{aligned}$$

Remark a general element in $V\otimes W$ is of the form $(v_1\otimes w_1+\dots v_i\otimes w_j+\dots+v_k\otimes w_k)$, which means that $e_1\otimes e_2+e_2\otimes e_2\in\mathbb{R}^2\otimes\mathbb{R}^2$ cannot be further simplified into a single expression, i.e. $e_1\otimes e_2+e_2\otimes e_2\neq v\otimes w$ for any $v,w\in\mathbb{R}^2$. we emphasize this fact since in vector space, we always have $(v_1,w_1)+(v_2,w_1) = (v_1+v_2,w_1)$

Exercise 28.1 take $v = \begin{pmatrix}a\\b\end{pmatrix}, w = \begin{pmatrix}c\\d\end{pmatrix}$, show that there is no a,b,c,d satisfying $v\otimes w = e_1\otimes e_1+e_2\otimes e_2$

Theorem 28.1 (universal property)

let $f: V\times W\to U$ be a bilinear map, then $\exists\Phi: V\otimes W\to U$ s.t. Φ is a linear transformation and $\Phi(v\otimes w) = f(v,w),\forall v\in V$ and $\forall w\in W$. recall $f(v,w)$ is not a linear transformation

Proof

Example 28.4applications

1. if V has a basis $\{v_1,v_2,\dots,v_n\}$ and W has a basis $\{w_1,\dots,w_m\}$, then $V\otimes W$ has basis $C = \{v_i\otimes w_i|1\leqslant n, 1\leqslant j\leqslant m\}$. in particular, $dim(V\otimes W) = dim(V)dim(W) = nm$. we check that C spans $V\otimes W$. (for linear independence, we need to apply the universal property.)
2. if $T: V\to V'$ and $S: W\to W'$ are linear transformations, we can construct a linear transformation $T\otimes S: V\otimes W\to V'\otimes W'$ as follows: we define a bilinear map: $f: V\times W\to V'\otimes W'$ by $f(v,w) := T(v)\otimes S(w)$. ($f$ is bilinear: we have $f(2v,w) = T(2V)\otimes S(w) = (2T(v))\otimes S(w) = 2(T(v)\otimes S(w)) = f(v,w)$ and we also have $f(v,2w) = 2f(v,w)$

   for example, we let $T:\mathbb{R}^2\to\mathbb{R}^2, S:\mathbb{R}^3\to\mathbb{R}^3, T(z) = \begin{pmatrix}a&b\\c&d\end{pmatrix}z$ and $S(y) = \begin{pmatrix}p&q&r\\s&t&u\\v&w&x\end{pmatrix}y$, now we want to find $T\otimes S:\mathbb{R}^2\otimes\mathbb{R}^3\to\mathbb{R}^2\otimes\mathbb{R}^3$: recall a basis of $\mathbb{R}^2\otimes\mathbb{R}^3$ is given by $C = \{e_1\otimes f_1, e_1\otimes f_2, e_1\otimes f_3, e_2\otimes f_1, e_2\otimes f_2, e_2\otimes f_3\}$. $(T\otimes S)_{CC}$ is a 6×6 matrix:

$$T\otimes S(e_1\otimes f_1) = T(e_1)\otimes S(f_1) = \begin{pmatrix}a\\c\end{pmatrix}\otimes\begin{pmatrix}p\\s\\v\end{pmatrix} = ap(e_1\otimes f_1)+as(e_1\otimes f_2)+av(e_1\otimes f_3)+cp(e_2\otimes f_1)+cs(e_2\otimes f_2)+cv(e_2\times f_3)$$

   similarly, we have

$$T\otimes S(e_1\otimes f_2) = T(e_1)\otimes S(f_2) = \begin{pmatrix}a\\c\end{pmatrix}\otimes\begin{pmatrix}q\\t\\w\end{pmatrix} = aq(e_1\otimes f_1)+at(e_1\otimes f_2)+aw(e_1\otimes f_3)+cq(e_2\otimes f_1)+ct(e_2\otimes f_2)+cw(e_2\times f_3)$$

$$T\otimes S(e_1\otimes f_3) = T(e_1)\otimes S(f_3) = \begin{pmatrix}a\\c\end{pmatrix}\otimes\begin{pmatrix}r\\u\\x\end{pmatrix} = ar(e_1\otimes f_1)+au(e_1\otimes f_2)+ax(e_1\otimes f_3)+cr(e_2\otimes f_1)+cu(e_2\otimes f_2)+cx(e_2\times f_3)$$

$$T\otimes S(e_2\otimes f_1) = T(e_2)\otimes S(f_1) = \begin{pmatrix} b \\ d \end{pmatrix}\otimes\begin{pmatrix} p \\ s \\ v \end{pmatrix} = bp(e_1\otimes f_1)+bs(e_1\otimes f_2)+bv(e_1\otimes f_3)+dp(e_2\otimes f_1)+ds(e_2\otimes f_2)+dv(e_2\times f_3)$$

$$T\otimes S(e_2\otimes f_2) = T(e_2)\otimes S(f_2) = \begin{pmatrix} b \\ d \end{pmatrix}\otimes\begin{pmatrix} q \\ t \\ w \end{pmatrix} = bq(e_1\otimes f_1)+bt(e_1\otimes f_2)+bw(e_1\otimes f_3)+dq(e_2\otimes f_1)+dt(e_2\otimes f_2)+dw(e_2\times f_3)$$

$$T\otimes S(e_2\otimes f_3) = T(e_2)\otimes S(f_3) = \begin{pmatrix} b \\ d \end{pmatrix}\otimes\begin{pmatrix} r \\ u \\ x \end{pmatrix} = br(e_1\otimes f_1)+bu(e_1\otimes f_2)+bx(e_1\otimes f_3)+dr(e_2\otimes f_1)+du(e_2\otimes f_2)+dx(e_2\times f_3)$$

and we must know that

$$(T \otimes S)_{CC} = \begin{pmatrix} ap & aq & ar & bp & bq & br \\ as & at & au & bs & bt & bu \\ av & aw & ax & bv & bw & bx \\ cp & cq & cr & dp & dq & dr \\ cs & ct & cu & ds & dt & du \\ cv & cw & cx & dv & dw & dx \end{pmatrix}$$

we can also write the matrix as

$$(T \otimes S)_{CC} = \left(\begin{array}{c|c} a\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} & b\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \\ \hline c\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} & d\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \end{array}\right)$$

which is called Kronecker product.

Chapter tutorial

29.1 tutorial3

✍ **Exercise 29.1** Find a counter example for each of the following statements:

(i) Let $V = W \oplus V_1 = W \oplus V_2$. If $v_1 + w_1 = v_2 + w_2$, where $v_1 \in V_1, v_2 \in V_2$ and $w_1, w_2 \in W$, then $v_1 = v_2$ and $w_1 = w_2$

(ii) Let $V = V_1 \oplus V_2$. Then for any $v \in V$, either $v \in V_1$ or $v \in V_2$;

(iii) Let $W \subset V$ be a subspace. If $v_1 + v_2 \in W$, then $v_1, v_2 \in W$.

✍ **Exercise 29.2** Find a linear transformation $T : V \to V$ such that:

(i) $T \neq \mathrm{id}_V$, but $T^2 = T$;

(ii) $T \neq 0_V$, but $T^2 = 0_V$;

(iii) $T^k \neq 0_V$, but $T^{k+1} = 0_V$.

Hint: Since $\mathrm{Hom}(V, W) \cong \mathrm{M}_{m\times n}$, you can use matrices to construct linear maps you want. Here $\dim V = n$ and $\dim W = m$.

✍ **Exercise 29.3** Let V be a finite dimensional vector space over $\mathbf{F}$, and $T : V \to V$ be a linear operator.

(a) Show that

$$\{0\} \leq \ker T \leq \ker T^2 \leq \cdots \leq \ker T^i \leq \ker T^{i+1} \leq \cdots$$

(b) Show that if $\ker\left(T^i\right) = \ker\left(T^{i+1}\right)$ for some i, then

$$\ker\left(T^i\right) = \ker\left(T^{i+1}\right) = \ker\left(T^{i+2}\right) = \cdots$$

Suppose $\dim V = n$, and $T^m = \mathbf{0}$ for some (unknown) positive integer m.

(c) Show that $T^n = \mathbf{0}$

✍ **Exercise 29.4** Let V_1, V_2 be finite dimensional vector spaces and let $W_1 \subseteq V_1, W_2 \subseteq V_2$ be isomorphic subspaces, via a linear transformation $S : W_1 \xrightarrow{\cong} W_2$. Prove that there exists a linear transformation $T : V_1 \to V_2$ such that $T|_{W_1} = S$

Remark:

$T|_{W_1}$ is called the restriction of T.

$$T|_{W_1} : W_1 \to V_2$$
$$T|_{W_1}(x) = T(x), \forall x \in W_1$$

For this question, we only need to prove:

$$T(x) = S(x), \forall x \in W_1.$$

29.2 tutorial4

✍ **Exercise 29.5** (a) Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by $T(p) = \begin{bmatrix} p(0) & p(1) \\ p'(0) & p'(1) \end{bmatrix}$. Find the matrix representation of T with respect to the standard bases $\beta = \{1, x, x^2\}$ and $\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ (we can write as $e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}$) of $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively.

Proof

(b) Let $S : \mathbb{R}^3 \to P_2(\mathbb{R})$ be defined by $S(a,b,c) = (a+b) + (a-2c)x + (a+b+c)x^2$. Find the matrix representation of T with respect to the standard bases $\alpha = \{(1,0,0),(0,1,0),(0,0,1)\}$ and β defined in (1) of $\mathbb{R}^3$ and $P_2(\mathbb{R})$ respectively. And verify that $[TS]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$

Proof

(c) Base on (1)(2), let us consider a more general case.Let $T : M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ be defined $T(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} X \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a given matrix in $M_{2\times 2}$. Find the matrix representation of T with respect to the basis $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$

Proof

✍ **Exercise 29.6** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, and the matrix representation of T with respect to the basis $\alpha = (2,1,2), (-1,1,0), (0,1,1)$ is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find the the matrix representation of T with respect to another basis $\beta = (1,0,0), (1,1,0), (1,1,1)$.

Proof

✍ **Exercise 29.7** Let V be a finite dimensional vector space, and W be a subspace of V.Prove that $\dim(V/W) = \dim(V) - \dim(W)$

✍ **Exercise 29.8** Let V be a vector space.Let $U, W \subseteq V$ be subspaces. Prove that $(U+W)/W \cong U/(U \cap W)$

Remark the notation $[T]_\beta^\gamma$ is corresponding to the notation $[T]_{\gamma\beta}$ and recall the formula we learned $[T]_{BA}[v]_A = [Tv]_B$

Remark review of the matrix representation: the j-th column of $[T]_\alpha^\beta$ is the coordinate of $T(\alpha_j)$ w.r.t. the basis β

Remark the matrix representation of T w.r.t. the basis α is $(T)_{\alpha\alpha}$

29.3 tutorial5

✍ **Exercise 29.9** Prove the Noether Isomorphism Theorem: If $\boldsymbol{V}$ is an $\mathbb{F}$-vector space and if $\boldsymbol{U}, \boldsymbol{W} \subseteq \boldsymbol{V}$ are subspaces, then

$$(\boldsymbol{U}+\boldsymbol{W})/\boldsymbol{W} \cong \boldsymbol{W}/(\boldsymbol{U} \cap \boldsymbol{W}).$$

✍ **Exercise 29.10** Let $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation. Assume that $\boldsymbol{V}' \subseteq \boldsymbol{V}, \boldsymbol{W}' \subseteq \boldsymbol{W}$ are subspaces and assume that $T(V') \subseteq W'$. Prove that the recipe

$$\overline{\boldsymbol{T}} : V/V' \to W/W'$$

given by $\bar{T}(v+V') = T(v) + W'$ is a well defined linear transformation.

✍ **Exercise 29.11** Let $W' \subseteq W \subseteq V$, which are all vector spaces. Prove that

$$(V/W')/(W/W') \cong V/W.$$

✍ **Exercise 29.12** Let $V \xrightarrow{T} W \xrightarrow{S} U$ be a sequence of linear transformations. We say that the sequence is exact (at W) if $T(V) = \ker(S)$.
(a) Show that the linear transformation $T : V \to W$ is injective if and only if the sequence $\{0\} \to V \xrightarrow{T} W$ is exact.
(b) Show that the linear transformation $T : V \to W$ is surjective if and only if the sequence $V \xrightarrow{T} W \to \{0\}$ is exact. (c) Use the fundamental homomorphism theorem to show that if the sequence

$$\{0\} \to V' \xrightarrow{T} V \xrightarrow{S} V'' \to \{0\}$$

is exact(at all possible places), then $V'' \cong V/T(V')$.

✍ **Exercise 29.13** Let V be a vector space over $\mathbb{F}$, and $f, g \in V^*$ such that $f(v) = 0$ if and only if $g(v) = 0$. Prove that $f = \lambda g$ for some $\lambda \in \mathbb{F}$.

29.4 tutorial6

✍ **Exercise 29.14** 1, Consider $\Phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V^*, W^*)$ defined by $\Phi(T) = T^t$. Prove: (i) Φ is an injective linear transformation (ii) If W is finite dimensional, Φ is an isomorphism Hint: $\forall$ distinct $v_1, v_2 \in V$, where V is a finite dimensional vector space, $\exists f \in V^*$ s.t. $f(v_1) \neq f(v_2)$

✍ **Exercise 29.15** 2, Suppose T: $V \to W$ is a linear transformation. Prove: (i) T is surjective if and only if T^* is injective (ii) T is injective if and only if T^* is surjective

29.5 tutorial7

✍ **Exercise 29.16** 1. Let V be the vector field over $\mathbb{F}$, the matrix representation of linear transformation T with respect to the ordered basis $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is $(T)_{\alpha,\alpha} = \begin{bmatrix} a & 1 & 0 & \cdots & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & a & 1 \\ 0 & 0 & 0 & \cdots & & a \end{bmatrix}$
(a) If α_n is in the T-invariant subspace W, prove that $W = V$.

Proof recall the definition of matrix representation of linear transformation, we know that $T\alpha_n = \alpha_{n-1} + a\alpha_n$, since α_n in W and W is $T - invariant$ subsapce we know that $T\alpha_n$ is also in W, so α_{n-1} is also in W. using the similar deduction we know that $\alpha_1, \alpha_2, , , , \alpha_n$ are in W, so W = V.

(b) Prove that α_1 is in any nonzero T-invariant subspace of V.

Proof let w be any nonzero T-invariant subspace, we show that α_1 is in W. $\forall\beta$ in W, we have $\beta = k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s$, W is T-invariant, we have $T\beta \in W, T\beta = T(k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s) = k_1T(\alpha_1) + k_2T(\alpha_2) + \dots + k_sT(\alpha_s) \in W \to k_1T(_1) + k_2T(\alpha_1 + a\alpha_2) + \dots + k_sT(\alpha_{s-1} + a\alpha_s) = a\beta + (k_2\alpha_1 + \dots + k_s\alpha_{s-1}) \in W$, we have $(k_2\alpha_1 + \dots + k_s\alpha_{s-1}) \in W$.
$T(k_2\alpha_1 + \dots + k_s\alpha_{s-1}) = k_2T(\alpha_1) + k_3T(\alpha_2) + \dots + k_sT(\alpha_{s-1}) = k_2(a\alpha_1) + k_3(\alpha_1 + a\alpha_2) + \dots + k_s(\alpha_{s-2} + a\alpha_{s-1}) = a(k_2\alpha_1 + k_3\alpha_2 + \dots + k_s\alpha_{s-1}) + (k_3\alpha_1 + \dots k_s\alpha_{s-2}) \in W$
we have $(k_3\alpha_1 + \dots k_s\alpha_{s-2}) \in W$ keep the process we finally have:$k_s\alpha_1 \in W$ and we are done.

(c) Prove that V can not be decomposed into two direct sum of non-trivial T-invariant subspaces.

Proof using the conclusion of (b), we know that any nontrivial T-invariant subspaces M and N, we have $\alpha_1 inM, \alpha_1 \in N$ and $\alpha_1 \in M \bigcap N$, so $V \neq M \oplus N$

(d) Find all the T-invariant subspaces.

Proof

✍ **Exercise 29.17** 2. Consider the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ real matrices. Consider the linear map $T : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$ given by $T(A) = A^T$ for all $A \in M_{n\times n}(\mathbb{R})$. Here A^T denotes the transpose of A.
(a) Find the minimal polynomial of T.

Proof we know that $T(A) = A^T$, recall the property of transpose, we have $T(T(A)) = A$, so $T^2 - I = 0$, so the minimal polynomial must divide $x^2 - 1 = (x + 1)(x - 1)$. since $T - I \neq 0$ and $T + I \neq 0$, we have $m_T(x) = x^2 - 1$

(b) Find the characteristic polynomial of T.

Proof

✍ **Exercise 29.18** 3. Let $T : V \longrightarrow V$ be a linear map on a finite dimensional vector space, $v \in V$ be a vector. $M_{T,v}$ is the minimal polynomial for a vector v. Prove that
(a) $m_T(x) = m_{T,v}(x)m_{T|W}(x)$, where $W = \operatorname{Im}\left(m_{T,v}(T)\right)$;

Proof let $p(x) := m_{T,v}(x)$, $q(x) := \frac{m_T(x)}{p(x)}$, then $m_T(x) = p(x)q(x)$ we show that $q(x) = m_{T|W}(x)$. the idea is to show that $1.m_{T|W}(x)|q(x)$ and $2.q(x)|m_{T|W}(x)$:
first we need to show that W is T-invariant(why???).
then we need to check that $q(x) := \frac{m_T(x)}{p(x)}$ is indeed a polynomial, i.e. $m_{T,v}(x)|m_T(x)$, which is true since $m_T(T) = 0 \in Hom(V, V)$, $[m_T(T)](v) = 0_V, \forall v\ in\ V \rightarrow m_{T,v}(x)|m_T(x)$, so $q(x)$ is a polynomial.
1. $m_{T|W}(x)|q(x)$: it is equivalent to show that $q(T|_W) = 0$. for any $w \in W = Im(m_{T,v}(T))$, $\exists v \in V$ s.t. w = p(T)(v), then $q(T|_W)(w) = q(T)(w) = q(T)p(T)(v) = m_T(T)(v) = 0 \rightarrow q(T|_W) = 0$
2. $q(x)|m_{T|W}(x)$: let $g(x) := m_{T|W}(x)$ and we have $g(T|_W) = 0$, $\forall w \in W$, recall $W = Im(m_{T,V})$, $\exists v \in V$ s.t. $m_{T,v}(T)(v) = w \in W$, $g(T|_W)(w) = 0$, for any $v \in V$, we have $g(T)p(T)(v) = 0$ why? $\rightarrow g(T)p(T) = 0 \rightarrow p(x)q(x) = m_T(x)|g(x)p(x) \rightarrow q(x)|g(x)$

(b) the degree of $m_{T,v}(T)$ equals the least integer k such that $\left\{v = T^0v, T^1v, T^2v, \ldots, T^kv\right\}$ is linearly dependent.

29.6 tutorial8

29.7 tutorial9

✍ **Exercise 29.19** 1, Let T : $V \rightarrow V$ be a linear map on a finite dimensional vector space. Prove: T is invertible if and only if 0 is not a root of $m_T(x)$

✍ **Exercise 29.20** 2, Let T : $V \rightarrow V$ be a linear map on a finite dimensional vector space. Suppose there are two T-invariant subspaces W_1 and W_2 such that $V = \mathrm{W}_1 \oplus W_2$. Prove:

$$m_T(x) = \operatorname{lcm}\left(m_{T|\mathrm{W}_1}(x), m_{T|\mathrm{W}_2}(x)\right),$$

where 1 cm denotes the least common multiple

Definition 29.1 (minimal polynomial of a vector relative to T)

Suppose V is a vector space. If $0 \neq v \in V$, let $m_{T,v}(x)$ be the monic polynomial of least degree such that

$$m_{T,v}(T)(v) = 0$$

♣

✍ **Exercise 29.21** 3, Extended readings from chapter 2.2 in the textbook (not required in the exam)
$m_{T,v}(x)$ has the following properties:
Property 1: $m_{T,v}(x) \mid m_T(x)$
Property 2: Let $v_1, v_2 \in V$. Suppose $m_{T,v_1}(x) = f_1(x); m_{T,v_2}(x) = f_2(x)$ If $f_1(x)$ and $f_2(x)$ are relatively prime. Then

$$m_{T,v_1+v_2}(x) = f_1(x)f_2(x)$$

You can learn more about the minimal polynomial of a vector relative to T from the textbook if you are interested in the content.

29.8 tutorial10

✍ **Exercise 29.22** 1. Write down the Jordan form for each of the following matrices: (i) $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

✍ **Exercise 29.23** 2. Let $T : V \to V$ be a linear map on a finite dimensional vector space. Suppose $m_T(x) = (x-\lambda_1)^{e_1}(x-\lambda_2)^{e_2}\ldots(x-\lambda_k)$ and $\chi_T(x) = (x-\lambda_1)^{f_1}(x-\lambda_2)^{f_2}\ldots(x-\lambda_k)^{f_k}$, where λ_i are distinct. For each $1 \le i \le k$, prove that (i) $\dim(\ker(T-\lambda_i I)^{e_i}) = f_i$; (ii) e_i = the order of the largest Jordan block corresponding to λ_i : (iii) $\dim(\ker(T-\lambda_i I))$ = the number of Jordan blocks corresponding to λ_i : (iv) let $d_l := \dim\left(\ker(T-\lambda_i I)^l\right), (0 \le l \le e_i)$, then the number of $l \times l$ Jordan blocks corresponding to $\lambda_i = 2d_l - d_{l+1} - d_{l-1}$.

✍ **Exercise 29.24** 3. Up to similarity, write down all possible Jordan normal forms for a matrix with characteristic polynomial x^5.

29.9 tutorial11

✍ **Exercise 29.25** 1. Let $\mathbb{P}_2[0,2]$ represent the set of polynomials with real coefficients and of degree less than or equal to 2 , defined on $[0,2]$. For $p = (p(t)) \in \mathbb{P}_2$ and $q = (q(t)) \in \mathbb{P}_2$, define

$$\langle p, q\rangle := p(0)q(0) + p(1)q(1) + p(2)q(2).$$

(a) Verify that $\langle p, q\rangle$ is an inner product.

Proof by definition, checking $\langle p, q\rangle$ is symmetric, bilinear and positive definite.

(b) Let T represent the linear transformation that maps an element $p \in \mathbb{P}_2$ to the closest element of the span of the polynomials 1 and t in the sense of the norm associated with the inner product. Find the matrix A of T in the standard basis of $\mathbb{P}_2$. (Note: the standard basis of $\mathbb{P}_2$ is $\{1, t, t^2\}$.)

Proof T is a orthogonal projection onto the subspace spanned by 1 and t. and we need to find the matrix A for T in the standard basis. note that T maps an element p to the closest element of the span of the polynomials 1 and t in the sense of the norm associated with the inner product. so we have $T(1) = 1$ and $T(t) = t$. we wonder what is $T(t^2)$? i.e. the orthogonal projection of t^2 onto the subspace spanned by 1 and t.
first we need to find an orthogonal basis using gram-schmidt process. we get $v_1(t) = 1$ and $v_2(t) = t - \frac{\langle t,1\rangle}{\langle 1,1\rangle}1 = t-1$. $\{v_1, v_2\}$ is an orthogonal basis.
we find the orthogonal projection of t^2 onto the subspace.

$$T(t^2) = \frac{\langle t^2.1\rangle}{\langle 1,1\rangle}1 + \frac{\langle t^2, t-1\rangle}{\langle t-1, t-1\rangle}(t-1) = \frac{5}{3} + 2(t-1) = -\frac{1}{3} + 2t$$

so we map the standard basis $\{1, t, t^2\}$ to $\{1, t-1, -\frac{1}{3}+2t\}$. we are easy to get A:

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Is A symmetric? Is T self-adjoint? Do these facts contradict each other?

Proof from (b) we find A is not symmetric.
to check T is self-adjoint: we observe that $p, q \in \mathbb{P}_2[0,2], Tp, Tq \in span(1,t)$ and $p-Tp, q-Tq$ satisfy that $\langle p, q-Tq\rangle = 0$ and $\langle q, p-Tp\rangle$. so we have:

$$\langle q, Tp\rangle = \langle q, p\rangle = \langle p, q\rangle = \langle p, Tq\rangle$$

this shows that T is indeed a self-adjoint transformation.
the two facts do not contradict each other since the standard basis $\mathcal{B} = \{1, t, t^2\}$ is not orthonormal and we do not have the result that $A^t = A, \quad T_{\mathcal{B}\mathcal{B}} = A$

(d) Find the minimal polynomial of T.

Proof we know that T is a projection and thus $T^2 - T = 0$, note $T \neq I$ and $T \neq 0$, so the minimal polynomial is not x nor $x-1$. so the minimal polynomial is $\mu_T(x) = x^2 - x$

✍ **Exercise 29.26** 2. A real $n \times n$ matrix A is an isometry if it preserves length: $\|Ax\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Show that the following are equivalent. (a) A is an isometry (preserves length).
(b) $\langle Ax, Ay\rangle = \langle x, y\rangle$, for all vectors x, y, so A preserves inner products.
(c) $A^{-1} = A^*$
(d) The columns of A are unit vectors that are mutually orthogonal.

Proof $(b) \to (a)$: recall we define the norm $||x|| := \sqrt{\langle x, x\rangle}$ and it is trivial
$(a) \to (b)$: assume A preserves lengths, let $x, y \in \mathbb{R}^n$, we have $||A(x+y)||^2 = ||x+y||^2$consider $||A(x+y)||^2 - ||(x+y)||^2 = 0$: we have: $LHS = \langle A(x+y), A(x+y)\rangle - \langle x+y, x+y\rangle = \langle A_x, A_x\rangle + \langle Ax, Ay\rangle + \langle Ay, Ax\rangle + \langle Ay, Ay\rangle - \langle x, x\rangle - \langle x, y\rangle = \langle y, x\rangle - \langle y, y\rangle$. we have $\langle Ax, Ay\rangle = \langle Ay, Ax\rangle$ and $\langle Ay, Ay\rangle = ||Ay||^2 = ||y||^2 = \langle y, y\rangle$. we obtain that $||A(x+y)||^2 - ||(x+y)||^2 = 2\langle Ax, Ay\rangle - 2\langle x, y\rangle$. since $RHS = 0$, we have $\langle Ax, Ay\rangle = \langle x, y\rangle$. we are done.
$(c) \to (b)$: $\langle Ax, Ay\rangle = \langle A^*Ax, y\rangle = \langle x, y\rangle$ we are done. notice the first equality is given by the definition of A^*
how to prove $(d) \to (c)$?

✍ **Exercise 29.27** 3. (a) Prove that a normal operator on a finite dimensional complex inner product space with real eigenvalues is self-adjoint.
(b) Let V be a finite dimensional real inner product space and let $T : V \longrightarrow V$ be a self-adjoint operator. Is it true that T must have a cube root? Explain. (A cube root of T is an operator $S : V \longrightarrow V$ such that $S^3 = T$.)

✍ **Exercise 29.28** 4. In this problem, R is the field of real numbers. Let $(u_1, u_2, \ldots, u_m)$ be an orthonormal basis for subspace $W \neq \{0\}$ of the vector space $V = \mathbb{R}^{n\times 1}$ (under the standard inner product), let U be the $n \times m$ matrix defined by $U = [u_1, u_2, \ldots, u_m]$, and let P be the $n \times n$ matrix defined by $P = UU^T$.
(a) Prove that if v is any given member of V, then among all the vectors w in W, the one which minimizes $\|v - w\|$ is given by $w = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \ldots + \langle v, u_m\rangle u_m$. (The vector w is called the projection of v onto W.)

Proof $\forall x \in W$, we know that

$$\langle v - w, x\rangle = 0$$

since we have:

$$\langle v - w, x\rangle = \langle (v - \langle v, u_1\rangle u_1 - \ldots - \langle v, u_m\rangle u_m), x\rangle = \langle v, x\rangle - \langle v, u_1\rangle\langle u_1, x\rangle - \ldots - \langle v, u_m\rangle\langle u_m, x\rangle = 0$$

the last equality holds since:

$$\forall x \in W, \quad x = \langle x, u_1\rangle u_1 + \ldots + \langle x, u_m\rangle u_m$$

and thus

$$\langle v, u_1\rangle\langle u_1, x\rangle + \ldots + \langle v, u_m\rangle\langle u_m, x\rangle = \langle v, x\rangle$$

now we have proved that $\langle v-w, x\rangle = 0$, consider

$$||v-x||^2 = ||(v-w)+(w-x)||^2 = \langle (v-w)+(w-x), (v-w)+(w-x)\rangle$$

$$= \langle (v-w), (v-w)\rangle + \langle (w-x), (w-x)\rangle + 2\langle v-w, w-x\rangle = ||v-w||^2 + ||w-x||^2 + 2\langle (v-w), (w-x)\rangle$$

and in above we have $\langle v-w, x\rangle = 0$, so we have

$$\langle v-w, w-x\rangle = 0$$

since $w - x \in W$. so we have

$$||v-x||^2 = ||v-w||^2 + ||w-x||^2$$

and the minimum for $||v-x||^2$ is $||v-w||^2$ and is realized when $x = w$.

(b) Prove: For any vector $v \in \mathbb{R}^{n\times 1}$, the projection w of x onto W is given by $w = Px$.

Proof we already know the relation between w and v is:

$$w = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \ldots + \langle v, u_m\rangle u_m$$

recall in V = $\mathbb{R}^n$, the inner product is defined by $\langle u, v\rangle = u^t v$ and thus we have

$$w = (u_1^t v)u_1 + \ldots + (u_m^t v)u_m = (u_1^t u_1 + u_2^t u_2 + u_m^t u_m)v = UU^t v$$

where $(u_i^t u_i)$ is a real number so u_i and v can commute.

(c) Prove: P is a projection matrix. (Recall that a matrix $P \in \mathbb{R}^{n\times n}$ is called a projection matrix if and only if P is symmetric and idempotent.

Proof we want to prove that P is symmetric and idempotent($p^2 = 0$): $P := UU^t = (UU^t)^t = P^t$ and $P^2 = (UU^t)(UU^t) = U(U^tU)U^t = UIU^t = UU^t = I$, where $UU^t = I$ is due to the condition that $(u_1, \ldots, u_m)$ is an orthonormal basis.

(d) If $V = \mathbb{R}^{3\times 1}$, and $W = \text{span}\left[(1,2,2)^T, (1,0,1)^T\right]$, find the projection matrix P described above and use it to find the projection of $(2,2,2)^T$ onto W.

Proof

29.10 tutorial12

Exercise 29.29 1. Let $V = \mathbb{P}(\mathbb{R})$, and let U be a subspace of V given by span $(1, x, x^2)$.
(a) Pick a basis for U, and find the corresponding dual basis.

Proof we choose the standard basis $\{1, x, x^2\}$. recall the dual basis ϕ_i in the dual space of U are functionals on U s.t. $\phi_i(u_j) = \delta_{ij}$. we let $\phi_1(p(x)) = p(0)$ and we have $\phi_1(1) = 1, \quad \phi_1(x) = 0, \quad \phi_1(x^2) = 0$, let $\phi_2(p(x)) = p'(0)$, we have $\phi_2(1) = 0, \quad \phi_2(x) = 1, \quad {}_2(x^2) = 0$, let $\phi_3(p(x)) = \frac{1}{2}p''(0)$, we have $\phi_3(1) = 0, \quad \phi_3(x) = 0, \quad \phi_3(x^2) = 1$. the basis ϕ_1, ϕ_2, ϕ_3 is indeed a dual basis.

(b) Given the inner product on $\langle p_1, p_2\rangle = \int_0^1 p_1(x)p_2(x)dx$, find the Riesz representers (in U) of the dual basis in part a). Recall that the Riesz representer is the unique vector u in V s.t., given a fixed ϕ in V', $\phi(v) = \langle v, u\rangle$.

Proof

Exercise 29.30

2. In this problem, you are asked to prove that two real symmetric matrices commute if and only if they are diagonalizable in a common orthonormal basis. We suggest the following path. Let A and B be two real symmetric matrices and show each of the following. [5 points each]
(a) If A and B are diagonalizable in a common orthonormal basis, then A and B commute.

Proof we have $A = VD_AV^T, B = VD_BV^T$ and V is a orthogonal matrix s.t. $VV^T = I$. we have $AB = VD_AV^T \cdot VD_BV^T = VD_A(VV^T)D_BV^T = VD_AD_BV^T = VD_BD_AV^T = VD_BV^TVD_AV^T = BA$

(b) If A and B commute, and if λ is an eigenvalue of A, then the eigenspace E_λ of A that is associated with the eigenvalue λ is invariant under B.

Proof let λ be an eigenvalue of A and E_λ is the eigenspace of A. let $x \in E_\lambda$, then $Ax = \lambda x$, we have $BAx = \lambda Bx = ABx$, we see that $Bx \in E_\lambda$ and thus E_λ is invariant under B.

(c) If A and B commute, then A and B have at least one common eigenvector.

Proof we already proved that E_λ is invariant under B. notice that B is real symmetric, so the restriction of B to E_λ is a real symmetric linear operator from E_λ to E_λ so it has a real eigenvalue, μ, with eigenvector v. (v, μ) is an eigencouple of the restriction of B to E_λ, so we have (v, μ) is an eigencouple of B with $v \in E_\lambda$, recall E_λ is the eigenspace of A associated with the eigenvalue λ. we have v being an eigenvector of A(why?)

(d) If A and B commute, then A and B are diagonalizable in a common orthonormal basis.

Exercise 29.31 3. Let $\mathcal{P}_n$ represent the real vector space of polynomials in x of degree less than or equal to n defined on $[0, 1]$. Given a real number a, we define $Q_n(a)$ the subset of $\mathcal{P}_n$ of polynomials that have the real number a as a root.
(a) Let a be a real number. Show that $Q_n(a)$ is a subspace of $\mathcal{P}_n$. Determine the dimension of that subspace and exhibit a basis.

Proof polynomials in $Q(x)$ can be written as $p(x) = (x - a)q(x)$ where $q(x)$ is a polynomial degree less than or equal to $n - 1$. a basis of $Q(n)$ is $\{(x - a), (x - a)^2, ..., (x - a)^n\}$, its dimension is n.

(b) Let the inner product in $\mathcal{P}_n$ be defined by $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Determine the orthogonal complement of the subspace $Q_2(1)$ of $\mathcal{P}_2$.

Proof for a polynomial in P_2,$a_0 + a_1(x - 1) + a_2(x - 1)^2$. we need a polynomial orthogonal to $(x - 1)$ and $(x - 1)^2$, so

$$\int_0^1 (a_0 + a_1(x - 1) + a_2(x - 1)^2)(x - 1)dx = 0$$
$$\int_0^1 (a_0 + a_1(x - 1) + a_2(x - 1)^2)(x - 1)^2dx = 0$$

and get

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = a_2 \begin{pmatrix} 3/10 \\ 6/5 \\ 1 \end{pmatrix}$$

and the orthogonal complement of the subspace $Q_2(1)$ is $\{3a_2 + 12a_2(x - 1) + 10a_2(x - 1)^2, \quad a_2 \in \mathbb{R}\}$

Exercise 29.32 4. (a) Prove that a normal operator on a finite dimensional complex inner product space Let V be an inner product space over $\mathbb{C}$, with inner product $\langle u, v \rangle$.
(a) Prove that any finite set S of nonzero, pairwise orthogonal vectors is linearly independent.

(b) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that all eigenvalues of T are real.

(c) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that the eigenvectors of T associated with distinct eigenvalues λ and μ are orthogonal.

✍ **Exercise 29.33** 5. Let V be a finite-dimensional real vector space. (a) Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that Range T^m = Range T^{m+1}. Prove that Range T^k = Range T^m for all $k > m$.

(b) Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then V = Null $T \oplus$ Range T.

(c) Prove that if $T \in \mathcal{L}(V)$, then V = Null $T^n \oplus$ Range T^n, where $n = \dim V$.

Chapter homework

30.1 homework3

Exercise 30.1

Suppose $T : V \to W$ is an injective linear transformation, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, show that $\{T\mathbf{v}_1, \ldots, T\mathbf{v}_k\}$ is linearly independent in W.

Exercise 30.2 Suppose $T : V \to W$ is an surjective linear transformation, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans V, show that $\{T\mathbf{v}_1, \ldots, T\mathbf{v}_k\}$ spans W.

Exercise 30.3 Show that if $\dim(V) < \infty$ and $T : V \to W$ is an isomorphism, then for all bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V, $\{T\mathbf{v}_1, \ldots, T\mathbf{v}_n\}$ is a basis of W.

Exercise 30.4 If $\dim(V) = \dim(W) < \infty$, show that there is an isomorphism $T : V \to W$.

Exercise 30.5 Show that there is no linear transformation $T : \mathbf{F}^5 \to \mathbf{F}^2$ such that its nullspace is equal to

$$\mathrm{Nul}(T) = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$$

Exercise 30.6 (Vector space structure of $\mathrm{Hom}_F(V, W)$)

Let V, W be two vector spaces over $\mathbf{F}$, and

$$\mathrm{Hom}_{\mathbf{F}}(V, W) = \{ \text{ all linear transformations } T : V \to W\}$$

(a) Show that if $T, S \in \mathrm{Hom}_{\mathbf{F}}(V, W)$, and $\alpha \in \mathbf{F}$ then

$$\alpha T, (T + S) \in \mathrm{Hom}_{\mathbf{F}}(V, W).$$

(b) What is the zero vector $\mathbf{0} \in \mathrm{Hom}_{\mathbf{F}}(V, W)$?

30.2 homework4

Exercise 30.7 Let V be a finite dimensional vector space with bases A and B. Consider the identity transformation $Id : V \to V$ given by

$$Id(\mathbf{v}) = \mathbf{v} \quad \forall \mathbf{v} \in V$$

Show that $(Id)_{BA} = C_{BA}$, where C_{BA} is the change of basis matrix.

Exercise 30.8 Let $V = M_{2\times 2}(F)$, and define a matrix $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$. Let $T : V \to V$ be given by $T(B) = AB$.

(a) Show that T is a linear transformation.

(b) Let $\mathscr{A} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ be an ordered basis of V, where e_{ij} is the 2×2 matrix with (i,j)-entry equal to 1 and other entries equal to 0 . Compute the matrix representation (T) $\mathscr{A}$. Is T an isomorphism?

✍ **Exercise 30.9** Let T be a linear operator on $\mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(a) What is the representation matrix (T) $\mathscr{A}\mathscr{A}$ for the ordered basis $\mathscr{A} = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$?

(b) Prove that for every real number c the operator $T - cI$ (I is the identity transformation) is invertible.

(c) Prove that if $\mathscr{B}$ is any ordered basis, with $M := (T)_{\mathscr{B}\mathscr{B}}$, then $M_{12}M_{21} \neq 0$.

✍ **Exercise 30.10** Let V, W are vector spaces, and $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, B = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are bases of V and W respectively. We have the representation map:

$$(\cdot)_{BA} : \mathrm{Hom}_F(V, W) \to M_{m\times n}(F)$$
$$T \mapsto (T)_{BA}$$

(a) Show that $(\cdot)_{BA}$ is a linear transformation, i.e.

$$(xT + yS)_{BA} = x(T)_{BA} + y(S)_{BA}$$

(b) Using the definition of $(\cdot)_{BA}$, show that $(\cdot)_{BA}$ is surjective. More precisely, for all $m \times n$ matrix $M = (M_{ij})$, show that there is a linear transformation T such that $(T)_{BA} = M$.

(c) Using the representation theorem, show that $(\cdot)_{BA}$ is injective. More precisely, show that if $(T)_{BA}$ is the zero matrix, then $T = \mathbf{0} \in \mathrm{Hom}_F(V, W)$.

(d) What is dim $(\mathrm{Hom}_F(V, W))$?

30.3 homework5

✍ **Exercise 30.11** let $T : V \to W$ be a linear transformation, with $V' \leqslant V$, $W' \leqslant W$. assume that $T(V') \leqslant W'$. prove that the map

$$\tilde{T} : V/V' \to W/W'$$

given by $\tilde{T}(v + V') := T(v) + W'$ is well-defined, and is a linear transformation

Proof We use the theorems proved in lecture: Consider the composition of linear transformations

$$S := \pi \circ T : V \xrightarrow{T} W \xrightarrow{\pi} W/W'$$

which is also a linear transformation. Now for all $\mathbf{v}' \in V'$,

$$S(\mathbf{v}') = \pi(T(\mathbf{v}')) = T(\mathbf{v}') + W' = \mathbf{0} + W'$$

the last line holds since $T(\mathbf{v}') \in T(V') \leq W' \Rightarrow T(\mathbf{v}') \in W'$. Therefore

$$V' \leq \ker(S)$$

and by Universal Mapping Theorem I???, there is a linear transformation

$$\tilde{S} : V/V' \to W/W'$$

satisfying $\tilde{S}(\mathbf{v} + V') = S(\mathbf{v}) = \pi(T(\mathbf{v})) = T(\mathbf{v}) + W'$.

✍ **Exercise 30.12** let $W \leqslant V$ be a vector subspace. suppose $A = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ is an ordered basis of V, with $B = \{v_1, ..., v_k\}$ and ordered basis of W. show that $C = \{v_{K+1} + W, ..., v_n + W\}$ is a basis of $V \setminus W$

Proof Spanning set : for all $v + W \in V/W$, by writing $v = a_1v_1 + \cdots + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n$, $a_1v_1 + \cdots + a_kv_k \in W$

$$\begin{aligned} v + W &= (a_1v_1 + \cdots + a_kv_k) + a_{k+1}v_{k+1} + \cdots + a_nv_n + W \\ &= \mathbf{0} + a_{k+1}v_{k+1} + \cdots + a_nv_n + W = a_{k+1}(v_{k+1} + W) + \cdots + a_n(v_n + W) \end{aligned}$$

Linear independence: Suppose $b_{k+1}(v_{k+1} + W) + \cdots + b_n(v_n + W) = \mathbf{0}_{V/W} \Leftrightarrow \quad b_{k+1}v_{k+1} + \cdots + b_nv_n + W = \mathbf{0}_V + W$ $\Leftrightarrow \quad b_{k+1}v_{k+1} + \cdots + b_nv_n \in W \Leftrightarrow \quad b_{k+1}v_{k+1} + \cdots + b_nv_n \in \text{Span}(v_1, \ldots, v_k) \Leftrightarrow \quad b_{k+1}v_{k+1} + \cdots + b_nv_n = c_1v_1 + \cdots + c_kv_k$ Therefore, $b_{k+1} = \cdots = b_n = c_1 = \cdots = c_k = 0$ as follows.

✍ **Exercise 30.13**

(IMPORTANT EXERCISE TO BE USED LATER) Let $T : V \to V$ be a linear operator, and $W \le V$ be a T-invariant subspace, i.e. $T(W) \le W$. Let $A = \{\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{v_{k+1}}, \ldots, \mathbf{v_n}\}$ be an ordered basis of V, with $B = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ an ordered basis of W as in Question 2. Assume that

$$T_{AA} = \begin{pmatrix} P & Q \\ O & R \end{pmatrix}$$

where $P \in M_{k\times k}(\mathbf{R}), Q \in M_{k\times(n-k)}(\mathbf{R}), O \in M_{(n-k)\times k}(\mathbf{R}), R \in M_{(n-k)\times(n-k)}(\mathbf{R})$. (a) Show that the entries of $O \in M_{(n-k)\times k}(\mathbf{R})$ are all zeros.
(b) Find the matrix representation $(T|_W)_{BB}$ of $T|_W : W \to W$.
(c) Using the basis C of V/W given in Question 3, compute the matrix representation $(\tilde{T})_{CC}$ for the linear transformation $\tilde{T} : V/W \to V/W$ defined in Question 2.

Proof 3. (a) The first k columns of $\begin{pmatrix} P & Q \\ O & R \end{pmatrix}$ says

$$T(\mathbf{v}_j) = \sum_{i=1}^{k} p_{ij}\mathbf{v}_i + \sum_{i=k+1}^{n} o_{ij}\mathbf{v}_i, \quad 1 \le j \le k$$

Since $T(\mathbf{v}_j) \in W = \text{Span}\{\mathbf{v}_1 \ldots, \mathbf{v}_k\}$, all the o_{ij} in the above equations are zeros. Hence the bottom left hand matrix O is the zero matrix. (b) By the above equation,

$$T(\mathbf{v}_j) = \sum_{i=1}^{k} p_{ij}\mathbf{v}_i, \quad 1 \le j \le k.$$

where $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an ordered basis of W. Hence $(T)_{BB} = P$. (c) Note that the last (n − k) columns of $\begin{pmatrix} P & Q \\ O & R \end{pmatrix}$ says

$$T(\mathbf{v}_j) = \sum_{i=1}^{k} q_{ij}\mathbf{v}_i + \sum_{i=k+1}^{n} r_{ij}\mathbf{v}_i, \quad k+1 \le j \le n$$

Let $C = \{\mathbf{v}_{k+1} + W, \ldots, \mathbf{v}_n + W\}$ be the basis of V/W given in Question 3 , and $T : V/W \to V/W$ is given by

$$T(\mathbf{v} + W) = T(\mathbf{v}) + W.$$

Then

$$T(\mathbf{v_j} + W) = T(\mathbf{v}_j) + W = (\overbrace{\sum_{i=1}^{k} q_{ij}\mathbf{v}_i}^{\in W} + \sum_{i=k+1}^{n} r_{ij}\mathbf{v}_i) + W = \sum_{i=k+1}^{n} r_{ij}\mathbf{v}_i + W = \sum_{i=k+1}^{n} r_{ij}(\mathbf{v}_i + W)$$

So the $(T)_{CC} = (r_{ij}) = R$.

✍ **Exercise 30.14** (a) For each $\mathbf{v} \in V$, let $\Phi_\mathbf{v} : V^* \to \mathbf{F}$ check that the map given by

$$\Phi_\mathbf{v}(\phi) := \phi(\mathbf{v})$$

(here $\phi \in V^*$) is a linear transformation, i.e. $\Phi_v \in (V^*)^*$. (b) Let $\mathbf{v} \ne 0$. Show that there exists $\phi \in V^*$ such that $\Phi_\mathbf{v}(\phi) \ne 0$.

(Hint: Choose a basis of V with $\mathbf{v} \in$, and consider $\phi \in \mathscr{B}^*$) (c) Let V be a finite dimensional vector space. Show that the map

$$f : V \to (V^*)^*$$

defined by

$$f(\mathbf{v}) = \Phi_{\mathbf{v}}$$

Using (b), show that $f : V \to (V^*)^*$ is an isomorphism (Side note: The isomorphism f is natural, i.e. the definition of f does not involve a choice of bases of V or $(V^*)^*$.

30.4 homework6

Exercise 30.15 1) Let $V = C[0, 1]$ and $f \in V$. (a) Let $\Phi_f : V \to \mathbf{R}$ be defined by

$$\Phi_f(g) := \int_0^1 f(x)g(x)dx$$

Show that $\Phi_f \in V^*$. (b) Consider the map $\Phi : V \to V^*$ defined by

$$\Phi(f) := \Phi_f,$$

where $\Phi_f \in V^*$ be as given in (a). Check that Φ is an injective linear transformation.

Exercise 30.16 2) Let $W, U \leq V$ be vector subspaces of a finite dimensional vector space V. Show that

$$\operatorname{Ann}(W \cap U) = \operatorname{Ann}(W) + \operatorname{Ann}(U)$$
$$\operatorname{Ann}(W + U) = \operatorname{Ann}(W) \cap \operatorname{Ann}(U)$$

Proof We prove part (b) first. (b) For any $f \in \operatorname{Ann}(W_1 + W_2)$, we have

$$f(w_1) = 0,$$
$$f(w_2) = 0,$$

for any $w_1 \in W_1, w_2 \in W_2$, since W_1 and W_2 are subspaces of W. Thus $f \in \operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2)$, and $\operatorname{Ann}(W_1 + W_2) \subseteq \operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2)$. Conversely for any $f \in \operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2)$, we have

$$f(w_1 + w_2) = f(w_1) + f(w_2) = 0,$$

for any $w_1 + w_2 \in W_1 + W_2$. Thus $f \in \operatorname{Ann}(W_1 + W_2)$, and $\operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2) \subseteq \operatorname{Ann}(W_1 + W_2)$. Therefore $\operatorname{Ann}(W_1 + W_2) = \operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2)$. (a) For any $f_1 + f_2 \in \operatorname{Ann}(W_1) + \operatorname{Ann}(W_2)$ and $w \in W_1 \cap W_2$, we have

$$(f_1 + f_2)(w) = f_1(w) + f_2(w) = 0,$$

which implies $f \in \operatorname{Ann}(W_1 \cap W_2)$, hence $\operatorname{Ann}(W_1)$+$\operatorname{Ann}(W_2) \subseteq \operatorname{Ann}(W_1 \cap W_2)$. It suffices to show $\dim(\operatorname{Ann}(W_1) + \operatorname{Ann}(W_2)) = \dim(\operatorname{Ann}(W_1 \cap W_2))$, which is verified by

$$\begin{aligned}
&\dim(\operatorname{Ann}(W_1) + \operatorname{Ann}(W_2)) \\
&= \dim(\operatorname{Ann}(W_1)) + \dim(\operatorname{Ann}(W_2)) - \dim(\operatorname{Ann}(W_1 + W_2)) \\
&= \dim(V) - \dim(W_1) - \dim(W_2) + \dim(W_1 + W_2) \\
&= \dim(V) - \dim(W_1 \cap W_2) \\
&= \dim(\operatorname{Ann}(W_1 \cap W_2)),
\end{aligned}$$
$$= \dim(\operatorname{Ann}(W_1)) + \dim(\operatorname{Ann}(W_2)) - \dim(\operatorname{Ann}(W_1) \cap \operatorname{Ann}(W_2))$$

where (1), (4) are implied by that $\dim(U+W) = \dim(U)+\dim(W)-\dim(U \cap W)$; (3), (5) are implied by $\dim(\operatorname{Ann}(W)) = \dim(V) - \dim(W)$.

✍ **Exercise 30.17** 3) Let $V = \mathbf{F}^n$ be a vector space, and $W \leq V$ be the set of all vectors

$$W := \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in V \mid x_1 + \ldots + x_n = 0 \right\}$$

(a) Prove that $\operatorname{Ann}(W)$ consists of all linear functionals $g \in V^*$ of the form

$$g\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) := k \sum_{j=1}^{n} x_i$$

for some $k \in \mathbf{F}$. (b) Show that there is an isomorphism of vector spaces

$$\Pi : V^*/\operatorname{Ann}(W) \to W^*$$

(Hint: consider the inclusion linear transformation $i : W \to V, \quad i(\mathbf{w}) := \mathbf{w}, \forall \mathbf{w} \in W$).

Proof 3 (a) For all $\phi_1, \phi_2 \in V^*$,

$$\Phi_v\left(a\phi_1 + b\phi_2\right) = \left(a\phi_1 + b\phi_2\right)(v) = a\phi_1(v) + b\phi_2(v) = a\Phi_v\left(\phi_1\right) + b\Phi_v\left(\phi_2\right)$$

Hence $\Phi_v \in (V^*)^*$.

✍ **Exercise 30.18** 4) Let $V = \mathbf{R}[x]$. Let $a, b \in \mathbf{R}$ be fixed, and $\varphi \in V^*$ be defined by $\varphi(p) = \int_a^b p(t)dt$. Consider the differentiation operator D: $V \to V$ given by $D(p) = p'$ on V with the transpose linear operator $D^t : V^* \to V^*$. What is $D^t(\varphi)$?

30.5 homework10

✍ **Exercise 30.19** 1. Let V be a finite dimensional inner product space. Recall the definition of orthogonal complement in 2040 (which is also applicable in 3040):

$$U^{\perp} := \{\mathbf{w} \in V \mid \langle \mathbf{w}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}$$

Show that the following holds: (i) $U^{\perp} \leq V$.
(ii) $U^{\perp} + W^{\perp} \subseteq (U \cap W)^{\perp}$.
(iii) $U \cap U^{\perp} = \{\mathbf{0}\}$.
(iv) $U \subseteq (U^{\perp})^{\perp}$

✍ **Exercise 30.20** 2. Let V be a finite dimensional inner product space, and $W \leq V$. Show that

$$V = W \oplus W^{\perp}$$

(Hint: Extend a basis of W to a basis of V, and then apply Gram-Schmidt process)

✍ **Exercise 30.21** 3. Let V be a finite dimensional inner product space, and $S, T : V \to V$ are linear operators with adjoint $S^*, T^* : V \to V$. Show that (i) $\quad (S+T)^* = S^* + T^*$
(ii) $\quad (ST)^* = T^* S^*$
(iii) $(T^*)^* = T$
(iv) $(\lambda T)^* = \bar{\lambda} T^*$
(v) $m_{T^*}(x) = \overline{m_T(x)}$.

✍ **Exercise 30.22** 4. Let V be an inner product space and $\mathbf{v} \in V$. (a) Suppose $T : V \to V$ is self-adjoint. Show that $T^2(\mathbf{v}) = \mathbf{0}$ implies $T(\mathbf{v}) = \mathbf{0}$, and hence $T^n(\mathbf{v}) = 0$ for some $n > 0$ implies $T(\mathbf{v}) = \mathbf{0}$.
(b) Suppose S and T are both self-adjoint. Show that ST is self-adjoint iff S and T commute, i.e. $ST = TS$.

✍ **Exercise 30.23** 5. Consider the inner product space of polynomials of degree ≤ 2 in Homework 9 . Let $T : V \to V$ be the linear operator given by $T\left(a + bx + cx^2\right) = bx$. (a) Show that $T^* \neq T$, i.e. is not self-adjoint.

(b) Note that using the basis $\mathscr{B} = \{f_0, f_1, f_2\}$, $(T)_{BB} = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which satisfies $\bar{A}^t = A$. Explain why this is not a contradiction.

30.6 homework11

✍ **Exercise 30.24** 1. Show that if T is an operator on a finite dimensional real or complex vector space V satisfying $T'T = I$, then for any basis $\mathscr{B}$ of V,

$$|\det((T))_{\mathscr{B}\mathscr{B}}| = 1.$$

✍ **Exercise 30.25** 2. Recall the spectral theorem of normal operators: If $T : V \to V$ is a normal operator on a finite dimensional complex inner product space, then there exists $P_1, \ldots, P_k : V \to V$ such that
(i) P_i is self-adjoint;
(ii) $P_i^2 = P_i$;
(iii) $P_i P_j = \mathbf{0}$ if $i \neq j$;
(iv) $I = P_1 + \cdots + P_k$; and
(v) $T = \lambda_1 P_1 + \cdots + \lambda_k P_k$.
This question is to finish the proof by guiding you to prove that P_i is self-adjoint.
(a) Recall $P_i = a_i(T)q_i(T) = \beta_l T^l + \cdots + \beta_1 T + \beta_0 I$ for some $\beta_j \in \mathbf{C}$. Using this formulation of P_i, show that if T is normal (i.e. $T^*T = TT^*$) then P_i is also normal.
(b) Using (ii), show that the eigenvalues of P_i can only be 0 or 1 . (Hint: What's $m_{P_i}(x)$?)
(c) Let $S : V \to V$ be any normal operator. Prove that S is self-adjoint if and only if all eigenvalues of S are real. (Consequently, $S = P_i$ is self-adjoint by (b))

✍ **Exercise 30.26** 3. Suppose V is a complex inner product space and T is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

✍ **Exercise 30.27** 4. Suppose V is a complex inner-product space. Prove that every normal operator $T : V \to V$ has a square root. (An operator $S : V \to V$ is called a square root of T if $S^2 = T$.)

Chapter review

31.1 review for midterm: from 10.26 to 11.2

1. tutorial5 29.3
2. prove that $Ann(S) = Ann(span(S))$ 16.1;

Proof we know that $S \subset span(s) \to Ann(S) \supset Ann(span(S))$; for the other inclusion: let $\phi \in Ann(S)$, for any $\alpha_1 S_1 + \alpha_2 S_2 + ... + \alpha_k S_k \in span(S)$, we have $\phi(\alpha_1 S_1 + \alpha_2 S_2 + ... + \alpha_k S_k) = \alpha_1 \phi(S_1) + \alpha_2 \phi(S_2) + ... \alpha_k \phi(S_k) = \alpha \cdot 0 + \alpha \cdot 0 + ... + \alpha_k \cdot 0 = 0 \to \phi \in Ann(span(S))$

3. prove that if V is finite dimensional, then $\Phi : Ann(W) \to (V/W)^*$ is an isomorphism16.1

Proof we know that $dim(Ann(W)) = dim((V/W)^*)$ and $\Phi : Ann(W) \to (V/W)^*$ is an injection and linear transformation, which leads to that Phi is also surjective. we have shown that Φ is bijective, and we are done.

4. prove that T^t is a linear transformation17.1

first recall the definition of $T^t : W^* \to V^*$, and $T^t(f)(v) := f(T(v)), \forall v \in V$

Proof we want to show that $T^t(af + bg) = aT^t(f) + bT^t(g), \forall f, g \in W^*$.
notice that: $T^t(af + bg)(v) = (af + bg)(T(v)) = af(T(v)) + bg(T(v)) = aT^t(f) + bT^t(g)$. so we are done.

5. how to check a linear transformation is well defined?11.2

check 11.1 for reference: a function is said to be well defined iff the function can only provides one output for one input.

6. prove that there exists a surjective linear transformation $\Pi : V \to V/W$ by $\Pi(v) = v + W$, and $ker(\Pi) = W$. 14.1

Proof a. show Π is a linear transformation
b. show that Π is surjective
c. show that $ker(\Pi) = W$

7. state basis extension theorem8.1 and basis complementation theorem8.2
8. tutorial6.29.4
9. prove the relations between kernel and injection; the relations between image and surjection11.1

10. state and prove the rank-nullity theorem10.1
11. state the definition of change of basis matrix and the related proposition and corollary12.1, 12.1
12. prove that 16.1 b. if $dim(V) < \infty$ and $W \leqslant V$, then $dim(Ann(W)) = dim(V) - dim(W)$