JACOBI–TRUDI FORMULA FOR REFINED DUAL STABLE GROTHENDIECK POLYNOMIALS

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ABSTRACT. In 2007 Lam and Pylyavskyy found a combinatorial formula for the dual stable Grothendieck polynomials, which are the dual basis of the stable Grothendieck polynomials with respect to the Hall inner product. In 2016 Galashin, Grinberg, and Liu introduced refined dual stable Grothendieck polynomials by putting additional sequence of parameters in the combinatorial formula of Lam and Pylyavskyy. Grinberg conjectured a Jacobi–Trudi type formula for refined dual stable Grothendieck polynomials. In this paper this conjecture is proved by using bijections of Lam and Pylyavskyy.

1. INTRODUCTION

In 1982 Lascoux and Schützenberger [10] introduced *Grothendieck polynomials*, which are representatives of the structure sheaves of the Schubert varieties in a flag variety. Fomin and Kirillov [3] studied Grothendieck polynomials combinatorially and introduced *stable Grothendieck polynomials*, which are stable limits of Grothendieck polynomials. Buch [2] found a combinatorial formula for stable Grothendieck polynomials using set-valued tableaux. Lam and Pylyavskyy [9] first studied *dual stable Grothendieck polynomials* $g_\lambda(x)$, which are the dual basis of the stable Grothendieck polynomials under the Hall inner product. They also found a combinatorial formula for $g_\lambda(x)$ in terms of reverse plane partitions. Their formula gives a combinatorial way to expand $g_\lambda(x)$ in terms of Schur functions $s_\mu(x)$.

Dual stable Grothendieck polynomials $g_\lambda(x)$ are inhomogeneous symmetric functions in variables $x = (x_1, x_2, \dots)$. Galashin, Grinberg, and Liu [6] introduced *refined dual stable Grothendieck polynomials* $\widetilde{g}_{\lambda/\mu}(x; t)$ by putting an additional sequence $t = (t_1, t_2, \dots)$ of parameters in the combinatorial formula of Lam and Pylyavskyy. They showed that $\widetilde{g}_{\lambda/\mu}(x; t)$ is also symmetric in x. Refined dual stable Grothendieck polynomials generalize both dual stable Grothendieck polynomials and Schur functions: if $t_i = 1$ for all $i \geq 1$, then $\widetilde{g}_{\lambda/\mu}(x; t) = g_{\lambda/\mu}(x)$, and if $t_i = 0$ for all $i \geq 1$, then $\widetilde{g}_{\lambda/\mu}(x; t) = s_{\lambda/\mu}(x)$. Galashin [5] found a Littlewood–Richardson rule to expand $\widetilde{g}_{\lambda/\mu}(x; t)$ in terms of Schur functions. Yeliussizov [15] further studied (dual) stable Grothendieck polynomials and showed the following Jacobi–Trudi type formula for $\widetilde{g}_\lambda(x; t)$ originally conjectured by Darij Grinberg [8]:

$$(1.1) \qquad \widetilde{g}_\lambda(x; t) = \det \left(e_{\lambda'_i - i + j}(x_1, x_2, \dots, t_1, t_2, \dots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n},$$

where $e_k(z_1, z_2, \dots) = \sum_{i_1 < i_2 < \dots < i_k} z_{i_1} z_{i_2} \dots z_{i_k}$ is the kth elementary symmetric function and we define $e_0(z_1, z_2, \dots) = 1$ and $e_k(z_1, z_2, \dots) = 0$ for $k < 0$. We also note that the Jacobi–Trudi formula (1.1) for the case $t_i = 1$ for all $i \geq 1$ was first proved by Shimozono and Zabrocki [13].

The main result of this paper is the following Jacobi–Trudi formula for the refined dual stable Grothendieck polynomial $\widetilde{g}_{\lambda/\mu}(x; t)$, which was also conjectured by Darij Grinberg [8, slide 72] in 2015. See Section 2 for the precise definitions.

Theorem 1.1. *Let λ and μ be partitions with $\ell(\lambda') \leq n$. Then*

$$\widetilde{g}_{\lambda/\mu}(x; t) = \det \left(e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \dots, t_{\mu'_j + 1}, t_{\mu'_j + 2}, \dots, t_{\lambda'_i - 1}) \right)_{1 \leq i, j \leq n},$$

where, if $\mu'_j + 1 > \lambda'_i - 1$, the (i, j) entry is defined to be $e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \dots)$.

Note that if $t_i = 0$ for all $i \geq 1$, then Theorem 1.1 reduces to the classical (dual) Jacobi–Trudi formula for the Schur function $s_{\lambda/\mu}(x)$. Theorem 1.1 gives another proof of the fact that $\widetilde{g}_{\lambda/\mu}(x; t)$ is symmetric in the variables x. If $t_i = 1$ for all i, we obtain a new Jacobi–Trudi formula for the dual stable Grothendieck polynomial $g_{\lambda/\mu}(x)$.

Corollary 1.2. *Let λ and μ be partitions with $\ell(\lambda') \leq n$. Then*

$$g_{\lambda/\mu}(x) = \det \left(\sum_{k \geq 0} \binom{\lambda_i' - \mu_j' - 1}{k} e_{\lambda_i' - \mu_j' - i + j - k}(x_1, x_2, \dots) \right)_{1 \leq i, j \leq n},$$

where we define $\binom{m}{k} = \delta_{k,0}$ if $m < 0$.

There is a standard combinatorial method to prove a Jacobi–Trudi type formula using the Lindström–Gessel–Viennot lemma [7, 12]. First interpret the determinant as a signed sum of n-paths, i.e., sequences $(p_1, \dots, p_n)$ of n paths in a certain lattice. If there are intersections among the n paths, choose an intersection in a controlled way and exchange the "tails" of the two paths through this intersection. This will give a sign-reversing involution on the total n-paths leaving only the non-intersecting n-paths as fixed points. Then one interprets the non-intersecting n-paths as the desired tableaux by a simple bijection. This is how Yeliussizov [15] proved (1.1).

However, the Jacobi–Trudi formula in Theorem 1.1 cannot be proved in this way. Because of the restriction of a path depending on the initial point, the usual method of exchanging tails is not applicable. In this paper we prove Theorem 1.1 by finding a sign-reversing involution on certain n-paths using two maps introduced by Lam and Pylyavskyy [9] as intermediate steps in their bijection between reverse plane partitions and pairs of semistandard Young tableaux and so-called elegant tableaux.

The remainder of this paper is organized as follows. In Section 2 we give necessary definitions and notation, and show that the determinant in Theorem 1.1 is a generating function for "semi-noncrossing" n-paths. In Section 3 we define vertical tableaux and give a bijection between them and n-paths. In Section 4 we define RSE-tableaux and review two bijections ϕ_- and ϕ_+ on RSE-tableaux due to Lam and Pylyavskyy. In Section 5, we extend the definition of RSE-tableaux to skew shapes and study properties of the maps ϕ_- and ϕ_+ on skew RSE-tableaux. In Section 6 we give a sign-reversing involution on semi-noncrossing n-paths using these maps and complete the proof of Theorem 1.1. In Section 7 we give a concrete example of the sign-reversing involution defined in Section 6.

We note that Amanov and Yeliussizov [1] proved Theorem 1.1 and Corollary 1.2 independently about the same time this paper was written. Their proof uses a sign-reversing involution on 3-dimensional lattice paths. It would be interesting to see whether there is a connection between their sign-reversing involution and ours.

2. Definitions and notation

In this section we give basic definitions and notation which will be used throughout this paper.

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. Each λ_i is called a *part* of λ. The *length* $\ell(\lambda)$ of λ is the number of parts. Sometimes we will append some zeros at the end of λ so that for example $(4, 3, 1)$ and $(4, 3, 1, 0, 0)$ are considered as the same partition, and $\lambda_i = 0$ whenever $i > \ell(\lambda)$.

The *Young diagram* of λ is defined to be the set $\{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. From now on we will identify λ with its Young diagram. The Young diagram of λ will be visualized by placing a unit square, called a *cell*, in the ith row and jth column for each $(i, j) \in \lambda$. The *conjugate* λ' of λ is defined to be the partition given by $\lambda' = \{(i, j) : (j, i) \in \lambda\}$, see Figure 1.

For two partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$. In this case the *skew shape* λ/μ is defined to be the set-theoretic difference $\lambda - \mu$ of their Young diagrams.

For a skew shape λ/μ and an integer $k \geq 1$, define $\mathrm{row}_{\leq k}(\lambda/\mu)$ (resp. $\mathrm{row}_{\geq k}(\lambda/\mu)$) to be the skew shape obtained from λ/μ by taking the rows of index $j \leq k$ (resp. $j \geq k$). Similarly, we define $\mathrm{col}_{\leq k}(\lambda/\mu)$ and $\mathrm{col}_{\geq k}(\lambda/\mu)$ using columns. See Figure 2.
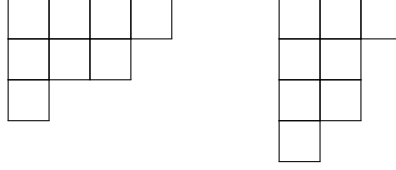
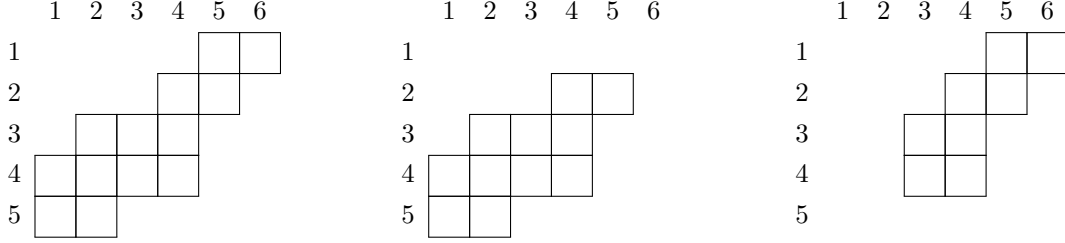FIGURE 1. The Young diagram of $\lambda = (4,3,1)$ on the left and its conjugate $\lambda' = (3,2,2,1)$ on the right.

FIGURE 2. The skew diagram λ/μ for $\lambda = (6,5,4,4,2)$ and $\mu = (4,3,1)$ on the left, $\mathrm{row}_{\geq 2}(\lambda/\mu)$ in the middle and $\mathrm{col}_{\geq 3}(\lambda/\mu)$ on the right. For visibility the row and column indices are written.
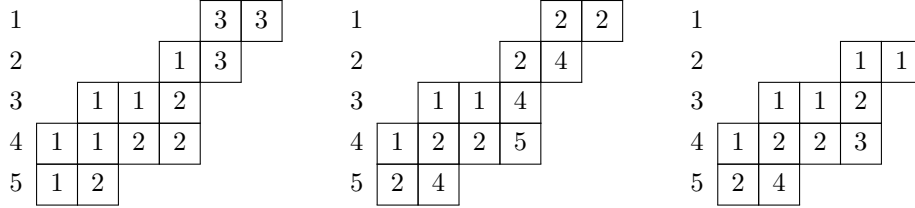

FIGURE 3. An RPP of shape $(6,5,4,4,2)/(4,3,1)$ on the left, an SSYT of shape $(6,5,4,4,2)/(4,3,1)$ in the middle and an elegant tableau of shape $(6,5,4,4,2)/(6,4,3,1)$ on the right. The row indices are written on the left of each diagram.

Let ρ be a finite subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A *tableau* of shape ρ is just a map $T : \rho \to Z$, where Z is a linearly ordered set. If $T : \rho \to Z$ is a tableau we write $\mathrm{sh}(T) = \rho$.

A *reverse plane partition (RPP)* of shape λ/μ is a tableau $R : \lambda/\mu \to \mathbb{Z}^+$ such that the entries weakly increase in each row and column, i.e., $R(i,j) \leq R(i,j+1)$ and $R(i,j) \leq R(i+1,j)$ whenever these values are defined. The set of RPPs of shape λ/μ is denoted by $\mathrm{RPP}(\lambda/\mu)$. For $R \in \mathrm{RPP}(\lambda/\mu)$, the *weight* of R is defined by

$$(2.1) \qquad \mathrm{wt}(R) = \prod_{i \geq 1} x_i^{a_i(R)} t_i^{b_i(R)},$$

where $a_i(R)$ is the number of columns containing an i and $b_i(R)$ is the number of cells (i,j) such that $R(i,j) = R(i+1,j)$. For example if R is the RPP in Figure 3, then $\mathrm{wt}(R) = x_1^4 x_2^3 x_3^2 t_1 t_3^2 t_4$.

A *semistandard Young tableau (SSYT)* is an RPP with the extra condition that the entries are strictly increasing in each column. The set of SSYTs of shape λ/μ is denoted by $\mathrm{SSYT}(\lambda/\mu)$. An *elegant tableau* is an SSYT E of a certain skew shape λ/ν such that $1 \leq E(i,j) \leq i-1$ for all $(i,j) \in \lambda/\nu$. See Figure 3. Elegant tableaux were first considered by Lenart [11, Theorem 2.7] and further studied by Lam and Pylyavskyy [9].

Note that if $R \in \mathrm{SSYT}(\lambda/\mu) \subseteq \mathrm{RPP}(\lambda/\mu)$, then R has no repeated entries in each column and therefore the weight of R defined in (2.1) is given by

$$\mathrm{wt}(R) = x_R = x_1^{c_1(T)} x_2^{c_2(T)} \cdots,$$

where $c_i(T)$ is the number of i's in R. For example, if R is the SSYT in Figure 3, then $\mathrm{wt}(R) = x_1^3 x_2^6 x_4^3 x_5$.

Let $x = (x_1, x_2, \dots)$ and $t = (t_1, t_2, \dots)$ be sequences of variables. The *refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}(x; t)$ is defined by

$$\widetilde{g}_{\lambda/\mu}(x; t) = \sum_{R \in \mathrm{RPP}(\lambda/\mu)} \mathrm{wt}(R).$$

Now we recall the main result, Theorem 1.1: if λ and μ are partitions with $\ell(\lambda') \le n$,

$$\widetilde{g}_{\lambda/\mu}(x; t) = \det \left(e_{\lambda_i' - \mu_j' - i + j}(x_1, x_2, \dots, t_{\mu_j' + 1}, t_{\mu_j' + 2}, \dots, t_{\lambda_i' - 1}) \right)_{1 \le i, j \le n}.$$

Note that if $\mu \not\subseteq \lambda$, by definition, $\widetilde{g}_{\lambda/\mu}(x; t) = 0$. It is easy to see that in this case the above determinant also vanishes because if $\mu \not\subseteq \lambda$, then $\lambda_r' < \mu_r'$ for some $1 \le r \le n$, which implies that the (i, j) entry is zero for all $r + 1 \le i \le n$ and $1 \le j \le r$. Moreover, it is also easy to see that if $\ell(\lambda') = m < n$, then the above determinant is equal to its principal minor consisting of the first m rows and columns. Therefore it is sufficient to show Theorem 1.1 for the case $\mu \subseteq \lambda$ and $\ell(\lambda') = n$. From now on we always assume that $\mu \subseteq \lambda$ and $\ell(\lambda') = n$.

Let ω be the smallest infinite ordinal number and let

$$\mathbb{N} = \{0, 1, 2, \dots\},$$
$$\mathbb{N}_\omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, 2\omega\},$$
$$G = \mathbb{N} \times \mathbb{N}_\omega,$$

where the numbers are ordered as usual by

$$0 < 1 < 2 < \cdots < \omega < \omega + 1 < \omega + 2 < \cdots < 2\omega.$$

We will also write $\omega + i$ as i^*.

A *path* from $(a, 0)$ to $(b, 2\omega)$ is a pair (s_1, s_2) of infinite sequences $s_j = ((u_0^{(j)}, v_0^{(j)}), (u_1^{(j)}, v_1^{(j)}), \dots)$, $j = 1, 2$, of points in G satisfying the following conditions:

- The steps $(u_{i+1}^{(j)}, v_{i+1}^{(j)}) - (u_i^{(j)}, v_i^{(j)})$, for $i \ge 0$ and $j = 1, 2$, consist of *up steps* $(0, 1)$ and *diagonal steps* $(1, 1)$.
- $(u_0^{(1)}, v_0^{(1)}) = (a, 0)$, $v_0^{(2)} = \omega$ and

$$\lim_{m \to \infty} u_m^{(1)} = u_0^{(2)}, \qquad \lim_{m \to \infty} u_m^{(2)} = b.$$

The *weight* of a path p is defined to be

$$\mathrm{wt}(p) = \prod_{i \ge 1} x_i^{a_i(p)} t_i^{b_i(t)},$$

where $a_i(p)$ (resp. $b_i(p)$) is the set of diagonal steps of p ending at height i (resp. $\omega + i$). See Figure 4.

Suppose that λ and μ are partitions with $\mu \subseteq \lambda$ and $\ell(\lambda') \le n$. Denote by $\mathcal{L}_{\lambda/\mu}(i, j)$ the set of all paths from $(\mu_i' + n - i, 0)$ to $(\lambda_j' + n - j, 2\omega)$ in which there is no diagonal step between the lines $y = \omega$ and $y = \omega + \mu_i'$ and no diagonal step above the line $y = \omega + \lambda_i' - 1$. See Figure 4 for a typical example of a path in $\mathcal{L}_{\lambda/\mu}(i, j)$. It is clear from the construction that

$$(2.2) \qquad \sum_{p \in \mathcal{L}_{\lambda/\mu}(i, j)} \mathrm{wt}(p) = e_{\lambda_j' - \mu_i' - j + i}(x_1, x_2, \dots, t_{\mu_i' + 1}, t_{\mu_i' + 2}, \dots, t_{\lambda_j' - 1}).$$

An *n-path* is an n-tuple of paths. Denote by $\mathfrak{S}_n$ the set of permutations on $\{1, 2, \dots, n\}$. For a permutation $\pi \in \mathfrak{S}_n$, let $\mathcal{L}_{\lambda/\mu}(\pi)$ denote the set of n-paths $\mathbf{p} = (p_1, \dots, p_n)$ such that
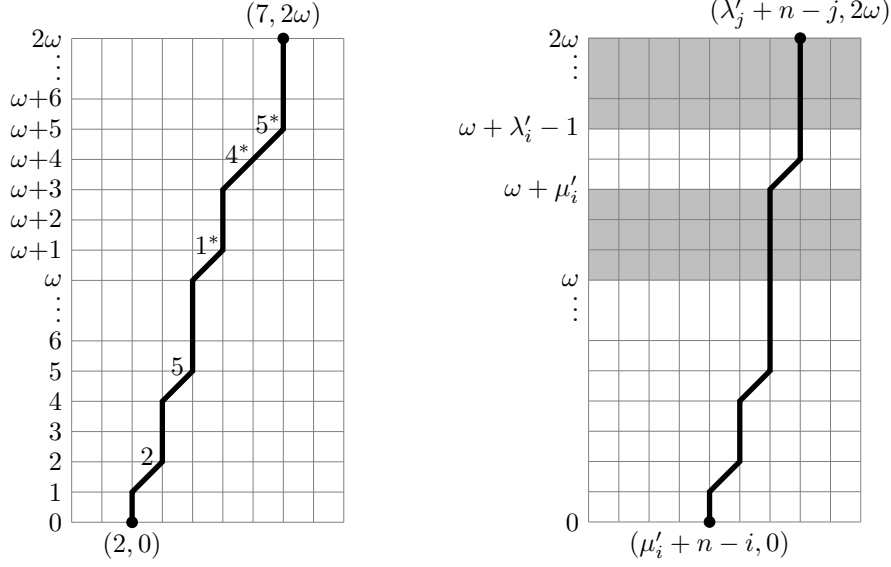
FIGURE 4. The left diagram is a path from $(2, 0)$ to $(7, 2\omega)$ with weight $x_2 x_5 t_1 t_4 t_5$. The height of the ending point of each diagonal step is shown. The right diagram illustrates a typical path in $\mathcal{L}_{\lambda/\mu}(i, j)$, which cannot have diagonal steps in the gray areas.

$p_i \in \mathcal{L}_{\lambda/\mu}(i, \pi(i))$ for all $1 \leq i \leq n$. Note that $\mathcal{L}_{\lambda/\mu}(\pi)$ may be empty for some π. Define

$$\mathcal{L}_{\lambda/\mu} = \bigcup_{\pi \in \mathfrak{S}_n} \mathcal{L}_{\lambda/\mu}(\pi).$$

The *type* of an n-path $\mathbf{p}$ in $\mathcal{L}_{\lambda/\mu}$, denoted type$(\mathbf{p})$, is the permutation π for which $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}(\pi)$. Note that type$(\mathbf{p})$ is uniquely determined because the starting points $(\mu_i' + n - i, 0)$ and the ending points $(\lambda_j' + n - j, 2\omega)$ are all distinct. The *weight* of $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu}$ is defined by

$$\mathrm{wt}(\mathbf{p}) = \mathrm{sign}(\mathrm{type}(\mathbf{p})) \, \mathrm{wt}(p_1) \cdots \mathrm{wt}(p_n).$$

The following notation will be used throughout this paper. See Figure 5 for an illustration.

Notation 2.1. Fix partitions λ and μ with $\mu \subseteq \lambda$, $\ell(\lambda) = \ell$, and $\ell(\lambda') = \lambda_1 = n$. Define $d_1 > d_2 > \cdots > d_r$ to be the distinct integers in $\{\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}\}$ and let $d_0 = n$ and $d_{r+1} = 0$. For $1 \leq i \leq r + 1$,

- m_i denotes the multiplicity of the part d_i in μ, where the multiplicity of $d_{r+1} = 0$ in μ is defined to be $\ell(\lambda) - \ell(\mu)$,
- $M_i = m_1 + \cdots + m_i$, and
- $D_i = \{d_i + 1, d_i + 2, \ldots, d_{i-1}\}$.

Note that for $1 \leq i, j \leq n$, we have $\mu_i' = \mu_j'$ if and only if $i, j \in D_k$ for some $1 \leq k \leq r$.

Suppose $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu}$. We say that $\mathbf{p}$ is *noncrossing* if p_i and p_j have no common points for all $i \neq j$, and that $\mathbf{p}$ is *semi-noncrossing* if p_i and p_j have no common points whenever i and j are distinct elements in D_k for some $1 \leq k \leq r$. Denote by $\mathcal{L}_{\lambda/\mu}^{\mathrm{NC}}$ (resp. $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$) the set of noncrossing (resp. semi-noncrossing) paths in $\mathcal{L}_{\lambda/\mu}$. Note that if $\mu = \emptyset$, then $\mathcal{L}_{\lambda/\mu}^{\mathrm{NC}} = \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$.

By expanding the determinant in Theorem 1.1 (more precisely, the determinant of the transpose of the matrix) using (2.2), we have

$$(2.3) \qquad \det \left(e_{\lambda_i' - \mu_j' - i + j}(x_1, x_2, \ldots, t_{\mu_j' + 1}, t_{\mu_j' + 2}, \ldots, t_{\lambda_i' - 1}) \right)_{1 \leq i, j \leq n} = \sum_{\mathbf{p} \in \mathcal{L}_{\lambda/\mu}} \mathrm{wt}(\mathbf{p}).$$

The standard method of the Lindström–Gessel–Viennot lemma [7, 12] interprets a determinant as a weighted sum of noncrossing n-paths via a sign-reversing involution which exchanges "tails" of
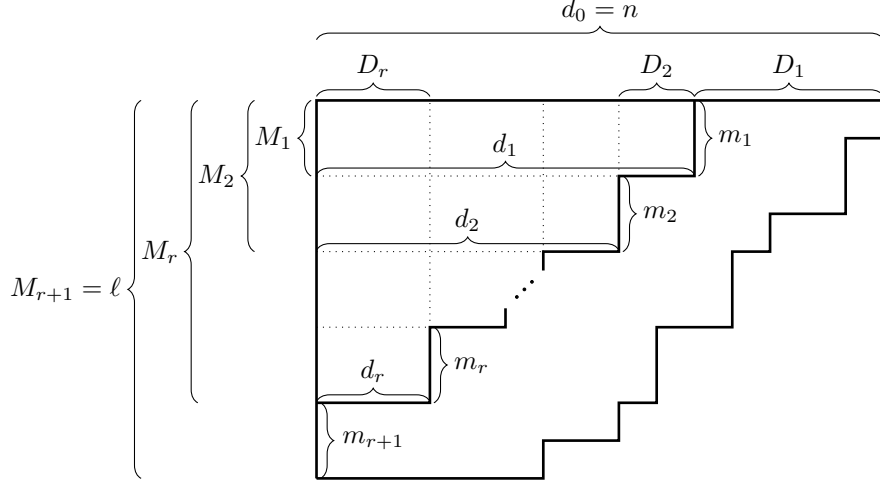
FIGURE 5. An illustration of Notation 2.1 for a given λ/μ. Every letter is an integer except D_i's, which are sets of column indices.
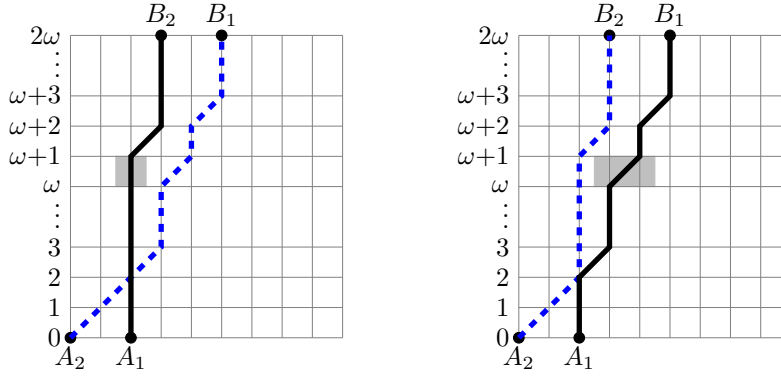

FIGURE 6. Let $\lambda = (2,2,2,1)$ and $\mu = (1)$ so that $\lambda'_1 = 4$, $\lambda'_2 = 3$ and $\mu'_1 = 1$. The left diagram shows a 2-path $(p_1, p_2) \in \mathcal{L}_{\lambda/\mu}$, where p_i is the path from A_i to B_i for $i = 1, 2$. If we switch the tails of p_1 and p_2 after their unique intersection, we obtain the resulting 2-path (p'_1, p'_2) as shown on the right. The gray area shows (part of) the restriction on the path whose starting point is $A_1 = (\mu_1 + 2 - 1, 0) = (2, 0)$. Since p'_1 has a diagonal step in the gray area, $(p'_1, p'_2) \notin \mathcal{L}_{\lambda/\mu}$.

intersecting paths. Roughly speaking, in order for this to work "local" changes of the steps in an n-path must be allowed. Such "local" changes are not allowed for an n-path $\mathbf{p} = (p_1, \ldots, p_n)$ in $\mathcal{L}_{\lambda/\mu}$ because each path p_i has the "global" restriction that there are no diagonal steps between the lines $y = \omega$ and $y = \omega + \mu'_i$. For example, see Figure 6.

However, it is possible to cancel all n-paths except for the semi-noncrossing n-paths.

Proposition 2.2. *Let λ and μ be partitions with $\mu \subseteq \lambda$, $\ell(\lambda) = \ell$, and $\ell(\lambda') = n$. Then*

$$\det\left(e_{\lambda'_i - \mu'_j - i + j}(x_1, x_2, \ldots, t_{\mu'_j+1}, t_{\mu'_j+2}, \ldots, t_{\lambda'_i-1})\right)_{1 \le i,j \le n} = \sum_{\mathbf{p} \in \mathcal{L}^{\mathrm{SNC}}_{\lambda/\mu}} \mathrm{wt}(\mathbf{p}).$$

Proof. By (2.3) it is sufficient to show that

$$(2.4) \qquad \sum_{\mathbf{p} \in \mathcal{L}_{\lambda/\mu}} \mathrm{wt}(\mathbf{p}) = \sum_{\mathbf{p} \in \mathcal{L}^{\mathrm{SNC}}_{\lambda/\mu}} \mathrm{wt}(\mathbf{p}).$$
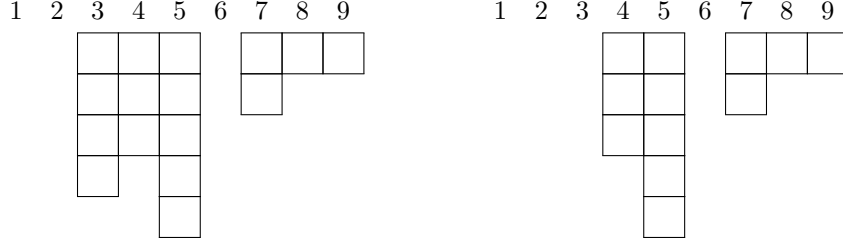
FIGURE 7. The vertical diagram $V(\alpha)$ on the left and the vertical diagram $\mathrm{col}_{\geq 4}(V(\alpha))$ on the right for the composition $\alpha = (0, 0, 4, 3, 5, 0, 2, 1, 1)$. For visibility the column indices are written above the diagrams.
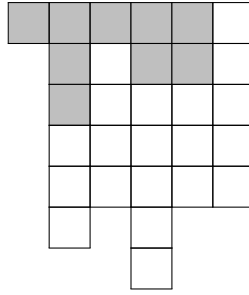

FIGURE 8. The diagram $V(\beta)/V(\alpha)$ for $\alpha = (1, 3, 1, 2, 2, 0)$ and $\beta = (1, 6, 5, 7, 5, 5)$ is shown with the white cells.

We will cancel all paths in $\mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ using the standard method of switching tails of two paths. More precisely, suppose $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$. Then we can find the smallest integer k such that p_i and p_j have common points for some $i \neq j$ in D_k. Choose such i and j so that (i, j) is the smallest in the lexicographic order. Let (a, b) be the last intersection of p_i and p_j. Let p_i' and p_j' be the paths obtained from p_i and p_j respectively by exchanging the subpaths after (a, b). If $\mathrm{type}(\mathbf{p}) = \pi$, then $p_i \in \mathcal{L}_{\lambda/\mu}(i, \pi(i))$ and $p_j \in \mathcal{L}_{\lambda/\mu}(j, \pi(j))$. Since $i, j \in D_k$, we have $\mu_i' = \mu_j'$. Therefore neither p_i nor p_j has diagonal steps between heights ω and $\omega + \mu_i' = \omega + \mu_j'$, which ensures that $p_i' \in \mathcal{L}_{\lambda/\mu}(i, \pi(j))$ and $p_j' \in \mathcal{L}_{\lambda/\mu}(j, \pi(i))$. Let $\mathbf{p}'$ be the n-path obtained from $\mathbf{p}$ by replacing p_i and p_j by p_i' and p_j' respectively. Then $\mathbf{p} \in \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ and $\mathrm{type}(\mathbf{p}') = \pi(i, j)$, where (i, j) is the transposition. Therefore $\mathrm{wt}(\mathbf{p}) = -\mathrm{wt}(\mathbf{p}')$. It is easily seen that this argument shows that $\sum_{\mathbf{p} \in \mathcal{L}_{\lambda/\mu} \setminus \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}} \mathrm{wt}(\mathbf{p}) = 0$, hence (2.4). $\square$

3. VERTICAL TABLEAUX AND n-PATHS

In this section we introduce a notion of vertical tableaux and give a simple bijection between them and certain n-paths.

A *composition* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers. The *vertical diagram* of a composition α is defined by

$$V(\alpha) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq n, 1 \leq i \leq \alpha_j\}.$$

Similarly to Young diagrams each element (i, j) in the vertical diagram is represented by a cell in row i and column j. Since $\lambda = V(\lambda')$ as subsets of $\mathbb{Z}^2$, we will also consider the Young diagram of λ as a vertical diagram. The notation used for Young diagrams is naturally extended to vertical diagrams. For example, for a vertical diagram V, define $\mathrm{col}_{\geq k}(V) = \{(i, j) \in V : j \geq k\}$, and for two vertical diagrams V_1 and V_2 with $V_1 \subseteq V_2$, define V_2/V_1 to be the set-theoretic difference $V_2 - V_1$. We say that V_1 and V_2 are the *inner shape* and the *outer shape* of V_2/V_1, respectively. See Figures 7 and 8.
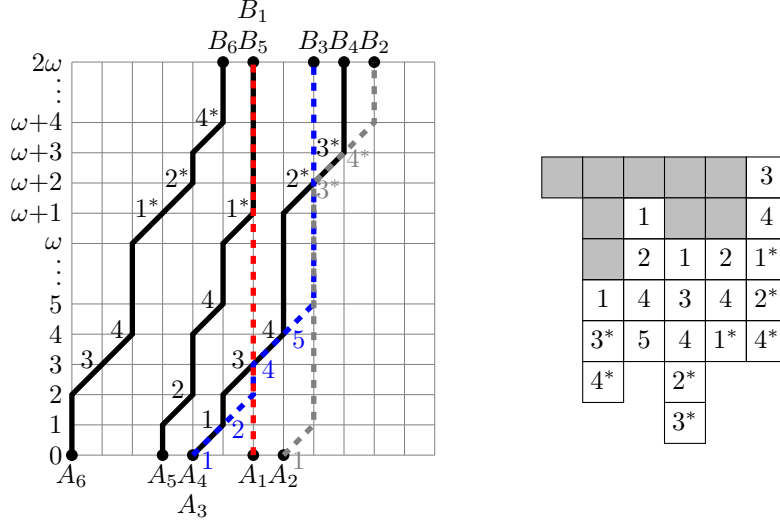
FIGURE 9. On the left is a 6-path $\mathbf{p} = (p_1, \ldots, p_6) \in \mathcal{L}(\alpha, \beta)$ for $\alpha = (1, 3, 1, 2, 2, 0)$ and $\beta = (1, 6, 5, 7, 5, 5)$. Each p_i is a path from $A_i = (\alpha_i + 6 - i, 0)$ to $B_i = (\beta_i + 6 - i, 2\omega)$. Its corresponding vertical tableau Tab($\mathbf{p}$) is shown on the right.

For vertical diagrams V_1 and V_2 with $V_1 \subseteq V_2$, a *vertical tableau* of shape V_2/V_1 is a filling of V_2/V_1 with numbers in $\{1 < 2 < \cdots < 1^* < 2^* < \cdots\}$ such that the entries are strictly increasing in each column. See the right diagram in Figure 9 for an example of a vertical tableau. Let VT(V_2/V_1) denote the set of vertical tableaux of shape V_2/V_1.

Definition 3.1. [The map Tab sending n-paths to vertical tableaux] Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be compositions with $V(\alpha) \subseteq V(\beta)$. Define $\mathcal{L}(\alpha, \beta)$ to be the set of n-paths $\mathbf{p} = (p_1, \ldots, p_n)$, where p_i is a path from $(\alpha_i + n - i, 0)$ to $(\beta_i + n - i, 2\omega)$.

For $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}(\alpha, \beta)$, define Tab($\mathbf{p}$) to be the vertical tableau $T \in \text{VT}(V(\beta)/V(\alpha))$ constructed as follows. For each diagonal step of p_i, if its ending point is (a, b), fill the $(a - n + i - 1, i)$-entry of T with b.

See Figure 9 for an example of the map Tab in Definition 3.1. The following proposition is straightforward to verify.

Proposition 3.2. *Following the notation in Definition 3.1, the map* Tab *is a bijection from* $\mathcal{L}(\alpha, \beta)$ *to* VT($V(\beta)/V(\alpha)$). *Moreover, if* Tab($\mathbf{p}$) = T, *then for every positive integer h the total number of diagonal steps in $\mathbf{p}$ ending at height h (resp. $\omega + h$) is equal to the number of times h (resp. h^*) appears in T.*

For a partition λ with $\ell(\lambda') = n$ and a permutation $\pi \in \mathfrak{S}_n$, we define $\pi(\lambda)$ to be the vertical diagram given by

$$\pi(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \le j \le n, 1 \le i \le \lambda'_{\pi_j} - \pi_j + j\}.$$

Note that if π is the identity permutation then $\pi(\lambda)$ is the Young diagram of λ. One may worry about the situation that $\lambda'_{\pi_j} - \pi_j + j < 0$ in the definition of $\pi(\lambda)$. Since we will only consider $\pi(\lambda)$ when VT($\pi(\lambda)/\mu$) is nonempty (or equivalently, when $\mu \subseteq \pi(\lambda)$) this will never occur, see the paragraph after the proof of Lemma 3.3.

The following lemma shows that the type of $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}$ is encoded in the outer shape of the vertical tableau Tab($\mathbf{p}$) while the inner shape of Tab($\mathbf{p}$) is always μ. See Figure 18 for an example.

Lemma 3.3. *For $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}$, we have* type($\mathbf{p}$) = π *if and only if* Tab($\mathbf{p}$) $\in$ VT($\pi(\lambda)/\mu$).

Proof. Suppose that $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu}$ has type($\mathbf{p}$) = π. Let α and β be the compositions given by $\alpha_i = \mu'_i$ and $\beta_i = \lambda'_{\pi_i} - \pi_i + i$. Then $V(\beta)/V(\alpha) = \pi(\lambda)/\mu$. Since p_i is a path from

R (left):

1	1	1	2	3
1	1	2	3	3
1	2	2	3	4	$\star$
2	3
3	4

E (middle, shape λ/ν):

		3*	3*
		4*	4*
3*	5*

$T = (R,E)$ (right):

1	1	1	2	3
1	1	2	3	3
1	2	2	3	4	$\star$
2	3	3*	3*
3	4	4*	4*
3*	5*

FIGURE 10. An RPP R of shape ν on the left with row 3 marked, an elegant tableau E of shape λ/ν in the middle, and the RSE-tableau $T = (R,E) \in \mathrm{RSE}_3(\lambda)$ of level 3 and shape λ on the right, where $\lambda = (5,5,5,4,4,2)$ and $\nu = (5,5,5,2,2)$. Note that $\mathrm{row}_{\geq 3}(R)$ is an SSYT.

$(\mu'_i + n - i, 0) = (\alpha_i + n - i, 0)$ to $(\lambda'_{\pi_i} + n - \pi_i, 2\omega) = (\beta_i + n - i, 2\omega)$, we have $\mathbf{p} \in \mathrm{Tab}(\beta/\alpha)$. By Proposition 3.2, $\mathrm{Tab}(\mathbf{p}) \in \mathrm{VT}(V(\beta)/V(\alpha)) = \mathrm{VT}(\pi(\lambda)/\mu)$.

Conversely, suppose that $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}$ satisfies $\mathrm{Tab}(\mathbf{p}) \in \mathrm{VT}(\pi(\lambda)/\mu)$. Let $\mathrm{type}(\mathbf{p}) = \sigma$. Then by what we just proved, we obtain $\mathrm{Tab}(\mathbf{p}) \in \mathrm{VT}(\sigma(\lambda)/\mu)$, which implies $\sigma(\lambda) = \pi(\lambda)$, or equivalently,

$$(\lambda'_{\pi_1} - \pi_1 + 1, \ldots, \lambda'_{\pi_n} - \pi_n + n) = (\lambda'_{\sigma_1} - \sigma_1 + 1, \ldots, \lambda'_{\sigma_n} - \sigma_n + n).$$

By subtracting i from the ith component we also have

$$(\lambda'_{\pi_1} - \pi_1, \ldots, \lambda'_{\pi_n} - \pi_n) = (\lambda'_{\sigma_1} - \sigma_1, \ldots, \lambda'_{\sigma_n} - \sigma_n).$$

Both sequences in the above equation are rearrangements of $(\lambda'_1 - 1, \ldots, \lambda'_n - n)$, which is a strictly decreasing sequence. Since there are no repeated entries in this sequence, the rearrangements must be identical and we obtain $\pi = \sigma$. Hence $\mathrm{type}(\mathbf{p}) = \pi$ and the proof is completed. $\square$

Note that in the proof of the above lemma if $\mathrm{type}(\mathbf{p}) = \pi$, we must have $\lambda'_{\pi_i} + n - \pi_i \geq \mu'_i + n - i$, or, equivalently, $\lambda'_{\pi_i} - \pi_i + i \geq \mu'_i \geq 0$. Hence in this case we always have $\lambda'_{\pi_j} - \pi_j + j \geq 0$ in the definition of $\pi(\lambda)$.

4. RSE-tableaux and bijections of Lam and Pylyavskyy

In this section we define RSE-tableaux and two maps ϕ_- and ϕ_+ on these objects. The notion of RSE-tableaux was introduced implicitly by Lam and Pylyavskyy [9, Proof of Theorem 9.8] in their bijection between RPPs and pairs of SSYTs and elegant tableaux. The maps ϕ_- and ϕ_+ described in this sections are intermediate steps in their bijection. We assume reader's familiarity with the RSK algorithm and its basic properties. See [4, Section 1.1] or [14, Section 7.11] for a standard reference. In particular, we will use Row Bumping Lemma and Proposition in [4, Section 1.1].

Definition 4.1. An *RSE-tableau of shape λ of level k* is a pair $T = (R, E)$ satisfying the following conditions:

- $R \in \mathrm{RPP}(\nu)$ with row k of R marked, for some partition $\nu \subseteq \lambda$ satisfying $\mathrm{row}_{\leq k}(\nu) = \mathrm{row}_{\leq k}(\lambda)$,
- $\mathrm{row}_{\geq k}(R)$ is an SSYT, and
- E is an elegant tableau of shape λ/ν such that every entry in E is at least k.

The set of RSE-tableaux of shape λ and level k is denoted by $\mathrm{RSE}_k(\lambda)$.

We will represent an RSE-tableau $T = (R, E)$ as the tableau obtained by drawing both R and E in which every entry i in E is written as i^*. See Figure 10 for an example of an RSE-tableau.

By definition, if $T = (R, E)$ is an RSE-tableau of level k, then $\mathrm{row}_{\leq k}(R)$, $\mathrm{row}_{\geq k}(R)$, and E are, respectively, an RPP, an SSYT, and an elegant tableau. The name "RSE" stands for the initials of these three tableaux.
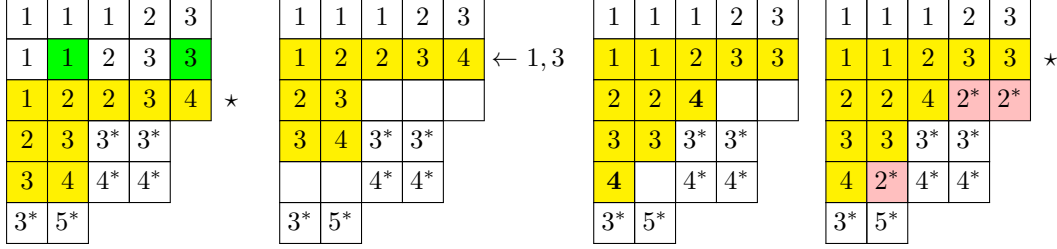
FIGURE 11. An illustration of the level-decreasing map ϕ_- applied to $T = (R, E) \in \mathrm{RSE}_3(\lambda)$, where $\lambda = (5, 5, 5, 4, 4, 2)$. The first diagram shows T where the novel entries are colored green and $\mathrm{row}_{\geq 3}(R)$ is colored yellow. If we remove row 2 and shift $\mathrm{row}_{\geq 3}(R)$ up by one, we get R' in the second diagram. If we insert the novel entries $1, 3$ into $\mathrm{row}_{\geq 2}(R')$, we get the third diagram, where the entries in the newly added cells are written in boldface. By filling the empty cells with 2^* we obtain the fourth diagram, which is $\phi_-(T) \in \mathrm{RSE}_2(\lambda)$.

Note that if $T = (R, E) \in \mathrm{RSE}_1(\lambda)$, then both R and E are SSYTs. Thus T can be considered as an SSYT whose entries are from $\{1, 2, \ldots, 1^*, 2^*, \ldots\}$. Using this observation the following proposition is easy to verify.

Proposition 4.2. *The map* Tab *is a weight-preserving bijection between* $\mathcal{L}^{\mathrm{NC}}_{\lambda/\emptyset}$ *and* $\mathrm{RSE}_1(\lambda)$.

If $T = (R, E) \in \mathrm{RSE}_\ell(\lambda)$, then $E = \emptyset$ and R is an RPP of shape λ with no extra conditions. Hence, we will identify $\mathrm{RSE}_\ell(\lambda)$ with $\mathrm{RPP}(\lambda)$.

The *weight* of $T = (R, E) \in \mathrm{RSE}_k(\lambda)$ is defined by

$$\mathrm{wt}(T) = \mathrm{wt}(R) t_E,$$

where $t_E = t_1^{c_1(E)} t_2^{c_2(E)} \cdots$ and $c_i(E)$ is the number of i's in E. For example, if $T = (R, E)$ is the RSE-tableau in Figure 10, then $\mathrm{wt}(R) = x_1^3 x_2^4 x_3^4 x_4^4 t_1^3 t_2^3$, $t_E = t_3^3 t_4^2 t_5$, and $\mathrm{wt}(T) = \mathrm{wt}(R) t_E = x_1^3 x_2^4 x_3^4 x_4^2 t_1^3 t_2^3 t_3^3 t_4^2 t_5$.

We now describe two maps ϕ_- and ϕ_+ on RSE-tableaux, where the level of an RSE-tableau is decreased by ϕ_- and increased by ϕ_+. These maps are due to Lam and Pylyavskyy [9] who used them as intermediate steps in their bijection between $\mathrm{RPP}(\lambda)$ and $\mathrm{RSE}_1(\lambda)$. See Figures 11 and 12 for illustrations of these maps.

Definition 4.3. [The level-decreasing map $\phi_- : \mathrm{RSE}_{k+1}(\lambda) \to \mathrm{RSE}_k(\lambda)$] Let λ be a partition with $\ell(\lambda) = \ell$ and let $T = (R, E) \in \mathrm{RSE}_{k+1}(\lambda)$ with $1 \leq k \leq \ell - 1$. Then $\phi_-(T)$ is defined as follows.

   Step 1: For $1 \leq j \leq \lambda_k$, the entry $R(k, j)$ is *novel* if $j > \lambda_{k+1}$ or $R(k, j) \neq R(k + 1, j)$. Let $a_1 \leq a_2 \leq \cdots \leq a_r$ be the novel entries. Let R' be the tableau obtained from R by removing row k and shifting $\mathrm{row}_{\geq k+1}(R)$ up by one (so that $\mathrm{row}_{\geq k}(R') = \mathrm{row}_{\geq k+1}(R)$). Then $H := \mathrm{sh}(R)/\mathrm{sh}(R')$ is a horizontal strip.

   Step 2: Update R' by inserting $a_1, a_2, \ldots, a_r$ in this order into $\mathrm{row}_{\geq k}(R')$ using the RSK algorithm. By the property of the RSK algorithm, if a_i was the novel entry in column j, then a_i bumps the (k, j)-entry of $\mathrm{row}_{\geq k}(R')$ (in case it exists) or a_i is simply placed at position (k, j) (in case $\mathrm{row}_{\geq k}(R')$ has no (k, j)-entry). Therefore row k of R' becomes the original row k of R and the newly created cells of R' lie in the horizontal strip H. Let E' be the union of E and the remaining empty cells in H, which we fill with k's. Finally, mark row k of R' as the level and define $\phi_-(T) = (R', E')$.

Definition 4.4. [The level-increasing map $\phi_+ : \mathrm{RSE}_k(\lambda) \to \mathrm{RSE}_{k+1}(\lambda)$] Let λ be a partition with $\ell(\lambda) = \ell$ and let $T = (R, E) \in \mathrm{RSE}_k(\lambda)$ with $1 \leq k \leq \ell - 1$. Then $\phi_+(T)$ is defined as follows.

   Step 1: Let $c_1 < c_2 < \cdots < c_r$ be the column indices j such that column j of E does not contain k. Let E' be the tableau obtained from E by removing the cells containing k. For
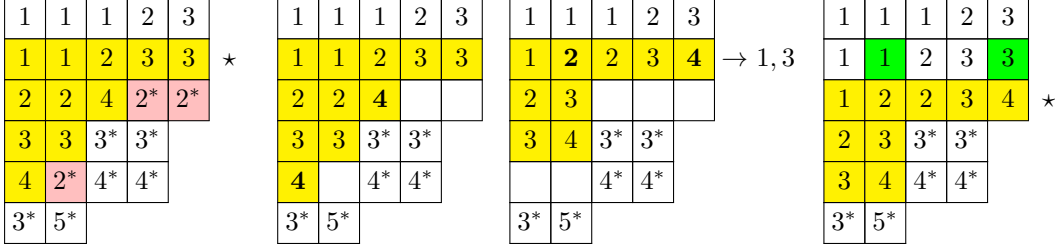
FIGURE 12. An illustration of the level-increasing map ϕ_+ applied to $T = (R, E) \in \mathrm{RSE}_2(\lambda)$, where $\lambda = (5, 5, 5, 4, 4, 2)$. The first diagram shows T, where $\mathrm{row}_{\geq 2}(R)$ is colored yellow and the cells with a 2^* are colored pink. If we remove the cells with a 2^* we get the second diagram, where the entry of the last cell in each column without a 2^* is written in boldface. If we perform the reverse RSK algorithm to $\mathrm{row}_{\geq 2}(R)$ starting from each boldface entry from right to left, we obtain R' in the third diagram, where 1 and 3 are bumped from columns 2 and 5 respectively. By shifting $\mathrm{row}_{\geq 2}(R')$ down by one, putting the bumped entries in the corresponding columns in row 2, and filling each empty cell with the same entry directly below it, we obtain the fourth diagram, which is $\phi_+(T) \in \mathrm{RSE}_3(\lambda)$.

$i = r, r - 1, \ldots, 1$ in this order, apply the reverse RSK algorithm to $\mathrm{row}_{\geq k}(R)$ starting from the last cell of column c_i and denote the resulting tableau by R_1. Let a_i and b_i be the integers such that the reverse RSK algorithm bumps a_i at position (k, b_i) at the end.

Step 2: Let R' be the tableau obtained from R by replacing $\mathrm{row}_{\geq k}(R)$ by R_1. Shift $\mathrm{row}_{\geq k}(R')$ down by one so that row k of R' is now empty. For each $1 \leq j \leq \lambda_k$, if $j = b_i$ for some i, then let $R'(k, j) = a_i$, and otherwise let $R'(k, j)$ equal $R'(k + 1, j)$. Finally, mark row $k + 1$ of R' as the level and define $\phi_+(T) = (R', E')$.

The following proposition is shown in [9, Proof of Theorem 9.8].

Proposition 4.5. *Let λ be a partition with $\ell(\lambda) = \ell$. For $1 \leq k \leq \ell - 1$, the maps $\phi_+ : \mathrm{RSE}_k(\lambda) \to \mathrm{RSE}_{k+1}(\lambda)$ and $\phi_- : \mathrm{RSE}_{k+1}(\lambda) \to \mathrm{RSE}_k(\lambda)$ are weight-preserving bijections and they are mutual inverses.*

As a corollary to Proposition 4.5, we obtain that the map $\phi_-^{\ell-1} : \mathrm{RSE}_\ell(\lambda) \to \mathrm{RSE}_1(\lambda)$ is a weight-preserving bijection. Since we can identify $\mathrm{RSE}_\ell(\lambda)$ with $\mathrm{RPP}(\lambda)$, it follows that

$$\widetilde{g}_\lambda(x; t) = \sum_{R \in \mathrm{RPP}(\lambda)} \mathrm{wt}(R) = \sum_{T \in \mathrm{RSE}_\ell(\lambda)} \mathrm{wt}(T) = \sum_{T \in \mathrm{RSE}_1(\lambda)} \mathrm{wt}(T).$$

Note that since $\mathcal{L}_{\lambda/\mu}^{\mathrm{NC}} = \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ for $\mu = \emptyset$, Proposition 2.2 shows

$$\det\left(e_{\lambda'_i - i + j}(x_1, x_2, \ldots, t_1, t_2, \ldots, t_{\lambda'_i - 1})\right)_{1 \leq i, j \leq n} = \sum_{\mathbf{p} \in \mathcal{L}_\lambda^{\mathrm{NC}}} \mathrm{wt}(\mathbf{p}).$$

By Proposition 4.2,

$$\sum_{\mathbf{p} \in \mathcal{L}_\lambda^{\mathrm{NC}}} \mathrm{wt}(\mathbf{p}) = \sum_{T \in \mathrm{RSE}_1(\lambda/\mu)} \mathrm{wt}(T).$$

Combining the above three equations we obtain (1.1). This proof is essentially the same as Yeliussizov's [15, §10.1].

5. SKEW RSE-TABLEAUX

In this section we extend the definition of RSE-tableaux to skew shapes and study properties of the maps ϕ_- and ϕ_+ on them. We first need to extend the definition of elegant tableaux.

Let μ be a partition. A μ-*elegant tableau* is an SSYT E of a certain skew shape λ/ν with $\mu \subseteq \nu$ such that $\mu'_j + 1 \leq E(i, j) \leq i - 1$ for all $(i, j) \in \lambda/\nu$. Note that the $\emptyset$-elegant tableaux are the usual elegant tableaux.

FIGURE 13. A skew RSE-tableau $T \in \mathrm{RSE}_2(\lambda/\mu)$ for $\lambda = (6, 5, 4, 4, 2)$ and $\mu = (4, 3, 1)$ on the left, and its corresponding tableau $\overline{T} \in \overline{\mathrm{RSE}}_2(\lambda/\mu)$ on the right, where the cells in μ are colored gray.

Definition 5.1. A *(skew) RSE-tableau of shape λ/μ of level k* is a pair $T = (R, E)$ satisfying the following conditions:

- $R \in \mathrm{RPP}(\nu/\mu)$ with row k of R marked, for some partition ν satisfying $\mu \subseteq \nu \subseteq \lambda$ and $\mathrm{row}_{\leq k}(\nu) = \mathrm{row}_{\leq k}(\lambda)$,
- $\mathrm{row}_{\geq k}(R)$ is an SSYT, and
- E is a μ-elegant tableau of shape λ/ν such that every entry in E is at least k.

The set of RSE-tableaux of shape λ/μ and level k is denoted by $\mathrm{RSE}_k(\lambda/\mu)$.

The *weight* of $T = (R, E) \in \mathrm{RSE}_k(\lambda/\mu)$ is defined in the same way by

$$\mathrm{wt}(T) = \mathrm{wt}(R)t_E.$$

For example, if T is the skew RSE-tableau in Figure 13, then $\mathrm{wt}(R) = x_1^4 x_2^3 x_3^2 t_1$, $t_E = t_3^2 t_4$, and $\mathrm{wt}(T) = \mathrm{wt}(R)t_E = x_1^4 x_2^3 x_3^2 t_1 t_3^2 t_4$.

We also define $\overline{\mathrm{RSE}}_k(\lambda)$ to be the set of RSE-tableaux (R, E) of shape λ and level k, where the entries of R are taken from

$$\tag{5.1} \{\overline{1} < \overline{2} < \cdots < 1 < 2 < \cdots\},$$

while the entries of E are still positive integers.

For $T = (R, E) \in \mathrm{RSE}_k(\lambda/\mu)$, let $\overline{T}$ be the RSE-tableau $(\overline{R}, E) \in \overline{\mathrm{RSE}}_k(\lambda)$, where $\overline{R}$ is obtained from R by filling the cells in row i of μ with $\overline{i}$'s for each $1 \leq i \leq \ell(\mu)$. We will identify T with $\overline{T}$ so that $\mathrm{RSE}_k(\lambda/\mu) \subseteq \overline{\mathrm{RSE}}_k(\lambda)$. By replacing each entry i in E by i^* and putting $\overline{R}$ and E together we will also consider $T = \overline{T} = (\overline{R}, E) \in \mathrm{RSE}_k(\lambda/\mu)$ as a tableau of shape λ whose entries are taken from

$$\tag{5.2} \{\overline{1} < \overline{2} < \cdots < 1 < 2 < \cdots < 1^* < 2^* < \cdots\}.$$

We call the elements in (5.2) the *extended integers*. See Figure 13 for an example of this correspondence. Sometimes we will also consider $T \in \mathrm{RSE}_k(\lambda/\mu)$ as an RPP of shape λ whose entries are extended integers. We call $\overline{i}$ a *negative entry* and i^* an *ω-entry*.

The following proposition is immediate from the definition of $\mathrm{RSE}_k(\lambda/\mu)$.

Proposition 5.2. *Let $T \in \overline{\mathrm{RSE}}_k(\lambda)$. Then $T \in \mathrm{RSE}_k(\lambda/\mu)$ if and only if the following conditions hold:*

(1) *The cells containing a negative entry are exactly those in μ.*
(2) *If $(i, j) \in \mu$, then $T(i, j) = \overline{i}$.*
(3) *If $T(i, j) = a^*$, then $\mu_j' + 1 \leq a \leq \lambda_j' - 1$.*

Note that if $T = (R, E) \in \mathrm{RSE}_1(\lambda/\mu)$ then we can regard T as an SSYT of shape λ/μ whose entries are from $\{1, 2, \ldots, 1^*, 2^*, \ldots\}$. Using this observation, similarly to Proposition 4.2, the following proposition is easy to verify.

Proposition 5.3. *The map* Tab *is a weight-preserving bijection between $\mathcal{L}_{\lambda/\mu}^{\mathrm{NC}}$ and $\mathrm{RSE}_1(\lambda/\mu)$.*

If $T = (R, E) \in \mathrm{RSE}_\ell(\lambda/\mu)$, then by definition of an RSE-tableau, we must have $E = \emptyset$ and R can be any RPP of shape λ/μ whose entries are positive integers. Hence, we will identify $\mathrm{RSE}_\ell(\lambda/\mu)$ with $\mathrm{RPP}(\lambda/\mu)$.

FIGURE 14. If $T_1 \in \mathrm{RSE}_2(\lambda/\mu)$ and $T_2 \in \mathrm{RSE}_1(\lambda/\mu)$ are the left and right diagrams respectively, then $\phi_-(T_1) = T_2$ and $\phi_+(T_2) = T_1$.

FIGURE 15. An RPP T_1 on the left, an RPP T_2 in the middle, and $T_1 \sqcup T_2$ on the right. Since $T_1 \sqcup T_2$ is also an RPP, we have $T_1 \leq T_2$.

FIGURE 16. An RSE-tableau $T \in \mathrm{RSE}_1((2,2)/(1))$ on the left and its image $\phi_+(T) \in \overline{\mathrm{RSE}}_2((2,2))$ on the right. Note that $\phi_+(T) \notin \mathrm{RSE}_2((2,2)/(1))$.

Using the ordering given by (5.1), the same maps ϕ_- and ϕ_+ are applied to $\overline{\mathrm{RSE}}_k(\lambda)$. By the identification $\mathrm{RSE}_k(\lambda/\mu) \subseteq \overline{\mathrm{RSE}}_k(\lambda)$, these maps ϕ_- and ϕ_+ are also applied to $\mathrm{RSE}_k(\lambda/\mu)$. See Figure 14 for an example.

The following definitions will be used frequently for the rest of this paper. See Figure 15 for an example.

Definition 5.4. Let T_1 and T_2 be RPPs whose entries are extended integers. Define $T_1 \sqcup T_2$ to be the tableau obtained by concatenating T_1 and T_2, i.e., $\mathrm{col}_{\leq k}(T_1 \sqcup T_2) = T_1$ and $\mathrm{col}_{\geq k+1}(T_1 \sqcup T_2) = T_2$, where k is the number of columns in T_1. Define $T_1 \leq T_2$ if $T_1 \sqcup T_2$ is also an RPP (with extended integers).

Note that if T_1 and T_2 are RSE-tableaux of levels k_1 and k_2, respectively, with $k_1 \leq k_2$ such that $T_1 \leq T_2$, then $T_1 \sqcup T_2$ with row k_2 marked is an RSE-tableau of level k_2.

In Lemma 5.5 below we will show that ϕ_- is a map from $\mathrm{RSE}_k(\lambda/\mu)$ to $\mathrm{RSE}_{k-1}(\lambda/\mu)$, i.e., for every $T \in \mathrm{RSE}_k(\lambda/\mu)$ we have $\phi_-(T) \in \mathrm{RSE}_{k-1}(\lambda/\mu)$. On the contrary, ϕ_+ does not always send an element in $\mathrm{RSE}_{k-1}(\lambda/\mu)$ to an element in $\mathrm{RSE}_k(\lambda/\mu)$, see Figure 16. We will find equivalent conditions for $T \in \mathrm{RSE}_{k-1}(\lambda/\mu)$ to satisfy $\phi_+(T) \in \mathrm{RSE}_k(\lambda/\mu)$ in Lemma 5.6.

Lemma 5.5. Let $T \in \mathrm{RSE}_{k+1}(\lambda/\mu)$. Then $\phi_-(T) \in \mathrm{RSE}_k(\lambda/\mu)$ and, for all $1 \leq s \leq \mu_k$,

$$(5.3) \qquad \phi_-(T) = \mathrm{col}_{\leq s}(T) \sqcup \phi_-(\mathrm{col}_{\geq s+1}(T)).$$

Proof. Let $T = (R, E)$. Since $T \in \mathrm{RSE}_{k+1}(\lambda/\mu) \subseteq \overline{\mathrm{RSE}}_{k+1}(\lambda)$, we have $\phi_-(T) \in \overline{\mathrm{RSE}}_k(\lambda)$. In order to show $\phi_-(T) \in \mathrm{RSE}_k(\lambda/\mu)$, we must show that $\phi_-(T)$ satisfies the three conditions in Proposition 5.2. To this end we first prove the following claim, which is equivalent to (5.3).

Claim: If $1 \leq s \leq \mu_k$, then $\mathrm{col}_{\leq s}(\phi_-(T)) = \mathrm{col}_{\leq s}(T)$ and $\mathrm{col}_{\geq s+1}(\phi_-(T)) = \phi_-(\mathrm{col}_{\geq s+1}(T))$.

The first s entries of row k in T are all $\overline{k}$ and every entry in row $k+1$ is either $\overline{k+1}$ or a positive integer. Thus the first s entries are novel entries. In the definition of ϕ_- we delete row k of R and insert the novel entries into $\text{row}_{\geq k+1}(R)$ (after shifting it up by one). Since $\overline{k}$ is smaller than every entry in $\text{row}_{\geq k+1}(R)$, each of the first s insertion paths is a straight vertical path. Since insertion paths never intersect, the first s columns are not changed after the insertion of the first s $\overline{k}$'s. This shows the first identity of the claim. The fact that the insertion paths starting from columns of index greater than μ_k never enter columns of index at least μ_k also implies the second identity of the claim.

We now show that $\phi_-(T)$ satisfies the three conditions in Proposition 5.2. For the first two conditions it is enough to show that the restrictions of T and $\phi_-(T)$ to μ are equal because ϕ_- preserves the total number of negative entries. Since μ is contained in $\text{row}_{\leq k}(\lambda) \cup \text{col}_{\leq \mu_k}(\lambda)$, this follows from the special case $\text{col}_{\leq \mu_k}(\phi_-(T)) = \text{col}_{\leq \mu_k}(T)$ of the claim and the fact $\text{row}_{\leq k}(\phi_-(T)) = \text{row}_{\leq k}(T)$. For the third condition note that in the construction of $\phi_-(T) = (R', E')$ from $T = (R, E)$, E' is obtained from E by adding some k^*'s. Suppose $\phi_-(T)(i,j) = a^*$. If $a \neq k$, then we must have $T(i,j) = a^*$. Since $T \in \text{RSE}_{k+1}(\lambda/\mu)$, in this case $\mu'_j + 1 \leq a \leq \lambda'_j - 1$. If $a = k$, we must have $j \geq \mu_k + 1$ since $\text{col}_{\leq \mu_k}(\phi_-(T)) = \text{col}_{\leq \mu_k}(T)$ and T has no k^*. But $j \geq \mu_k + 1$ implies that $\mu'_j < k$, and $\phi_-(T) \in \overline{\text{RSE}}_k(\lambda/\mu)$ implies $k \leq \lambda'_j - 1$. Hence the third condition also holds and the proof is completed. $\square$

Lemma 5.6. *Let $T \in \text{RSE}_k(\lambda/\mu)$. Then the following are equivalent:*

(1) $\phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$,
(2) $T \in \phi_-(\text{RSE}_{k+1}(\lambda/\mu))$,
(3) $\phi_+(T) = \text{col}_{\leq s}(T) \sqcup \phi_+(\text{col}_{\geq s+1}(T))$, *for all* $1 \leq s \leq \mu_k$, *and*
(4) $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$.

Proof. We will prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Let $T' = \phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$. Then $T = \phi_-(T') \in \phi_-(\text{RSE}_{k+1}(\lambda/\mu))$.

$(2) \Rightarrow (3)$: Suppose $T = \phi_-(T')$ for some $T' \in \text{RSE}_{k+1}(\lambda/\mu)$. Then $T' = \phi_+(T)$. We need to show that for $1 \leq s \leq \mu_k$,

$$\text{col}_{\leq s}(T') = \text{col}_{\leq s}(T),$$
$$\text{col}_{\geq s+1}(T') = \phi_+(\text{col}_{\geq s+1}(T)).$$

By Lemma 5.5, $T = \phi_-(T') = \text{col}_{\leq s}(T') \sqcup \phi_-(\text{col}_{\geq s+1}(T'))$. This shows that $\text{col}_{\leq s}(T) = \text{col}_{\leq s}(T')$, which is the first equality, and $\text{col}_{\geq s+1}(T) = \phi_-(\text{col}_{\geq s+1}(T'))$, which is equivalent to the second equality after applying ϕ_+.

$(3) \Rightarrow (4)$: The fact that $\phi_+(T) = \text{col}_{\leq \mu_k}(T) \sqcup \phi_+(\text{col}_{\geq \mu_k+1}(T))$ is an RPP (with extended integers as entries) shows that $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$.

$(4) \Rightarrow (1)$: Suppose that $T = (R, E) \in \text{RSE}_k(\lambda/\mu)$ satisfies $\text{col}_{\leq \mu_k}(T) \leq \phi_+(\text{col}_{\geq \mu_k+1}(T))$. To show $\phi_+(T) \in \text{RSE}_{k+1}(\lambda/\mu)$ we must show that $\phi_+(T)$ satisfies the three conditions in Proposition 5.2. Since the restriction of $\phi_+(T)$ to the ω-entries is exactly the same as that of T with k^* deleted and $T \in \text{RSE}_{k+1}(\lambda/\mu)$ satisfies the third condition, so does $\phi_+(T)$. For the first two conditions, it is enough to show that the restrictions of T and $\phi_+(T)$ to μ are the same. Since μ is contained in $\text{row}_{\leq k}(\lambda) \cup \text{col}_{\geq \mu_k+1}(\lambda)$ and $\text{row}_{\leq k}(\phi_+(T)) = \text{row}_{\leq k}(T)$, it suffices to prove the following equality:

(5.4) $$\text{col}_{\leq \mu_k}(\phi_+(T)) = \text{col}_{\leq \mu_k}(T).$$

To show (5.4) we investigate the construction of $\phi_+(T)$ in Definition 4.4. Let $c_1 < c_2 < \cdots < c_r$ be the column indices j such that column j of E does not contain k. Since $(R, E) \in \text{RSE}_k(\lambda/\mu)$, E is a μ-elegant tableau. Thus for every cell (i,j) of E with $1 \leq j \leq \mu_k$ we have $E(i,j) \geq \mu'_j + 1 \geq k + 1$. This shows that E has no entries equal to k in the first μ_k columns, i.e., $c_j = j$ for all $1 \leq j \leq \mu_k$.

Recall that in the definition of $\phi_+(T)$ we apply the reverse RSK algorithm to $\text{row}_{\geq k}(R)$ starting from the last cell of column c_j for $j = r, r-1, \ldots, 1$ in this order. We denote by P_j each inverse bumping path.

Claim: $P_{\mu_k}, P_{\mu_k-1}, \ldots, P_1$ are straight vertical paths.

By the construction of $\phi_+(T)$ the claim implies (5.4). Hence it suffices to prove the claim. To this end let Q be the tableau obtained from $\mathrm{row}_{\geq k}(R)$ by applying the reverse RSK algorithm to the last cell of column c_j for $j = r, r-1, \ldots, \mu_k + 1$. Note that the same process is applied to $\mathrm{row}_{\geq k}(\mathrm{col}_{\geq \mu_k+1}(T))$ when we compute $(R_1, E_1) = \phi_+(\mathrm{col}_{\geq \mu_k+1}(T))$. Since $\mathrm{row}_{\geq k+1}(R_1)$ has been shifted down in Step 2 of the definition of ϕ_+, we have

$$\text{(5.5)} \qquad\qquad \mathrm{row}_{\geq k+1}(R_1) = \mathrm{row}_{\geq k}(Q).$$

Suppose that $R_1(k, \mu_k + 1) < R_1(k+1, \mu_k + 1)$. This means that the entry $R_1(k, \mu_k + 1)$ was bumped during the applications of the reverse RSK algorithm. Since column $\mu_k + 1$ is the leftmost nonempty column of $\mathrm{col}_{\geq \mu_k+1}(T)$, the reverse bumping path that pushed this cell must be a straight vertical path and we have $c_{\mu_k+1} = \mu_k + 1$. Since each path P_j, for $1 \leq j \leq \mu_k + 1$, starts from column j and P_{μ_k+1} is a straight vertical path, by the non-intersecting property of the reverse bumping paths, the claim follows.

Suppose now that $R_1(k, \mu_k + 1) = R_1(k+1, \mu_k + 1)$. By the same reasoning as in the previous paragraph, it is sufficient to show that P_{μ_k} is a straight vertical path. By the assumption $\mathrm{col}_{\leq \mu_k}(T) \leq \phi_+(\mathrm{col}_{\geq \mu_k+1}(T))$, we have $\mathrm{col}_{\leq \mu_k}(R) \leq R_1$. This together with (5.5) implies that $R(i, \mu_k) \leq R_1(i, \mu_k + 1) = Q(i-1, \mu_k + 1)$ for all $i \geq k+1$. These inequalities and the assumption $R_1(k, \mu_k + 1) = R_1(k+1, \mu_k + 1)$ ensure that P_{μ_k} is a straight vertical path. This completes the proof of the claim. $\qquad\square$

The following two lemmas will be useful in the next section.

Lemma 5.7. *Suppose* $T \in \mathrm{RSE}_b(\mathrm{col}_{\geq c+1}(V/\mu))$, *where* b, c *are nonnegative integers,* V *is a vertical diagram, and* μ *is a partition with* $\mu \subseteq V$ *such that* $\mathrm{col}_{\geq c+1}(V/\mu)$ *is a skew shape and* $1 \leq c \leq \mu_b$. *Then for all* $1 \leq d \leq b-1$ *and* $1 \leq j \leq \mu_{b-1}$, *we have*

$$\text{(5.6)} \qquad\qquad \phi_-^d(T) \in \mathrm{RSE}_{b-d}(\mathrm{col}_{\geq c+1}(V/\mu)),$$

$$\text{(5.7)} \qquad\qquad \phi_-^d(T) = \mathrm{col}_{\leq j}(T) \sqcup \phi_-^d(\mathrm{col}_{\geq j+1}(T)).$$

In other words (5.7) *means that applying* ϕ_-^d *to* T *is the same thing as applying* ϕ_-^d *only to the columns of index greater than* j *while keeping* $\mathrm{col}_{\leq j}(T)$ *unmodified.*

Proof. Since $\mathrm{col}_{\geq c+1}(V/\mu)$ is a skew shape, applying Lemma 5.5 repeatedly gives (5.6).

We prove (5.7) by induction on d. If $d = 1$, it is just Lemma 5.5. Suppose that (5.7) is true for $1 \leq d \leq b-2$ and consider the $d+1$ case. Using Lemma 5.5 with (5.6) and the inequality $j \leq \mu_{b-1} \leq \mu_{b-d-1}$, we have

$$\text{(5.8)} \qquad\qquad \phi_-(\phi_-^d(T)) = \mathrm{col}_{\leq j}(\phi_-^d(T)) \sqcup \phi_-(\mathrm{col}_{\geq j+1}(\phi_-^d(T))).$$

Note that the induction hypothesis (5.7) for the d case is equivalent to

$$\mathrm{col}_{\leq j}(\phi_-^d(T)) = \mathrm{col}_{\leq j}(T), \qquad \mathrm{col}_{\geq j+1}(\phi_-^d(T)) = \phi_-^d(\mathrm{col}_{\geq j+1}(T)).$$

Hence (5.8) can be written as

$$\phi_-^{d+1}(T) = \mathrm{col}_{\leq j}(T) \sqcup \phi_-(\phi_-^d(\mathrm{col}_{\geq j+1}(T))) = \mathrm{col}_{\leq j}(T) \sqcup \phi_-^{d+1}(\mathrm{col}_{\geq j+1}(T)),$$

which is (5.7) for the $d+1$ case. This completes the proof. $\qquad\square$

Lemma 5.8. *Suppose* $T \in \phi_-^a(\mathrm{RSE}_b(\mathrm{col}_{\geq c+1}(V/\mu)))$, *where* a, b, c *are nonnegative integers,* V *is a vertical diagram, and* μ *is a partition with* $\mu \subseteq V$ *such that* $\mathrm{col}_{\geq c+1}(V/\mu)$ *is a skew shape and* $1 \leq c \leq \mu_b$. *Then for all* $1 \leq d \leq a$ *and* $1 \leq j \leq \mu_{b-a+d-1}$, *we have*

$$\text{(5.9)} \qquad\qquad \phi_+^d(T) \in \mathrm{RSE}_{b-a+d}(\mathrm{col}_{\geq c+1}(V/\mu)),$$

$$\text{(5.10)} \qquad\qquad \phi_+^d(T) = \mathrm{col}_{\leq j}(T) \sqcup \phi_+^d(\mathrm{col}_{\geq j+1}(T)).$$

In other words (5.10) *means that applying* ϕ_+^d *to* T *is the same thing as applying* ϕ_+^d *only to the columns of index greater than* j *while keeping* $\mathrm{col}_{\leq j}(T)$ *unmodified.*

Proof. Since ϕ_- and ϕ_+ are inverses of each other, the assumption $T \in \phi_-^a(\mathrm{RSE}_b(\mathrm{col}_{\geq c+1}(V/\mu))$ together with Lemma 5.7 implies that for all $1 \leq d \leq a$,

$$\phi_+^d(T) \in \phi_-^{a-d}(\mathrm{RSE}_b(\mathrm{col}_{\geq c+1}(V/\mu)) \subseteq \mathrm{RSE}_{b-a+d}(\mathrm{col}_{\geq c+1}(V/\mu)),$$

which shows (5.9). For the second statement, let $T' = \phi_+^d(T)$. Then (5.10) is equivalent to

(5.11) $$\mathrm{col}_{\leq j}(T') = \mathrm{col}_{\leq j}(T), \qquad \mathrm{col}_{\geq j+1}(T') = \phi_+^d(\mathrm{col}_{\geq j+1}(T)).$$

Since $T' = \phi_+^d(T) \in \mathrm{RSE}_{b-a+d}(\mathrm{col}_{\geq c+1}(V/\mu))$, by Lemma 5.7,

$$T = \phi_-^d(T') = \mathrm{col}_{\leq j}(T') \sqcup \phi_-^d(\mathrm{col}_{\geq j+1}(T')).$$

This shows that

$$\mathrm{col}_{\leq j}(T) = \mathrm{col}_{\leq j}(T'), \qquad \mathrm{col}_{\geq j+1}(T) = \phi_-^d(\mathrm{col}_{\geq j+1}(T')).$$

By taking ϕ_+^d in each side of the second equation we obtain (5.11), completing the proof. $\square$

Recall that at the end of the previous section we showed that $\phi_-^{\ell-1} : \mathrm{RSE}_\ell(\lambda) \to \mathrm{RSE}_1(\lambda)$ is a weight-preserving bijection and

$$\widetilde{g}_\lambda(x;t) = \sum_{R \in \mathrm{RPP}(\lambda)} \mathrm{wt}(R) = \sum_{T \in \mathrm{RSE}_\ell(\lambda)} \mathrm{wt}(T) = \sum_{T \in \mathrm{RSE}_1(\lambda)} \mathrm{wt}(T).$$

For the skew shape case the map $\phi_- : \mathrm{RSE}_k(\lambda/\mu) \to \mathrm{RSE}_{k-1}(\lambda/\mu)$ is not a bijection but just an injection.

Proposition 5.9. *Let λ and μ be partitions with $\mu \subseteq \lambda$ and $\ell(\lambda) = \ell$. Then, for $2 \leq k \leq \ell$, the map $\phi_- : \mathrm{RSE}_k(\lambda/\mu) \to \mathrm{RSE}_{k-1}(\lambda/\mu)$ is a weight-preserving injection. In other words, $\phi_- : \mathrm{RSE}_k(\lambda/\mu) \to \phi_-(\mathrm{RSE}_k(\lambda/\mu))$ is a weight-preserving bijection.*

Proof. This follows from Proposition 4.5 and Lemma 5.5. $\square$

The fact that $\phi_- : \mathrm{RSE}_k(\lambda/\mu) \to \mathrm{RSE}_{k-1}(\lambda/\mu)$ is an injection can still be used to give a different expression for $\widetilde{g}_{\lambda/\mu}(x;t)$.

Proposition 5.10. *Let λ and μ be partitions with $\mu \subseteq \lambda$ and $\ell(\lambda) = \ell$. Then*

$$\widetilde{g}_{\lambda/\mu}(x;t) = \sum_{T \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))} \mathrm{wt}(T).$$

Proof. By the identification of $\mathrm{RPP}(\lambda/\mu)$ and $\mathrm{RSE}_\ell(\lambda/\mu)$,

$$\widetilde{g}_{\lambda/\mu}(x;t) = \sum_{R \in \mathrm{RPP}(\lambda/\mu)} \mathrm{wt}(R) = \sum_{T \in \mathrm{RSE}_\ell(\lambda/\mu)} \mathrm{wt}(T).$$

Applying Proposition 5.9 repeatedly we obtain that $\phi_-^{\ell-1} : \mathrm{RSE}_\ell(\lambda/\mu) \to \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))$ is a weight-preserving bijection. Therefore

$$\sum_{T \in \mathrm{RSE}_\ell(\lambda/\mu)} \mathrm{wt}(T) = \sum_{T \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))} \mathrm{wt}(T),$$

and the proof follows. $\square$

By Propositions 2.2 and 5.10, in order to prove Theorem 1.1 it is sufficient to show the following proposition whose proof will be given in the next section.

Proposition 5.11. *Let λ and μ be partitions with $\mu \subseteq \lambda$, $\ell(\lambda) = \ell$, and $\ell(\lambda') = n$. Then*

$$\sum_{\mathbf{p} \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}} \mathrm{wt}(\mathbf{p}) = \sum_{T \in \phi_-^{n-1}(\mathrm{RSE}_\ell(\lambda/\mu))} \mathrm{wt}(T).$$

6. Sign-reversing involution

In this section we define a sign-reversing involution on $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ to prove Proposition 5.11. Recall Notation 2.1. For any tableau Q denote by $\mathrm{col}_{D_k}(Q)$ the part of Q consisting of column j for all $j \in D_k$. The following proposition is an immediate consequence of Proposition 5.3.

Proposition 6.1. *Let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{L}_{\lambda/\mu}$ and $T = \mathrm{Tab}(\mathbf{p})$. Then $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ if and only if each $\mathrm{col}_{D_k}(T)$ (with row 1 marked) is an RSE-tableau of level 1 (and of some skew shape).*

We now define the sign-reversing involution Φ on $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$. See Section 7 for a concrete example of the map Φ.

Definition 6.2 (The sign-reversing involution Φ on $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$). Let $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$. Then $\Phi(\mathbf{p})$ is defined as follows. Here we use the letters defined in Notation 2.1.

Step 1: Suppose $T = \mathrm{Tab}(\mathbf{p}) \in \mathrm{VT}(\pi(\lambda)/\mu)$ and write $T = T_{r+1} \sqcup \cdots \sqcup T_2 \sqcup T_1$, where each $T_i = \mathrm{col}_{D_i} T$ is considered as an RSE-tableau of level 1.

Step 2: Let $U_1 = T_1$. For $i = 1, 2, \ldots, r$, if U_i has been defined and $T_{i+1} \leq \phi_+^{m_i}(U_i)$, define U_{i+1} to be the RSE-tableau $T_{i+1} \sqcup \phi_+^{m_i}(U_i)$ with level $M_i + 1$.

Step 3: If U_{r+1} is defined, set $\Phi(\mathbf{p}) = \mathbf{p}$. Otherwise, let k be the smallest integer such that $T_{k+1} \not\leq \phi_+^{m_k}(U_k)$. In order to define $\Phi(\mathbf{p})$ we proceed as follows.

Step 3-1: Let $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ be the partition defined by

$$\gamma_i = \begin{cases} \lambda_i, & \text{if } 1 \leq i \leq M_k, \\ \min(\lambda_i, d_k), & \text{if } M_k + 1 \leq i \leq \ell. \end{cases}$$

Considering $\mathrm{row}_{\geq M_k+1}(T_{k+1} \sqcup \phi_+^{m_k}(U_k))$ as an element in $\mathrm{VT}(\pi(\lambda)/\gamma)$, let $\mathbf{q} = (q_1, \ldots, q_n)$ be the n-path such that

$$\mathrm{Tab}(\mathbf{q}) = \mathrm{row}_{\geq M_k+1}(T_{k+1} \sqcup \phi_+^{m_k}(U_k)).$$

Step 3-2: Let s be the largest integer such that column s of $\mathrm{row}_{\geq M_k+1}(\phi_+^{m_k}(U_k))$ is nonempty. Choose the intersection point (a, b) of q_i and q_j for $d_{k+1} + 1 \leq i, j \leq s$ in such a way that (b, a) is the largest in the lexicographic order. Let q_i' and q_j' be the paths obtained from q_i and q_j, respectively, by exchanging their subpaths after the intersection (a, b). Define $\mathbf{q}'$ to be the n-path $\mathbf{q}$ in which q_i and q_j are replaced by q_i' and q_j', respectively.

Step 3-3: Note that $\mathrm{row}_{D_k}(\mathrm{Tab}(\mathbf{q})) = \mathrm{row}_{\geq M_k+1}(T_{k+1})$ and $\mathrm{row}_{\geq d_k+1}(\mathrm{Tab}(\mathbf{q})) = \mathrm{row}_{\geq M_k+1}(\phi_+^{m_k}(U_k))$. Let $\widetilde{T}_{k+1}$ be the RSE-tableau of level 1 obtained from T_{k+1} by replacing $\mathrm{row}_{D_k}(\mathrm{Tab}(\mathbf{q}))$ by $\mathrm{row}_{D_k}(\mathrm{Tab}(\mathbf{q}'))$. Let $\widetilde{U}_k$ be the RSE-tableau of level $M_k + 1$ obtained from $\phi_+^{m_k}(U_k)$ by replacing replacing $\mathrm{row}_{\geq d_k+1}(\mathrm{Tab}(\mathbf{q}))$ by $\mathrm{row}_{\geq d_k+1}(\mathrm{Tab}(\mathbf{q}'))$. See Figure 17 for an illustration of the construction of $\widetilde{T}_{k+1}$ and $\widetilde{U}_k$. Let

$$T' = T_r \sqcup \cdots \sqcup T_{k+2} \sqcup \widetilde{T}_{k+1} \sqcup \phi_-^{M_k}(\widetilde{U}_k).$$

Finally, define $\Phi(\mathbf{p})$ to be the n-path $\mathbf{p}'$ satisfying $\mathrm{Tab}(\mathbf{p}') = T'$.

The main theorem in this section is as follows.

Theorem 6.3. *The map Φ is a sign-reversing involution on $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ whose fixed point set is*

$$\left\{ \mathbf{p} \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}} : \mathrm{Tab}(\mathbf{p}) \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu)) \right\}.$$

Note that Theorem 6.3 immediately implies Proposition 5.11, and hence completes the proof of Theorem 1.1. The rest of this section is devoted to proving Theorem 6.3. We will constantly use the notation in Definition 6.2.

We first show that Φ is a well-defined map on $\mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$. The only thing that needs to be checked is Step 3-3 in the construction of $\Phi(\mathbf{p})$. More precisely we must check the three assertions in the following lemma. One can see that these three assertions imply $\Phi(\mathbf{p}) = \mathbf{p}' \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ as follows. By the second assertion, we obtain that $\phi_-^{M_k}(\widetilde{U}_k)$ is an RSE-tableau of level 1. This together with
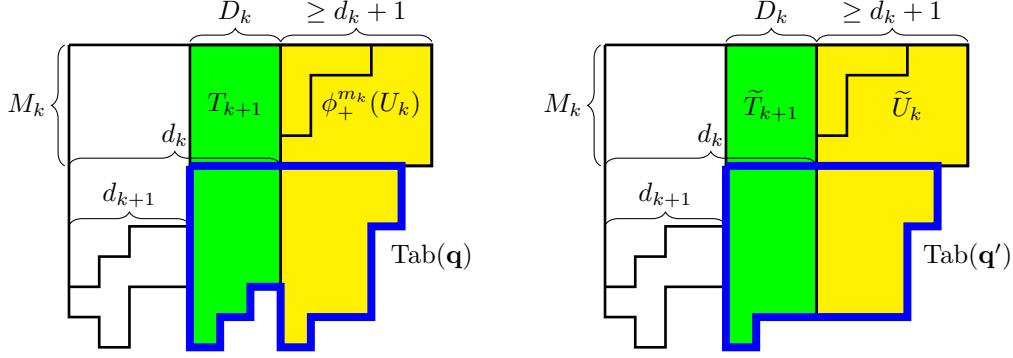
FIGURE 17. The construction of $\widetilde{T}_{k+1}$ and $\widetilde{U}_k$. The tableau $\widetilde{T}_{k+1} \sqcup \widetilde{U}_k$ is obtained from $T_{k+1} \sqcup \phi_+^{m_k}(U_k)$ by replacing $\mathrm{Tab}(\mathbf{q})$ by $\mathrm{Tab}(\mathbf{q}')$.

the first assertion implies that $\mathrm{col}_{D_k}(T')$ is an RSE-tableau of level 1 for all $1 \le k \le r+1$. Then Propositions 3.2, 6.1 and the third assertion imply that $\mathbf{p}' \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$.

Lemma 6.4. *Let $\mathbf{p} \in \mathcal{L}_{\lambda/\mu}^{\mathrm{SNC}}$ with $\mathrm{type}(\mathbf{p}) = \pi$. Suppose that U_{r+1} is not defined in the construction of $\Phi(\mathbf{p})$. Then*

   (1) *$\widetilde{T}_{k+1}$ is an RSE-tableau of level 1,*
   (2) *$\widetilde{U}_k$ is an RSE-tableau of level $M_k + 1$, and*
   (3) *$T' \in \mathrm{VT}(\pi'(\lambda)/\mu)$, where $\pi' = \pi(i,j)$ and (i,j) is the transposition exchanging i and j.*

Proof. Recall that $\mathbf{q}' = (q_1', \ldots, q_n')$ and $\mathbf{q} = (q_1, \ldots, q_n)$ differ only by the ith and jth paths, where q_i' and q_j' are obtained from q_i and q_j by exchanging the subpaths after the intersection (a,b). Suppose $i < j$ so that $i \in D_k$ and $j \ge d_k + 1$. The choice of the intersection point (a,b) in Step 3-2 guarantees that both $\{q_l' : d_{k+1} + 1 \le l \le d_k\}$ and $\{q_l' : d_k + 1 \le l \le s\}$ are non-intersecting. This implies that $\mathrm{col}_{D_k}(\mathrm{Tab}(\mathbf{q}'))$ and $\mathrm{col}_{\ge d_k+1}(\mathrm{Tab}(\mathbf{q}'))$ are SSYTs (whose entries are extended integers). Moreover, since $i < j$, the initial point $(\mu_j' + n - j, 0)$ of q_j is to the left of the initial point $(\mu_i' + n - i, 0)$ of q_i. Therefore the intersection (a,b) of q_i and q_j must occur after the first diagonal step of q_j. This shows that row $M_k + 1$ of $\mathrm{col}_{\ge d_k+1}(\mathrm{Tab}(\mathbf{q}'))$ is the same as that of $\mathrm{col}_{\ge d_k+1}(\mathrm{Tab}(\mathbf{q}))$.

Note that

$$\mathrm{row}_{\le M_k}(\widetilde{T}_{k+1}) = \mathrm{row}_{\le M_k}(T_{k+1}),$$
$$\mathrm{row}_{\le M_k}(\widetilde{U}_k) = \mathrm{row}_{\le M_k}(\phi_+^{m_k}(U_k)),$$
$$\mathrm{row}_{\ge M_k+1}(\widetilde{T}_{k+1}) = \mathrm{col}_{D_k}(\mathrm{Tab}(\mathbf{q}')),$$
$$\mathrm{row}_{\ge M_k+1}(\widetilde{U}_k) = \mathrm{col}_{\ge d_k+1}(\mathrm{Tab}(\mathbf{q}')),$$

where everything in the right-hand side is an SSYT (whose entries are extended integers) except $\mathrm{row}_{\le M_k}(\phi_+^{m_k}(U_k))$, which is an RPP. Thus checking the first and second assertions reduces to checking the following:

   (1) Rows M_k and $M_k + 1$ of $\widetilde{T}_{k+1}$ form an SSYT.
   (2) Rows M_k and $M_k + 1$ of $\widetilde{U}_k$ form an RPP.

The first statement is true because row M_k of $\widetilde{T}_{k+1}$ is empty (or equivalently filled with negative entries $\overline{M_k}$'s). The second statement is also true because rows M_k and $M_k + 1$ of $\widetilde{U}_k$ are identical with those of $\mathrm{col}_{\ge d_k+1}(\phi_-^{m_k}(U_k))$ by the last sentence of the previous paragraph.

For the final assertion recall that $\mathrm{Tab}(\mathbf{p}) \in \mathrm{VT}(\pi(\lambda)/\mu)$ and $\mathrm{Tab}(\mathbf{q}) \in \mathrm{VT}(\pi(\lambda)/\gamma)$. By Lemma 3.3, $\mathrm{type}(\mathbf{p}) = \mathrm{type}(\mathbf{q}) = \pi$. Since $\mathbf{q}'$ is obtained from $\mathbf{q}$ by changing the tails of q_i and q_j, we have $\mathrm{type}(\mathbf{q}') = \pi'$. Hence, by Lemma 3.3, $\mathrm{Tab}(\mathbf{q}') \in \mathrm{VT}(\pi'(\lambda)/\gamma)$. Since T' has the same inner shape as T and the same outer shape as $\mathrm{Tab}(\mathbf{q}')$, the third assertion follows. $\qquad\square$

The following lemma gives a more direct way of computing U_i.

Lemma 6.5. *Using the notation in Definition 6.2, for $1 \le i \le r+1$, if U_i is defined, then*

$$\phi_-^{M_{i-1}}(U_i) = \mathrm{col}_{\ge d_i+1}(T),$$

or equivalently,

$$U_i = \phi_+^{M_{i-1}}(\mathrm{col}_{\ge d_i+1}(T)),$$

where $M_0 = 0$.

Proof. We proceed by induction on i. It is true for the case $i = 1$, which is $U_1 = T_1$. Assume true for the case $i \ge 1$ and consider the case $i + 1$. Since $U_{i+1} \in \mathrm{RSE}_{M_i+1}(\mathrm{col}_{\ge d_{i+1}+1}(\pi(\lambda)/\mu))$ and $\mu_{M_i} = d_i$, by Lemma 5.8, $\phi_-^{m_i}(U_{i+1}) \in \mathrm{RSE}_{M_{i-1}+1}(\mathrm{col}_{\ge d_{i+1}+1}(\pi(\lambda)/\mu))$ and

$$\phi_-^{m_i}(U_{i+1}) = \mathrm{col}_{\le d_i}(U_{i+1}) \sqcup \phi_-^{m_i}(\mathrm{col}_{\ge d_i+1}(U_{i+1})).$$

Using the construction of $U_{i+1} = T_{i+1} \sqcup \phi_+^{m_i}(U_i)$ the above equation can be rewritten as

$$\phi_-^{m_i}(U_{i+1}) = T_{i+1} \sqcup \phi_-^{m_i}(\phi_+^{m_i}(U_i)) = T_{i+1} \sqcup U_i,$$

which implies

(6.1) $$\mathrm{col}_{\le d_i}(\phi_-^{m_i}(U_{i+1})) = T_{i+1}, \qquad \mathrm{col}_{\ge d_i+1}(\phi_-^{m_i}(U_{i+1})) = U_i.$$

Since $\phi_-^{m_i}(U_{i+1}) \in \mathrm{RSE}_{M_{i-1}+1}(\mathrm{col}_{\ge d_{i+1}+1}(\lambda/\mu))$ and $d_i = \mu_{M_i} < \mu_{M_{i-1}}$, by using Lemma 5.8 and (6.1) we obtain

$$\phi_-^{M_{i-1}}(\phi_-^{m_i}(U_{i+1})) = \mathrm{col}_{\le d_i}(\phi_-^{m_i}(U_{i+1})) \sqcup \phi_-^{M_{i-1}}(\mathrm{col}_{\ge d_i+1}(\phi_-^{m_i}(U_{i+1}))) = T_{i+1} \sqcup \phi_-^{M_{i-1}}(U_i).$$

By the induction hypothesis, $\phi_-^{M_{i-1}}(U_i) = \mathrm{col}_{\ge d_i+1}(T)$. Hence the above equation can be rewritten as

$$\phi_-^{M_i}(U_{i+1}) = T_{i+1} \sqcup \mathrm{col}_{\ge d_i+1}(T) = \mathrm{col}_{\ge d_{i+1}+1}(T),$$

which is the desired statement for the $i + 1$ case. This completes the proof. $\qquad\square$

The following lemma gives an equivalent condition for U_i to be defined.

Lemma 6.6. *Using the notation in Definition 6.2, for $1 \le i \le r+1$, U_i is defined if and only if*

(6.2) $$\mathrm{col}_{\ge d_i+1}(T) \in \phi_-^{M_{i-1}}(\mathrm{RSE}_{M_{i-1}+1}(\mathrm{col}_{\ge d_i+1}(\rho))).$$

where $\rho = \mathrm{sh}(T) = \pi(\lambda)/\mu$.

Proof. Suppose that U_i is defined. Then (6.2) follows immediately from Lemma 6.5 since $U_i \in \mathrm{RSE}_{M_{i-1}+1}(\mathrm{col}_{\ge d_i+1}(\rho))$.

Conversely suppose that (6.2) holds. We will prove by induction that U_s is defined for all $1 \le s \le i$. The case $s = 1$ is trivial. Assume that U_s is defined and $1 \le s \le i - 1$. Then by Lemma 6.5,

(6.3) $$U_s = \phi_+^{M_{s-1}}(\mathrm{col}_{\ge d_s+1}(T)).$$

Recall that U_{s+1} is defined if $T_{s+1} \le \phi_+^{m_s}(U_s)$. To show this let $Q = \mathrm{col}_{\ge d_i+1}(T)$. Since $Q \in \phi_-^{M_{i-1}}(\mathrm{RSE}_{M_{i-1}+1}(\mathrm{col}_{\ge d_i+1}(\rho)))$, by Lemma 5.8 with $a = M_{i-1}, b = M_{i-1} + 1, d = M_{s-1}$, and $j = \mu_{b-a-d-1} = \mu_{M_{s-1}} = d_s$, we have

(6.4) $$\phi_+^{M_s}(Q) = \mathrm{col}_{\le d_s}(Q) \sqcup \phi_+^{M_s}(\mathrm{col}_{\ge d_s+1}(Q)) \in \mathrm{RSE}_{M_s+1}(\mathrm{col}_{\ge c+1}(\rho)).$$

This implies that $\mathrm{col}_{\le d_s}(Q) \le \phi_+^{M_s}(\mathrm{col}_{\ge d_s+1}(Q))$. Since the leftmost columns of $\mathrm{col}_{\le d_s}(Q)$ and T_{s+1} coincide, we also have

$$T_{s+1} \le \phi_+^{M_s}(\mathrm{col}_{\ge d_s+1}(Q)).$$

On the other hand, by (6.3),

$$\phi_+^{M_s}(\mathrm{col}_{\ge d_s+1}(Q)) = \phi_+^{M_s}(\mathrm{col}_{\ge d_s+1}(T)) = \phi_+^{m_s}(U_s).$$

The above two equations show that $T_{s+1} \le \phi_+^{m_s}(U_s)$ and hence U_{s+1} is defined. Therefore by induction U_i is also defined, which completes the proof. $\qquad\square$

The following lemma shows that Φ has the desired fixed points.

Lemma 6.7. *We have* $\Phi(\mathbf{p}) = \mathbf{p}$ *if and only if* $T = \mathrm{Tab}(\mathbf{p}) \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))$.

Proof. Suppose $\Phi(\mathbf{p}) = \mathbf{p}$. Then U_{r+1} is defined and therefore by Lemma 6.5,

$$(6.5) \qquad\qquad T = \mathrm{col}_{\geq d_{r+1}+1}(T) \in \phi_-^{M_r}(\mathrm{RSE}_{M_r+1}(\lambda/\mu)).$$

Thus $T = \phi_-^{M_r}(Q)$ for some $Q \in \mathrm{RSE}_{M_r+1}(\lambda/\mu)$. To obtain $T \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))$ it is enough to show that $Q \in \phi_-^{m_{r+1}-1}(\mathrm{RSE}_\ell(\lambda/\mu))$ because it would imply that there is $Q' \in \mathrm{RSE}_\ell(\lambda/\mu)$ satisfying

$$T = \phi_-^{M_r}(Q) = \phi_-^{M_r}(\phi_-^{m_{r+1}-1}(Q')) = \phi_-^{\ell-1}(Q').$$

Since $Q = \phi_+^{m_{r+1}-1}(Q')$ if and only if $\phi_-^{m_{r+1}-1}(Q) = Q'$, the condition $Q \in \phi_-^{m_{r+1}-1}(\mathrm{RSE}_\ell(\lambda/\mu))$ is equivalent to $\phi_+^{m_{r+1}-1}(Q) \in \mathrm{RSE}_\ell(\lambda/\mu)$. We will show by induction that

$$(6.6) \qquad\qquad \phi_+^i(Q) \in \mathrm{RSE}_{M_r+1+i}(\lambda/\mu), \qquad 0 \leq i \leq m_{r+1}-1,$$

which is true for $i = 0$ by assumption. Let $0 \leq i \leq m_{k+1}-2$ and suppose (6.6) is true for i. Now we apply Lemma 5.6 with $T = \phi_+^i(Q)$ and $k = M_r + 1 + i$. Since $\mu_{M_r+1+i} = 0$, the fourth condition of the lemma trivially holds. Hence the first condition of the lemma also holds and we obtain $\phi_+(\phi_+^i(Q)) \in \mathrm{RSE}_{M_r+1+i+1}(\lambda/\mu)$, which is exactly (6.6) with $i+1$. By induction (6.6) is true for all $0 \leq i \leq m_{r+1}-1$, and in particular we obtain $\phi_+^{m_{r+1}-1}(Q) \in \mathrm{RSE}_\ell(\lambda/\mu)$ as desired.

Conversely, suppose that $T \in \phi_-^{\ell-1}(\mathrm{RSE}_\ell(\lambda/\mu))$. To show $\Phi(\mathbf{p}) = \mathbf{p}$, we must show that U_{r+1} is defined, which is by Lemma 6.5 equivalent to (6.5). By the assumption there is $Q \in \mathrm{RSE}_\ell(\lambda/\mu)$ with $T = \phi_-^{\ell-1}(Q)$. Let $Q' = \phi_-^{m_{r+1}-1}(Q)$ so that

$$T = \phi_-^{\ell-1}(Q) = \phi_-^{M_r}(\phi_-^{m_{r+1}-1}(Q)) = \phi_-^{M_r}(Q').$$

By applying Lemma 5.5 repeatedly, we obtain $Q' = \phi_-^{m_{r+1}-1}(Q) \in \mathrm{RSE}_{M_r+1}(\lambda/\mu)$, which shows (6.5). $\qquad\square$

The following lemma shows that Φ is indeed an involution.

Lemma 6.8. *If* $\Phi(\mathbf{p}) = \mathbf{p}'$ *and* $\mathbf{p} \neq \mathbf{p}'$, *then* $\Phi(\mathbf{p}') = \mathbf{p}$.

Proof. In this proof we will use the notation $X(\mathbf{p})$ to indicate the object X, for example $X = U_i$ or $X = \widetilde{U}_k$, in Definition 6.2 when we apply Φ to $\mathbf{p}$.

Let $\rho = \mathrm{sh}(T(\mathbf{p}'))$. Then $\widetilde{U}_k(\mathbf{p}) \in \mathrm{RSE}_{M_k+1}(\mathrm{col}_{\geq d_k+1}(\rho))$, and by Lemma 5.5, we have $\phi_-^{m_k}(\widetilde{U}_k(\mathbf{p})) \in \mathrm{RSE}_{M_{k-1}+1}(\mathrm{col}_{\geq d_k+1}(\rho))$. Since

$$\mathrm{col}_{\geq d_k+1}(T(\mathbf{p}')) = \phi_-^{M_k}(\widetilde{U}_k(\mathbf{p})) = \phi_-^{M_{k-1}}(\phi_-^{m_k}(\widetilde{U}_k(\mathbf{p}))) \in \phi_-^{M_{k-1}}(\mathrm{RSE}_{M_{k-1}+1}(\mathrm{col}_{\geq d_k+1}(\rho))),$$

by Lemma 6.6, $U_k(\mathbf{p}')$ is defined. Then by Lemma 6.5,

$$U_k(\mathbf{p}') = \phi_+^{M_{k-1}}(\mathrm{col}_{\geq d_k+1}(T(\mathbf{p}'))) = \phi_+^{M_{k-1}}(\phi_-^{M_k}(\widetilde{U}_k(\mathbf{p}))) = \phi_-^{m_k}(\widetilde{U}_k(\mathbf{p})),$$

or equivalently,

$$\widetilde{U}_k(\mathbf{p}) = \phi_+^{m_k}(U_k(\mathbf{p}')).$$

By the construction we have $\widetilde{T}_{k+1}(\mathbf{p}) \not\leq \widetilde{U}_k(\mathbf{p})$. Since $T_{k+1}(\mathbf{p}') = \widetilde{T}_{k+1}(\mathbf{p})$, we obtain

$$T_{k+1}(\mathbf{p}') \not\leq \phi_+^{m_k}(U_k(\mathbf{p}')).$$

Therefore in the construction of $\Phi(\mathbf{p}')$, $U_1(\mathbf{p}'), U_2(\mathbf{p}'), \ldots, U_k(\mathbf{p}')$ are defined but not $U_{k+1}(\mathbf{p}')$. Observe that in Step 3-1 we compute

$$\mathrm{Tab}(\mathbf{q}(\mathbf{p}')) = \mathrm{row}_{\geq M_k+1}(T_{k+1}(\mathbf{p}') \sqcup \phi_+^{m_k}(U_k(\mathbf{p}'))) = \mathrm{row}_{\geq M_k+1}(\widetilde{T}_{k+1}(\mathbf{p}) \sqcup \widetilde{U}_k(\mathbf{p}))) = \mathrm{Tab}(\mathbf{q}'(\mathbf{p})).$$

Since the map in Step 3-2 sending $\mathbf{q}$ to $\mathbf{q}'$ is easily seen to be an involution, we obtain that $\mathrm{Tab}(\mathbf{q}'(\mathbf{p}')) = \mathrm{Tab}(\mathbf{q}(\mathbf{p}))$ and therefore

$$\widetilde{T}_{k+1}(\mathbf{p}') = T_{k+1}(\mathbf{p}), \qquad \widetilde{U}_k(\mathbf{p}') = \phi_+^{m_k}(U_k(\mathbf{p})).$$
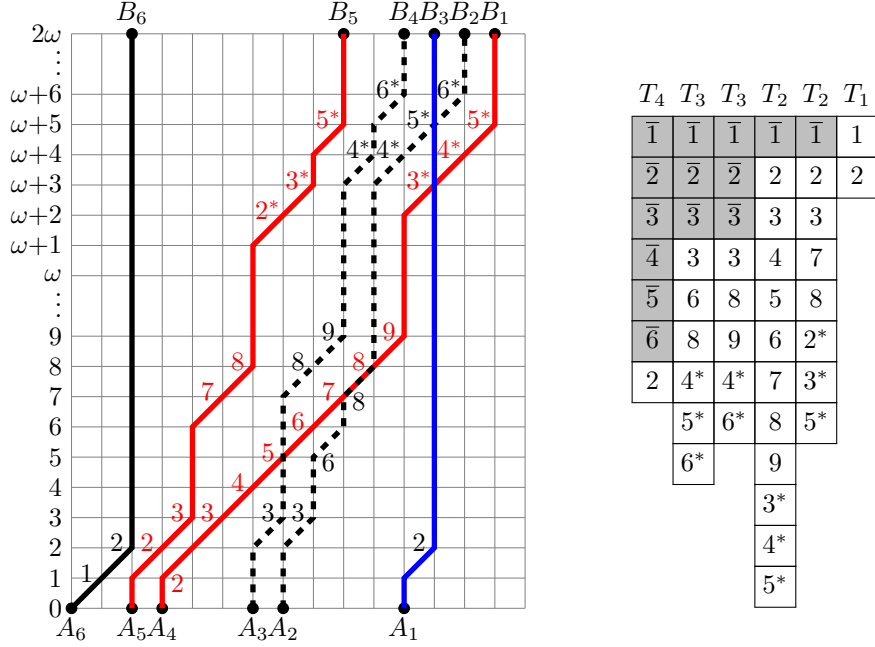
FIGURE 18. An n-path $\mathbf{p} \in \mathcal{L}^{\text{SNC}}_{\lambda/\mu}$ on the left and the vertical tableau $T = \text{Tab}(\mathbf{p}) \in \text{VT}(\pi(\lambda)/\mu)$ on the right, where $n = 6$, $\lambda = (6, 6, 5, 5, 5, 5, 5, 5, 4)$, $\mu = (5, 3, 3, 1, 1, 1)$, and $\pi = (3, 2, 4, 1, 5, 6)$. Each p_i is a path from $A_i = (\mu'_i + n - i, 0)$ to $B_{\pi_i} = (\lambda'_{\pi_i} + n - \pi_i)$. The gray cells are those in μ. At the top of each column is written the tableau T_i containing that column.

Then

$$T'(\mathbf{p}') = T_r(\mathbf{p}') \sqcup \cdots \sqcup T_{k+2}(\mathbf{p}') \sqcup \widetilde{T}_{k+1}(\mathbf{p}') \sqcup \phi^{M_k}_-(\widetilde{U}_k(\mathbf{p}'))$$

$$= T_r(\mathbf{p}) \sqcup \cdots \sqcup T_{k+2}(\mathbf{p}) \sqcup T_{k+1}(\mathbf{p}) \sqcup \phi^{M_{k-1}}_-(U_k(\mathbf{p})).$$

By Lemma 6.5, $\phi^{M_{k-1}}_-(U_k(\mathbf{p})) = \text{col}_{\geq d_k+1}(T(\mathbf{p}))$, and we obtain $T'(\mathbf{p}') = T(\mathbf{p})$. This means that $\Phi(\mathbf{p}') = \mathbf{p}$ as desired. □

Now we can prove Theorem 6.3 easily. By Lemmas 6.7 and 6.8, Φ is an involution on $\mathcal{L}^{\text{SNC}}_{\lambda/\mu}$ with the desired fixed point set. Suppose that $\Phi(\mathbf{p}) = \mathbf{p}'$ and $\mathbf{p} \neq \mathbf{p}'$. Lemma 3.3 and the third assertion of Lemma 6.4 show that $\text{sign}(\text{type}(\mathbf{p}')) = -\text{sign}(\text{type}(\mathbf{p}))$. Note that $\text{wt}(\mathbf{p}) = \text{sign}(\text{type}(\mathbf{p}))\, \text{wt}(T)$ and $\text{wt}(\mathbf{p}') = \text{sign}(\text{type}(\mathbf{p}'))\, \text{wt}(T')$. By the construction of Φ, we have $\text{wt}(T) = \text{wt}(T')$, and hence $\text{wt}(\mathbf{p}') = -\text{wt}(\mathbf{p})$, which completes the proof.

7. AN EXAMPLE OF THE INVOLUTION

In this section we give a concrete example of the involution Φ applied to $\mathbf{p}$ which is not a fixed point.

Let $n = 6$, $\lambda = (6, 6, 5, 5, 5, 5, 5, 5, 4)$ and $\mu = (5, 3, 3, 1, 1, 1)$. Then $\lambda' = (9, 9, 9, 9, 8, 2)$ and $\mu' = (6, 3, 3, 1, 1)$. Let $A_i = (\mu'_i + n - 1, 0)$ and $B_i = (\lambda'_i + n - i, 2\omega)$. Consider the n-path $\mathbf{p} \in \mathcal{L}^{\text{SNC}}_{\lambda/\mu}$ in Figure 18. Note that $\text{type}(\mathbf{p}) = \pi = (3, 2, 4, 1, 5, 6)$. We construct $\Phi(\mathbf{p}) = \mathbf{p}'$ as follows.

In Step 1, we find $T = \text{Tab}(\mathbf{p})$ and express $T = T_4 \sqcup T_3 \sqcup T_2 \sqcup T_1$ as shown in Figure 18. Then $T \in \text{VT}(\pi(\lambda)/\mu)$.

In Step 2, we find U_1 and $\phi^{m_1}_+(U_1) = \phi_+(U_1)$ as in Figure 19, and $U_2 = T_2 \sqcup \phi^{m_1}_+(U_1)$ and $\phi^{m_2}_+(U_2)$ as in Figure 20. Observe that T_3, which is columns 2 and 3 in the right diagram of Figure 20, does not satisfy $T_3 \leq \phi^{m_2}_+(U_2)$. Therefore U_3 is not defined and Step 2 is finished.
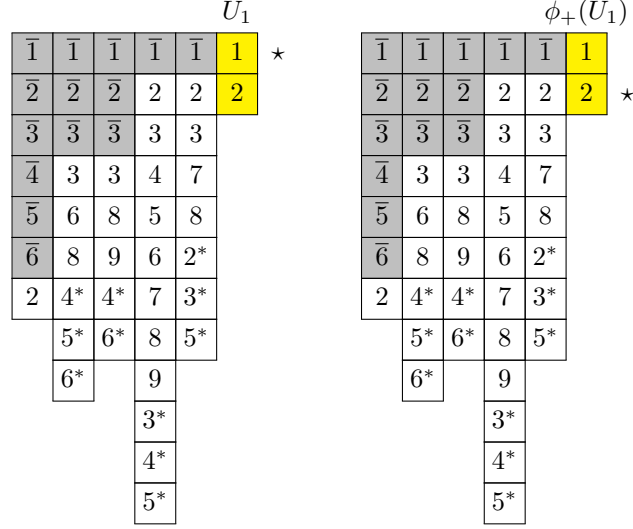
U_1 $\phi_+(U_1)$

1̄	1̄	1̄	1̄	1̄	1
2̄	2̄	2̄	2	2	2
3̄	3̄	3̄	3	3	
4̄	3	3	4	7	
5̄	6	8	5	8	
6̄	8	9	6	2*	
2	4*	4*	7	3*	
	5*	6*	8	5*	
	6*		9		
			3*		
			4*		
			5*		

$U_1 = T_1$ is the last column in the left diagram. $\star$ at row 1.

$\phi_+(U_1)$:

1̄	1̄	1̄	1̄	1̄	1
2̄	2̄	2̄	2	2	2
3̄	3̄	3̄	3	3	
4̄	3	3	4	7	
5̄	6	8	5	8	
6̄	8	9	6	2*	
2	4*	4*	7	3*	
	5*	6*	8	5*	
	6*		9		
			3*		
			4*		
			5*		

$\star$ at row 2.

FIGURE 19. $U_1 = T_1$ is the last column in the left diagram and $\phi_+^{m_1}(U_1) = \phi_+(U_1)$ is the last column in the right diagram. The level of each RSE-tableau is marked by a star.

U_2

1̄	1̄	1̄	1̄	1̄	1
2̄	2̄	2̄	2	2	2
3̄	3̄	3̄	3	3	
4̄	3	3	4	7	
5̄	6	8	5	8	
6̄	8	9	6	2*	
2	4*	4*	7	3*	
	5*	6*	8	5*	
	6*		9		
			3*		
			4*		
			5*		

$\star$ at row 2.

$\phi_+(U_2)$

1̄	1̄	1̄	1̄	1̄	1
2̄	2̄	2̄	2	2	2
3̄	3̄	3̄	2	3	
4̄	3	3	3	4	
5̄	6	8	5	7	
6̄	8	9	6	8	
2	4*	4*	7	3*	
	5*	6*	8	5*	
	6*		9		
			3*		
			4*		
			5*		

$\star$ at row 3.

$\phi_+^2(U_2)$

1̄	1̄	1̄	1̄	1̄	1
2̄	2̄	2̄	2	2	2
3̄	3̄	3̄	2	3	
4̄	3	3	2	3	
5̄	6	8	3	4	
6̄	8	9	5	7	
2	4*	4*	6	8	
	5*	6*	7	5*	
	6*		8		
			9		
			4*		
			5*		

$\star$ at row 4.
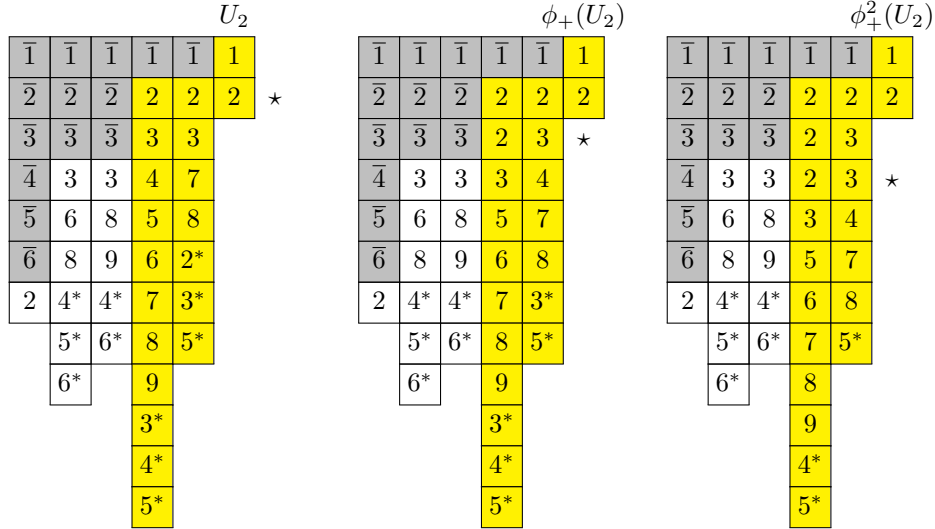
FIGURE 20. $U_2 = T_2 \sqcup \phi_+^{m_1}(U_1)$ is the last three columns in the left diagram, $\phi_+(U_2)$ is the last three columns in the middle diagram, and $\phi_+^{m_2}(U_2) = \phi_+^2(U_2)$ is the last three columns in the right diagram.

In Step 3, $k = 2$ is the smallest integer such that $T_{k+1} \not\preceq \phi_+^{m_k}(U_k)$.

In Step 3-1, we have $s = 5$ and $\gamma = (6, 6, 5, 1, 1, 1, 1)$. The vertical tableau $\mathrm{row}_{\geq M_k+1}(T_k \sqcup \phi_+^{m_k}(U_{k-1})) \in \mathrm{VT}(\pi(\lambda)/\gamma)$ and its corresponding n-path $\mathbf{q}$ are shown in Figure 21.

In Step 3-2, there are 7 intersections among $\{q_i : d_{k+1} + 1 \leq i \leq s\} = \{q_2, q_3, q_4, q_5\}$ and the intersection $(a, b) = (8, 8)$ of $q_i = q_3$ and $q_j = q_5$ is chosen. Then $\mathbf{q}'$ is obtained from $\mathbf{q}$ by exchanging the subpaths of q_3 and q_5 after $(8, 8)$. See Figure 22 for $\mathbf{q}'$ and its corresponding vertical tableau.

In Step 3-3, $\widetilde{T}_{k+1} \sqcup \widetilde{U}_k = \widetilde{T}_3 \sqcup \widetilde{U}_2$ is obtained from $T_{k+1} \sqcup \phi_+^{m_k}(U_k) = T_3 \sqcup \phi_+^2(U_2)$ by replacing the part equal to $\mathrm{Tab}(\mathbf{q})$ by $\mathrm{Tab}(\mathbf{q}')$, see Figure 23. We then compute $\phi_-^{M_k}(\widetilde{U}_k) = \phi_-^3(\widetilde{U}_2)$ as in
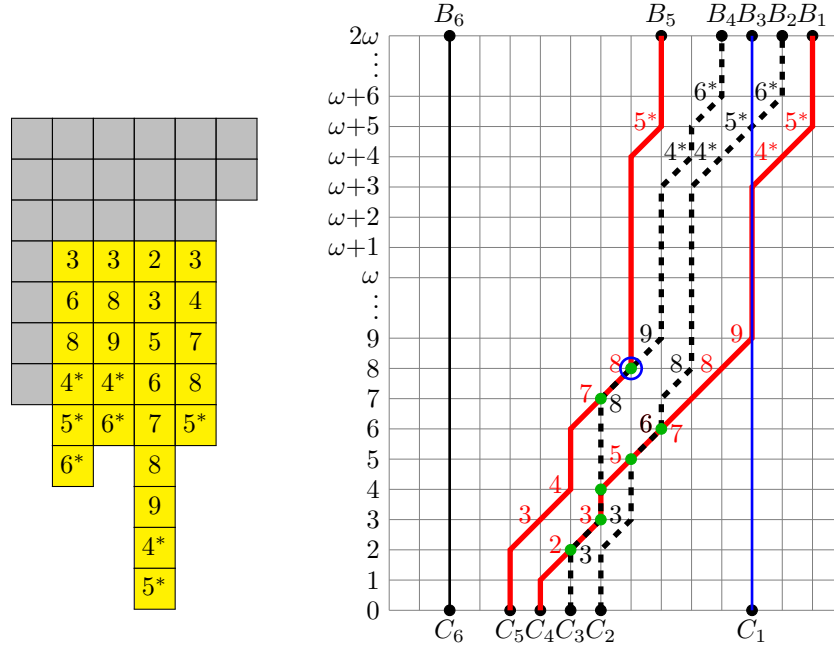
FIGURE 21. The vertical tableau $\mathrm{row}_{\geq M_2+1}(T_3 \sqcup \phi_+^{m_2}(U_2)) \in \mathrm{VT}(\pi(\lambda)/\gamma)$ and the corresponding n-path $\mathbf{q}$. The chosen intersection (a, b) is circled.
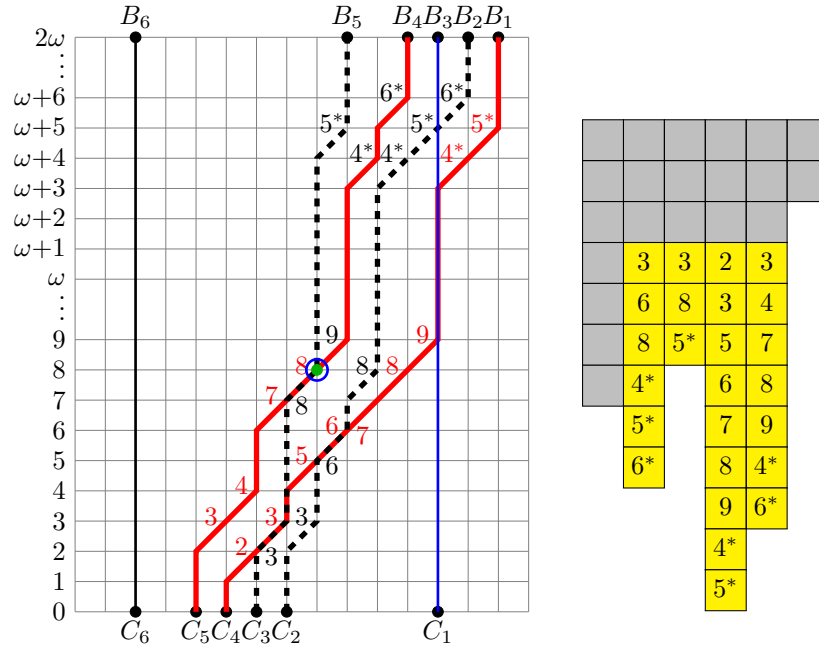

FIGURE 22. The n-path $\mathbf{q}'$ and the corresponding vertical tableau $\mathrm{Tab}(\mathbf{q}') \in \mathrm{VT}(\pi'(\lambda)/\gamma)$, where $\pi' = \pi(4, 5)$.

Figure 24. Finally, the tableau $T' = T_{r+1} \sqcup \cdots \sqcup T_{k+2} \sqcup \widetilde{T}_{k+1} \sqcup \phi_+^{M_k}(\widetilde{U}_k) = T_4 \sqcup \widetilde{T}_3 \sqcup \phi_+^3(\widetilde{U}_2)$ and the corresponding n-path $\mathbf{p}'$ are obtained as in Figure 25.

FIGURE 23. The left diagram is $T_{k+1} \sqcup \phi_+^{m_k}(U_k) = T_3 \sqcup \phi_+^2(U_2)$, where the part equal to Tab($\mathbf{q}$) is colored yellow. The right diagram is $\widetilde{T}_{k+1} \sqcup \widetilde{U}_k = \widetilde{T}_3 \sqcup \widetilde{U}_2$, where the part equal to Tab($\mathbf{q}'$) is colored yellow.

FIGURE 24. The RSE-tableaux $\widetilde{U}_k = \widetilde{U}_2, \phi_-(\widetilde{U}_2), \phi_-^2(\widetilde{U}_2)$ and $\phi_-^3(\widetilde{U}_2) = \phi_-^{M_k}(\widetilde{U}_k)$ from left to right. The level of each RSE-tableau is marked by a star.
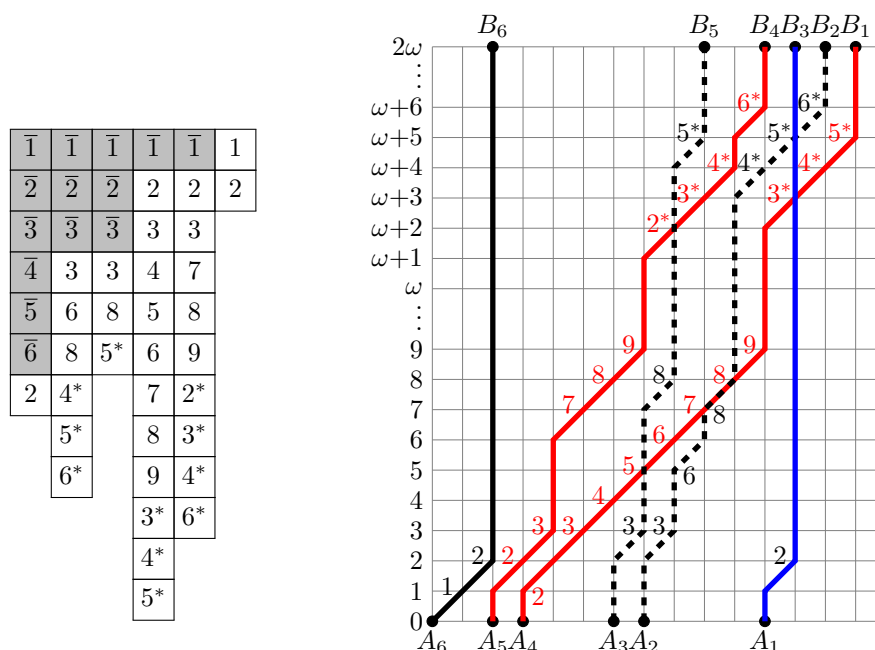
ACKNOWLEDGMENTS

FIGURE 25. The tableau $T' = T_{r+1} \sqcup \cdots \sqcup T_{k+2} \sqcup \widetilde{T}_{k+1} \sqcup \phi_+^{M_k}(\widetilde{U}_k) = T_4 \sqcup \widetilde{T}_3 \sqcup \phi_+^3(\widetilde{U}_2)$ and the corresponding n-path $\mathbf{p}'$.

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