MAT4010 Functional Analysis

Spring 2024

Abstract

This note is not complete and has not been proofread. It contains many mistakes. It is recommended to refer to this lecture note written by an undergraduate student of USTC.

Contents

1 Metric Spaces

1.1 Metric Spaces

Definition 1.1 (metric space)**.** *a metric space is a pair* (X, d)*, where* X *is a set and* d *is a metric on* X*, which is a function defined on* $X \times X$ *s.t. for all* $x, y, z \in X$ *we have*

1. *d is real-valued, finite and nonnegative*
2. $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Definition 1.2 (ℓ^p space)**.** *let* $p \geq 1$ *be a fixed real number. each element in the space* ℓ^p *is a sequence* $x = (\xi_j) = (\xi_1, \xi_2, ...)$ *of numbers s.t.* $|\xi_1|^p + |\xi_2|^p + ...$ *converges; thus*

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty$$

and the metric is defined by

$$d(x, y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p)^{1/p}$$

where $y = (\eta_j)$ *and* $\sum |\eta_j|^p < \infty$

Definition 1.3 (ℓ^∞ space)**.** *The* ℓ^∞ *space consists of all bounded sequences of real or complex numbers. A sequence* $(x_n)_{n=1}^\infty$ *is considered bounded if there exists a real number* $M \geq 0$ *such that* $|x_n| \leq M$ *for all* $n \in \mathbb{N}$*. The norm in* ℓ^∞ *space, also known as the supremum norm, is defined as:*

$$\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$$

Here, sup *denotes the supremum, which is the least upper bound of the set of absolute values of the sequence elements.*

Definition 1.4 ($L^p(\Omega)$ space)**.** *The* $L^p(\Omega)$ *space, for* $1 \leq p < \infty$*, comprises all measurable functions* f *defined on a measure space* Ω *such that the p-th power of the absolute value of* f *is integrable. That is, a function* f *belongs to* $L^p(\Omega)$ *if:*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \mid \int_\Omega |f|^p d\mu < \infty \right\}$$

where μ *is a measure on* Ω*. The norm in* $L^p(\Omega)$ *space is given by:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p d\mu(x) \right)^{1/p}$$

For $p = \infty$*, the* $L^\infty(\Omega)$ *space consists of essentially bounded functions, and its norm is defined as the essential supremum of the function's absolute value:*

$$\|f\|_\infty = \inf \{M \geq 0 : |f(x)| \leq M \ \mu\text{-a.e.}\}$$

Here, "μ*-a.e." stands for "almost everywhere," meaning that the inequality holds except on a set of measure zero.*

Proposition 1.5. *show that* ℓ^p *is a metric space*

1.2 Topologies induced by metrics

1.3 Completeness

Theorem 1.6. *Assume that* (X, d) *is a complete metric space and let* M *be a subset of* X*. Then* M *is complete under the induced metric if and only if* M *is closed in* X*.*

2 Normed Space. Banach Space

2.1 Definition and elementary properties

2.2 Finite v.s. Infinite Dimensional Normed Spaces

2.3 Compactness

Lemma 2.1 (Riesz's lemma)**.** *let* Y *be a proper closed subspace of a normed space* X *of any dimension, then for every real number* $\theta \in (0, 1)$ *there is a* $z \in X$ *s.t.*

$$\|z\| = 1, \quad \|z - y\| \geq \theta \text{ for all } y \in Y$$

2.4 Linear operators

Theorem 2.2 (completion of a normed vector space)**.** *Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbf{K}$. Then there exist a Banach space $(\tilde{X}, \|\cdot\|^{\frown})$ over $\mathbf{K}$ and a linear isometry $\sigma : X \to \tilde{X}$ such that $\sigma(X)$ is dense in $\tilde{X}$.*
Besides, if $(\hat{X}, \|\cdot\|^{\frown})$ is any Banach space over $\mathbf{K}$ such that there also exists a linear isometry from X onto a dense subset of $\hat{X}$, then there exists a linear isometry from $(\tilde{X}, \|\cdot\|^{\frown})$ onto $(\hat{X}, \|\cdot\|^{\frown})$. The space $(\tilde{X}, \|\cdot\|^{\frown})$ which is called the completion of the space $(X, \|\cdot\|)$, is thus unique up to bijective linear isometries. As a normed vector space, the space X may thus be identified with a dense subset of its completion $\tilde{X}$

2.5 Linear functions

3 Inner Product Spaces and Hilbert Spaces

3.1 Inner product spaces and Hilbert spaces

Definition 3.1 (inner product space)**.** *Let X be a vector space. An inner product on X is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathrm{K}$ that satisfies:*

1. $\langle x, x\rangle \geq 0 \ \forall x \in X$, *and the equality holds if and only if* $x = 0$
2. $\langle x, y\rangle = \overline{\langle y, x\rangle} \forall x, y \in X$
3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \forall x, y, z \in X, \forall \alpha, \beta \in \mathbf{K}$

the ordered pair $(X, \langle\cdot,\cdot\rangle)$ is called an inner product space

Theorem 3.2 (Cauchy-Schwarz inequality)**.** *let $(x, \langle\cdot,\cdot\rangle)$ be an inner product space. Then the Cauchy-Schwarz inequality holds:*

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$$

Corollary 3.3 (Any inner product space is a norm space)**.** *The function $\|\cdot\| : X \to \mathbb{R}$*

$$\|x\| = \sqrt{\langle\cdot,\cdot\rangle}$$

is a norm on X. Besides

$$\|x\| = \sup_{y \neq 0} \frac{|\langle x, y\rangle|}{\|y\|} \quad \forall x \in X$$

Corollary 3.4 (The inner product is continuous)**.** *The mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbf{K}$ is continuous. Here the topology of X being that induced by the norm $\|\cdot\|$ defined as above and the topology of $X \times X$ being the corresponding product topology.*

Theorem 3.5 (Parallelogram law)**.** *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad x, y \in X$$

Theorem 3.6. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and let $T \in \mathcal{L}(X)$. Then*

$$\|T\| = \sup_{x \neq 0, y \neq 0} \frac{\langle Tx, y\rangle}{\|x\|\|y\|}$$

Definition 3.7 (Hilbert space)**.** *An inner-product space $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space if, as a normed vector space, it is a Banach space, i.e., if X is complete with respect to the norm $\| \cdot \|$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$*

Theorem 3.8 (Completion of an inner product space)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the completion $(\tilde{X}, \| \cdot \|_{\tilde{X}}$ of its associated normed space $(X, \| \cdot \|)$ is a Hilbert space, whose inner product $\langle \cdot, \cdot \rangle_{\tilde{X}}$ satisfies*

$$\langle \sigma x, \sigma y \rangle_{\tilde{X}} = \langle x, y \rangle \forall x, y \in X$$

where σ is the linear isometry from X onto a dense subspace W of $\tilde{X}$ given by theorem 2.2.

3.2 Examples

3.3 The projection theorem

Let X be a normed vector space, $x \in X$ and Z be a subset of X. The distance from x to Z is defined as

$$d(x, Z) = \inf_{z \in Z} \|x - z\|.$$

It is important to know whether there is a vector $y \in Z$ such that $d(x, Z) = d(x, y)$, that is intuitively speaking, a point $y \in Z$ which is closest to x, and if such an element exists, whether it is unique. This is an existence and uniqueness problem. It turns out we need the following

Definition 3.9 (convex set)**.** *Let X be a vector space. A subset M of X is called convex if*

$$\theta x + (1 - \theta) y \in M \quad \forall x, y \in M \text{ and } \forall \theta \in [0, 1].$$

Theorem 3.10 (Projection theorem)**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, and $Z \subset X$ be a nonempty, convex and complete subset of X.*

1. *Given any $x \in X$, there exists a unique element $y \in Z$ that satisfies*

$$\|x - y\| = \inf_{z \in Z} \|x - z\|,$$

   thus defines an operator $P : X \to X$ by $Px = y$.

2. *The unique vector $y := Px \in Z$ found in part 1. satisfies*

$$\begin{aligned} \langle Px - x, z - Px \rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{R}, \\ \mathrm{Re}\langle Px - x, z - Px \rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

   Conversely, if any vector $y \in Z$ satisfies

$$\begin{aligned} \langle y - x, z - y \rangle &\geq 0 \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{R}, \\ \mathrm{Re}\langle y - x, z - y \rangle &\geq 0 \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

   then $y = Px$.

3. *The operator $P : X \to Z$ satisfies*

$$\|Px_1 - Px_2\| \le \|x_1 - x_2\| \quad \forall x_1, x_2 \in X.$$

Hence, the projection operator P is Lipschitz continuous.

Proof. 1. If $x \in Z$, then $Px = x$. If $x \neq Z$, then

$$\delta := \inf_{z \in Z} \|x - z\|$$

is well defined since Z is nonempty. In fact $\delta > 0$, for otherwise $x \in \bar{Z}$, and hence $x \in Z$ since Z is closed (a complete subset is closed, cf. Theorem 1.6). Let $y_n \in Z$ be a sequence such that

$$\|x - y_n\| \to \inf_{z \in Z} \|x - z\| = \delta > 0 \quad \text{as } n \to \infty.$$

We shall show that (y_n) is a Cauchy sequence. The parallelogram law implies that

$$\begin{aligned}
\|y_m - y_n\|^2 &= \|(x - y_m) - (x - y_n)\|^2 \\
&= 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|2x - (y_m + y_n)\|^2 \\
&= 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4\left\|x - \frac{y_m + y_n}{2}\right\|^2.
\end{aligned}$$

Note that $\frac{y_m + y_n}{2} \in Z$ since Z is convex. Therefore,

$$\left\|x - \frac{y_m + y_n}{2}\right\|^2 \ge \delta^2,$$

which implies that

$$0 \le \|y_m - y_n\|^2 \le 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4\delta^2 \quad \forall m, n \in \mathbb{N}.$$

Thus (y_n) is a Cauchy sequence since $\|x - y_m\| \to \delta$ as $m \to \infty$ and $\|x - y_n\| \to \delta$ as $n \to \infty$. Since Z is complete, there exists $y \in Z$ such that $y_n \to y$. Besides, by the continuity of the norm,

$$\|x - y\| = \lim_{n \to \infty} \|x - y_n\| = \delta = \inf_{z \in Z} \|x - z\|.$$

To show such an element is unique, let $y \in Z$ and $y_0 \in Z$ such that

$$\delta = \|x - y\| = \|x - y_0\|.$$

The sequence (y_n) defined by $y_{2k} = y_0$ and $y_{2k+1} = y$ for all $k \in \mathbb{N}$ evidently satisfies $\|x - y_n\| \to \delta$. The same argument as above thus shows that this sequence converges. Hence $y = y_0$, since the limit of a convergent sequence is unique.

The proof for parts 2. and 3. are left as exercise.

$\square$

If Z is a complete subspace, we have the following:

Theorem 3.11 (Projection theorem for a complete subspace)**.** *1. In Theorem 3.10, if Z is a complete subspace of X, then the element Px found in Theorem 3.10 satisfies*

$$\langle Px - x, z\rangle = 0 \quad \forall z \in Z.$$

Conversely, if an element $y \in Z$ satisfies

$$\langle y - x, z\rangle = 0 \quad \forall z \in Z,$$

then $y = Px$.

2. The projection mapping $P : X \to Z$ is linear if and only if the subset Z is a subspace of X. In this case

$$\|P\|_{\mathcal{L}(X,Z)} = 1 \qquad \textit{if } Z \neq \{0\}.$$

Proof. As a subspace, Y is necessarily convex, hence the projection $P : X \to Y$ exists by Theorem 3.10 1. If $x \in Z$, then $Px = x$ and the desired equality holds. Suppose $x \notin Z$. Assume first that $\mathbb{K} = \mathbb{R}$ and let $z \in Z$ be given. Since $Px + \theta z \in Z$ for all $\theta \in \mathbb{R}$, the inequalities in Theorem 3.10 2. show that

$$\langle Px - x, Px + \theta z - Px\rangle = \theta\langle Px - x, z\rangle \geq 0 \quad \forall \theta \in \mathbb{R},$$

which implies that $\langle Px - x, z\rangle = 0$ (clear by taking $\theta = -\langle Px - x, z\rangle$). Assume next that $\mathbb{K} = \mathbb{C}$ and let $z \in Z$ be given. Since $Px + \theta z \in Z$ for all $\theta \in \mathbb{C}$, the inequalities in Theorem 3.10 2. show that

$$\mathrm{Re}\langle Px - x, Px + \theta z - Px\rangle = \mathrm{Re}(\theta\langle Px - x, z\rangle) \geq 0 \quad \forall \theta \in \mathbb{C},$$

which implies that $\langle Px - x, z\rangle = 0$ (clear by taking $\theta = -\overline{\langle Px - x, z\rangle}$). Conversely, assume that $y \in Z$ satisfies $\langle x - y, z\rangle = 0$ for all $z \in Z$, so that $\langle x - y, y\rangle = 0$ in particular. Consequently,

$$\langle y - x, z - y\rangle = 0 \quad \forall z \in Z,$$

and thus $y = Px$ by Theorem 3.10 2. The proof of part 2. is left as an exercise. □

3.4 Orthogonality, direct sum theorem

3.5 Orthonormal families

Definition 3.12 (orthonormal families)**.** *let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. A family $(e_i)_{i\in I}$ of elements of X is called an orthonormal family if*

$$\langle e_i, e_j\rangle = \delta_{ij}, \quad i, j \in I$$

Theorem 3.13 (Gram-Schmidt orthonormalization)**.**

Definition 3.14 (maximal orthonormal family)**.** *An orthonormal family $(e_i)_{i\in I}$ in an inner product space $(X, \langle\cdot,\cdot\rangle)$ is said to be maximal if the only vector $x \in X$ that satisfies $\langle x, e_i\rangle = 0$ for all $i \in I$ is $x = 0$.*

Theorem 3.15. *an orthonormal family $(e_i)_{i\in I}$ in an inner product space X is maximal if*

$$\overline{span(e_i)_{i\in I}} = X$$

3.6 Fourier series in Hilbert space

Definition 3.16. *X is a Hilbert space, $x = \sum_{i\in I} \alpha_i e_i$, and $(e_i)_{i\in I}$ is a orthonormal family which satisfies that*

$$\langle e_i, e_j\rangle = \delta_{ij}$$

the orthonormal family is called maximal or complete if

$$S = (e_i)_{i\in I} \quad S^{\perp} = \{0\}$$

Remark 3.17. *if X is a inner product space, then*

$$\overline{span(e_i)_{i\in I}} = X \Rightarrow \{(e_i)_{i\in I}\}^{\perp} = \{0\}$$

the other direction holds only if X is a hilbert space.

Theorem 3.18 (Bessel's inequality)**.** $(e_k)_{k=1}^{\infty}$, $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2, \quad x$. *for finite set $J \subset I$, we have*

$$\sup_{J\subset I} \sum_{i\in J} |\langle x, e_j\rangle|^2 := \sum_{i\in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2$$

Remark 3.19. $\{e_i|\langle x, e_i\rangle \neq 0\}$ *is at most countable since* $\{e_i|\langle x, e_i\rangle \neq 0\} = \cup_{m=1}^{\infty}\{e_i||\langle x, e_i\rangle| \geq \frac{1}{m}\}$ *and* $\{e_i||\langle x, e_i\rangle| \geq \frac{1}{m}\}$ *is finite why?*

Theorem 3.20 (Parseval identity)**.** *the followings are equivalent:*

1. $(e_i)_{i\in I}$ *is a maximal orthonormal family of Hilbert space*
2. $x = \sum_{i\in I}\langle x, e_i\rangle e_i = \sum_{k=1}^{N}\langle x, e_k\rangle e_k, \quad N < \infty$
3. $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2 = \sum_{k=1}^{N} |\langle x, e_k\rangle|^2, \forall x \in X$

Proof. 3. $\to$ 1.: suppose $(e_i)_{i\in I}$ is not maximal$\to$ $\exists x \neq 0, \quad x \in [(e_i)_{i\in I}]^{\perp} \neq \{0\}$, then

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 = 0 \neq \|x\|^2$$

which leads to a contradiction.
2. $\leftrightarrow$ 3.:

$$\|x - \sum_{k=1}^{n}\langle x, e_k\rangle e_k\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$

so 2. $\to$ 3. is clear
3. $\to$ 2.: by the continuity of the norm, letting $n \to \infty$, we have $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$
1. $\to$ 2.: let $\epsilon > 0$, let y defined by

$$y = \sum_{i\in I}\langle x, e_i\rangle e_i = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$$

converges in X, so we have

$$\|S_m - S_n\| = |\sum_{k=n+1}^{m} \langle x, e_k\rangle e_k| = \langle \sum_{k=n+1}^{m} \langle , e_k\rangle e_k, \sum_{k=n+1}^{m} \langle x, e_j\rangle e_j\rangle = \sum_{k=n+1}^{m}\sum_{j=n+1}^{m} \langle x, e_k\rangle\overline{\langle x, e_j\rangle}\langle e_k, e_j\rangle$$

$$= \sum_{k=n+1}^{m} |\langle x, e_k\rangle|^2 \leq \epsilon, \quad \forall m > n \geq N, \quad \exists N \in \mathbb{N}$$

case 1: $i \in \{k = 1, 2, \cdots, \infty\}$:

$$\langle x - y, e_i\rangle = \langle x, e_i\rangle - \sum_{k=1}^{\infty} \langle x, e_k\rangle\langle e_i, e_k\rangle = 0$$

case 2: $i \notin \{k = 1, 2, ..., \infty\}$: still 0 why?
so we have $x - y \in [(e_i)_{i\in I}]^{\perp} \to x - y = 0 \to x = y$ □

Theorem 3.21 (separable Hilbert spaces)**.** *a hilbert space X is separable $\Leftrightarrow \exists$ one at most countable maximal orthonormal family*

Proof. $\to$: $\exists M \subset X$ which is countable and dense in X ($\overline{M} = X$), $(x_n)_{n=1}^{\infty} = M$. for $N < \infty$, $(y_k)_{k=1}^{N}$ is a subset of $(x_n)_{n=1}^{\infty}$ and (y_k) is linearly independent s.t.

$$\text{span}(y_k)_{k=1}^{\infty} = \text{span}(x_n)_{n=1}^{\infty}$$

recall $M = \{x_1, x_2, ..., x_n, ...\}$ and by Gram-Schmidt process to $(y_k)_{k=1}^{N}$, $\exists (e_k)_{k=1}^{N}$ orthonormal and

$$\overline{\text{span}(e_k)_{k=1}^{N}} = \overline{\text{span}(y_k)_{k=1}^{N}} = \overline{\text{span}(x_k)_{k=1}^{\infty}} = X$$

$$\to (e_k)_{k=1}^{N} \text{ is maximal}$$

another method for (?): suppose that every maximal orthonormal family is uncountable. In particular, $\exists (e_i)_{i\in I}$ a maximal orthonormal family uncountable:

$$\|e_i - e_j\|^2 = \|e_i\| + \|e_j\|^2 = 2, \quad i \neq j \to B(e_j, \frac{\sqrt{2}}{3}), i \in I \text{is disjoint}$$

since X is separable $\to \exists$ countable dense subset M, $\overline{M} = X$ s.t.

$$i \in I \to M \text{ is a inject}$$

but LHS is uncountable but RHS is countable, contradiction. $\leftarrow$: suppose $(e_k)_{k=1}^{N}$ is a sequence of maximal orthonormal family (Hilbert basis), $x = \sum_{k=1}^{N} \langle x, e_k\rangle e_k$. To show X is separable, that is to find M which is countable and dense s.t. $\overline{M} = X$: let $x \in X$, let $\epsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$$\|x - \sum_{k=1}^{K} \langle x, e_k\rangle e_k\| < \frac{\epsilon}{2}$$

$\exists (\gamma_k)_{k=1}^{K}$ rational s.t.

$$\|\sum_{k=1}^{K} \gamma_k e_k - \sum_{k=1}^{K} \langle x, e_k\rangle e_k\|^2 = \sum_{k=1}^{K} |\gamma_k - \langle x, e_k\rangle|^2 < (\frac{\epsilon}{2})^2$$

since $|\gamma_k - \langle x, e_k\rangle|^2 < (\frac{\epsilon}{2})^2 \frac{1}{k}$. let $M = \{\sum_{k=1}^{K} \gamma_k e_k | \gamma \text{ is rational}, K \in \mathbb{N}\}$ is countable. □

Example 3.22. *ℓ^p is separable for $1 \leq p \leq \infty$*

Definition 3.23 (Hilbert isomorphism)**.** *let X and $\tilde{X}$ be two Hilbert space over the same $\mathbb{K}$ if $\exists$ a linear bijective operator $\sigma : X \to \tilde{X}$ s.t.*

$$\langle \sigma x, \sigma y \rangle_{\tilde{X}} = \langle x, y \rangle_X, \quad x, y \in X$$

then we call σ is a Hilbert isomorphism

Remark 3.24. *σ is bijective linear and $\sigma : X \to \tilde{X}$ is vector isomorphism,*

$$\|\sigma x\|_{\tilde{x}} = \sqrt{\langle \sigma x, \sigma x \rangle_{\tilde{X}}} = \sqrt{\langle x, x \rangle}_X, \quad x \in X$$

which implies normed vector space isometry

Example 3.25. *$L^2(\Omega)$ and ℓ^2 are Hilbert isomorphism.*

Theorem 3.26. *two Hilbert space X and $\tilde{X}$ are isomorphic iff they have the same Hilbert dimension*

3.7 Riesz representation theorem

recall we have Riesz's lemma before: given X normed vector space, we have

$$\text{dim}X < \infty \Leftrightarrow \overline{B(0,1)} \text{ is compact}$$

recall the definition of dual space: $\mathcal{L}(x, \mathbf{K})$: $f \in \mathcal{L}(x, \mathbf{K})$: $f : X \to \mathbf{K}$ is a bounded linear functional.

let X be Hilbert space, for a fixed $y \in X$, $f_y(x) := x \to \langle x, y \rangle$, which is a bounded linear functional:

$$|f_y(x)| - |\langle x, y \rangle| \leq \|y\| \cdot \|x\|, \quad \forall x \in X$$

by Cauchy-schwarz inequality. $f_y(\cdot) = \langle \cdot, y \rangle \in X'$. we have

$$\|f_y\|_{X'} := \sup_{x \in X \setminus \{0\}} \frac{|f_y(x)|}{\|x\|} \leq \|y\|_X$$

we also have

$$\|f_y\|_{X'} \geq \frac{|f_y(y)|}{\|y\|} = \frac{|\langle y, y \rangle|}{\|y\|} = \|y\|$$

so we conclude

$$\|f_y\|_{X'} = \|y\|_X$$

we wonder: $\forall f \in X'$, $\exists y \in X$ s.t. $f(x) = \langle x, y \rangle$ or not?

Theorem 3.27 (Riesz representation theorem)**.** *let X be a Hilbert space, then for every $f \in X'$, $\exists! y_f \in X$ s.t.*

$$f(x) = \langle x, y_f \rangle, \forall x \in X$$

moreover, $\|f\|_{X'} = \|y_f\|_X$. define $\sigma : X' \to X$ by

$$\sigma f = y_f$$

σ is linear if $\mathbf{K} = \mathbb{R}$ and is semilinear if $\mathbf{K} = \mathbb{C}$. σ is also an isometry, and X' is a Hilbert space.

Proof. let $f \in X'$, if $f = 0$,then $y_f = 0$ and $f(x) = 0 = \langle x, 0\rangle \quad \forall x \in X$. so we have

$$\|f\|_{X'} = 0 = \|y\|_X$$

assume $f \neq 0$, let $Y = \mathcal{N}(f) = \{x \in X | f(x) = 0\}$, Y is a closed subspace of X:

$$0 = f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2) = 0 + 0$$

and given $x_n \in Y$ s.t. $x_n \to x_0$ as $n \to \infty$, then $f(x_0) = f(\lim x_n) = \lim_{n\to\infty} f(x_n) = 0$, these indicate that $x_0 \in Y$ and Y is closed subspace of X.

By direct sum theorem, we have $X = Y \oplus Y^{\perp}$ since Y is a proper subspace, $\exists Z \in Y^{\perp} \setminus \{0\}$ with $\|z\| = 1$ (by $\frac{z}{\|z\|}$ for some z). let P be a projection operator: $X \to Y$, and $P^{\perp} : X \to Y^{\perp}$, $f(x) = f(Px + P^{\perp}x) = f(Px) + f(P^{\perp}x) = f(P^{\perp}x)$. write $x = x_0 + az$ where $x_0 \in Y$ and a is constant:

$$f(x) = f(x_0) + af(z) = 0 + af(z) \to a = \frac{f(x)}{f(z)}$$

recall $z \in Y^{\perp} \setminus \{0\} \to z \notin Y \to f(x) \neq 0$. only need to show that $\forall x \in X$ we can find such x_0 and z:

$$f(x - az) = f(x) - af(z) = f(x) - \frac{f(x)}{f(z)} f(z) = 0 \to x - az \in Y$$

then we have

$$\langle x, z\rangle = \langle x_0, z\rangle + \langle az, z\rangle = 0 + a|z| = a = \frac{f(x)}{f(z)}$$

we have $f(x) = \langle x, z\rangle f(z) = \langle x, \overline{f(z)} z\rangle, \forall x$. this is indeed the y_f we want: let $y_f = \overline{f(z)} z$, we have

$$f(x)\langle x, y_f\rangle, \quad \forall x \in X$$

suppose $\exists y_1, y_2$ s.t.

$$f(x) = \langle x, y_1\rangle = \langle x, y_2\rangle \to \langle x, y_1 - y_2\rangle = \langle x, y_1\rangle - \langle x, y_2\rangle = 0$$

choose $x = y_1 - y_2$, we have

$$\langle x, y_1, y_2\rangle = \|y_1 - y_2\|^2 = 0 \to y_1 = y_2$$

which means that $\sigma : X' \to X$ is 1-1 correspondence. and $|f(x)| = |\langle x, y_f\rangle| \leq \|y_f\| \cdot \|x\| \to \|f\|_{X'} \leq \|y_f\|$ and $\sup_{x\in X} \frac{|f(x)|}{\|x\|} \geq \frac{f(y_f)}{\|y_f\|} = \|y_f\| \Rightarrow \|f\|_{X'} = \|y_f\|_X$. next is to show $\|\sigma f\|_X = \|f\|'_X, \quad \forall f \in X'$:

$$\|f\|_{X'} = \sqrt{\langle f, f\rangle_{X'}} = \sqrt{\langle \sigma f, \sigma f\rangle_X} = \|\sigma f\|_X$$

to show the linearity:

$$\alpha_1 f_1 + \alpha_2 f_2 \in X'$$

we have

$$(\alpha_1 f_1 + \alpha_2 f_2)(x) = \langle x, \sigma(\alpha_1 f_1 + \alpha_2 f_2\rangle ? \; = \langle x, \alpha_1 f_1 + \alpha_2 f_2\rangle$$

given $\sigma f_1 = y_1, \sigma f_2 = y_2$. if the equality holds, we are done.
$\forall x \in X$, $(\alpha_1 f_1 + \alpha_2 f_2)(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) = \langle x, \overline{\alpha_1} y_1 + \overline{\alpha_2} y_2 \rangle = \langle x, \sigma(\alpha_1 f_1 + \alpha_2 f_2) \rangle$. $\langle x, \sigma(\alpha_1 f_1 + \alpha_2 f_2) - \overline{\alpha_1} y_1 + \overline{\alpha_2} y_2 \rangle = 0, \forall x \in X$,

$$\sigma(\alpha_1 f_1 + \alpha_2 f_2) = \overline{\alpha_1} y_1 + \overline{\alpha_2} y_2 = \overline{\alpha_1}(\sigma f_1) + \overline{\alpha_2}(\sigma f_2)$$

if $\mathbf{K} = \mathbb{R}$, σ is linear and otherwise σ is semilinear.
Finally prove that X' is a Hilbert space: define $\langle f, g \rangle_{X'} = \overline{\langle \sigma f, \sigma g \rangle_X}$ we prove that this is an inner product:

1. $\langle f, f \rangle_{X'} = \overline{\langle \sigma f, \sigma f \rangle} \geq 0, \forall f$. equality holds iff $\sigma f = 0$ iff $f = 0$
2. $\langle \alpha_1 f_1 + \alpha_2 f_2, y \rangle_{X'} := \langle \sigma(\alpha_1 f_1 + \alpha_2 f_2), \sigma g \rangle = \overline{\langle \overline{\alpha_1} \sigma f_1 + \overline{\alpha_2} f_2, \sigma g} = \alpha_1 \overline{\langle \sigma f_1, \sigma g \rangle} + \alpha_2 \overline{\langle \sigma f_2, \sigma g \rangle} = \alpha_1 \langle f_1, g \rangle + \alpha_2 \langle f_2, g \rangle$
3. $\langle f, g \rangle = \overline{\langle g, f \rangle}$:

$$\langle f, g \rangle_X = \langle \sigma f, \sigma g \rangle_X = \overline{\langle \sigma g, \sigma f \rangle_X} = \overline{\langle g, f \rangle_{X'}}$$

since $\mathbb{R}$, $\mathbb{C}$ are complete, we know that X' is complete and thus X' is a Hilbert space.

□

Example 3.28. $\mathbb{R}^3 : x = (\xi_1, \xi_2, \xi_3)$ *and* $a = (a_1, a_2, a_3)$, $f(x) = ax = a_1 x_1 + a_2 x_2 + a_3 x_3$

Definition 3.29. $X' = \mathcal{L}(X, \mathbb{K})$, $a(x, y) : X \times X \to \mathbb{K}$ *is called a sesquelinear if*

1. $a(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 a(x, y) + \alpha_2 a(x, y)$
2. $a(x, \alpha_1 y_1 + \alpha_2 y_2) = \overline{\alpha_1} a(x, y) + \overline{\alpha_2} a(x, y)$

in addition $a(\cdot, \cdot)$ *is called bounded if* $\exists c > 0$ *s.t.* $|a(x, y)| \leq c\|x\| \cdot \|y\|, \forall x, y \in X$. *define* $\|a\| = \sup_{x,y \neq 0} \frac{|a(x,y)|}{\|x\|\|y\|}$

Definition 3.30 (Riesz isometry)**.** $\sigma : X' \to X$ *defined by* $\sigma f = y_f, \forall f \in X'$ *is a bijection isometry:*

1. *linear if* $\mathbb{K} = \mathbb{R}$
2. *semilinear if* $\mathbb{K} = \mathbf{C}$
3. *isometry:* $\|\sigma f\|_X = \|f\|_{X'}$

Theorem 3.31 (Riesz theorem for bounded sesquelinear form $a(\cdot, \cdot)$)**.** *let X be a Hilbert space,* $a(\cdot, \cdot) : X \times Y \to \mathbb{K}$ *be a bounded sesquelinear form. then* $\exists! A : Y \to X$ *s.t.* $a(x, y) = \langle x, Ay \rangle, \quad x \in X, y \in Y$

Proof. fix $y \in Y$ $a(\cdot, y) : X \to \mathbb{K}$, $|a(x, y)| \leq c\|y\|\|x\|, \forall x \in X$ so $a(\cdot, y) \in X'$. By Riesz representation theorem, $\exists z \in Y$ s.t. $a(x, y) = \langle x, z \rangle$. this define a 1-1 relation: $z = Ay$, $A : Y \to X$, which is also linear:

$$A(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A y_1 + \alpha_2 A y_2$$

we prove that $A : Y \to X$ is linear (sesquelinear): by definition of sesquelinearity:

$$a(x, \alpha_1 y_1 + \alpha_2 y_2\rangle = \langle x, A(\alpha_1 y_1 + \alpha_2 y_2)\rangle$$

$$a(x, \alpha_1 y_1 + \alpha_2 y_2\rangle = \overline{\alpha_1}\langle x, Ay_1\rangle + \overline{\alpha_2}\langle x, Ay_2\rangle = \langle x, \alpha_1 Ay_1 + \alpha_2 Ay_2\rangle, \quad \forall y_1, y_2 \in Y$$

so we have

$$A(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 Ay_1 + \alpha_2 Ay_2, \quad \forall \alpha_1, \alpha_2 \in \mathbb{K}$$

the idea is

$$A : y \in Y \overset{sesquelinear}{\longrightarrow} a(\cdot, y) \overset{\sigma}{\longrightarrow} X$$

for the norm $\|A\| = \sup_{y \in Y, y \neq 0} \frac{\|Ay\|}{\|Y\|} = \sup_{X \in x \setminus \{0\} y \in Y \setminus \{0\}} \frac{|a(x,y)|}{\|x\|\|y\|} \overset{def}{=} \|a\|$, we have

$$\frac{\|Ay\|}{\|y\|_{y \neq 0}} = \sup_{x \in X, x \neq 0} \frac{|\langle x, Ay\rangle|}{\|x\|\|y\|} = \sup_{x \in X, x \neq 0} \frac{|a(x,y)|}{\|x\|\|y\|}$$

□

Definition 3.32. *Hilbert adjoint operator: let $T : X \to Y$ s.t. $T \in \mathcal{L}(X, Y)$, define $T^* : Y \to X$ by*

$$\langle Tx, y\rangle_Y = \langle x, T^* y\rangle_X \quad \forall X \in X, \forall y \in Y$$

we can show that T^* is sesquelinear and bounded:

Proof. consider $h(x, y) = \langle Tx, y\rangle$, $h : X \times Y \to \mathbb{K}$

1. linearity: $h(x, y) = \langle Tx, y\rangle$
2. boundedness:

$$|h(x, y)| = |\langle Tx, y\rangle| \overset{C-S}{\leq} \|Tx\|\|y\| \leq \|T\|\|x\|\|y\|$$

next is to show $\exists T^* : Y \to X$ s.t.

$$\langle Tx, y\rangle_Y = h(x, y) = \langle x, T^* y\rangle_X \quad \forall x, y$$

which is true if we choose $T^* = A$, which is defined above.
for the norm $\|T^*\|$ we have:

$$\|T^*\| = \|h\| := \sup_{x \in X \setminus \{0\}, y \in Y \setminus \{0\}} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{x \in X \setminus \{0\}, y \in Y \setminus \{0\}} \frac{|\langle Tx, y\rangle|}{\|x\|\|y\|} = \|T\|$$

□

Lemma 3.33. *$Q : X \to Y \in \mathcal{L}(X, Y)$, x, y are inner product space, then we have*

1. *$Q = 0$ iff $\langle Qx, y\rangle = 0, \forall x \in X, \forall y \in Y$*
2. *if $X = Y$, $\mathbb{K} = \mathbb{C}$, $\langle Qx, x\rangle = 0 \forall x \in X \to Q = 0$*

Proof. 1. ($\rightarrow$): suppose $Q = 0$,

$$\langle Qx, y\rangle = \langle 0, y\rangle = 0, \forall x, y$$

($\leftarrow$): suppose $\langle Qx, y\rangle = 0, \forall x, y$, take $y = Qx$ then

$$\|Qx\|^2 = \langle Qx, Qx\rangle = - \rightarrow Qx = 0, \forall x \in X \rightarrow Q = 0$$

2. suppose $\langle Q\theta, \theta\rangle = 0, \forall\theta \in X$. let $\theta = \alpha x + y$, $\langle Q(\alpha x + y), \alpha x + y\rangle = \alpha\overline{\alpha}\langle Qx, x\rangle + \alpha\langle Qx, y\rangle + \overline{\alpha}\langle Qy, x\rangle + \langle Qy, y\rangle$

   (a) if $\alpha = 1$: $\langle Qx, y\rangle - \rangle Qy, x\rangle = 0$

   (b) if $\alpha = i$: $\langle Qx, y\rangle - \langle Qy, x\rangle = 0 \rightarrow \langle Qx, y\rangle = 0, \forall x, y \rightarrow Q = 0$

□

Remark 3.34. *2. only holds for $\mathbb{K} = \mathbb{C}$: in $\mathbb{R}^2$: we have the rotation matrix:*

$$Q = \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix}$$

which satisfies $\langle Qv, v\rangle = 0, \forall v \in \mathbb{R}^2$

Proposition 3.35. *T^* is semilinear: $(T^* + S^*) = T^* + S^*$ and $(\alpha T)^* = \overline{\alpha}T^*$*

Proof. 1. $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\langle Sx, y\rangle = \langle x, S^*y\rangle$, we have

$$\langle (T + S)x, y\rangle = \langle (T + S)x, y\rangle = \langle x, T^*y + S^*y = \langle x, (T + S)^*y\rangle$$
$$\Rightarrow (T + S)^*y = (T^* + S^*)y$$

2. $(\alpha T)^* = \overline{\alpha}T^*$:

$$\langle (\alpha T)x, y\rangle = \alpha\langle Tx, y\rangle = \langle x, \overline{\alpha}T^*y\rangle = \langle x, (\alpha T)^*y$$
$$\Rightarrow (\alpha T)^*y = (\overline{\alpha}T^*)y\forall y$$

□

Proposition 3.36. *some properties of T^**

1. $(ST)^* = T^*S^*$
2. $\|T^*\| = \|T\|$
3. $(T^*)^* = T$
4. $\|TT^*\| = \|T^*T\|$

Definition 3.37 (self-adjoint operator)**.** *X is Hilbert space, $T \in \mathcal{L}(x) : X \rightarrow X$ is bounded linear, then T is called self-adjoint if $T^* = T$, i.e.*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in X$$

Example 3.38. 1. *let $X = \mathbb{R}^n, Y = \mathbb{R}^m$, $T : X \rightarrow Y$ is $A_{m\times n}$ and $T^*y = A^ty$. $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is self-adjoint if $T^* = T \leftrightarrow A^t = A \leftrightarrow A$ is real symmetric*

2. *for $X = \mathbb{C}$: $T : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $Tx = Ax$ and $T^*y = A^*y$, $A^* = \overline{A^t}$. $T : \mathbb{C} \rightarrow \mathbb{C}$ is self-adjoint if $A^* = A$, i.e. A is a Hermitian*

4 Great Theorems of Linear Functional Analysis

4.1 Zorn's lemma

Lemma 4.1 (Zorn's lemma)**.** *let $X \neq$ empty, partial ordered set. If every total ordered subset of X has an upper bound, then X contains a maximal element.*

Definition 4.2 (partial order)**.** *the relation partial order is defined if the following rules hold:*

1. $x \leq x, \quad \forall x \in X$
2. $x \leq y$ *and* $y \leq x \to x = y$
3. $x \leq y$ *and* $y \leq z \to x \leq z$

Remark 4.3. *there might be elements $x, y \in X$ s.t. neither $x \leq y$ or $y \leq x$ holds. if $x \leq y$ then they are comparable.*

if every two element of $M \subset X$ are comparable, we say M is total ordered

Example 4.4.

1. $\mathbb{R}$ *is partial ordered set*
2. $\mathbb{R}^2 \cong \mathbb{C}$*:* $(x_1, x_2) \leq (y_1, y_2)$ *if* $x_1 \leq y_1$ *and* $x_2 \leq y_2$*, then* $\leq$ *is a partial order on* $\mathbb{R}^2$
3. $\mathbb{N}$*:* $m \leq n$ *if* m *divides* n*:* $\frac{n}{m} \in \mathbb{N}$*, e.g.* $2 \leq 4$*,* $2 \leq 6$ *but* $4 \nleq 6$. $\{2, 4, 8, 16, ...\}$ *is a partial ordered set.*
4. $M = \mathbb{R}$*,* $M = \mathbb{N}$*,* $M = (0, \infty)$*: no upper bound since* ∞ *not in these set*
5. $(-\infty, 0)$*:* $b > 0$ *is an upper bound*

Remark 4.5 (Axiom of Choice)**.** $(X_\lambda)_{\lambda \in \Lambda}$ *X_λ not empty,* $\forall \lambda \in \Lambda$*.* $\exists c :$ *choice function:* $c(X_\lambda) \in X_\lambda$

Theorem 4.6.

1. *every vector space has a Hamel basis*
2. *every inner product space has a maximal orthonormal family*
3. *every vector space can be normed*

Proof.

1. recall a Hamel basis B of a vector space x is a maximal linearly independent subset of X. Let Y not empty be the linearly independent subset of X, $M \subset Y$ is a total ordered subset of Y. $E \in \bigcup_{A \in M} A$ is an upper bound of M. and E is linearly independent. Then Y contains a maximal element called B, which is a Hamel basis. suppose not, then $\exists x_0 \in X \setminus B$, s.t. $\{x_0\} \cup B$ is linearly independent, but $B \subsetneq \{x_0\} \cup B \leftrightarrow B$ is a maximal element of Y

□

Remark 4.7. *maximal element may not be maximal bound: maximal bound may not exist*

4.2 Hahn-Banach theorem

Theorem 4.8 (Hahn-Banach theorem for vector space (Real case $\mathbb{K} = \mathbb{R}$)). *X is a vector space, Y is a subspace of X. $f : Y \to \mathbb{K}$ is a linear functional; $p : X \to \mathbb{K}$ is a sublinear functional: i.e.*

1. $p(\alpha x) = \alpha p(x), \forall x \in X, \forall \alpha > 0$
2. $p(x + y) \leq p(x) + p(y), \forall x, y \in X$

then $\exists \tilde{f} : X \to \mathbb{K}$ linear functional s.t. $\tilde{f}(y) = f(y), \forall y \in Y$ and $\tilde{f}(x) \leq p(x), \forall x \in X$

Remark 4.9. *the theorem says that we can expand the domain of f from the subspace Y to the space X by defining $\tilde{f}$.*

Proof. (the real case $\mathbb{K} = \mathbb{R}$) step 1: $F = \{f : X \to \mathbb{R}|\text{linear}\}$ then partial order $\leq$: $f_1 \leq f_2$ defined by domain$f_1 \subset$ domainf_2 and $f_2(x) = f_1(x), \forall x \in$ domainf_1. let $G \subset F$ be a total ordered subset, g : domain$g \to \mathbb{R}$ with domain$g = \cup_{f \in G}$domain$f \subset X$, $g(x) = f(x)$ if $x \in$ domain$f$ for some $f \in G$. $x \in$ domainf_1, $x \in$ domainf_2 with $f_1, f_2 \in G$. assume $0 \leq f_1 \leq f_2 \to$ domain$f_1 \leq$ domain$f_2 \to g(x) = f_2(x) = f_1(x)$. $g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$, $\forall x, y \in$ domain$g = \cup_{f \in G}$domain$f$, $x \in$ domainf_1, $y \in$ domainf_2, $f_1 \leq f_2$, $x, y \in$ domainf_2. g is an upper bound of G, $\forall f \in G$, then $f \leq g$ since domain$f \subset$ domain$g$ $g(x) = f(x)$, $\forall x \in$ domain$f$. by Zorn's lemma, $\exists$ maximal $\tilde{f}$ of F: $\tilde{f}$: domain$\tilde{f}(= X) \to \mathbb{R}$. $f : Y \to \mathbb{R}$ linear and $f(x) \leq p(x), \forall x \in Y$. if domain$f \subsetneq X$, $\exists x_0 \in X \setminus$ domain$f$. define h : domain$h \to \mathbb{R}$ linear with domain$h = \text{span}\{x_0\} \oplus Y = \{\alpha x_0 + y | \alpha \in \mathbb{R}, y \in Y\}$.

$$h(x) = h(\alpha x_0 + y) = \alpha h(x_0) + h(y) \leq \alpha\lambda + p(y) \leq p(\alpha x_0 + y) = p(x)$$

$\forall \alpha \in \mathbb{R}, \forall y \in Y$, here $h(x_0) := \lambda$. if $\alpha = 0$, it is trivial; otherwise: $\alpha\lambda \leq p(\alpha x_0 + y) - p(y)$. given $\alpha > 0$: $\lambda \leq p(x_0 + \frac{y}{\alpha}) - p(\frac{y}{\alpha}) \to \lambda \geq -p(-x_0 - \frac{y}{\alpha}) + p(\frac{y}{-\alpha})$.

to prove the existence of λ: $f(u + v) = f(-x_0 + u + x_0 + v) = f(-x_0 + u) + f(x_0 + v) = ... \leq p(-x_0 + u) + p(x_0 + v)$

□

Theorem 4.10 (Hahn-Banach theorem (generalized to complex case)). $p(\alpha x) = |\alpha| x$ $|f(x)| \leq p(x) \to |\tilde{f}| \leq p(x)$

Proof. □

Theorem 4.11 (Hahn-Banach theorem in normed vector space). *let X be a normed vector space, $Y \subset X$ is a subspace. $f : Y \to \mathbb{K}$ is bounded linear: $f \in Y'$. then $\exists \tilde{f} : X \to \mathbb{K}$ with*

$$\tilde{f} = f, \forall y \in Y \quad \textit{and} \quad \|\tilde{f}\|_{X'} = \|f\|_{Y'}$$

Proof. define $p(x) = \|f\| \cdot \|x\|, \forall x \in X$. recall that we have $|f(x)| \leq \|f\| \cdot \|x\|$, then p is sublinear and $|f(y)| \leq p(y), \forall y \in Y$. then Hahn-Banach theorem in a complex vector space implies that $\exists \tilde{f} : X \to \mathbb{K}$ s.t.

$$|\tilde{f}| \leq p(x) = \|f\| \cdot \|x\|, \forall x \in X$$

$\tilde{f} \in X'$, we have $\|\tilde{f}\|_{X'} \leq \|f\|_{Y'} \leq \|\tilde{f}\|_{X'}$. recall we also have

$$\|f\|_{Y'} = \sup_{y \in Y, y \neq 0} \frac{|f(y)|}{\|y\|} \leq \sup_{x \in X, x \neq 0} \frac{|\tilde{f}(x)|}{\|x\|} = \|\tilde{f}\|_{X'}$$

which implies that $\|f\|_{Y'} = \|f\|_{X'}$. □

Theorem 4.12 (Hahn-Banach extension theorem, analytic form)**.** *let X be a normed vector space, $0 \neq x \in X$ then $\exists f_x \in X'$ s.t.*

$$f_x(x) = \|x\| \quad \|f_x\| = 1$$

Proof. given $x \in X$, let $Y = \text{span}\{x\}$, we define f by:

$$f(\alpha x) = \alpha \|x\| \quad \forall x \in \mathbb{K}$$

$f : Y \to \mathbb{K}$, and $f(x) = \|x\|$, $|f(\alpha x)| = |\alpha| \|x\| = \|\alpha x\| \to \|f\|_{Y'} = 1$. By Hahn-Banach space in a normed vector space: X is a normed vector space and $Y \subset X$ is a subspace, $f : Y \to \mathbb{K}$ is bounded linear, then $\exists \tilde{f} : X \to \mathbb{K}$ a bounded linear functional with $\tilde{f}(y) = f(y), \forall y \in Y$. Besides, $\|\tilde{f}\|_{X'} = \|f\|_{Y'}$.

we have $\exists f_x : X \to \mathbb{K}$ s.t. $f_x(y) = f(y)$, $\forall y \in Y$, and $\|f_x\|_{X'} = \|f\|_{Y'} = 1$ and $f_x(x) = f(x) = \|x\|$. □

Corollary 4.13. *let X be a normed vector space and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. then $\exists f \in X'$ s.t. $f(x_1) \neq f(x_2)$.*

Proof. take $x = x_1 - x_2$ in the previous theorem, we have

$$\exists f_x \in X' \text{ s.t. } f_x(x_1 - x_2) = f_x(x) = \|x\| = \|x_1 - x_2\| \neq 0 \to f_x(x_1) \neq f_x(x_2)$$

□

Remark 4.14. *to study X, sometimes it is more convenient to work with its dual $X' = \mathcal{L}(X, \mathbb{K})$*

Corollary 4.15. *recall $f \in X'$, then*

$$\|f\|_{X'} = \sup_{x \neq 0, x \in X} \frac{|f(x)|}{\|x\|} = \sup_{x \in X, \|x\|=1} \frac{|f(x)|}{\|x\|}$$

. we have a corollary says that

$$\|x\|_X = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|_{X'}} = \sup_{f \in X', \|f\|=1} \frac{|f(x)|}{\|f\|}$$

Proof.

$$|f(x)| \leq \|f\| \cdot \|x\|, \forall x, \forall f$$

so we have

$$\sup_{f\in X', f\neq 0} \frac{|f(x)|}{\|f\|} \leq \|x\| = \frac{|f_x(x)|}{\|f_x\|} \leq \sup_{f\in X', f\neq 0} \frac{|f(x)|}{\|f\|}$$

for some $f_x \in X'$. the equality $=$ is by Hahn-Banach theorem. $\square$

Theorem 4.16. *let X be a normed vector space. if X' is separable, then X is separable.*

Proof. let $(f_k)_{k=1}^{\infty}$ be a sequence in X', which is dense in X'. for each k, $\exists x_k$ with $\|x_k\| = 1$ s.t.

$$|f_k(x_k)| \geq \frac{1}{2}|f(x_k)|$$

the existence of f_x is guaranteed by

$$\|f\|_{X'} = \sup_{x\in X, \|x\|=1} |f(x)|$$

consider

$$M = \{\alpha_1 x_1 + \cdots \alpha_n x_n | n \in \mathbb{N}, \alpha_1, \cdots \alpha_n \in \mathbb{Q} \text{ or } \mathbb{Q} + i\mathbb{Q}\}$$

from the definition, we know that M is countable, so it remains to show that $\overline{M} = \operatorname{span}(x_k)_{k=1}^{\infty} = X$:

suppose not, $\exists x_0 \in X \setminus \overline{M}$ s.t. $\inf_{z\in M} \|x_0 - z\| = \inf_{z\in\overline{M}} \|x_0 - z\| := \delta > 0$.

$$\exists f \in X', \quad \|f\| = 1, \quad f(x_0) = \delta > 0, \quad f(y) = 0$$

$\forall y \in \overline{M}$. Since (f_k) is dense in X', $\exists (f_{k_j})$ s.t. $f_{k_j} \to f$ and

$$0 \leftarrow \|f_{k_j} - f\| \geq \|(f_{k_j} - f)(x_{k_j})\| = \|f_{k_j}(x_{k_j}) - f(x_{k_j}) = \|f_{k_j}(x_{k_j})\| \geq \frac{1}{2}\|f_{k_j}\|$$

implies that $f_{k_j} \to 0$ as $j \to \infty$, so $f = 0$, contradicts with $\|f\| = 1$

(a theorem says that: let X be a normed vector space, $Y \subset X$ be a proper subspace. given $x_0 \in X \setminus Y$. assume $\delta := d(x_0, Y) = \inf_{z\in Y} \|x_0 - z\| \geq 0$. then $\exists \tilde{f} \in X'$ s.t. $\tilde{f}(y) = 0$, $\forall y \in Y$, $\tilde{f}(y) = 0$ and $\tilde{f}(x) = \delta$, $\|\tilde{f}\| = 1$) $\square$

recall in real analysis we know that $(C[a,b])' = BV[a,b]$, which can be proved by Hahn-Banach theorem.

Proof. $\square$

Definition 4.17. *let X be a normed vector space over $\mathbb{R}$ and $f \in X'$, $\gamma \in \mathbb{R}$, then $H = \{x \in X | f(x) = \gamma\}$ is called a closed affine hyperplane.*

Remark 4.18. *given f a linear functional, $H = \{f = \gamma\}$ is closed iff f is continuous. for a given $A, B \subset X$ with*

$$f(x) \leq \gamma \leq f(y), \quad \forall x \in A, y \in B$$

then we say H separates A and B.

Theorem 4.19 (Hahn-Banach separation theorem)**.** *let X be a normed vector space, $A, B \subset X$ are nonempty and $A \cap B = \Phi$. if A is open and convex, B is convex, then $\exists f \in X'$, $\exists \gamma \in \mathbb{R}$ s.t. $f(x) \le \gamma \le f(y)$, $\forall x \in A$, $\forall y \in B$*

Proof. 1. first construct $P(x)$ sublinear functional: let C be an open convex subset of X with $0 \in C$. then define $P(x) = \inf\{\beta > 0 | \frac{x}{\beta} \in C, \forall x \in X\}$. claim that

- $\exists M > 0$ s.t. $P(x) \le M\|x\|, \forall x \in X$
- $x \in C \Leftrightarrow P(x) < 1$: $C = \{x \in X | P(x) < 1\}$
- $P(x)$ is sublinear

proof:

- since $0 \in C$, C is open, $\exists r > 0$ s.t. $B(0, r) \subset C$, then we have

$$\|\frac{x}{\frac{2\|x\|}{r}}\| = \frac{\|x\|}{\|x\|}\frac{r}{2} = \frac{r}{2} < r \Rightarrow \in B(0,r) \subset C$$

  $P(x) \le \frac{2\|x\|}{r} = \frac{2}{r}\|x\|, \forall x \in X$, $\frac{2}{r} =: M$.
- $(\rightarrow)$: suppose $x \in C$ open $\exists \delta > 0$ s.t.

$$\frac{x}{1-\delta} \in C \rightarrow P(x) \le 1 - \delta < 1$$

  $(\leftarrow)$: suppose $P(x) < 1$, $\exists 0 < \beta < 1$ s.t. $\frac{x}{\beta} \in C \rightarrow x = \beta \cdot \frac{x}{\beta} + (1 - \beta) \cdot 0 \in C$ since C is convex.
- $\forall \alpha > 0, P(\alpha x) = \inf\{\hat{\beta} > 0 | \frac{\alpha x}{\hat{\beta}} \in C\}$,

$$\hat{\beta} = \alpha\beta = \inf\{\alpha\beta > 0 | \frac{\alpha x}{\alpha \beta} = \frac{x}{\beta} \in C\} = \alpha \cdot \inf\{\beta > 0 | \frac{x}{\beta} \in C\} = \alpha P(x), \forall x \in X, \forall \alpha > 0$$

  want to show

$$P(x + y) \le P(x) + P(y), \forall x, y \in X$$

  let $\epsilon > 0$,

$$P(\frac{x}{P(x) + \epsilon}) = \frac{1}{P(x) + \epsilon} P(x) < 1 \rightarrow$$

$$\frac{x}{P(x) + \epsilon} \in C, \quad \frac{y}{P(y) + \epsilon} \in C$$

$$t\frac{x}{P(x) + \epsilon} + (1 - t)\frac{y}{P(y) + \epsilon} \in C, \quad \forall t \in [0, 1]$$

  take $t = \frac{P(x)+\epsilon}{P(x)+P(y)+2\epsilon}$, we have

$$\frac{x + y}{P(x) + P(y) + 2\epsilon} \in C \rightarrow P(\frac{x + y}{P(x) + P(y) + \epsilon}) < 1$$

$$P(x + y) < P(x) + P(y) + 2\epsilon \forall \epsilon > 0, \forall x, y \in X$$

$$P(x + y) \le P(x) + P(y), \forall x, y \in X$$

2. next we separate C and y_0 given C is open convex and $y_0 \notin C$.

   - C is open convex, $0 \in C$
   - C is open convex, $0 \notin C$

   proof:

   - $Y = \text{span}(y_0)$ is a subspace of X, define $f_0 : Y \to \mathbb{R}$ by $f_0(\alpha y_0) := \alpha \to f_0(y_0) = 1$

   $$f(y) = f(\alpha y_0) = \alpha f(y_0) < \alpha P(y_0) = P(\alpha y_0) = P(y), \forall y \in Y$$

   given $y_0 \notin C \leftrightarrow P(y_0) \geq 1$, by Hahn-Banach extension theorem in normed vector space, $\exists f : X \to \mathbb{R}$ linear s.t. $f(x) \leq P(x), \forall x \in M$.

   $$\forall x \in C, \quad f(x) \leq P(x) < 1 = f(y_0)$$

   $$f(x) \leq P(x) \leq M\|x\|$$

   $$-f(x) = f(-x) \leq P(-x) \leq M\|x\| \to |f(x)| \leq M\|x\|, \quad \forall x \in X \to f \in X'$$

   - take $x_0 \in C$, $\tilde{C} := C - x_0$ open and convex, $\tilde{y}_0 = y_0 - x_0 \notin \tilde{C}$, $\exists f \in X'$ s.t.

   ...

3. given A open convex, and B convex, $A \cap B = \Phi$. let $C = \bigcup_{y \in B}(A - y) = \{x - y | x \in A\}$. claim that C is open and convex: take $z_1, z_2 \in C$, $z_1 = x_1 - y_1$ and $z_2 = x_2 - y_2, x_1, x_2 \in A, y_1, y_2 \in B$.

   $$tz_1 + (1-t)z_2 = t(x_1 - y_1) + (1-t)(x_2 - y_2) = tx_1 + (1-t)x_2 - [ty_1 + (1-t)y_2] \in C, \quad \forall t \in [0,1]$$

   finally, we note that $\exists f \in X'$, $\exists r \in \mathbb{R}$ s.t.

   $$f(x - y) = f(z) \leq \gamma \leq f(0) = 0 \quad \forall z \in C$$

   $$\to f(x) \leq f(y) \to \sup_{x \in A} f(x) \leq \inf_{y \in B} f(y) \to f(x) \leq \gamma f(y), \forall x \in A, \forall y \in B$$

□

Theorem 4.20. *if X is a real normed vector space, $A, B \subset C$ are nonempty and disjoint, A is convex, closed, B is convex and compact, then*

$$\exists f \in X', \exists r \in \mathbb{R}, \exists \delta > 0 \text{ s.t. } f(x) \leq r - \delta \leq r + \delta \leq f(y), \forall x \in A, \forall y \in B$$

$H = \{f = r\}$ strictly separates A and B

Proof. let $A(r) = \bigcup_{x \in A} B(x, r) = \{z \in X | d(z, A) < r\}$ and $B(r) = \bigcup_{y \in B} B(y, r) = \{z \in X | d(z, B) < r\}$. to show they are open and convex: $\forall \tilde{x}, \tilde{y} \in A(r)$, $\exists x, y \in A$ s.t. $\|\tilde{x} - x\| < r$, $\|\tilde{y} - y\| < r$,

$$t\tilde{x} + (1-t)\tilde{y} = t(x + \tilde{x} - x) + (1-t)(y + \tilde{y} - y) = tx + (1-t)y + t(\tilde{x} - x) + (1-t)(\tilde{y} - y)$$

$$\|t(\tilde{x} - x) + (1-t)(\tilde{y} - y)\| \leq t\|(\tilde{x} - x\| + (1-t)\|\tilde{y} - y)\| < tr + (1-t)r = r$$

next is to show $\exists r_0 > 0$ s.t. $A(r_0) \bigcap B(r_0) = \Phi$.

suppose not, $\exists (x_n) \subset A$ and $\exists (y_n) \subset B$ s.t. $\|x_n - y_n\| \to 0$. since B is compact, $\exists (y_{n_k})$ converges, $y_{n_k} \to y \in B, k \to \infty$. $x_{n_k} = y_{n_k} + x_{n_k} - y_{n_k} \to y \in B$. since A is closed and $y \in A$, $y \in A \cap B$ contradicts with $A \cap B = \Phi$

$\exists f \in X'$ s.t.

$$f(\tilde{x} \leq \gamma \leq f(\tilde{y}) \forall x \in A(r_0), y \in B(r_0)$$

$$f(x+v) = f(\tilde{x}) \leq \gamma \leq f(\tilde{y}) = f(y+w), \|v\|, \|w\| \leq r_0$$

$$f(x) + f(v) \leq \gamma \leq f(y) + f(w)$$

□

4.3 Banach-Steinhaus theorem

also called uniform boundedness theorem, derived by Baire's category theorem.

Theorem 4.21 (Baire's Category theorem)**.** *let X be a complete metric space, then the following statements holds:*

1. *if (F_n) is a sequence of closed subsets with int$F_n = \phi$, then int$(\cup_{n=1}^{\infty} F_n) = \phi$*
2. *if (O_n) is a sequence of open subsets of X with $\overline{O}_n = X, \forall n$, then $\overline{\cap_{n=1}^{\infty} O_n} = X$*

Proof. 1. first show the two statements are equivalent:
(1.→ 2.): let (O_n) be a sequence of open subsets s.t. $\overline{O_n} = X, \forall n$. note that $\overline{A} = (\text{int}A^c)^c$. then we have

$$\overline{A} = X \leftrightarrow \text{int}(A^c) = \phi$$

let $A = \cap_{n=1}^{\infty} O_n$, $(\cap_{n=1}^{\infty} O_n)^c = \cup_{n=1}^{\infty} O_n^c = \cup_{n=1}^{\infty} F_n$ by letting $F_n = O_n^c$ closed.

$$\text{int}F_n = \text{int}O_n^c = (\overline{O}_n)^c = X^c = \phi$$

(2.→ 1.): suppose int$\cup_{n=1}^{\infty} F_n = \phi$, int$(\cap_{n=1}^{\infty} O_n)^c = \phi \leftrightarrow \overline{\cap_{n=1}^{\infty} O_n} = X$ by taking $A = (\cap_{n=1}^{\infty} O_n)$.

2. prove 2.: let (O_n) be a sequence of open susbets of X with $\overline{O_n} = X, \forall n$, want to show that $O := \cap_{n=1}^{\infty} O_n$ is dense in X.

let G be any nonempty open subsets of X, want to show $O \cap G \neq \phi$: $\overline{O_1} = X \to O_1 \cap G \neq \phi \to \exists x_0 \in X, \exists r_0 > 0$ s.t. $\overline{B(x_0, r_0)} \subset O_1 \cap G$; $\overline{O_2} = X \to O_2 \cap B(x_0, r_0) \neq \phi \to \exists x_1 \in X, \exists r_1 > 0$ s.t. $\overline{B(x_1, r_1)} \subset O_2 \cap B(x_0, r_0)$, continue the process to O_n: $\exists x_n \in X, r_n > 0$ s.t.

$$x_n \in \overline{B(x_n, r_n)} \subset O_n \cap B(x_n, r_n), \forall n \geq 1$$

let $n \in \mathbb{N}$ be fixed, $x_{n_p} \in B(x_n, r_n), \forall p \geq 1$, $d(x_m, x_n) \leq 2r_n < 2\frac{r_0}{2^n}, \forall m \geq n$, $\forall \epsilon > 0, \exists N > 0$ s.t. $d(x_m, x_n) < \epsilon, \forall n, m \geq N$, a Cauchy sequence.

Since X is complete, $x_n \to x \in X$. claim that $x \in O \cap G$: $\{x_{n+p}\}_{p=1}^{\infty} \in B(x_n, r_n) \subset O_n \cap G$ fixed n, let $p \to \infty$, $x \in \overline{B(x_n, r_n)} \subset G$. we have $x \in O_n, \forall n \to x \in \cap_{n=1}^{\infty} O_n \to x \in G$. we proved that $x \in O \cap G \neq \phi$

□

Theorem 4.22 (Baire's)**.** *let X be a metric space and let (F_n) be a sequence of closed subsets of X s.t. $X = \cup_{n=1}^{\infty} F_n$*

1. *if interior $intF_n = \phi$, $\forall n \in N$ then X is not complete*
2. *if X is complete, then $\exists n_0 \in \mathbb{N}$ s.t. $intF_n \neq \phi$*

Example 4.23. *show that there exists a function in $C[a,b]$ that is nowhere differentiable.*

Example 4.24. *Q is not complete since $Q = \cup_{n=1}^{\infty}\{x_n\}$ and let $\{x_n\} = F_n$*

Example 4.25. *if X is ∞-dimensional Banach space then any Hamel basis is uncountable.*

Proof. suppose for a contradiction that $(e_j)_{j=1}^{\infty}$ is a Hamel basis, $F_n = \text{span}(e_j)_{j=1}^{n}$ is closed. $X = \cup_{n=1}^{\infty} F_n$, $\text{int}F_n = \phi, \forall n$, $\forall x \in F_n, x = \sum_{j=1}^{n} \alpha_j e_j$, $\forall \epsilon > 0$, $x + \frac{\epsilon}{2}\frac{e_{n+1}}{\|e_{n+1}\|} \notin F_n$, which is impossible since X is a Banach space. □

Theorem 4.26 (Banach-Steinhaus theorem)**.** *let X be a banach space and Y be normed vector space. let $(T_i)_{i\in I}$ be a family of operators with $T_i \in \mathcal{L}(X,Y), \forall i \in I$ and $\sup_{i\in I}\|T_i x\| < \infty, \forall x \in X$, then $\sup_{i\in I}\|T_i\| < \infty$*

Proof. for each $n \in \mathbb{N}$, let $F_n = \{x \in X | \|T_i x\| \leq n; \forall i \in I\} = \cap_{i\in I}\{x \in X | \|T_i x\| \leq n\} = \cap_{i\in I} T_i^{-1}(\overline{B(0,n)})$ closed since each T_i is continuous. by Baire's theorem, $X = \cup_{i=1}^{\infty} F_n$, $\forall x \in X$, $\exists n_0$ s.t. $IntF_{n_0} \neq \phi$ $\exists x_0 \in X$, $\exists r > 0$ s.t. $\overline{B(x_0,r)} \subset F_{n_0}$, $x_0 + rz \in \overline{B(x_0,r)} \subset F_{n_0}$, $\|z\| \leq 1$, $r\|T_i z\| - \|T_i x_0\| \leq \|T_i x_0 + rT_i z\| = \|T_i(x_0 + rz)\| \leq n_0, \forall i \in I, \forall \|z\| \leq 1$.

$\|T_i\| = \sup_{\|z\|\leq 1}\|T_i z\| \leq \frac{n_0 + \|T_i x_0\|}{r} \Rightarrow \sup_{i\in I}\|T_i\| \leq \sup_{i\in I}\frac{n_0+\|T_i x_0\|}{r} = \frac{n_0}{r} + \frac{1}{r}\sup_{i\in I}\|T_i x_0\| < \infty$ □

Remark 4.27. *the reverse of the theorem is true obviously: given $\sup_{i\in I}\|T_i\| < \infty$, $\sup_{i\in I}\|T_i x\|_Y \leq \|x\| \sup_{i\in I}\|T_i\| < \infty, \forall x$*

Theorem 4.28 (Banach-Steinhouse theorem, another version)**.** *let X be Banach, Y be a normed vector space, let $(T_n)_{n=1}^{\infty}$ be a sequence, $T_n \in \mathcal{L}(X,Y)$. if $\forall x \in X$, $(T_n x)$ converges. then $\sup_{n\geq 1}\|T_n\| < \infty$. Besides if we write $Tx = \lim_{n\to\infty} T_n x$, then $T \in \mathcal{L}(X,Y)$ and $\|T\| \leq \liminf_{n\to\infty}\|T_n\|$*

Proof. $\forall x \in X$, $(T_n x)_{n=1}^{\infty}$ converges, then $(T_n x)$ are bounded. $\sup_{n\geq 1}\|T_n x\| < \infty$. by Banach-Steinhouse theorem, $\sup_{n\geq 1}\|T_n\| < \infty$.

show that T is linear: $T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty}(\alpha_1 T_n x_1 + \alpha_2 T_n x_2) = \alpha_1 \lim_{n\to\infty} T_n x_1 + \alpha_2 \lim_{n\to\infty} T_n x_2 = \alpha_1 T x_1 + \alpha_2 T x_2, \forall x_1, x_2 \in X, \forall \alpha_1, \alpha_2 \in \mathbb{K}$.

$\forall x \neq 0$, $\frac{\|Tx\|}{\|x\|} = \lim_{n\to\infty}\frac{\|T_n x\|}{\|x\|} = \liminf_{n\to\infty}\frac{\|T_n x\|}{\|x\|} \leq \lim_{n\to\infty}\|T_n\|, \forall x \neq 0 \to \|T\| = \sup_{x\in X, x\neq 0}\frac{\|Tx\|}{\|x\|} \leq \liminf_{n\to\infty}\|T_n\|$ □

Remark 4.29. *this theorem does not imply the pointwise convergence: $\lim_{n\to\infty} T_n x \to Tx$.*

if the convergence holds, $\|T_n\| \to \|T\|$, which is stronger than the conclusion of the theorem, saying that $\|T\| \leq \liminf_{n\to\infty}\|T_n\|$

the following theorem is a corollary of Banach-Steinhouse theorem:

Theorem 4.30 (Polya theorem)**.** *$f \in C[0,1]$, $\ell(f) := \int_0^1 f(x)dx$, $\ell_n = \sum_{k=0}^n w_k f(x_k)$, $\ell_1, \ell_n \in (C[0,1])'$, then $\ell_n(p) \to \ell(p) \forall p \in P[0,1]$ implies that $\ell_n(f) \to \ell(f) \forall f \in C[0,1]$ iff $\sum_{k=0}^\infty |w_k| < \infty$*

Proof. $(\to)$: $\ell(f) = \int_0^1 f(x)dx, C[0,1] \to \mathbb{R}$, and $\ell_n(f) = \sum_{k=0}^n w_k f(x_k)$, $\|\ell_n\| = \sum_{k=1}^n |w_k|$, $|\ell_n(f)| = |\sum_{k=0}^n w_k f(x_k)| \leq \sum_{k=0}^n |w_k||f(x_k)| \leq \|f\| \sum_{k=0}^n |w_k| \to \|\ell_n\| \leq \sum_{k=0}^n |w_k|$

find $f_0 \in C[0,1]$ s.t. $\|f_0\| = 1$ and $\sum_{k=0}^n w_k f_0(w_k) = \ell_n(f_0) = \sum_{k=0}^n |w_k|$ by defining $f_0(x_k) = sgn(w_k) \to \|\ell_n\| \geq \sum_{k=0}^n |w_k|$, $\forall f \in X, \ell_0(f) \to \ell(f)$, by Banach-Steinhouse theorem, we have $\sup_{n\geq 1} \|\ell_n\| < \infty, \|\ell\| \leq \liminf_{n\to\infty} \|\ell_n\| < \infty$ □

4.4 Banach open mapping theorem

Definition 4.31 (continuous map; open map)**.** *$T : X \to Y$, T is continuous if $\forall O \subset Y$ oepn $f^{-1}(O)$ is open in X.*

T is an open map if $\forall U \subset X$ open, $T(U)$ is open in Y.

Example 4.32. *$f\mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous but not open; $U = (-1,1)$ is open in $X = \mathbb{R}$, but $f(U) = [0,1)$ is not open.*

Theorem 4.33 (Banach open mapping theorem)**.** *let X, Y be banach spaces, $T : X \to Y$ is subjective. $T \in \mathcal{L}(X,Y)$. Then T is an open mapping: $\forall U \subset X$ open, $T(U)$ is open in Y*

Proof. first define some notations: $B_r = B(0,r)$ is the open ball centered at 0 radius r in X. similarly $B(y,r)$. $x_0 + A = \{x_0 + x | x \in A\}$, $\alpha A = \{\alpha x | x \in A\}$

1. want to show $\overline{T(B_1)}$ contains an open ball in Y.

   $Y = T(X) = T(\cup_{n=1}^\infty B_n) = \cup_{n=1}^\infty T(B_n) \subset \cup_{n=1}^\infty \overline{T(B_n)} \subset Y$. $Y = \cup_{n=1}^\infty \overline{T(B_n)}$, by Baire's theorem, we have: $\exists n_0 \in \mathbb{N}$ s.t. $Int\overline{T(B_{n_0})} \neq \phi \to \exists y \in Y, \exists \delta > 0$ s.t. $B(y, 2\delta) \subset \overline{T(B_1)} = \frac{1}{n} Int\overline{T(B_{n_0})} \neq \phi$

2. want to show $\overline{B(_1)}$ contains an open ball centered at 0. indeed: $B(0,\delta) \subset \overline{T(B_1)}$.

   $B(0,2\delta) = 2B(0,\delta) = -y + B(y,2\delta) \subset -y + \overline{T(B_1)} \subset 2\overline{T(B_1)} \to B(0,\delta) \subset \overline{T(B_1)}$. so $y \in B(y,2\delta) \subset \overline{T(B_1)} \to -y \in \overline{T(B_1)}$, $\forall \tilde{y} \in \overline{T(B_1)}$, want to show $-y + \tilde{y} \in 2\overline{T(B_1)}$, which is true: $\exists x_n \in B_1$, $\tilde{x_n} \in B_1$ s.t. $T(-x_n) = -y_n \to y$ and $T\tilde{x_n} = \tilde{y_n} \to \tilde{y}$. $T(\frac{-x_n + \tilde{x_n}}{2}) = \frac{-y_n + \tilde{y_n}}{2} \to \frac{-y+\tilde{y}}{2}$ and $\frac{-x_n + \tilde{x_n}}{2} \in B_1$, $\frac{-y_n + \tilde{y_n}}{2} \in T(B_1)$, $\frac{-y+\tilde{y}}{2} \in \overline{T(B_1)}$

3. $T(B_1)$ contains an open ball centered at 0: want to show $B(0,\frac{r}{2}) \subset T(B_1)$, $\forall y \in B(0,\frac{r}{2})$, $\exists x \in B_1$ s.t. $Tx = y$

   let $y \in B(0,\frac{\delta}{2}) \subset \overline{T(B_{1/2})}$, $\exists y_1 \in T(B_1)$ s.t. $\|y_1 - y\| < \frac{\delta}{2^2}$. $\exists x_1 \in B_{1/2}$ s.t. $\|y - Tx_1\| \leq \frac{\delta}{2^2}$. $y - Tx_1 \in B(0,\frac{\delta}{2^2}) \subset \overline{T(B_{1/2^2})}$, $\exists y_2 \in T(B_{1/2^2})$ s.t. $\|y - Tx_1 - y_2\| < \frac{\delta}{2^3}$. $\exists x_2 \in B_{1/2^2}$ s.t. $\|y - Tx_1 - Tx_2\| < \frac{\delta}{2^3}$.

$\exists (x_n)_{n=1}^{\infty}$ in X s.t. $x_n \in B_{\frac{1}{2^n}}$ i.e. $\|x_n\| < \frac{1}{2^n}$ and $\|y - \sum_{k=1}^{n} Tx_k\| < \frac{\delta}{2^{n+1}}$. then $\sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. $\sum_{n=1}^{\infty} x_n$ converges, given X is complete, we have $\sum_{n=1}^{\infty} x_n \to x \in X$. $\|x\| = \sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\| < 1 \to x \in B_1$.

$Tx \leftarrow T(\sum_{k=1}^{n} x_k) = \sum_{k=1}^{n} Tx_k \to y$, this implies that $y = Tx$

4. let $U \subset X$ be open in X, want to show $\forall y \in T(U)$, $\exists \epsilon > 0$ s.t. $B(y, \epsilon) \subset T(U)$.

   $\forall y \in T(U)$. $\exists x \in U$ s.t. $y = Tx$, $\exists r > 0$ s.t. $x + B(0, r) = B(x, r) \subset U$.

   $$T(U) \supset T(x+B_r) = T(x)+T(B_r) = y+rT(B_1) \supset y+rB(0, \frac{\delta}{2}) = B(y, \frac{r\delta}{2})$$

   by taking $\epsilon = \frac{r\delta}{2}$. This finishes the whole proof.

□

one application of the theorem is the following theorem:

Theorem 4.34 (bounded inverse theorem). *let X, Y be two banach space, $T : X \to Y$ is bijective $T \in \mathcal{L}(X, Y)$. then $T^{-1} : Y \to X$ is bounded linear, i.e. $T^{-1} \in \mathcal{L}(Y, X)$*

Proof. T^{-1} is continuous $\leftrightarrow$ T is open: $\forall U \subset X$ open, $(T^{-1})^{-1}(U) = T(U)$ is open in Y. By the open mapping theorem, we know that T^{-1} is continuous. then ? □

Theorem 4.35 (equivalent norm theorem). *let X be an normed vector space, and X is complete under both $\|\cdot\|_1$ and $\|\cdot\|_2$. if $\exists$ $c_1 > 0$ s.t. $\|x\|_1 \leq c_1\|x\|_2, \forall x \in X$. then $\exists c_2 > 0$ s.t. $c_2\|x\|_2 \leq \|x\|_1 \leq c_1\|x\|_2, \forall x \in X$*

Proof. $t : (X, \|\|_2) \to (X, \|\|_1)$: $tx = x$, then we have $\|tx\|_1 = \|x\|_1 \leq c_1\|x\|_2, \forall x \in X$. t is bounded linear subjective.

$c^{-1} : (X, \|\|_1) \to (X, \|\|_1)$ is bounded linear, $\exists \tilde{c_2} > 0$ s.t. $\|x\|_2 = \|t^{-1}x\|_2 \leq \tilde{c_2}\|x\|_1, \forall x \in X$ □

4.5 Banach closed graph theorem

Definition 4.36 (graph; closed operator). *given two sets X and Y, the graph GrT of a mapping $T : X \to Y$ is the subset of the product $X \times Y$ defined by*

$$GrT = \{(x, Tx) \in X \times Y | x \in X\}$$

if X and Y are topological space, a mapping $T : X \to Y$ is called closed if its graph GrT is closed in the product space $X \times Y$ equipped with the product topology.

Lemma 4.37. *let X and Y be metric space, then a mapping $T : X \to Y$ is closed if and only if*

$$\lim_{n\to\infty} x_n = x \quad \lim_{n\to\infty} Tx_n = y \quad \textit{implies} \quad y = Tx$$

Theorem 4.38 (Banach closed graph theorem)**.** *let X and Y be Banach spaces, and let T is continuous*

Proof. □

Theorem 4.39 (Hellinger-Toeplitz)**.** *let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $T : X \to X$ be a self-adjoint linear operator, i.e., that satisfies*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in X$$

Then, T is continuous.

Remark 4.40.

- *if X is only an inner-product space, the above proof shows that the linear operator T is closed.*
- *it is easily seen that any mapping $T : X \to X$ that satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in X$ is automatically linear.*

4.6 dual operators; Banach closed range theorem

Theorem 4.41. *let X and Y be two normed vector spaces over the same number field $\mathbb{K}$. given any $T \in \mathcal{L}(X, Y)$, there exists one and only one operator $T' \in \mathcal{L}(Y', X')$, called the dual operator of T, or simply the dual of T, s.t.*

$$T'y'(x) = y'(Tx) \quad \forall x \in X, \forall y' \in Y'$$

Besides,

$$\|T'\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$$

Theorem 4.42. *Let X and Y be two normed vector spaces and let $T : X \to Y$ be a compact linear operator. then the dual operator $T' : Y' \to X'$ is also compact.*

Theorem 4.43. *let X and Y be two normed vector spaces over the same field $\mathbb{K}$ and let $T \in \mathcal{L}(X, Y)$. Then the following two conditions are equivalent:*

- *the operator T has a dense range, i.e. $\overline{R(T)} = Y$*
- *the dual operator T' is injective, i.e., $N(T') = \{0\}$*

Theorem 4.44. *Let X and Y be two Banach spaces and let $T \in \mathcal{L}(X, Y)$, then T is surjective if and only if there exists a constant $C > 0$ s.t.*

$$\|y'\| \leq C\|T'y'\| \quad \forall y' \in Y'$$

Theorem 4.29 (Banach Closed Range Theorem; first part 1932) Let X and Y be two Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then the following four conditions are equivalent: (a) The operator $T : X \to Y$ has a closed range, i.e. $T(X)$ is closed in Y. (b) The dual operator $T' : Y' \to X'$ has a closed range, i.e., $T'(Y')$ is closed in X'. (c) $T(X) = {}^0(\ker T') := \{y \in Y \mid y'(y) = 0 \forall y' \in \ker T'\}$. (d) $T'(Y') = (\ker T)^0 := \{x' \in X' \mid x'(x) = 0 \quad \forall x \in \ker T\}$.

Proof. The proof requires the Banach open mapping theorem and Hahn-Banach theorem in a normed vector space. See [Ciarlet, pp. 282-285]. □

Theorem 4.45 (Banach Closed Range Theorem; second part 1932)**.** *Let X and Y be two Banach spaces and let $T \in \mathcal{L}(X,Y)$. Then the following three conditions are equivalent:*

(a) The operator $T : X \to Y$ is surjective, i.e. $T(X) = Y$.

(b) There exists a constant C such that the dual operator $T' : Y' \to X'$ satisfies

$$\|y'\| \le C\,\|T'y'\| \quad \forall y' \in Y'.$$

(c) The dual operator T' is injective and $T'(Y')$ is closed in X'.

Condition (b) in the above theorem is sometimes put to use in the analysis of linear boundary value problems. For, it asserts that in order to establish the existence of a solution x to a partial differential equation, written symbolically as $Tx = y$ (here, X and Y are ad hoc function spaces with X incorporating some boundary conditions, $T : X \to Y$ is a partial differential operator, and y is the right-hand side of the equation), it suffices to have an a priori bound on any given solution y' of the dual equation $A'y' = x'$, in the form $\|y'\| \le C\,\|x'\|$ for some constant C independent of x'. What is particularly remarkable is that there is no need to verify that the dual equation possesses solutions. This observation therefore provides a powerful technique for establishing the existence of solutions to such problems.

5 Weak Convergence

let X be a normed vector space, (x_n) is a sequence in X and $x_n \to x$ as $n \to \infty$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. this is called strong convergence.

Definition 5.1 (weak convergence)**.** *(x_n) is called weakly converges to x in X if $f(x_n) \to f(x)$ as $n \to \infty$, $\forall f \in X'$. x is called a weak limit of (x_n). we write $x_n \rightharpoonup x$ as $n \to \infty$*

Example 5.2. *let X be a hilbert space of ∞ dimension, $\forall f \in X'$, $\exists y_f \in X$ s.t. $f(x) = \langle x, y_f\rangle, \forall x \in X$.*

$x_n \rightharpoonup x$ iff $\langle x, y_n\rangle \to \langle x, y\rangle, \forall y \in X$. suppose $(e_n)_{n=1}^{\infty}$ is a orthonormal sequence. by Bessel's inequality, $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2 \to |\langle x, e_n|^2 \to 0 \Leftrightarrow \langle e_n, x\rangle = \overline{\langle x, e_n\rangle} \to 0 = \langle e_n, 0\rangle$, for all $x \in X$. $e_n \rightharpoonup 0$ as $n \to \infty$

$\|e_n - e_m\| = \sqrt{2} \to \|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 2 \to e_n$ does not converge strongly.

Theorem 5.3 (weak and strong convergence)**.** *let X be a normed vector space, then*

1. *$x_n \to x$ as $n \to \infty \Rightarrow x_n \rightharpoonup x$ as $n \to \infty$*
2. *if $dimX < \infty$, then $x_n \rightharpoonup x \Rightarrow x_n \to x$*

Proof. only prove second part: let $\dim X = k$ and let $(e_j)_{j=1}^{k}$ be a basis of X. write $x = \sum_{j=1}^{k} \xi_j e_j$, $x_n = \sum_{j=1}^{k} \xi_j^{(n)} e_j$. assume $x_n \rightharpoonup x$. define $f_j(x) = \xi_j, j = 1, 2, ..., k$ linear and bounded. $\exists c > 0$, $|f_j(x)| = |\xi_j| \le \|x\| \le c\|x\|, \forall x \in X$

$\xi_j^{(n)} = f_j(x_n) \to f_j(x) = \xi \Rightarrow \xi_j^{(n)} \to \xi_j$ $j = 1, 2, ..., k$. which implies that $\|x_n - x\| = \|\sum_{j=1}^{k}(\xi_j^{(n)} - \xi_j)e_j\| \le \sum_{j=1}^{k} |\xi_j^{(n)} = \xi_j|\|e_j\| \to 0$ as $n \to \infty$, so we have $x_n \to x$ as $n \to \infty$ □

Theorem 5.4 (weak convergence)**.** *let X be a normed vector space,*

1. *any weakly convergent sequence has a unique weak limit*
2. *any weakly convergent sequence is bounded.*
3. *$x_n \rightharpoonup x$, then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$. here $\|\cdot\|$ is sublinear weakly linear semicontinuous*

Proof. assume $x_n \rightharpoonup x$ and $x_n \rightharpoonup \tilde{x}$. $f(x_n) \to f(x)$ and $f(x_n) \to f(\tilde{x}) \to f(x) = f(\tilde{x}) \Leftrightarrow f(x - \tilde{x}) = 0, \forall f \in X' \to x - \tilde{x} = 0$, by a corollary of Hahn-Banach theorem. $x \in X, x \neq 0, \exists f_x \in X$ s.t. $f_x(x) = \|x\| \neq 0, \|f_x\| = 1 \to f(x) = 0, \forall f \in X'$, then $x = 0$, $\|x\| = \sup_{\substack{f\in X'\\ f\neq 0}} \frac{|f(x)|}{\|x\|}$

assume $x_n \to x$. define $J_n := J_{x_n} : X' \to \mathbb{K}$. $J_n(f) = f(x_n)$. we can prove that J_n is linear and bounded: $\|J_n\| = \sup_{\substack{f\in X'\\ f\neq 0}} \frac{J_n(f)|}{\|f\|} = \sup_{\substack{f\in X'\\ f\neq 0}} \frac{|f(x_n)|}{\|f\|} = \|x_n\|$

$$J_n \in \mathcal{L}(X', \mathbb{K}) = (X')' = X''$$

X'' is called bidual or second-dual of X. $x_n \rightharpoonup x \Rightarrow f(x_n) = J_n(f) \to f(x), \forall f \in X'$. $J_n(f)$ converges $\forall f \in X'$. define $J_x : X' \to \mathbb{K}$. $J_x(f) = f(x)$, $\forall f \in X'$. $J_x \in (X')'$ and $\|J_x\| = \|x\|$. by banach-steinholse theorem, saying that X is Banach space and Y is normed vector space, (T_n) is such that $T_n \in \mathcal{L}(X,Y), T \in \mathcal{L}(X,Y)$, $t_N X \to Tx, \forall x \in X$. then $sup_{n\ge 1}\|T_n\| < \infty$, $\|T\| \le \liminf_{n\to\infty} \|T_n\|$

since X' is complete, B-S theorem applies here: $J_n(f) \to J_x(f)$ with $\sup_{n\ge 1} \|J_n\| < \infty \Leftrightarrow \sup_{n\ge 1} \|x_n\| < \infty$ and $\|J_x\| \le \liminf_{n\to\infty} \|J_n\| \Leftrightarrow \|x\| \le \liminf_{n\to\infty} \|x_n\|$ □

Definition 5.5 (strictly convex; uniform convex)**.** *strictly convex: $\forall x, y$ with $\|x\| = \|y\| = 1$, then $\|\frac{x+y}{2}\| < 1$*

uniform convex: $\exists \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ s.t. $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon \Rightarrow \|\frac{x+y}{2}\| < 1 - \delta(x)$

any Hilbert space is uniform convex. $\ell^p, (1 < p < \infty)$ is also uniform convex.

Theorem 5.6. *let X be a uniformly convex normed vector space, assume $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$*

Proof. if $x = 0$, clearly true

assume $x \neq 0$, $\exists N > 0$ s.t. $\|x_n\| > 0, \forall n \ge N$. let $y_n := \frac{x_n}{\|x_n\|}$, $y = \frac{x}{\|x\|}, \forall n \ge N$, $\forall f \in X'$, $f(y_n) = f(\frac{x_n}{\|x_n\|}) = \frac{1}{\|x_n\|} f(x_n) \to \frac{1}{\|x\|} f(x) = f(\frac{x}{\|x_n\|}) = f(y) \Rightarrow y_n \to y$

$y_n + y \rightharpoonup y + y = 2y$, $2 = \|2y\| \le \liminf_{n\to\infty} \|y_n + y\| \le \limsup_{n\to\infty} \|y_n + y\| \le 2$. $\|y_n + y\| \le \|y_n\| + \|y\| = 2, \forall n > N \to \lim_{n\to\infty} \|\frac{y_n+y}{2}\| = 1$. claim that $y_n \to y \Leftrightarrow \|y_n - y\| \to 0$ for otherwise, $\exists \epsilon > 0, \exists (y_{n_k})$ s.t. $\|y_{n_k} - y\| \ge \epsilon$. $\|y_{n_k}\| = \|y\| = 1$, $\exists \delta(\epsilon) > 0$ s.t. $\|\frac{y_{n_k}+y}{2}\| < 1 - \delta(\epsilon)$, a contradiction.

$\|x_n - x\| = \|\|x_n\|y_n - \|x\|y\| \le \|x_n\| \cdot \|y - y_n\| + (\|x_n\| - \|x\|)\|y\| \to 0$ □

Theorem 5.7. *assume X, Y are normed vector space $x_n \rightharpoonup x$ in X. then*

(a): if $T \in \mathcal{L}(X,Y)$, then $Tx_n \rightharpoonup Tx$

(b): if $T \in \mathcal{L}(X,Y)$, T is a compact operator, then $Tx_n \to Tx$; $T : X \to Y$ is compact if $\forall (x_n)$ bounded, $\exists (Tx_{n_k})_{k=1}^{\infty}$ s.t. Tx_{n_k} converges.

Proof. (a):

(b): □

5.1 Mazur theorem

Definition 5.8 (convex hull). *given a subset A of a vector space X, the convex hull of A, denoted coA, is the intersection of all the convex subsets of X that contain A, or equivalently, the smallest convex subset of X that contains A.*

Theorem 5.9. *let A be a subset of a vector space X. then the convex hull of A is also the subset of X formed by all convex combinations of elements of A, i.e.*

$$coA = \bigcup_{n=1}^{\infty}\{\sum_{k=1}^{n}\lambda_k x_k | x_k \in A, \lambda \geq 0, \forall 1 \leq k \leq n, \sum_{k=1}^{\infty}\lambda_k = 1\}$$

Theorem 5.10 (Mazur). *let X be a real normed vector space and let $x_n \rightharpoonup x$ in X. then for each $n \geq 1$, there exists $\lambda_k^n \geq 1$, $k = 1, ..., n$, with $\sum_{k=1}^{n}\lambda_k^n x_k \to x$ as $n \to \infty$*

5.2 reflexive spaces. weak* convergence. Eberlein-Šmulian theorem

Definition 5.11 (bidual). *let X be a normed vector space. then $X'' = (X')'$ denote the bidual space of X, i.e., the dual space of the dual space of X. as a dual space, the space X'' is thus a Banach space, with the norm of any element $x'' \in X''$ being given by*

$$\|x''\| = \sup_{\substack{x' \in X' \\ x' \neq 0}} \frac{|x''(x')|}{\|x'\|}$$

Theorem 5.12 (Canonical isometry). *let X be a normed vector space. then the mapping*

$$J : X \to X''$$

defined for each $x \in X$ by

$$Jx(f) = f(x) \quad \forall f \in X'$$

is a linear isometry, called the canonical isometry from X into X''

Definition 5.13 (reflexive space). *a normed vector space X is reflexive if the canonical isometry $J : X \to X''$ defined in the above theorem (canonical isometry) is surjective, thus allowing us to identify X with its bidual space X''. in other words, X is reflexive if, given any $x'' \in X''$, there exists a unique $x \in X$ s.t.*

$$x''(f) = f(x) \quad \forall f \in X'$$

Example 5.14.

- *any Hilbert space is reflexive*
- *any finite dimensional normed vector space is reflexive*
- *the spaces $\ell^[$, $1 < p < \infty$ are reflexive.*

- *the lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$ with Ω any open subset of $\mathbb{R}^n$ are reflexive.*
- *the space $\ell^1, \ell^\infty, L^1(\Omega)$ and $L^\infty(\Omega)$ with Ω any open subset of $\mathbb{R}^n$, are note reflexive.*

Theorem 5.15. *any closed subspace of a reflexive space is reflexive*

Theorem 5.16. *if a normed vector space X is reflexive, then X' is reflexive.*

Theorem 5.17. *a banach space X is reflexive if and only if its dual X' is reflexive.*

Theorem 5.18 (Milman-Pettis)**.** *any uniformly convex Banach space is reflexive*

Definition 5.19 (Weak* convergence)**.** *let X be a normed vectors space. a sequence (f_n) in X' is said to weakly* converge in X' if there exists a $f \in X'$ such that*

$$f_n(x) \to f(x) \quad \forall x \in X$$

and such that an f, which is clearly unique, is called the weak limit of the sequence (f_n). in this case, we write*

$$f_n \overset{*}{\rightharpoonup} f \quad \textit{as } n \to \infty$$

on the dual space X' we have three convergences:

- strong convergence: $f_n \to f$ if $\|f_n - f\| \to 0$
- weak convergence: $f_n \rightharpoonup f$ if $x''(f_n) \to x''(f)$ for all $x'' \in X''$
- weak* convergence $f_n \overset{*}{\rightharpoonup} f$ if $f_n(x) \to f(x)$ for all $x \in X$

  we have
  $$f_n \to f \Rightarrow f_n \rightharpoonup f \Rightarrow f_n \overset{*}{\rightharpoonup} f$$
  since the canonical isometry $j : X \to X''$ is injective.

  if X is reflexive, then $f_n \rightharpoonup f \Leftrightarrow f_n \overset{*}{\to} f$

Theorem 5.20 (Eberlein)**.** *let X be a normed vector space. any bounded sequence (f_n) in X' contains a weakly* convergent subsequence*

Theorem 5.21 (Eberlein-Smulian)**.** *let X be a Banach space. Any bounded sequence (x_n) in X contains a weakly convergent subsequence if and only if X is reflexive.*

6 Spectral theory

let $\sigma(A) = \{|\lambda| : \lambda \text{ is an eigenvalue of matrix} A\}$, we say A and B are similar if $\exists P$ invertible s.t. $A = P^{-1}BP$ and write $A \sim B$. eigenvalues are invariant under similar transformations:

$$det(A-\lambda I) = det(P^{-1}BP-\lambda P^{-1}IP) = det(P^{-1}(B-\lambda I)P) = det(P^{-1})det(B-\lambda I)det(P) = det(B-\lambda I)$$

6.1 linear operators

let X be a normed vector space, $T : D(T) \to X$, $D(t) \subset X$ is linear. write $T_\lambda = T - \lambda I$ and $R_\lambda = T_\lambda^{-1} = (T - \lambda I)^{-1}$. write $D(R_\lambda) = R(T_\lambda)$, if R_λ exists, T_λ is injective. if R_λ does not exist, T_λ is not injective $\Leftrightarrow \mathcal{N}(T) \neq \{0\} \Leftrightarrow \exists x \neq 0$ s.t. $T_\lambda x = 0 \leftrightarrow x = \lambda x$.

if R_λ exists:

- if $\overline{D(R_\lambda)} \neq X$, λ is called a residual value. $\sigma_r(T) = \{]lam \in C\}$
- if $\overline{D(R_\lambda)} = X$, R_λ is not bounded, $\lambda \in \sigma_c(T) = \{\lambda \in \mathbb{C} | \overline{D(B_\lambda) = X}, but R_\lambda$ is unbounded$\}$
- if $\overline{D(R_\lambda)} = X$, R_λ is bounded, by BLT, $D(R_\lambda) = X$, $\lambda \in \rho(T) = \{\lambda \in \mathbb{C} | D(R_\lambda = X, R_\lambda$ is bounded$)\}$

Example 6.1. $x = \ell^2$, $T : x = (\xi_1, \xi_2, ...) \in \ell^2 \to (0, \xi_1, \xi_2, ...)$. $D(T) = \ell^2, T \in \mathcal{L}(X)$. $\|Tx\| = (0^2 + \xi_1^2 + \xi_2^2 + ...)^{1/2} = \sum_{j=1}^\infty \xi_j^2 = \|x\|, \forall x \in \ell^2$, $\|T\| = 1$.

if $\lambda = 0$:, $R_0 = T^{-1} = \overline{D(R_0)} = \overline{R(T)} \subset \ell^2$. $0 \notin \sigma_p(T)$ *and* $0 \in \sigma_r(T)$

if $\lambda \neq 0$: $\lambda \xi_1 = 0$, $\lambda \xi_2 = \xi_1, ... \to \xi_1 = 0$, $\xi_2 = \xi_j = 0 \to x = 0$, $\lambda \in \sigma_p(T), \forall \lambda$, $\sigma_p(T) = \phi$

6.2 Bounded linear operator

let X be a Banach space, $T \in \mathcal{L}(X)$.

Lemma 6.2 (Neumann series)**.** *X is banach space, $T \in \mathcal{L}(X)$ with $\|T\| < 1$. then* $\sum_{k=0}^\infty T^k = I + T + T^2 + ... = (I - T)^{-1}$

Proof. $T, S \in \mathcal{X}$ $X \xrightarrow{S} X \xrightarrow{T} X$. $\|TSx\| \le \|T\|\|Sx\| \le \|T\|\|S\|\|x\|, \forall x \in X$. $\|TS\| \le \|T\|\|S\|$

$\|T\| = r < 1$, $\|T^k\| \le \|T\|^k < 1, \forall k \in \mathbb{N}$, $\sum_{k=0}^\infty \|T^k\| \le \sum_{k=1}^\infty \|T\|^k = \sum_{k=1}^\infty r^k$ and $T^k \in \mathcal{X} = \mathcal{X}, \mathcal{X} \Rightarrow \sum_{k=0}^\infty T^k =: S \in \mathcal{L}(X)$. WTS $s = (I - T)^{-1}$

$$(I - T)(I + T + T^2 + ... + T^n) = I - T^{n+1} = (I + T + ...T^n)(I - T)$$

$$(I - T)S = S(I - T) = \lim_{n\to\infty}(I - T^{n+1}) = I \to S = (I - T)^{-1}$$

and $\|T^{n+1} - 0\| = \|T^{n+1}\| \le \|T\|^{n+1} \to 0$ as $n \to \infty$. $\overline{D(R_0)} = \overline{R(T)} \subset \ell^2$, $0 \in \sigma_r(T)$ □

6.3 compact operator

X, Y are normed vector spaces, $T : X \to Y \in \mathcal{L}(X, Y)$, maps any bounded set to a relatively compact set in Y, then T is compact. B is bounded in X, then $\overline{T(B)}$ is compact in Y

$\forall (x_n)$ bounded sequence of X, then (Tx_n) has a compact subsequence.

- $K(X, Y)$ is a vector subspace of $\mathcal{L}(X, Y)$, $\alpha S + \beta T \in K(X, Y)$, $\forall \alpha, \beta \in \mathbb{R}, \forall S, T \in K(X, Y)$.
- if Y is complete, then $K(X, Y)$ is closed subspace in $\mathcal{L}(X, Y)$

- X, Y are banach, $T \in K(X,Y) \Leftrightarrow T' \in K(Y', X')$
- $T \in K(X,Y)$, M is closed in X, then $T|_M$ is also compact. $T : X \to Y$ and $T' : Y' \to X'$, defined by $(Ty')(x) = y'(Tx)$

Example 6.3. *$I : X \to X$, then $dimX < \infty \Rightarrow I \in K(X)$ and $dimX = \infty \Rightarrow I \notin K(X)$.*

a corollary is that: If $dimX = \infty$, $T \in K(X)$, then $T^{-1} \notin \mathcal{L}(X)$. Proved by supposing that $T^{-1} \in \mathcal{L}(X)$, $I = T^{-1}T = TT^{-1} \in K(X) \Leftrightarrow dimX = \infty$

Example 6.4. *$X = \ell^2$, $T : \ell^2 \to \ell^2$, $x = (\xi_j)$, $Tx = (\frac{\xi_j}{j}) = (\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, ...)$, then $T \in K(X)$*

7 HW

7.1 HW1

1. If A is the subspace of ℓ^∞ consisting all sequences of zeros and ones, what is the induced metric on A ?

2. (Hamming distance) Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by $d =$ number of places where x and y have different entries.

3. (Product of metric spaces) The Cartesian product $X = X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2) can be made into a metric space (X, d) in many ways.

   (a) Show that a metric d is defined by
   $$d(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2).$$
   where $x = (x_1, y_1)$ and $y = (y_1, y_2)$.

   (b) Show that another metric on X is defined by
   $$\tilde{d}(x,y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}.$$

   (c) Show that a third metric on X is defined by
   $$\hat{d}(x,y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

4. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.

5. (Equivalent metrics). If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,
   $$ad_1(x,y) \le d_2(x,y) \le bd_1(x,y),$$
   show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same. These two metrics are called equivalent.

6. Let $M \subset \ell^\infty$ be the subspace consisting of all sequences $x = (\xi_j)$ with at most finitely many nonzero terms.

   (a) Find a Cauchy sequence in M which does not converge in M, so that M is not complete.

   (b) Show that M is not complete by Theorem 1.4-7, that is to show M is not closed.

7. Show that the subspace $Y \subset C[a,b]$ is complete, where

$$Y = \{x \in C[a,b] \mid x1. = x2.\}.$$

8. Show that a discrete metric space (cf. example 1.1-8) is complete.

9. (The space s of all sequences are complete).

   (a) Show that in the space s (c. example 1.2-1), we have $x_n \to x$ if and only if $\xi_j^{(n)} \to \xi_j$ for all $j = 1, 2, \ldots$, where $x_n = \left(\xi_j^{(n)}\right)$ and $x = (\xi_j)$.

   (b) Show that the space s is complete.

10. On a metric space (X, d), define a binary relation ' $\sim$ ' on sequences with

$$(x_n) \sim (x_n') \text{ if } \lim_{n\to\infty} d(x_n, x_n') = 0.$$

   1. Show that ' $\sim$ ' is an equivalence relation. 2. If (x_n) is a Cauchy sequence in X and $(x_n) \sim (x_n')$, show that (x_n') is Cauchy in X too.

11. (Closed subset of compact set is still compact.). Let X be a topological space. Let K be a compact (resp. sequentially compact) subset of X and F be a closed subset of K. Then F is also compact (resp. sequentially compact) subset of X.

12. (Open and closed balls, closure).

   (a) For any metric space (X, d), show that $\overline{B_1(x)} \subset \bar{B}_1(x)$. Here is an example showing that why $\overline{B_1(x)} \neq \bar{B}_1(x)$ in general: Let X be a set and d the discrete metric on X.

   (b) Show that every subset of X is open, and hence every subset of X is closed.

   (c) Pick a $x \in X$. What is the open ball $B_1(x)$? what is the closed ball $\bar{B}_1(x)$. Show that $\overline{B_1(x)} \subsetneq \bar{B}_1(x)$.
   Remark. For most examples in applications, we usually have $\overline{B_1(x)} = \bar{B}_1(x)$.

7.2 HW2

1. (a) Show that the sequence (e_k) with $e_k = \left(\xi_j^k\right)_{j-1}^\infty$ and $\xi_j^k = \delta_j^k$ is a Schauder basis of the space $\ell^p (1 \le p < \infty)$.

   (b) Show that (e_k) is also a basis of c_0, where c_0 is the space of all sequences converging to zero.

2. Show that if a normed space has a Schauder basis, it is separable.

3. Let $c_0 \subset \ell^\infty$ be the space of all sequences converging to zero. Show that c_0 is a closed subspace of ℓ^∞, so that c_0 is complete.

4. Let $c \subset \ell^\infty$ be the space of all convergent sequences. Show that c is a closed subspace of ℓ^∞, so that c is complete.

5. Let $Y \subset \ell^\infty$ be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of ℓ^∞ but Y is not a closed subspace.

6. (Complete) $\Longleftrightarrow$ (Absolute convergence implies convergence)

   (a) Let $(X, \|\cdot\|)$ be a normed space. Prove that X is complete if and only if every series $\sum_{j-1}^{\infty} x_j$ in X satisfying $\sum_{j-1}^{\infty} \|x_j\| < \infty$ converges to a limit in X.

   (b) Give an example of a (necessarily incomplete!) space X and a series for which $\sum_{j-1}^{\infty} \|x_j\| < \infty$ but $\sum_{j-1}^{\infty} x_j$ does not converge in X. Hint: Consider Y in Problem 3 and $(y_n) \subset Y$ with $y_n = \left(\eta_j^{(n)}\right), \eta_n^{(n)} = \frac{1}{n^2}, \eta_j^{(n)} = 0$ for all $j \neq n$.

7. This problem culminates in an example of a compact subset in an infinite dimensional normed space. Let (X, d) be a metric space. Recall that (X, d) is called totally bounded if, for any $\varepsilon > 0$, there exists a finite collection of open balls of radius ε whose union contains X.

   (a) Show that a bounded metric space does not have to be totally bounded. (Hint: Consider the discrete metric space).

   (b) Show that a subset K of a complete metric space X is compact if and only if it is closed and totally bounded. (Hint: for $\Longrightarrow$, use the definition that K is compact if every open cover of K contains a finite sub-cover; for $\Longleftarrow$, prove K is sequentially compact and use the pigeonhole principle to find the convergent subsequence.)

   (c) Consider $c_0 \subset \ell^\infty$ the space of all sequences converging to zero. Fix a sequence $x \in c_0$ and let

   $$S_x = \{y \in c_0 | |y_n| \leq |x_n \,|\}.$$

   Show that S_x is compact subset of c_0. Hint.To show S_x is closed and totally bounded, similar to finding a finite cover for the box $[-x_1, x_1] \times [-x_2, x_2] \times \cdots \times [-x_N, x_N]$ for fixed N.

8. (Non-equivalent norms).

   (a) Show that the ℓ^p and ℓ^q norms on the space ℓ^1 are not equivalent for $1 \leq p < q \leq \infty$. Hint. Consider the sequence $x_n = (1, \ldots, 1, 0, 0, \ldots)$ with first n-terms being 1 .

   (b) Show that the L^p and L^q norms on $C[0, 1]$ are not equivalent for $1 \leq p < q \leq \infty$. Hint. Consider the sequence $x_n(t) = \iota^n$.

9. Assume X is a normed space and Y is a finite-dimensional, proper subspace of X. Show that $\alpha = 1$ also works in the F. Riesz's Lemma.

7.3 HW3

1. Consider the space $X = c_0$ equipped with the norm $\|\cdot\|_\infty$. Let

$$Y = \left\{ \ x = (\xi_j) \in c_0 \ \middle| \ \sum_{j=1}^{\infty} \frac{\xi_j}{2^j} = 0 \ \right\}.$$

   (a) Show that Y is a closed subspace of c_0.

   (b) Show that α can not equal to 1 in the Riesz lemma for this case.

$$\|z - y\| \geq \|z\| \to 0 \leq \|z - y\| - \|z\| \leq -\|y\|$$

$$0 \leq -\|y\| \to 0 \geq \|y\|$$

   since y is arbitrary, this is impossible

   (c) Let $x_0 = (2, 0, 0, \cdots)$. Show that

$$\inf_{y \in Y} \|x_0 - y\| = 1$$

   but

$$\|x_0 - y\| > 1 \quad \forall y \in Y.$$

2. Show that a linear operator from a normed space X into a normed linear space Y is bounded if and only if it maps bounded sets onto bounded sets.

   solution. ($\Rightarrow$) Suppose $T \in \mathcal{L}(X, Y)$. Let $B \subset X$ be a bounded set. Then there exists a constant $C > 0$ such that $\|x\| \leq C$ for all $x \in B$. This leads to $\|Tx\| \leq \|T\|\|x\| \leq \|T\|C$ for all $x \in B$, hence the image $T(B)$ is also a bounded set in Y.

   ($\Leftarrow$) Suppose T is a linear operator from X to Y that maps bounded sets onto bounded sets. For the bounded set B_1, the closed unit ball in X, there exists a $M > 0$ such that $\|Tx\| \leq M$ for all $\|x\| \leq 1$. Hence, $\left\| \frac{T}{\|x\|} x \right\| \leq M$ for all $x \in X \setminus \{0\}$, and therefore $\|Tx\| \leq M\|x\|$ for all $x \in X$, and $T \in L(X, Y)$, completing the proof.

   □

3. Assume X and Y are normed vector spaces and $T \in \mathcal{L}(X, Y)$. Show that (i) $\|T\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|$. (ii) $\|T\| = \sup_{\substack{x \in \times \\ \|x\| < 1}} \|Tx\|$.

   solution. (a)

$$\sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\| \geq \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\| = \|T\|$$

   and

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \geq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \frac{\|Tx\|}{\|x\|} \geq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|$$

(b) from part (a) we see that

$$\|T\| = \sup_{\substack{x\in X\\ \|x\|\le 1}} \|Tx\| \ge \sup_{\substack{x\in X\\ \|x\|<1}} \|Tx\|$$

for any $\|x\| = 1$ and $\epsilon > 0$, we have $\|\frac{x}{1+\epsilon}\| < 1$ and

$$\|Tx\| = \|(1+\epsilon)T\frac{x}{1+\epsilon}\| \le (1+\epsilon) \sup_{\substack{x\in X\\ \|x\|<1}} \|Tx\|$$

which implies that

$$\|T\| = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\| \le (1+\epsilon) \sup_{\substack{x\in X\\ \|x\|<1}} \|Tx\|, \quad \epsilon > 0$$

sending $\epsilon \to 0^+$ in the last equation yields

$$\|T\| \le \sup_{\substack{x\in X\\ \|x\|<1}} \|Tx\|$$

□

4. Let T be a bounded linear operator from a normed space X onto a normed space Y. If there exists a constant $B > 0$ such that

$$\|x\| \le B\|Tx\| \quad \forall x \in X.$$

Then the inverse $T^{-1} : Y \to X$ exists and is bounded.

solution. Assume $x_1, x_2 \in X$ with $x_1 \ne x_2$. Then we have

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \ge \frac{1}{B}\|x_1 - x_2\| > 0$$

which leads to $Tx_1 \ne Tx_2$ and thus T is injective and, since $T : X \to Y$ is onto, a bijective. Therefore, the inverse operator $T^{-1} : Y \to X$ exists.

To see T^{-1} is bounded, write $x = T^{-1}y$ for any $y \in Y$, we then have

$$\|T^{-1}y\| = \|x\| \le B\|Tx\| = B\|y\| \quad \forall y \in Y.$$

Therefore, T^{-1} is bounded.

Remark. It is also clear that the inverse operator of any linear operator, if it exists, must be linear.

□

5. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded.

solution. Consider the two normed vector spaces $X = C[0,1]$ and $Y = L^1[0,1]$ and the operator $T : X \to Y$ to be the inclusion map:

$$Tx = x \quad \forall x \in X = C[0,1].$$

It is clear that T is linear. It is known that

$$\|Tx\|_Y = \int_0^1 |x(t)|dt \le \max_{0\le t\le 1} |x(t)| = \|x\|_X \quad \forall x \in X = C[0,1].$$

Therefore, $\|T\| \le 1$ and $T \in L(X,Y)$. The range $R(T) = C[0,1] \subset Y$. The inverse operator $T^{-1} : R(T) \to X$ is, however, unbounded by checking the sequence $x_n(t)$ defined by

$$x_n(t) = \begin{cases} 1 - nt & 0 \le t < \frac{1}{n} \\ 0 & \frac{1}{n} \le t \le 1 \end{cases}.$$

Then,

$$\frac{\|T^{-1}x_n\|_X}{\|x_n\|_{\mathcal{R}(T)}} = \frac{1}{1/2n} \to \infty \quad \text{given} \quad \|x_n\|_{\mathcal{R}(T)} = \frac{1}{2n},$$

thus T^{-1} is unbounded. $\square$

6. Let $X = L^1[a,b]$ and $Y = C[a,b]$, and let the operator T be defined as

$$(Tx)(t) = \int_a^t x(s)ds.$$

(i) Assume $T \in \mathcal{L}(X,Y)$. Find the operator norm $\|T\|$. (ii) Assume $T \in \mathcal{L}(X,X)$. Find the operator norm $\|T\|$.

solution. (a) $T : X \to Y$,

$$\|Tx\|_Y = \max_{a\le t\le b} |\int_a^t x(s)ds| \le \max_{a\le t\le b} \int_a^t |x(s)|ds \le \int_a^b |x(s)|ds = \|x\|_X$$

hence $\|T\| \le 1$

taking $x_0(t) \equiv 1$ gives

$$\|x_0\|_X = \int_a^b |x_0(t)|dt = b - a$$

and

$$Tx_0 = \max_{a\le t\le b} |\int_a^t ds = \max_{a\le t\le b} |t-a| = b - a$$

which implies that

$$\|T\| = \sup_{\substack{x\in X \\ x\ne 0}} \frac{\|Tx\|}{\|x\|} \ge \frac{\|Tx_0\|}{\|x_0\|} = \frac{b-a}{b-a} = 1$$

therefore we have $\|T\|_{\mathcal{L}(X,Y)} = 1$

(b)

$$\|Tx\|_X = \int_a^b |(Tx)(t)|dt = \int_a^b |\int_a^t x(s)ds|dt \leq \int_a^b \int_a^t |x(s)|ds \leq \int_a^b \int_a^b |x(s)|ds = (b-a)\|x\|_X$$

consider a sequence x_n defined by

$$x_n(t) = \begin{cases} n & \text{if } a \leq t \leq a + \frac{1}{n} \\ 0 & \text{if } a + \frac{1}{n} < t \leq b \end{cases}$$

then

$$\|x_n\|_X = \int_a^b |x_n(t)|dt = 1$$

and

$$\|Tx_n\|_X = \int_a^b |Tx_n(t)|dt = \int_a^b |\int_a^t x_n(s)|dt = \int_a^{a+\frac{1}{n}} n(t-a)dt + \int_{a+\frac{1}{n}}^b 1dt = b-a-\frac{1}{2n}$$

therefore

$$\|T\| \geq \frac{\|Tx_n\|}{\|x_n\|} = b - a - \frac{1}{2n}$$

by sending $n \to \infty$, which yields,

$$\|T\| \geq b - a$$

hence we have $\|T\|_{\mathcal{L}(X,X)} = b - a$

$\square$

7. Assume X is a normed space with $\dim X = \infty$. Show that there exists unbounded linear functional on X; thus $X' \subsetneq X^*$.

solution. Let B be a Hamel basis of X, then choose a sequence $B' = (e_k)$ from B. Define a linear functional $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{\text{finite k}} k\xi_k \|e_k\|$$

where x is expressed as a finite sum as follows:

$$x = \sum_{\text{finite } k} \xi_k e_k + \sum_{\substack{\text{finite}\lambda \\ e_\lambda \notin B'}} \xi_\lambda e_\lambda$$

it is clear that f is linear. To see f is unbounded, note that

$$\frac{|f(e_k)|}{\|e_k\|} = k \to \infty \quad \text{as } k \to \infty$$

therefore f is a linear functional that is unbounded. $\square$

8. Let X be a normed vector space and $f \in X'$. Show that

$$\|f\| = \sup_{\substack{x \in X \\ \|x\|=1}} f(x).$$

solution. note that

$$\sup_{\substack{x \in X \\ \|x\|=1}} f(x) \leq \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)| = \|f\|$$

for the other direction, we take $\alpha = \text{sgn} f(x)$. then given $\|x\| = 1$,

$$|f(x)| = \alpha f(x) = f(\alpha x) \leq \sup_{\substack{y \in X \\ \|y\|=1}} f(y)$$

which leads to

$$\|f\| = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)| \leq \sup_{\substack{y \in X \\ \|y\|=1}} f(y) = \sup_{\substack{x \in X \\ \|x\|=1}} f(x)$$

□

9. Assume $y \in C[0,1]$, and define a linear functional $\int \in (C[0,1])'$ by

$$f(x) = \int_0^1 x(\iota)y(\iota)d\iota.$$

Show that

$$\|f\| = \int_0^1 |y(t)|dt := \|y\|_{L^1}$$

10. Show that the dual of c_0 is isomorphic to ℓ^1. Here, $c_0 \subset \ell^\infty$ is the space of all sequences converging to zero.

7.4 HW4

1. (Polarization identity). Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, whose norm satisfies the parallelogram law. Show that X is also an inner product space with the inner product given by

$$\langle x, y \rangle = \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & \text{if } \mathbb{K} = \mathbb{R}, \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

In addition, the inner product satisfies $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$. Remark. This result can be used to show that any inner product space $(X, \langle \cdot, \cdot \rangle)$ can be completed as a Hilbert space.

solution. Complex case:

(a) $\langle x, x \rangle = \frac{1}{4}(\|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2) = \|x\|^2 \geq 0.$

(b)

$$\langle y, x\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2\right)$$
$$\overline{\langle x, y\rangle} = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)$$

we have $\langle y, x\rangle = \overline{\langle x, y\rangle}$ since $\|y+ix\| = \|x-iy\|$ and $\|x+iy\| = \|y-ix\|$

(c)

$$\langle (x+y), z\rangle = \frac{1}{4}(\|(x+y)+z\|^2 - \|(x+y)-z\|^2 + i\|(x+y)+iz\|^2 - i\|(x+y)-iz\|^2$$
$$\langle x, z\rangle = \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2 + i\|(x+y)+iz\|^2 - i\|x-iz\|^2$$
$$\langle y, z\rangle = \frac{1}{4}(\|y+z\|^2 - \|y-z\|^2 + i\|(x+y)+iz\|^2 - i\|y-iz\|^2$$

we prove the real part: want to prove

$2\|(x+y)+z\|^2 - 2\|(x+y)-z\|^2 = 2\|x+z\|^2 - 2\|x-z\|^2 + 2\|y+z\|^2 - 2\|y-z\|^2$

$2(\|(x+y)+z\|^2 + \|x-z\|^2) - 2(\|x+y-z\|^2 + \|x+z\|^2) = 2\|y+z\|^2 - 2\|y-z\|^2$

By parallelogram law we have

$LHS = \|2x+y\|^2 + \|2z+y\|^2 - (2x+y\|^2 + \|y-2z\|^2) = \|2z+y\|^2 - \|y-2z\|^2$

By parallelogram law again:

$\|2z+y\|^2 + \|y\|^2 - (\|y-2z\|^2 + \|y\|^2) = \|2z\|^2 + \|2z+2y\|^2 - (\|2z\|^2 + \|2y-2z\|^2) = 2\|y+z\|^2 - 2\|y-z\|^2$

the proof of imaginary part is similar.

(d) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle : \langle \alpha x, y\rangle = \frac{1}{4}\left(\|\alpha x+y\|^2 - \|\alpha x-y\|^2 + i\|\alpha x+iy\|^2 - i\|\alpha x-iy\|^2\right)$. Note in (c) we have prove that $\|2z+y\|^2 = 2\|y+z\|^2$ by parallelogram law. Here we repeat the proof and get $\|\alpha x+y\|^2 = \alpha\|x+y\|^2$ and $\|\alpha x-y\|^2 = \alpha\|x-y\|^2$. So we have $Re(\langle \alpha x, y\rangle) = Re(\alpha\langle x, y\rangle)$ and the imaginary part is similar. So we proved that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

□

2. Prove that $C[0,1]$ equipped with the sup-norm is not an inner product space.

solution. by the parallelogram law, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in C[0,1]$$

here $\|f(t)\| = \sup_{t\in[0,1]} |f(t)|$. So we have

$$(\sup|x+y|)^2 + (\sup|x-y|)^2 = 2\sup|x|^2 + 2\sup|y|^2 \quad \forall x, y \in C[0,1]$$

consider $x(t) = t$ and $y(t) = t^2$, we have

$$RHS = 2\cdot 1 + 2\cdot 1 = 4$$
$$LHS = (\sup|t-t^2|)^2 + (\sup|t+t^2|)^2 = \frac{1}{16} + 4 \neq RHS$$

so we conclude that the pair is not an inner product space. □

3. On $L^2[0,T]$, define a functional

$$f(x) = \left| \int_0^T e^{-(T-t)} x(t) dt \right| .$$

Show that f attains its maximum on the unit sphere of $L^2[0,T]$ and find the maximum value. Hint. Use the Cauchy-Schwarz inequality.

solution. The associated norm is defined by

$$\|x\|_{L^2[0,T]} = \left(\int_0^T |x(t)|^2 dt \right)^{1/2}$$

On the unit sphere of $L^2[0,T]$, we have $\|x\|_{L^2[0,T]} = 1$ The inner product is defined by

$$\langle h, g \rangle = \int_0^T h(t)g(t)dt \quad \forall h, g \in L^2[0,T]$$

let $(L^2[0,T], \langle \cdot, \cdot \rangle)$ be an inner product space. Then the Cauchy-Schwarz inequality holds:

$$|\langle x, y \rangle| \le \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$$

$$\left| \int_0^T h(t)g(t)dt \right| \le \sqrt{\int_0^T h(t)^2 dt} \sqrt{\int_0^T g(t)^2 dt}$$

let $h(t) = e^{-(T-t)}$ and $g(t) = x(t)$, we have

$$f(x) = \left| \int_0^T e^{-(T-t)} x(t) dt \right| \le \sqrt{\int_0^T e^{-2(T-t)} dt} \sqrt{\int_0^T x(t)^2 dt}$$

$$= \sqrt{\int_0^T e^{-2(T-t)} dt} = \sqrt{\frac{1 - e^{-2T}}{2}}$$

The equality holds when $g(t) = \lambda h(t)$ for some λ, so we have $x(t) = \lambda e^{-(T-t)}$, and recall $\left(\int_0^T |x(t)|^2 dt \right)^{1/2} = 1$. solve the equation we get $\lambda = \pm \frac{\sqrt{2} e^T}{\sqrt{e^{2T} - 1}}$, and $x = \pm \frac{\sqrt{2} e^t}{\sqrt{e^{2T} - 1}}$

□

4. (Lemma 3.3-7). Assume H is a Hilbert space and M is subset of H. Show that

$$M^{\perp\perp} = \overline{\operatorname{span} M}$$

and

$$\overline{\operatorname{span} M} = H \iff M^{\perp} = \{0\}.$$

solution. by the linearity of the inner product we have

$$M^{\perp} = (\text{span}M)^{\perp}$$

and by theorem 3.7 in the lecture note, we have

$$M^{\perp} = (\text{span}M)^{\perp} = (\overline{\text{span}M})^{\perp}$$

by the lemma 3.1 which says that a subspace Y of a Hilbert space X is closed if and only if $Y^{\perp\perp} = Y$, we have

$$\overline{\text{span}M} = ((\overline{\text{span}M})^{\perp})^{\perp} = (M^{\perp})^{\perp} = M^{\perp\perp}$$

$\square$

5. (Projection theorem, continued). Recall the projection theorem: Let Z be a nonempty, convex and complete subset of a real or complex inner product space $(X, \langle\cdot,\cdot\rangle)$. Then given any $x \in X$, there exists a unique vector $y := Px \in Z$ satisfying

$$\|x - Px\| = \inf_{z \subset Z} \|x - z\|.$$

Now, prove the following:

(a) The unique vector $y := Px \in Z$ found above satisfies

$$\begin{aligned} \langle Px - x, z - Px\rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{R}, \\ \text{Re}\langle Px - x, z - Px\rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

Conversely, if any vector $y \in Z$ satisfies

$$\begin{aligned} \langle y - x, z - y\rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{R}, \\ \text{Re}\langle y - x, z - y\rangle &\geq 0 \quad \forall z \in Z \quad \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

then $y = Px$.

solution. Using the identity

$$\|x + y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$$

we have

$$\|(Px-x)+(z-Px)\|^2 = \|Px-x\|^2+2Re\langle Px-x, z-Px\rangle+\|z-Px\|^2$$

Notice that $LHS = \|z - x\|^2$ and $\|x - Px\| = \inf_{z\in Z} \|x - z\|$ we have

$$2Re\langle Px - x, z - Px\rangle + \|z - Px\|^2 \geq 0$$

Since $z \in Z$ is arbitrary and $Px \in Z$, we select $z = Px$ and get $\|z - Px\| = 0$, then we have

$$Re\langle Px - x, z - Px\rangle + \|z - Px\|^2 \geq 0$$

The case $\mathbb{K} = \mathbb{R}$ is similar. $\square$

(b) The mapping $P : X \to Z$ satisfies

$$\|Px_1 - Px_2\| \leq \|x_1 - x_2\| \quad \forall x_1, x_2 \in X.$$

Hence, the projection mapping is Lipschitz continuous.

solution. Since Z is convex, for all $z \in Z$ and $\lambda \in (0, 1)$, we have

$$\|x - Px\|^2 \leq \|x - (\lambda z + (1-\lambda)Px)\|^2 = \|x - Px - \lambda(z - Px)\|^2$$

Using the identity

$$\|x + y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$$

we have

$$\langle x - Px, z - Px\rangle \leq 0 \quad \forall z \in Z$$

for x_1 and x_2, we have

$$\langle x_1 - Px_1, z - Px_1\rangle \leq 0 \quad \forall z \in Z$$

$$\langle x_2 - Px_2, z - Px_2\rangle \leq 0 \quad \forall z \in Z$$

let $z = Px_2$ in first inequality and let $z = Px_1$ in second inequality:

$$\langle x_1 - Px_1, Px_2 - Px_1\rangle \leq 0$$

$$\langle x_2 - Px_2, Px_1 - Px_2\rangle \leq 0 \Rightarrow \langle x_2 - Px_2, Px_2 - Px_1\rangle \geq 0$$

we have

$$\langle x_1 - Px_1, Px_2 - Px_1\rangle - (\langle x_2 - Px_2, Px_2 - Px_1\rangle) \leq 0$$

$$\Rightarrow \langle x_1 - Px_1 - (x_2 - Px_2), Px_2 - Px_1\rangle \leq 0$$

thus

$$\|Px_2 - Px_1\|^2 \leq \langle x_1 - x_2, Px_2 - Px_1\rangle \leq \|x_2 - x_1\|\|Px_2 - Px_1\|$$

and hence $\|Px_2 - Px_1\| \leq \|x_2 - x_1\|$ □

(c) Recall that, in the lecture, we have shown: If Z is a complete subspace of X, then

$$\langle x - Px, z\rangle = 0 \quad \forall z \in Z,$$

that is $x - Px \in Z^\perp$. Show that, conversely, if an element $y \in Z$ satisfies $x - y \in Z^\perp$, i.e., if

$$\langle x - y, z\rangle = 0 \quad \forall z \in Z,$$

then $y = Px$.

solution. suppose that $\exists y' \in Z$ s.t. $\langle x - y', z\rangle = 0, \forall z \in Z$, since $y = Px$ satisfies that

$$\|x - y\| = \inf_{z \in Z} \|x - z\| \Rightarrow \|x - y'\| \geq \|x - y\|$$

we have

$$\|x - y\| + \|y - y'\| \geq \|x - y'\|$$

which denotes that $\|y - y'\| = 0 \Rightarrow y' = y = Px$ □

(d) The projection mapping $P : X \to Z$ is linear if and only if the subset Z is a subspace of X. In this case

$$\|P\|_{B(X,Z)} = 1 \quad \text{if } Z \neq \{0\}.$$

6. Let H be a Hilbert space.

(a) Let Z be a closed subspace. Show that the projection operator $P : H \to Z$ posses the following properties: $\|P\| = 1$ except if $Z = \{0\}$; P is idempotent: $P^2 = P$ and P is symmetric: $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in H$.

solution. we prove $P^2 = P$: $\forall x \in H$, let $y = Px \in Z$, $P^2x = P(Px) = Py$, let $k = Py \in Z$, by definition, $\|k - y\| = \inf_{z\in Z} \|z - y\| = 0$ since $y \in Z$, so we have $\|k - y\| = 0 \Rightarrow k = y \Rightarrow Py = P(Px) = Px$, which denotes that $P^2 = P$ since x is arbitrary.
To prove $\|P\| = 1$:

$$\|Px\|^2 = |\langle Px, Px\rangle| = |\langle x, Px\rangle| \leq \|x\|\cdot\|Px\| \Rightarrow \frac{\|Px\|}{\|x\|} \leq 1 \Rightarrow \|P\| \leq 1$$

also notice that

$$\|Px\| = \|P^2x\| = \|P(Px)\| \leq \|P\| \cdot \|Px\| \Rightarrow \|P\| \geq 1$$

so we have $\|P\| = 1$
To prove $\langle Px, y\rangle = \langle x, Py\rangle$:

$$\langle Px, y\rangle - \langle x, Py\rangle = \langle Px, y\rangle - \langle x, y\rangle + \langle x, y\rangle - \langle x, Py\rangle$$

$$\langle Px - x, y\rangle + \langle x, y - Py\rangle$$

recall that a closed subspace of a complete space is complete. By theorem 3.6, we have

$$\langle Px - x, y\rangle = \langle x, y - Py\rangle = 0$$

so we have

$$\langle Px, y\rangle = \langle x, Py\rangle$$

□

(b) Let $Q : H \to H$ be a continuous linear operator that is idempotent and symmetric. Show that $Q(H)$ is a closed subspace of H and that Q is the projection operator of H onto $Q(H)$.

(c) Let $Q : H \to H$ be a continuous linear operator that is idempotent and satisfies $\|Q\| \leq 1$. Show that $Q(H)$ is a closed subspace of H and that Q is the projection operator of H onto $Q(H)$.

7. Let Y be a nonempty convex and closed subset of a real Hilbert space X and let $b \in X \backslash Y$. Show that there exist $a \in X$ and $\alpha \in \mathbb{R}$ such that

$$\langle b, a \rangle < \alpha < \langle y, a \rangle \quad \forall y \in Y.$$

Remark. This property expresses that the hyperplane $\{x \in X \mid x - a = \alpha\}$ strictly separates the convex sets Y and $\{b\}$; which is a special case of the Hahn-Banach separation theorem.

solution. since $\langle \cdot, \cdot \rangle \geq 0$, we only need to prove that $\exists a \in X$ and $\alpha > 0$ s.t.

$$\alpha < \langle y, a \rangle - \langle b, a \rangle \Rightarrow \alpha < \langle y - b, a \rangle$$

By corollary 3.2, which says that the inner product is a continuous mapping. Given $b \in X \setminus Y$, $\forall y \in Y$, $\|b - y\| > 0$. Suppose that we cannot find such α and a, then we have: $\forall a \in X$, we have $\langle y - b, a \rangle = 0$, which is impossible since a is arbitrary and $\|y - b\| > 0$. □

8. (Least squares). Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^m$.

(a) Let A be a given $m \times n$ matrix and $c \in \mathbb{R}^m$ be a given vector. Show that the least square problem: Find $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$|Ax - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

has at least one solution.

solution. suppose $\exists x' \in \mathbb{R}^n$ s.t.

$$|Ax' - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

By definition, $\forall x \in \mathbb{R}^n$, $|Ax - c| \geq \inf_{y \in \mathbb{R}^n} |Ay - c|$, $\forall n > 0$, $\exists x_n \in \mathbb{R}^n$, $|Ax_n - c| < \inf_{y \in \mathbb{R}^n} |Ay - c| + \frac{1}{n}$,

$$|Ax_n| \leq |Ax_n - c| + |c| \leq \inf_{y \in \mathbb{R}^n} |Ay - c| + \frac{1}{n} + |c| \leq \inf_{y \in \mathbb{R}^n} |Ay - c| + 1 + |c|$$

so $\{x_n\}$ is bounded. $\mathbb{R}^n$ is finitely dimensional, and we have $\{x_{n_k}\} \subset \{x_n\}$ and $\{x_{n_k}\} \to x' \in \mathbb{R}^n$

$$\inf_{y \subset \mathbb{R}^n} |Ay - c| \leq |Ax_n - c| \leq \inf_{y \subset \mathbb{R}^n} |Ay - c| + \frac{1}{n_k}$$

as $k \to \infty$ we have

$$|Ax - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

□

(b) Show that a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfies the above least square problem if and only if $\boldsymbol{x}$ is a solution of the normal equation:

$$A^T A \boldsymbol{x} = A^T c.$$

solution. $\Rightarrow$: x is the solution of the least square problem $\to$ x is the minimum point of the function $f(x) = (Ax - c)^T(Ax - c)$. Differentiating $f(x)$ with respect to x we get $2A^TAx - 2A^Tc = 0 \to A^TAx = A^Tc$
$\Leftarrow$:

$$A^TA\boldsymbol{x} = A^Tc \to A^T(Ax - c) = 0 \to Ax = c$$

we find $x \in \mathbb{R}^n$ s.t. $|Ax - c| = 0 \leq \inf_{y\in\mathbb{R}^n} |Ay - c|$ and $\inf_{y\in\mathbb{R}^n} \leq |Ax - c|$ since $x \in \mathbb{R}^n$, so we have $|Ax - c| = \inf_{y\in\mathbb{R}^n} |Ay - c|$ $\square$

(c) Show that the solution to the above normal equation is unique if and only if $\operatorname{rank} A = n$ (which of course implies that $n \leq m$), or equivalently, if and only if the symmetric matrix A^TA is positive definite.

9. Show that a subspace Y of a Hilbert space H is closed if and only if $Y^{\perp\perp} = Y$.

solution. Let X be a Hilbert space and Y a subspace of X. suppose Y is closed and by theorem 3.8 which says that $X = Y \oplus Y^{\perp}$. Then we have

$$X = Y \oplus Y^{\perp} = (Y^{\perp})^{\perp} \oplus Y^{\perp}$$

which implies that $Y^{\perp\perp} = Y$.
Suppose $Y^{\perp\perp} = Y$, by theorem 3.7 which says that if Z is a nonempty subset of an inner product space X, then $Z^{\perp}$ is a closed subspace of X. So we conclude that Y is closed. $\square$

10. Let Y be the subset of the Hilbert space ℓ^2 defined by

$$Y := \left\{x = (\xi_i) \in \ell^2 \mid \xi_{2k-1} = \xi_{2k} \quad \forall k \in \mathbb{N}\right\}.$$

(a) Show that Y is a closed subspace of ℓ^2.

solution. The space ℓ^2 is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j}$$

and the norm is defined by

$$\|x\| = \langle x, x\rangle^{1/2} = (\sum_{j=1}^{\infty} |\xi_j|^2)^{1/2}$$

linearity: $\forall \alpha, \beta \in \mathbb{R}$, and $y_1 = (\xi^1), y_2 = (\xi^2) \in Y$, we have

$$\alpha y_1 + \beta y_2 = ((\alpha\xi^1 + \beta\xi^2)_i)$$

$$(\alpha\xi^1 + \beta\xi^2)_{2k-1} = \alpha\xi^1_{2k-1} + \beta\xi^2_{2k-1} = \alpha\xi^1_{2k} + \beta\xi^2_{2k} = (\alpha\xi^1 + \beta\xi^2)_{2k}$$

so we have

$$\alpha y_1 + \beta y_2 \in Y$$

closedness: suppose $\{x_n\}$ is a sequence in Y and $\{x_n\} \to x$ and want to prove that $x \in Y$: given $\{x_n\} \in Y$, for $x_n := (\xi_i)_n$, consider $\{\xi_{2k-1}\}_n \to \xi_{2k-1}$ and $\{\xi_{2k}\}_n \to \xi_{2k}$, since $x_n \in Y$, we have $\xi^{(n)}_{2k-1} = \xi^{(n)}_{2k}, \forall n \in \mathbf{N}$ and thus $\lim_{n\to\infty} \xi^{(n)}_{2k-1} = \lim_{n\to\infty} \xi^{(n)}_{2k} \Rightarrow \xi_{2k} = \xi_{2k-1}$. so $x \in Y$ and Y is closed. $\square$

(b) Identify the orthogonal complement of Y in ℓ^2.

solution. recall the definition of orthogonal complement:

$$Z^\perp = \{x \in X : \langle x, z\rangle = 0 \quad \forall z \in Z\}$$

consider the subset

$$Z = \left\{y = (\eta_i) \in \ell^2 \mid \eta_{2k-1} = -\eta_{2k} \quad \forall k \in \mathbb{N}\right\}$$

we have

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j} = \sum_{\substack{j=2k-1\\k=1}}^{\infty} \xi_j^2 - \sum_{\substack{j=2k\\k=1}}^{\infty} \xi_j^2 = 0$$

$\square$

(c) Identify the projection operators $P : \ell^2 \to Y$ and $P^\perp : \ell^2 \to Y^\perp$.

solution. for $x = (\xi_j)$, consider the operator

$$P : P(x) = (\xi') = \begin{cases} \xi'_j = \xi_j & \text{if } j \text{ is even} \\ \xi'_j = \xi_{j+1} & \text{if } j \text{ is odd} \end{cases}$$

and

$$P^\perp : P(x) = (\xi') = \begin{cases} \xi'_j = \xi_j & \text{if } j \text{ is even} \\ \xi'_j = -\xi_{j+1} & \text{if } j \text{ is odd} \end{cases}$$

$\square$

7.5 HW5

1. A normed vector space X is called uniformly convex if: for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| > \varepsilon \quad \text{implies that} \quad \left\|\frac{x+y}{2}\right\| < 1 - \delta(\varepsilon).$$

Show that for an inner product space X is uniformly convex. Hint. Use the parallelogram law.

solution. convex: $x, y \in A \to tx+(1-t)y \in A$; uniform convex: $\|x-y\| > \epsilon \to \|\frac{x+y}{2}\| < 1 - \delta(\epsilon)$
by parallelogram law, we have

$$\|\frac{x+y}{2}\|^2 = 2(\|\frac{x}{x}\|^2 + \|\frac{y}{2}\|^2) - \|\frac{x-y}{2}\|^2 = 1 - \frac{1}{4}\|x-y\|^2 < 1 - \frac{\epsilon^2}{4}$$

$$\|\frac{x+y}{2}\| < \sqrt{1 - \frac{\epsilon^2}{4}} = 1 - (1 - \sqrt{1 - \frac{\epsilon^2}{4}}) \to \delta(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$$

$\square$

2. Let (e_n) and (f_n) be two orthonormal sequences in a Hilbert space X such that
$$\sum_{n=1}^{\infty} \|e_n - f_n\| < 1.$$
Show that (e_n) is maximal if and only if (f_n) is maximal.

solution. suppose e_n is maximal: suppose f_n is not maximal: $\exists x \neq 0$: $\langle x, f_n\rangle = 0, \forall n \in N$.

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle x, e_n - f_n\rangle|^2 \leq \sum_{n=1}^{\infty} \|x\|^2 \cdot \|e_n - f_n\|^2$$

$$\leq \|x\|^2 \sup_n \|e_n - f_n\| \cdot \sum_{n=1}^{\infty} \|e_n - f_n\| < \|x\|^2$$

since $\sup_n \|e_n - f_n\| < 1, \sum_{n=1}^{\infty} \|e_n - f_n\| < 1$. contradiction. □

3. Let X be an inner product space and (e_k) be an orthonormal sequence. Show that
$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle \langle y, e_k\rangle| \leq \|x\|\|y\| \quad \forall x, y \in X.$$

solution. let $a_k = \langle x, e_k\rangle$, $b_k = \langle y, e_k\rangle$. $\|x\|^2 = \sum_{k=1}^{\infty} a_k^2 < \infty$ since $a_k, b_k \in \ell^2$orthonormal

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle \langle y, e_k\rangle| \leq (\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2)^{\frac{1}{2}} \cdot (\sum_{k=1}^{\infty} |\langle y, e_k\rangle|^2)^{\frac{1}{2}} \leq \|x\| \cdot \|y\|$$

by holder's inequality □

4. (Sobolev spaces). Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain (thus an open set) in $\mathbb{R}^n$. Let $C_0^m(\Omega)$ denote the set of functions with m-th continuous derivatives and has a compact support in Ω, that is u vanishes in a neighbourhood of $\partial\Omega$:
$$C_0^m(\Omega) = \{u \in C^\infty(\Omega) \mid \operatorname{supp} u \subset \Omega\}.$$

   (a) Show that
$$\langle u, v\rangle_m = \sum_{|\alpha|=m} \int_\Omega \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx$$
   and
$$\langle u, v\rangle = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx$$
   are two inner products defined on $C_0^m(\Omega)$.

solution. i. linearity: $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, which is true by the linearity of differentiation operator and integration operator

ii. $\langle x, y\rangle = \overline{\langle y, x\rangle}$: which is also true by observing that $\overline{\overline{v(x)}} = v(x)$ for all $v(x) \in C_0^m(\Omega)$

iii. $\langle M, M\rangle_m = 0 \Leftrightarrow M = 0$:

$$\int_\Omega \partial^\alpha M \cdot \overline{\partial^\alpha M} dx = 0, \quad \forall |\alpha| = m : \alpha_1 + \alpha_2 + ... + \alpha_n = m \Rightarrow$$

$$\int_\Omega |\partial^\alpha M| dx = 0 \to \partial^\alpha M = 0 \to \partial^{\alpha-1} M = c = 0 \to \partial^{\alpha-2} M = c' = 0 \to \langle M, M\rangle = 0$$
$$\to M = 0$$

□

(b) Prove the Poincaré inequality:

$$\sum_{|\alpha|<m} \int_\Omega |\partial^\alpha u(x)|^2 \, dx \le C \sum_{\alpha=m} \int_\Omega |\partial^\alpha u(x)|^2 \, dx \quad \forall u \in C_0^m(\Omega),$$

where C is a positive constant only depends on Ω and m.
Remark. The Poincare inequality means the two norms induced by the two inner products

$$\|u\|_m = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u(x)|^2 \, dx\right)^{1/2}$$

and

$$\|u\| = \left(\sum_{|\alpha|\le m} \int_\Omega |\partial^{\alpha x} u(x)|^2 \, dx\right)^{1/2}$$

are equivalent. The completion of $(C_0^m(\Omega), \langle\cdot,\cdot\rangle)$ is usually denoted by $H_0^m(\Omega)$, which is a Sobolev space. Sobolev spaces $H_0^m(\Omega)$ and $H^m(\Omega)$ (without vanishing boundary condition) are infinite dimensional separable Hilbert spaces, which play central roles in the modern theory of partial differential equations.

solution. make a continuation supp$M \subset \Omega$ s.t. $M(0) = 0$

$$|M(x)| \le \int_0^{x_1} |\frac{\partial M}{\partial t}(t, x_2, ..., x_n)| dt \le x_1^{\frac{1}{2}} [\int_0^{x_1} |\frac{\partial}{\partial t}(t, x_2, ..., x_n)|^2 dt]^{1/2}$$
$$\le \alpha^{1/2} [\int_0^\alpha |\frac{\partial M}{\partial t}(t, x_2, ..., x_n)|^2 dt]^{1/2} = \alpha^{1/2} [\int_0^\alpha |\frac{\partial M}{\partial x_1}|^2 dx_1]^{1/2}$$

$$(\int_\Omega |M(x)^2| dx) \le \int_\Omega \alpha (\int_0^\alpha |\frac{\partial M}{\partial x_1}|^2 dx_1) dx \le \alpha \int_0^\alpha (\int_\Omega |\frac{\partial M}{\partial x_1}|^2 dx) dx_1$$
$$\le \alpha^2 \sum_{i=1}^n \|\frac{\partial M}{\partial x_i}\|_{L^2(\Omega)}^2 \le \alpha^2 \sum_{?} \|\partial^\alpha M\|_{L^2(\Omega)}$$

□

5. (Legendre polynomials). For the Hilbert space $L^2[-1,1]$, applying the Gram-Schmidt process to a linearly independent sequence $(f_n)_{n=0}^{\infty}$ with

$$f_n(x) = x^n \quad n = 0, 1, 2, \dots,$$

we can generate an orthonormal family $(e_n)_{n=0}^{\infty}$. (a), (b) see textbook

(a) Show that $\overline{\text{span}\,(e_n)_{n=0}^{\infty}} = L^2[-1,1]$, thus $(e_n)_{n=0}^{\infty}$ form an orthonormal basis of $L^2[-1,1]$. Hint. For any $L^2[-1,1]$ function x, first approximate it by a continuous function y (a known result in Real Analysis), then approximate y by a polynomial (the Weierstrass Λ aproximation Theorem).

solution. want to show that (e_n) is total in $L^2[-1,1]$: by theorem 3.2-3 the set $W = A(X)$ is dense in $L^2[-1,1]$. Hence for any fixed $x \in L^2[-1,1]$ and given $\epsilon > 0$ there is a continuous function y defined on $[-1,1]$ such that

$$\|x - y\| < \frac{\epsilon}{2}$$

for this y there is a polynomial z s.t. for all $t \in [-1,1]$

$$|y(t) - z(t)| < \frac{\epsilon}{2\sqrt{2}}$$

this follow from the weirerstrass approximation theorem and implies

$$\|y - z\|^2 = \int_{-1}^{1} |y(t) - z(t)|^2 dt < 2(\frac{\epsilon}{2\sqrt{2}})^2 = \frac{\epsilon^2}{4}$$

together, by the triangle inequality

$$\|x - z\| \leq \|x - y\| + \|y - z\| < \epsilon$$

the definition of the Gram-Schmidt process, we have $z \in \text{span}\{e_0, ...e_m\}$ for sufficiently large m. since $x \in L^2[-1,1]$ and $\epsilon > 0$ were arbitrary, this proves totality of (e_n) $\square$

(b) Show that

$$e_n(x) = \sqrt{n + \frac{1}{2}} P_n(x),$$

with

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1\right)^n \right], \quad n = 0, 1, 2, \dots,$$

where $P_n(x)$ is known as the Legendre polynomials, and the above formula is called Rodrigues' formula for Legendre polynomials.

solution. given $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1\right)^n \right]$, by applying the binomial theorem to $(t^2 - 1)^n$ and differentiating the result n times term by term we obtain that

$$P_n(t) = \sum_{j=0}^{N} (-1)^j \frac{(2n - 2j)!}{2^n j!(n - j)!(n - 2)!} t^{n-2j}$$

where $N = n/2$ if n is even and $N = (N - 1)/2$ if n is odd. To prove $\sqrt{n + \frac{1}{2}} P_n(x) = e_n(x)$

i. we first show that (P_n) is an orthogonal sequence in the space $L^2[-1,1]$: i.e. $\langle P_m, P_n\rangle = 0$ where $0 \le m < n$. Since P_m is a polynomial, it suffices to prove that $\langle x_m, P_n\rangle = 0$ for $m < n$, where x_m is defined by $x_j(t) = t^j$. By integration by parts:

$$2^n!\langle x_m, P_n\rangle = \int_{-1}^{1} t^m (u^n)^{(n)} dt = t^m (u^n)^{(n-1)}|_{-1}^{1} - m\int_{-1}^{1} t^{m-1}(u^n)^{(n-1)} dt$$

$$= \cdots = (-1)^m m! \int_{-1}^{1} (u^n)^{(n-m)} dt = (-1)^m m! (u^n)^{(n-m-1)}|_{-1}^{1} = 0$$

ii. we prove that

$$\|P_n\| = [\int_{-1}^{1} P_n^2(t)]^{1/2} = \sqrt{\frac{2}{2n+1}}$$

which is true since

$$(2^n!)\|P_n\|^2 = \int_{-1}^{1} (u^n)^{(n)}(u^n)^{(n)} dt = (u^n)^{n-1}(u^n)^{(n)}|_{-1}^{1} - \int_{-1}^{1} (u^n)^{(n-1)}(u^n)^{(n+1)} dt$$

$$= \cdots = (-1)^n (2n)! \int_{-1}^{1} u^n dt = 2(2n)! \int_{0}^{1} (1-t^2)^n dt = 2(2n)! \int_{0}^{\pi/2} \cos^{2n+1}\tau d\tau$$
$$= \frac{2^{2n+1}(n!)^2}{2n+1}$$

where $u = t^2 - 1$. and division by $(2^n n!)^2$ implies that $\|P_n\| = \sqrt{\frac{2}{2n+1}}$

the above two steps imply that (P_n) is an orthogonal sequence in the space $L^2[-1,1]$. denote $\sqrt{n + \frac{1}{2}P_n(t)}$ and let $Y_n = \text{span}\{e_0, ..., e_n\} = \text{span}\{x_0, ..., x_n\} = \text{span}\{y_0, ..., y_n\}$ here the second equality sign follows from the algorithm of the Gram-Schmidt process and the last equality sign from $\dim Y_n = n+1$ together with the linear independence of $\{y_0, ..., y_n\}$, we know that

$$y_n = \sum_{j=0}^{n} \alpha_j e_j$$

now by the orthogonality,

$$y_n \perp Y_{n-1} = \text{span}\{y_0, ..., y_{n-1}\} = \text{span}\{e_0, ..., e_{n-1}\}$$

this implies that for $k = 0, ..., n-1$ we have

$$0 = \langle y_n, e_k\rangle = \sum_{j=0}^{n} \alpha_j \langle e_j, e_k\rangle = \alpha_k$$

hence we have $y_n = \alpha_n e_n$. here $|\alpha_n| = 1$ since $\|y_n\| = \|e_n\| = 1$. $y_n(t) > 0$ for sufficiently large t since the coefficient of t^n is positive. also $e_n(t) > 0$ for sufficiently large t. hence $\alpha = 1$ and $y_n = e_n$. $\square$

(c) Show that $P_n(x)$ satisfies the Legendre differential equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+n(n+1)y(x)=0.$$

solution. given $P_n(x)=\sum_{j=0}^{N}(-1)^j\frac{(2n-2j)!}{2^n j!(n-j)!(n-2)!}t^{n-2j}$, we check every term satisfies the Legendre differential equation: let $c_n = (-1)^j\frac{(2n-2j)!}{2^n j!(n-j)!(n-2)!}$, we have

$$\frac{dP_n(x)}{dx}\sim c_n x^{n-2j-1}$$

$$\frac{d^2P_n(x)}{dx}\sim (n-2j-1)c_n x^{n-2j-2}$$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]\sim (1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}=c_n(n-2j-1)(x^{n-2j-2}-x^{n-2j})-2c_nx^{n-2j}$$

$$n(n+1)P_n(x)\sim c_n x^{n-2j}$$

□

Remark. This implies that Legendre polynomials are the eigenfunctions of the differential operator L :

$$Lu(x)=-\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right],\quad -1\le x\le 1$$

in the sense that

$$LP_n(x)=\lambda_n P_n(x)\qquad \text{with } \lambda_n=n(n+1).$$

Hence, the orthogonality of $(P_n(x))$ follows from the Hilbert-Schmidt theory (will be discussed at the end of this course), which saying that the eigenvectors associated to distinct eigenvalues of a self-adjoint compact operator L are orthogonal, (a result extending the linear algebra case for real symmetric or complex Hermitian matrices.) Indeed, for this particular Legendre polynomial, we can do:

(d) Using the Legendre differential equation to prove that

$$\int_{-1}^{1}P_n(x)P_m(x)dx=0\qquad \text{if } m\ne n.$$

solution.

$$-[(1-x)^2P_n']=\lambda_nP_n\to-[(1-x)^2P_n']P_m=\lambda_nP_nP_m$$

$$-[(1-x^2)P_m']'P_n=\lambda_mP_mP_n$$

$$(\lambda_n-\lambda_m)\int_{-1}^{1}P_nP_mdx=\int_{-1}^{1}[(1-x^2)P_n']'P_mdx+\int_{-1}^{1}[(1-x^2)P_m']'P_ndx$$

$$=-(1-x^2)P_n'P_m|_{-1}^{1}+\int_{-1}^{1}(1-x^2)P_n'P_m'=0$$

□

(e) Let X be an inner product space. An operator $T \in \mathcal{L}(X)$ is called self-adjoint if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in X.$$

Show that the Legendre differential operator L is self-adjoint on the real $L^2[-1,1]$ space.

6. (classical Fourier series). Consider the functions $(e_k(x))$ defined by

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\cos nx, \quad \frac{1}{\sqrt{\pi}}\sin nx, \quad n = 1, 2, \ldots$$

(a) Show that $(e_k(x))$ form an orthonormal family on $L^2(0, 2\pi)$.

solution. i. check $\|e_k\| = 1$

ii. check $\langle x_m, x_n\rangle = 0$

$\square$

(b) Show that $(e_k(x))$ are maximal, thus form an orthonormal basis on $L^2(0, 2\pi)$.
Hint. Using the fact that $C^\infty(0, 2\pi)$ is dense in $L^2(0, 2\pi)$ (a fact from Real Analysis) and the Weierstrass trigonometric polynomial approximation theorem: the space of trigonometric polynomials is dense in $C^\infty(0, 2\pi)$.

solution. want to show that the only element x in $L^2(0, 2\pi)$ such that $\langle x, e_k\rangle = 0$ is 0: from the hint we know that every element in $L^2(0, 2\pi)$ can be approximated by a element in $C^\infty(0, 2\pi)$ and also approximated by a trigonometric polynomial: $\forall x \in L^2(0, 2\pi)$, $\langle x, e_k\rangle = \langle \sum_{n=1}^\infty a_n \cos nx + b_n \sin nx, e_k\rangle$. from (a) we know that this inner product is 0. $\square$

(c) Show that $(e_k(x))$ are the eigenfunctions of the following operator

$$Lu = -\frac{d^2}{dx^2}$$

in the sense that

$$Le_k(x) = \lambda_k e_k(x).$$

solution. $Le_k(x) = -\frac{d^2 e_k}{dx}$: $L\cos nx = \frac{d^2 \cos nx}{dx} = -n^2 \cos nx$ and $L \sin nx = \frac{d^2 \sin nx}{dx} = -n^2 \sin nx$. Let $\lambda_n = n^2$, we are done. $\square$

(d) Thus, for any function $f \in L^2(0, 2\pi)$, we have the classical Fourier expansion

$$f(x) = \sum_{k=0}^\infty \langle f, e_k\rangle\, e_k := \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx),$$

where the convergence is in the $L^2(0, 2\pi)$ space. Show that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx, \quad n = 0, 1, 2, \ldots$$

solution. from (a) we know that $f(x) \in L^2(0, 2\pi) \to f(x) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_n a_n \frac{1}{\sqrt{\pi}} \cos nx + \sum_n b_n \frac{1}{\sqrt{\pi}} \sin nx$. let the basis related to a_n be e_a:

$$\langle f, e_a \rangle = \langle a_0 \frac{1}{\sqrt{2\pi}} + \sum_n a_n \frac{1}{\sqrt{\pi}} \cos nx + \sum_n b_n \frac{1}{\sqrt{\pi}} \sin nx, e_a \rangle = \langle a_n \frac{1}{\sqrt{\pi}} \cos nx, e_a \rangle$$

$$= \langle a_n \frac{1}{\sqrt{\pi}} \cos nx, a_n \frac{1}{\sqrt{\pi}} \cos nx \rangle = \|a_n \frac{1}{\sqrt{\pi}} \cos nx\|^2 = \int_0^{2\pi} a_n^2 \frac{1}{\pi} \cos nx dx$$

$$\langle f, e_a \rangle = \int_0^{2\pi} f(x) a_n \frac{1}{\sqrt{\pi}} \cos nx dx$$

compare two integral we conclude that $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$. the case of $sinnx$ is similar. □

(e) Explain why

$$\lim_{n\to\infty} \int_0^{2\pi} f(x) \cos nx dx = 0, \quad \lim_{n\to\infty} \int_0^{2\pi} f(x) \sin nx dx = 0?$$

These are known as the Riemann-Lebesgue lemma.

7. This problem provides an example of a nonseparable Hilbert space and of an uncountably infinite orthonormal family.

(a) Let the subspace Y of the complex vector space $C(\mathbb{R}; \mathbb{C})$ be defined as

$$Y = \text{span}\,(e_\lambda)_{\lambda\in\mathbb{R}} \quad \text{where } e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R}.$$

Show that

$$\langle f, g \rangle = \frac{1}{2T} \int_{-T}^{T} f(x) \overline{g(x)} dx$$

defines an inner product over Y.

solution.

i. $\langle f, f \rangle = \frac{1}{2T} \int_{-T}^{T} f(x) \overline{f(x)} dx = \frac{1}{2T} \int_{-T}^{T} (Ref)^2 + (Imf)^2 dx \geq 0$

ii. $\langle f, g \rangle = \overline{\langle g, f \rangle}$: obvious by the definition of $\langle f, g \rangle$

iii. linearity: obvious by the linearity of integration.

□

(b) Show that $(e_\lambda)_{\lambda\in\mathbb{R}}$ is an orthonormal family in the space $(Y, \langle \cdot, \cdot \rangle)$.

solution. check that $\langle e_i, e_j \rangle = \delta_{ij}$: recall that $e^{i\lambda x} = \cos \lambda x + i \sin \lambda x$, $\langle e_i, e_j \rangle = \frac{1}{2T} \int_{-T}^{T} e^{i\lambda x} \overline{e^{j\lambda x}} = \frac{1}{2T} \int_{-T}^{T} (\cos \lambda ix + i \sin \lambda ix)(\cos \lambda jx - \sin \lambda jx) dx = \frac{1}{2T} \int_{-T}^{T} dx = 1$ □

(c) Show that $(e_\lambda)_{\lambda\in\mathbb{R}}$ is an maximal orthonormal family in the space $(Y, \langle \cdot, \cdot \rangle)$.

solution. check that the only element y in Y such that $\langle y, e_i\rangle = 0$ is 0:

$$\langle 0, e_i\rangle = \frac{1}{2T}\int_{-T}^{T} 0 \cdot \overline{e^{i\lambda x}} dx = 0$$

suppose $\langle y, e_i\rangle = 0$:

$$\frac{1}{2T}\int_{-T}^{T} y\overline{e^{j\lambda x}} dx = \frac{1}{2T}\int_{-T}^{T} (Rey + iImy)(\cos j\lambda x - i\sin j\lambda x) dx = 0$$

we have

$$\int_{-T}^{T} Rey\cos j\lambda x + Imy\sin j\lambda x dx = 0$$

$$\int_{-T}^{T} -Rey\sin j\lambda x + Imy\cos j\lambda x dx = 0$$

the only possibility for y and $\forall j$ we conclude that $y = 0$ $\square$

(d) Show that the completion X of Y is a Hilbert space that is not separable.

7.6 HW6

1. (Existence of orthonormal basis). Let $X \neq \{0\}$ be an inner product space. Show that there exists a maximal orthonormal family $M = (e_i)_{i\in I}$. Hint. Use Zorn's lemma. Remark. Recall that if X is a Hilbert space, then a maximal orthonormal family is an orthonormal basis.

   solution. let $\mathcal{M} = \{\text{all orthonormal family in } X\}$, we know that for all total order chain $\ell \subset \mathcal{M}$, let $\tilde{U} = \bigcup_{u\in\ell} u$. $\forall x_i, x_j \in \tilde{U}, x_i \in c_i$, $x_j \in c_j, c_i, c_j \in \ell$. WLOG, $c_i \subset c_j \to x_i \in c_j$ and $x_j \in c_j \to x_i \perp x_j \to \tilde{U} \in \mathcal{M} \to \tilde{U}$ is an upper bound of ℓ. $\exists$ maximal element v of $\mathcal{M}$, claim that V is a maximal orthonormal family. if not, $\exists x \in X$ s.t. $x \in V^{\perp}$ and $x \neq 0$. then $V \bigcup\{\frac{x}{\|x\|}\}$ is still orthonormal. contradiction with the fact that V is maximal element

   $\square$

2. (B.L.T. (Bounded Linear Transformation) Theorem). Prove the following theorem. Let X be a normed vector space and Y be a Banach space. Suppose $T \in \mathcal{L}(X, Y)$. Then T can be uniquely extended to a bounded linear transformation $\widetilde{T} \in \mathcal{L}(\widetilde{X}, Y)$, where $\widetilde{X}$ is the completion of X. Besides,

   $$\|T\| = \|\widetilde{T}\|.$$

   Hint. For any $x \in \widetilde{X}$, there exists a sequence $x_n \in X$ with $x_n \to x$. Define

   $$\widetilde{T}x = \lim_{n\to\infty} Tx_n.$$

   (Why the above limit exists?)

solution. consider $\{x_n\}$ and $\{z_n\} \in X$ s.t. $x_n \to x$ and $z_n \to x$ WTS $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tz_n$, i.e. $\tilde{T}$ is well-defined: since T is bounded $\{Ty_n\}$ is Cauchy sequence and we have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tz_n$

since $\tilde{X}$ is the completion of X, for any $x \in \tilde{X}$, there exists a sequence $x_n \in X$ with $x_n \to x$. given $T \in \mathcal{L}(X, Y)$, a bounded linear transformation, consider a Cauchy sequence in X: $\{x_n\}$,

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\| \to 0$$

as $n, m \to \infty$, which is also a Cauchy sequence. note that Y is complete, so $\{Tx_n\}$ converges and the limit is denoted as $\tilde{T}x$. in this way we have defined a map $\tilde{T} : \tilde{X} \to Y$ and next we show that $\tilde{T}$ is the linear transformation that extended by T and satisfies that $\|T\| = \|\tilde{T}\|$

(a) linearity: $\tilde{T}(x + cy) = \lim_{n\to\infty}(Tx_n + cTy_n) = \lim_{n\to\infty} Tx_n + c\lim_{n\to\infty} Ty_n = \tilde{T}x + c\tilde{T}y$

(b) uniqueness: is guaranteed by the uniqueness of the limit

(c) expansion: is to show that $\forall x \in X$, $Tx = \tilde{T}x$: $\forall x \in X$, consider the convergent sequence $\{x, x, ..., x, ...\} \subset X$, we have $\tilde{T}x = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx = Tx$

(d) norm: $\|\tilde{T}\| = \sup_{\substack{x\neq 0 \\ x\in\tilde{X}}} \frac{\|\tilde{T}x\|}{\|x\|} \geq \sup_{\substack{x\neq 0 \\ x\in X}} \frac{\|\tilde{T}x\|}{\|x\|} = \sup_{\substack{x\neq 0 \\ x\in X}} \frac{\|Tx\|}{\|x\|} = \|T\|$ since $\tilde{X} \supset X$

for the other side: $\|\tilde{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \|T\| \cdot \lim_{n\to\infty} \|x_n\| = \|T\|\|x\|$

$\square$

3. (Hahn-Banach theorem in a Hilbert space). Prove the Hahn-Banach theorem in a Hilbert space use the projection theorem and Riesz's representation theorem.

Let X be a Hilbert space over $\mathbb{K}$, let Y be a subspace of X, and let $f : Y \to \mathbb{K}$ be a continuous linear functional on Y. Then there exists a continuous linear form $\tilde{f} : X \to \mathbb{K}$ that satisfies

$$\tilde{f}(y) = f(y) \quad \forall y \in Y, \quad \text{and} \quad \|\tilde{f}\| = \|f\|.$$

Besides, such an extension is unique.
Hint. First apply the B.L.T. theorem to extend f to $\hat{f} : \bar{Y} \to \mathbb{K}$, and then apply the direction sum theorem to $\bar{Y} : X = \bar{Y} \oplus Y^{\perp}$. Define

$$\tilde{f}(x) = \tilde{f}(y) + \tilde{f}\left(y^{\perp}\right) := \tilde{f}(y) \quad \vee x \in X,$$

where $x = y + y^{\perp}$ with $y \in \bar{Y}$ and $y^{\perp} \in Y^{\perp}$.

solution. By the Bounded Linear Transformation (BLT) theorem, f can be uniquely extended to a continuous linear functional $\hat{f}$ on the closure $\bar{Y}$ of Y such that $\|\hat{f}\| = \|f\|$.

Consider the Hilbert space X and observe that it can be written as the direct sum of $\bar{Y}$ and its orthogonal complement $Y^{\perp}$, that is, $X = \bar{Y} \oplus Y^{\perp}$.

For every $x \in X$, there exists a unique decomposition $x = y + y^{\perp}$ where $y \in \bar{Y}$ and $y^{\perp} \in Y^{\perp}$.

To define $\tilde{f}$ on X, we use the Riesz Representation Theorem to represent $\hat{f}$ as an inner product with some vector in $\bar{Y}$, say $\hat{f}(y) = \langle y, z\rangle$ for some $z \in \bar{Y}$. We extend $\hat{f}$ to $\tilde{f}$ on X by setting

$$\tilde{f}(x) = \tilde{f}(y + y^{\perp}) = \hat{f}(y) + \hat{f}(y^{\perp}) = \langle y, z\rangle + \langle y^{\perp}, z\rangle.$$

Since $y^{\perp}$ is orthogonal to all elements of $\bar{Y}$, including z, we have $\tilde{f}(x) = \langle y, z\rangle$ for all $x \in X$, which ensures that $\tilde{f}$ is a linear extension of f.

The extension is unique because any other extension that agrees with f on Y and has the same norm must coincide with $\tilde{f}$ on the dense set $\bar{Y}$ and hence everywhere on X due to continuity. Furthermore, the norm of $\tilde{f}$ equals the norm of f because the extension was constructed to preserve the inner product representation, ensuring $\|\tilde{f}\| = \|f\|$.

□

4. Let X and Y be two Hilbert spaces and let T^* denote the Hilbert adjoint of $T \in \mathcal{L}(X, Y)$. Show that the following relations hold

$$(R(T))^{\perp} = N(T^*) \quad \text{ and } \quad (R(T^*))^{\perp} = N(T),$$
$$Y = N(T^*) \oplus \overline{R(T)} \text{ and } X = N(T) \oplus \overline{T^*}.$$

solution. given $x = Px + P^{\perp}x$ and $P : X \to Y$, $\tilde{f}(x) = \tilde{f}(Px) + \tilde{f}(P^{\perp}x) = \tilde{f}(Px) = f(Px)$. assume $\exists \tilde{f}_1$, $\tilde{f}_2$ s.t. $\tilde{f}_1 \neq \tilde{f}_2$, by RRT: $\tilde{f}_1(x) = \langle x, z_1\rangle$, $\tilde{f}_2(x) = \langle x, z_2\rangle$, $z_1, z_2 \in X$. consider $y \in Y$, $\tilde{f}_1(y) = \tilde{f}_2(y)$:

$$\tilde{f}_1(y) = \langle y, z_1\rangle = \langle y, Pz_1\rangle, \quad P^{\perp}z_1 \in Y^{\perp}, \langle y, P^{\perp}z_1\rangle = 0$$

$$\tilde{f}_2(y) = \langle y, z_2\rangle = \langle y, Pz_2\rangle, \quad P^{\perp}z_2 \in Y^{\perp}, \langle y, P^{\perp}z_2\rangle = 0$$

take $y = Pz_1 - Pz_2$, we have $Pz_1 = Pz_2$. claim that $z = Pz$: $f(x) = \langle x, z\rangle = \langle x, Pz + P^{\perp}z\rangle = \langle x, Pz\rangle \to \|f\| = \|Pz\|$.

$$z = Pz + P^{\perp}z \to \|z\|^2 = \|Pz\|^2 + \|P^{\perp}z\|^2 \to \|P^{\perp}z\| = 0 \to z = Pz$$

given $\|z\|^2 = \|Pz\|^2$ □

5. (Lax-Milgram Theorem). Let X be a (real or complex) Hilbert space and let $a : X \times X \to \mathbb{K}$ be a sesquilinear form, that is bounded and coercive, i.e., there exist $M > 0$ and $c > 0$ such that

$$\text{(bounded) } |a(x, y)| \leq M\|x\|\|y\| \quad \forall x, y \in X$$
$$\text{(coercive) } |a(x, x)| \geq c\|x\|^2 \quad \forall x \in X.$$

Show that there exists a unique $A \in B(X, X)$ such that

$$a(x, y) = \langle x, Ay\rangle \quad \vee\, x, y \in X$$

and its inverse A^{-1} exists and $A^{-1} \in B(X, X)$ with

$$\|A^{-1}\| \leq \frac{1}{c}.$$

Hint. 1) Use the Riesz Representation Theorem for sesequelinear forms to show the uniqueness and existence of A. 2) Prove A is injective. 3) Prove that $R(A)$ is closed and $R(A)^{\perp} = \{0\}$, thus A is a surjective.

solution. given $a(x, y)$ is a bounded linear functional on $X \times X$, by Riesz representation theorem for sesquelinear form, there exists $!A : X \to X$ s.t. $a(x, y) = \langle x, Ay \rangle, \forall x, y \in X$, which shows the existence and uniqueness of $A \in B(X, X)$.

to show A is injective: $\forall y_1, y_2 \in X, y_1 \neq y_2$, $a(y_1 - y_2, y_1 - y_2) = \langle y_1 - y_2, A(y_1 - y_2) \rangle = \langle y_1 - y_2, Ay_1 \rangle - \langle y_1 - y_2, Ay_2 \rangle$. by the coercive property of $a(x, y)$: we know that $\langle y_1 - y_2, Ay_1 \rangle - \langle y_1 - y_2, Ay_2 \rangle \geq c\|x - y\|^2 \geq 0$ and the equality only holds when $y_1 - y_2 = 0$. for $y_1 \neq y_2$, we know that $\langle y_1 - y_2, Ay_1 \rangle - \langle y_1 - y_2, Ay_2 \rangle > 0$ and we conclude that $Ay_1 \neq Ay_2$

to show $a(x, y)$ is subjective: first is to show that $R(A)$ the range is closed: consider a Cauchy sequence $\{y_n\} \subset X$, given $a(x, y)$ is bounded, we know that $A : X \to X$ is also bounded, so the sequence Ay_n satisfies that

$$\|Ay_n - Ay_m\| \leq \|A\| \cdot \|y_n - y_m\|$$

Ay_n is also a Cauchy sequence. X is a Hilbert space, so Ay_n converges to its limit $Ay \in X$. this shows that $R(A)$ is closed.

suppose $x \neq 0$, $x \in R(A)^{\perp}$, $Ax \in R(A)$, we have $a(x, x) = \langle x, Ax \rangle = 0$. by the coercive property, we have $|a(x, x)| \geq c\|x\|^2 \Rightarrow \|x\| = 0 \Leftrightarrow x = 0$, this contradiction implies that $x \neq R(A)^{\perp}$ and thus $R(A) = \{0\}$

we have proved that A is a bijective, so A^{-1} exists and $A^{-1} \in B(X, X)$ for any $y \in X$, we have $c\|A^{-1}y\|^2 \leq |a(A^{-1}y, A^{-1}y)| = |\langle A^{-1}y, y \rangle|$, which implies that $\|A^{-1}y\|^2 \leq \frac{1}{c}\|\langle A^{-1}y, y \rangle\| \leq \frac{1}{c}\|A^{-1}y\|\|y\|$. Thus, $\|A^{-1}\| \leq \frac{1}{c}$, completing the proof. $\square$

6. (Existence and uniqueness of weak solutions of Poisson equation). Let Ω be a domain in $\mathbb{R}^n$ and let $f \in L^2(\Omega)$. Consider the Dirichlet problem of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

(i) Let $u \in C^2(\bar{\Omega})$ be solution of (1). Show that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad \vee\, v \in C_0^{\infty}(\Omega).$$

(ii) Note that in (2), u is not required to be twice differentiable. Thus, we say $u \in I_0^1(\Omega)$ is a weak solution of (1) if

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad \forall v \in \Pi_0^1(\Omega).$$

Here, $H_0^1(\Omega)$ is a Sobolev space. Show that there exists a unique weak solution of (1).

supplementary knowledge:

(a) Weak derivative: $\forall \phi \in D(\Omega) = C_0^\infty(\Omega)$, $\int_\Omega v\phi dx = (-1)^{|\alpha|}\int_\Omega u(\partial^\alpha \phi)dx$, $\Omega \in \mathbb{R}^n$ is a domain, $v = u'$ is called the weak derivative of u

(b) sobolev space: $W^{m,p}(\Omega) := \{v \in L^p(\Omega) \text{ s.t. } \partial^\alpha v \in L^p(\Omega), 1 \leq |\alpha| \leq m\}$. $\|v\|_{w^{m,p}(\Omega)} = \sum_{0\leq|\alpha|<m} \|\partial^\alpha v\|_p$. Sobolev space is banach, separable for $1 < p < \infty$, reflexive: $X \cong x''$.
$H^1(\Omega) = w^{1,2}(\Omega) := \{v \in L^2(\Omega), \nabla v \in L^2(\Omega)\}$, here ∇v is the weak derivative ?

(c) DBVP of Possion equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

the weak solution u is defined by

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \quad u_{|\partial\Omega=0}$$

$\exists! u \in H_0^1(\Omega)$ s.t. u is a weak solution

solution. (a) Utilizing Green's first identity, we have:

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega (\Delta u) v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds,$$

where $\frac{\partial u}{\partial n}$ is the derivative of u normal to the boundary $\partial\Omega$.
Since $u = 0$ on $\partial\Omega$ and $v = 0$ on $\partial\Omega$ (because $v \in C_0^\infty(\Omega)$, which denotes that v is smooth and has compact support within Ω), the boundary integral term is zero. By substituting $-\Delta u = f$ into the equation, derived from the given Poisson equation, we obtain:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

(b)

$$\|u\|_{H_0^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$$

let $a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$, by Cauchy-Schawarz inequality, $|a(u,v)| \leqslant \|\nabla u\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)} \leqslant \|u\|_{H_0^1(\Omega)} \cdot \|v\|_{H_0^1(\Omega)}$
by Poincare inequality, we have $|a(u,u)| = \|\nabla u\|_{L^2(\Omega}^2 \geqslant c \cdot \|u\|_{H_0^1(\Omega)}^2$
$\therefore \exists A : H_0^1(\Omega) \to H_0^1(\Omega) : a(u,v) = \langle Au, v\rangle$ is a bijective
let $f(v) = \int_\Omega f v dx, \quad f \in H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$ by Riesz Representation Theorem: $f(v) = (\tilde{f}, v), \tilde{f} \in H_0'(n)$. $u = A^{-1}\tilde{f}$ is a weak sol: $a(u,v) = \langle Au, v\rangle = \langle \tilde{f}, v\rangle = f(v) \cdot \forall v \in H_0'(\Omega)$ uniqueness by bijective of A

$\square$

7. (Reproducing kernel). Let S be a nonempty set and let X be a Hilbert space over $\mathbb{K}$ whose elements are functions $x : S \to \mathbb{K}$. Assume that, for each $t \in S$, there exists a constant $C(t) > 0$ such that

$$|x(t)| \leq C(t)\|x\| \quad \vee_x \in X.$$

(i) Prove that there exists a function $K : X \times X \to \mathbb{K}$ called reproducing kernel of X, such that, for each $t \in S$,
1) $K(\cdot, t) : S \to \mathbb{K}$ is an element of X;
2) $x(t) = \langle x, K(\cdot, t)\rangle$ for all $x \in X$.
(ii) Assume that $\mathbb{K} = \mathbb{C}$ and $(e_j)_{j=1}^{\infty}$ is an orthonormal basis of X. Show that

$$K(s,t) = \sum_{j=1}^{\infty} e_j(s)\overline{e_j(t)} \quad \forall s, t \in S.$$

(iii) Let $S = \mathbb{N}$ and $X = \ell^2$. Find the reproducing kernel $K(m, n)$.

solution. (a) Given that for each $t \in S$, there exists a constant $C(t) > 0$ such that

$$|x(t)| \leq C(t)\|x\| \quad \forall x \in X.$$

This condition implies the boundedness of the evaluation functional $L_t : X \to \mathbb{K}$ defined by $L_t(x) = x(t)$, as $\|L_t(x)\| = |x(t)| \leq C(t)\|x\|$ for all $x \in X$. By the Riesz Representation Theorem, since L_t is a continuous linear functional on a Hilbert space X, there exists a unique element in X, denoted by $K(\cdot, t)$, such that

$$L_t(x) = \langle x, K(\cdot, t)\rangle \quad \forall x \in X.$$

This directly satisfies condition 2. Additionally, since $K(\cdot, t)$ is an element of X for each t, condition 1 is also satisfied. Hence, $K(\cdot, t)$ is the reproducing kernel of X.

(b) Given any $x \in X$, it can be expressed as

$$x = \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j.$$

Then, for any $t \in S$,

$$x(t) = \left(\sum_{j=1}^{\infty} \langle x, e_j\rangle e_j\right)(t) = \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j(t).$$

Using the reproducing property, $x(t) = \langle x, K(\cdot, t)\rangle$. Therefore,

$$\langle x, K(\cdot, t)\rangle = \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j(t) = \sum_{j=1}^{\infty} \left(\sum_{j=1}^{\infty} x_j \overline{e_j}\right) e_j(t) = \sum_{j=1}^{\infty} x_j \left(\sum_{j=1}^{\infty} \overline{e_j} e_j(t)\right),$$

implying

$$K(\cdot, t) = \overline{\sum_{j=1}^{\infty} \overline{e_j} e_j(t)} = \sum_{j=1}^{\infty} \overline{e_j(t)} e_j.$$

Evaluating at s yields

$$K(s,t) = \sum_{j=1}^{\infty} e_j(s)\overline{e_j(t)}.$$

(c) Given $S = \mathbb{N}$ and $X = \ell^2$, we need to identify the reproducing kernel $K(m,n)$.

In ℓ^2, an orthonormal basis is represented by vectors where each vector has all components as zero except for one component which is 1, i.e., for the j-th vector, $e_j(n) = \delta_{jn}$ where δ_{jn} is the Kronecker delta.

Substituting into the formula for $K(s,t)$ from part (ii),

$$K(m,n) = \sum_{j=1}^{\infty} e_j(m)\overline{e_j(n)} = \sum_{j=1}^{\infty} \delta_{jm}\delta_{jn} = \delta_{mn},$$

which serves as the reproducing kernel for ℓ^2 when $S = \mathbb{N}$.

$\square$

8. (Hahn-Banach Theorem in a complex vector space)). Prove the following Hahn-Banach theorem.

Let X be a complex vector space and let $p : X \to \mathbb{R}$ be a sublinear functional on X. Let Y be a subspace of X and let $f : Y \to \mathbb{C}$ be a linear functional on Y such that $|f(y)| \le p(y)$ for all $y \in Y$. Then there exists a linear functional $\tilde{f} : X \to \mathbb{C}$ such that

$$\tilde{f}(y) = f(y) \quad \vee\, y \in Y, \quad \text{and} \quad |\tilde{f}(x)| \le p(x) \quad \vee\, x \in X.$$

Hint. For each $y \in Y$, write $f(y) = \operatorname{Re} f(y) - i \operatorname{Re}(f(iy))$. Then apply the Hahn Banach theorem for real vector space to the linear functionals $y \in Y \to \operatorname{Re}(f(y))$ and $y \in Y \to \operatorname{Re}(f(iy))$.

solution. To prove this theorem, we start by considering the real and imaginary parts of f. For each $y \in Y$, we can write $f(y) = \operatorname{Re} f(y) + i \operatorname{Im} f(y)$. However, since f is linear and $\mathbb{C}$ is a complex field, we use the hint provided:

$$f(y) = \operatorname{Re} f(y) - i \operatorname{Re}(f(iy)).$$

This decomposition allows us to handle the real and imaginary parts of $f(y)$ in terms of the real part of f applied to y and iy.

We then apply the real version of the Hahn-Banach theorem separately to the real-valued linear functionals defined on Y: $\operatorname{Re}(f(y))$ and $\operatorname{Re}(f(iy))$. The sublinear functional p serves as a common dominator for both extensions because $|f(y)| \le p(y)$ ensures that both the real and imaginary components of f (as expressed above) are bounded by $p(y)$ in magnitude.

By the Hahn-Banach theorem for real vector spaces, there exist extensions $\tilde{f}_1 : X \to \mathbb{R}$ and $\tilde{f}_2 : X \to \mathbb{R}$ of $\operatorname{Re}(f(y))$ and $\operatorname{Re}(f(iy))$, respectively, such that for all $x \in X$,

$$|\tilde{f}_1(x)| \le p(x) \quad \text{and} \quad |\tilde{f}_2(x)| \le p(x).$$

Now, define $\tilde{f} : X \to \mathbb{C}$ by

$$\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_2(x).$$

This $\tilde{f}$ is a linear functional on X that extends f and satisfies $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$, completing the proof.

$\square$

9. Prove the following theorem: Let Z be a proper subspace of a normed vector space X and suppose $x_0 \in X$ satisfies

$$\delta = d(x_0, Z) = \inf_{z \in Z} \|z - x_0\| > 0.$$

Then there is an $f \in X'$ such that $\|f\| = 1, f(x_0) = \delta$ and $f(z) = 0$ for $z \in Z$. Hint. Consider the bounded linear functional $g : Y \to \mathbb{K}$ with $Y = \operatorname{span}\{x_0\} \oplus Z$ and

$$g(\alpha x_0 + z) = \alpha\delta \quad \vee\, \alpha \in \mathbb{K}, \quad \vee z \in Z,$$

then extends it to the whole X.

solution. given g is a bounded linear functional, we can apply the Hahn-Banach theorem if Y is a subspace of the normed vector space X. the expansion $f(x)$ satisfies that $\|f\| = \|g\|$ and $f(x_0) = g(x_0)$, $g(z) = f(z)$ since x_0 and $z \in Y$. so we only need to check that $g(x_0) = \delta$, $g(z) = 0$ and $\|g\| = 1$

(a) $g(z) = \alpha\delta|_{x_0=0} = 0$ since Z is a proper subspace of X and $0 \in Z$

(b) $g(x_0) = \alpha\delta|_{\alpha=1} = \delta$

(c) $\|g\| : g(y) = g(\alpha x_0 + z) = \alpha g(x_0) + g(z) = \alpha\delta$, we have $y \notin Z$ since $\delta > 0$: $\|g\| = \sup_{y \neq 0} \frac{|g(y)|}{\|y\|} = \sup_{\substack{\forall z \in Z, \\ x_0 \notin Z}} \frac{|\alpha\delta|}{\|\alpha x_0 + z\|} = \frac{|\alpha\delta|}{\|\alpha x_0\|} = 1$

$\square$

10. Let $X = P[0,1]$ be equipped with the sup-norm $\|\cdot\|$, let $Y = P_3[0,1]$ denote the space of polynomials of degree not exceeding 3, and let the linear functional $f : Y \to \mathbb{R}$ be defined by

$$f(p) = \frac{1}{6}\left(p(0) + 4p\left(\frac{1}{2}\right) + p(1)\right) \quad \vee\, p \in P_3[0,1].$$

(a) Show that f is continuous and $\|f\|_{Y'} = 1$.

(b) Define two linear functional $\tilde{f}_1, \tilde{f}_2 : X \to \mathbb{R}$ by

$$\tilde{f}_1(p) = \frac{1}{6}\left(p(0) + 4p\left(\frac{1}{2}\right) + p(1)\right) \quad \text{and} \quad \tilde{f}_2(p) = \int_0^1 p(t)dt, \quad \vee p \in P[0,1].$$

Show that

$$\widetilde{f}_j(p) = f(p) \quad \vee\, p \in P_3[0,1] \quad \text{and} \quad \left\|\widetilde{f}_j\right\| = \|f\| = 1, \quad \vee j = 1,2$$

solution. (a) To show that f is continuous, we demonstrate that there exists a constant $C > 0$ such that for all $p \in Y$, the absolute value of $f(p)$ is less than or equal to C times the sup-norm of p, $\|p\|$.

Given $f(p) = \frac{1}{6}(p(0) + 4p(\frac{1}{2}) + p(1))$, the sup-norm $\|p\|$ is the maximum absolute value of $p(t)$ for $t \in [0, 1]$. Thus,

$$|f(p)| \le \frac{1}{6}(|p(0)| + 4|p(\frac{1}{2})| + |p(1)|) \le \frac{1}{6}(6\|p\|) = \|p\|.$$

This demonstrates that f is continuous with $C = 1$, showing that $\|f\|_{Y'} \le 1$.

To show that $\|f\|_{Y'} = 1$, we find a polynomial $p \in Y$ such that $|f(p)| = \|p\|$. Consider $p(t) = 1$, which is in Y. We have $\|p\| = 1$ and

$$f(p) = \frac{1}{6}(p(0) + 4p(\frac{1}{2}) + p(1)) = 1.$$

Therefore, $|f(p)| = \|p\|$, proving that $\|f\|_{Y'} = 1$.

(b) To show $\widetilde{f}_j(p) = f(p) \forall p \in P_3[0, 1]$ and $\|\widetilde{f}_j\| = \|f\| = 1, \forall j = 1, 2$, we note that:

i. For $\tilde{f}_1(p)$ and $\tilde{f}_2(p)$, the definitions exactly match with $f(p)$ for any polynomial $p \in P_3[0, 1]$, directly showing that $\widetilde{f}_j(p) = f(p) \forall p \in P_3[0, 1]$.

ii. The argument made for f regarding continuity and norm applies equally to $\tilde{f}_1$, thus $\|\tilde{f}_1\| = 1$.

iii. For $\tilde{f}_2$, considering f was specifically constructed to be an exact evaluation of the integral of polynomials up to degree 3 by Simpson's rule. Hence, $\tilde{f}_2$ also has norm 1 for $P_3[0, 1]$ as it yields the exact integral value, matching $f(p)$.

This concludes the proofs for the properties of f, $\tilde{f}_1$, and $\tilde{f}_2$.

□

7.7 HW7

1. If $X \ne \{0\}$ and Y are real normed vector spaces and $\mathcal{L}(X, Y)$ is complete, show that Y is complete.

   Hint. Use Hahn-Banach theorem.

   Remark. Previously, we have show Y is complete implies $\mathcal{L}(X, Y)$ is complete. Thus, we have

   $$Y \text{ is complete } \iff \mathcal{L}(X, Y) \text{ is complete.}$$

solution. fix $x_0 \in X$ s.t. $\|x_0\| = 1$, $\exists f \in X'$ s.t. $\|f\| = 1$, $f(x_0) = \|x_0\|$. let $T_n x = f(x) y_n \in Y$, where T_n is a sequence in $\mathcal{L}(X, Y)$, $\|T_n x\| = \|f(x) y_n\| \le |f(x)| \cdot \|y_n\| \le \|f\| \cdot \|y_n\| \|x\| \to T_n$ is bounded.

$\|T_n - T_m\| \le \|f\| \cdot \|y_n - y_m\| \to T_n$ is Cauchy sequence $\to T_n \to T \in \mathcal{L}(X, Y)$. $y_n = \|x_0\| y_n = f(x_0) y_n = T_n x_0 T x_0$ so $y_n \to T x_0 \in Y$ and Y is complete. □

2. (Taylor-Foguel theorem). A normed vector space is called strictly convex if

$$x \neq y \text{ and } \|x\| = \|y\| = 1 \text{ implies } \left\| \frac{x+y}{2} \right\| < 1.$$

Prove the following theorem.

Let X be a real normed vector space. Then all the continuous linear functionals defined on subspaces of X have a unique norm-preserving extension to X if and only if the dual space X' of X is strictly convex.

solution. We split the proof into two parts, showing both the necessary and sufficient conditions.

$\leftarrow$: Assume the dual space X' is strictly convex. Consider a subspace Y of X and a continuous linear functional $f : Y \to \mathbb{R}$. By the Hahn-Banach Theorem, there exists at least one extension $\tilde{f} : X \to \mathbb{R}$ of f preserving the norm, i.e., $\|\tilde{f}\| = \|f\|$.

Suppose there exist two such extensions $\tilde{f}$ and $\tilde{g}$ with $\tilde{f}|_Y = f = \tilde{g}|_Y$. Consider $\frac{\tilde{f}+\tilde{g}}{2}$, which also extends f and is a linear functional. The strict convexity of X' and the assumption that $\|\tilde{f}\| = \|f\| = \|\tilde{g}\|$ imply that

$$\left\| \frac{\tilde{f}+\tilde{g}}{2} \right\| < \max(\|\tilde{f}\|, \|\tilde{g}\|) = \|f\|,$$

leading to a contradiction. Thus, the extension must be unique.

$\rightarrow$: Assume that every continuous linear functional defined on any subspace Y of X can be uniquely extended to X while preserving its norm. To show that X' is strictly convex, consider distinct functionals $f, g \in X'$ with $\|f\| = \|g\| = 1$. The uniqueness of the norm-preserving extension implies that if $f \neq g$, then

$$\left\| \frac{f+g}{2} \right\| < 1,$$

since any other situation would violate the uniqueness condition. Hence, X' is strictly convex.

This completes the proof that the dual space X' is strictly convex if and only if every continuous linear functional on any subspace of X has a unique norm-preserving extension to X.

□

Remark. Tayor-Foguel theorem also holds for a complex normed vector space.

3. Let X be a normed vector space. Show that a linear functional $f : X \to \mathbb{K}$ is continuous if and only if the hyperplane $H = \{x \in X \mid f(x) = 0\}$ is closed.

solution. $(\rightarrow)$: x_n is a sequence in X s.t. $f(x_n) = 0$, and $x_n \to x$ then we have $f(x) = \lim f(x_n) = \lim 0 = 0$ since f is continuous. then H is closed.

$(\leftarrow)$: H^c is open. suppose $x_0 \in X$ s.t. $f(x_0) < 0$ and $x_0 \in H^c \to \exists r > 0$ s.t. $B_r(x_0) \in H^c$. claim that $f(x) < 0, \forall x \in B_r(x_0)$:

prove by contradiction: suppose not, then $\exists \lambda \in (0,1)$ s.t. $f(\langle x_0 + (1-\lambda)x) = \lambda f(x_0) + (1-\lambda)f(x) = 0$ since $f(x) > 0$ and $f(x_0) < 0$ which contradicts with the fact that $\lambda x_0 + (1-\lambda)x \in B_r(x_0)$. write $B_r(x_0) = \{x_) + rz | z \in X, \|z\| < 1, f(x_0 + rz) < 0, \forall z, \|z\| < 1\}$, $\|f\| = \sup_{\|z\|<1} f(z) < -\frac{f(x_0)}{r} < \infty$, this implies that f is continuous. $\square$

4. Let X be a normed vector space and Y a subspace of X. Show that $\bar{Y} = X$ if and only if the only continuous linear functional f that satisfies $f(y) = 0$ for all $y \in Y$ is $f = 0$. Hint. $(\Leftarrow)$ Use Problem 9 of Assignment 6.

solution. $(\rightarrow)$: given $\overline{Y} = X, \forall x \in X, \exists y_n \in Y$ s.t. $f(x) = \lim_{y_n \to x} f(y_n)$, we know that $f(y) = 0$ for all $y \in Y$, so $f(x) = \lim_{y_n \to x} f(y_n) = \lim 0 = 0$, which implies that $f = 0$

$(\leftarrow)$: suppose that $\overline{Y} \neq X$, $\exists x_0 \notin X \setminus \overline{Y}$ and $x_0 \neq 0$, by problem 9 of assignment 6, we know that $\exists f \in X'$ s.t. $\|f\| = 1$ and $f(x_0) = \delta$ and $f(y) = 0$ for $y \in Y$, so we conclude that $f = 0$ is the only continuous linear functional f that satisfies $f(y) = 0$. $\square$

5. Use Baire's theorem to show that:

   An infinite-dimensional Banach space cannot have a countably infinite Hamel basis.

   In particular, the space of all polynomials of one, or several, variables cannot be equipped with a norm that would make it a Banach space. Hint. Suppose (e_j) is a countable Hamel basis of X. Show that $F_n = \operatorname{span}(e_j)_{j=1}^n$ is closed and has empty interior for every $n \in \mathbb{N}$.

Proof. suppose for a contradiction that $(e_j)_{j=1}^\infty$ is a Hamel basis, $F_n = \operatorname{span}(e_j)_{j=1}^n$ is closed. $X = \cup_{n=1}^\infty F_n$, $\operatorname{int} F_n = \phi, \forall n$, $\forall x \in F_n, x = \sum_{j=1}^n \alpha_j e_j$, $\forall \epsilon > 0$, $x + \frac{\epsilon}{2} \frac{e_{n+1}}{\|e_{n+1}\|} \notin F_n$, which is impossible since X is a Banach space. $\square$

6. Let P denote the space of all real polynomials.

   (i) Given a polynomial $x(t) = \sum_{k=0}^m \alpha_k t^k$. Let $\|x\| = \max_{0 \le k \le m} |\alpha_k|$. Show that $\|\cdot\|$ is a norm on P.

   (ii) Define $T_n x = \sum_{k=0}^{\min\{m,n\}} \alpha_k$. Show that (T_n) is a sequence of bounded linear functional, with

$$\sup_{n \ge 0} \|T_n x\| < \infty \quad \forall x \in P$$

but

$$\sup_{n\geq 0}\|T_n\| = \infty.$$

Remark. This shows that The Banach-Steinhaus theorem fails when the space X is not complete. [This space is not complete by Problem 5].

solution. (i): recall the definition of norm:

(a) $\|x\| = 0 \Leftrightarrow x = 0$: $\|x\| = 0 \Leftrightarrow \max_{0\leq k\leq m}|\alpha_k| = 0 \Leftrightarrow \alpha_k = 0 \Leftrightarrow x(t) = 0$

(b) $\|\alpha x\| = |\alpha|\|x\|$: $\alpha x = \alpha\sum_{k=0}^m \alpha_k t^k = \sum_{k=0}^m \alpha\alpha_k t^k \to \|\alpha x\| = \max_{0\leq k\leq m}|\alpha\alpha_k| = |\alpha|\cdot\max_{0\leq k\leq m}|\alpha_k| = |\alpha|\|x\|$

(c) $\|x+y\| \leq \|x\| + \|y\|$: let $y = \sum_{k=0}^m \beta_k t^k$, $x + y = \sum_{k=0}^m(\alpha_k + \beta_k)t^k$, $\|x+y\| = \|\sum_{k=0}^m(\alpha_k+\beta_k)t^k\| \leq \|\sum_{k=0}^m \alpha_k t^k\| + \|\sum_{k=0}^m \beta_k t^k\| = \|x\| + \|y\|$

(ii): T_n is bounded since $T_n x = \sum_{k=0}^{\min\{m,n\}} \alpha_k \leq \sum_{k=0}^m \alpha_k = x(1)$, which is bounded since $\alpha_k \in \mathbb{R}$. to show T_n is linear: $T_n(ax+y) = \sum_{k=0}^{\min\{m,n\}} a\alpha_k + \beta_k = a\sum_{k=0}^{\min\{m,n\}}\alpha_k + \sum_{k=0}^{\min\{m,n\}}\beta_k = aT_n x + T_n y$.
prove $\sup_{n\geq 0}\|T_n x\| < \infty$: note that $\|T_n x\| \leq \sum_{k=0}^m \alpha_k \leq m\max_{0\leq k\leq m}|\alpha_k| = m\|x\| < \infty$, we know that $\sup_{n\geq 0}\|T_n x\| < \infty$

prove $\sup_{n\geq 0}\|T_n\| = \infty$: recall the definition of $\|T\|$: $\|T_n\| = \sup_{\|x\|=1}\frac{\|T_n x\|}{\|x\|} = \sup_{\|x\|=1}\|T_n x\| = \sup_{\|x\|=1}\|\sum_{k=0}^{\min\{m,n\}}\alpha_k\| \geq \min\{m,n\}\max_{0\leq k\leq m}\|\alpha_k\| = \min\{m,n\}$, $\sup_{n\geq 0}\|T_n\| = \sup_{n\geq 0}\min\{m,n\} = n$, which will go to ∞ as $n\to\infty$. $\square$

7. (i) Let X be a Banach space, let Y and Z be normed space, and let $B: X\times Y \to Z$ be a bilinear mapping that is "separately continuous" in the sense that

for each $y\in Y$, $\lim_{n\to\infty} x_n = x$ in X implies $\lim_{n\to\infty} B(x_n, y) = B(x,y)$ in Z,
for each $x\in X$, $\lim_{n\to\infty} y_n = y$ in Y implies $\lim_{n\to\infty} B(x, y_n) = B(x,y)$ in Z.

Using the Banach Steinhaus theorem, show that B is continuous; i.e., that, for each $(x,y)\in X\times Y$,

$$\lim_{n\to\infty} x_n = x \text{ in } X \text{ and } \lim_{n\to\infty} y_n = y \text{ in } Y \text{ implies } \lim_{n\to\infty} B(x_n, y_n) = B(x,y) \text{ in } Z.$$

(ii) Given an example of normed spaces X, Y, Z and of a separately continuous bilinear mapping $B: X\times Y\to Z$ that is not continuous.

solution. (i): $\|B(x_n,y_n) - B(x,y)\| = \|B(x_n,y_n) - B(x_n,y) + B(x_n,y) - B(x,y)\| \leq \|B(x_n,y_n) - B(x_n,y)\| + \|B(x_n,y) - B(x,y)\|$. given $\lim_{n\to\infty} B(x_n,y) = B(x,y)$, we only focus on $\|B(x_n,y) - B(x,y)\|$. $\forall y\in Y$, $T_y(x_n) := B(x_n,y)$ is linear and converges to $T_y(x)$, by Banach Steinhaus theorem, we know that $\sup_{n\geq 1}\|T_y\| < \infty$, then $\|B(x_n,y) - B(x,y)\| = \|T_y(x_n) - T_y(x)\| \leq \|T_y\|\cdot\|x_n - x\|$. given T_y bounded and let $x_n\to x$, we proved that B is continuous. $\square$

8. Given a weight function $w \in L^1(0,1)$, let there be given a sequence of bounded linear functionals $\ell_n : C[0,1] \to \mathbb{R}$, of the form

$$\ell_n(x) = \sum_{j=0}^{n} \omega_j^n x\left(t_j^n\right), \qquad \text{where } 0 \leq t_0^n < t_1^n < \cdots < t_n^n \leq 1,$$

with the following property:

$$\lim_{n\to\infty} \left| \int_0^1 p(t)w(t)dt - \ell_n(p) \right| = 0 \quad \forall\, p \in P[0,1].$$

(i)

$$\lim_{n\to\infty} \left| \int_0^1 x(t)w(t)dt - \ell_n(x) \right| = 0 \quad \forall\, x \in C[0,1]$$

if and only if

$$\sup_{n\geq 0} \left(\sum_{j=0}^{n} \left|\omega_j^n\right| \right) < \infty.$$

This is Polya's theorem

Hint. It is directly to have $\|\ell_n\| \leq \sum_{k=0}^n |\omega_k^n|$. To show that $\|\ell_n\| \geq \sum_{k=0}^n |\omega_k^n|$, consider the piecewise linear function $x_0 \in C[0,1]$ such that

$$x_0(0) = \operatorname{sgn}\omega_0^n, \quad x_0\left(t_j\right) = \operatorname{sgn}\omega_k^n, \quad 0 \leq k \leq n, \quad x_0(1) = \operatorname{sgn}\omega_n^n.$$

($\Rightarrow$) by the Banach Steinhaus Theorem. ($\Leftarrow$) by the Weierstrass Approximation Theorem.

(ii) Prove Steklov's theorem: Show that, if $\omega_k^n \geq 0$ for all $n \geq 0$ and all $0 \leq k \leq n$, then

$$\lim_{n\to\infty} \left| \int_0^1 x(t)w(t)dt - \ell_n(x) \right| = 0 \quad \forall\, x \in C[0,1].$$

solution. let $x_0(t_j^n) = sgn w_j^n, \|x_0\| = 1,\ \|l_n\| \geq |l_n(x_0)| = \sum_{j=0}^n |w_j^n| \to \|l_n\| = \sum_{j=0}^n |w_j^n| \Rightarrow \in C[0,1], \lim_{n\to\infty} l_n(x) = \int_0^1 w(t)x(t)dt.$

$\sup_{n\geq 0} |l_n(x)| < \infty \Rightarrow \sup_{n\geq\infty} \|l_n\| = \sup_{n\geq 0}(\sum_{j=0}^n |w_j^n|) < \infty$

$(\Leftarrow) : \|l_n\| \leq M, \forall n \geq 0,\ \exists p \in P[0,1],\ \|x-p\|_{L^\infty[0,1]} < \min\{\frac{\epsilon}{3\|N\|_{L^1}} \frac{\epsilon}{3M}\}$

$$|l_n(x) - \int_0^1 w(t)x(t)dt| \leq |l_n(x) - l_n(p)| + |l_n(p) - \int_0^1 w(t)p(t)dt| + |\int_0^1 w(t)p(t)dt - \int_0^1 w(t)x(t)dt|$$

(a) $(I) \leq \|l_n\| \cdot \|x-p\| \leq m\frac{\epsilon}{3m} = \frac{\epsilon}{3}$

(b) $(II) < \epsilon, \exists N, \forall n \geq N$

(c) $(III) \leq \|w\|_{L^1(0,1)} \cdot \|x-p\|_{L^\infty(0,1)} < \frac{\epsilon}{3}$

subproblem (ii): want to show $\sup_{n\geq 0}(\sum_{j=0}^n w_j^n) < \infty$, $p(t) = 1$, $l_n(p) = \sum_{j=0}^n w_j^n \to \int_0^1 w(t)dt$, $w_j^n = a_n$ $\square$

9. (Lagrange interpolation). For each integer $n \geq 0$, let $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$ be any set of $(n+1)$ distinct nodes, and let the interpolating operator $L_n : C[0,1] \to P[0,1]$ be defined by

$$L_n x\left(t_k\right) = x\left(t_k\right), \quad 0 \leq k \leq n, \quad \text{and} \quad \deg L_n x \leq n.$$

This is the Lagrange interpolating polynomial of $x(t)$ of degree $\leq n$ with the $(n+1)$ nodes $(t_k)_{k=0}^{n}$, which is given as

$$L_n x(t) = \sum_{k=0}^{n} x\left(t_k\right) p_k(t),$$

where

$$p_k(t) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(t - t_j)}{(t_k - t_j)}$$

It is known that $L_n p = p$ for all polynomials p of degree $\leq n$.

(i) Show that the operator L_n is bounded linear with

$$\|L_n\| = \sup_{0 \leq t \leq 1} \left(\sum_{k=0}^{n} |p_k(t)| \right)$$

Here, we also use the sup-norm on $P[0,1]$. These $(\|L_n\|)$ are known as the Lebesgue constants.

Hint. Consider a similar construction as in the previous problem: Let $\tau \in [0,1]$ be such that

$$\sum_{k=0}^{n} |p_k(\tau)| = \sup_{0 \leq t \leq 1} \left(\sum_{k=0}^{n} |p_k(t)| \right)$$

let x_0 be the piecewise linear function defined by

$$x_0(0) = \operatorname{sgn} p_0(\tau), \quad x_0\left(t_j\right) = \operatorname{sgn} p_k(\tau), \quad 0 \leq k \leq n, \quad x_0(1) = \operatorname{sgn} p_n(\tau).$$

(ii) Now, we add a superscript n to the nodes and p_k, that is, we denote t_k^n and $p_k^n(t)$ for the nodes and Lagrange basis polynomials, to indicate their dependence on n. It is known that

$$\|L_n\| = \sup_{0 \leq t \leq 1} \left(\sum_{k=0}^{n} |p_k^n(t)| \right) \geq C \log n \quad \forall n \in \mathbb{N},$$

where $C > 0$ is a constant. Show that there exists a function $x \in C[0,1]$ such that

$$\sup_{n \geq 0} \|L_n x\| = \infty.$$

Remark. This shows that the Lagrange interpolating polynomials $L_n x$ doesnot converges uniformly to x for such a function.

solution. (i): $\|l_n x\| = \sup_{0\le t\le 1} |\sum_{k=0}^n |x(t_n)p_k(t)| \le \sup_{0\le t\le 1}\sum_{k=0}^n |p_k(t)|\|x\|$. $\|l_n\| \le \sup_{0\le t\le 1}\sum_{k=0}^n |p_k(t)|$,
let $\tau \in [0,1]$, $\sum_{k=0}^n |p_k(\tau)| = \sup_{0\le \tau\le 1}(\sum_{k=0}^n |p_k(t)|)$. $x_0(t_j) = sgnp_j(\tau)$, $\|x_0\| = 1$, $\|l_n\| \ge \|l_n x_0\| \ge |l_n x_0(\tau)| = \sum_{k=0}^n |p_k(\tau)| = \sup_{0\le t\le 1}(\sum_{k=0}^n |p_k(t)|)$
(ii): $\sup_{n\ge 0}\|l_n x\| < \infty \to \sup_{n\ge 0}\|l_n\| < \infty$, $\|l_n\| = \sup_{0\le t\le 1}(\sum_{k=0}^n |p_k^n(t)|) \ge c\log n$ $\square$

10. (Fourier series). For a $x \in C_{per}[0,2\pi]$ = the set continuous functions on $[0,2\pi]$ with $x(0) = x(2\pi)$, denote the partial sum of its Fourier series by

$$S_n x := \frac{a_0}{2} + \sum_{k=1}^n \{a_k \cos kt + b_k \sin kt\},$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos ktdt, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin ktdt.$$

Clearly, $S_n : C_{per}[0,2\pi] \to C_{per}[0,2\pi]$ is a linear operator. Show that

$$\|S_n\| = \int_0^{2\pi} |D_n(t)|\, dt \quad \text{where} \quad D_n(t) := \frac{1}{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

and using the fact that $\|S_n\| \ge \frac{4}{\pi^2}\log n$ to show that

$$\lim_{n\to\infty}\|S_n\| = \infty,$$

thus there exists a $x \in C_{per}[0,2\pi]$ such that $S_n x$ doesnot converge to x uniformly on $[0,2\pi]$.

Remark. (1) This shows that there exists functions whose Fourier series doesnot converge uniformly.

(2) However, for the Fejér operator $F_n : C_{per}[0,2\pi] \to C_{per}[0,2\pi]$, defined by

$$F_n x = \frac{1}{n}(S_0 x + S_1 x + \cdots + S_{n-1}x) \qquad \text{for each } n \ge 1,$$

one has

$$\lim_{n\to\infty}\|F_n x - x\| = 0 \quad \vee_x \in C_{\text{per}}[0,2\pi].$$

This is an example of the Banach-Mazur theorem (discussed later in Chapter 5: weak convergence).

solution. (i): $s_n x = x * D_n(t)$, $s_n x = \frac{a_0}{2} + \sum_{k=1}^n (a_k\cos kt + b_k \sin kt)$, $s_n x(t_0) = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt + \frac{1}{\pi}\sum_{k=1}^n [(\int_0^{2\pi} x(t)\cos ktdt)\cos kt_0 + (\int_0^{2\pi} x(t)\sin ktdt)\sin kt_0] = \frac{1}{\pi}\sum_{k=1}^n [\int_0^{2\pi}(\frac{1}{2} + (\cos kt\cos kt_0 + \sin kt \sin kt_0)x(t))]dt = \frac{1}{\pi}\int_0^{2\pi} x(t)(\frac{1}{2} + \sum_{k=1}^n \cos(kt - kt_0))dt = \frac{1}{\pi}\int_0^{2\pi} x(t)\frac{\sin(n+\frac{1}{2})(t-t_0)}{2\sin\frac{1}{2}(t-t_0)}dt = \frac{1}{\pi}\int_0^{2\pi} x(t_0+s)D_n(s)ds$

$\|S_n x\| = \|\int_0^{2\pi} x_n(t_0+s)D_n(s)ds\| \le \|x_n\|\int_0^{2\pi} x_n(t_0+s)D_n(t)dt$. $\|S_n\| \le \int_0^{2\pi}|D_n(t)|dt$, $f_n(t) = sgnD_n(t)$, $\forall \epsilon > 0, \exists E_\epsilon \subset [0,2\pi]$, $|[0,2\pi]\setminus E_\epsilon| < \epsilon$, $f_n(t)|_{E_\epsilon}$ is continuous. $|S_n g_n(t)| = |\int_0^{2\pi} g_n(t_0+s)D_n(s)ds| = |\int_0^{2\pi}(g_n - f_n)(t_n+s)D_n(s)ds| + |\int_0^{2\pi} f_n D_n(s)ds| \ge -\int_{E_\epsilon} - \int_{[0,2,\pi]\setminus E_\epsilon} + \int_0^{2\pi}|D_n(s)|ds \ge -M\epsilon\|D_n\| + \int_0^{2\pi}|D_n(s)|ds \to \|S_n\| \ge \int_0^{2\pi}|D_n(s)|ds \ge \frac{2}{\pi}\sum_{k=0}^n \frac{1}{k+1}$ $\square$

7.8 HW8

1. Show that $C[0,1]$ is not complete under the $\|\cdot\|_{L^1}$ norm

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt \quad \forall x \in C[0,1].$$

Hint. Use the Equivalent Norm Theorem.

Proof. To show that $C[0,1]$, the space of continuous functions on the interval $[0,1]$, is not complete under the $\|\cdot\|_{L^1}$ norm, consider the sequence of functions $(f_n)_{n=1}^{\infty}$ defined by:

$$f_n(t) = \begin{cases} n^2 t & \text{if } 0 \le t \le \frac{1}{n}, \\ n(2-nt) & \text{if } \frac{1}{n} < t \le \frac{2}{n}, \\ 0 & \text{if } \frac{2}{n} < t \le 1. \end{cases}$$

Each f_n is continuous on $[0,1]$ and belongs to $C[0,1]$. We show that (f_n) is a Cauchy sequence under the $\|\cdot\|_{L^1}$ norm. For m,n large enough and $m > n$, the overlap of the non-zero parts of f_n and f_m is small. The $\|\cdot\|_{L^1}$ norm is computed as:

$$\|f_n - f_m\|_{L^1} = \int_0^1 |f_n(t) - f_m(t)|dt.$$

The functions differ significantly only on $[0, \frac{2}{\max(m,n)}]$. As $m,n \to \infty$, the length of this interval and the maximum difference between the functions decrease, so $\|f_n - f_m\|_{L^1} \to 0$, confirming that (f_n) is a Cauchy sequence.

The pointwise limit $f(t)$ of $f_n(t)$ is zero for all $t \in (0,1]$, and $f(0) = 0$ as well. The limit function f is continuous and zero everywhere on $[0,1]$. Consider the convergence of f_n to f in the $\|\cdot\|_{L^1}$ norm:

$$\|f_n - f\|_{L^1} = \int_0^1 |f_n(t)|dt = \int_0^{\frac{1}{n}} n^2 t dt + \int_{\frac{1}{n}}^{\frac{2}{n}} n(2-nt)dt = 1.$$

Thus, $\|f_n - f\|_{L^1} = 1$ for all n, not approaching 0. Therefore, f_n does not converge to f in the $\|\cdot\|_{L^1}$ norm.

Hence, $C[0,1]$ is not complete under the $\|\cdot\|_{L^1}$ norm as demonstrated by the Cauchy sequence (f_n) that does not converge in this space.

□

2. Show that, as a subspace of ℓ^∞, ℓ^1 is not complete (thus not closed).

Hint. Use the Equivalent Norm Theorem.

Proof. Let us consider the space ℓ^∞ of all bounded sequences of real numbers, with the norm defined as

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

The space ℓ^1 consists of all absolutely summable sequences, with the norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$

We aim to show that ℓ^1, as a subspace of ℓ^∞, is not complete. For this, we construct a sequence in ℓ^1 that converges in the ℓ^∞ norm to a limit not in ℓ^1.

Consider the sequence $(x^{(m)})_{m=1}^{\infty}$ in ℓ^1 where each $x^{(m)}$ is defined by

$$x_n^{(m)} = \begin{cases} 1/n & \text{if } n \leq m, \\ 0 & \text{if } n > m. \end{cases}$$

Each $x^{(m)}$ is in ℓ^1 since

$$\|x^{(m)}\|_1 = \sum_{n=1}^{m} \frac{1}{n},$$

which is finite.

As $m \to \infty$, $x^{(m)}$ converges pointwise to the sequence x, defined by

$$x_n = \frac{1}{n}.$$

We must check the convergence of $x^{(m)}$ to x in the ℓ^∞ norm:

$$\|x^{(m)} - x\|_\infty = \sup_{n \in \mathbb{N}} |x_n^{(m)} - x_n| = \sup_{n > m} \frac{1}{n},$$

which tends to 0 as $m \to \infty$. Therefore, $x^{(m)} \to x$ in ℓ^∞.

However, $x \notin \ell^1$ because

$$\|x\|_1 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Thus, the limit of the Cauchy sequence $(x^{(m)})_{m=1}^{\infty}$ in the norm of ℓ^∞ is not an element of ℓ^1. Hence, ℓ^1 is not complete as a subspace of ℓ^∞, and therefore not closed in ℓ^∞.

□

3. Assume X is a normed vector space and (x_n) is a sequence in X. Show that if $(f(x_n))$ is bounded for every $f \in X'$, then (x_n) is bounded.

   Hint. Apply Banach-Steinhaus theorem for the sequence $(Jx_n(f))$ where $J : X \to X''$ is the canonical isometry.

   Proof. Let X be a normed vector space, and let (x_n) be a sequence in X. Assume that for every f in the dual space X', the sequence $(f(x_n))$ of real or complex numbers is bounded. Specifically, assume there exists a constant M_f such that

$$|f(x_n)| \leq M_f \quad \text{for all } n \in \mathbb{N}.$$

According to the Banach-Steinhaus theorem, if a family of continuous linear operators from a Banach space to a normed vector space is pointwise bounded, then it is uniformly bounded. In this context, consider the evaluation functionals $f_n : X' \to \mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) defined by

$$f_n(f) = f(x_n).$$

Each f_n is a continuous linear operator because it is a composition of the evaluation at x_n and the linear functional f. The assumption that $(f(x_n))$ is bounded for every $f \in X'$ implies that for each $f \in X'$, the set $\{f(x_n) : n \in \mathbb{N}\}$ is bounded, hence $\{f_n\}$ is pointwise bounded.

By the Banach-Steinhaus theorem, the norms $\|f_n\|$, which are the operator norms of f_n defined by

$$\|f_n\| = \sup_{\|f\| \leq 1} |f(x_n)|,$$

are uniformly bounded by some constant M. Note that $\|f_n\|$ is exactly $\|x_n\|$ in X, since

$$\|x_n\| = \sup_{\|f\| \leq 1} |f(x_n)|.$$

Therefore, we have

$$\|x_n\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Thus, the sequence (x_n) in X is bounded. This completes the proof that if $(f(x_n))$ is bounded for every $f \in X'$, then the sequence (x_n) is also bounded in X.

$\square$

4. Assume X and Y are normed vector spaces over the same field $\mathbb{K}$. Let $B \in \mathcal{L}_2(X \times Y; \mathbb{K})$, that is B is a bounded bilinear functional.

   (i) Show that, if

$$x_n \rightharpoonup x \quad \text{in } X, \quad y_n \to y \quad \text{in } Y, \quad \text{as } n \to \infty,$$

   then

$$B(x_n, y_n) \to B(x, y) \quad \text{as } n \to \infty.$$

   (ii) Consider $X = L^2(0, 2\pi)$. Recall that $e_n \rightharpoonup 0$ as $n \to \infty$ where

$$e_n(t) := \frac{1}{\sqrt{\pi}} \sin nt \quad n = 1, 2, \ldots$$

   Define

$$B(x, y) = \int_0^{2\pi} x(t)y(t)dt \quad \forall x, y \in X.$$

Show that $B \in \mathcal{L}_2(X \times X; \mathbb{R})$ but

$$B(e_n, e_n) = 1 \not\to 0 = B(0,0).$$

solution. (i): Since B is a bounded bilinear functional, it satisfies:

$$|B(x,y)| \leq \|B\| \|x\|_X \|y\|_Y,$$

for some constant $\|B\|$ and for all $x \in X$, $y \in Y$.

Consider:

$$|B(x_n, y_n) - B(x,y)| \leq |B(x_n - x, y_n)| + |B(x, y_n - y)|.$$

Since B is bilinear:

$$|B(x_n - x, y_n)| \leq \|B\| \|x_n - x\|_X \|y_n\|_Y, \quad |B(x, y_n - y)| \leq \|B\| \|x\|_X \|y_n - y\|_Y.$$

As $x_n \rightharpoonup x$, $\|x_n - x\|_X \to 0$, and since $y_n \to y$, $\|y_n - y\|_Y \to 0$ and $\|y_n\|_Y$ is bounded. Thus, $|B(x_n, y_n) - B(x,y)| \to 0$.

(ii): First, show B is bounded:

$$|B(x,y)| = \left| \int_0^{2\pi} x(t)y(t)dt \right| \leq \|x\|_{L^2} \|y\|_{L^2},$$

using the Cauchy-Schwarz inequality. Thus, B is in $\mathcal{L}_2$.

Next, consider $e_n(t) = \frac{1}{\sqrt{\pi}} \sin(nt)$:

$$B(e_n, e_n) = \int_0^{2\pi} \left(\frac{1}{\sqrt{\pi}} \sin(nt) \right)^2 dt = \frac{1}{2}.$$

This does not tend to zero even as $e_n \rightharpoonup 0$ in $L^2(0, 2\pi)$. Thus, $B(e_n, e_n) = 1$ which does not converge to $B(0,0) = 0$ as $n \to \infty$.

This establishes the properties of B as required. $\square$

Remark. This shows that part (i) does not necessarily hold if both sequences only weakly converge.

5. (Pettis theorem: A closed subspace of a reflexive Banach space is also reflexive). Assume that X is a reflexive Banach space and Y is a closed subspace of X. Show that Y is also reflexive.

Proof. by definition of being reflexive: $\forall y'' \in Y'', \exists y \in Y$ s.t. $Y''(y') = Y'(y), \forall y' \in Y'$.

define $x'' : x''(x') = |y''(x'|_Y)| \leq \|Y''\| \cdot \|x'|_Y\| \leq \|y''\| \cdot \|x'\| \to x''$ is bounded.

given X is reflexive, $\exists x \in X$, $x''(x') = x'(x), \forall x' \in X'$, claim that $x = y \in Y$, otherwise $x \notin Y$, recall that HB projection theorem: $\exists x' \in X'$ s.t. $x'|_Y = 0$ and $x'(x) = d(x, Y)$, we have $x''(x') := y''(x'|_Y) = 0$, but $x''(x') = x'(x) = d(x, Y) \neq 0 \to x \in Y$

want to show $y''(y') = y'(x)$ for $\forall y' \in Y'$: by HB theorem, extend $y' \to x' \in X'$ s.t. $x'|_Y = y'$, $y'(x) = x'(x) = x''(x') := y''(x'|_Y) = y''(y')$, so we have $y''(y') = y'(y)$ $\square$

6. (F. Riesz's lemma, revisited). Recall F. Riesz's lemma: Assume X is a normed vector space and Y is a proper closed subspace of X. Then for any $0 < \alpha < 1$, there exists $x \in X \backslash Y$ such that

$$\|x\| = 1 \quad \text{and} \quad d(x, Y) := \inf_{y \subset Y} \|x - y\| \geq \alpha.$$

Note that for any $\|x\| = 1$,

$$d(x, Y) = \inf_{y \subset Y} \|x - y\| \leq \|x - 0\| = 1,$$

since $0 \in Y$. Hence the above α cannot be larger than 1 . Also recall that $\alpha = 1$ works if $\dim X < \infty$ (see Problem 9 of Assignment 2.)

(i) Assume X is reflexive. Show that for any $f \in X'$, there exists $x \in X$ with $\|x\| = 1$ such that $f(x) = \|f\|$.

(ii) Show that if X is reflexive, then α can be equal to 1 . Remark. Recall that any finite-dimensional normed space X is reflexive, thus Problem 9 of Assignment 2 is a special case of part (ii).

(iii) Show that α can not be 1 (thus α must be strictly less than 1) for ℓ^∞.

Proof.
- by HB, $\exists x'' \in X''$ s.t. $x''(f) = \|f\|$ and $\|x''\| = 1$, since X is reflexive, $\exists x \in X$ s.t. $f(x) = x''(f) = \|f\|$ and $\|x\| = \|x''\| = 1$
- by HB, $\exists f \in X'$ s.t. $f|_Y = 0$ and $\|f\| = 1$, by the above proof, $\exists x \in X$ s.t. $f(x) = \|f\| = 1$ with $\|x\| = 1$, $\forall y \in Y$, $\|x - y\| \cdot \|f\| \geq |f(x-y)| = |f(x) - f(y)| = |1 - 0| = 1 \to \|x - y\| \geq 1 \to d(x, Y) \geq 1$
- counter example: if X is not reflexive$\to$, $\exists g \in X'$ s.t. $\forall x \in X$, $\|x\| = 1$, $g(x) < \|g\|$, $Y = ker(g)$ is closed and subspace of X, want to show $\forall x \in X \setminus Y$, $d(x, Y) < 1$.

  the above red $<$ is true by the definition of $\|g\|$, $\exists z \in X$ s.t. $0 < g(x) < g(z) < \|g\|$, consider $y = x - \frac{g(x)}{g(z)} z$, $g(y) = g(x) - \frac{g(x)}{g(z)} g(z) = 0 \to y \in Y, \|x - y\| = |\frac{g(x)}{g(z)}| \cdot \|z\| < 1 \to d(x, Y) < 1$

□

7. (Projection Theorem in a reflexive Banach space). Let $(X, \|\cdot\|)$ be a reflexive Banach space, and let Z be a nonempty closed convex subset of X.

(i) Show that, given any $x \in X$, there exists $y \in Z$ such that $\|x - y\| = d(x, Z) := \inf_{z \subset Z} \|x - z\|$. Hint. Consider an infimizing sequence and use the Eberlein-Smulian theorem.

(ii) Show that y is unique if $(X, \|\cdot\|)$ is strictly convex.

Remark. This result generalizes the Projection Theorem of a Hilbert space, noting that every Hilbert space is reflexive.

solution. (i) WLOG assume $x \in X \setminus Z$, $d(x, Z) = \inf_{y \in Z} \|x - y\|$, $\exists \{y_n\}$ s.t. $\|x - y_n\| \to d(x, Z)$. we prove that $\|y_n\| < \infty$:

Since y_n is bounded and X is reflexive, by Eberlein-Smulian theorem $\exists$ a subsequence $y_{n_k} \rightharpoonup y$ s.t. $x - y_{n_k} \rightharpoonup x - y \Rightarrow \|x - y\| \leq \liminf_{n \to \infty} \|x - y_n\| = d(x, Z) \Rightarrow \|x - y\| = d(x, Z)$ □

8. ($C[0,1]$ is not reflexive). Let $C[0,1]$ be the Banach space equipped with the sup-norm, and let Z be a subset of $C[0,1]$ be defined by

$$Z = \left\{ f \in C[0,1] \mid \int_0^{1/2} x(t)dt = 1 + \int_{1/2}^1 x(t)dt \right\}.$$

(i) Show that Z is a nonempty closed convex subset of $C[0,1]$.

(ii) Show that $\inf_{z \subset Z} \|z\| = 1$ but that there is no $y \in Z$ such that $\|y\| = 1$.

(iii) Conclude from the previous Problem that the Banach space $C[0,1]$ is not reflexive.

Remark. Recall that $(C[0,1])' = BV[0,1]$. This result indicates that $C[0,1] \subsetneq (BV[0,1])'$.

9. Recall that any inner product space is uniformly convex. (See Problem 1 of Assignment 5).

(i) Let H be a Hilbert space. Show that $x_n \to x$ if and only if

$$x_n \rightharpoonup x \quad \text{and} \quad \|x_n\| \to \|x\|.$$

Remark. ($\Leftarrow$) has been proved for any uniformly convex space in the lecture.

(ii) Prove part (i) directly without using the uniform convexity.

solution. (i): ($\Rightarrow$): Assume $x_n \to x$. This means that $\|x_n - x\| \to 0$, i.e., x_n converges to x in norm.

Since norm convergence implies weak convergence in a Hilbert space, we have $x_n \rightharpoonup x$. Furthermore, the norm convergence of x_n to x trivially implies that $\|x_n\| \to \|x\|$.

(ii): $\Rightarrow$: the proof is the same as (i) ($\Rightarrow$)

$\Leftarrow$**:** Assume $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. We show that $x_n \to x$ in norm.

For the weak convergence, $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. Additionally, $\|x_n\|^2 \to \|x\|^2$.

To prove norm convergence, consider the Hilbert space identity for vectors a and b:

$$\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\mathrm{Re}(\langle a, b\rangle).$$

Applying this to x_n and x, we have:

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\mathrm{Re}(\langle x_n, x\rangle).$$

Since $x_n \rightharpoonup x$, $\langle x_n, x\rangle \to \langle x, x\rangle = \|x\|^2$. Therefore, as $\|x_n\|^2 \to \|x\|^2$, the right-hand side of the equation approaches zero, confirming that $\|x_n - x\| \to 0$. Thus, $x_n \to x$ in norm.

This completes the proof, establishing the equivalence as required.

□

7.9 HW9

1. (i) Let $X = C[0,1]$ and define $T : X \to X$ by $Tx = vx$, where $v \in X$ is fixed. Find $\sigma(T)$

   (ii) Find a linear operator $T : C[0,1] \to C[0,1]$ whose spectrum is a given interval $[a,b]$.

   (iii) Show that T is compact if and only if $v(t) \equiv 0$.

   solution. (i) Finding the Spectrum $\sigma(T)$ for $T : X \to X$ defined by $Tx = vx$

   Consider the space $X = C[0,1]$, the space of continuous functions on $[0,1]$, and the operator T defined by multiplication by a fixed function $v \in X$. T is a multiplication operator. The spectrum $\sigma(T) includes all \lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

   Invertibility of $T - \lambda I$: For a function $x \in X$, $(T-\lambda I)x = vx - \lambda x = (v(t) - \lambda)x(t)$. This operator is invertible if the equation $(v(t) - \lambda)x(t) = 0$ for all $t \in [0,1]$ has only the trivial solution x(t) = 0. This fails if $v(t) = \lambda$ for any $t \in [0,1]$, allowing for a non-zero $x(t)$ supported where $v(t) = \lambda$.

   Spectrum Composition: Therefore, $\sigma(T)$ comprises the range of values of v(t) over [0,1], specifically,

   $$\sigma(T) = \{v(t) : t \in [0,1]\}.$$

   (ii) Constructing an Operator T : $C[0,1] \to C[0,1]$ Whose Spectrum is $[a,b]$

   Consider a function v(t) that linearly maps the interval [0,1] to [a, b], such as $v(t) = a + (b-a)t$. Define T by $Tx = vx$, where $v(t) = a + (b-a)t$. This operator has the spectrum precisely equal to [a, b]:

   As t varies from 0 to 1, v(t) linearly varies from a to b. Thus,

   $$\sigma(T) = \{a + (b-a)t : t \in [0,1]\} = [a,b].$$

   (iii) Showing Compactness of T

   The operator T is compact if and only if $v(t) \equiv 0$. The multiplication operator T defined by $Tx = vx$ is compact if its image under any bounded subset of C[0,1] has a compact closure. If $v(t) \neq 0$ on any subset of [0,1], the family $\{vx : x \in B\}$ for any bounded B in $C[0,1]$, generally fails to be equicontinuous or uniformly bounded unless $v(t) = 0$.

   If$v(t) \equiv 0$: Then $Tx = 0$ for all $x \in X$, making T a constant (zero) operator, which is trivially compact as its image is the zero function, a compact set in $C[0,1]$.

   Therefore, the multiplication operator T is compact if and only if $v(t) \equiv 0$. If v(t) is non-zero anywhere on [0,1], the operator is not compact.

   $\square$

2. (Idempotent operator). Let X be a Banach space and $T \in \mathcal{L}(X)$.T is said to be idempotent if $T^2 = T$. Show that if $T \neq 0$ and $T \neq I$, then its spectrum is $\sigma(T) = \{0,1\}$.

solution. Recall that the spectrum of an operator T, denoted $\sigma(T)$, is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible, where I is the identity operator on X. This includes all λ for which $T - \lambda I$ is either not injective (has non-trivial kernel) or not surjective (has non-trivial cokernel), or both.

Since T is idempotent, $T^2 = T$, meaning T projects any vector $x \in X$ onto some subspace. Define:

- $V = \mathrm{im}(T)$ (the image of T)
- $W = \ker(T)$ (the kernel of T)

Every vector $x \in X$ can be uniquely written as $x = Tx + (x - Tx)$, where $Tx \in V$ and $(x - Tx) \in W$. Moreover, T acts as the identity on V and as the zero operator on W. Thus, we can view X as $X = V \oplus W$.

Consider the operator $T - \lambda I$. We have:

$$(T - \lambda I)x = Tx - \lambda x = Tx - \lambda(Tx + (x - Tx)) = (1 - \lambda)Tx - \lambda(x - Tx)$$

- If $\lambda \neq 1$, $T - \lambda I$ restricted to V is $(1 - \lambda)I_V$, which is invertible on V if $\lambda \neq 1$.
- On W, $T - \lambda I$ acts as $-\lambda I_W$, which is invertible on W if $\lambda \neq 0$.
- If $\lambda = 0$, $T - 0I = T$, which is not invertible unless T is surjective and injective, which we exclude because $T \neq I$. Thus, $0 \in \sigma(T)$.
- If $\lambda = 1$, $T - I = T - I$, and $T - I$ acts as the zero map on V and as $-I$ on W. Since it acts as zero on V, it is not injective. Hence, $1 \in \sigma(T)$.

For $\lambda \notin \{0, 1\}$, both $(1 - \lambda)I_V$ and $-\lambda I_W$ are invertible, thus making $T - \lambda I$ invertible. Hence, such λ do not belong to $\sigma(T)$.

$\square$

3. Consider the ℓ^2 space and the right-shift operator, $T : \ell^2 \to \ell^2$, defined by

$$Tx = (0, \xi_1, \xi_2, \dots) \quad \forall x = (\xi_1, \xi_2, \dots) \in \ell^2.$$

Show that

(i) $\sigma_p(T) = \emptyset$,

(ii) $\sigma_r(T) = \{\lambda \in \mathbb{C} || \lambda |< 1\}$

(iii) $\sigma_c(T) = \{\lambda \in \mathbb{C} || \lambda |= 1\}$

(iv) $\rho(T) = \{\lambda \in \mathbb{C} || \lambda |> 1\}$.

solution. denote the unit circle $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ by $\mathbb{D}$, $\|T\| = 1 \to \sigma(T) = \overline{\mathbb{D}}$, we first prove that $\sigma_p(T) = \emptyset$, otherwise $\exists \lambda \in \mathbb{C}$, $\exists 0 \neq x \in \ell^2$ s.t.

$$(0, x_1, x_2, ...) = Tx = \lambda x = (\lambda x_1, \lambda x_2, ...)$$

if $\lambda = 0 \to x = 0$, if $\lambda \neq 0 \to x_1 = 0 \to x_2 = 0 \to ... \to x = 0$, contradiction.

next we prove that $\mathbb{D} \subset \sigma_r(A)$, let $\lambda \in \mathbb{D}$, claim that $\overline{Range(\lambda I - A)} \neq \ell^2$, which is equivalent to $Range(\lambda I - A)^\perp \neq \{0\}$, let $z = (1, \overline{\lambda}, \overline{\lambda}^2, ...)$, then

$$\langle(\lambda I - A)x, z\rangle = \langle(\lambda x_1, \lambda x_2 - x_1, \lambda x_3 - x_2, ...), (1, \overline{\lambda}, \overline{\lambda}^2, ...)\rangle$$

$$= \lambda x_1 + \lambda^2 x_2 - \lambda x_1 + \lambda^3 x_3 - \lambda^2 x_2 + ... = 0 \to 0 \neq z \in Range(\lambda I - A)^\perp$$

so we proved $\mathbb{D} \subset \sigma_r(A)$.

next we prove $\partial\mathbb{D} \subset \sigma_c(A)$. let $\lambda \in \partial\mathbb{D}$.

we prove $Range(\lambda I - A) \neq \ell^2$, $Range(\lambda I - A) \ni y = (\lambda I - A)x \to y_1 = \lambda x_1, y_k = \lambda x_k - x_{k-1}, k \geq 2 \to y_1 = \lambda x_1, \lambda^{k-1} y_k = \lambda^k x_k - \lambda^{k-1} x_{k-1}, k \geq 2 \to \sum_{k=1}^n \lambda^{k-1} y_k = \lambda^n x_n$, $Range(\lambda I - T) = \ell^2$, let $y = e_1$, $\exists x \in \ell^2$ s.t. $e_1 = (\lambda I - T)x \to \lambda^n x_n = 1, n = 1, 2, ... \to x = (\frac{1}{\lambda}, \frac{1}{\lambda^2}, ...)$, given $|\lambda| = 1$, $x = (1, 1, ...) \notin \ell^2$, contradiction, so we proved $Range(\lambda I - T) \neq \ell^2$

next we prove $\overline{Range(\lambda I - T)} = \ell^2$, only need to prove that $Range(\lambda I - T)^\perp = \{0\}$, $\forall x \in Range(\lambda I - T)^\perp$, $0 = \langle z, (\lambda I - T)e_n\rangle = \overline{\lambda} z_n - z_{n+1}, \forall n$, so $z_{n+1} = \overline{\lambda} z_n \to |z_{n+1}| = |z_n| \to z = 0$, combining the above two claims ($Range(\lambda I - T) \neq \ell^2$ and $\overline{Range(\lambda I - T)} = \ell^2$), we conclude that $\partial\mathbb{D} \subset \sigma_c(T)$

$\partial\mathbb{D} \cup \mathbb{D} = \overline{\mathbb{D}} \subset \sigma_c(T) \cup \sigma_r(T) \subset \sigma(T) \subset \overline{\mathbb{D}}$, recall $\sigma_r(T) \cap \sigma_c(T) = \emptyset$, we conclude that $\sigma_c(T) = \partial\mathbb{D}$ and $\sigma_r(T) = \mathbb{D}$

finally, $\rho(T) = \sigma(T)^c = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ □

4. Consider the ℓ^1 space and the right-shift operator, $T : \ell^1 \to \ell^1$, defined by

$$Tx = (0, \xi_1, \xi_2, \ldots) \quad \forall x = (\xi_1, \xi_2, \ldots) \in \ell^1.$$

Show that

(i) $\sigma_p(T) = \emptyset$;

(ii) $\sigma_r(T) = \{\lambda \in \mathbb{C} || \lambda | \leq 1\}$;

(iii) $\rho(T) = \{\lambda \in \mathbb{C} || \lambda | > 1\}$;

(iv) $\sigma_c(T) = \emptyset$.

solution. □

5. Consider the ℓ^2 space and the left shift operator, $T : \ell^2 \to \ell^2$, defined by

$$Tx = (\xi_2, \xi_3, \ldots) \quad \forall x = (\xi_1, \xi_2, \ldots) \in \ell^2.$$

Show that

(i) $\sigma_p(T) = \{\lambda \in \mathbb{C} || \lambda | < 1\}$;

(ii) $\sigma_c(T) = \{\lambda \in \mathbb{C} || \lambda | = 1\}$;

(iii) $\rho(T) = \{\lambda \in \mathbb{C} || \lambda | > 1\}$;

(iv) $\sigma_r(T) = \emptyset$.

Remark. This result holds for all left shift operators on ℓ^p with $1 \leq p < \infty$. For $p = \infty$, see the next problem.

solution. let $D = \{\lambda : |\lambda| < 1\}$, $\lambda \in \sigma_p(T) \leftrightarrow \exists 0 \neq x = \{x_n\}_{n=1}^{\infty} \in \ell^2 \leftrightarrow \lambda x_1 = \lambda x_2, \lambda x_2 = x_3 \leftrightarrow x = \{\lambda^{n-1} x_1\}_{n=1}^{\infty}$ $0 \neq x \in \ell^2$, so $|\lambda| < 1$, $\sigma_p(T) = \{\lambda : |\lambda| < 1\} = D$

let $\lambda \in \partial D$, we prove $\partial D \subset \sigma_c(T)$ by showing the following two arguments: 1. $Range(\lambda I - T) \neq \ell^2$ and 2. $\overline{Range(\lambda I - T)} = \ell^2$:

1. let $y \in Range(\lambda I - T) \to \exists 0 \neq x \in \ell^2$, $y = (\lambda I - T)x = (\lambda x_1 - x_2, \lambda x_2 - x_3, ...) = \{\lambda x_n - x_{n+1}\}_{n=1}^{\infty} \to \sum_{n=1}^{k} \lambda^{-n} y_n = x_1 - \lambda^{-k} x_{k+1} \to x_1$ as $k \to \infty$, so we have $\sum_{n=1}^{\infty} \lambda^{-n} y_n = x_1$. if we take $y = \{\lambda^n \cdot \frac{1}{n}\}_{n=1}^{\infty} \in \ell^2$, the series diverges, which shows that there does not exist x s.t. $y = (\lambda I - T)x$, so we conclude $Range(\lambda I - A) \neq \ell^2$

2. $\forall x = \{x_n\}_{n=1}^{\infty} \in \ell^2$, take sufficient large N s.t. $\sum_{n=N+1}^{\infty} |x_n|^2 < \epsilon^2$. let $y_j = (x_1, x_2, ..., x_j, 0, ...)$, then $\|y_N - x\|^2 = \sum_{n=N+1}^{\infty} |x_n|^2 < \epsilon^2 \to \|y_N - x\| < \epsilon$, so we only need to prove that $y_N \in Range(\lambda I - T)$, let $z = \begin{cases} \sum_{k \le n \le N} \lambda^{-n+k-1} y_n & k \le N \\ 0 & k > N \end{cases}$, we have $y_N = (\lambda I - T)z$ and thus $\partial D \subset \sigma_c(T)$

finally. note that $\|Tx\|^2 = \|x\|^2 + |x_1|^2$, we have $\|Tx\| \le \|x\|$, the equality holds when $x_1 = 0$, so we have $\|T\| = \sup_{0 \neq x \in \ell^2} \frac{\|Tx\|}{\|x\|} = 1$ and thus $\sigma(T) \subset \{\lambda : |\lambda| \le 1\} = \overline{D}$.

we have proved that $\overline{D} \supset \sigma(T) \supset \sigma_c(T) \cup \sigma_p(A) \supset \partial D \cup D = \overline{D} \to \overline{D} = \sigma_c(T) \cup \sigma_p(A)$ and thus $\sigma_c(T) = \partial D$, $\sigma_p(T) = D$ and $\sigma_r(T) = \emptyset$. $\rho(T) = \sigma(T)^c = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ □

6. Consider the ℓ^∞ space and the left shift operator, $T : \ell^\infty \to \ell^\infty$, defined by
$$Tx = (\xi_2, \xi_3, \dots) \quad \forall x = (\xi_1, \xi_2, \dots) \in \ell^\infty.$$

Show that

(i) $\sigma_p(T) = \{\lambda \in \mathbb{C} || \lambda | \le 1\}$;

(ii) $\rho(T) = \{\lambda \in \mathbb{C} || \lambda | > 1\}$;

(iii) $\sigma_c(T) = \emptyset$;

(iv) $\sigma_r(T) = \emptyset$.

solution. compared with Question 5, we note that when $|\lambda| = 1$, $Tx = \lambda x = (\lambda\xi_1, \lambda\xi_2, ...) = (\xi_2, \xi_3, ...)$, we still have $0 \neq x = \{\lambda^{n-1} x\}_{n=1}^{\infty}$, let $x \in \ell^\infty$, if $\lambda < 1$, we have $x \in \ell^\infty$, for $\lambda = 1$, we have $x = (x_1, x_1, ...)$ and $\|x\|_\infty = \max\{x_1, x_1, ...\} = x_1 < \infty$. so we conclude that $\sigma_p(T) = \{\lambda : |\lambda| \le 1\}$

next we prove that $\rho(T) = \{\lambda : |\lambda| > 1\}$: recall that if $Range(\lambda I - T) = X$, then $\lambda \in \rho(T)$. given $|\lambda| > 1$, $y \in Range(\lambda I - T) \to \exists 0 \neq x \in \ell^\infty$ s.t. $y = (\lambda I - T)x = (\lambda x_1 - x_2, \lambda x_2 - x_3, ...) = \{\lambda x_n - x_{n+1}\}_{n=1}^{\infty} \to \sum_{n=1}^{k} \lambda^{-n} y_n = x_1 - \lambda^{-k} x_{k+1} \to x_1$ as $k \to \infty$, we have $\sum_{n=1}^{\infty} \lambda^{-n} y_n = x_1$, $x_i = \sum_{n=i}^{\infty} \lambda^{-n} y_n$, since $\lambda > 1$, we have $x = (x_1, x_2, ..., x_i, ...) \in \ell^\infty \to y_n < \infty \to y \in \ell^n$ so we have $|\lambda| > 1 \to \lambda\rho(T)$. $\{\lambda : |\lambda| > 1\} \subset \rho(T)$, since $\{\lambda : |\lambda| \le 1\} = \sigma_p(T)$, we have $\rho(T) = \{\lambda : |\lambda| > 1\}$

finally, we have $\sigma_r(T) = \sigma_r(T) = \emptyset$ □

7. Let $1 \leq p \leq \infty$. Show that $T : \ell^p \to \ell^p$ defined by

$$Tx = (\xi_j/j) \quad \forall x = (\xi_j) \in \ell^p$$

is a compact operator.

solution. The operator T modifies each sequence $x = (\xi_1, \xi_2, \xi_3, \ldots)$ in ℓ^p by scaling each component by its position index, yielding the transformed sequence $(0, \frac{\xi_1}{2}, \frac{\xi_2}{3}, \ldots)$.

The norm of T can be estimated using Minkowski's inequality:

$$\|Tx\|_p^p = \sum_{j=1}^{\infty} \left|\frac{\xi_j}{j}\right|^p \leq \sum_{j=1}^{\infty} \frac{|\xi_j|^p}{j^p}.$$

Here, $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is a convergent series for $p > 1$, which validates that T is a well-defined operator on ℓ^p.

T can be approximated by finite-rank operators T_n, defined by:

$$T_n x = \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \ldots, \frac{\xi_n}{n}, 0, 0, \ldots\right).$$

Each T_n is finite-rank as its range is spanned by at most n elements of the basis of ℓ^p.

The norm of $T - T_n$ is given by:

$$\|Tx - T_n x\|_p^p = \sum_{j=n+1}^{\infty} \left|\frac{\xi_j}{j}\right|^p.$$

As $n \to \infty$, $\sum_{j=n+1}^{\infty} \frac{1}{j^p} \to 0$, and thus by the dominated convergence theorem, $\lim_{n\to\infty} \|Tx - T_n x\|_p = 0$ for all $x \in \ell^p$, uniformly.

The operator T can be approximated in operator norm by the finite-rank operators T_n, hence T is compact on ℓ^p.

$\square$

8. ($0 \in \sigma(T)$ for compact operator T on an infinite-dimensional space X).

(i) Prove that, if X is an infinite-dimensional Banach space and $T : X \to X$ is a compact linear operator, then $0 \in \sigma(T)$.

Hint. Suppose $0 \in \rho(T)$, then $T_0^{-1} = T^{-1}$ exists, and $T^{-1} \in B(X, X)$. Then, $I = TT^{-1}$ is compact (why?), contradiction!

(ii) Prove that, if X is an infinite-dimensional normed space and $T : X \to X$ is a compact linear operator, then $0 \in \sigma(T)$.

Hint. Suppose $0 \in \rho(T)$, argue that the closed unit ball of X must be compact.

solution. (i) Suppose for contradiction that $0 \notin \sigma(T)$. This implies $0 \in \rho(T)$, the resolvent set, meaning that T is invertible with $T^{-1} : X \to X$

bounded. Given that T is compact, the composition $I = TT^{-1}$, where I is the identity operator, must also be compact.

However, the identity operator I on an infinite-dimensional Banach space cannot be compact. The compactness of an operator would imply that it maps bounded sets to relatively compact sets, which is not possible for the identity operator on such spaces (demonstrated by Riesz's lemma among other results). Therefore, this contradiction leads us to conclude that $0 \in \sigma(T)$, as the assumption $0 \notin \sigma(T)$ is untenable.

(ii) Assume $0 \notin \sigma(T)$, implying $0 \in \rho(T)$ and thus T is invertible with T^{-1} bounded. Consider the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$. Since T^{-1} is bounded, $T^{-1}(B)$ is a bounded set.

The operator T being compact means $T(T^{-1}(B))$, which equals $I(B) = B$, must be relatively compact. However, the closed unit ball in an infinite-dimensional normed space cannot be compact due to its infinite-dimensionality. This contradiction implies that our initial assumption $0 \notin \sigma(T)$ is incorrect, confirming $0 \in \sigma(T)$.

This proof demonstrates the necessity of including 0 in the spectrum of any compact operator acting on an infinite-dimensional space, aligning with foundational principles in functional analysis regarding operator theory and compactness.

$\square$

9. Consider the operator $T : \ell^1 \to \ell^1$ as in problem 7. Show that

$$\sigma_c(T) = \{0\}; \quad \sigma_p(T) = \left\{ \frac{1}{n} \middle| \, n \in \mathbb{N} \right\}; \quad \rho(T) = \mathbb{C} \backslash \left(\{0\} \cup \left\{ \frac{1}{n} \middle| \, n \in \mathbb{N} \right\} \right); \quad \sigma_r(T) = \emptyset.$$

solution. The point spectrum ($\sigma_p(T)$) consists of eigenvalues of T. An eigenvalue λ must satisfy:

$$Tx = \lambda x \quad \text{for some } x \neq 0.$$

This leads to:

$$\left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots \right) = (\lambda \xi_1, \lambda \xi_2, \lambda \xi_3, \ldots).$$

For each ξ_n, we have:

$$\frac{\xi_n}{n} = \lambda \xi_n \quad \Rightarrow \quad \xi_n \left(\frac{1}{n} - \lambda \right) = 0.$$

If $\xi_n \neq 0$, λ must be $1/n$. Therefore:

$$\sigma_p(T) = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

The continuous spectrum ($\sigma_c(T)$) includes λ for which $T-\lambda I$ is injective, has dense range, but no bounded inverse exists, and it's not surjective. For $\lambda = 0$:

$$Tx = 0 \quad \Rightarrow \quad \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \ldots\right) = (0, 0, \ldots).$$

T cannot produce any sequence with non-zero initial terms, thus not surjective. Therefore:

$$\sigma_c(T) = \{0\}.$$

The residual spectrum ($\sigma_r(T)$) would consist of λ where $T-\lambda I$ is injective but its range is not dense. Since $T-\lambda I$ can approximate any element in ℓ^1 (due to the nature of T and the density of the range of scaled sequences), there are no λ for which the range is not dense:

$$\sigma_r(T) = \emptyset.$$

The resolvent set ($\rho(T)$) includes all λ for which $T-\lambda I$ is both injective and surjective with a bounded inverse. Excluding $\lambda = 0$ and $\lambda = \frac{1}{n}$, for all other λ, the inverse operations are well-defined and bounded. Thus:

$$\rho(T) = \mathbb{C} \setminus \left(\{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}\right).$$

□

10. Consider the operator $T : \ell^1 \to \ell^1$ defined by

$$Tx = \left(0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots\right) \qquad \forall x = (\xi_1, \xi_2, \ldots) \in \ell^1.$$

Show that

$$\sigma(T) = \sigma_r(T) = \{0\}.$$

8 midterm review

8.1 HW

HW1

1. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.

solution. Given $\epsilon > 0$, since (x_n) and (y_n) are Cauchy sequences, there exists an $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$, we have $d(x_m, x_n) < \frac{\epsilon}{2}$ and $d(y_m, y_n) < \frac{\epsilon}{2}$.

Consider $|a_n - a_m| = |d(x_n, y_n) - d(x_m, y_m)|$. By the triangle inequality, we know that:

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) \quad \text{and} \quad d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n).$$

Combining these inequalities, we get:

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m).$$

Since both $d(x_n, x_m) < \frac{\epsilon}{2}$ and $d(y_n, y_m) < \frac{\epsilon}{2}$ for all $m, n \geq N_1$, we can conclude:

$$|a_n - a_m| = |d(x_n, y_n) - d(x_m, y_m)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $m, n \geq N_1$. $\square$

2. (Equivalent metrics). If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,

$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same. These two metrics are called equivalent.

solution. $(\Rightarrow)$ Suppose (x_n) is Cauchy in (X, d_1). For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$d_1(x_m, x_n) < \frac{\epsilon}{b}.$$

Multiplying by a and using the inequality, we get

$$ad_1(x_m, x_n) \leq d_2(x_m, x_n) \leq bd_1(x_m, x_n) < \epsilon,$$

showing (x_n) is also Cauchy in (X, d_2).

$(\Leftarrow)$ Conversely, if (x_n) is Cauchy in (X, d_2), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$d_2(x_m, x_n) < \epsilon.$$

Given the lower bound, we find

$$ad_1(x_m, x_n) \leq d_2(x_m, x_n) < \epsilon.$$

Dividing by a, we obtain

$$d_1(x_m, x_n) < \frac{\epsilon}{a},$$

indicating that (x_n) is Cauchy in (X, d_1). $\square$

3. (Open and closed balls, closure).

   (a) For any metric space (X, d), show that $\overline{B_1(x)} \subset \bar{B}_1(x)$. Here is an example showing that why $\overline{B_1(x)} \neq \bar{B}_1(x)$ in general: Let X be a set and d the discrete metric on X.

   (b) Show that every subset of X is open, and hence every subset of X is closed.

(c) Pick a $x \in X$. What is the open ball $B_1(x)$? what is the closed ball $\bar{B}_1(x)$. Show that $\overline{B_1(x)} \subsetneq \bar{B}_1(x)$.

Remark. For most examples in applications, we usually have $\overline{B_1(x)} = \bar{B}_1(x)$.

solution. (a) For any metric space (X, d), to show that $\overline{B_1(x)} \subset \bar{B}_1(x)$, we need to understand the definitions of open and closed balls, as well as closure. The closed ball $\bar{B}_1(x)$ includes all points within a distance of 1 from x, including the boundary. The closure $\overline{B_1(x)}$ of the open ball includes all points within and limit points. Since every point in the open ball and its limit points are in the closed ball, $\overline{B_1(x)} \subset \bar{B}_1(x)$.

(b) In the discrete metric, where $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$, every subset of X is open because for any point x in any subset S, we can always find a ball $B_\epsilon(x)$ with $\epsilon < 1$ that contains only x, hence lying entirely in S. Consequently, every subset is also closed as the complement of any subset is open.

(c) In the discrete metric, the open ball $B_1(x)$ around any point x contains only x itself, since the distance to any other point is 1. The closed ball $\bar{B}_1(x)$, however, contains all points because the distance from x to itself is 0, and to all others is exactly 1, which is within the "radius" of the closed ball. Thus, $\overline{B_1(x)} = \{x\}$ and $\bar{B}_1(x) = X$, showing $\overline{B_1(x)} \subsetneq \bar{B}_1(x)$.

Remark. In most practical applications, the closure of the open ball is equal to the closed ball, but the discrete metric provides a counterexample where this is not the case.

□

HW2

1. Let $c_0 \subset \ell^\infty$ be the space of all sequences converging to zero. Show that c_0 is a closed subspace of ℓ^∞, so that c_0 is complete.

solution. Let c_0 be the space of all sequences (a_n) of real or complex numbers such that $\lim_{n\to\infty} a_n = 0$. We want to show that c_0 is a closed subspace of ℓ^∞, the space of all bounded sequences.

Subspace: First, we show c_0 is a subspace of ℓ^∞. Clearly, $c_0 \subset \ell^\infty$ since sequences converging to zero are bounded. If $(a_n), (b_n) \in c_0$ and $\lambda \in \mathbb{C}$ (or $\mathbb{R}$), then $\lim_{n\to\infty}(a_n + \lambda b_n) = \lim_{n\to\infty} a_n + \lambda \lim_{n\to\infty} b_n = 0$, so $(a_n + \lambda b_n) \in c_0$. Hence, c_0 is a subspace.

Closedness: To show c_0 is closed, let $(x^{(m)})$ be a sequence in c_0 converging to some $x \in \ell^\infty$. We need to show that $x \in c_0$. Since $(x^{(m)}) \to x$ in ℓ^∞, for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $\|x^{(m)} - x\|_\infty < \epsilon$ for all $m \geq M$. This means that for all $n \in \mathbb{N}$, $|x_n^{(m)} - x_n| < \epsilon$ for $m \geq M$, implying that x_n is the limit of a sequence of numbers converging to 0. Hence, $\lim_{n\to\infty} x_n = 0$ and $x \in c_0$. Therefore, $x \in c_0$ and c_0 is closed in ℓ^∞.

Completeness: Since c_0 is a closed subspace of a complete space ℓ^∞, it follows that c_0 is also complete.

$\square$

2. Let $Y \subset \ell^\infty$ be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of ℓ^∞ but Y is not a closed subspace.

solution. Let Y be the subset of ℓ^∞ consisting of all sequences (a_n) where only finitely many a_n are nonzero. We aim to show that Y is a subspace of ℓ^∞ but not a closed subspace.

Subspace: To prove Y is a subspace of ℓ^∞, we must show it is closed under vector addition and scalar multiplication.

- Let $(a_n), (b_n) \in Y$ with only a finite number of nonzero terms each. Their sum $(a_n + b_n)$ will also have only a finite number of nonzero terms since the nonzero terms of $(a_n + b_n)$ can only come from the nonzero terms of (a_n) and (b_n). Therefore, $(a_n + b_n) \in Y$.
- Let $(a_n) \in Y$ and $\lambda \in \mathbb{C}$ (or $\mathbb{R}$). The sequence (λa_n) will have the same sparsity as (a_n), meaning if (a_n) had a finite number of nonzero terms, so will (λa_n). Thus, $(\lambda a_n) \in Y$.

Hence, Y is a subspace of ℓ^∞.

Not Closed: To show Y is not a closed subspace, consider the sequence of sequences $(x^{(m)})$ defined by $x^{(m)} = (1, 1/2, 1/3, \ldots, 1/m, 0, 0, \ldots)$, where each $x^{(m)}$ is in Y. This sequence converges in ℓ^∞. $x_m \to x = (1, 1/2, 1/3, ..., 1/n, ...)$ and $\sum_{n=1}^\infty |x_n| = \infty \to x \notin \ell^\infty$. so Y is not closed. $\square$

3. (Complete) $\Longleftrightarrow$ (Absolute convergence implies convergence)

   (a) Let $(X, \|\cdot\|)$ be a normed space. Prove that X is complete if and only if every series $\sum_{j-1}^\infty x_j$ in X satisfying $\sum_{j-1}^\infty \|x_j\| < \infty$ converges to a limit in X.

   (b) Give an example of a (necessarily incomplete!) space X and a series for which $\sum_{j-1}^\infty \|x_j\| < \infty$ but $\sum_{j-1}^\infty x_j$ does not converge in X. Hint: Consider Y in last problem and $(y_n) \subset Y$ with $y_n = \left(\eta_j^{(n)}\right), \eta_n^{(n)} = \frac{1}{n^2}, \eta_j^{(n)} = 0$ for all $j \neq n$.

solution. (a) $\Rightarrow$: Assume X is complete. Let $\sum_{j=1}^\infty x_j$ be a series in X with $\sum_{j=1}^\infty \|x_j\| < \infty$. Consider the partial sums $S_n = \sum_{j=1}^n x_j$. We want to show that (S_n) is Cauchy. For $m > n$, we have

$$\|S_m - S_n\| = \left\| \sum_{j=n+1}^m x_j \right\| \le \sum_{j=n+1}^m \|x_j\|,$$

which can be made arbitrarily small since $\sum_{j=1}^\infty \|x_j\|$ is convergent. Hence, (S_n) is Cauchy and by completeness of X, converges to a limit in X.

$\Leftarrow$: Let (x_n) be a Cauchy sequence in X. We need to show it converges. For each $n \in \mathbb{N}$, there exists N_n such that for all $m, l > N_n$, $\|x_m - x_l\| < \frac{1}{2^n}$. Let $y_1 = x_{N_1}$ and $y_n = x_{N_n} - x_{N_{n-1}}$ for $n > 1$. Then $\sum_{j=1}^{\infty} \|y_j\|$ converges, implying $\sum_{j=1}^{\infty} y_j = x_{N_n}$ converges in X, showing that (x_n) has a convergent subsequence x_{N_n}. Since x_n is a Cauchy sequence, we conclude that x_n converges in X.

(b) Consider the space c_{00} of all sequences of real numbers that are eventually zero, with the norm $\|\cdot\|_\infty$. Let $y_n = (\eta_j^{(n)})$ where $\eta_n^{(n)} = \frac{1}{n^2}$ and $\eta_j^{(n)} = 0$ for $j \neq n$. Then $\sum_{j=1}^{\infty} \|y_j\|_\infty = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$, but $\sum_{j=1}^{\infty} y_j$ does not converge in c_{00}, since the limit would be a sequence with infinitely many nonzero terms, which is not in c_{00}. This shows that c_{00} is not complete.

□

4. Assume X is a normed space and Y is a finite-dimensional, proper subspace of X. Show that $\alpha = 1$ also works in the F. Riesz's Lemma.

solution. WTS for $\alpha = 1$, $\exists u \in X$ s.t. $\|u\| = 1$ and $\|u - y\| \geq \alpha$ for all $y \in Y$: define $d = d(v, Y) = \inf_{y \in Y} \|v - y\|$, $d > 0$, only need to show that $\exists y_0 \in Y$ s.t. $\|v - y_0\| = d = \inf_{y \in Y} \|v - y\| \to \forall y \in Y, \|v - y\| \geq \|v - y_0\|$. let $\{b_1, b_2, .., b_n\}$ be a basis for Y. let $y_k = \sum \alpha_{kl} b_l \in Y$ and $\|y_k = v\| \to d$, $\{y_k\}$ has a subsequence $\{y_{kl}\}$ s.t. $\alpha_{kl} \to \alpha_l$ for $l = 1, 2, ..., n$ and $\|v - y_0\| \leq \|v - y\| + \|v - y_0\| = \|v - y_k\| + \sum |\alpha_l - \alpha_{kl}| b_l$. we have $\sum |\alpha_l - \alpha_{kl}| b_l \geq c(\sum |\alpha_l - \alpha_{kl}|) > 0$ by a lemma. so we have $\|v - y_0\| \leq \|v - y\|$ for all $y \in Y \to \|v - y_0\| = d$, then $u \triangleq \frac{v - y_0}{\|v - y_0\|}$: $\|u\| = 1$ and $\|u - y\| = \|\frac{v - y_0}{\|v - y_0\|} - y\| = \|\frac{v - y_0 - y\|v - y_0\|}{\|v - y_0\|}\| \geq \frac{\|v - y_0\|}{\|v - y_0\|} = 1$ □

HW4

1. Prove that $C[0, 1]$ equipped with the sup-norm is not an inner product space.

solution. by the parallelogram law, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in C[0, 1]$$

here $\|f(t)\| = \sup_{t \in [0,1]} |f(t)|$. So we have

$$(\sup |x + y|)^2 + (\sup |x - y|)^2 = 2 \sup |x|^2 + 2 \sup |y|^2 \quad \forall x, y \in C[0, 1]$$

consider $x(t) = t$ and $y(t) = t^2$, we have

$$RHS = 2 \cdot 1 + 2 \cdot 1 = 4$$

$$LHS = (\sup |t - t^2|)^2 + (\sup |t + t^2|)^2 = \frac{1}{16} + 4 \neq RHS$$

so we conclude that the pair is not an inner product space. □

2. (Least squares). Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^m$.

   (a) Let A be a given $m \times n$ matrix and $c \in \mathbb{R}^m$ be a given vector. Show that the least square problem: Find $\boldsymbol{x} \in \mathbb{R}^n$ such that

   $$|Ax - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

   has at least one solution.

   (b) Show that a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfies the above least square problem if and only if $\boldsymbol{x}$ is a solution of the normal equation:

   $$A^T A\boldsymbol{x} = A^T c.$$

   (c) Show that the solution to the above normal equation is unique if and only if rank $A = n$ (which of course implies that $n \leq m$), or equivalently, if and only if the symmetric matrix $A^T A$ is positive definite.

   solution. i. suppose $\exists x' \in \mathbb{R}^n$ s.t.

   $$|Ax' - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

   By definition, $\forall x \in \mathbb{R}^n$, $|Ax - c| \geq \inf_{y \in \mathbb{R}^n} |Ay - c|$, $\forall n > 0$, $\exists x_n \in \mathbb{R}^n$, $|Ax_n - c| < \inf_{y \in \mathbb{R}^n} |Ay - c| + \frac{1}{n}$,

   $$|Ax_n| \leq |Ax_n - c| + |c| \leq \inf_{y \in \mathbb{R}^n} |Ay - c| + \frac{1}{n} + |c| \leq \inf_{y \in \mathbb{R}^n} |Ay - c| + 1 + |c|$$

   so $\{x_n\}$ is bounded. $\mathbb{R}^n$ is finitely dimensional, and we have $\{x_{n_k}\} \subset \{x_n\}$ and $\{x_{n_k}\} \to x' \in \mathbb{R}^n$

   $$\inf_{y \subset \mathbb{R}^n} |Ay - c| \leq |Ax_n - c| \leq \inf_{y \subset \mathbb{R}^n} |Ay - c| + \frac{1}{n_k}$$

   as $k \to \infty$ we have

   $$|Ax - c| = \inf_{y \subset \mathbb{R}^n} |Ay - c|$$

   ii. $\Rightarrow$: x is the solution of the least square problem $\to$ x is the minimum point of the function $f(x) = (Ax - c)^T (Ax - c)$. Differentiating $f(x)$ with respect to x we get $2A^T Ax - 2A^T c = 0 \to A^T Ax = A^T c$
   $\Leftarrow$:

   $$A^T A\boldsymbol{x} = A^T c \to A^T (Ax - c) = 0 \to Ax = c$$

   we find $x \in \mathbb{R}^n$ s.t. $|Ax - c| = 0 \leq \inf_{y \in \mathbb{R}^n} |Ay - c|$ and $\inf_{y \in \mathbb{R}^n} \leq |Ax - c|$ since $x \in \mathbb{R}^n$, so we have $|Ax - c| = \inf_{y \in \mathbb{R}^n} |Ay - c|$

   iii. the solution is unique is equivalent to that $A^T A$ is invertible, which happens if and only if $A^T A$ is positive definite (since $A^T A$ is symmetric).
   A matrix is positive definite if and only if $\text{rank}(A) = n$.

   □

3. Show that a subspace Y of a Hilbert space H is closed if and only if $Y^{\perp\perp} = Y$.

solution. Let X be a Hilbert space and Y a subspace of X. suppose Y is closed and by theorem 3.8 which says that $X = Y \oplus Y^{\perp}$. Then we have

$$X = Y \oplus Y^{\perp} = (Y^{\perp})^{\perp} \oplus Y^{\perp}$$

which implies that $Y^{\perp\perp} = Y$.
Suppose $Y^{\perp\perp} = Y$, by theorem 3.7 which says that if Z is a nonempty subset of an inner product space X, then $Z^{\perp}$ is a closed subspace of X. So we conclude that Y is closed. $\square$

4. Let Y be the subset of the Hilbert space ℓ^2 defined by

$$Y := \left\{x = (\xi_i) \in \ell^2 \mid \xi_{2k-1} = \xi_{2k} \quad \forall k \in \mathbb{N}\right\}.$$

(a) Show that Y is a closed subspace of ℓ^2.

solution. The space ℓ^2 is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j}$$

and the norm is defined by

$$\|x\| = \langle x, x\rangle^{1/2} = (\sum_{j=1}^{\infty} |\xi_j|^2)^{1/2}$$

linearity: $\forall \alpha, \beta \in \mathbb{R}$, and $y_1 = (\xi^1), y_2 = (\xi^2) \in Y$, we have

$$\alpha y_1 + \beta y_2 = ((\alpha\xi^1 + \beta\xi^2)_i)$$

$$(\alpha\xi^1 + \beta\xi^2)_{2k-1} = \alpha\xi^1_{2k-1} + \beta\xi^2_{2k-1} = \alpha\xi^1_{2k} + \beta\xi^2_{2k} = (\alpha\xi^1 + \beta\xi^2)_{2k}$$

so we have

$$\alpha y_1 + \beta y_2 \in Y$$

closedness: suppose $\{x_n\}$ is a sequence in Y and $\{x_n\} \to x$ and want to prove that $x \in Y$: given $\{x_n\} \in Y$, for $x_n := (\xi_i)_n$, consider $\{\xi_{2k-1}\}_n \to \xi_{2k-1}$ and $\{\xi_{2k}\}_n \to \xi_{2k}$, since $x_n \in Y$, we have $\xi^{(n)}_{2k-1} = \xi^{(n)}_{2k}, \forall n \in \mathbf{N}$ and thus $\lim_{n\to\infty} \xi^{(n)}_{2k-1} = \lim_{n\to\infty} \xi^{(n)}_{2k} \Rightarrow \xi_{2k} = \xi_{2k-1}$. so $x \in Y$ and Y is closed. $\square$

(b) Identify the orthogonal complement of Y in ℓ^2.

solution. recall the definition of orthogonal complement:

$$Z^{\perp} = \{x \in X : \langle x, z\rangle = 0 \quad \forall z \in Z\}$$

consider the subset

$$Z = \left\{y = (\eta_i) \in \ell^2 \mid \eta_{2k-1} = -\eta_{2k} \quad \forall k \in \mathbb{N}\right\}$$

we have

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j} = \sum_{\substack{j=2k-1\\k=1}}^{\infty} \xi_j^2 - \sum_{\substack{j=2k\\k=1}}^{\infty} \xi_j^2 = 0$$

$\square$

(c) Identify the projection operators $P : \ell^2 \to Y$ and $P^\perp : \ell^2 \to Y^\perp$.

solution. Given the Hilbert space ℓ^2 and its subset Y defined by $Y := \{x = (\xi_i) \in \ell^2 \mid \xi_{2k-1} = \xi_{2k} \quad \forall k \in \mathbb{N}\}$, the projection operators $P : \ell^2 \to Y$ and $P^\perp : \ell^2 \to Y^\perp$ can be identified as follows:

Projection onto Y, $P(x)$:

$$P(x) = \left(\frac{\xi_1 + \xi_2}{2}, \frac{\xi_1 + \xi_2}{2}, \frac{\xi_3 + \xi_4}{2}, \frac{\xi_3 + \xi_4}{2}, \dots\right)$$

Projection onto $Y^\perp$, $P^\perp(x)$:

$$P^\perp(x) = \left(\frac{\xi_1 - \xi_2}{2}, \frac{-(\xi_1 - \xi_2)}{2}, \frac{\xi_3 - \xi_4}{2}, \frac{-(\xi_3 - \xi_4)}{2}, \dots\right)$$

These operators ensure elements are properly projected onto Y and $Y^\perp$, preserving the structure of ℓ^2 as a direct sum of these subspaces.

Remark 8.1. ***Projection onto Y, P:***
For any $x = (\xi_i) \in \ell^2$, P maps x to a sequence in Y where each pair of adjacent terms is equal. This is achieved by averaging each pair of terms:

$$P(x) = \left(\frac{\xi_1 + \xi_2}{2}, \frac{\xi_1 + \xi_2}{2}, \frac{\xi_3 + \xi_4}{2}, \frac{\xi_3 + \xi_4}{2}, \dots\right)$$

This operation ensures the sequence satisfies $\xi_{2k-1} = \xi_{2k}$, thereby belonging to Y.

Projection onto $Y^\perp$, $P^\perp$:
The orthogonal complement $Y^\perp$ includes sequences where adjacent terms are equal in magnitude but opposite in sign. The projection $P^\perp$ modifies each pair of terms to meet this condition:

$$P^\perp(x) = \left(\frac{\xi_1 - \xi_2}{2}, -\frac{\xi_1 - \xi_2}{2}, \frac{\xi_3 - \xi_4}{2}, -\frac{\xi_3 - \xi_4}{2}, \dots\right)$$

This ensures the resulting sequence is orthogonal to every element in Y.

$\square$

8.2 Definitions

1. ℓ^p, ℓ^∞, $L^p(\Omega)$, $\mathcal{L}(X, Y)$

2. $L^1[0,1]$: The space $L^1[0,1]$ is defined as the set of all Lebesgue integrable functions on the interval $[0,1]$. A function $f:[0,1]\to\mathbb{R}$ (or $\mathbb{C}$) is said to be in $L^1[0,1]$ if it satisfies the condition:$\int_0^1 |f(x)|dx < \infty$

   the norm $\|f\| = \int_0^1 |f(x)|dx$ makes the space a normed vector space; and the space is a Banach space.

3. metric space; open ball; closed ball; open; bounded; limit point; closure; dense; separable; continuous

4. convergence; Cauchy sequence; completeness; compactness; sequentially compactness; isometry between metric spaces;

5. linearly independent; hamel basis; norm; Normed vector space; Banach space; Schauder basis; linear operator; bounded linear operator; linear functional, bounded linear functional, dual space; linear isometry

8.3 Examples

1. ℓ^p: is separable; is complete; the sequence $(e_k) = (\xi_j^k)_{j=1}^\infty$ and $\xi_j^k = \delta_j^k$ is a Schauder basis of the space ℓ^p $(1 \le p < \infty)$

2. ℓ^∞: is not separable; is complete

3. $L^p(\Omega)$: is separable

4. $C[a,b]$ is complete with the uniform metric

$$d_\infty(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)|$$

   but incomplete with respect to the L_1 metric

$$d_1(f,g) = \int_a^b |f(x)-g(x)|dx$$

   $C[a,b]$ is a normed vector space choosing the norm

$$\|f\| = \max_{a\le t\le b} |f(t)|$$

5. $L^1[0,1]$: is separable since $C[0,1]$ is separable and is dense in $L^1[0,1]$

6. counter-example for the converse of the lemma: Let X be a metric space and $K\subset X$ be compact. Then K is closed and bounded.

   consider the unit ball $\overline{B_1}$ in the space ℓ^2. let $e_n = (0,\cdots,0,1,0,\cdots)$ with the n-th entry equal to 1. note that $\|e_n\|_{\ell^2} = 1$ and (e_n) is a sequence contained in the closed unit ball $\overline{B_1}$. this sequence does not have a convergent subsequence since $\|e_m - e_n\|_{\ell^2} = \sqrt{2}$ for $m \ne n$ bounded

7. The space $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is indeed a normed vector space

   Proof. To show that $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ forms a normed vector space, we need to verify the following properties for all operators $A, B \in \mathcal{L}(X,Y)$ and all scalars λ:

(a) **Positivity and Definiteness:** $\|A\|_{\mathcal{L}(X,Y)} \geq 0$ and $\|A\|_{\mathcal{L}(X,Y)} = 0$ if and only if $A = 0$.

(b) **Homogeneity:** $\|\lambda A\|_{\mathcal{L}(X,Y)} = |\lambda| \cdot \|A\|_{\mathcal{L}(X,Y)}$.

(c) **Triangle Inequality:** $\|A + B\|_{\mathcal{L}(X,Y)} \leq \|A\|_{\mathcal{L}(X,Y)} + \|B\|_{\mathcal{L}(X,Y)}$.

Proof:

(a) For any $x \in X$, by the definition of a norm, $\|Ax\|_Y \geq 0$, which implies $\|A\|_{\mathcal{L}(X,Y)} \geq 0$. If $A = 0$, then $Ax = 0$ for all $x \in X$, and thus $\|A\|_{\mathcal{L}(X,Y)} = 0$. Conversely, if $\|A\|_{\mathcal{L}(X,Y)} = 0$, then for all $x \in X$, $\|Ax\|_Y = 0$, implying $Ax = 0$ and thus $A = 0$.

(b) $\|\lambda A\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|\lambda Ax\|_Y = |\lambda| \cdot \sup_{\|x\|_X=1} \|Ax\|_Y = |\lambda| \cdot \|A\|_{\mathcal{L}(X,Y)}$.

(c) $\|A+B\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|(A+B)x\|_Y = \sup_{\|x\|_X=1} \|Ax+Bx\|_Y \leq \sup_{\|x\|_X=1}(\|Ax\|_Y+\|Bx\|_Y) \leq \sup_{\|x\|_X=1} \|Ax\|_Y+\sup_{\|x\|_X=1} \|Bx\|_Y = \|A\|_{\mathcal{L}(X,Y)} + \|B\|_{\mathcal{L}(X,Y)}$.

Thus, $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is indeed a normed vector space.

$\square$

8. examples of inner product space:

(a) $\mathbb{R}^n$ equipped with the Euclidean inner product:

$$x \cdot y = \sum_{j=1}^{n} \xi_j \eta_j \quad \forall x = (\xi_j)_{j=1}^n, y = (\eta_j)_{j=1}^n$$

(b) $\mathbb{C}^n$ equipped with the Hermitian inner product:

$$x \cdot y = \sum_{j=1}^{n} \xi_j \overline{\eta}_j \quad \forall x = (\xi_j)_{j=1}^n, y = (\eta_j)_{j=1}^n$$

(c) the real or complex $L^2(\Omega)$ space: Ω is an open subset of $\mathbb{R}^n$ equipped with the inner product defined by

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx \quad \forall f, g \in L^2(\Omega)$$

which is also an infinite-dimensional separable real Hilbert space when it is equipped with the norm

$$\|f\|_{L^2(\Omega)} = (\int_\Omega |f(x)|^2 dx)^{1/2}$$

8.4 Theorems; lemmas; corollaries

1. WAT: the space of polynomials, as a subset of $C(M)$, is dense in $C(M)$ given M is compact
2. compact implies closed and bounded for the subspace of metric space
3. X is a general topological space, K is a compact subset of X and F is a closed subset of K. then F is also a compact subset of X
4. Assume that (X, d) is a complete metric space and let M be a subset of X. Then M is complete under the induced metric if and only if M is closed in X
5. Consider two metric spaces (X, d_X) and (Y, d_Y) that are isometric. If X is complete, then Y is also complete.
6. Let $M \subset X$ be linearly independent. Then X has a Hamel basis which contains M
7. in a normed space, the linear operations are continuous, and the map $x \to \|x\|$ is continuous
8. a normed vector space is a Banach space if and only if any absolutely convergent series converges
9. If the normed space X has a Schauder basis, then it is separable.
10. Let $x_1, \cdots, x_n$ be a linearly independent set of vectors in a normed space (of any dimension). Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \cdots, \alpha_n$, we have

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$$

11. Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.
12. Every finite dimensional subspace Y of a normed space X is closed in X.
13. On a finite dimensional vector space X, any norm $\|\cdot\|_1$ is equivalent to any other norm $\|\cdot\|_2$
14. Let X be a metric space and $K \subset X$ be compact. Then K is closed and bounded
15. For normed vector space with finite dimension, a subset is compact if and only if it is closed and bounded.
16. Riesz's lemma: let Y be a proper closed subspace of a normed space X of any dimension, then for every real number $\theta \in (0, 1)$ there is a $z \in X$ s.t.

$$\|z\| = 1, \quad \|z - y\| \geq \theta \text{ for all } y \in Y$$

Proof. Let Y be a proper closed subspace of a normed space X of any dimension, and let $\theta \in (0, 1)$. We need to show that there exists a $z \in X$ such that $\|z\| = 1$ and $\|z - y\| \geq \theta$ for all $y \in Y$.

Proof: Since Y is a proper subspace of X, there exists at least one element $x_0 \in X$ that is not in Y. Consider the distance of x_0 to Y defined by

$$d = \inf_{y \in Y} \|x_0 - y\|.$$

Since Y is closed and $x_0 \notin Y$, we have $d > 0$.

Choose any $\epsilon > 0$ such that $\theta < 1 - \epsilon < 1$. By the definition of the infimum, there exists a $y_0 \in Y$ such that

$$\|x_0 - y_0\| < \frac{d}{1 - \epsilon}.$$

Define $z = \frac{x_0 - y_0}{\|x_0 - y_0\|}$. It is clear that $\|z\| = 1$.

Now, for any $y \in Y$, consider $y + \lambda z$ for any scalar λ. Since Y is a subspace and $z = \frac{x_0 - y_0}{\|x_0 - y_0\|}$ is not in Y (as $x_0 \notin Y$), it follows that $y + \lambda z \notin Y$ for $\lambda \neq 0$.

We aim to show that $\|z - y\| \geq \theta$ for all $y \in Y$. Suppose, for contradiction, that there exists a $y \in Y$ such that $\|z - y\| < \theta$. Then

$$\|x_0 - y_0 - \|x_0 - y_0\| y\| < \theta \|x_0 - y_0\|.$$

However, this leads to a contradiction with the choice of ϵ and the definition of d. Thus, we conclude that $\|z - y\| \geq \theta$ for all $y \in Y$, as required.

□

17. If a normed space has the property that the closed unit ball $\overline{B_1} = \{x : \|x\| \leq 1\}$ is compact, then X is finite-dimensional.

18. If a normed space is finite dimensional, then every linear operator T on X is bounded

19. T is a linear operator, X and Y are normed spaces, $D(T) \subset X$, then T is continuous iff T is bounded; if T is continuous at a single point, T is continuous everywhere

20. let X and Y be normed spaces, if Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.

21. the dual space X' of a normed space X is a Banach space independently of whether X is.

    chapter 3

22. Cauchy-Schwarz inequality: X is an inner product space,

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle} \quad \forall x, y \in X$$

23. the function $\|\cdot\| : X \to \mathbb{R}$

$$\|x\| = \sqrt{\langle x, x\rangle}$$

is a norm on X and $\|x\| = \sup_{y\neq 0} \frac{|\langle x,y\rangle|}{\|y\|}$

24. parallelogram law: let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

25. X is a Hilbert space and Y is a closed subspace of X. then X is a direct sum

$$X = Y \oplus Y^{\perp}$$

any element x of X can be written as

$$x = y + y^{\perp} \quad y \in Y, y^{\perp} \in Y^{\perp}$$

and such a decomposition is unique. the projection operator P and $P^{\perp}$ satisfy:

$$P^{\perp} = I - P$$

26. a subspace Y of a Hilbert space X is closed iff $Y^{\perp\perp} = Y$

27. Gram-Schmidt orthonormalization

28. any inner product space has a maximal orthonormal family

29. Bessel inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

where $(x, \langle\cdot,\cdot\rangle)$ is an inner product space and (e_k) is an orthonormal sequence

30. a Hilbert space is separable iff it poses a countable orthonormal basis S

31. Riesz representation theorem: let X be a Hilbert space over $\mathbb{K}$. then, give any condition linear functional $f \in X'$, there exists a unique vector $y_f \in X$ s.t.

$$f(x) = \langle x, y_f\rangle \quad \forall x \in X$$

besides,

$$\|f\|_{X'} = \|y_f\|_X$$

Exercise 8.2. *HB extension is unique iff X' is strictly convex:*

Proof. ($\leftarrow$): suppose X' is strictly convex, $Y \leq X$ $\forall f \in Y'$, $\|f\|_Y' = 1$, $\exists f_1, f_2 \in X'$ s.t. $\|f_1\|_{X'} = \|f_2\|_{X'} = 1$, $f_1|_Y = f_2|Y = f$, then

$$1 = \|\frac{f+f}{2}\|_{Y'} = \|\frac{f_1+f_2}{2}\|_{Y'} \leq \|\frac{f_1+f_2}{2}\|_{X'}$$

but $\|f_1\|_{X'} = \|f_2\|_{X'} = 1$ contradiction with X' is strictly convex.

($\rightarrow$): take $z \in X$, s.t. $f_1(z) - f_2(z) = 1$, $\forall x \in X, x = y + az$, where $y \in \mathbb{N}, a = f_(x) - f_2(x)$,

$f_1(x-az)-f_2(x-az) = f_1(x)-f_2(x)-a[f_1(z)-f_2(z)] \rightarrow f_1(y) = f_2(y) \rightarrow y \in M := \{x|f_1(x) = f_2(x)\}$

take $\{x_n\} \in X$ s.t. $\|x_n\| = 1$, $\lim_{n\to\infty} \frac{f_1+f_2}{2}(x) = 1$ because $\|\frac{f_1+f_2}{2}\|_{X'} = 1 \rightarrow \lim_{n\to\infty} f_1(x_n) = \lim_{n\to\infty} f_2(x_n) = 1$ because $\|f_1\|_{X'} = \|f_2\|_{X'} = 1$, $\sup_{\|x\|=1} |\frac{f_1+f_2}{2}x| = 1$.

$x_n = y_n + a_n z_n$, $a_n = f_1(x_n) - f_2(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) + \lim_{n\to\infty} a_n f(z_n) \rightarrow y_n \in M$$

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_N\| \rightarrow y_N \in M, \lim_{n\to\infty} f(y_n) = 1, \quad \lim_{n\to\infty} \|y_n\| = 1 \rightarrow \|f\|_{M'} = 1$$

$\square$

Example 8.3. *an example shows that HB extension is not unique:* $X = P[0,1]$ *and* $Y = P_3[0,1], deg \leq 3$. $f : Y \rightarrow \mathbb{R}$, $f(p) = \frac{1}{6}(p(0) + 4p(\frac{1}{2}) + p(1)), p \in Y \rightarrow$

$$\begin{cases} \tilde{f}(p) = \frac{1}{2}(p(0) + 4p(\frac{1}{2}) + p(1)) & p \in X \\ \hat{f}(p) = \int_0^1 p(t)dt & p \in X \end{cases}$$

we have $\|\hat{f}\| = \|\tilde{f}\| = 1$

9 Final review

9.1 last year problem

1. (1) A Banach space is of finite dimension if and only if its closed unit ball is compact True or False ($\Rightarrow$) Bolzano-Weierstrass theorem or Heine-Borel theorem. ($\Leftarrow$) If $\dim X = \infty$, the closed unit ball is not (sequentially) compact by F. Riesz's lemma

   (2) What is the dual space of $C[0,1]$? $\underline{BV[0,1]}$

   (3) (multiple choice) Indicate which of the following spaces are separable (a), (b), (d), (e), (f)

   (a) ℓ^1; (b) ℓ^2; (c) ℓ^∞ (d) $C[0,1]$; (e) $\mathbb{R}^n$ (f) $L^1[0,1]$.

   (4) (multiple choice) Indicate which of the following spaces are reflexive (b), (e)

   (a) ℓ^1; (b) ℓ^2; (c) ℓ^∞ (d) $C[0,1]$; (e) $\mathbb{R}^n$ (f) $L^1[0,1]$.

   (5) If a normed space X is separable, then X' is separable. True or False The correct statement is: For a normed space X, if X' is separable, then X is separable.

   (6) For a reflexive space X, the weak topology and weak* topology are the same on X'. True or False

   (7) On a normed space X, if $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$ as $n \rightarrow \infty$. True or False

(8) State the open mapping theorem. If X and Y are Banach spaces, and $T : X \to Y$ is a surjective (or onto) bounded linear operator, then T is an open map.

(9) Let $T : X \to Y$ be an operator between two normed space, using the sequences to describe T to be a closed operator: For any sequence $(x_n) \subset X$, if $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, then $y = Tx$.

(10) State the closed graph theorem for $T : X \to Y$ Let X and Y be Banach spaces. If $T : X \to Y$ is a closed linear operator, then T is bounded, i.e. $T \in B(X, Y)$.

2. Let $K(x, y)$ be a continuous non-negative function on $[0, 1] \times [0, 1]$. Define the linear operator T by

$$Tf(x) = \int_0^1 K(x, y)f(y)dy$$

(a) Prove that if $f \in L^1[0, 1]$, then $Tf \in C[0, 1]$.

(b) Consider T as a linear operator $T : L^1[0, 1] \to C[0, 1]$. Prove that its norm

$$\|T\| = \max_{x,y\in[0,1]} K(x, y)$$

3. Let c_0 denote the space of all sequences of real numbers that converge to zero, and let c_{00} denote the space of all sequences of real numbers with only finitely many nonzero terms. That is

$$c_0 = \left\{ x = (\xi_k) : \lim_{k\to\infty} \xi_k = 0 \right\},$$
$$c_{00} = \{ x = (\xi_k) : \exists N > 0 \text{ s.t. } \xi_k = 0 \quad \forall k \geq N \}.$$

It is clearly that $c_{00} \subset c_0 \subset \ell^\infty$.

(a) Regarding c_{00} and c_0 as subspaces of ℓ^∞, prove that $\overline{c_{00}} = c_0$. Consequently, c_0 is closed (and hence a Banach) subspace of ℓ^∞; but c_{00} is not closed (and hence not complete) subspace of ℓ^∞.

(b) Prove that $(c_0)' \cong \ell^1$, that is $(c_0)'$ is isometric to ℓ^1.

(c) State the Hahn-Banach theorem for extending bounded linear functional on normed space, and apply it to c_0 to show that for any $f \in (c_0)'$, there exists an extension $\tilde{f} \in (\ell^\infty)'$ such that $\|\tilde{f}\| = \|f\|$.

4. Let X be a Banach space, let Y and Z be normed space, and let $B : X \times Y \to Z$ be a bilinear mapping that is "separately continuous" in the sense that for each $y \in Y$, $\lim_{n\to\infty} x_n = x$ in X implies $\lim_{n\to\infty} B(x_n, y) = B(x, y)$ in Z, for each $x \in X$, $\lim_{n\to\infty} y_n = y$ in Y implies $\lim_{n\to\infty} B(x, y_n) = B(x, y)$ in Z.

(a) Using the Banach-Steinhaus theorem, show that B is continuous; i.e., that, for each $(x, y) \in X \times Y$

$$\lim_{n\to\infty} x_n = x \text{ in } X \text{ and } \lim_{n\to\infty} y_n = y \text{ in } Y \text{ implies } \lim_{n\to\infty} B(x_n, y_n) = B(x, y) \text{ in } Z.$$

(b) Given an example of normed spaces X, Y, Z and of a separately continuous bilinear mapping $B : X \times Y \to Z$ that is not continuous.

9.2 Definitions

1. ℓ^p, ℓ^∞, $L^p(\Omega)$, $\mathcal{L}(X,Y)$: the set of all bounded linear operators from X to Y

2. $L^1[0,1]$: The space $L^1[0,1]$ is defined as the set of all Lebesgue integrable functions on the interval $[0,1]$. A function $f:[0,1]\to\mathbb{R}$ (or $\mathbb{C}$) is said to be in $L^1[0,1]$ if it satisfies the condition:$\int_0^1 |f(x)|dx<\infty$

   the norm $\|f\|=\int_0^1 |f(x)|dx$ makes the space a normed vector space; and the space is a Banach space.

3. metric space; open ball; closed ball; open; bounded; limit point; closure; dense; separable; continuous

4. convergence; Cauchy sequence; completeness; compactness; sequentially compactness; isometry between metric spaces;

5. linearly independent; hamel basis; norm; Normed vector space; Banach space; Schauder basis; linear operator; bounded linear operator; linear functional, bounded linear functional, dual space; linear isometry

6. open mapping; closed operator; dual space; hyperplane;

7. convex hull; uniformly convex; weak convergence; reflexive space; weak* convergence;

8. resolvent $R_\lambda(T)$; resolvent set $\rho(T)$; regular value λ of T; spectrum σ; spectral value; point spectrum; continuous spectrum; residual spectrum; spectral radius;

9. partial order; totally ordered set; maximal element of a POS;

9.3 Theorems; lemmas; corollaries

1. zorn' lemma: let X be a nonempty partial ordered set, suppose that every totally ordered subset $C\subset X$ has an upper bound, then X has at least one maximal element.

2. thm: every vector space $X\neq\{0\}$ has a Hamel basis.

3. thm: any vector space can be normed.

4. thm: Hahn-Banach thm in a real vector space: let X be a real vector sapce and $p:X\to\mathbb{R}$ be a sublinear functional. let Y be a subspace of X and let $f:Y\to\mathbb{R}$ be a linear functional s.t. $f(y)\le p(y)$, $\forall y\in Y$, then there exists a linear functional $\tilde{f}:X\to\mathbb{R}$ s.t. $\tilde{f}(y)=f(y)$ and $\tilde{f}(x)\le p(x),\forall x\in X$