1 control barrier function

Definition 1.1 (forward invariant; safety)**.** *let $u = k(x)$ be a feedback controller s.t. the resulting dynamical system*

$$\dot{x} = f_{cl}(x) := f(x) + g(x)k(x)$$

is locally Lipschitz.

the set $\mathcal{C}$ is forward invariant if for every $x_0 \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for $x(0) = x_0$ and all $t \in I(x_0)$. the system is safe with respect to the set $\mathcal{C}$ if the set $\mathcal{C}$ is forward invariant. $I(x_0) = [0, \tau_{max})$

Definition 1.2 (control barrier function)**.** *let $\mathcal{C} \subset D \subset \mathbb{R}^n$ be the superlevel set of a a continuously differentiable function $h : D \to \mathbb{R}$, then h is a control barrier function if there exists an extended class $\mathcal{K}_\infty$ function α s.t. for the control system*

$$\dot{x} = f(x) + g(x)u$$

we have

$$\sup_{u \in U} [L_f h(x) + L_g h(x) u] \geq -\alpha(h(x))$$

for all $x \in D$

consider the set consisting of all control values that render $\mathcal{C}$ safe:

$$K_{cbf}(x) = \{u \in U : L_f h(x) + L_g h(x) u + \alpha(h(x)) \geq 0\}$$

we can quantify the set of all control inputs at a point $x \in D$ that keep the system safe.

the next theorem says that the existence of a control barrier function implies that the control system is safe:

Theorem 1.3. *let $\mathcal{C} \in \mathbb{R}^n$ be a set defined as the superlevel set of a continuously differentiable function $h : D \subset R^n \to R$. if h is a control barrier function on D and $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \partial\mathcal{C}$, then any Lipschitz continuous controller $u(x) \in K_{cbf}(x)$ for the system*

$$\dot{x} = f(x) + g(x)u$$

renders the set $\mathcal{C}$ safe. additionally, the set C is asymptotically stable in D

finally, we note that control barrier functions provide the strongest possible conditions for safety in that they are necessary and sufficient given reasonable assumptions on $\mathcal{C}$

Theorem 1.4. *let $\mathcal{C}$ be a compact set that is the superlevel set of a continuously differentiable function $h : D \to \mathbb{R}$ with the property that $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \partial\mathcal{C}$. if there exists a control law $u = k(x)$ that renders $\mathcal{C}$ safe, i.e., $\mathcal{C}$ is forward invariant with respect to*

$$\dot{x} = f_{cl}(x) := f(x) + g(x)k(x)$$

then $h|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ is a control barrier function on $\mathcal{C}$