MAT4220 Partial Differential Equation Note

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Contents

Chapter chapter1 where PDEs come from

1.1 what is a pde

Question 1.1.1. *what is a PDE? 1. it has more than one independent variable x,y,..*
2. it has a dependent variable u(x,y,...)
3.we can take partial derivatives of u w.r.t. independent variables e.g. u_x

Question 1.1.2. *what is the order of PEDs? the highest derivative that appears in the equation.*

Question 1.1.3. *what is a solution of a PDE? a function u(x,y,...) that satisfies the PDE.*

Question 1.1.4. *what is the principle of superposition?*

1.2 first order linear equations

first case: constant coefficient equation:

Question 1.2.1. *how to solve the pde:* $au_x + bu_y = 0$*? using geometric method or coordinate method?*

Question 1.2.2. *how to understand the geometric method? (directional derivative)*

Question 1.2.3. *how to understand the coordinate method? (change of variable)*

second case: variable coefficient equation:

Question 1.2.4. *how to solve the pde:* $u_x + yu_y = 0$*?*

using geometric method, we treat LHS as the directional derivative and hence the direction is (1,y)

Question 1.2.5. *what is the characteristic curves?*

geometric understanding is that u(x,y) is constant on each characteristic curve. hence we can treat the pde as an ode on each curve.

1.3 flows, vibrations, and diffusions

the wave equation is

$$u_{tt} = c^2 u_{xx}$$

.
the diffusion equation is

$$u_t = k u_{xx}$$

1.4 initial and boundary conditions

for the wave equation, if we can determine the initial velocity $\psi(x) = \frac{\partial u}{\partial x}(x, t_0)$ and the initial position $\phi(x) = u(x, t_0)$, we can show that the position function $u(x,t)$ can be determined.
there are several types of boundary conditions:
1. Dirichlet condition: u is specified

2. Neumann condition: the normal derivative $\frac{\partial u}{\partial n}$ is specified.
3. Robin condition:

1.5 well-posed problems

Definition 1.1

a problem consisting of a pde is called well-posed if it enjoies the following properties:
1. existence: there exists at least one solution u(x,t) satisfying all the conditions
2. uniqueness: there is at most one solution
3. stability: the unique solution u(x,t) depends in a stable manner on the data of the problem. this means that if the data are changed a little, the corresponding solution changes only a little.

♣

1.6 types of second-order equations

we consider the pde:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Theorem 1.1

by a linear transformation of the independent variables, the equation can be reduced to one of three forms:
1. elliptic case: $det(A) := a_{11}a_{22} - a_{12}^2 > 0$, *then we can reduce the above pde into the form:*

$$u_{xx} + u_{yy} + ... = 0$$

where ... denotes terms of order 1 or 0
2.parabolic case: $det(A) = 0$, *we can reduce the pde into form:*

$$u_{xx} + ... = 0$$

(it is important to notice the case when $a_{11} = a_{12} = a_{22} = 0$, *which fail be counted as a parabolic case) 3. hyperbolic case:* $det(A) < 0$, *we can reduce the pde into the form:*

$$u_{xx} - u_{yy} + ... = 0$$

♡

Proof change of variable

Chapter chapter2 waves and diffusions

Introduction

- *wave equation*
- *using d'Alembert formula to solve homogeneous wave equation*
- *diffusion equation*
- *diffusion equation's maximal principle*

2.1 the wave equation

Definition 2.1

the wave equation is defined to be:

$$u_{tt} = c^2 u_{xx}$$

for $-\infty < x < \infty$

Exercise 2.1 show that the general solution to the wave equation is

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are two arbitrary twice differentiable functions of a single variable.

we can prove this result using two methods. we now prove that given the initial velocity and initial position we can determine the solution u(x,t) completely.

Theorem 2.1 (d'Alembert formula for wave equation)

given the initial velocity $\psi(x) = \frac{\partial u}{\partial t}(x, t_0)$ *and the initial position* $\phi(x) = u(x, t_0)$*, show that*

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

where $\phi \in C^2$ *and* $\psi \in C^1$ *and* $u \in C^2$

Proof

2.2 causality and energy

now we introduce the law of conservation of energy:

Theorem 2.2

for wave equation, if we let $c^2 = \frac{T}{\rho}$*, we have*$\rho u_{tt} = T u_{xx}$*, then* $E = \frac{1}{2}\int_{-\infty}^{+\infty}(\rho u_t^2 + T u_x^2)dx$ *is a constant independent of t.*

Proof

2.3 the diffusion equation

the one-dimensional diffusion equation is

$$u_t = k u_{xx}$$

.

Theorem 2.3 (the maximal principle)

if u(x,t) satisfies the diffusion equation in a rectangle ($0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T$) in space-time, then the maximum value of u(x,t) is assumed either initially ($t = 0$) or on the lateral sides(x=0 or x=l)
in one sentence, u(x,t) has its maximum on the bottom or sides.

Proof from calculus, at an interior maximum, the first derivatives ($u_t = 0$) vanish and the second derivatives satisfies that $u_x x \leqslant 0$. if $u_x x \neq 0$, $u_x x < 0$ and $u_t = 0$, which violate the diffusion equation. but $u_x x$ may be 0, so we need the following discussion:
let M denote the maximum of u(x,t) on the parabolic boundary, we aim to show that u(x,t) ¡ M throughout the rectangle domain.
let ϵ be a positive constant and let $v(x,t) = u(x,t) + \epsilon x^2$, our goal is to show that $v(x,t) < M + \epsilon l^2$, then $u(x,t) \leqslant M + \epsilon(l^2 - x^2)$, since $0 \leqslant x \leqslant l$ and ϵ is arbitrary, we have $u(x,t) \leqslant M$.

2.4 diffusion on the whole line

our purpose is to solve the problem $u_t = ku_{xx}, (-\infty < x < \infty, 0 < t < \infty), u(x,0) = \phi(x)$.
our method is to solve it for a particular $\phi(x)$ and then build the general solution from this particular one. we'll use five basic invariance properties of the diffusion equation:

Proposition 2.1

a. the translate u(x-y,t) of any solution u(x,t) is another solution, for any fixed y.

Proposition 2.2

b. any derivative (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

Proposition 2.3

c. a linear combination of solutions of $u_t = ku_{xx}$ is again a solution of the equation.

Proposition 2.4

d. an integral of solutions is again a solution, thus if S(x,t) is a solution of $u_t = ku_{xx}$, then so is S(x-y,t) and so is $v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy$ for any function g(y), as long as this improper integral converges appropriately.

Proposition 2.5

e. if u(x,t) is a solution of $u_t = ku_{xx}$, so is the dilated function $u(\sqrt{a}x, at)$, for any $a > 0$, we can prove that by chain rule.

our goal is to find a particular solution of $u_t = ku_{xx}$, and then construct all the other solutions using property d. the particular solution we will look for is this one, denoted $Q(x,t)$, which satisfies the special initial condition

$$Q(x,0) = 1, \text{for } x > 0, Q(x,0) = 0, \text{for } x < 0$$

step 1 we will look for Q(x,t) of the special form:

$$Q(x,t) = g(p), \text{ where } p = \frac{x}{\sqrt{4kt}}$$

g is a funciton of only one variable, where $\sqrt{4k}$ is a factor.
we expect Q ot have this form because property e. says that the equation $u_t = ku_{xx}$ does not change with the dilation $x \to \sqrt{a}x$ and $t \to at$.

step2 using $Q(x,t) = g(p)$, where $p = \frac{x}{\sqrt{4kt}}$ we convert the initial pde into an ode:

$$g'' + 2pg' = 0$$

Proof

solve the ode we get $Q(x,t) = g(p) = c_1 \int e^{-p^2} dp + c_2$
step3 using the boundary conditions, we get

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2} dp$$

step4 having found Q, we define $S := \partial Q/\partial x$. by property b., S is also a solution of $u_t = ku_{xx}$. given any function ϕ, we also define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \text{for t ¿ 0}$$

u is another solution of $u_t = ku_{xx}$.

Claim 2.4.1. *u is the unique solution of $u_t = ku_{xx}$, $u(x,0) = \phi(x)$, with the assumption that $\phi(y)$ is zero for —y— large.* $\rightarrow u(x,0) = \phi(x)$

Proof

so we conclude that $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$for t ¿ 0 is our solution formula, $S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/4kt}$ for $t > 0$. so

$$u(x,t) = \frac{1}{\sqrt{4kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)dy$$

S(x,t) is called the source function or the diffusion kernel or the fundamental solution.

Question 2.4.2. *what happens when t = 0?*

Definition 2.2 (error function)

$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$

♣

✍ **Exercise 2.2** solve the diffusion equation with the initial condition

$$\phi(x) = 1, \text{ for } |x| < l \text{ and } \phi(x) = 0 \text{ for } |x > l|$$

write the answer in terms of $\mathscr{E}rf(x)$

✍ **Exercise 2.3** solve the diffusion equation with the initial condition

$$\phi(x) = 1, \text{ for } x > 0 \text{ and } \phi(x) = 0 \text{ for } x < 0$$

write the answer in terms of $\mathscr{E}rf(x)$

✍ **Exercise 2.4** prove properties a. to e. of the diffusion equation $u_t = ku_{xx}$

Chapter chapter3 reflections and sources

Introduction

- solve diffusion/wave equation on the half line
- using duhamel's principle to solve inhomogeneous diffusion equation
- solve inhomogeneous wave equation (wave equation with a source)

3.1 diffusion on the half-line

we first consider diffusion equation on the half line with dirichlet condition:

$$\partial_t u = k\partial_{xx}u,\ u(x,0) = \phi(x),\ u(0,t) = 0,\ 0 < x < \infty,\ 0 < t < \infty$$

we consider diffusion equation on the half line with neumann condition:

$$\partial_t u = k\partial_{xx}u,\ u(x,0) = \phi(x),\ u_x(0,t) = 0,\ 0 < x < \infty,\ 0 < t < \infty$$

what we will deal with next is the case when dirichlet condition is inhomogeneous:

$$\partial_t u = k\partial_{xx}u,\ u(x,0) = \phi(x),\ u(0,t) = f(t),\ 0 < x < \infty,\ 0 < t < \infty$$

and when neumann condition is inhomogeneous:

$$\partial_t u = k\partial_{xx}u,\ u(x,0) = \phi(x),\ u_x(0,t) = f(t),\ 0 < x < \infty,\ 0 < t < \infty$$

3.2 reflections of waves

3.3 diffusion with a source

we solve the inhomogeneous diffusion equation on the whole line

$$u_t - ku_{xx} = f(x,t)$$
$$u(x,0) = \phi(x)$$
$$(-\infty < x < \infty, 0 < t < \infty))$$

with f(x,t) and $\phi(x)$ arbitrary given functions.
we will show that the solution is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$$

where S(x,t) is the source function defined above

first consider the ODE: $\frac{d}{dt}u + Au = f(t)$, A is a constant, we have:

$$\frac{d}{dt}(e^{tA}u) = e^{tA}f(t)$$
$$e^{tA}u(t) - u(0) = \int_0^t e^{As}f(s)ds$$
$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)}f(s)ds$$

if A is a $n \times n$ matrix, u = $(u_1, u_2, ...u_n)$,f = $(f_1, f_2, ...)$, we still have

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A}f(s)ds$$

where $e^{tA} = 1 + tA + \frac{1}{2}(tA)^2 + \ldots$

we turn to the diffusion equation, let A be $-k\Delta$, we repeat the similar step:

$$\frac{d}{dt}(e^{-kt\Delta}u) = (-k\Delta u)e^{-kt\Delta} + u_t e_{-kt\Delta} = e^{-kt\Delta}(u_t - ku_{xx}) = e^{-kt\Delta}f(x,t)$$

we take integration on both sides and get:

$$e^{-kt\Delta}u(x,t) = u(x,0) + \int_0^t [e^{-ks\Delta}f(x,s)]ds$$

$$u(x,t) = e^{kt\Delta}u(x,0) + e^{kt\Delta}\int_0^t [e^{-ks\Delta}f(x,s)]ds = e^{kt\Delta}u(x,0) + \int_0^t [e^{k(t-s)\Delta}f(x,s)]ds$$

if we acknowledge that

$$e^{kt\Delta}\phi = \Phi * \phi = \frac{1}{\sqrt{4k\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy := \int_{\mathbb{R}} \Phi(x-y,t)\phi(y)dy$$

(why? the idea is that

$$e^{tA}\phi \text{ is the solution to the ODE: } \frac{d}{dt}u - Au = 0, u(0) = \phi$$

we have

$$e^{kt\Delta\phi} \text{ is the solution to the PDE: } \partial_t u - ku = 0, u(0,x) = \phi(x)$$

recall the general solution of this PDE($u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy$), we get the conclusion.)
we get the final formula:

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)\phi(y,0)dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)\phi(y,s)dyds$$

we introduce the duhamel's principle for providing another (more formal) way to get the solution to the inhomogeneous diffusion equation

Theorem 3.1 (duhamel's principle for diffusion equation)

suppose U = U(t,x;s) satisfies: $\partial_t U = k\partial_x^2 U, x \in \mathbb{R}$ *and* $U(s,x;s) = f(x,s)$*, where s is a parameter, then*

$$u(x,t) := \int_0^t U(t,x;s)ds$$

solves the PDE:

$$\partial_t u = k\partial_x^2 u + f(x,t) \text{ and } u(x,0) = 0$$

Proof

we have $\partial_t u = U(t,x;t) + \int_0^t \partial_t U(t,x;s)ds$ since

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} \partial_t f(t,s)ds = f(t,\beta(t)) - f(t,\alpha(t)) + \int_{\alpha(t)}^{\beta(t)} \partial_t f(t,s)ds$$

and

$$\partial_x^2 u = \int_0^t \partial_x^2 U(t,x,s)ds \Rightarrow \partial_t u - k\partial_x^2 u = f(x,t) + \int_0^t (\partial_t u - k\partial_x^2 u)(t,x;s)ds$$

and $u(x,0) = \int_0^0 Uds$

Proof using the theorem, we can prove the formula of inhomogeneous diffusion equation.
we want $w(x,t) = U(t+s,x;s)$, then $\partial_t w = k\partial_x^2 w$ and $w(x,0) = U(s,x;s) = f(x,s)$ (thus we can use the theorem) and also want

$$w(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)f(y,s)dy$$

(thus $w(x,t)$ is the solution to the homogeneous diffusion equation)
to satisfy the above two conditions, we must have $U(t,x;s) = w(x,t-s) = \int_{\mathbb{R}} \Phi(x-y,t-s)f(y,s)dy$, using the theorem,

we have

$$v(x,t) := \int_0^t U(t,x;s)ds = \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)f(y,s)dyds$$

and $v(x,t)$ solves the system:

$$\partial_t u = k\partial_x^2 u + f(x,t) \text{ and } u(x,0) = 0$$

combine the above results together, we note that $u(x,t) := (w+v)(x,t)$ is the solution to the inhomogeneous diffusion equation since

$$\partial_t u = \partial_t(v+w) = k\partial_x^2(v+w) + f(x,t) = k\partial_x^2 u + f(x,t)$$

and

$$u(x,0) = v(x,0) + w(x,t) = \phi(x) + 0$$

next we study the source on a half-line:
we first solve dirichlet boundary problem:

$$v_t - kv_{xx} = f(x,t) \text{ for } 0 < x < \infty, \text{ for } 0 < t < \infty$$
$$v(0,t) = h(t)$$
$$v(x,0) = \phi(x)$$

to use odd extension, we need $v(0,t) = 0$, so we do the subtraction device: let $V(t) = V(x,t) - h(t)$, then we construct a new PDE system:

$$V_t - kV_{xx} = f(x,t) - h'(t) \text{ for } 0 < x < \infty, \text{ for } 0 < t < \infty$$
$$V(0,t) = 0$$
$$V(x,0) = \phi(x)$$

the new system can be solved with the method of reflection, and after getting V(x,t), v(x,t) = V(x,t)+h(t).
for neumann problem on the half-line:

$$v_t - kv_{xx} = f(x,t) \text{ for } 0 < x < \infty, \text{ for } 0 < t < \infty$$
$$v_x(0,t) = h(t)$$
$$v(x,0) = \phi(x)$$

the method is the same and we only need to subtract off the function $xh(t)$, $V(x,t) = v(x,t) - xh(t)$, then $V_X(0,t) = 0$ and we can do even extension.

3.4 waves with a source

we wish to solve the equation:

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

on the whole line, with the usual initial conditions:

$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

we first give the final solution to the PDE:

Theorem 3.2
the unique solution of the wave equation with a source is:
$$u(x,t) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi + \frac{1}{2c}\int_0^t\int_{x-c(t-x)}^{x+c(t-s)} f(y,s)dyds$$
♡

Remark we usually write:
$$\iint_\Delta = \int_0^t\int_{x-c(t-x)}^{x+c(t-s)} f(y,s)dyds$$

Proof (method of characteristic coordinates) we introduce the usual characteristic coordinates $\xi = x+ct, \eta = x-ct$,

Proof (method using green's theorem) in this method we integrate f over the past history triangle Δ. thus

Proof notice that $v(x,t,s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t+s)} f(y,s)dy$ solves the IVP:
$$v_{tt} = c^2 v_{xx}, v(x,s;s) = 0, v_t(x,s;s) = f(x,s)$$
and we can prove that $u(x,t) := \int_0^t v(x,t;s)ds$ is a solution of
$$u_{tt} - c^2u_{xx} = f(x,t), u(x,0) = 0, u_t(x,0) = 0$$
thus we have:
$$u_0(x,t) = \frac{1}{2c}\int_0^t v(x,t;s)ds = \frac{1}{2c}\int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t+s)} f(y,s)dyds$$
recall that the homogeneous problem is that: initial velocity $\psi(x) = \frac{\partial u_1}{\partial t}(x,t_0)$ and the initial position $\phi(x) = u_1(x,t_0)$,
$$u_1(x,t) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$$
we have $u = u_0 + u_1 = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \frac{1}{2c}\int_0^t\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t+s)} f(y,s)dyds$

Chapter chapter4 Boundary problems

4.1 separation of variables, the dirichlet condition

we consider the homogeneous dirichlet conditions for the wave equation in a finite interval:

$$u_{tt} = c^2 u_{xx}, \text{ for } 0 < x < l$$
$$u(0,t) = 0 = u(l,t)$$

with initial conditions

$$u(x,0) = \phi(x), u_t(x,t) = \psi(x)$$

we wish to find the so-called separated solution of the form:

$$u(x,t) = X(x)T(t)$$

we plug the separated form into the wave equation and get:

$$X(x)T''(t) = c^2 X''(x)T(t)$$
$$-\frac{T''}{c^2T} = -\frac{X''}{X} =: \lambda$$

we claim that λ must be a constant:

Proof by definiton, we have $-\frac{T''}{c^2T} = -\frac{X''}{X} =: \lambda$ and we divide it into two equations: $-\frac{T''}{c^2T} = \lambda$ and $-\frac{X''}{X} = \lambda$, we take partial derivative with respective to x and get

$$0 = -\frac{\partial}{\partial x}\frac{T''}{c^2T}(t) = \frac{\partial \lambda}{\partial x}$$

and similarly

$$0 = -\frac{\partial}{\partial t}\frac{X''}{X}(x) = \frac{\partial \lambda}{\partial t}$$

and these show that λ does not depend on x or t, λ must be a constant.

we assume that $\lambda > 0$ (we will show that later), $\lambda =: \beta^2, \beta > 0$, then the equations above are a pair of separate ordinary differential equations for $X(x), T(t)$:

$$X'' = \beta^2 X = 0 \text{ and } T'' + C^2\beta^2 T = 0$$

we can solve the two ODE

$$X(x) = C\cos\beta x + D\sin\beta x$$

$$T(t) = A\cos\beta ct + B\sin\beta ct$$

A, B, C, D are constants.

Proof 1. we prove that $\lambda > 0$
if $\lambda > 0$, we have $X'' = 0$, so that $X(x) = C + Dx$, recall $X(0) = X(l) = 0$ and thus $C = D = 0$, so that $X(x) \equiv 0$, therefore $\lambda = 0$ is not an eigenvalue.
if $\lambda < 0$, we write $\lambda = -\gamma^2$, and $X'' = \gamma^2 X \to X(x) = C\cosh\gamma x_D \sinh\gamma x \to 0 = X(0) = C$ and $0 = X(l) = D\sinh\gamma l$, notice $l \neq 0 \to \sinh\gamma l \neq 0 \to D = 0$, similarly, $\lambda < 0$ is not an eigenvalue.
if λ is a complex number, let γ be either one of the two square roots of $-\lambda$, the other one is $-\gamma$, then $X(x) = Ce^{\gamma x} + De^{-\gamma x}$, the boundary condition yield $0 = X(0) = C + D$ and $0 = Ce^{\gamma l} + De^{\gamma l} \to Re(\gamma) = 0, 2l \times Im(\gamma) = 2\pi n, n \in \mathbb{Z} \to \lambda = \gamma^2 > 0$.
do we need to check whether $\text{Im}(\gamma) = 0$?
2. we prove the formula of the solutions to the ODEs
???TBD

the second step is to impose the boundary conditions on the separated solution:

$$X(0) = 0 = X(l)$$

thus

$$0 = X(0) = C \text{ and } 0 = X(l) = D \sin \beta l$$

we do not want $C = D = 0$ since it leads to $X(x) = 0$. so we must have $\beta l = n\pi$

$$\beta_n = \frac{n\pi}{l}, \lambda_n = \frac{n\pi}{l}^2, X_n(x) = C \cos \beta x + D \sin \beta x = \sin \frac{n\pi x}{l}, n = 1, 2, \ldots$$

(notice we let $\beta > 0$ and n is positive integer. since A, B, C, D are arbitrary, we can accept the assumption that $\beta > 0$) this says that there are infinitely many separated solution of the homogeneous dirichlet wave equation:

$$u_n(x,t) = X(x)T(t) = (A_n \cos \beta ct + B_n \sin \beta ct) \sin \frac{n\pi x}{l}, n = 1, 2, \ldots$$

where A_n, B_n are arbitrary constants. and the sum of solutions is again a solution, so any finite sum

$$u(x,t) = \sum_n (A_n \cos \beta ct + B_n \sin \beta ct) \sin \frac{n\pi x}{l}$$

is also a solution of the wave equation with dirichlet condition.
if we let $t = 0$ in the above equation, we get:

$$\phi(x) = u(x,0) = \sum_n A_n \sin \frac{n\pi x}{l}$$

and

$$\psi(x) = u_t(x,0) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$$

in summary, what we have done above is saying that: if $\phi(x)$ and $\psi(x)$ in the initial problem can be written into the series form, then

$$u(x,t) = \sum_n (A_n \cos \beta ct + B_n \sin \beta ct) \sin \frac{n\pi x}{l}$$

is a solution of the initial wave equation with dirichlet condition.
let's check the problem for diffusion:

$$DE : U_t = ku_{xx}, (0 < x < l, 0 < t < \infty)$$

$$BC : u(0,t) = u(l,t) = 0$$

$$IC : u(x,0) = \phi(x)$$

we separate the variables $u = T(t)X(x)$, and we get:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = constant$$

similarly we solve two ODE and get the result:

$$-X'' = \lambda X \text{ in } 0 < x < l \text{ with } X(0) = X(l) = 0$$

which is the same as the equation in the wave case and we omit here. the form of $T(t)$ is given by:

$$T' = \lambda kT$$

and

$$T = Ae^{-\lambda kt}$$

so the solution to the initial diffusion problem is:

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}$$

if we have:

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

there are some definitions:

Definition 4.1 (eigenvalue; eigenfunction)

the number $\lambda_n = (n\pi/l)^2$ are called eigenvalues and the functions $X(x) = \sin(n\pi x/l)$ are called eigenfunctions ♣

Remark we notice that eigenfunctions and eigenvalues satisfy the following equation:

$$-\frac{d^2}{dx^2}X = \lambda X,\ X(0) = X(l) = 0$$

which is similar to the equation in linear algebra:

$$AX = \lambda X$$

an eigenfunction is a solution $X \not\equiv 0$ of this euqation and an eigenvalue is a number λ for which there exists a solution $X \not\equiv 0$

4.2 the neumann condition

4.2.1 diffusion with neumann condition

the neumann condition is

$$u_x(0, t) = 0 = u_x(l, t)$$

the eigenfunctions are the solutions $X(x)$ of

$$-X'' = \lambda X,\ X'(0) = X'(l) = 0$$

the trivial solution $X(x) \equiv 0$ is not regarded as an eigenfunction.
we first consider the case where $\lambda = \beta^2 > 0$: given $X''(x) = \beta^2 X$, from 4.1 we know that

$$X(x) = C\cos\beta x + D\sin\beta x$$

take first derivative w.r.t. x and we have

$$X'(x) = -C\beta\sin\beta x + \beta D\cos\beta x$$

we use the neumann boundary condition and get:

$$0 = X'(l) = -C\beta\sin\beta l + \beta D\cos\beta l$$
$$0 = X'(0) = \beta D \to D = 0$$

and we plug $D = 0$ in the first equation and get:

$$-C\beta\sin\beta l = 0$$

we do not want C = 0 (which leads to trivial solution $X(x) = 0$) and we get

$$\beta = \frac{k\pi}{l},\ k = 1, 2, \ldots$$

the eigenvalue $\lambda = \beta^2 = (\frac{k\pi}{l})^2$ and the eigenfunction is

$$X(x) = C\cos\beta x + D\sin\beta x = C\cos\frac{n\pi x}{l}$$

we may let $C = 1$ since in the series solution we will add a coefficient A to this solution. so the eigenfunction has the form:

$$X(x) = \cos\frac{n\pi x}{l},\ n = 1, 2, \ldots$$

we consider the case where $\lambda = 0$. from

$$-X'' = \lambda X$$

we have

$$X'' = 0 \to X' \equiv D \to X = C + Dx, \text{ where C, D are constants}$$

the neumann condition requires that

$$X'(l) = D = 0 \rightarrow x \equiv C$$

the eigenfunction is

$$x \equiv C, \text{C can be any number.}$$

if $\lambda < 0$ or λ is a complex number, we can show that λ is not an eigenvalue: one way is shown in 4.1 when we encounter dirichlet boundary condition and another way will be introduced in 5.3
in general we know that the eigenvalues have the form:

$$\lambda = (\frac{n\pi}{l})^2, n = 0, 1, 2, \ldots$$

recall that for diffusion problem, $T(t) = Ae^{-\lambda kt}$, so the general solution of diffusion problem with neumann condition is

$$u(x,t) = \sum_{n=0}^{\infty} X(x)T(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 kt} \cos\frac{n\pi x}{l}$$

where we write the constant solution (the case λ is 0) in $\frac{1}{2}A_0$ for later convenience.

4.2.2 wave equation with neumann condition

we next consider neumann condition for wave euqation.
if $\lambda = 0$, following the same process we know that $X \equiv C, T''(t) = \lambda c^2 T(t) \rightarrow T(t) = A + Bt$, A, B are contstants.
if $\lambda > 0$, from 4.1 we have the $\lambda > 0$ case that

$$u_n(x,t) = (A_n \cos\frac{n\pi ct}{l} + B_n \frac{n\pi ct}{l}) \sin\frac{n\pi x}{l}$$

we plug in what we have for wave equation

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}) \cos\frac{n\pi x}{l}$$

the coefficient $\frac{1}{2}$ is justified later.
the initial data must satisfy

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}$$

4.2.3 mixed boundary condition

a mixed boundary condition would be dirichlet at one and and neumann at the other. one example is that

$$-X'' = \lambda X, X(0) = X'(l) = 0$$

for more details, see exercise 1 and 2 in 4.2

4.3 the robin condition

the robin condition means that we solving

$$-X'' = \lambda X$$

with the boundary conditions

$$X' - a_0 X = 0 \text{ at } x = 0$$

$$X' + a_l X = 0 \text{ at } x = l$$

the two constants a_0 and a_1 are given.
we are going to solve the ode

$$-X'' = \lambda X$$

with the robin boundary conditions.

4.3.1 positive eigenvalues

Our task now is to solve the ODE $-X'' = \lambda X$ with the boundary conditions. First let's look for the positive eigenvalues

$$\lambda = \beta^2 > 0.$$

As usual, the solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x$$

so that

$$X'(x) \pm aX(x) = (\beta D \pm aC)\cos\beta x + (-\beta C \pm aD)\sin\beta x.$$

At the left end $x = 0$ we require that

$$0 = X'(0) - a_0 X(0) = \beta D - a_0 C.$$

So we can solve for D in terms of C. At the right end $x = l$ we require that

$$0 = (\beta D + a_l C)\cos\beta l + (-\beta C + a_l D)\sin\beta l.$$

they are equivalent to the matrix equation

$$\begin{pmatrix} -a_0 & \beta \\ a_l\cos\beta l - \beta\sin\beta l & \beta\cos\beta l + a_l\sin\beta l \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, substituting for D, we have

$$0 = (a_0 C + a_l C)\cos\beta l + \left(-\beta C + \frac{a_l a_0 C}{\beta}\right)\sin\beta l.$$

We don't want the trivial solution $C = 0$. We divide by $C\cos\beta l$ and multiply by β to get

$$(\beta^2 - a_0 a_l)\tan\beta l = (a_0 + a_l)\beta$$

we discuss the following cases for $\beta > 0$:
1. if $a_0 a_l > 0$ and $a_0 + a_l > 0$: we have

$$n^2\frac{\pi^2}{l^2} < \lambda_n < (n+1)^2\frac{\pi}{l}, n = 0, 1, 2, 3, ...$$

there are infinitely many eigenvalues. and we have $\lim_{n\to\infty}\beta_n = n\frac{\pi}{l}$
2. if $a_0 a_l > 0, a_0 + a_l < 0$: we have the similar result with 1. (we only need to flip the image of $\frac{(a_0+a_l)\beta}{\beta^2 - a_0 a_l}$ around β-axis)
3. if $a_0 a_l < 0, a_0 + a_l > 0$: we have $\lim_{n\to\infty}\beta_n = n\frac{\pi}{l}$; we also notice that there is an eigenvalue $0 < \lambda_0 < (\pi/\lambda)^2$ if and only if $a_0 + a_l > -a_0 a_l l$ by checking the image of two functions.
4. if $a_0 a_l < 0, a_0 + a_l < 0$: we have the similar result with 3.

4.3.2 zero eigenvalues

we let $\lambda = \beta^2 = 0$: we can show that there is a zero eigenvalue iff

$$a_0 + a_l = -a_0 a_l l$$

since $X'' = 0$, we assume$X(x) = Ax + B \to X'(0) = a_0(A \cdot 0 + B) \to A = a_0 B$. using the robin condition at $x = l$, we have $A = -a_l(Al + B) = -a_l Al - a_l B \to (1 + a_l l)A = -a_l B \to (1 + a_l l)a_0 B = -a_l B, b \neq 0$ the eigenfunction is in the form:

$$X(x) = a_0 x + 1$$

as we let $B = 1$

4.3.3 negative eigenvalue

we let $\lambda = -\beta^2$, solve the equations:

$$X''(x) = \beta^2 X(x)$$

$$X'(0) = a_0 X(0), X'(l) = a_l X(l)$$

and we have

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x)$$

then we have two equations about $X'(0)$ and $X'(l)$ and we finally get

$$\tanh(\beta l = \frac{-\beta(a_l + a_0)}{a_0 a_l + \beta^2})$$

we discuss the following cases for $\beta > 0$:
1. $a_0 + a_l > 0, a_0 a_l > 0$: we claim that there is no intersection and no β satisfies the equation $\tanh(\beta l = \frac{-\beta(a_l+a_0)}{a_0 a_l+\beta^2})$
2. $a_0 + a_l > 0, a_0 a_l < 0$: there may be one eigenvalue or no eigenvalue: if $a_0 < 0$ and $a_l > -a_0$ and $a_0 + a_l < -a_0 a_l l$, then we have exactly one negative eigenvalue. otherwise there is no eigenvalue.
3. $a_0 + a_l < 0, a_0 a_l > 0$: there may be 0 or 1 or 2 eigenvalues
4. $a_0 + a_l < 0, a_0 a_l < 0$: there is exactly one eigenvalue.

4.3.4 summary

given the relation among a_0 l, and a_l, you should know how many negative eigenvalues and positive eigenvalues and zero eigenvalues are there. then you can use the fourier series technique to get the expression of the solution $u(x, t)$ without knowing the exact expressions of β or γ.
1. $a_0 > 0, a_l > 0$: infinitely many positive eigenvalues
2. $a_0 < 0, a_l > 0, a_0 + a_l > 0, a_0 + a_l > -a_0 a_l l$: infinitely many positive eigenvalues
3. $a_0 < 0, a_l > 0, a_0 + a_l > 0, a_0 + a_l = -a_0 a_l l$: zero is an eigenvalue, all the rest are positive
4. $a_0 < 0, a_l > 0, a_0 + a_l > 0, a_0 + a_l < -a_0 a_l l$: one negative eigenvalue, all the rest are positive.

Chapter chapter5 Fourier series

Introduction

- *find the fourier full/sine series of a periodic function*
- *determine the convergence of fourier series (3 kinds of convergence)*

5.1 the coefficients

the fourier sine series for ϕ is defined to be:

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

in the interval $(0, l)$

the first que is to find the coefficients A_n for a given function ϕ. the following property of sine function helps a lot to find A_n:

$$\int_0^l sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0 \text{ if } m \neq n$$

where m and n are positive integers.

Proof

now we use this property to find the expression of A_n:

from our hypothesis we have

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

we multiply both sides by $\sin(m\pi x/l)$ and integrate the series term by term to get:

$$\int_0^l \phi(x) \sin \frac{m\pi x}{l} dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx$$

we use the property of sine function and get that all but one term in this sum vanishes, namely the one with $n = m$. therefore, we are left with the singel term

$$A_m \int_0^l \sin^2 \frac{m\pi x}{l} dx = \frac{1}{2} l A_m$$

so we have

$$A_m = \frac{2}{l} \int_0^l \phi(x) \sin \frac{m\pi x}{l} dx$$

using this formula we can get the coefficient A_n.

now we apply the result to solve PDEs. recall that in 4.1 we have shown that the wave equation with dirichlet boundary conditions satisfies that

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l}$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$$

we have got A_n in above. and similarly we have

$$\frac{m\pi c}{l} B_m = \frac{2}{l} \int_0^l \psi(x) \sin \frac{m\pi x}{l} dx$$

the diffusion equation with dirichlet conditions only concerns $\phi(x)$ and we have the

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{l}$$

we can similarly get the formula for A_n.
we consider neumann boundary conditions on (0,l)). we have shown that the series expression of $\phi(x), \psi(x)$ involve cosine function so we first study fourier cosine series:
we want to find the coefficients in the formula:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

we can verify that we still have

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0 \text{ if } m \neq n$$

where m, n are nonnegative integers

Proof

then we follow what we have done in the case of sine function and we have:

$$\int_0^l \phi(x)\cos\frac{m\pi x}{l}dx = A_m\int_0^l \cos^2\frac{m\pi x}{l}dx = \frac{1}{2}lA_m, \text{ if m} \neq 0$$

for $m = 0$: we have

$$\int_0^l \phi(x)\cdot 1dx = \frac{1}{2}A_0\int_0^l 1^2dx = \frac{1}{2}A_0 l$$

which is consistent to the case $m \neq 0$ **Remark** why we need to discuss the $m = 0$case separately? it is due to the form of $\phi(x)$, you may find that in the expression $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ the $m = 0$ term is separated from other terms.

for nonnegative m, we have the formula for the coefficients of the cosine series

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}dx$$

and this fourier cosine series is corresponding to the diffusion equation with the neumann boundary condition.

we next introduce the full fourier series:

Definition 5.1 (full fourier series)

the full fourier series of $\phi(x)$ on the interval $-l < x < l$ is defined as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l})$$

♣

the full fourier series has the similar property as the sine and cosine fourier series, that is:

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \qquad \text{for all } n, m$$

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l}1\cdot\cos\frac{n\pi x}{l}dx = 0 = \int_{-l}^{l}1\cdot\sin\frac{m\pi x}{l}dx.$$

which can be interpreted as: multiply any two different eigenfunctions and integrate over the interval and you get zero.
if the eigenfunctions are the same, we have

$$\int_{-l}^{l}\cos^2\frac{n\pi x}{l}dx = l\int_{-l}^{l}\sin^2\frac{n\pi x}{l}dx$$

$$\int_l^l 1^2 dx = 2l$$

with the property we can calculate A_n and B_n in the full fourier series:

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}dx,\ \text{n = 0,1,2,...}$$
$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}dx,\ \text{n = 0,1,2,...}$$

note that these formulas are different from the formula in the sine and cosine fourier series.

5.2 even, odd, periodic, and complex functions

first there is a introduction about parity and periodic functions, which is familiar.

5.2.1 fourier series and boundary conditions

we notice that fourier sine series can be regarded as an expansion of an arbitrary function that is odd and has period 2l defined on the whole line $-\infty < 0 < \infty$ and we have the following relations:
a. $u(0,t) = u(l,t) = 0$: dirichlet BCs correspond to the odd extension
b. $u_x(0,t) = u_x(l,t) = 0$: neumann BCs correspond to the even extension
c. $u(l,t) = u(-l,t), u_x(l,t) = u_x(-l,t)$: periodic BCs correspond to the periodic extension.

5.2.2 the complex form of the full fourier series

using DeMoivre formulas, we can express sine and cosine in terms of the complex exponentials $e^{in\pi x/l}$, $-\infty < n < \infty$. for a given function $\phi(x)$, we want to find coefficients c_n s.t.

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

notice that we still have the following property for this formula: if $n \neq m$, we have

$$\begin{aligned}\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l} dx &= \int_{-l}^{l} e^{i(n-m)\pi x/l} dx \\ &= \frac{l}{i\pi(n-m)}[(-1)^{n-m} - (-1)^{m-n}] = 0\end{aligned} \tag{5.1}$$

if $m = n$, we have

$$\int_{-l}^{l} e^{i(n-n)\pi x/l dx = \int_{-l}^{l} 1 dx = 2l}$$

with this property, we can get the formula for the coefficient c_n. (using the same method we developed in 5.1) we finally have

$$c_n = \int_{-l}^{l} \phi(x) e^{-in\pi x/l} dx$$

5.3 orthogonality and general fourier series

Definition 5.2 (inner product)

f(x) and g(x) are two real-valued continuous functions defined on an intervl $a \leqslant x \leqslant b$, we define their inner product to be the integral of their product:

$$(f,g) \equiv \int_a^b f(x)g(x)dx$$

♣

Definition 5.3 (orthogonal)

we call f(x) and g(x) are orthogonal if (f,g) = 0 ♣

Remark there is no function is orthogonal to itself except $f(x) \equiv 0$

in the cases we discussed in 5.1, every eigenfunction is orthogonal to every other eigenfunction. next we explain the reason.

we study the operator $A = -d^2/dx^2$ with some boundary conditions. let $X_1(x)$ and $X_2(x)$ be two different eigenfunctions. thus we have:

$$-X_1'' = \frac{-d^2X_1}{dx^2} = \lambda_1 X_1 \tag{5.2}$$

$$-X_2'' = \frac{-d^2X_2}{dx^2} = \lambda_1 X_2$$

where both functions satisfy the boundary conditions (maybe dirichlet or neumann or...). let's assume that $\lambda_1 \neq \lambda_2$
We now verify the identity

$$-X_1''X_2 + X_1X_2'' = \left(-X_1'X_2 + X_1X_2'\right)'.$$

(Work out the right side using the product rule and two of the terms will cancel.) We integrate to get

$$\int_a^b \left(-X_1''X_2 + X_1X_2''\right) dx = \left(-X_1'X_2 + X_1X_2'\right)\Big|_a^b \tag{5.3}$$

This is sometimes called Green's second identity. On the left side of (3) we now use the differential equations (2). On the right side we use the boundary conditions to reach the following conclusions:
Case 1: Dirichlet. This means that both functions vanish at both ends: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$. So the right side of (3) is zero.
Case 2: Neumann. The first derivatives vanish at both ends. It is once again zero.
Case 3: Periodic. $X_j(a) = X_j(b), X_j'(a) = X_j'(b)$ for both $j = 1, 2$. Again you get zero!
Case 4: Robin. Again you get zero! See Exercise 8. Thus in all four cases, (3) reduces to

$$(\lambda_1 - \lambda_2)\int_a^b X_1X_2 dx = 0.$$

Therefore, X_1 and X_2 are orthogonal! This completely explains why Fourier's method works (at least if $\lambda_1 \neq \lambda_2$)!

The right side of (3) isn't always zero. For example, consider the different boundary conditions: $X(a) = X(b), X'(a) = 2X'(b)$. Then the right side of (3) is $X_1'(b)X_2(b) - X_1(b)X_2'(b)$, which is not zero. So the method doesn't always work; the boundary conditions have to be right.

5.3.1 symmetric boundary conditions

any pair of boundary conditions can be written in the form:

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta X'(b) = 0$$

$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta X'(b) = 0$$

there are eight real constants

Definition 5.4 (symmetric boundary conditions)

for any $f(x)$ and $g(x)$ satisfy the pair of boundary conditions above, if they satisfy that

$$f'(x)g(x) - f(x)g'(x)\big|_{x=a}^{x=b} = 0$$ ♣

Definition 5.5 (eigenfunction)

an eigenfunction is a solution of $-X'' = \lambda X$ that satisfies the above pair of boundary conditions. ♣

Definition 5.6 (orthogonal)

if $(f, g) = 0$, f and g are called orthogonal ♣

Theorem 5.1

if you have symmetric boundary conditions, then any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. therefore, if any function is expanded in a series of these eigenfunctions, the coefficients are determined. ♡

Proof

Remark two eigenfunctions have the same eigenvalues, then they do not need to be orthogonal, but if they are not orthogonal, they can be made so by the Gram-Schmidt orthogonalization procedure.

Question 5.3.1. *(The Gram-Schmidt orthogonalization procedure) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vectorspace with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define*

$$Y_2 = X_2 - (X_2, Z_1)\, Z_1 \quad \text{and } Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)\, Z_2 - (X_3, Z_1)\, Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on.
(a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.
(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

5.3.2 complex eigenvalues

Definition 5.7 (inner product)

if $f(x)$ and $g(x)$ are two complex-valued functions, we define the inner product on (a,b) as

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx$$

if two functions have $(f, g) = 0$, they are called orthogonal. ♣

Definition 5.8 (symmetric)

we have the boundary conditions above with 8 real constants, they are called symmetric if

$$f'(x)\overline{g(x)} - f(x)\overline{g'(x)}|_a^b = 0$$

♣

Theorem 5.2

under the same conditions as the above theorem, that is with symmetric boundary conditions, we have all eigenvalues are real numbers ♡

Proof

5.3.3 negative eigenvalues

Theorem 5.3

with symmetric boundary conditions, if

$$f(x)f'(x)|_{x=a}^{x=b} \leqslant 0$$

for all real-valued functions f(x) satisfying the pair of boundary conditions, then there are no negative eigenvalue. (we can show that dirichlet, neumann, periodic boundary conditions lead to the condition, and robin condition with $a_0 > 0, a_1 > 0$ also leads to the condition. see Q11 in 5.3)

Proof Q13 in 5.3

5.4 completeness

we state the basic theorems about the convergence of fourier series. there are three senses of convergence of functions and we state sufficient conditions on a function $f(x)$ that its fourier series converge to it in these three senses. at the end of this chapter we define the notion of completeness using the mean-square convergence.
consider the eigenvalue problem

$$X'' + \lambda X = 0 \quad \text{with symmetric boundary conditions on (a,b) in 5.3}$$

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta X'(b) = 0$$

$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta X'(b) = 0$$

in 5.2 we have proved that all the eigenvalues λ are real.

Theorem 5.4

for the above eigenvalue problem, there are an infinite number of eigenvalues. they form a sequence $\lambda_n \to +\infty$. we can assume that the eigenfunction $X_n(x)$ are pairwise orthogonal and real valued. we may list the eigenvalues as

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \to +\infty$$

with the corresponding eigenfunctions

$$X_1, X_2, \ldots$$

which are paiwise orthogonal.

Definition 5.9 (fourier coefficients and fourier series)

for any function satisfies the boudnary conditions on (a, b), the fourier coefficients are defined as

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x)\overline{X_n(x)}dx}{\int_a^b |X_n(x)|^2 dx}$$

and the fourier seires of the function is defined to be

$$\sum_n A_n X_n(x)$$

5.4.1 three notions of convergence

Definition 5.10 (pointwise convergence)

we say that an infinite series $\sum_{n=1}^{\infty} f_n(x)$ *converges to* $f(x)$ *pointwise in (a,b) if it converges to* $f(x)$ *for each* $a < x < b$. *that is for each* $a < x < b$ *we have*

$$|f(x) - \sum_{n=1}^{N} f_n(x)| \to 0 \quad \textit{as } N \to \infty$$

♣

Definition 5.11 (uniform convergence)

we say that series converges uniformly to $f(x)$ *in [a,b] if*

$$\max_{a \leqslant x \leqslant b} |f(x) - \sum_{n=1}^{N} f_n(x)| \to 0 \quad \textit{as } N \to \infty$$

♣

Definition 5.12 (mean-square sense convergence)

we say the series converges in the mean-square (or L^2*) sense to* $f(x)$ *in* (a, b) *if*

$$\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \to 0 \quad \textit{as } N \to \infty$$

♣

we point out that uniform convergence is stronger than both pointwise and L^2 convergence.

Example 5.1 $f_n(x) = (1 - x)x^{n-1}$ on the interval $0 < x < 1$, show that the series converges pointwise to the function $f(x) \equiv 1$. also show that the function does not converge uniformly but converge in mean-square.

Example 5.2 $f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2}$ in the interval $0 < x < l$. show that the series converges pointwise to the sum $f(x) \equiv 0$. and show that the series dose not converge in the mean-square sense nor in the uniform sense.

5.4.2 convergence theorem

Theorem 5.5 (uniform convergence)

the fourier series $\sum A_n X_n(x)$ *converges uniformly on* $[a, b]$ *if the following two conditions holds:*

a.

b.

♡

Theorem 5.6 (L^2 convergence)

the fourier series $\sum A_n X_n(x)$ *converges in square sense in* (a, b) *if*

$$\int_a^b |f(x)|^2 dx \textit{ is finite}$$

♡

before introduce the next theorem, we provide two definitions:

Definition 5.13 (jump discontinuity)

...at $x = c$ *we have* $f(c+) \neq f(c-)$*...*$f(c)$ *may be not defined...*

♣

Definition 5.14 (piecewise continuous)

...a finite number of jump discontinuities...on $[a, b]$

♣

Theorem 5.7 (pointwise convergence of classical fourier series)

the theorem says two things:
a. ...on (a, b)...$f(x)$ is continuous on $[a, b]$ and $f'(x)$ is piecewise continuous on $[a, b]$..
b. if $f(x)$ is only piecewise continuous on $[a, b]$ and $f'(x)$ is piecewise continuous on $[a, b]$, then the fourier series converges at every point x. the sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}[f(x+) + f(x-)] \text{ for all } a < x < b.$$

for not differentiable function, we can only use the theorem with L^2 convergence.

Example 5.3 the fourier sine series of the function $f(x) \equiv 1$ on the interval $(0, \pi)$ is

$$\sum_{\text{n odd}} \frac{4}{n\pi} \sin nx$$

show that it converge pointwisely at each point but does not converge uniformly on $[0, \pi]$.
hint: sine series is related to dirichlet boundary conditions.

Example 5.4 for the last example, can we take derivative term by term? hint is using the n-th term test for divergence.

5.4.3 the L^2 theory

Definition 5.15 (L^2 norm)

the L^2 norm of f is defined as

$$||f|| = (f, f)^{1/2} = [\int_a^b |f(x)^2|dx]^{1/2}$$

we can restate the theorem 5.6 as follows:
if $\{x_n\}$ are the eigenfunctions associated with a set of symmetric boundary conditions and $||f|| < \infty$, then

$$||f - \sum_{n \leqslant N} || \to 0 \text{ as N} \to \infty$$

Theorem 5.8 (least-square approximation)

$\{X_n\}$ is any orthogonal set of functions. $||f|| < \infty$...N is a fixed positive integer...the choice of $c_{1,2}, ..., c_n$ that minimizes

$$||f - \sum_{n=1}^{N} c_n X_n||$$

is $c_1 = A_1, c_2 = A_2, ..., c_n = A_n$. recall that $A_n := \frac{(f, X_n)}{||X_n||^2}$

Proof the key is to complete the square. and we denote the error by $E_n := ||f - \sum_{n=1}^{N} c_n X_n||^2$

Proposition 5.1 (bessel's inequality)

after you finish the proof above, one observation is that

$$0 \leqslant E_n = ||f||^2 - \sum_{n \leqslant N} \frac{(f, X_n)^2}{||X_n||^2} = ||f||^2 - \sum_{n \leqslant N} A_n^2 ||X_n||^2$$

we have

$$\sum_{n \leqslant N} A_n^2 \int_a^b |X_n(x)|^2 dx \leqslant \int_a^b |f(x)|^2 dx$$

note that the LHS is the partial sums of a series of positive terms with bounded partial sums, the partial sum *converges and its infinite series satisfies*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx \leqslant \int_a^b |f(x)|^2 dx$$

this is called bessel's inequality and it is valid as long as RHS is finite.

note mean-square convergence means that the error $E_N \to 0$ as $N \to \infty$, which leads to the following theorem.

Theorem 5.9 (parsevals's equality)

the fourier series of f(x) converges to f(x) in the mean-square sense if and only if

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

Definition 5.16 (complete)

the infinite orthogonal set of functions ??? is called complete if parsevals' equality is true for all f with $\int_a^b |f|^2 dx < \infty$

the theorem5.6 asserts that the set of eigenfunctions coming from ??? is always complete. thus we have the following conclusion:

Corollary 5.1

if $\int_a^b |f|^2 dx$ *is finite, then the parseval equality is true.*

Example 5.5 recall that we have $1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx$ so we have

$$\sum_{n \text{ odd}} (\frac{4}{n\pi})^2 \frac{\pi}{2} = \pi$$

show the procedure!

5.5 completeness and the gibbs phenomenon

5.5.1 proof of pointwise convergence of full fourier series

what we want to prove is that: the full fourier series of a C^1 function f(x) on the whole line of period 2l converges to f(x), and l is chosen to be π. to be more clear, we want to show that

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx) \to f(x) \textbf{ as N} \to \infty$$

where

$$A_n = \int_{-\pi}^{\pi} f(y) \cos ny \frac{dy}{\pi}, \textbf{n = 0,1,2...}$$

$$B_n = \int_{-\pi}^{\pi} f(y) \sin ny \frac{dy}{\pi}, \textbf{n = 0,1,2...}$$

pointwise convergence means that x is kept fixed as we take the limit.

Proof we define the dirichlet kernel:

Definition 5.17 (dirichlet kernel)

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta$$

it is remarkable that

$$K_N(\theta) = \frac{\sin[(N+\frac{1}{2})\theta]}{\sin\frac{1}{2}\theta}$$

one can show that

$$S_N(x) = \int_{-\pi}^{\pi} K_N(x-y) f(y) \frac{dy}{2\pi}$$

we do the change of variable and compute $S_N(x) - f(x)$:

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} K_N(x)[f(x+\theta) - f(x)] \frac{d\theta}{2\pi}$$

let $g(\theta) = \frac{f(x+\theta)-f(x)}{\sin\frac{1}{2}\theta}$, we have

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} g(\theta) \sin[(N+\frac{1}{2})\theta] \frac{d\theta}{2\pi}$$

let $\phi_N(\theta) = \sin[(N+\frac{1}{2})\theta]$, N = 1,2,3.... we can show that $\phi_N(\theta)$ is a sequence of orthogonal eigenfunctions on the interval $(-\pi,\pi)$ and we can apply bessel's inequality:

$$\sum_{n \leqslant N}^{\infty} \frac{|(g,\phi_N)|^2}{||\phi_N||^2} \leqslant ||g||^2$$

we can calculate that $||\phi_N||^2 = \int_{\pi}^{\pi} (\phi_N)^2 d\theta = \pi$. if $||g|| < \infty$, the series on the left hand side is convergent and thus its terms tend to zero. so $(g,\phi_N) \to 0$ and we are done. the only remaining thing is to check that $||g|| < \infty$:...

5.5.2 proof for discontinuous functions

5.5.3 proof of uniform convergence

what we want to prove is that given f and f', f'' are continuous functions of period 2π, the fourier series $\sum A_n X_n$ converges uniformly to the function

Proof $A_n := \int_{-\pi}^{\pi} f(x) \cos nx \frac{dx}{\pi}$ and $B_n := \int_{-\pi}^{\pi} f(x) \sin nx \frac{dx}{\pi}$
we notice that $A_n = -\frac{1}{n} B_n'$ for $n \neq 0$ and $B_n = \frac{1}{n} A_n'$.
we have

$$|\sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx| \leqslant \sum_{n=1} (|A_n| + |B_n|) = \sum_{n=1}^{\infty} \frac{1}{n}(|B_n'| + |A_n'|) \leqslant (\sum_{n=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}} [\sum_{n=1}^{\infty} 2(|A_n'|^2 + |B_n'|^2)]^{\frac{1}{2}}$$

here we use the Schwarz's inequality. $RHS < \infty$ and this means that the fourier series converges absolutely. we know that $f(x) = \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$

$$max|f(x) - S_N(x)| \leqslant max \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx| \leqslant (|A_n| + |B_n|) < \infty$$

the last sum is the tail of a convergent series of numbers so that it tends to zero as $N \to \infty$. therefore the fourier series converges to f(x) both absolutely and uniformly.

5.5.4 the gibbs phenomenon

The Gibbs Phenomenon is what happens to Fourier series at jump discontinuities. The key is that it happens when the convergence is only pointwise, not uniform. Gibbs showed that near the jump discontinuity, the partial sum $S_N(f)$ always differs from f by an overshoot of about 9 percent. Though the width of the overshoot goes to 0 as N goes to ∞, the extra height remains at 9 percent of the jump. That is to say

$$\lim_{N\to\infty} |S_N(f)(x) - f(x)| \neq 0,$$

when x is near the jump discontinuity. However, for x that does not jump, $S_N(f)(x) \to f(x)$. Consider a concrete example given in the textbook.

5.5.5 fourier series solutions

Chapter chapter6 harmonic functions

6.1 laplace's equation

first recall what is laplace equation? what is a harmonic function? what is poisson's equation? consider these definitions in 1 dimension, 2 dimension and in 3 dimension.

Definition 6.1

1. *laplace equation*
2. *harmonic function*
3. *poisson's equation*
4. *dirichlet problem for lapalce equation*

♣

6.1.1 maximum principle

recall the conditions for maximum principle:
a. the domain D is required to be ...
b. the harmonic function u is required to be ...
then we have ... (notice it is the strong maximum principle)

Proof

6.1.2 uniqueness of the dirichlet problem

Remark the dirichlet problem is refer to

$$\Delta u = f \text{ in D}$$

$$u = h \text{ on bdy D}$$

we can use the maximum principle to prove the uniqueness of the dirichlet problem.

6.1.3 invariance in two dimensions

we want to prove that laplace equation is invariant under transformation and rotation
transformation:

$$x' = x + a \text{ and } y' = y + b$$

rotation:

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

we can show that

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$$

the rotation invariance implies that should take a particular simple form in polar coordinates. we let

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

by jacobian matrix and chain rule we have

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

the final result is that

$$\Delta_2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and since u is rotational invariant, we know that u only depends on r, i.e.

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r \to (ru_r)_r = 0, ru_r = c_1, u = c_1 \log r + c_2$$

6.1.4 invariance in three dimensions

we use vector-matrix notation to proceed the proof. any rotation in three dimensions is given by

$$x' = Bx$$

where B is an orthogonal matrix ($BB^T = B^T B = I$) The laplacian is $\Delta u = \Sigma_{i=1}^3 u_{ii} = \Sigma_{i,j=1}^3 \delta_{ij} u_{ij}$ where the subscripts on u denote partial derivatives. Therefore,

$$\begin{aligned}\Delta u &= \sum_{k,l}\left(\sum_{i,j} b_{ki}\delta_{ij}b_{lj}\right)u_{k'l'} = \sum_{k,l}\delta_{kl}u_{k'l'} \\ &= \sum_k u_{k'k'}\end{aligned}$$

because the new coefficient matrix is

$$\sum_{i,j} b_{ki}\delta_{ij}b_{lj} = \sum_i b_{ki}b_{li} = \left(B^t B\right)_{kl} = \delta_{kl}.$$

So in the primed coordinates Δu takes the usual form

$$\Delta u = u_{x'x'} + u_{y'y'} + u_{z'z'}.$$

For the three-dimensional laplacian

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

it is natural to use spherical coordinates (r, θ, ϕ) (see Figure 3). We'll use the notation

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2} \\ s &= \sqrt{x^2 + y^2} \\ x &= s\cos\phi \quad z = r\cos\theta \\ y &= s\sin\phi \quad s = r\sin\theta. \end{aligned}$$

the chain of variables is $(x, y, z) \to (s, \phi, z) \to (r, \theta, \phi)$. By the two-dimensional Laplace calculation, we have both

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

and

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

We add these two equations, and cancel u_{ss}, to get

$$\begin{aligned}\Delta_3 &= u_{xx} + u_{yy} + u_{zz} \\ &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.\end{aligned}$$

In the last term we substitute $s^2 = r^2\sin^2\theta$ and in the next-to-last term

$$\begin{aligned} u_s &= \frac{\partial u}{\partial s} = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial \theta}{\partial s} + u_\phi\frac{\partial \phi}{\partial s} \\ &= u_r \cdot \frac{s}{r} + u_\theta \cdot \frac{\cos\theta}{r} + u_\phi \cdot 0. \end{aligned}$$

This leaves us with

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right]$$

which may also be written as

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Finally, let's look for the special harmonic functions in three dimensions which don't change under rotations, that is, which depend only on r. we have

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = 0 \to (r^2 u_r)_r = 0 \to r^2 u_r = c_1 \to u = -c_1\frac{1}{r} + c_2$$

6.2 rectangles and cubes

instead of copying the content of textbook, I decide to do the exercises of this chapter here.

Question 6.2.1. *1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a, 0 < y < b$ with the following boundary conditions:*

$$u_x = -a \quad \text{on } x = 0 \quad u_x = 0 \quad \text{on } x = a$$
$$u_y = b \quad \text{on } y = 0 \quad u_y = 0 \quad \text{on } y = b.$$

(Hint: Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

Question 6.2.2. *2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.*

Proof recall the definition of being orthogonal: we want to prove that $\forall m, n, m', n' \in \mathbb{Z}$, we have

$$\int_{-\pi}^{\pi} (\sin my \sin nz)(\sin m'y \sin n'z)dydz = 0$$

Question 6.2.3. *3. Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions: $u_y = 0$ for $y = 0$ and for $y = \pi$, $u = 0$ for $x = 0$ and $u = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$ for $x = \pi$*

Proof by separation of variables we have

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Longrightarrow$$
$$X'' - \lambda X = 0 \quad and \quad Y'' + \lambda Y = 0$$

check the boundary conditions:

$$X(0) = 0, \quad X(\pi) = \cos^2 y, \quad Y'(0) = Y'(\pi) = 0$$

we need to discuss the cases of eigenvalues for X and Y. (if you can remember the result) we finally get:

$$X_0(x) = C_1 x, \quad X_n(x) = \sinh nx, n = 1, 2, \ldots$$
$$Y_0(y) = C_2, \quad Y_n(y) = \cos ny, n = 1, 2, \ldots$$

we have

$$u(x, y) = X_0Y_0 + X_1Y_1 + \ldots = C_3 x + \sum_{n=1}^{\infty} C_n \cos ny \sinh nx$$

finally we apply the boundary condition $X(\pi) = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$: we have

$$C_3\pi = \frac{1}{2} \quad and \quad C_2 \cos 2y \sinh 2\pi = \frac{1}{2\cos 2y}$$

the final answer is

$$u(x, y) = \frac{x}{2\pi} + \frac{1}{2\sinh 2\pi}\sinh 2x \cos 2y$$

Question 6.2.4. *4. Find the harmonic function in the square* $\{0 < x < 1, 0 < y < 1\}$ *with the boundary conditions* $u(x, 0) = x, u(x, 1) = 0, u_x(0, y) = 0, u_x(1, y) = y^2$.

Proof by separation of variables we have

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Longrightarrow$$
$$X'' - \lambda X = 0 \quad and \quad Y'' + \lambda Y = 0$$

check the boundary conditions:

$$Y(0) = x \quad Y(1) = 0 \quad X'(0) = 0 \quad X'(1) = y^2$$

there are 2 boundary conditions are not homogeneous, then...

Question 6.2.5. *5. Solve Example 1 in the case* $b = 1, g(x) = h(x) = k(x) = 0$ *but* $j(x)$ *an arbitrary function.*

Question 6.2.6. *6. Solve the following Neumann problem in the cube* $\{0 < x < 1, 0 < y < 1, 0 < z < 1\} : \Delta u = 0$ *with* $u_z(x, y, 1) = g(x, y)$ *and homogeneous Neumann conditions on the other five faces, where* $g(x, y)$ *is an arbitrary function with zero average.*

Proof

Question 6.2.7. *7. (a) Find the harmonic function in the semi-infinite strip* $\{0 \le x \le \pi, 0 \le y < \infty\}$ *that satisfies the "boundary conditions":*

$$u(0, y) = u(\pi, y) = 0, u(x, 0) = h(x), \lim_{y\to\infty} u(x, y) = 0.$$

(b) What would go away if we omitted the condition at infinity?

6.3 poisson's formula

we study the dirichlet problem for a circle.

$$u_{xx} + u_{yy} = 0 \quad \text{for } x^2 + y^2 < a^2$$
$$u = h(\theta) \quad \text{for } x^2 + y^2 = a^2$$

we finally get

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

from the knowledge of full fourier series, we know

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi) \cos n\phi d\phi$$
$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi) \sin n\phi d\phi$$

plug A_n and B_n into the solution of u, we have

$$u(r, \theta) = \int_0^{2\pi} h(\phi)\{1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta - \phi)\}\frac{d\phi}{2\pi}$$

we can sum the series in braces:

$$1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta - \phi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}$$

Definition 6.2 (poisson's formula)

u is a solution of the dirichlet problem for a circle, u is given by the following poisson's formula:

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{d\phi}{2\pi}$$

♣

The Poisson formula can be written in a more geometric way as follows. Write $\mathbf{x} = (x, y)$ as a point with polar coordinates (r, θ). We could also think of $\mathbf{x}$ as the vector from the origin $\mathbf{0}$ to the point (x, y). Let $\mathbf{x}'$ be a point on the boundary.

$$\mathbf{x} : \text{polar coordinates } (r, \theta)$$

$$\mathbf{x}' : \text{polar coordinates } (a, \phi).$$

The origin and the points $\mathbf{x}$ and $\mathbf{x}'$ form a triangle with sides $r = |\mathbf{x}|$, $a = |\mathbf{x}'|$, and $|\mathbf{x} - \mathbf{x}'|$. By the law of cosines

$$|\mathbf{x} - \mathbf{x}'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi).$$

The arc length element on the circumference is $ds' = a d\phi$. Therefore, Poisson's formula takes the alternative form

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} ds'$$

for $\mathbf{x} \in D$, where we write $u(\mathbf{x}') = h(\phi)$. This is a line integral with respect to arc length $ds' = a d\phi$, since $s' = a\phi$ for a circle. For instance, in electrostatics this formula expresses the value of the electric potential due to a given distribution of charges on a cylinder that are uniform along the length of the cylinder.

Theorem 6.1

let the boundary of D be a circle c and h is a continuous function on C, then the poisson formula provides the only harmonic function in D for which we have

$$\lim_{x \to x_0} u(x) = h(x_0) \quad \forall x_0 \in C$$

♡

Proof we want to prove that u given by the poisson formula is harmonic and prove the equation: $\lim_{x \to x_0} u(x) = h(x_0) \quad \forall x_0 \in C$ holds.

to show u is harmonic, recall if the integrand function and its derivative are continuous, we can differentiate under the integral sign. we use the possion formula:

$$1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta - \phi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} := P(r, \theta - \phi)$$

we conclude that $P(r, \theta - \phi)$ is continuous and we assume $h(\phi)$ is continuous, and their derivative w.r.t r are also continuous (since $h(\phi)$ is irrelevant with r).

to prove the limit, we claim and prove $P(r, \theta)$ has the following 3 properties.

(i) $P(r, \theta) > 0$ for $r < a$. This property follows from the observation that $a^2 - 2ar\cos\theta + r^2 \geq a^2 - 2ar + r^2 = (a - r)^2 > 0$.

(ii)

$$\int_0^{2\pi} P(r, \theta)\frac{d\theta}{2\pi} = 1.$$

This property follows from the second part of (17) because $\int_0^{2\pi} \cos n\theta d\theta = 0$ for $n = 1, 2, \ldots$.

(iii) $P(r, \theta)$ is a harmonic function inside the circle. This property follows from the fact that each term $(r/a)^n \cos n\theta$ in the series is harmonic and therefore so is the sum.

Now we can differentiate under the integral sign (as in Appendix A.3) to get

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta r} &= \int_0^{2\pi} \left(P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\oplus\oplus}\right)(r, \theta - \phi)h(\phi)\frac{d\phi}{2\pi} \\ &= \int_0^{2\pi} 0 - h(\phi)d\phi = 0 \end{aligned}$$

for $r < a$. So u is harmonic in D. So it remains to prove (15). To do that, fix an angle θ_0 and consider a radius r near a. Then we will estimate the difference

$$u(r, \theta_0) - h(\theta_0) = \int_0^{2r} P(r, \theta_0 - \phi)\left[h(\phi) - h(\theta_0)\right]\frac{d\phi}{2\pi}$$

by Property (ii) of P. But $P(r, \theta)$ is concentrated near $\theta = 0$. This is true in the precise sense that, for $\delta \leq \theta \leq 2\pi - \delta$,

$$|P(r, \theta)| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a - r)^2 + 4ar\sin^2(\theta/2)} < \epsilon$$

for r sufficiently close to a. Precisely, for each (small) $\delta > 0$ and each (small) $\epsilon > 0$, (19) is true for r sufficiently close to a. Now from Property (i), (18), and (19), we have

$$|u(r, \theta_0) - h(\theta_0)| \leq \int_{\epsilon_0 - \hbar}^{\theta_0 + \xi} P(r, \theta_0 - \phi)\,\epsilon\frac{d\phi}{2\pi} + \epsilon\int_{|\phi - \epsilon_0| > \delta} |h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi}$$

for r sufficiently close to a. The ϵ in the first integral came from the continuity of h. In fact, there is some $\delta > 0$ such that $|h(\phi) - h(\theta_0)| < \epsilon$ for $|\phi - \theta_0| < \delta$. Since the function $|h| \leq H$ for some constant H, and in view of Property (ii), we deduce from (20) that

$$|u(r, \theta_0) - h(\theta_0)| \leq (1 + 2H)\epsilon$$

provided r is sufficiently close to a. This is relation (15).

6.3.1 mean value property

u is a harmonic function on a disk D and is continuous in $\overline{D}$, then the value of u at the center of D equals the average of u on its circumference ($|x|$ = radius of the disk)

Proof we let x = 0 in the above formula and get

$$u(0) = \frac{a^2}{2\pi a}\int_{|x'|=a} \frac{u(x')}{|0 - x'|^2}ds' = \frac{1}{2\pi a}\int_{|x'|=a} u(x')ds'$$

and RHS is the average of the value of u on its circumference.

6.3.2 maximum principle

we prove the maximum principle mentioned above using mean value property.

6.3.3 differentiability

u is a harmonic function in any open set D of the plane. we prove that $u(x, y)$ possesses all partial derivatives of all orders in D.

6.4 circles, wedges, and annuli

a wedge is the region: $\{0 < \theta < \theta_0, 0 < r < a\}$
an annulus is the region: $\{0 < a < r < b\}$
the exterior of a circle is: $\{a < r < \infty\}$

Example 6.1 we solve the laplace equation with the homogeneous dirichlet condition on the stright sides and the inhomogeneous neumann condition on the curved side:

$$\Delta u = 0, \quad u(r, 0) = 0 = u(r, \beta), \quad \frac{\partial u}{\partial r}(a, \theta) = h(\theta)$$

recall in the last chapter we have proved that the solutions satisfy using separation of variables:

$$\Theta'' + \lambda\Theta = 0 \quad r^2R'' + rR' - \lambda R = 0$$

solve the boundary condition problem:

$$\Theta'' + \lambda\Theta = 0 \quad \Theta(0) = \Theta(\beta) = 0$$

you can memorize that the solution is:

$$\lambda = (\frac{n\pi}{\beta})^2, \quad \Theta(\theta) = \sin\frac{n\pi\theta}{\beta}$$

solve the ode:

$$r^2R'' + rR' - \lambda R = 0,$$

we have

$$R(r) = r^{\alpha}, \alpha = \sqrt{\lambda} \text{ or } -\sqrt{\lambda}$$

and the negative one is impossible since we require $u(r,\theta)$ to be continuous at origin, so we have the solution is:

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}$$

we also need the boundary condition $h(\theta)$ to determine the coefficient A_n, that is:

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{-1+n\pi/\beta} \sin\frac{n\pi\theta}{\beta}$$

and thus

$$A_n = a^{1-n\pi/\beta}\frac{2}{n\pi}\int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta}d\theta$$

Chapter chapter7 green's identities and green's functions

Introduction

- find the green's function of the laplacian operator
- properties of green's function

7.1 green's first identity

7.1.1 green's first identity

Proposition 7.1 (green's first identity)

$$\iint_{bdyD} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v\Delta u dx$$

recall the product rule, we have

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\Delta u$$

recall the divergence theorem

$$\iiint_D \operatorname{div} F dx = \iint_{bdyD} F \cdot n dS$$

where $F = (F_1, F_2, F_3)$ and div $F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z_3}$ integrate and use the divergence theorem on the left side to get:

$$\iint_{bdyD} v\frac{\partial u}{\partial n}dS = \iint_{bdyD} (v\nabla u) \cdot n dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v\Delta u dx$$

where $\frac{\partial u}{\partial n} = n \cdot \nabla u$ is the directional derivative in the outward normal direction. the identity is called green's first identity. one of its application is to check whether the given problem is well-defined or not. e.g. ???

7.1.2 mean value property

Proposition 7.2 (mean value property)

the average value of any harmonic function over any sphere equals its value at the center:

$$\frac{1}{\text{area of } S} \iint_S u dS = u(0)$$

where S is the boundary of a ball D $\{|\mathbf{x}|^2 = x^2 + y^2 + z^2 < a^2\}$

Proof the idea is to prove that the average expression is independent of r.
using the second green's identity with v = 1, we have

$$\iint_{bdyD} \frac{\partial u}{\partial n}dS = \iiint_D \Delta u dx$$

since $\Delta u = 0$ in D, we have: $\iiint_D \Delta u dx = 0$, and we can also prove that

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$$

so we have

$$\iint_{bdyD} \frac{\partial u}{\partial r}dS = 0$$

using spherical coordinates, (r, θ, ϕ), we have

$$\int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)a^2\sin\theta d\theta d\phi = 0$$

we consider the function w.r.t r:

$$\frac{\partial}{\partial r}[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)d\theta d\phi] = 0$$

which says that

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta d\theta d\phi$$

is independent of r, let $r \longrightarrow 0$, we get

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(0)\sin\theta d\theta d\phi = u(0)$$

we are done.

7.1.3 maximum principle

Proposition 7.3

we want to deduce the maximum principle in 3d case, which says that: if D is any solid region, a nonconstant harmonic function in D cannot take its maximum value inside D, but only on bdy D.

7.1.4 uniqueness of dirichelet's problem

recall we have proved the uniqueness before using maximum principle. now we give another proof by the energy method. the main idea is using the green's first identity to get:

$$\iint_{bdyD} u\frac{\partial u}{\partial n}dS = \iiint_D |\nabla u|^2 dx$$

and using the first vanishing theorem and the observation that a function with vanishing gradient must be a constant provided D is connected.

7.1.5 dirichlet's principle

Proposition 7.4

let $u(x)$ be the unique harmonic function in D that satisfies the dirichlet's boundary condition:

$$w = h(x) \text{ on bdy } D$$

then we have

$$E[w] \geqslant E[u]$$

where E is defined as

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 dx$$

Proof (1)let v = u-w, using green's first identity we can show that

$$E[w] = E[u] + E[v]$$

(2)another proof says that we let $u(x)$ be a function satisfies the dirichlet's boundary condition, we are going to show that $\Delta u = 0$ so u is a harmonic function, and by the uniqueness we know that there is only function satisfying the boudnary condition and minimizing the energy.

$$E[u] \leqslant E[u+\epsilon v] = E[u] - \epsilon\iiint_D \Delta u v dx + \epsilon^2 E[v]$$

if $\epsilon = 0$, we have the minimum. consider

$$\iiint_D \Delta u v dx = 0$$

which is valid for all functions v in D, for any strict subdomain of D, that is $D' \subset D$, let $v(x) = 1$ for $x \in D'$, we have

$$\iiint_D \Delta dx = 0 \quad \text{for all D'}$$

and we apply the second vanishing theorem. we conclude that

$$\Delta u = 0 \quad x \in D$$

we are done. why we need to consider D'?

7.2 green's second identity

Proposition 7.5 (green's second identity)

$$\iiint_D (u\nabla v - v\nabla u)dx = \iint_{bdyD} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS$$

Proof recall the green's first identity, we have

$$\iint_{bdyD} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v\nabla u dx$$

we switch u and v and get

$$\iint_{bdyD} u\frac{\partial v}{\partial n}dS = \iiint_D \nabla u \cdot \nabla v dx + \iiint_D u\nabla v dx$$

substract:

$$\iiint_D (u\nabla v - v\nabla u)dx = \iint_{bdyD} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS$$

this is the green's second identity.

we use green's second identity to define symmetric boundary condition:

Definition 7.1 (symmetric boundary condition)

a boundary condition is called symmetric for the operator Δ if the right side of green's second identity vanishes for all pairs of functions u, v that satisfy the boundary condition.

7.2.1 representation formula

Proposition 7.6

the representation formula represents any harmonic function as an integral over the boundary.
D is any bounded and connected region (domain) in $\mathbb{R}^3$, u is a harmonic function, if $\Delta u = 0$ in D, we have

$$u(x_0) = \iint_{bdyD} [-u(x)\frac{\partial}{\partial n}(\frac{1}{|x-x_0|}) + \frac{1}{|x-x_0|}\frac{\partial u}{\partial n}]\frac{dS}{4\pi}$$

Proof we need the green's second identity with $v(x) = -\frac{1}{4\pi|x-x_0|}$. also note that v is not defined at x_0, so we focus on the domain D_ϵ, which is D minus a ball centered at x_0, the boundary of $D_\epsilon = \partial D$ and $\partial r = \epsilon$
note that $\Delta u = 0$ and $\Delta v = 0$, so by green's second identity, we have

$$\iint_{D\bigcup\{r=\epsilon\}} [u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n}]dS = \iint_D [u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n}]dS + \iint_{\{r=\epsilon\}} [u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n}]dS = 0$$

on $\{r = \epsilon\}$ we have

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$$

we write $r = \frac{1}{4\pi r}$ by WLOG assume x_0 at origin. so we have

$$-\iint_{bdyD}(u\frac{\partial}{\partial n}\frac{1}{r}-\frac{\partial u}{\partial n}\frac{1}{r})dS = -\iint_{r=\epsilon}(u\frac{\partial}{\partial r}\frac{1}{r}-\frac{\partial u}{\partial r}\frac{1}{r})dS$$

what we only need to show is that RHS approaches $4\pi u(0)$ as $\epsilon \to 0^+$
note RHS is

$$\frac{1}{\epsilon^2}\iint_{r=\epsilon} udS + \frac{1}{\epsilon}\iint_{r=\epsilon}\frac{\partial u}{\partial r}dS = 4\pi\overline{u} + 4\pi\epsilon\overline{\frac{\partial u}{\partial r}}$$

by continuity of u and boundedness of $\frac{\partial u}{\partial r}$, we conclude that

$$4\pi\overline{u} + 4\pi\epsilon\overline{\frac{\partial u}{\partial r}} \to 4\pi u(0) + 0, \quad \text{as } \epsilon \to 0^+$$

7.3 green's functions

the representation formula uses two properties of the function $v(x) = \frac{1}{-4\pi|x-x_0|}$ that it is harmonic except at x_0 and it has a certain singularity there. we want to modify the function to only keep one term in the formula. the modified function is called the green's function.

Definition 7.2 (green's function in $\mathbb{R}^3$)

the green's function $G(X)$ for the operator $-\Delta$ and the domain D at the point $x_0 \in D$ is a function defined for $x \in D$ s.t.
1. $G(x)$ possess continuous second derivatives and $\Delta G = 0$ in D, except at the point $x = x_0$
2. $G(x) = 0$ for $x \in bdyD$
3. the function $G(x) + \frac{1}{4\pi|x-x_0|}$ is finite at x_0 and has continuous second derivatives everywhere and is harmonic at x_0.

notice that in 3. $\frac{1}{4\pi|x-x_0|}$ is a solution to the function$\Delta u(r) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$
for green's function in $\mathbb{R}^2$, we need to change conditions
1. $G(x)$ is smooth and harmonic on the domain except at x_0
2.$G(x) = 0$ on the boundary
3. into: $G(x) - \frac{1}{2\pi}\log|x-x_0|$ is smooth and harmonic on the whole domain
for green's function in $\mathbb{R}^1$, we have:
1. $G''(x) = 0$ for $x \neq 0$
2. $G(0) = G(l) = 0$
3. $G(x) + \frac{1}{2}|x-x_0|$ is harmonic at x_0 and is continuous

Theorem 7.1

if $G(x, x_0)$ is the green's function, then the solution of the dirichlet problem is given by the formula:

$$u(x_0) = \iint_{bdyD} u(x)\frac{\partial G(x,x_0)}{\partial n}dS$$

Theorem 7.2 (symmetry of green's function)

for any region D we have a green's function $G(x, x_0)$. it is always symmetric

$$G(x,x_0) = G(x_0,x)$$

Theorem 7.3

the solution of the problem

$$\Delta u = f \text{ in } D \quad u = h \text{ on bdy } D$$

is given by

$$u(X_0) = \iint_{bdyD} h(x)\frac{\partial G(x,x_0)}{\partial n}dS + \iiint_D f(x)G(x,x_0)dx$$

7.4 half-space and sphere

we using the method of reflection to find the green's function for the half-space $\Omega = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$

Proposition 7.7

the function G defined as

$$G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|}$$

is the desired green's function in Ω of the point x_0.

now we can use the green's function to solve the dirichlet problem

$$\Delta u = 0, \quad z > 0$$
$$u(x,y,0) = h(x,y)$$

the solution is given by the theorem above:

$$u(x_0) = \iint_{bdyD} u(x)\frac{\partial G(x,x_0)}{\partial n}dS$$

and we have $\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z} = -\frac{1}{4\pi}\frac{1}{|x-x_0|^2}\partial_z|x-x_0| + \frac{1}{4\pi}\frac{1}{|x-x_0^*|^2}\partial_z|x-x_0^*| = -\frac{z-z_0}{4\pi|x-x_0|^3} + \frac{z-z_0^*}{4\pi|x-x_0^*|}$ and we have $z_0^* = -z_0$, we have

$$\frac{\partial G}{\partial n} = \frac{1}{2\pi}\frac{z_0}{|x-x_0|^3}$$

the solution is

$$u(x_0) = \frac{z_0}{2\pi}\iint_{\partial D}\frac{h(x)}{|x-x_0|^3 dS}$$

for sphere, we want to construct the green's function for the ball $D = \{|x| < a\}$ of radius a. similarly we use the reflection method.

one important observation is that for $x_0 \in D, X_0^* \notin D$, we cannot find a point$x_0^*$ outside D s.t.

$$|x - x_0^*| = |x - x_0| \quad \forall x \in \partial D$$

. so we wonder whether it is possible to find a C and a fixed point x_0 s.t.

$$|x - x_0| = C|x - x_0^*|$$

and we claim that $C = \frac{a}{|x_0|}$ and $x_0^* = \frac{a^2}{|x_0|^2}x_0$ and we have

$$\frac{1}{|x-x_0|} = \frac{a}{x_0}\frac{1}{|x-x_0^*|} \quad \forall x \in \partial D$$

therefore we set $G(x,x_0) := -\frac{1}{4\pi|x-x_0|} + \frac{a}{|x_0|}\frac{1}{4\pi|x-x_0^*|}$and this function G satisfies the second condition for green's function.
checking condition 1 and 3: ...

Proof [proof for the claim] it is due to some geometry knowledge. we can find

$$|x-x_0^*| = |x - \frac{a^2}{|x_0|^2}x_0| = \frac{a}{|x_0|}|\frac{|x_0|}{a}x - \frac{a}{|x_0|}x_0| = \frac{a}{|x_0|}|x-x_0|$$

thus we have

$$\frac{1}{|x-x_0|} = \frac{a}{|x_0|}\frac{1}{|x-x_0^*|}, \quad \forall x \in \partial D$$

we want to use the green's function to find a formula for the solution of the dirichlet problem in a ball:

$$\Delta u = 0 \forall |x| < a, \quad u(x) = h(x) \quad \forall |x| = a$$

the conclusion is that

$$u(x_0) = \frac{a^2 - |x_0|^2}{4\pi a} \iint_{|x|=a} \frac{h(x)}{|x - x_0|^3} dS$$

Proof [proof for the formula] $\frac{\partial G}{\partial n} = n \cdot \nabla G$ and on $|x| = a$, we have $n = \frac{x}{a}$, and $\nabla G = \frac{x - x0)}{4\pi |x - x_0|^3} - \frac{a}{|x_0|} \frac{x - x_0^*}{4\pi |x - x_0^*|} = \frac{1}{4\pi |x - x_0|^3}[x - (|x_0|/a)^2 x]$thus

$$\frac{\partial G}{\partial n} = \frac{x}{a} \nabla G = \frac{a^2 - |x_0|^2}{4\pi |x - x_0|^3}$$

and

$$u(x_0) = \frac{a^2 - |x_0|^2}{4\pi a} \iint_{|x|=a} \frac{h(x)}{|x - x_0|^3} dS$$

Chapter chapter12 distributions and transforms

Introduction

- ❏ *find the fourier transform of a given function*

8.1 distributions

we want to study the so-called approximate delta functions. so what is a delta function like? it have integral 1:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

we would call it a distribution rather a function. a distribution is a rule that assigns numbers to functions. and the delta function is the rule that number $\phi(0)$ to the function $\phi(x)$. we need to say what kinds of $\phi(x)$ are used. a test function $\phi(x)$ is a real C^∞ function (a function all of whose derivatives exist) that vanishes outside a finite interval. so $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable for all $-\infty < x < \infty$ and $\phi(x) \equiv 0$ for x large. we use $\mathcal{D}$ denote the collection of all test functions.

Definition 8.1 (distribution)

a distribution f is a functional(rule): $\mathcal{D} \to \mathbb{R}$ which is linear and continuous in the following sense. if $\phi \in \mathcal{D}$ is a test function, then we denote the corresponding real number by (f, ϕ).
by linearity we mean that

$$(f, a\phi + b\psi) = a(f, \phi) + b(f, \psi)$$

for all constants a, b and all test functions ϕ, ψ.
by continuity we mean the following. if $\{\phi_n\}$ is a sequence of test functions that vanish outside a common interval and converge uniformly to a test function ϕ, and if all their derivatives do as well, then

$$(f, \phi) \to (f, \phi) \quad as\ n \to \infty$$

♣

Definition 8.2 (weak convergence)

if $f_N(x)$ is a sequence of distributions and f is another distribution, we say that f_N converges weakly to f if

$$(f_N, \phi) \to (f, \phi) \quad as\ N \to \infty$$

for all test functions ϕ.

♣

Example 8.1 let $f(x)$ be any ordinary integrable function. it corresponds to the distribution

$$\phi \to \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

we can show that this map is a distribution. and we usually use the notation:

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0)$$

the source function for the diffusion equation on the whole line is

$$S(x,t) = 1/\sqrt{4\pi kt}e^{-x^2/4kt} \text{ for } t > 0$$

we have proved that

$$\int_{-\infty}^{\infty} S(x,t)\phi(x)dx \to \phi(0) \quad \text{as } t \to 0$$

we have

$$S(x,t) \to \delta(x) \quad as \quad t \to 0$$

Example 8.2 prove that

$$\int_{-\pi}^{\pi} K_N(\theta)\phi(\theta)d\theta \to 2\pi\phi(0) \quad as \quad N \to \infty$$

for any periodic C^1 function $\phi(x)$. and thus we have

$$K_N(\theta) \to 2\pi\delta(\theta) \quad \text{weakly as } N \to \infty \text{ in the interval } (-\pi, \pi)$$

we next study the derivative of a distribution, which always exists and is another distribution. the idea is due to the following observation: recall the definition of $\phi(x)$ saying that $\phi(x)$ vanishes for large $|x|$, so we have

$$\int_{-\infty}^{\infty} f'(x)dx\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

by integration by parts.

Definition 8.3 (derivative f')

fro any distribution f, we define its derivative f' by the formula:

$$(f', \phi) = -(f, \phi) \text{ for all test functions } \phi$$

we can check that f' satisfies the linearity and continuity properties. and if $f_N \to f$ weakly, we have $f'_N \to f$ weakly.

♣

Example 8.3 the derivatives of the delta function are

$$(\delta', \phi) = -(\delta, \phi) = -\phi'(0)$$
$$(\delta'', \phi) = -(\delta', \phi') = (\delta, \phi'') = \phi''(0)$$

8.2 green's functions, revisited

8.3 fourier transforms

consider a function $f(x)$ defnied on the interval $(-l, l)$. its fourier series in complex notation is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where the coefficients are

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-in\pi y/l}dy$$

the fourier integral comes from letting $l \to \infty$. let $k = n\pi/l$, and substitute the coefficients into the seires, we get

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-l}^{l} f(y)e^{-iky}dy\right]e^{ikx}\frac{\pi}{l}$$

as $l \to \infty$, the interval expands to the whole line and the points k get closer together. in the limit we should exect k to become a continuous variable, and the sum to become an integral. the distrance between two successive k's is $\Delta k = \frac{\pi}{l}$, which we may think of as becoming dk in the limit. therefore, we expect the result

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)e^{-iky}dy\right]e^{ikx}dk$$

here we do not provide a rigorous proof. it is a continuous version of the completeness property of fourier series.

Definition 8.4 (fourier transform)

$F(k)$ is called the fourier transform of $f(x)$ if we have

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}\frac{dk}{2\pi}$$

where $F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$. notice the two equations only differ a minus sign in the exponent and the 2π factor. ♣

there are some important transforms:

	$f(x)$	$F(k)$
Delta function	$\delta(x)$	1
Square pulse	$H(a-\lvert x\rvert)$	$\frac{2}{k}\sin ak$
Exponential	$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}\quad (a>0)$
Heaviside function	$H(x)$	$\pi\delta(k)+\frac{1}{ik}$
Sign	$H(x)-H(-x)$	$\frac{2}{ik}$
Constant	1	$2\pi\delta(k)$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}e^{-k^2/2}$

let $F(k)$ be the transform of $f(x)$ and let $G(k)$ be the transform of $g(x)$. then we have the following table:

	Function	Transform
(i)	$\frac{df}{dx}$	$ikF(k)$
(ii)	$xf(x)$	$i\frac{dF}{dk}$
(iii)	$f(x-a)$	$e^{-iak}F(k)$
(iv)	$e^{iax}f(x)$	$F(k-a)$
(v)	$af(x)+bg(x)$	$aF(k)+bG(k)$
(vi)	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{k}{a}\right)\quad (a\neq 0)$

next we discuss the convolution:

Definition 8.5 (convolution)

the convolution of $f(x)$ and $g(x)$ is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

♣

Proposition 8.1

let the fourier transform of $f(x)$ and $g(x)$ be $F(k)$ and $G(k)$, notice the fourier transform of $f*g$ is

$$\int_{-\infty}^{\infty}(f*g)(x)e^{-ikx}dx = \iint f(x-y)g(y)dye^{-ikx}dx$$

let $z = x-y$, we have

$$\iint f(z)e^{-ik(y+z)}dzg(y)dy = \int f(z)e^{-ikz}dz\cdot\int g(y)e^{-iky}dy = F(k)\cdot G(k)$$

that is

$$(f*g) = F(k)\cdot G(k)$$

♠

we can define the fourier transform in three dimensions:

Definition 8.6 (fourier transform in three dimensions)

$$F(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ik\cdot x}dx$$

where $x = (x, y, z)$, $k = (k_1, k_2, k_3)$, and $k \cdot x = xk_1 + yk_2 + zk_3$. we can recovers $f(x)$ from the formula:

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k)e^{ik\cdot x}\frac{dk}{(2\pi)^3}$$

♣

8.4 source functions

diffusion

the source function is defined as the unique solution of the problem

$$S_t = S_{xx} \quad (-\infty < x < \infty, 0 < t < \infty), \quad S(x,0) = \delta(x)$$

note the diffusion constant is taken to be 1.

we find the fourier transform of $S(x,t)$ as a distribution in x, for each t:

$$\hat{S}(k,t) = \int_{-\infty}^{\infty} S(x,t)e^{-ikx}dx$$

we have

$$\frac{\partial S}{\partial t} = \frac{\partial^2 \hat{S}}{\partial x^2} = (ik)^2\hat{S} = -k^2\hat{S}, \quad \hat{S}(k,0) = 1$$

for each k this is an ode that is easy to solve. the solution is

$$\hat{S}(k,t) = e^{-k^2 t}$$

to find $S(x)$, we only need to check the table above with $\hat{S}(k,t) = e^{-k^2 t}$

waves

we want to find the source function satisfying

$$S_{tt} = c^2 S_{xx}, \quad S(x,0) = 0, \quad S_t(x,0) = \delta(x)$$

we have

$$\frac{\partial^2 \hat{S}}{\partial t^2} = -c^2k^2\hat{S}, \quad \hat{S}(k,0) = 0, \quad \frac{\partial \hat{S}}{\partial t}(k,0) = 1$$

the ode has the solution

$$\hat{S}(k,t) = \frac{1}{kc}\sin kct = \frac{e^{ikct} - e^{-ikct}}{2ikc}$$

and

$$S(x,t) = \int_{-\infty}^{\infty} \frac{e^{ik(x+ct)} - e^{ik(x-ct)}}{4\pi ikc}dk = \frac{sgn(x+ct) - sgn(x-ct)}{4c} = \frac{H(c^2t^2 - x^2)}{2c}$$

laplace's equation in a half-plane

We use the Fourier transform to rework the problem of Section 7.4,

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{in the half-plane } y > 0, \\ u(x,0) &= \delta(x) && \text{on the line } y = 0. \end{aligned}$$

We cannot transform the y variable, but can transform x because it runs from $-\infty$ to ∞. Let

$$U(k,y) = \int_{-\infty}^{\infty} e^{-ikx}u(x,y)dx$$

be the Fourier transform. Then U satisfies the ODE

$$-k^2U + U_{yy} = 0 \quad \text{for } y > 0, \quad U(k,0) = 1.$$

The solutions of the ODE are $e^{\pm yk}$. We must reject a positive exponent because U would grow exponentially as $|k| \to \infty$ and would not have a Fourier transform. So $U(k,y) = e^{-y|k|}$. Therefore,

$$u(x,y) = \int_{-\infty}^{\infty} e^{ikx}e^{-y|k|}\frac{dk}{2\pi}.$$

This improper integral clearly converges for $y > 0$. It is split into two parts and integrated directly as

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi(ix-y)} e^{ikx-ky}\Big|_0^\infty + \frac{1}{2\pi(ix+y)} e^{ikx+ky}\Big|_{-\infty}^0 \\ &= \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) = \frac{y}{\pi\left(x^2+y^2\right)}, \end{aligned}$$

in agreement with Exercise 7.4.6.

Chapter review

9.1 review for midterm: 10.28-10.30

0. formulae:

separation method

$$\frac{T''}{C^2T} = \frac{X''}{X} = -\lambda$$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

robin condition

$$u_x(0,t) - a_0u(0,t) = 0$$

$$u_x(l,t) + a_lu(l,t) = 0$$

wave equation (d'Alembert Formula):

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\phi(s)ds + \frac{1}{2}(\phi(x-ct)+\phi(x+ct))$$

diffusion equation

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy$$

energy of wave equation:

$$E = \frac{1}{2}\int_{\mathbb{R}}(u_t^2 + c^2u_x^2)dx$$

energy of diffusion equation

$$E = \frac{1}{2}\int_{\Omega}u^2dx$$

diffusion equation with a source

$$u(x,t) = \int_{\mathbb{R}}\Phi(x-y,t)\phi(y)dy + \int_0^t\int_{\mathbb{R}}\Phi(x-y,s)dyds, \quad \Phi(x-y,t) = \frac{1}{\sqrt{4\pi kt}}e^{\frac{(x-y)^2}{4\pi kt}}$$

wave equation with a source

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \int_0^t\int_{x-c(t-s)}^{x+c(t-s)}f(y,s)dyds$$

fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}, \quad A_m = \frac{2}{l}\int_0^l\phi(x)\sin\frac{m\pi x}{l}dx$$

fourier cosine series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}, \quad A_m = \frac{2}{l}\int_0^l\phi(x)\cos\frac{m\pi x}{l}dx$$

fourier full series

$$\phi(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l})$$

$$A_n=\frac{1}{l}\int_{-l}^{l}\phi(x)\cos\frac{n\pi x}{l}dx,\quad B_n=\frac{1}{l}\int_{-l}^{l}\phi(x)\sin\frac{n\pi x}{l}dx$$

the following formulas are not difficult to remember, you need to know that we use separation method and with dirichlet condition there are only positive eigenvalues and with neumann there are zero eigenvalue and positive eigenvalue. wave equation with dirichlet boundary condition

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi ct}{l}+B_n\sin\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}$$

diffusion equation with dirichlet boundary condition

$$u(x,t)=\sum_{n=1}^{\infty}A_ne^{-(n\pi/l)^2kt}\sin\frac{n\pi x}{l}$$

wave equation with neumann boundary condition

$$u(x,t)=\frac{1}{2}A_0+\frac{1}{2}B_0t+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi ct}{l}+B_n\sin\frac{n\pi ct}{l})\cos\frac{n\pi x}{l}$$

diffusion equation with neumann boundary condition

$$u(x,t)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_ne^{-i(n\pi/l)^2t}\cos\frac{n\pi x}{l}$$

1. energy method and uniqueness for heat equation

Question 9.1.1. *The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem for the diffusion equation. That is, there is at most one solution of*

$$\begin{aligned} u_t-ku_{xx}&=f(x,t)\quad \textit{for } 0<x<l \textit{ and } t>0\\ u(x,0)&=\phi(x)\\ u(0,t)&=g(t)\qquad u(l,t)=h(t)\end{aligned}$$

for four given functions f,ϕ,g, and h.

Proof Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (3). Let $w=u_1-u_2$ be their difference. Then $w_t-kw_{xx}=0, w(x,0)=0, w(0,t)=0, w(l,t)=0$. Let $T>0$. By the maximum principle, $w(x,t)$ has its maximum for the rectangle on its bottom or sides-exactly where it vanishes. So $w(x,t)\le 0$. The same type of argument for the minimum shows that $w(x,t)\ge 0$. Therefore, $w(x,t)\equiv 0$, so that $u_1(x,t)\equiv u_2(x,t)$ for all $t\ge 0$.

Proof Here is a second proof of uniqueness for problem (3), by a very different technique, the energy method. Multiplying the equation for $w=u_1-u_2$ by w itself, we can write

$$0=0\cdot w=(w_t-kw_{xx})(w)=\left(\frac{1}{2}w^2\right)_t+(-kw_xw)_x+kw_x^2.$$

(Verify this by carrying out the derivatives on the right side.) Upon integrating over the interval $0<x<l$, we get

$$0=\int_0^l\left(\frac{1}{2}w^2\right)_t dx-kw_xw|_{x=0}^{x=l}+k\int_0^l w_x^2dx.$$

Because of the boundary conditions ($w = 0$ at $x = 0, l$),

$$\frac{d}{dt}\int_0^l \frac{1}{2}[w(x,t)]^2 dx = -k\int_0^l [w_x(x,t)]^2\, dx \leq 0,$$

where the time derivative has been pulled out of the x integral (see Section A.3). Therefore, $\int w^2 dx$ is decreasing, so

$$\int_0^l [w(x,t)]^2 dx \leq \int_0^l [w(x,0)]^2 dx$$

for $t \geq 0$. The right side of (4) vanishes because the initial conditions of u and v are the same, so that $\int [w(x,t)]^2 dx = 0$ for all $t > 0$. So $w \equiv 0$ and $u_1 \equiv u_2$ for all $t \geq 0$.

2. well-posedness que

Question 9.1.2. *(a) (5 points) State the definition of a well-posed PDE problem. (b) (5 points) Is the following problem well-posed? Why?*

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & x^2 + y^2 < 1 \\ \frac{\partial u}{\partial \vec{n}}(x,y) = 0, & x^2 + y^2 = 1, \vec{n} \text{ is the unit outnorm of } x^2 + y^2 = 1 \end{cases}$$

(c) (10 points) Verifying that $u_n(x,t) = \frac{1}{n}\sin nx e^{-n^2 t}$ solves the following probem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < \pi, \quad -\infty < t < +\infty \\ u(0,t) = u(\pi,t) = 0, & -\infty < t < \infty \\ u(x, t = 0) = \frac{1}{n}\sin nx, & 0 \leq x \leq \pi \end{cases}$$

for all positive integer n. How does the energy change when $t \longrightarrow \pm\infty$? (recall the definition of energy: for wave equation, we have $E = \frac{1}{2}\int_{-\infty}^{+\infty}(\rho u_t^2 + T u_x^2)dx$ is a constant independent of t.)
(d) (10 points) Is the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < \pi, \quad t < 0 \\ u(0,t) = u(\pi,t) = 0, & t < 0 \\ u(x, t = 0) = 0, & 0 < x < \pi \end{cases}$$

well-posed? Why?

3. maximum principle

Question 9.1.3. *Is there a maximum principle for the Cauchy problem for the 1-dimensional wave equation? Explain why?*
(Cauchy problem: A Cauchy problem can be an initial value problem or a boundary value problem)

Proof Consider the following Cauchy problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & -\infty < x < +\infty, \quad t > 0 \\ u(x, t = 0) = 0, & \partial_t u(x, t = 0) = \sin x, \quad -\infty < x < +\infty \end{cases}$$

And the unique solution is given by d'Alembert formula:

$$u(x,t) = \frac{1}{2}\cos(x+t) - \cos(x-t) = -\sin x \sin t, \quad -\infty < x < \infty, t > 0$$

Then $u(x,t)$ attains its maximum 1 only at the interior points $\left(\frac{\pi}{2} \pm 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ or $\left(\frac{3\pi}{2} \pm 2n\pi, \frac{\pi}{2} + 2n\pi\right)$ for $n = 0, 1, 2, \cdots$. However, $u(x,t) = 0$ on the boundary $\{(x,t) : t = 0\}$. Therefore there is no maximum principle for the Cauchy problem for the 1-dimensitonal wave equation. Remark: The key is to find an counterexample.

Question 9.1.4 (generalized maximum principle). *(a) Prove the following generalized maximum principle: if $\partial_t u - \partial_x^2 u \leq 0$ on $R \equiv [0,l] \times [0,T]$, then*

$$\max u(x,t) = \max_{\partial R} u(x,t)$$

where $\partial R = \{(x,t) \in R \mid$ such that either $x = 0$, or $x = l$, or $t = 0\}$.

(b) Show that if v solves the following problem

$$\begin{aligned}
&\partial_t v = \partial_x^2 v + f(x,t), \quad 0<x<l, \quad 0<t<T \\
&v(x,0) = 0 \\
&v(0,t) = 0 = v(l,t), \quad 0 \le x \le T
\end{aligned}$$

then

$$v(x,t) \le t \max_R |f(x,t)|$$

(hint, applying the result in (a) to $u(x,t) = v(x,t) - t\max_R |f(x,t)|$)

Proof (a) Let $v(x,t) = u(x,t) + \epsilon x^2$, then v satisfies

$$\partial_t v - \partial_x^2 v = \partial_t u - \partial_x^2 u - 2\epsilon < 0$$

First, claim that v attains its maximum on the parabolic boundary R. Let $\max_R v(x,t) = M = v(x_0,t_0)$. Suppose on the contrary, then either i. $0 < x_0 < l, 0 < t_0 < T$. In this case, $v_t(x_0,t_0) = v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \le 0$. Thus $\partial_t v - \partial_x^2 v\big|_{(x_0,t_0)} \ge 0$, which is impossible. ii. $0 < x_0 < l, t_0 = T$. In this case, $v_t(x_0,t_0) \ge 0, v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \le 0$. Thus $\partial_t v - \partial_x^2 v\big|_{(x_0,t_0)} \ge 0$, which is impossible. Hence

$$\max_R v(x,t) = \max_{\partial R} v(x,t).$$

Then for any $(x,t) \in R$,

$$u(x,t) \le u(x,t) + \epsilon x^2 \le \max_{\partial R} v(x,t) \le \max_{\partial R} u(x,t) + \epsilon l^2$$

Letting $\epsilon \longrightarrow 0$ gives $u(x,t) \le \max_{\partial R} u(x,t)$ for any $(x,t) \in R$. Hence $\max_R u(x,t) = \max_{\partial R} u(x,t)$

4. characteristic line method/geometric method

Question 9.1.5. *1. (20 points) (a) (8 points) Find the general solutions to*

$$u_x - 2u_y + 2u = 0$$

(b) (12 points) Solve the problem:

$$\begin{cases} y\partial_x u + 3x^2 y \partial_y u = 0 \\ u(x=0,y) = y^2 \end{cases}$$

Find the region in the xy-plane so that the solution is uniquely determined.

Question 9.1.6. *(a) (8 points) Solve the following problem*

$$\begin{cases} \partial_t u + 4\partial_x u - 2u = 0 \\ u(x,t=0) = x^2 \end{cases}$$

(b) (12 points) Solve the problem

$$\begin{cases} 2\partial_x u + y\partial_y u = 0 \\ u(x=0,y) = y \end{cases}$$

What are characteristic curves of this equation?

Proof (a) Method 1:Coordinate Method: Use the following new coordinates

$$t' = t + 4x, x' = 4t - x$$

Hence $\partial_t u + 4\partial_x u = 17\partial_{t'} u = 2u$. Thus the solution is $u(t',x') = f(x')e^{\frac{2}{17}t'}$ with function f to be determined. Therefore, the general solutions are

$$u(t,x) = f(4t-x)e^{\frac{2}{17}(t+4x)}.$$

Moreover, the initial condition implies that

$$u(x,t=0) = f(-x)e^{\frac{8x}{17}} = x^2,$$

or equivalently,

$$f(x) = x^2 e^{\frac{8x}{17}}.$$

Finally,

$$u(t,x) = (4t - x)^2 e^{2t}.$$

Method 2: Geometric Method: The corresponding characteristic curves are

$$\frac{dt}{1} = \frac{dx}{4}$$

that is, $x = 4t + C$ where C is an arbitrary constant. Then

$$\frac{d}{dt}u(t, 4t + C) = u_t(t, 4t + C) + 4u_x(t, 4t + C) = 2u(t, 4t + C).$$

Hence $u(t, 4t + C) = f(C)e^{2t}$, where f is an arbitrary function. Therefore,

$$u(t,x) = f(x - 4t)e^{2t}.$$

While the initial condition shows that

$$u(x, t = 0) = x^2 = f(x)$$

thus

$$u(x,t) = (x - 4t)^2 e^{2t}$$

(b) The characteristic equations are

$$\frac{dx}{2} = \frac{dy}{y}$$

thus the characteristic curves are given by

$$y = Ce^{\frac{x}{2}}$$

where C is an arbitrary constant. Then

$$\frac{d}{dx}u\left(x, Ce^{\frac{x}{2}}\right) = u_x + \frac{C}{2}e^{\frac{x}{2}}u_y = u_x + \frac{y}{2}u_y = 0$$

Hence $u\left(x, Ce^{\frac{x}{2}}\right) = f(C)$ where f is an arbitrary function. Thus

$$u(x,y) = f\left(ye^{-\frac{x}{2}}\right)$$

Besides, the auxiliary condition gives that $y = u(x = 0, y) = f(y)$. Hence, the solution is

$$u(x,y) = ye^{-\frac{x}{2}}$$

5. duhamel principle and inhomogeneous problem
suppose U = U(t,x;s) satisfies: $\partial_t U = k\partial_x^2 U, x \in \mathbb{R}$ and $U(s, x; s) = f(x, s)$, where s is a parameter, then

$$u(x,t) := \int_0^t U(t, x; s)ds$$

solves the PDE:

$$\partial_t u = k\partial_x^2 u + f(x,t) \text{ and } u(x,0) = 0$$

Question 9.1.7. *we can use the duhamel principle to prove the formula of inhomogeneous diffusion equation*

6. method of reflection

Question 9.1.8. *6. (10 points) Derive the solution formula for the following initial-boundary value problem*

$$\begin{cases} \partial_t u = \partial_x^2 u, \quad 0 < x < +\infty, \quad t > 0 \\ u(x, t = 0) = \phi(x) \quad 0 < x < +\infty \\ \partial_x u(x = 0, t) = 0, \quad t > 0 \end{cases}$$

by the method of reflection.

7. comparison principle

Question 9.1.9. *(2.3 que 6) Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$*

8. parity

Question 9.1.10. *4. (20 points) (a) (10 points) Consider the following problem*

$$\begin{cases} \partial_t u = \partial_x^2 u + f(x,t), -\infty < x < +\infty, \quad t > 0 \\ u(x, t = 0) = \varphi(x) \end{cases}$$

Prove that if $\varphi(x)$ and $f(x,t)$ are even functions of x, then the solution $u(x,t)$ to above solution must be even in x. (b) (10 points) Apply the result in (a) to solve the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u + e^{-x^2}, \quad x > 0, \quad t > 0 \\ u(x, t = 0) = \cos x, \quad x > 0 \\ \partial_x u(x = 0, t) = 0 \end{cases}$$

9. fourier series
10. types of PDEs: $det > 0$ is elliptic
11. cases of eigenvalues of robin conditions4.3.4
12. eigenvalues and eigenfunctions

Question 9.1.11. *1. (5 points) Can the eigenvalue problem*

$$\begin{cases} -X''(x) = \lambda X(x), \quad 0 < x < 1 \\ X'(0) = 0, \quad X(1) = 0 \end{cases}$$

have nonpositive eigenvalues? Prove your statements. Write down all the eigenvalues and corresponding eigenfunctions.

13. characteristic coordinate method

(refer to 2.1) we have an important result is that the solution of the wave equation can be written into the form

$$u(x,t) = f(x + ct) + g(x - ct)$$

given $u_{tt} = c^2 u_{xx}$

9.2 review for second quiz

we review chapter 6 and 7 here.

1. formulae

   (a). divergence theorem

   $$\iiint_D \text{div } F dx = \iint_{bdyD} F \cdot ndS$$

   where $F = (F_1, F_2, F_3)$ and div $F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z_3}$

   (b). bessel's inequality

   $$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx \leq \int_a^b |f(x)|^2 dx$$

   which holds as long as the integral of $|f|^2$ is finite (c). parseval's equality

$$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

which is valid iff the fourier series of $f(x)$ converges to $f(x)$ in MS sense.

(d). laplace equation is rotational invariant:

$$\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

the solution is $u = c_1 \log r + c_2$

$$\Delta_3 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$$

(e). the solution is $u(r) = -c_1 \frac{1}{r} + c_2$ mean value property: we have the mean value property for:
case1: harmonic function on a disk

$$u(0) = \frac{1}{2\pi a} \int_{|x'|=a} u(x') ds'$$

case2: harmonic function over any sphere

$$\frac{1}{\text{area of S}} \iint_S u dS = u(0)$$

(f). possion's formula (for dirichlet problem for a circle of laplace equation):

$$u(r,\theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

(g). dirichlet kernel: for fourier series:

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

we plug in the coefficients A_n and B_n and will get the n-th sum:

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) = \int_{-\pi}^{\pi} [1 + 2\sum_{n=1}^{N} (\cos ny \cos nx + \sin ny \sin nx)] f(y) \frac{dy}{2\pi}$$

and dirichlet kernel is defined by

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta = \frac{\sin (N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}$$

(h). green's first identity:

$$\iint_{bdyD} \frac{\partial u}{\partial n} dS = \iiint_D \Delta u dx$$

(i). green's second identity

$$\iint_D (u\Delta v - v\Delta u) dx = \iint_{bdyD} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) dS$$

(j). representation formula:

$$(3D) \quad u(x_0) = \iint_{bdyD} [-u(x)\frac{\partial}{\partial n}(\frac{1}{|x-x_0|}) + \frac{1}{|x-x_0|}\frac{\partial u}{\partial n}]\frac{dS}{4\pi}$$

$$(2D) \quad u(x_0) = \frac{1}{2\pi}\int_{bdyD} [u(x)\frac{\partial}{\partial n}(\log|x-x_0|) - \frac{\partial u}{\partial n}\log|x-x_0|]ds$$

2. poisson's formula

we can prove that dirichlet problem for a disk $\{r < a\}$ of laplace equation has a fourier full series solution. (this is from 6.3) check HW7 Q2, Q4 (Q4 tells that if we consider the exterior, we get different fourier series solution!) actually the most important thing is that the solution to the lapalce equation is:

$$u(r,\theta) = E + F\log(r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

3. wedges and annuli and their combination

the idea is to use different boundary condition to get the solution of $u(r,\theta)$ note it is different from the possion's formula because the boundary condition is different. (check HW7 Q13)

4. convergence theorems:

pointwise convergence
uniform convergence
mean square convergence

5. dirichlet principle: which says that among all the functions $w(x)$ in D that satisfies the dirichlet boundary condition

$$w = h(x) \quad \text{on bdy D}$$

the lowest energy orrcus for the harmonic function satisfying the boundary condition. and the energy is defined as

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 dx$$

6. green's function:

definition:

(a). $G(x)$ posseses continuous second derivatives and $\Delta G = 0$ in D, except at the point $x = x_0$

(b). $G(x) = 0$ for $x \in bdyD$

(c). in $\mathbb{R}^3$, we need $G(x) + \frac{1}{4\pi|x-x_0|}$ is finite at x_0 and has continuous second derivatives everywhere and is harmonic at x_0

formulae for green's formula:

(a). green's function in $\mathbb{R}^3$: $G(x,x_0) = -\frac{1}{4\pi|x-x_0|}$

(b). green's function in $\mathbb{R}^2$: $G(x,x_0) = \frac{1}{2\pi}\ln|x-x_0|$

(c). green's function in $\mathbb{R}$: $G(x,x_0) = -\frac{1}{4\pi|x-x_0|}$

(d). green's function for the half-plane $\{(x,y) : y > 0\}$ $G(x,x_)) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|}$

(e). green's function for the ball $\{|\mathbf{x}| < a\}$

$$G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|r_0 x/a - a x_0/r_0|}$$

or

$$G(X,X_0) = -\frac{1}{4\pi\rho} + \frac{a}{|x_0|}\frac{1}{4\pi\rho^*}$$

(f). green's function for the circle $\{|\mathbf{x}| < a\}$:

$$G(x, x_0) = \frac{1}{2\pi} \log \rho - \frac{1}{2\pi} \log(\frac{r_0}{a} \rho^*)$$

we can use green's function to get the dirichlet problem:

$$u(x_0) = \iint_{bdyD} u(x) \frac{\partial G(x, x_0)}{\partial n} dS$$

7. solve the problem: $X'' + \lambda X = 0 \quad X(0) = X'(a) = 0$ the solutions are: $X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{a}$
8. solve the problem: $Y'' - \lambda Y = 0 \quad Y'(0) + Y(0) = 0$

Chapter contents from MATH4220, 2022

10.1 homework1

Question 10.1.1. *solve the pde:* $u_x + u_y + u = e^{x+2y}$ *with u(x,0)=0*

Question 10.1.2. *consider the equation:* $u_x + yu_y = 0$ *with the boundary condition* $u(x,0) = \phi(x)$
(a) for $\phi(x) \equiv x$*, show that no solution exists.*
(x) for $\phi(x) \equiv 1$ *show that there are many solutions.*

10.2 homework2

Question 10.2.1. *solve* $u_x x - 3u_x t - 4u_t t = 0, u(x,0) = x^2, u_t(x,0) = e^x$.
hint: factor the operator

Question 10.2.2. *use the energy conservation of the wave equation to prove that the only solution with* $\phi \equiv 0$ *and* $\psi \equiv 0$

Question 10.2.3. *consider the diffusion equation* $u_t = u_{xx}$ *in* $0 < x < 1, 0 < t < \infty$ *with* $u(0,t) = u(1,t) = 0$ *and* $u(x,0) = 4x(1-x)$*:*
(a) show that $0 < u(x,t) < 1$ *for all* $t > 0$ *and* $0 < x < 1$
(b) show that $u(x,t) = u(1-x,t)$ *for all* $t \geqslant 0$ *and* $0 \leqslant x \leqslant 1$
(c) use the energy method to show that $\int_0^1 u^2 dx$ *is a strictly decreasing function of t.*

Question 10.2.4. *prove the maximum principle for a smooth function u on a parabolic domain* Ω_T *satisfying the inequality* $\partial_t u - k\Delta u \leqslant 0$

10.3 homework3

✍ **Exercise 10.1** solve the diffusion equation if $\phi(x) = e^{3x}$

Proof recall the formula for diffusion formula:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy, t > 0$$

and $\int_{\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$

Question 10.3.1. *solve* $u_{tt} = c^2 u_{xx} + e^{ax}, u(x,0) = 0, u_t(x,0) = 0$

Proof this problem is not difficult to think about, but the calculation is tedious. only need to recall that formula for nonhomogeneous wave equations

Question 10.3.2. *solve the homogeneous wave equation* $u_{xx} = c^2 u_{xx}$ *on the half-line* $(0,\infty)$ *with zero initial data* $(u(x,0) = \phi(x) = 0$ *and* $u_t(x,0) = \psi(x) = 0)$ *and with the neumann boundary condition* $u_x(0,t) = k(t)$

Proof well...a hard problem indeed
we first write what we are going to solve:

$$u_{tt} = c^2 u_{xx}, 0 < x < \infty$$

$$u(x,0) = \phi(x) = 0$$

$$u_t(x,0) = \psi(x) = 0$$

$$u_x(0,t) = k(t)$$

the neumann boundary condition is inhomogeneous, let $v(x,t) = u(x,t) - xk(t)$, we have:

$$v_{tt} - c^2 v_{xx} = -k''(t)x =: f(x,t), 0 < x < \infty$$
$$v(x,0) = -xk(0) =: \tilde{\phi}(x)$$
$$v_t(x,0) = -xk'(0) =: \tilde{\psi}(x)$$
$$v_x(0,t) = 0$$

since this a neumann boundary condition, we do even extension:

$$\tilde{\phi}_{even}(x) = \begin{cases} \tilde{\phi}(x) = -xk(0), & x > 0 \\ \tilde{\phi}(-x) = xk(0), & x < 0 \end{cases} \quad \tilde{\psi}_{even}(x) = \begin{cases} \tilde{\psi}(x) = -xk'(0), & x > 0 \\ \tilde{\psi}(-x) = xk'(0), & x < 0 \end{cases}$$
$$f_{even}(x) = \begin{cases} f(x) = f(x,t) = -k''(t)x, & x > 0 \\ f(-x) = f(-x,t) = k''(t)x, & x < 0 \end{cases}$$

now we can apply the formula of wave equation with a source 3.2 and get:

$$v(x,t) = \frac{1}{2}[\tilde{\phi}_{even}(x+ct) + \tilde{\phi}_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}_{even}(y)dy + \frac{1}{2c}\iint_{\Delta} f_{even}(y,s)dyds$$

we need to discuss several cases:
case1: $x + ct > 0, x - ct > 0$:

$$v(x,t) = \frac{1}{2}[-(x+ct)k(0) + (-(x-ct))k(0)] + \frac{1}{2c}\int_{x-ct}^{x+ct} (-y)k'(0)dy + \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} -yk''(s)dy$$
$$v(x,t) = -xk(t)$$

case2: $x + ct > 0, x - ct < 0$:

$$v(x,t) = \frac{1}{2}[-(x+ct)k(0) + (x-ct)k(0)] + \frac{1}{2c}\int_0^{x+ct} (-y)k'(0)dy$$
$$+ \frac{1}{2c}\int_{x-ct}^0 yk'(0)dy + \frac{1}{2c}\iint_{\Delta} -yk''(s)dy$$

we first do not compute the last term:

$$\frac{1}{2}[-(x+ct)k(0) + (x-ct)k(0)] + \frac{1}{2c}\int_0^{x+ct} (-y)k'(0)dy + \frac{1}{2c}\int_{x-ct}^0 yk'(0)dy =$$
$$-ctk(0) + \frac{-k'(0)}{2c}\cdot\frac{(ct+x)^2}{2} + \frac{k'(0)}{2c}[-\frac{-(ct+x)^2}{2}] = -ctk(0) - \frac{(2x^2+2c^2t^2)k'(0)}{4c}$$

we compute the integral:

$$\frac{1}{2c}\iint_{\Delta} -yk''(s)dy = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} -yk''(s)dy = \frac{1}{2c}\int_0^{t-\frac{x}{c}} [\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy]ds$$
$$+ \frac{1}{2c}\int_{t-\frac{x}{c}}^t [\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy]ds$$

if $0 < s < t - \frac{x}{c}$, we have $x - c(t-s) < 0$:

$$\frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_{x-c(t-s)}^0 f(y,s)dyds + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_0^{x+c(t-s)} f(y,s)dyds = ??$$

if $s > t - \frac{x}{c}$, we have $x - c(t-s) > 0$:

$$\frac{1}{2c}\int_{t-\frac{x}{c}}^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \frac{1}{2c}\int_{t-\frac{x}{c}}^t \int_{x-c(t-s)}^{x+c(t-s)} -k''(s)dyds =$$
$$-\frac{1}{2c}\int_{t-\frac{x}{c}}^t \frac{[x+c(t-s)^2] - [x-c(t-s)^2]}{2}k''(s)ds = -\frac{1}{2c}\int_{t-\frac{x}{c}}^t k''(s)2xc(t-s)ds = ??$$

finally we add up every term and get

$$v(x,t) = -k(t)x - \int_0^{t-\frac{x}{c}} k(s)ds, x - ct < 0$$

the final answer is that

$$u(x,t) = \begin{cases} 0, & x - ct > 0 \\ -\int_0^{t-\frac{x}{c}} k(s)ds, & x - ct > 0 \end{cases}$$

case3: $x + ct < 0, x - ct < 0$: this case does not exist since $x > 0$ and $t > 0$

10.4 homework5

Question 10.4.1. *1. (a) Use (5) to prove that if $\phi(x)$ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.*
(b) Also, if $\phi(x)$ is an even function, its full Fourier series on $(-l, l)$ has only cosine terms. (Hint: Don't use the series directly. Use the formulas for the coefficients to show that every second coefficient vanishes.)

Question 10.4.2. *2. (a) Prove that differentiation switches even functions to odd ones, and odd functions to even ones.*
(b) Prove the same for integration provided that we ignore the constant of integration.

Question 10.4.3. *3. (a) Let $\phi(x)$ be a continuous function on $(0, l)$. Under what conditions is its odd extension also a continuous function?*
(b) Let $\phi(x)$ be a differentiable function on $(0, l)$. Under what conditions is its odd extension also a differentiable function?
(c) Same as part (a) for the even extension.
(d) Same as part (b) for the even extension.

Question 10.4.4. *4. (The Gram-Schmidt orthogonalization procedure) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define*

$$Y_2 = X_2 - (X_2, Z_1)\, Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)\, Z_2 - (X_3, Z_1)\, Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on. (a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.
(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

Question 10.4.5. *5. $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is a geometric series. (a) Does it converge pointwise in the interval $-1 < x < 1$?*
(b) Does it converge uniformly in the interval $-1 < x < 1$?
(c) Does it converge in the L^2 sense in the interval $-1 < x < 1$? (Hint: You can compute its partial sums explicitly.)

Question 10.4.6. *6. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n\left(\frac{1}{2}\right) = 0$, $f_n(x) = \gamma_n$ in the interval $\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right)$, let $f_n(x) = -\gamma_n$ in the interval $\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right]$ and let $f_n(x) = 0$ elsewhere. Show that: (a) $f_n(x) \to 0$ pointwise.*
(b) The convergence is not uniform.
(c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.
(d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n = n$.

Question 10.4.7. *7. Let*

$$\phi(x) = \begin{cases} -1 - x & \text{for } -1 < x < 0 \\ +1 - x & \text{for } 0 < x < 1 \end{cases}$$

(a) Find the full Fourier series of $\phi(x)$ in the interval $(-1, 1)$.
(b) Find the first three nonzero terms explicitly.
(c) Does it converge in the mean square sense?
(d) Does it converge pointwise?
(e) Does it converge uniformly to $\phi(x)$ in the interval $(-1, 1)$?

10.5 homework6

Question 10.5.1. *1. Let $\phi(x) \equiv 1$ for $0 < x < \pi$. Expand*

$$1 = \sum_{n=0}^{\infty} B_n \cos\left[\left(n + \frac{1}{2}\right)x\right]$$

(a) Find B_n.
(b) Let $-2\pi < x < 2\pi$. For which such x does this series converge? For each such x, what is the sum of the series? [Hint: Think of extending $\phi(x)$ beyond the interval $(0, \pi)$.]
(c) Apply Parseval's equality to this series. Use it to calculate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

Proof (a) directly applying fourier cosine series formula
(b) review the convergence theorems: for pointwise convergence, we need f(x) to be continuous and $f'(x)$ to be piecewise continuous
(c) recall the parseval's equality, we only need to calculate the integral: $\int_0^\pi |f(x)|^2 dx$ and $\int_0^\pi |X_n(x)|^2 dx$

Question 10.5.2. *2. (a) Solve the problem $u_t = ku_{xx}$ for $0 < x < l, u(x, 0) = \phi(x)$, with the unusual boundary conditions*

$$u_x(0, t) = u_x(l, t) = \frac{u(l, t) - u(0, t)}{l}.$$

Assume that there are no negative eigenvalues. (Hint: See Exercise 4.3.12.)
(b) Show that as $t \to \infty$,

$$\lim u(x, t) = A + Bx,$$

assuming that you can take limits term by term.
(c) Use Green's first identity and Exercise 3 to show that there are no negative eigenvalues.
(d) Find A and B. (Hint: A + Bx is the beginning of the series. Take the inner product of the series for $\phi(x)$ with each of the functions 1 and x. Make use of the orthogonality.)

Question 10.5.3. *3. Consider the diffusion equation on $[0, l]$ with Dirichlet boundary conditions and any continuous function as initial condition. Show from the series expansion that the solution is infinitely differentiable for $t > 0$. (Hint: Use the general theorem at the end of Section A. 2 on the differentiability of series, together with the fact that the exponentials are very small for large n. See Section 3.5 for an analogous situation.)*

Question 10.5.4. *4. Prove that the classical full Fourier series of $f(x)$ converges uniformly to $f(x)$ if merely $f(x)$ is continuous of period 2π and its derivative $f'(x)$ is piecewise continuous. (Hint: Modify the discussion of uniform convergence in this chapter.)*

Question 10.5.5. *5. Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.*

Proof $u_{xx}+u_{yy} = u_{rr}+\frac{1}{r}u_r = 1, ru_{rr}+u_r = r \Rightarrow (ru_r)' = r \Rightarrow u_r = \frac{1}{2}r+\frac{c}{r}, u = \frac{1}{4}r^2+C\ln r+D$, C,D are constants, on the boundary, we have $u(a) = \frac{1}{4}a^2+C\ln a+D = 0 \Rightarrow D = -\frac{1}{4}a^2-C\ln a$, so we have $u(r) = \frac{1}{4}r^2+c\ln r-\frac{1}{4}a^2-c\ln a, 0 < r < a$

Question 10.5.6. *6. Show that there is no solution of*

$$\Delta u = f \quad \text{in } D, \quad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$

in three dimensions, unless

$$\iiint_D f dxdydz = \iint_{Bdy\,(D)} g dS.$$

(Hint: Integrate the equation.) Also show the analogue in one and two dimensions.

Proof $\iiint_D f dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla \cdot (\nabla u) dxdydz = \iint_{bdyD} \nabla u \cdot n dS = \iint_{bdyD} g dS$. here we use the divergence theorem for the third equality.

Question 10.5.7. *7. A function $u(\mathbf{x})$ is subharmonic in D if $\Delta u \geq 0$ in D. Prove that its maximum value is attained on bdy D. [Note that this is not true for the minimum value.]*

Proof if $\Delta u = 0$, we can use the maximum principle to conclude that its maximum value is attained on bdy D.
if $\Delta u > 0$, suppose there is $x_0 \in D \setminus bdyD$ a maximum point, we have $u_{xx}(x_0) \leqslant 0$ and $u_{yy}(x_0) \leqslant 0$, which violate the hypothesis that $\Delta u = u_{xx} + u_{yy} > 0$, so $u(x)$ only attains its maximum value on bdy D.

10.6 homework7

Question 10.6.1. *7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.*

Proof we recall that laplace equation is rotational invariant and we have the form for $\mathbb{R}^3$:$u_{rr} + \frac{2}{r}u_r = 1$, solve the ODE, we have $r^2 u_{rr} + 2ru_r = r^2$, $(r^2 u_r)' = r^2 \Rightarrow r^2 u_r = \frac{1}{3}u^3 + C \Rightarrow u_r = \frac{1}{3}r + r^{-2}C \Rightarrow u = \frac{1}{6}r^2 - r^{-1}C + D$. we plug in the boundary conditions: $u(a) = u(b) = 0$ and get $C = \frac{\frac{1}{6}b^2 - \frac{1}{6}a^2}{\frac{1}{a} - \frac{1}{b}}$ and $D = \frac{\frac{1}{6a}b^2 - \frac{1}{6b}a^2}{\frac{1}{a} - \frac{1}{b}}$ the result is

$$u(x, y, z) = \frac{1}{6}r^2 - \frac{1}{r}\left(\frac{\frac{1}{6}b^2 - \frac{1}{6}a^2}{\frac{1}{a} - \frac{1}{b}}\right) + \left(\frac{\frac{1}{6a}b^2 - \frac{1}{6b}a^2}{\frac{1}{a} - \frac{1}{b}}\right) = \frac{1}{6}(x^2 + y^2 + z^2) - \frac{1}{\sqrt{x^2 + y^2 + z^2}}\frac{\frac{1}{6}b^2 - \frac{1}{6}a^2}{\frac{1}{a} - \frac{1}{b}} + \frac{\frac{1}{6a}b^2 - \frac{1}{6b}a^2}{\frac{1}{a} - \frac{1}{b}}$$

Question 10.6.2. *6. Solve the following Neumann problem in the cube $\{0 < x < 1, 0 < y < 1, 0 < z < 1\} : \Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.*

Proof

Question 10.6.3. *2. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition*

$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$

Proof recall that $u(r, \theta)$ have full fourier series solution (also possion's formula). just plug in it. (in case you forget what I am saying, check 6.3, page 167)
we recall that in the deduction of the poisson's formula, we have:

$$u(r, \theta) = E + F\log(r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

notice that the domain is $\{r < a\}$, so at the origin, we require the formula is well-defined, which means that $F = D = 0$ and $u(r, \theta) = E + \sum_{n=1}^{\infty} C_n r^n (A_n \cos n\theta + B_n \sin n\theta)$, plug in $r = a$, we have

$$u(a, \theta) = 1 + 3\sin\theta =: h(\theta) = E + \sum_{n=1}^{\infty} C_n r^n (A_n \cos n\theta + B_n \sin n\theta)$$

we replace E by $\frac{1}{2}A_0$, using the knowledge of fourier full series, we know that

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi d\phi$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi d\phi$$

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} h(\phi)d\phi$$

recall the orthogonality of sin and cos, we find $B_n = 0, n \geqslant 2$ and $A_n = 0, n \geqslant 1$, and $A_0 = 2$, $B_1 = \frac{3}{a}$, we conclude that $u(r,\theta) = 1 + \frac{3r}{a}\sin\theta$

Question 10.6.4. *4. Derive Poisson's formula (9) for the exterior of a circle.*

Proof the procedure is similar for the interior of a circle. still we have

$$u(r,\theta) = E + F\log r + \sum_{n=1}(Cr^n + Dr^{-n})(A\cos n\theta + B\sin n\theta)$$

the difference is that let $r \to \infty$, we have u is bounded and we can eliminate some parameter in the above formula: $C = F = 0$ and thus we have

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n\sin n\theta)$$

Question 10.6.5. *13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta, u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.*

Proof consider θ: solve the boundary condition problem:

$$\Theta'' + \lambda\Theta = 0 \quad \text{and } \Theta(\alpha) = \Theta(\beta) = 0$$

the result is

$$\Theta(\theta) = \sin\frac{m\pi}{\beta-\alpha}(\theta-\alpha) \text{ the parameter is omitted}$$

solve the boundary condition problem:

$$r^2R''(r) + rR' - \lambda R = 0$$

we have

$$R(r) = Ar^{\frac{m\pi}{\beta-\alpha}} + Br^{-\frac{m\pi}{\beta-\alpha}}$$

10.7 homework8

Question 10.7.1. *1. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among all real-valued functions $w(\mathbf{x})$ on D the quantity*

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 d\mathbf{x} - \iint_{\text{bdy } D} hwdS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D, \quad \frac{\partial u}{\partial n} = h(\mathbf{x}) \quad \text{on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11). Notice three features of this principle: (i) There is no constraint at all on the trial functions $w(\mathbf{x})$. (ii) The function $h(\mathbf{x})$ appears in the energy. (iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$. (Hint: Follow the method in Section 7.1.)

Proof dirichlet's principle says that the harmonic function minimizes $E[w] := \frac{1}{2}\iiint_D |\nabla w|^2 dx - \iint_{bdyD} hwdS$ among all functions that satisfy the boundary condition.
let u be the harmonic function in D satisfying neumann boundary condition and let w be any function in D that satisfying neumann boundary condition. let $v = u - w$,

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 dx - \iint_{bdyD} hwdS = \frac{1}{2}\iiint_D |\nabla u - \nabla v|^2 dx - \iint_{bdyD} h(u-v)dS$$
$$= \frac{1}{2}\iiint_D [|\nabla u|^2 - 2\nabla u\cdot\nabla v + |\nabla v|^2]dx - \iint_{bdyD} hudS + \iint_{bdyD} hvdS = E[u] + \frac{1}{2}\iiint_D |\nabla v|^2 dx - \iint_{bdyD} hvdS - \frac{1}{2}\iiint_D 2\nabla u \cdot$$

we focus on the last term $\frac{1}{2}\iiint_D 2\nabla u \cdot \nabla v dx$: notice that by green's first identity, we have

$$\iint_{bdyD} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v\Delta u dx$$

we have

$$\iiint_D \nabla u \cdot \nabla v dx = \iint_{bdyD} v\frac{\partial u}{\partial n}dS - \iiint_D v\Delta u dx = \iint_{bdyD} v\frac{\partial u}{\partial n}dS - 0 = \iint_{bdyD} v \cdot h(x)dS$$

plug the result back into the above equation we have

$$E[w] = E[u] + \frac{1}{2}\iiint_D |\nabla v|^2 dx + \iint_{bdyD} hvdS - \iint_{bdyD} hvdS = E[u] + \frac{1}{2}\iiint_D |\nabla v|^2 dx$$

since $\iiint_D |\nabla v|^2 dx \geq 0$, we must have $E[w] \geq E[u]$. we are done.

Question 10.7.2. *2. (Rayleigh-Ritz approximation to the harmonic function u in D with $u = h$ on bdy D.) Let $w_0, w_1, \ldots, w_n$ be arbitrary functions such that $w_0 = h$ on bdy D and $w_1 = \cdots = w_n = 0$ on bdy D. The problem is to find constants $c_1, \ldots, c_n$ so that $w_0 + c_1 w_1 + \cdots + c_n w_n$ has the least possible energy. Show that the constants must solve the linear system*

$$\sum_{k=1}^{n} (\nabla w_j, \nabla w_k)\, c_k = -\left(\nabla w_0, \nabla w_j\right) \quad \textit{for } j = 1, 2, \ldots, n$$

Remark here $(\nabla w_j, \nabla w_k) := \iiint_D \nabla w_j, \nabla w_k dx$

Proof let $w := w_0 + c_1 w_1 + ... + c_n w_n$ and let $F(c_1, c_2, ..., c_n) := E(w) = \frac{1}{2}\iiint_D |\nabla w_0 + \nabla c_1 w_1 + ... + \nabla c_n w_n|^2 dx$, we are asked to find $c_0, c_1, ..., c_n$ that minimize $F(c_0, c_1, ..., c_n)$, by knowledge of calculus, we know that $\frac{\partial F}{\partial c_i} = 0$.

$$\frac{\partial F}{\partial c_i} = \iiint_D [\nabla w_0 + \nabla c_1 w_1 + ... + \nabla c_n w_n]\nabla w_i dx = 0$$

which is

$$\iiint_D \nabla w_0 \cdot \nabla w_i dx + c_1 \iiint_D \nabla w_1 \cdot \nabla w_i dx + ... + c_n \iiint_D \nabla w_n \nabla w_i dx = 0$$

we move the term of ∇w_0 to the other side and get:

$$-(\nabla w_0, \nabla w_j) = \sum_{n=1}^{n} (\nabla w_j, \nabla w_k) c_k$$

Question 10.7.3. *3. Let $\phi(\mathbf{x})$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that*

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.

Proof using green's second identity, let $v(x) = -\frac{1}{|x|}$ and $u(x) = \phi(x)$, we have

$$\iiint \frac{1}{|x|}\Delta\phi(x)dx = -\iiint v\Delta u dx = \iint (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS - \iiint (u\Delta v)dx$$

notice that $\Delta v = (\partial_{rr} + \frac{1}{r}\partial_r)v = 0$, so we have

$$-\iiint v\Delta u dx = \iint (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS = \iint \phi(x)\frac{\partial}{\partial n}(-\frac{1}{|x|}) - (-\frac{1}{|x|})\frac{\partial \phi}{\partial n}dS$$

let D be a sphere centered at $x = 0$ with radius R, $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ and $\frac{1}{|x|} = \frac{1}{r}$. so we have

$$\iiint v\Delta u dx = \iint \phi(x)\frac{\partial}{\partial n}\frac{1}{|x|} - \frac{1}{|x|}\frac{\partial \phi}{\partial n}dS = \iint_{bdyD} (\phi(x)\frac{\partial}{\partial r}\frac{1}{r})dS - \iint_{bdyD} \frac{1}{r}\frac{\partial \phi}{\partial r}dS = \iint_{bdyD} \phi(x)(-\frac{1}{R^2})dS - \frac{1}{R}\iint_{bdyD} \frac{\partial \phi}{\partial r}dS$$

$$\iiint_D \frac{1}{r}\Delta\phi(x)dx = \iint_{bdyD} \phi(x)(-\frac{1}{R^2})dS - \frac{1}{R}\iint_{bdyD} \frac{\partial \phi}{\partial r}dS$$

$$-\frac{1}{4\pi}\iiint_D \frac{1}{r}\Delta\phi(x)dx = \frac{1}{\text{area of S}}\iint_{bdyD} \phi(x)dS + \frac{R}{\text{area of S}}\iint_{bdyD} \frac{\partial \phi}{\partial r}dS$$

let $R \longrightarrow 0$, we have

$$\lim_{R\to 0}\frac{1}{\text{area of S}}\iint_{bdyD}\phi(x)dS = \phi(0), \quad \lim_{R\to 0}\frac{R}{\text{area of S}}\iint_{bdyD}\frac{\partial\phi}{\partial r}dS = 0$$

and

$$-\lim_{R\to 0}\frac{1}{4\pi}\iiint_D \frac{1}{r}\Delta\phi(x)dx = -\iiint \frac{1}{|x|}\Delta\phi(x)\frac{dx}{4\pi}$$

Question 10.7.4. *4. Find the one-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be restated as follows. (i) It solves $G''(x) = 0$ for $x \neq x_0$ ("harmonic"). (ii) $G(0) = G(l) = 0$. (iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.*

Question 10.7.5. *5. (a) Find the Green's function for the half-plane $\{(x, y) : y > 0\}$. (b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$. (c) Calculate the solution with $u(x, 0) = 1$.*

Proof recall the green's function in 2 dimension is

$$G(x, x_0) = \frac{1}{2\pi}\int_{bdyD}[u(x)\frac{\partial}{\partial n}(\ln|x - x_0|) - \frac{\partial u}{\partial n}\ln|x - x_0|]ds$$

in textbook page 191, we use the reflection method and get the green's function of half-space,

10.8 homework9

Question 10.8.1. *1. Verify directly from the definition that $\phi \mapsto \int_{-\infty}^{\infty} f(x)\phi(x)dx$ is a distribution if $f(x)$ is any function that is integrable on each bounded set.*

Question 10.8.2. *5. Verify, directly from the definition of a distribution, that the discontinuous function $u(x, t) = H(x - ct)$ is a weak solution of the wave equation.*

Question 10.8.3. *2. Let f be any distribution. Verify that the functional f' defined by $(f', \phi) = -(f, \phi')$ satisfies the linearity and continuity properties and therefore is another distribution.*

10.9 homework10

Question 10.9.1. *1. Use the Fourier transform directly to solve the heat equation with a convection term, namely, $u_t = \kappa u_{xx} + \mu u_x$ for $-\infty < x < \infty$, with an initial condition $u(x, 0) = \phi(x)$, assuming that $u(x, t)$ is bounded and $\kappa > 0$.*

Question 10.9.2. *2. Use the Fourier transform in the x variable to find the harmonic function in the half-plane $\{y > 0\}$ that satisfies the Neumann condition $\partial u/\partial y = h(x)$ on $\{y = 0\}$.*

10.10 tutorial5

Question 10.10.1. *let $\phi(x)$ be a continuous function s.t. $|\phi(x)| \leqslant Ce^{ax^2}$, show that the solution of the diffusion equation*

$$\frac{1}{\sqrt{4\pi kt}}\int_{\mathbb{R}} e^{-\frac{(x-s)^2}{4kt}}\phi(s)ds$$

makes sense for $0 < t < \frac{1}{4ak}$, but not necessarily for large t.

Question 10.10.2. *solve the following wave equation on the half-line:*

$$\partial_t^2 u - c^2\partial_x^2 u = 0, \quad t \in \mathbb{R}, \quad x > 0$$

$$u(x, 0) = \phi(x), \quad \partial_t u(x, 0) = \psi(x)$$

$$u_x(0, t) = e^t$$

Proof transform it into a homogeneous equation and use reflection method

Question 10.10.3. *solve the following wave equation on the half-line with inhomogeneous dirichlet boundary condition:*

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad u(0,t) = t^2$$
$$u(x,0) = x, \quad u_t(x,0) = 0$$

Proof transform the dirichlet boundary condition into a homogeneous euqation. and use reflection method.

Remark notice that when applying the reflection method, you need to discuss the relation between x and t since in different region of x, the odd/even extensions have different expressions.

Question 10.10.4. *consider the following problem with a robin boundary condition:*

$$u_t = k u_{xx}, \quad u(x,0) = f(x)$$
$$u_x(0,t) - 2u(0,t) = 0, \quad x = 0$$

let $f(x) = x, x > 0$ and $f(x) = x + 1 - e^{2x}, x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)dy$$

show that v(x,t) satisfies the PDE problem for $x > 0$ assuming uniqueness, deduce the solution of the PDE problem is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)dy$$

10.11 tutorial6

Question 10.11.1. *Consider waves in a resistant medium that satisfy the problem*

$$\begin{cases} u_{tt} = c^2 u_{xx} - r u_t, \quad 0 < x < I \\ u(0,t) = 0; \quad u(l,t) = 0 \\ u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x) \end{cases}$$

where r is a constant, $0 < r < 2\pi c/$. (a). Write down the series expansion of the solution. (b). (exercise) Do the same for the case $2\pi c/I < r < 4\pi c/I$.

Question 10.11.2. *Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < 1$, with the boundary conditions $u_x(0,t) = 0$ and $u(I,t) = 0$ (Neumann at the left, Dirichlet at the right). (a). Show that the eigenfunctions are*

$$\cos\left(\frac{n + \frac{1}{2}}{l}\pi x\right).$$

(b). Write the series expansion for a solution $u(x,t)$.

Proof we can still use separation method to deal with the inhomogeneous problem.

Question 10.11.3. *For the Robin BC s, show that*

$$E_R := \frac{1}{2}\int_0^1 \left(c^{-2}u_t^2 + u_x^2\right) dx + \frac{1}{2}a_l u^2(l,t) + \frac{1}{2}a_0 u^2(0,t).$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is 'lost' to the boundary if a_0 and a_l are positive and 'gained' from the boundary if a_0 and a_l are negative.

10.12 tutorial10

Question 10.12.1. *1. Derive the representation formula for harmonic functions (7.2.5) in two dimensions.*

Question 10.12.2. *2. Let $\phi(\mathbf{x})$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that*

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.

Question 10.12.3. *3. Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and $\mathbf{x}_0$ its center in the representation formula (1).*

Chapter contents from MATH4220, 2016

11.1 tutorial0

Question 11.1.1. *under what condition can we have $\partial_{xy} = \partial_{yx}$?*
review MATH2007

Question 11.1.2. *What is the Green's formula? how to derive it?*
review MATH2007
same questions for Divergence theorem.

Question 11.1.3. *under what conditions can we exchange the order of integration and taking derivative in multivariable calculus?*
if the upper and lower indice are constants, ∞, or variables?

Question 11.1.4. *how to perform change of coordinates formula?*

The followings should be taught in an ode course, like MATH2002 in CHUK(SZ).

Question 11.1.5. *what is a first order ode?*
what is the general form of a first order linear equation and what is the general solution?
the integrating factor $\mu(x) = e^{\int p(t)dt}$, where p(t) is the coefficient of x in the general form: $\frac{dy}{dx} + p(t)x = q(t)$

11.2 tutorial2

✍ **Exercise 11.1Exercise 4 on P27** Consider the Neumann problem

$$\Delta u = f(x, y, z) \text{ in } D$$
$$\frac{\partial u}{\partial n} = 0 \text{ on bdy } D$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness. (b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)dxdydz = 0$$

is a necessary condition for the Neumann problem to have a solution. (c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Proof (a) Adding a constant C to a solution will give another solution, so we do not have uniqueness if there is a solution:
(b) Integrating $f(x, y, z)$ on D and using the divergence theorem, we obtain

$$\iiint_D f(x, y, z)dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla \cdot \nabla u dxdydz = \iint_D \nabla u \cdot ndS = 0$$

(c) For the heat flow, the equation which is independent of time t shows that the temperature u of the object reaches a steady state when there is an heat source or sink $f(x, y, z)$. At the same time, the Nuemann boundary condition means that the object is insulated,thus there is no heat flows in or out across the boundary. The part (b) shows that in order to make the PDE and the boundary condition hold simultaneously, we need $\iiint_D f(x, y, z)dxdydz = 0$, that is, the total heat source or sink on the domain D should be 0 since the heat is steady and no heat flows in and out across the boundary.(otherwise it won't be steady if $\iiint_D f(x, y, z)dxdydz \neq 0$). The part (a) means that if a given heat distribution is a steady state then rising or lowering the heat uniformly at every point is also a possible steady heat distribution. The difference between the steady state u and $u + C$ is that they have the different total heat energy.
For diffusion, the eqution means that the concentration u of the substance reaches a steady state when there is an external source or sink of the substance $f(x, y, z)$. The Nuemann condition means that the container is impermeable, i.e,

no substance escape or enter across the boundary. The part (b) shows that in order to make the PDE and the boundary condition hold simultaneously, we need $\iiint_D f(x,y,z)dxdydz = 0$, that is, the total external source or sink of the substance should be 0 since the state of the substance is steady and no substance escape or enter across the boundary.(otherwise it won't be steady if $\iiint_D f(x,y,z)dxdydz \neq 0$). The part (a) means that if a given concentration distribution is a steady state then rising or lowering the concentration uniformly at every point is also a possible steady concentration distribution. The difference between the steady state u and $u + C$ is that they have the different total mass.

11.3 tutorial3

Exercise 11.2

Exercise 8 on **P**46 Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "radiating" or "dissipative".)

Proof

For the diffusion equation $u_t - ku_{xx} = 0$,

$$\begin{aligned}\frac{d}{dt}\int_0^l u^2 dx &= \int_0^l 2uu_t dx = \int_0^l 2kuu_{xx}dx = 2kuu_x|_0^l - \int_0^l 2ku_x^2 dx \\ &= 2k\left[u(l,t)u_x(l,t) - u(0,t)u_x(0,t)\right] - \int_0^l 2ku_x^2 dx.\end{aligned}$$

Then the Robin boundary conditions imply

$$\frac{d}{dt}\int_0^l u^2 dx = -2k\left[a_l u(l,t)^2 + a_0 u(0,t)^2\right] - \int_0^l 2ku_x^2 dx \leq 0,$$

where $-2k\left[a_l u(l,t)^2 + a_0 u(0,t)^2\right] \leq 0$ shows that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)dx$.

Exercise 11.3

Exercise 8 on P52 Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \leq |x| < \infty} S(x,t) \to 0 \quad \text{as } t \to 0.$$

[This means that thee tail of $S(x,t)$ is "uniformly small".]

Proof

By the definition of $S(x,t)$,

$$\max_{\delta \leq |x| < \infty} S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt},$$

so

$$\lim_{t\to 0^+}\max_{\delta \leq |x| < \infty} S(x,t) = \lim_{t\to 0^+}\frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt} = \lim_{x\to +\infty}\frac{\sqrt{x}}{\sqrt{4\pi k}} e^{-x\delta^2/4k} = 0.$$

11.4 tutorial4

Exercise 11.4 1. Using reflection method to solve the following problem

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \quad x > 0, t > 0$$

$$u(x, t = 0) = \phi(x), \partial_t u(x, t = 0) = \psi(x), x > 0$$

$$\partial_x u(x = 0, t) = 0, t > 0$$

Proof Solution: Use the reflection method, and first consider the following Cauchy Problem:

$$\partial_t^2 v - c^2 \partial_x^2 v = 0, \quad -\infty < x < \infty, t > 0$$

$$v(x, t = 0) = \phi_{\text{even}}(x), \partial_t v(x, t = 0) = \psi_{\text{even}}(x), -\infty < x < \infty$$

where $\phi_{\text{even}}(x)$ and $\psi_{\text{even}}(x)$ are even extension of ϕ and ψ.Then the unique solution is given by d'Alembert formula:

$$v(x, t) = \frac{1}{2}\left[\phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(y) dy$$

And since $\phi_{\text{even}}(x)$ and $\psi_{\text{even}}(x)$ are even, so is $v(x, t)$ for $t > 0$, which implies

$$\partial_x v(x = 0, t) = 0, t > 0$$

Set $u(x, t) = v(x, t), x > 0$, then $u(x, t)$ is the unique solution of Neumann Problem on the halfline.More presicely, if $x > ct$,

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y) dy$$

if $0 < x < ct$,

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(ct - x)] + \frac{1}{2c}\left\{\int_0^{ct-x} \psi(y) dy + \int_0^{x+ct} \psi(y) dy\right\}$$

11.5 tutorial5

Well-posedness of the following diffusion equations:

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x, t), -\infty < x < \infty, t > 0 \\ u(x, t = 0) = \phi(x), -\infty < x < \infty \end{cases}$$

The well-posedness has three ingredients:

(a) Existence: there exists at least one solution. We have shown that

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) dy ds$$

is a solution, where $S(z, \tau)$ is the heat kernel.

(b) Uniqueness: there exists at most one solution. And we have shown that the above solution is unique.

(c) Stability: if the data changes a little, then u also changes only a little.And we will show that the above problem is stable in the sense of uniform norm. Suppose that $u_i(x, t)$ is the solution with the source $f_i(x, t)$ and the initial data $\phi_i(x), i = 1, 2$. Set $u(x, t) = u_1(x, t) - u_2(x, t)$, $f(x, t) = f_1(x, t) - f_2(x, t)$ and $\phi(x) = \phi_1(x) - \phi_2(x)$, thus $u(x, t)$ is a solution of

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x, t), -\infty < x < \infty, t > 0 \\ u(x, t = 0) = \phi(x), -\infty < x < \infty \end{cases}$$

thus

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) dy ds$$

by the solution formula. For any $-\infty < x < \infty, 0 \le t \le T$, then

$$\begin{aligned}
|u(x, t)| &\le \int_{-\infty}^{\infty} S(x - y, t)|\phi(y)| dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s)|f(y, s)| dy ds \\
&\le \int_{-\infty}^{\infty} S(x - y, t) \max_{-\infty<y<\infty} |\phi(y)| dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) \max_{-\infty<y<\infty, 0\le t\le T} |f(y, s)| dy ds \\
&\le \|\phi\| \int_{-\infty}^{\infty} S(x - y, t) dy + \|f\|_T \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) dy ds \\
&\le \|\phi\| + \|f\|_T \int_0^t ds \quad \text{by} \int_{-\infty}^{\infty} S(x, t) dx = 1 \\
&\le \|\phi\| + T\|f\|_T
\end{aligned}$$

Hence

$$\|u\|_T \le \|\phi\| + T\|f\|_T$$

So if $\|f\|_T$ and $\|\phi\|$ are small, then $\|u\|_T$ is small.

11.6 tutorial6

Exercise 11.5 2. Using the method of separation of variables to solve the problem:

$$\begin{cases} u_t - ku_{xx} = 0, 0 < x < l, t > 0 \\ u_x(0,t) = 0, u(l,t) = 0, \\ u(x, t = 0) = \phi(x) \end{cases}$$

Proof Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

Actually, λ is positive. Therefore, $T(t)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T(t) = Ae^{-\lambda kt}$. Furthermore,

$$-X'' = \lambda X, X'(0) = X(l) = 0.$$

So by solving the above DE, the eigenvalues are $[\frac{(n+\frac{1}{2})\pi}{l}]^2$, the eigenfunctions are $X_n(x) = \cos\frac{(n+\frac{1}{2})\pi x}{l}$ for $n = 0, 1, 2, \cdots$, and the solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-[\frac{(n+\frac{1}{2})\pi}{l}]^2 kt} \cos\frac{\left(n+\frac{1}{2}\right)\pi x}{l}.$$

provided that

$$\phi(x) = \sum_{n=0}^{\infty} A_n \cos\frac{\left(n+\frac{1}{2}\right)\pi x}{l}.$$

11.7 tutorial12

Question 11.7.1. *The solution of the problem*

$$\Delta u = f \text{ in } D \quad u = h \text{ on } \partial D$$

is given by

$$u(\mathbf{x}_0) = \iint_{\partial D} h(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} dS + \iiint_D f(\mathbf{x})G(\mathbf{x},\mathbf{x}_0)\, d\mathbf{x}$$

Proof Let $v(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \mathbf{x} \neq \mathbf{x}_0$, then $\Delta v(\mathbf{x}) = 0, \mathbf{x} \neq \mathbf{x}_0$. Let D_ϵ be the region D with a ball (of radius ϵ and the center $\mathbf{x}_0$) excised. Applying Green's Second Identity to v and u on D_ϵ, we have

$$\iiint_{D_*} -vf\, d\mathbf{x} = \iiint_{D_*} u\Delta v - v\Delta u d\mathbf{x} = \iint_{\partial D_\epsilon} \left[u \cdot \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \cdot v\right] dS$$

Noting that ∂D_ϵ consists of two parts and on $\{|\mathbf{x}-\mathbf{x}_0| = r = \epsilon\}$, $\frac{\partial}{\partial\pi} = -\frac{\partial}{\partial r}$, we have

$$\iint_{r=\epsilon} u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}vdS = -\iint_{r-\epsilon} u\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}vdS = -\frac{1}{4\pi\epsilon^2}\iint_{r-\epsilon} udS - \frac{1}{4\pi\epsilon}\iint_{r=\epsilon}\frac{\partial u}{\partial r}dS = -\bar{u} - \epsilon\overline{\frac{\partial u}{\partial r}}$$

where $\bar{u}$ denotes the average value of u on the sphere $\{r = c\}$, and $\overline{\frac{\partial u}{\partial r}}$ denotes the average value of $\frac{\partial u}{\partial r}$. on this sphere. Since u is continuous and $\frac{\partial u}{\partial r}$ is bounded, we have

$$-\bar{u} - \epsilon\overline{\frac{\partial u}{\partial r}} \longrightarrow -u(\mathbf{x}_0) \quad \text{as } \epsilon \longrightarrow 0.$$

So let ϵ tend to 0 and then we have

$$\iiint_D -vf\, d\mathbf{x} = \iint_{\partial D}\left[u \cdot \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \cdot v\right] dS - u(\mathbf{x}_0)$$

Suppose $G(\mathbf{x},\mathbf{x}_0)$ is the Green's function for $-\Delta$, then $H = G - v$ is a harmonic function on D, and $G = 0$ on ∂D.

Applying the second Green's Identity to u and H on D, we have

$$\iiint_D -Hf\,d\mathbf{x} = \iiint_D u\Delta H - H\Delta u\,d\mathbf{x} = \iint_{\partial D}\left[u\cdot\frac{\partial H}{\partial n} - \frac{\partial u}{\partial n}\cdot H\right]dS$$

Adding (2) and (3) and using $G = H + v$ in D, $G = 0$ on ∂D, we get

$$\iiint_D -Gf\,d\mathbf{x} = \iint_{\partial D}\left[u\cdot\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n}\cdot G\right]dS - u\left(\mathbf{x}_0\right) = \iint_{\partial D} h\frac{\partial G}{\partial n}dS - u\left(\mathbf{x}_0\right)$$

That is,

$$u\left(\mathbf{x}_0\right) = \iint_{\partial D} h\frac{\partial G}{\partial n}dS + \iiint_D Gf\,d\mathbf{x}$$

Remark the proof is quite similar to the proof of representation formula in $\mathbb{R}^3$

11.8 quiz1

Question 11.8.1. *solve the equation $\partial_x u + x\partial_y u = 0$ with the following two conditions:*
(a)$u(0, y) = y^2$
(b)$u(x, 0) = x^2$
hint: take care of the domain of the solution.