Study Path of Control Theory

Yingtai XU

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Abstract

Contents

1 non-linear system behaviors

1.1 limit cycle

Example 1.1. *the second-order nonlinear differential equation*

$$mx'' + 2c(x^2 - 1)x' + kx = 0$$

where m, c, k are positive constants, is called the Van der Pol equation. It describes a mass-spring-damper system with a position-dependent damping coefficient $2c(x^2-1)$. For large values of x. the damping coefficient is positive and the damper removes energy from the system. This implies that the system motion has a convergent tendency. However, for small values of x, the damping coefficient is negative and the damper adds energy into the system. This suggests that *the system motion has a divergent tendency. Therefore. because the nonlinear damping varies with x, the system motion can neither grow unboundedly nor decay to zero. This so-called limit cycle is sustained by periodically releasing energy into and absorbing energy from the environment, through the damping term.*

Limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects. First. the amplitude of the self-sustained excitation is independent of the initial condition. Second, marginally stable linear systems are very sensitive to changes in system parameters (with a slight change capable of leading either to stable convergence or to instability), while limit cycles are not easily affected by parameter changes.

1.2 bifurcations

As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does in linear systems) and so can the number of equilibrium points. Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values.

1.3 Chaos

For stable linear systems, small differences in initial conditions can only cause small differences in output. Nonlinear systems, however. can display a phenomenon called chaos.

1.4 Other behaviors

Other interesting types of behavior, such as jump resonance, subharmonic generation, asynchronous quenching, and frequency-amplitude dependence of free vibrations, can also occur and become important in some system studies. However, the above description should provide ample evidence that nonlinear systems can have considerably richer and more complex behavior than linear systems.

2 Lyapunov theory

2.1 Nonlinear Systems and Equilibrium Points

Definition 2.1 (nonlinear dynamic system)**.** *A nonlinear dynamic system can usually be represented by a set of nonlinear differential equations in the form*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{f}$ is a $n \times 1$ nonlinear vector function, and $\mathbf{x}$ is the $n \times 1$ state vector. The number of states n is called the order of the system. The above equation can represent the closed-loop dynamics of a feedback control system, with the control input being a function of state $\mathbf{x}$ and time t. Specifically, if the plant dynamics is

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

and some control law has been selected

$$\mathbf{u} = \mathbf{g}(\mathbf{x}, t)$$

then the closed-loop dynamics is

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x}, t), t)$$

Definition 2.2 (autonomous system)**.** *The nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x}, t)$ is said to be autonomous if $\mathbf{f}$ does not depend explicitly on time, i.e., if the system's state equation can be written*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

otherwise, the system is called non-autonomous.

The fundamental difference between autonomous and non-autonomous systems lies in the fact that the state trajectory of an autonomous system is independent of the initial time. This difference requires us to consider the initial time explicitly in defining stability concepts for non-autonomous systems, and makes the analysis more difficult than that of autonomous systems.

Definition 2.3 (equilibrium state/equilibrium point)**.** *a state $\mathbf{x}^*$ is an equilibrium state or equilibrium point of the system if once $\mathbf{x}(t)$ is equal to $\mathbf{x}^*$, it remains equal to $\mathbf{x}^*$ for all future time. It means that the constant vector $\mathbf{x}^*$ satisfies*

$$\mathbf{0} = \mathbf{f}(\mathbf{x}^*)$$

Equilibrium points can be found by solving the nonlinear algebraic equations.

A nonlinear system can have several (or infinitely many) isolate equilibrium points. The following example involves a familiar physical system.

Example 2.4. *consider the pendulum system, whose dynamics is given by the following nonlinear autonomous equation*

$$MR^2\theta'' + b\theta' + Mg\sin\theta = 0$$

where R is the pendulum's length, M its mass, b the friction coefficient at the hinge, and g the gravity constant. Letting $x_1 = \theta$, $x_2 = \theta'$, the corresponding state-space equation is

$$x_1' = x_2$$
$$x_2' = -\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

The equilibrium points are given by

$$x_2 = 0, \quad \sin x_1 = 0$$

Physically, these points correspond to the pendulum resting exactly at the vertical up and down positions.

Definition 2.5 (nominal motion)**.** *In some practical problems, we are not concerned with stability around an equilibrium point, but rather with the stability of a motion, i.e., whether a system will remain close to its original motion trajectory if slightly perturbed away from it. We can show that this kind of motion* *stability problem can be transformed into an equivalent stability problem around an equilibrium point, although the equivalent system is non-autonomous.*

Let $\mathbf{x}^*(t)$ *be the solution of equation*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, t)$$

i.e., the nominal motion trajectory, corresponding to initial condition $\mathbf{x}^*(0) = \mathbf{x}_0$*. Let us now perturb the initial condition to be* $\mathbf{x}(0) = x_0 + \delta\mathbf{x}_0$ *and study the associated variation of the motion error*

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$$

since both $\mathbf{x}^*(t)$ *and* $\mathbf{x}(t)$ *are solutions of* $\mathbf{x}' = \mathbf{f}(\mathbf{x}, t)$*, we have*

$$\mathbf{x}^{*\prime} = \mathbf{f}(\mathbf{x}^*) \quad \mathbf{x}(0) = x_0$$

$$\mathbf{x} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0 + \delta\mathbf{x}_0$$

then $\mathbf{e}(t)$ *satisfies the following non-autonomous differential equation*

$$\mathbf{e}' = \mathbf{f}(\mathbf{x}^* + \mathbf{e}, t) - \mathbf{f}(\mathbf{x}^*, t) = \mathbf{g}(\mathbf{e}, t)$$

with initial condition $\mathbf{e}(0) = \delta\mathbf{x}_0$*. Since* $\mathbf{g}(\mathbf{0}, t) = 0$*, the new dynamic system, with* $\mathbf{e}$ *as state and* $\mathbf{g}$ *in place of* $\mathbf{f}$*, has an equilibrium point at he origin of the state space. Note the perturbation dynamics is non-autonomous, due to the presence of the nominal trajectory* $\mathbf{x}^*(t)$ *on the right-hand side.*

Example 2.6. *Consider the autonomous mass-spring system*

$$mx'' + k_1 x + k_2 x^3 = 0$$

which contains a nonlinear term reflecting the hardening effect of the spring. Let us study the stability of the motion $x^*(t)$ *which starts from initial position* x_0*.*

Assume that we slightly perturb the initial position to be $x(0) = x_0 + \delta x_0$*. The resulting system trajectory is denoted as* $x(t)$*. Proceeding as before, the equivalent different equation governing the motion error* e *is*

$$me'' + k_1 e + k_2[e^3 + 3e^2 x^*(t) + 3ex^*(t)^2] = 0$$

which is a non-autonomous system.

2.2 Concepts of stability

We introduce the intuitive notion of stability as a kind of well-behavedness around a desired operating point. However, since nonlinear systems may have much more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion. A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed.

Let $\mathbf{B}_R$ denote the spherical region (or ball) defined by $\|x\| < R$ in state-space, and $\mathbf{S}_R$ the sphere itself, defined by $\|x\| = R$.

Definition 2.7 (stable)**.** *The equilibrium state* $\mathbf{x} = \mathbf{0}$ *is said to be stable if, for any* $R > 0$*, there exists* $r > 0$*, such that if* $\|x(0)\| < r$*, then* $\|x(t)\| < R$ *for all* $t \geq 0$*. Otherwise, the equilibrium point is unstable.*

Essentially, stability (also called stability in the sense of Lyapunov, or Lyapunov stability) means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it. Mathematically, the definition can be written as

$$\forall R > 0, \exists r > 0, \|\mathbf{x}(0)\| < r \quad \Rightarrow \quad \forall t \geq 0, \|x(t)\| < R$$

It is important to point out the qualitative difference between instability and the intuitive notion of "blowing up" (all trajectories close to origin move further and further away to infinity). In linear systems, instability is equivalent to blowing up, because unstable poles always lead to exponential growth of the system states. However, for nonlinear systems, blowing up is only one way of instability. The following example illustrates this point.

Example 2.8 (Instability of the Van der Pol Oscillator)**.** *The Van der Pol oscillator is described by*

$$x_1' = x_2$$

$$x_2' = -x_1 + (1 - x_1^2)x_2$$

show that the system has an equilibrium point at the origin.

System trajectories staring from any non-zero initial states all asymptotically approach a limit cycle. This implies that, if we choose R in the definition 2.7 to be small enough for the circle of radius R to fall completely within the closed-curve of the limit cycle, then system trajectories starting near the origin will eventually get out of this circle. This implies instability of the origin.

Even though the state of the system does remain around the equilibrium point in a certain sense, it cannot stay arbitrarily close to it. This is the fundamental distinction between stability and instability.

In many engineering applications, Lyapunov stability is not enough. For example, when a satellite's attitude is disturbed from its nominal position, we not only want the satellite to maintain its attitude in a range determined by the magnitude of the disturbance, i.e., Lyapunov stability, but also require that the attitude gradually go back to its original value. This type of engineering requirement is captured by the concept of asymptotic stability.

Definition 2.9 (asymptotically stable)**.** *An equilibrium point* $\mathbf{0}$ *is asymptotically stable if it is stable and if in addition there exists some* $r > 0$ *such that* $\|x(0)\| < r$ *implies that* $\mathbf{x}(t) \to 0$ *as* $t \to \infty$

Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to 0 actually converge to 0 as time t goes to infinity. The ball $\mathbf{B}_r$ is called a domain of attraction of the equilibrium point (where the domain of attraction of the equilibrium point refers to the largest such region, i.e., to the set of all points such that trajectories initiated at these points eventually converge to the origin). An equilibrium point which is Lyapunov stable but not asymptotically stable is called marginally stable.

Definition 2.10 (exponentially stable)**.** *An equilibrium point* $\mathbf{0}$ *is exponential stable if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \quad \|\mathbf{x}(t) \leq \alpha\|\mathbf{x}(0)\|e^{-\lambda t}$$

in some ball $\mathbf{B}_r$ *around the origin.*

The positive number λ is often called the rate of exponential convergence.

The above definitions are formulated to characterize the local behavior· of systems, i.e., how the state evolves after starting near the equilibrium point. Local properties tell little about how the system will behave when the initial state is some distance away from the equilibrium. Global concepts are required for this purpose.

Definition 2.11 (globally stable)**.** *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.*

2.3 Linearization and local stability

Lyapunov's linearization method is concerned with the local stability of a nonlinear system. It is a formalization of the intuition that a nonlinear system should behave similarly to its linearized approximation for small range motions. Because all physical systems are inherently nonlinear, Lyapunov's linearization method serves as the fundamental justification of using linear control techniques in practice, i.e., shows that stable design by linear control guarantees the stability of the original physical system locally.

Definition 2.12 (Linearization)**.** *consider the autonomous system 2.1, and assume* $\mathbf{f}(\mathbf{x})$ *is continuously differentiable. Then the system dynamics can be written as*

$$\mathbf{x}' = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \mathbf{f}_{h.o.t}(\mathbf{x})$$

where $\mathbf{f}_{h.o.t}$ *stands for higher-order terms in* $\mathbf{x}$*. Note that the above Taylor expansion starts directly with the first-order term, due to the fact that* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$*, since* $\mathbf{0}$ *is an equilibrium point.*

Let $\mathbf{A} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\mathbf{x}=\mathbf{0}}$*, then the system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is called the linearization of the original nonlinear system at the equilibrium point $\mathbf{0}$*.*

In practice, finding a system's linearization is often most easily done simply by neglecting any term of order higher than 1 in the dynamics.

The following result makes precise the relationship between the stability of the linear system and that of the original nonlinear system.

Theorem 2.13. *we have the following conclusions:*

1. *If the linearized system is strictly stable (i.e, if all eigenvalues of* A *are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*

2. *If the linearized system is unstable (i.e, if at least one eigenvalue of A is strictly in the right-half complex plane), then the equilibrium point is unstable (for the nonlinear system).*

3. *If the linearized system is marginally stable (i.e, all eigenvalues of A are in the left-half complex plane, but at least one of them is on the jro axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system)*

Remark 2.14. *A summary of the theorem is that it is true by continuity. If the linearized system is strictly stable, or strictly unstable, then, since the approximation is valid "not too far" from the equilibrium, the nonlinear system itself is locally stable, or locally unstable. However, if the linearized system is marginally stable, the higher-order terms in 2.12 can have a decisive effect on whether the nonlinear system is stable or unstable.*

2.4 Lyapunov's Direct Method

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total energy of a mechanical (or electrical) system is continuously dissipated, then the system, whether linear or nonlinear, must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single scalar function.

Example 2.15. *Let us consider the nonlinear mass-damper-spring system in figure 3.6, whose dynamic equation is*

$$mx'' + bx'|x'| + k_0 x + k_1 x^3 = 0 \tag{1}$$

with $bx'|x'|$ representing nonlinear dissipation or damping, and $(k_0 x + k_1 x^3)$ representing a nonlinear spring term. Assume that the mass is pulled away

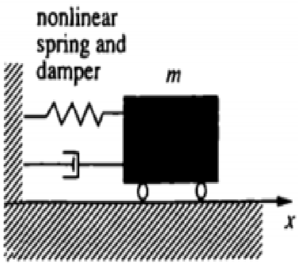

Figure 1: A nonlinear mass-damper-spring system

from the natural length of the spring by a large distance, and then released. Will the resulting motion be stable? It is very difficult to answer this question using the definitions of stability, because the general solution of this nonlinear equation is unavailable. The linearization method cannot be used either because *the motion starts outside the linear range (and in any case the system's linear approximation is only marginally stable). However, examination of the system energy can tell us a lot about the motion pattern. The total mechanical energy of the system is the sum of its kinetic energy and its potential energy.*

$$V(\mathbf{x}) = \frac{1}{2}mx'^2 + \int_0^x (k_0 x + k_1 x^3)dx = \frac{1}{2}mx'^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4 \tag{2}$$

Comparing the definitions of stability and mechanical energy, one can easily see some relations between the mechanical energy and the stability concepts described earlier:

1. zero energy corresponds to the equilibrium point ($\mathbf{x} = 0, \mathbf{x}' = 0$)
2. asymptotic stability implies the convergence of mechanical energy to zero
3. instability is related to the growth of mechanical energy

These relations indicate that the value of a scalar quantity, the mechanical energy, indirectly reflects the magnitude of the state vector; and furthermore, that the stability properties of the system can be characterized by the variation of the mechanical energy of the system.

The rate of energy variation during the system's motion is obtained easily by differentiating 2 and using 1

$$V'(\mathbf{x}) = mx'x'' + (k_0 x + k_1 x^3)x' = x'(-bx'|x'|) = -b|x'|^3 \tag{3}$$

Equation 3 implies that the energy of the system, starting from some initial value, is continuously dissipated by the damper until the mass settles down, i.e., until $x' = 0$. Physically, it is easy to see that the mass must finally settle down at the natural length of the spring, because it is subjected to a non-zero spring force at any position other than the natural length.

The direct method of Lyapunov is based on a generalization of the concepts in the above mass-spring-damper system to more complex systems. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate a scalar "energy-like" function for the dynamic system, and examine the time variation of that scalar function. In this way, conclusions may be drawn on the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

2.5 Positive Definite Functions and Lyapunov Functions

The energy function in 2 has two properties. The first is a property of the function itself: it is strictly positive unless both state variables x and x' are zero. The second property is a property associated with the dynamics 1: the function is monotonically decreasing when the variables x and x' vary according to1. In Lyapunov's direct method, the first property is formalized by the notion of positive definite functions, and the second is formalized by the so-called Lyapunov functions.

Definition 2.16. *A scalar continuous function $V(x)$ is said to be locally positive definite if $V(\mathbf{0}) = 0$ and, in a ball $\mathbf{B}_{R_0}$*

$$\mathbf{x} \neq 0 \rightarrow V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ and the above property holds over the whole state space, then $V(x)$ is said to be globally positive definite.

A few related concepts can be defined similarly, in a local or global sense, i.e., a function $V(x)$ is negative definite if $-V(x)$ is positive definite; $V(x)$ is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$; $V(x)$ is negative semi-definite if $-V(x)$ is positive semi-definite. The prefix "semi" is used to reflect the possibility of V being equal to zero for $\mathbf{x} \neq 0$. These concepts can be given geometrical meaning similar to the ones given for positive definite functions.

With x denoting the state of the system 2.5, a scalar function $V(\mathbf{x})$ actually represents an implicit function of time t. Assuming that $V(\mathbf{x})$ is differentiable, its derivative with respect to time can be found by the chain rule,

$$V' = \frac{dV(\mathbf{x})}{dt} = \frac{\partial V}{\partial \mathbf{x}}\mathbf{x}' = \frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})$$

We see that, because x is required to satisfy the autonomous state equations 2.5, V only depends on x. It is often referred to as "the derivative of V along the system trajectory". For the system 1, $V(x)$ is computed in 3 and found to be negative. Functions such as V in that example are given a special name because of their importance in Lyapunov's direct method.

Definition 2.17 (Lyapunov function)**.** *If, in a ball $\mathbf{B}_{R_0}$, the function $V(x)$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system 2.5 is negative semi-definite, i.e.,*

$$V'(\mathbf{x}) \leq 0$$

then $V(\mathbf{x})$ is said to be a Lyapunov function for the system 2.5. A Lyapunov

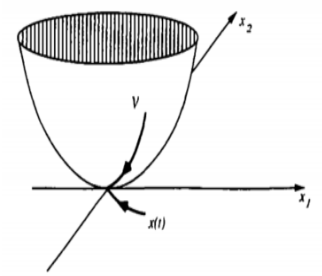

Figure 2: Illustrating definition 2.17 for $n = 2$

function can be given simple geometrical interpretations. In Figure 2, the point denoting the value of $V(x_1, x_2)$ is seen to always point down a bowl. In Figure 3, the state point is seen to move across contour curves corresponding to lower and lower values of V.

2.6 Equilibrium point theorem

The relations between Lyapunov functions and the stability of systems are made precise in a number of theorems in Lyapunov's direct method. Such theorems usually have local and global versions. The local versions are concerned with stability properties in the neighborhood of equilibrium point and usually involve a locally positive definite function.

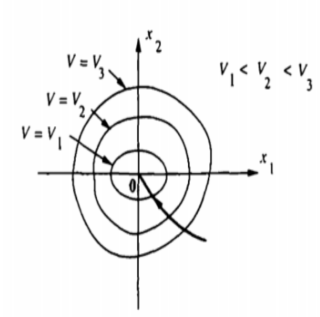

Figure 3: Illustrating definition 2.17 for $n = 2$ using contour curves

Theorem 2.18 (local stability)**.** *If, in a ball $\mathbf{B}_{R_0}$, there exists a scalar function $V(x)$ with continuous first partial derivatives such that*

1. *$V(x)$ is positive definite (locally in $\mathbf{B}_{R_0}$)*
2. *$V(x)$ is negative semi-definite (locally in $\mathbf{B}_{R_0}$)*

then the equilibrium point $\mathbf{0}$ is stable. If, actually, the derivative $V(\mathbf{x})$ locally negative definite in $\mathbf{B}_{R_0}$, then the stability is asymptotic.

Example 2.19 (local stability)**.** *A simple pendulum with viscous damping is described by*

$$\theta'' + \theta' + \sin\theta = 0$$

Consider the following scalar function

$$V(\mathbf{x}) = (1 - \cos\theta) + \frac{\theta'^2}{2}$$

One easily verifies that this function is locally positive definite. As a matter of fact, this function represents the total energy of the pendulum, composed of the sum of the potential energy and the kinetic energy. Its time-derivative is easily found to be

$$V'(\mathbf{x}) = \theta' \sin\theta + \theta'\theta'' = -\theta'^2 \leq 0$$

Therefore, by invoking the above theorem, one concludes that the origin is a stable equilibrium point. In fact, using physical insight, one easily sees the reason why $V'(\mathbf{x}) \leq 0$, namely that the damping term absorbs energy. Actually. V' is precisely the power dissipated in the pendulum. However, with this Lyapunov function, one cannot draw conclusions on the asymptotic stability of the system, because $V'(x)$ is only negative semi-definite.

Example 2.20 (Asymptotic stability)**.** *Let us study the stability of the nonlinear system defined by*

$$x_1' = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$x_2' = x_2(x_1^2 + x_2^2 - 2) + 4x_1^2 x_2$$

around its equilibrium point at the origin. Given the positive definite function

$$V'(x_1, x_2) = x_1^2 + x_2^2$$

its derivative V along any system trajectory is

$$V' = 2(X_1^2 + X_2^2)(X_1^2 + X_2^2 - 2)$$

Thus, V' is locally negative definite in the 2-dimensional ball $\mathbf{B}_2$, i.e., in the region defined by $x_1 + x_2 < 2$. Therefore, the above theorem indicates that the origin is asymptotically stable.

2.7 Lyapunov theorem for global stability

The above theorem applies to the local analysis of stability. In order to assert global asymptotic stability of a system, one might naturally expect that the ball $\mathbf{B}_{R_0}$ in the above local theorem has to be expanded to be the whole state-space. This is indeed necessary, but it is not enough. An additional condition on the function V has to be satisfied: $V(x)$ must be radially unbounded, by which we mean that $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ (in other words, as $\mathbf{x}$ tends to infinity in any direction). We then obtain the following powerful result:

Theorem 2.21 (Global stability)**.** *Assume that there exists a scalar function V of the state $\mathbf{x}$, with continuous first order derivatives such that*

1. *$V(\mathbf{x})$ is positive definite*
2. *$\dot{V}(\mathbf{x})$ is negative definite*
3. *$V(\mathbf{x}) \to \infty$ as $\|x\| \to \infty$*

then the equilibrium at the origin is globally asymptotically stable.

Example 2.22. *Consider the system*

$$x_1' = x_2 - x_1(x_1 + x_2^2)$$

$$x_2' = -x_1 - x_2(x_1^2 + x_2^2)$$

The origin of the state-space is an equilibrium point for this system. Let V be the positive definite function

$$V(\mathbf{x}) = x_1^2 + x_2^2$$

The derivative of V along any system trajectory is

$$V'(\mathbf{x}) = 2x_1 x_1' + 2x_2 x_2' = -2(x_1^2 + x_2^2)^2$$

which is negative definite. Therefore, the origin is a globally asymptotically stable equilibrium point. Note that the globalness of this stability result also implies that the origin is the only equilibrium point of the system.

Remark 2.23. *Many Lyapunov functions may exist for the same system. For instance, if V is a Lyapunov function for a given system, so is*

$$V_1 = \rho V^\alpha$$

where ρ is any strictly positive constant and α is any scalar (not necessarily an integer) larger than 1. Indeed, the positive-definiteness of V implies that of V_1 , the positive definiteness (or positive semi-definiteness) of $-V'$ implies that of $-V_1'$, and (the radial unboundedness of V (if applicable) implies that of V_1

Along the same lines, it is important to realize that the theorems in Lyapunov analysis are all sufficiency theorems. If for a particular choice of Lyapunov function candidate V, the conditions on V' are not met, one cannot draw any conclusions on the stability or instability of the system - the only conclusion one should draw is that a different Lyapunov function candidate should be tried.

2.8 Invariant Set Theorems

Asymptotic stability of a control system is usually a very important property to be determined. However, the equilibrium point theorems just described are often difficult to apply in order to assert this property. The reason is that it often happens that V, the derivative of the Lyapunov function candidate, is only negative semi-definite. as seen in 3. In this kind of situation, fortunately, it is still possible to draw conclusions on asymptotic stability, with the help of the powerful invariant set theorems, attributed to La Salle. This section presents the local and global versions of their variant set theorems.

Definition 2.24 (invariant set)**.** *A set G is an invariant set for a dynamic system if every system trajectory which starts from a point in G remains in G for all future time.*

For instance, any equilibrium point is an invariant set. The domain of attraction of an equilibrium point is also an invariant set. A trivial invariant set is the whole statespace. For an autonomous system, any of the trajectories in state-space is an invariant set. Since limit cycles are special cases of system trajectories (closed curves in the phase plane), they are also invariant sets.

The invariant set theorems reflect the intuition that the decrease of a Lyapunov function V has to gradually vanish (i.e., V has to converge to zero) because V is lower bounded. A precise statement of this result is as follows.

Theorem 2.25 (Local invariant set theorem)**.** *Consider an autonomous system of the form 2.5, with f continuous, and let $V(\mathbf{x})$ be a scalar function with continuous first partial derivatives. Assume that*

1. *for some $l > 0$, the region Ω_l defined by $V(\mathbf{x}) < l$ is bounded*
2. *$V'(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega_l$*

Let $\mathbf{R}$ be the set of all points within Ω_l where $V'(\mathbf{x}) = 0$, and $\mathbf{M}$ be the largest invariant set in $\mathbf{R}$. Then every solution $\mathbf{X}(t)$ originating in Ω_l tends to $\mathbf{M}$ as $t \to \infty$

Let us now illustrate applications of the invariant set theorem using some examples. The first example shows how to conclude asymptotic stability for problems which elude the local Lyapunov theorem. The second example shows how to determine a domain of attraction, an issue which was not specifically. addressed before. The third example shows the convergence of system trajectories to a limit cycle.

Example 2.26 (Asymptotic stability of the mass-damper-spring system). *For the system 1. one can only draw conclusion of marginal stability using the energy function 2 in the local equilibrium point theorem, because V' is only negative semi-definite according to 3. Using the invariant set theorem. however. we can show that the system is actually asymptotically stable. To do this, we only have to show that the set* $\mathbf{M}$ *contains only one point.*

The set $\mathbf{R}$ *is defined by* $\mathbf{x}' = 0$, *i.e., the collection of states with zero velocity or the whole horizontal axis in the phase plane* $(\mathbf{x}, \mathbf{x}')$. *Let us show that the largest invariant set* $\mathbf{M}$ *in this set* $\mathbf{R}$ *contains only the origin. Assume that* $\mathbf{M}$ *contains a point with a nonzero position* $\mathbf{x_1}$, *then the acceleration at that point is* $x'\prime = -(k_0/m)x - (k_1/m)x^3 \neq 0$. *This implies that the trajectory will immediately move out of the set* $\mathbf{R}$ *and thus also out of the set* $\mathbf{M}$. *a contradiction to the definition.*

Example 2.27 (Domain of Attraction). *Consider again the system in Example 2.20. For* $l = 2$. *the region* Ω_2 *defined by* $V(x) = x_1^2 + x_2^2 < 2$, *is bounded. The set* $\mathbf{R}$ *is simply the origin* $\mathbf{0}$, *which is an invariant set (since it is an equilibrium point). All the conditions of the local invariant set theorem are satisfied and, therefore. any trajectory starting within the circle converges to the origin. Thus, a domain of attraction is explicitly determined by the invariant set theorem.*

Example 2.28 (Attractive Limit Cycle).

$$x_1' = x_2 - x_1^7[x_1^4 + 2x_2 - 10]$$

$$x_2' = -x_1^3 - 3x_2^5[x_1^4 + 2x_2^2 - 10]$$

notice that the set defined by $x_1^4 + 2x_2^2 = 10$ *is invariant, since*

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -(4x_1^{10} + 12x_2^6)(x_1^4 + 2x_2 - 10)$$

which is zero on the set. The motion on this invariant set is described by either of the equations

$$x_1' = x_2$$

$$x_2' = -x_1^3$$

Therefore, we see that the invariant set actually represents a limit cycle, along which the state vector moves clockwise.

Is this limit cycle actually attractive? Let us define as a Lyapunov function candidate

$$V = (x_1^4 + 2x_2^2 - 10)^2$$

which represents a measure of the distance to the limit cycle. For any arbitrary positive number l, *the region* Ω_l, *which surrounds the limit cycle, is bounded. Using our earlier calculation, we immediately obtain*

$$V' = -8(x_1^{10} + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$$

Thus V' is strictly negative, except if

$$x_1^4 + 2x_2^2 = 10 \quad or \quad x_1^{10} + 3x_2^6 = 0$$

in which case $V' = 0$. The first equation is simply that defining the limit cycle, while the second equation is verified only at the origin. Since both the limit cycle and the origin are invariant sets, the set M simply consists of their union. Thus, all system trajectories starting in Ω_l converge either to the limit cycle, or to the origin.

Moreover, the equilibrium point at the origin can actually be shown to be unstable. However, this result cannot be obtained from linearization, since the linearized system ($X_1' = X_2, X_2' = 0$) is only marginally stable. Instead, and more astutely, consider the region Ω_{100} and note that while the origin 0 does not belong to Ω_{100}, every other point in the region enclosed by the limit cycle is in Ω_{100} (in other words, the origin corresponds to a local maximum of V). Thus, while the expression of V is the same as before, now the set M is just the limit cycle. Therefore, reapplication of the invariant set theorem shows that any state trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle. In particular, this implies that the equilibrium point at the origin is unstable.

Corollary 2.29. *Consider the autonomous system 2.5, with $\mathbf{f}$ continuous, and let $V(\mathbf{x})$ be a scalar function with continuous partial derivatives. Assume that in a certain neighborhood Ω of the origin*

1. *$V(\mathbf{x})$ is locally positive definite*
2. *V' is negative semi-definite*
3. *the set $\mathbf{R}$ defined by $V(\mathbf{x}) = 0$ contains no trajectories of 2.5 other than the trivial trajectory $x \equiv 0$*

Then, the equilibrium point $\mathbf{0}$ is asymptotically stable. Furthermore, the largest connected region of the form Ω_l (defined by $V(\mathbf{x}) < l$) within Ω is a domain of attraction of the equilibrium point.

Theorem 2.30 (global invariant set theorems)**.** *consider the autonomous system 2.5, with f continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. assume that*

1. *$\dot{V}(x) \leq 0$ over the whole state space*
2. *$V(x) \to \infty$ as $\|x\| \to \infty$*

let R be the set of all points where $\dot{V}(x) = 0$, and M be the largest invariant set in R. then all solutions globally asymptotically converge to M as $t \to \infty$

Example 2.31 (A class or second-order nonlinear systems)**.** *Consider a second-order system of the form*

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

where b and c are continuous functions verifying the sign conditions

$$\dot{x}b(\dot{x}) > 0 \quad for \quad \dot{x} \neq 0$$

$$\dot{x}c(x) > 0 \quad for \quad x \neq 0$$

The dynamics of a mass-damper-spring system with nonlinear damper and sring can be described by equations of this form. with the above sign conditions simply indicating that the otherwise arbitrary functions b and c actually represent "damping" and "spring" effects. A nonlinear R-L-C (resistor-inductor-capacitor) electrical circuit can also be represented by the above dynamic equation (Figure 4). Note that if the functions b and c are actually linear ($b(\dot{x}) = \alpha_1\dot{x}$. $c(x) = \alpha_0 x$), the above sign conditions are simply the necessary and sufficient conditions for the system's stability (since they are equivalent to the conditions $\alpha_1 > 0$, $\alpha_0 > 0$).

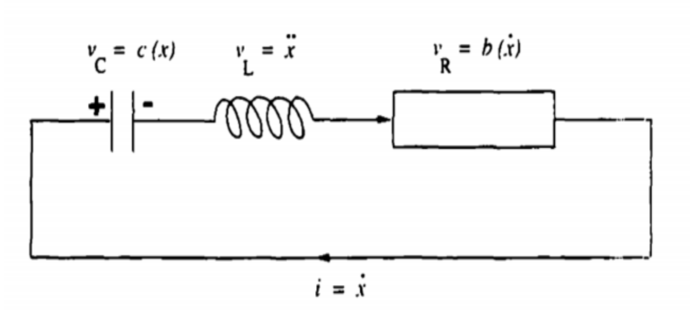

Figure 4: a nonlinear R-L-C circuit

Together with the continuity assumptions, the sign conditions on the function b and c imply that $b(0) = 0$ and $c(0) = 0$ (Figure 5). A positive definite function for this system is

$$V = \frac{1}{2}\dot{x}^2 + \int_0^x c(y)dy$$

which can be thought of as the sum of the kinetic and potential energy of the system.

differentiating V, we obtain

$$\dot{V} = \dot{x}\ddot{x} + c(x)\dot{x} = -\dot{x}b(\dot{x}) - \dot{x}c(x) + c(x)\dot{x} = -\dot{x}b(\dot{x}) \leq 0$$

which can be thought of as representing the power dissipated in the system. furthermore, by hypothesis, $\dot{x}b(\dot{x}) = 0$ only if $\dot{x} = 0$. now $\dot{x} = 0$ implies that

$$\ddot{x} = -c(x)$$

which is nonzero as long as $x \neq 0$. thus the system cannot get stuck at an equilibrium value other than $x = 0$; in other words, with R being the set defined by $\dot{x} = 0$, the largest invariant set M in R contains only one point, namely $[x = 0, \dot{x} = 0]$. use of the local invariant set theorem indicates that the origin is a locally asymptotically stable point. furthermore, if the integral $\int_0^x c(r)dr$ is unbounded as $|x| \to \infty$, then V is a radially unbounded function and the equilibrium point at the origin is globally asymptotically stable according to the global invariant set theorem.

As noticed earlier, several Lyapunov functions may exist for a given system, and therefore several associated invariant sets may be derived. The system then

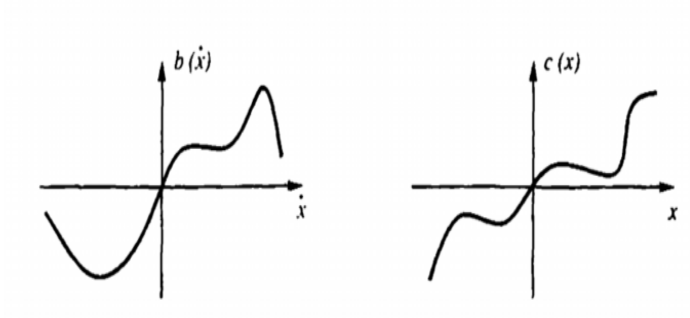

Figure 5: the function $\dot{b}(x)$ and $c(x)$

converges to the (necessarily non-empty) intersection of the invariant sets M_i, which may give a more precise result than that obtained from any of the Lyapunov functions taken separately. Equivalently, one can notice that the sum of two Lyapunov functions for a given system is also a Lyapunov function, whose set R is the intersection of the individual sets R_i

2.9 Lyapunov functions for linear time-invariant systems

given a linear system of the form $\dot{x} = Ax$, let us consider a quadratic Lyapunov function candidate

$$V = x^T P x$$

where P is a given symmetric positive definite matrix. Differentiating the positive definitive function V along the system trajectory yields another quadratic form

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = -x^T Q x$$

where

$$A^T P + PA = -Q \quad \text{(Lyapunov equation)}$$

the question is to determine whether the symmetric matrix Q defined by the so-called Lyapunov equation above is itself p.d. if this is the case, then V satisfies the conditions of the basic theorem2.21, and the origin is globally asymptotically stable. however, this natural approach may lead to inconclusive result, i.e., Q may be not positive definitive even for stable systems.

Example 2.32. *check the second-order linear system whose A is*

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

if we take $P = I$, then

$$-Q = \begin{bmatrix} 0 & -4 \\ -4 & -24 \end{bmatrix}$$

by Sylvester theorem we know that Q is not positive definitive as $Q_{11} = 0$. so this Lyapunov function cannot let us know whether the system is stable or not. a more useful way of studying a given linear system using scalar quadratic functions is to derive a positive definite matrix P from a given positive definite matrix Q, i.e.,

1. *choose a positive definite matrix Q*
2. *solve for P from the Lyapunov equation2.9*
3. *check whether P is p.d.*

if P is p.d., then $x^T Px$ is a Lyapunov function for the linear system and global asymptotical stability is guaranteed. Unlike the previous approach of going from a given P to Q, this technique of going from a given Q to a matrix P always leads to a conclusive results for stable linear systems, as seen from the following theorem

Theorem 2.33. *a necessary and sufficient condition for a LTI system $\dot{x} = Ax$ to be strictly stable is that, for any symmetric p.d. matrix Q, the unique matrix P solution of the Lyapunov equation2.9 be symmetric positive definite.*

this theorem says that any positive definite matrix Q can be used to determine the stability of a linear system. A simple choice of Q is the identity matrix.

2.10 Krasovskii's method

we move back to the problem of finding Lyapunov functions for general nonlinear systems. Krasovskii's method suggests a simple form of Lyapunov function candidate for autonomous nonlinear systems of the form 2.5, namely, $V = f^T f$. the basic idea of the method is simply to check whether this particular choice indeed leads to a Lyapunov function

Theorem 2.34 (Krasovskii)**.** *consider the autonomous system defined by 2.5, with the equilibrium point of interest being the origin. Let $A(x)$ denote the Jacobian matrix of the system, i.e.,*

$$A(x) = \frac{\partial f}{\partial x}$$

if the matrix $F = A + A^T$ is negative definite in a neighborhood Ω, then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is

$$V(x) = f^T(x)f(x)$$

if Ω is the entire state space and, in addition, $V(x) \to \infty$ as $\|x\| \to \infty$, then the equilibrium point is globally asymptotically stable.

we use the following example to show the use of the Krasovskii's theorem

Example 2.35. *consider the nonlinear system*

$$\dot{x}_1 = -6x_1 + 2x_2$$

$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

we have

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 6x_2^2 \end{bmatrix} \quad F = A + A^T = \begin{bmatrix} -12 & 4 \\ 4 & -12 - 12x_2^2 \end{bmatrix}$$

the matrix F is easily shown to be negative definite over the whole state space. therefore, the origin is asymptotically stable, and a Lyapunov function candidate is

$$V(x) = f^T(x)f(x) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

since $V(x) \to \infty$ as $\|x\| \to \infty$, the equilibrium state at the origin is globally asymptotically stable.

the above Krasovskii theorem's applicability is limited in practice, because the Jacobians of many systems do not satisfy the negative definiteness requirement. in addition, for systems of high order, it is difficult to check the negative definiteness of the matrix F for all x. And an immediate generalization of Krasovskii's theorem is as follows:

Theorem 2.36 (generalized Krasovskii theorem)**.** *Consider the autonomous system defined by 2.5, with the equilibrium point of interest being the origin, and let $A(x)$ denote the Jacobian matrix of the system. Then, a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices P and Q, such that $\forall x \neq 0$, the matrix*

$$F(x) = A^T P + PA + Q$$

is negative semi-definite in some neighborhood Ω of the origin. The function $V(x) = f^T P f$ is then a Lyapunov function for the system. If the region Ω is the whole state space, and if in addition, $V(x) \to \infty$ as $\|x\| \to \infty$, then the system is globally asymptotically stable.

2.11 The variable gradient method

The variable gradient method is a formal approach to constructing Lyapunov functions. It involves assuming a certain form for the gradient of an unknown Lyapunov function, and then finding the Lyapunov function itself by integrating the assumed gradient. For low order systems, this approach sometimes leads to the successful discovery of a Lyapunov function.
we note that a scalar function $V(x)$ is related to its gradient ∇V by the integral relation

$$V(x) = \int_0^x \nabla V dx$$

where $\nabla V = [\partial V/\partial x_1, \ldots, \partial V/\partial x_n]^T$. in order to recover a unique scalar function V from the gradient ∇V, the gradient function has to satisfy the so-called curl condition

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_j} \quad (i, j = 1, 2, ..., n)$$

note that the $i-th$ component ∇V_i is simply the directional derivative $\partial V/\partial x_i$. for instance, in the case $n = 2$, the above simply means that

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

the principle of the variable gradient method is to assume a specific form for the Lyapunov function V itself. a simple way is to assume that the gradient function is of the form

$$\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$$

where the a_{ij}'s are coefficients to be determined. this leads to the following procedure for seeking a Lyapunov function V:

1. assume that ∇V is given by $\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$ or another form
2. solve for the coefficient a_{ij} so as to satisfy the curl equations
3. restrict the coefficients in $\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$ so that $\dot{V}$ is negative semi-definite (at least locally)
4. compute V from ∇V by integration
5. check whether V is positive definite

since satisfaction of the curl conditions implies that the above integration result is independent of the integration path, it is usually convenient to obtain V by integrating along a path which is parallel to each axis in turn, i.e.,

$$V(x) = \int_0^{x_1} \nabla V_1(x_1, 0, ..., 0) dx_1 + \int_0^{x_2} \nabla V_2(x_1, x_2, 0, ..., 0) dx_2 + ... + \int_0^{x_n} \nabla V_n(x_1, x_2, ..., x_n) d_n$$