JACOBI–TRUDI FORMULAS FOR FLAGGED REFINED DUAL STABLE GROTHENDIECK POLYNOMIALS

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ABSTRACT. Recently Galashin, Grinberg, and Liu introduced the refined dual stable Grothendieck polynomials, which are symmetric functions in $x = (x_1, x_2, \dots)$ with additional parameters $t = (t_1, t_2, \dots)$. The refined dual stable Grothendieck polynomials are defined as a generating function for reverse plane partitions of a given shape. They interpolate between Schur functions and dual stable Grothendieck polynomials introduced by Lam and Pylyavskyy in 2007. Flagged refined dual stable Grothendieck polynomials are a more refined version of refined dual stable Grothendieck polynomials, where lower and upper bounds are given for the entries of each row or column. In this paper Jacobi–Trudi-type formulas for flagged refined dual stable Grothendieck polynomials are proved using plethystic substitution. This resolves a conjecture of Grinberg and generalizes a result by Iwao and Amanov–Yeliussizov.

1. INTRODUCTION

The (skew) Schur functions $s_{\lambda/\mu}(x)$ are a central object in algebraic combinatorics. They are symmetric functions in the variables $x = (x_1, x_2, \dots)$ and can be defined combinatorially as a generating function for semistandard Young tableaux of shape λ/μ. The Jacobi–Trudi formula and its dual formula express $s_{\lambda/\mu}(x)$ as a determinant in terms of the complete homogeneous symmetric functions $h_k(x)$ and the elementary symmetric functions $e_k(x)$, respectively:

$$(1.1) \qquad s_{\lambda/\mu}(x) = \det \left(h_{\lambda_i - \mu_j - i + j}(x) \right)_{1 \le i, j \le \ell(\lambda)},$$

$$(1.2) \qquad s_{\lambda'/\mu'}(x) = \det \left(e_{\lambda_i - \mu_j - i + j}(x) \right)_{1 \le i, j \le \ell(\lambda)},$$

where $\ell(\lambda)$ is the number of parts in λ and λ' is the transpose of λ.

The row-flagged and column-flagged Schur functions $s_{\lambda/\mu}^{\mathrm{row}(\alpha, \beta)}(x)$ and $s_{\lambda/\mu}^{\mathrm{col}(\alpha, \beta)}(x)$ are defined as a generating function for semistandard Young tableaux of shape λ/μ in which entries in each row or column have lower and upper bounds specified by α and β. Flagged Schur functions were introduced by Lascoux and Schützenberger [10] in their study of Schubert polynomials. See [2, 12, 14] and references therein for more details on flagged Schur functions. Jacobi–Trudi formulas for flagged Schur functions were discovered by Gessel [4] and Wachs [14].

Theorem 1.1. [14, Theorems 3.5 and 3.5*] *Let* $\alpha = (\alpha_1, \dots, \alpha_n)$ *and* $\beta = (\beta_1, \dots, \beta_n)$ *be sequences of nonnegative integers and let* λ *and* μ *be partitions with at most n parts.*

If $\alpha_i \le \alpha_{i+1}$ and $\beta_i \le \beta_{i+1}$ whenever $\mu_i < \lambda_{i+1}$, then

$$(1.3) \qquad s_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x) = \det\left(h_{\lambda_i-\mu_j-i+j}(x_{\alpha_j+1},\dots,x_{\beta_i})\right)_{1\le i,j\le n},$$

where $(x_{\alpha_j+1},\dots,x_{\beta_i})$ is the empty list if $\beta_i \le \alpha_j$.

If $\alpha_i - \mu_i \le \alpha_{i+1} - \mu_{i+1} + 1$ and $\beta_i - \lambda_i \le \beta_{i+1} - \lambda_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$, then

$$(1.4) \qquad s_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x) = \det\left(e_{\lambda_i-\mu_j-i+j}(x_{\alpha_j+1},\dots,x_{\beta_i})\right)_{1\le i,j\le n}.$$

Lam and Pylyavskyy [9] introduced dual stable Grothendieck polynomials $g_{\lambda/\mu}(x)$, which originate from the K-theory of Grassmannians. They showed that $g_{\lambda/\mu}(x)$ is a generating function for reverse plane partitions of shape λ/μ. The refined dual stable Grothendieck polynomials $\widetilde{g}_{\lambda/\mu}(x;t)$ are power series in variables $x = (x_1, x_2, \dots)$ and $t = (t_1, t_2, \dots)$ introduced by Galashin, Grinberg, and Liu [3]. Similarly to dual stable Grothendieck polynomials, $\widetilde{g}_{\lambda/\mu}(x;t)$ are defined as a generating function for reverse plane partitions of shape λ/μ with more refined weight system. The refined dual stable Grothendieck polynomials interpolate between Schur functions and dual stable Grothendieck polynomials. If $t_i = 0$ for all i, then $\widetilde{g}_{\lambda/\mu}(x;t)$ becomes the Schur function $s_{\lambda/\mu}(x)$, and if $t_i = 1$ for all i, then $\widetilde{g}_{\lambda/\mu}(x;t)$ becomes the dual stable Grothendieck polynomial $g_{\lambda/\mu}(x)$.

The following theorem was conjectured by Grinberg [5] and proved independently by Amanov and Yeliussizov [1], and the author [8].

Theorem 1.2. [1, 8] *For partitions λ and μ, we have*

$$\widetilde{g}_{\lambda/\mu}(x;t) = \det\left(e_{\lambda_i'-\mu_j'-i+j}(x_1, x_2, \dots, t_{\mu_j'+1}, t_{\mu_j'+2}, \dots, t_{\lambda_i'-1})\right)_{1\le i,j\le \ell(\lambda')},$$

where, if $\mu_j' + 1 > \lambda_i' - 1$, the (i,j) entry is defined to be $e_{\lambda_i'-\mu_j'-i+j}(x_1, x_2, \dots)$.

Since there are two Jacobi–Trudi formulas for $s_{\lambda/\mu}(x)$ in (1.1) and (1.2), a natural question is whether there is a Jacobi–Trudi formula for $\widetilde{g}_{\lambda/\mu}(x;t)$ in terms of h_k's. For the case of dual stable Grothendieck polynomials, equivalently the case that all $t_i = 1$, Amanov and Yeliussizov [1, Theorem 14], and Iwao [7, Proposition 5.2] found the following formula.

Theorem 1.3. [1, 7] *For partitions λ and μ, we have*

$$g_{\lambda/\mu}(x) = \widetilde{g}_{\lambda/\mu}(x;(1,1,\dots)) = \det\left(\phi^{i-j}h_{\lambda_i-\mu_j-i+j}(x)\right)_{1\le i,j\le \ell(\lambda)},$$

where $\phi^k h_n = \sum_{i=0}^n \binom{k+i-1}{i} h_{n-i}$.

In this paper we give a Jacobi–Trudi formula for $\widetilde{g}_{\lambda/\mu}(x;t)$ in terms of h_k's using plethystic substitution. We also give an equivalent version of Theorem 1.2 using plethystic substitution. More generally, we prove Jacobi–Trudi formulas for *flagged* refined dual stable Grothendieck polynomials $\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t)$ and $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t)$, which are generating functions for reverse plane partitions in which each column and row has lower and upper bounds given by α and β. See Section 2 for the precise definitions.

Our main results are the two Jacobi–Trudi-type formulas in the following theorem.

Theorem 1.4. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be sequences of nonnegative integers and let λ and μ be partitions with at most n parts.*
  If $\alpha_i \leq \alpha_{i+1} + 1$ and $\beta_i \leq \beta_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$, then

$$(1.5) \qquad \widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = \det\left(e_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i)} + T_{\lambda_i - 1} - T_{\mu_j}]\right)_{1 \leq i,j \leq n},$$

where $X_{(i,j)} = x_{i+1} + x_{i+2} + \cdots + x_j$ for $i < j$ and $X_{(i,j)} = 0$ for $i \geq j$, and $T_i = t_1 + t_2 + \cdots + t_i$ for $i \geq 1$ and $T_0 = 0$.
  If $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i+1}$ whenever $\mu_i < \lambda_{i+1}$, then

$$(1.6) \qquad \widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t) = \det\left(h_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i)} + T_{i-1} - T_{j-1}]\right)_{1 \leq i,j \leq n}.$$

Note that the assumption on α and β in our formula (1.5) is different from that in the formula (1.4). In fact (1.5) is not true under the assumptions for (1.4), see Remark 3.4.

The basic idea of proof of (1.5) and (1.6) is to show that both sides of the equation satisfy the same recurrence relation. We also show that (1.5) is equivalent to the following formula, which was conjectured by Grinberg (private communication).

Theorem 1.5. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be sequences of nonnegative integers and let λ and μ be partitions with at most n parts. If $\alpha_i \leq \alpha_{i+1} + 1$ and $\beta_i \leq \beta_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$, then*

$$(1.7) \qquad \widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = \det\left(e_{\lambda_i - \mu_j - i + j}(x_{\alpha_j + 1}, \ldots, x_{\beta_i}, t_{\mu_j + 1}, \ldots, t_{\lambda_i - 1})\right)_{1 \leq i,j \leq n}.$$

Note that Theorem 1.2 follows from Theorem 1.5. As a corollary of Theorem 1.4 we obtain two Jacobi–Trudi formulas for the refined dual stable Grothendieck polynomials.

Corollary 1.6. *Let λ and μ be partitions with at most n parts. Then*

$$(1.8) \qquad \widetilde{g}_{\lambda'/\mu'}(x;t) = \det\left(e_{\lambda_i - \mu_j - i + j}[X + T_{\lambda_i - 1} - T_{\mu_j}]\right)_{1 \leq i,j \leq n},$$

$$(1.9) \qquad \widetilde{g}_{\lambda/\mu}(x;t) = \det\left(h_{\lambda_i - \mu_j - i + j}[X + T_{i-1} - T_{j-1}]\right)_{1 \leq i,j \leq n},$$

where $X = x_1 + x_2 + \cdots$ and $T_i = t_1 + t_2 + \cdots + t_i$ for $i \geq 1$ and $T_0 = 0$.

It can also be shown that the formula (1.8) is equivalent to Theorem 1.2. Note that Corollary 1.6 reproves the symmetry of $\widetilde{g}_{\lambda/\mu}(x;t)$ in the x variables.

We note that Motegi and Scrimshaw [13] also proved (1.9) using difference operators.

The remainder of this paper is organized as follows. In Section 2 we give basic definitions. In Section 3 we restate our main results and give some remarks. In the last two sections we prove the main results.

2. PRELIMINARIES

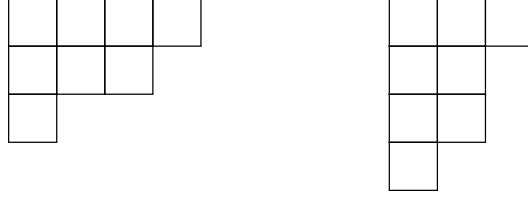In this section we give necessary definitions to prove the main results.

FIGURE 1. The Young diagram of $\lambda = (4, 3, 1)$ on the left and its transpose $\lambda' = (3, 2, 2, 1)$ on the right.
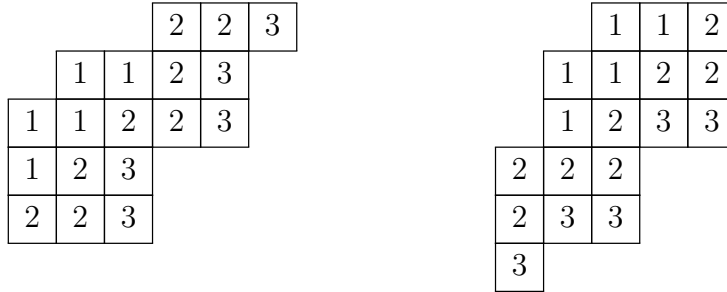

FIGURE 2. An RPP R of shape $(6, 5, 5, 3, 3)/(3, 1)$ on the left and its transpose R' on the right.

2.1. **Basic definitions.** Denote by $\mathbb{N}$ the set of nonnegative integers. For $n \in \mathbb{N}$, we denote $[n] = \{1, 2, \ldots, n\}$. If $\alpha \in \mathbb{N}^n$, the ith entry of α is denoted by α_i, i.e., $\alpha = (\alpha_1, \ldots, \alpha_n)$. For $\alpha, \beta \in \mathbb{N}^n$, we write $\alpha < \beta$ (resp. $\alpha \le \beta$) if $\alpha_i < \beta_i$ (resp. $\alpha_i \le \beta_i$) for all $1 \le i \le n$.

An element $\alpha \in \mathbb{N}^n$ is called a *partition* if $\alpha_1 \ge \cdots \ge \alpha_n$. Denote by Par_n the set of partitions in $\mathbb{N}^n$.

Let $\lambda \in \mathrm{Par}_n$. The *Young diagram* of λ is the set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 1 \le i \le n \text{ and } 1 \le j \le \lambda_i\}$. We will identify λ with its Young diagram. Therefore a partition is considered as a sequence of nonnegative integers and also as a set of pairs of positive integers. Each element $(i, j) \in \lambda$ is called a *cell*. The Young diagram λ will be visualized as an array of squares where we place a square in row i and column j for each $(i, j) \in \lambda$ using the matrix coordinates. The *transpose* λ' of λ is the partition given by $\lambda' = \{(j, i) : (i, j) \in \lambda\}$. Note that if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{Par}_n$, then $\lambda' \in \mathrm{Par}_{\lambda_1}$. See Figure 1.

Note that for two partitions λ and μ, we have $\mu \subseteq \lambda$ (as Young diagrams) if and only if $\mu \le \lambda$ (as elements in $\mathbb{N}^n$). We will mostly use the notation $\mu \subseteq \lambda$ since this emphasizes that μ and λ are Young diagrams.

For two partitions λ and μ with $\mu \subseteq \lambda$, the *skew shape* λ/μ is the set-theoretic difference $\lambda - \mu$ of their Young diagrams. In other words, if $\lambda, \mu \in \mathrm{Par}_n$ satisfy $\mu \subseteq \lambda$, then λ/μ is the set of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $1 \le i \le n$ and $\mu_i + 1 \le j \le \lambda_i$. A *reverse plane partition (RPP)* of shape λ/μ is a filling of λ/μ with positive integers such that the entries are weakly increasing in each row and column. If R is an RPP of shape λ/μ, the (i, j) entry of R is denoted by $R(i, j)$. The *transpose* of R is the RPP R' of shape λ'/μ' given by $R'(i, j) = R(j, i)$ for all $(i, j) \in \lambda'/\mu'$. See Figure 2.

For $\lambda, \mu \in \mathbb{N}^n$, the set of RPPs of shape λ/μ is denoted by $\mathrm{RPP}_{\lambda/\mu}$. If $\mu \notin \mathrm{Par}_n$, $\lambda \notin \mathrm{Par}_n$, or $\mu \not\subseteq \lambda$, then $\mathrm{RPP}_{\lambda/\mu}$ is defined to be the empty set. For $R \in \mathrm{RPP}_{\lambda/\mu}$, define

$$\mathrm{wt}(R) = \prod_{i \geq 1} x_i^{a_i(R)} t_i^{b_i(R)},$$

where $a_i(R)$ is the number of columns of R containing an i and $b_i(R)$ is the number of cells (i, j) such that $(i, j), (i+1, j) \in \lambda/\mu$ and $R(i, j) = R(i+1, j)$. For example, if R is the RPP shown in Figure 2 on the left, then $\mathrm{wt}(R) = x_1^3 x_2^5 x_3^3 t_1^3 t_2^3 t_3 t_4^2$ and $\mathrm{wt}(R') = x_1^3 x_2^5 x_3^5 t_1^2 t_2 t_3 t_4$.

Let $x = \{x_1, x_2, \dots\}$ and $t = \{t_1, t_2, \dots\}$ be sets of variables. For $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, define

$$X_{(r,s]} = x_{r+1} + x_{r+2} + \cdots + x_s,$$

where empty sums are zero, i.e., $X_{(r,s]} = 0$ if $r \geq s$. In other words, $X_{(r,s]}$ is the sum of the variables x_i for the integers i in the interval $(r, s] = \{u \in \mathbb{R} : r < u \leq s\}$. We define $T_{(r,s]}$ in the same way using the variables t_i. For integers i, we also define

$$X_i = x_1 + x_2 + \cdots + x_i, \qquad T_i = t_1 + t_2 + \cdots + t_i,$$

where empty sums are zero, i.e., $X_i = T_i = 0$ if $i \leq 0$. Note that if $0 \leq r \leq s$, then $X_{(r,s]} = X_s - X_r$.

Let $z = \{z_i : i \in I\}$ be a set of variables, where $I \subseteq \mathbb{N}$. The *elementary symmetric function* $e_n(z)$ and the *complete homogeneous symmetric function* $h_n(z)$, for $n \geq 1$, are defined by

$$e_n(z) = \sum_{i_1 < \cdots < i_n \text{ in } I} z_{i_1} \cdots z_{i_n},$$

$$h_n(z) = \sum_{i_1 \leq \cdots \leq i_n \text{ in } I} z_{i_1} \cdots z_{i_n}.$$

We define $e_0(z) = h_0(z) = 1$ and $e_k(z) = h_k(z) = 0$ for $k < 0$. Note that $e_k(z) = 0$ whenever $z = \{z_i : i \in I\}$ satisfies $|I| < k$.

2.2. Flagged refined dual stable Grothendieck polynomial. For $\lambda, \mu \in \mathbb{N}^n$, the *refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}(x; t)$ is defined by

$$\widetilde{g}_{\lambda/\mu}(x; t) = \sum_{R \in \mathrm{RPP}_{\lambda/\mu}} \mathrm{wt}(R).$$

The polynomials $\widetilde{g}_{\lambda/\mu}(x; t)$ were introduced by Galashin, Grinberg, and Liu [3]. They showed that $\widetilde{g}_{\lambda/\mu}(x; t)$ is symmetric in the variables x but not in the variables t.

For $\alpha, \beta, \lambda, \mu \in \mathbb{N}^n$, define $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}$ to be the set of RPPs R of shape λ/μ such that $\alpha_i + 1 \leq R(i, j) \leq \beta_i$ for all $(i, j) \in \lambda/\mu$. Similarly, for $\alpha, \beta \in \mathbb{N}^n$ and partitions λ, μ with $\lambda', \mu' \in \mathbb{N}^n$, define $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}$ to be the set of RPPs R of shape λ/μ such that $\alpha_j + 1 \leq R(i, j) \leq \beta_j$ for all $(i, j) \in \lambda/\mu$.

The *row-flagged refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t)$ and the *column-flagged refined dual stable Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}(x;t)$ are defined by

$$\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t) = \sum_{R\in\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}} \mathrm{wt}(R),$$

$$\widetilde{g}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}(x;t) = \sum_{R\in\mathrm{RPP}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}} \mathrm{wt}(R).$$

For simplicity we will sometimes omit $(x;t)$ and write $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}$ and $\widetilde{g}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}$.

2.3. **Plethystic substitution.** Let $\Lambda = \Lambda_{\mathbb{Q}}$ denote the ring of symmetric functions with rational coefficients. The *power sum symmetric functions* $p_k(x) = x_1^k + x_2^k + \cdots$ generate Λ as a $\mathbb{Q}$-algebra. Let $\mathbb{Q}[[a_1, a_2, \dots]]$ denote the ring of formal power series in variables $a_1, a_2, \dots$ with rational coefficients. Once $A \in \mathbb{Q}[[a_1, a_2, \dots]]$ is fixed, the *plethystic substitution* $f[A]$ for $f \in \Lambda$ is defined by the following rules:

- for $k \geq 1$, $p_k[A]$ is obtained from A by replacing each a_i by a_i^k,
- the map $f \mapsto f[A]$ is a ring homomorphism from Λ to $\mathbb{Q}[[a_1, a_2, \dots]]$.

If $A = a_1 + \cdots + a_n$, then $p_k[A] = a_1^k + \cdots + a_n^k = p_k(a_1, \dots, a_n)$, which implies $f[A] = f(a_1, \dots, a_n)$ for all $f \in \Lambda$. We refer the reader to [11] for more details on plethystic substitution. We need the following well known properties of the plethystic substitution.

Proposition 2.1. *Let $A, B \in \mathbb{Q}[[a_1, a_2, \dots]]$ and $f \in \Lambda$. Then*

$$f[A + B] = \sum_{(f)} f_{(1)}[A]f_{(2)}[B],$$

$$f[-A] = (S(f))[A],$$

where the Sweedler notation is used and S is the antipode of the Hopf algebra of symmetric functions. See [6] for more details on the Sweedler notation and the antipode.

In this paper we only need to compute $f[A]$ and $f[A - B]$ when $f = e_k$ or $f = h_k$, and both A and B are sums of variables taken from $x = (x_1, x_2, \dots)$ and $t = (t_1, t_2, \dots)$. If A and B are any formal power series with integer coefficients, since $Sh_k = (-1)^k e_k$ and $Se_k = (-1)^k h_k$, we have

$$(2.1) \qquad\qquad h_k[-A] = (-1)^k e_k[A],$$

$$(2.2) \qquad\qquad e_k[-A] = (-1)^k h_k[A],$$

$$(2.3) \qquad\qquad h_k[A - B] = \sum_{i=0}^{k} h_{k-i}[A](-1)^i e_i[B],$$

$$(2.4) \qquad\qquad e_k[A - B] = \sum_{i=0}^{k} e_{k-i}[A](-1)^i h_i[B].$$

3. Main results

In this section we restate our main results, Theorem 1.4 in the introduction, as two separate theorems, Theorems 3.2 and 3.3, and prove their corollaries. The main results will be proved in the next two sections.

The following theorem is Theorem 1.5 in the introduction, which is equivalent to one of the main results.

Theorem 3.1. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \lambda \in \mathrm{Par}_n$. If $\alpha_i \leq \alpha_{i+1} + 1$ and $\beta_i \leq \beta_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$, then*

$$\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = \det\left(e_{\lambda_i - \mu_j - i + j}(x_{\alpha_j+1}, \ldots, x_{\beta_i}, t_{\mu_j+1}, \ldots, t_{\lambda_i-1})\right)_{1 \leq i,j \leq n}.$$

The following theorem is the first main result in this paper. Using simple determinant evaluation techniques we show that Theorem 3.1 is equivalent to this theorem, see Proposition 4.19.

Theorem 3.2. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \lambda \in \mathrm{Par}_n$. If $\alpha_i \leq \alpha_{i+1} + 1$ and $\beta_i \leq \beta_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$, then*

$$\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = \det\left(e_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i]} + T_{\lambda_i - 1} - T_{\mu_j}]\right)_{1 \leq i,j \leq n}.$$

Note that the (i, j) entry of the matrix in Theorem 3.1 can be written as

$$e_{\lambda_i - \mu_j - i + j}(x_{\alpha_j+1}, \ldots, x_{\beta_i}, t_{\mu_j+1}, \ldots, t_{\lambda_i-1}) = e_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i]} + T_{(\mu_j, \lambda_i - 1]}].$$

If we replace $T_{(\mu_j, \lambda_i - 1]}$ by $T_{\lambda_i - 1} - T_{\mu_j}$, we obtain Theorem 3.2. However, unlike the t variables, we cannot replace $X_{(\alpha_j, \beta_i]}$ by $X_{\beta_i} - X_{\alpha_j}$.

The following theorem is the second main result, which is a dual version of Theorem 3.2.

Theorem 3.3. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \lambda \in \mathrm{Par}_n$. If $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i+1}$ whenever $\mu_i < \lambda_{i+1}$, then*

$$\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t) = \det\left(h_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i]} + T_{i-1} - T_{j-1}]\right)_{1 \leq i,j \leq n}.$$

Theorems 3.2 and 3.3 combined yield Theorem 1.4. Note that, for the variables in the plethystic substitution in Theorem 3.3, there are four ways of choosing the x and t variables from $\{X_{(\alpha_j, \beta_i]}, X_{\beta_i} - X_{\alpha_j}\}$ and $\{T_{(j-1, i-1]}, T_{i-1} - T_{j-1}\}$, respectively. In contrast to Theorem 3.2, the choice in Theorem 3.3 is the only one that gives a correct formula.

Remark 3.4. Recall that in the formula (1.4) the assumption is $\alpha_i - \mu_i \leq \alpha_{i+1} - \mu_{i+1} + 1$ and $\beta_i - \lambda_i \leq \beta_{i+1} - \lambda_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$. On the other hand the assumption in Theorem 3.2 is $\alpha_i \leq \alpha_{i+1} + 1$ and $\beta_i \leq \beta_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$. We cannot replace the assumption in Theorem 3.2 by that in (1.4). For example, if $\lambda = (3, 3)$, $\mu = (2)$, $\alpha = (2, 0)$, and $\beta = (2, 2)$, then $\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = 0$ but

$$\det\left(e_{\lambda_i - \mu_j - i + j}[X_{(\alpha_j, \beta_i]} + T_{\lambda_i - 1} - T_{\mu_j}]\right)_{1 \leq i,j \leq n} = \det\begin{pmatrix} 0 & x_1 x_2 t_1 t_2 \\ 1 & e_3(x_1, x_2, t_1, t_2) \end{pmatrix} \neq 0.$$

Remark 3.5. In Theorem 3.3 the assumption $\alpha_i \leq \alpha_{i+1}$ and $\beta_i \leq \beta_{i+1}$ whenever $\mu_i < \lambda_{i+1}$ is necessary. For example, if $\lambda = (2,2)$, $\mu = (1)$, $\alpha = (1,0)$, and $\beta = (1,1)$, then $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)} = 0$ but

$$\det\left(h_{\lambda_i-\mu_j-i+j}[X_{(\alpha_j,\beta_i]} + T_{i-1} - T_{j-1}]\right)_{1\leq i,j\leq n} = \det\begin{pmatrix} 0 & x_1^3 - x_1^2 t_1 \\ 1 & x_1^2 \end{pmatrix} \neq 0.$$

Moreover, the assumption $\lambda, \mu \in \mathrm{Par}_n$ is also necessary. If $\lambda = (1,1)$, $\mu = (0,1)$, $\alpha = (0,0)$, and $\beta = (1,1)$, then $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)} = 0$ but

$$\det\left(h_{\lambda_i-\mu_j-i+j}[X_{(\alpha_j,\beta_i]} + T_{i-1} - T_{j-1}]\right)_{1\leq i,j\leq n} = \det\begin{pmatrix} x_1 & x_1 - t_1 \\ 1 & 1 \end{pmatrix} \neq 0.$$

Let (a^n) denote the sequence $(a, a, \ldots, a)$ consisting of n a's. If we set $\alpha = (0^n)$ and $\beta = (b^n)$ and let $b \to \infty$ in Theorem 3.3, we obtain the formula (1.9) in the introduction, which we state again.

Corollary 3.6. *For any* $\lambda, \mu \in \mathrm{Par}_n$, *we have*

$$\widetilde{g}_{\lambda/\mu}(x;t) = \det\left(h_{\lambda_i-\mu_j-i+j}[X + T_{i-1} - T_{j-1}]\right)_{1\leq i,j\leq n}.$$

Proof. By definition of $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t)$ and Theorem 3.3,

$$\begin{aligned}
\widetilde{g}_{\lambda/\mu}(x;t) &= \lim_{b\to\infty} \widetilde{g}_{\lambda/\mu}((x_1,\ldots,x_b,0,0,\ldots);t) \\
&= \lim_{b\to\infty} \widetilde{g}_{\lambda/\mu}^{\mathrm{row}((0^n),(b^n))}(x;t) \\
&= \lim_{b\to\infty} \det\left(h_{\lambda_i-\mu_j-i+j}[X_{(0,b]} + T_{i-1} - T_{j-1}]\right)_{1\leq i,j\leq n} \\
&= \det\left(h_{\lambda_i-\mu_j-i+j}[X + T_{i-1} - T_{j-1}]\right)_{1\leq i,j\leq n}.
\end{aligned}$$

$\square$

Similarly, Theorem 1.2 follows from Theorem 3.1, and (1.8) follows from (1.5).

Theorem 1.3 is the special case $t_i = 1$ of Corollary 3.6. Amanov and Yeliussizov [1] showed Theorem 1.3 using Theorem 1.2 and an involution $\tau : \Lambda \to \Lambda$ satisfying $\tau(\widetilde{g}_{\lambda/\mu}(x;t)) = \widetilde{g}_{\lambda'/\mu'}(x;t)$ when all t_i are equal to 1. Unfortunately there is no such map for the general t. To see this suppose that there were an algebra homomorphism $\psi : \Lambda \to \Lambda$ satisfying $\psi(\widetilde{g}_{\lambda/\mu}(x;t)) = \widetilde{g}_{\lambda'/\mu'}(x;t)$. Then it must satisfy

$$\psi(h_k(x)) = \psi(\widetilde{g}_{(k)}(x;t)) = \widetilde{g}_{(1^k)}(x;t) = e_k[X + T_{k-1}].$$

Since $h_k(x) = \widetilde{g}_{(k)}(x;t) = \widetilde{g}_{(k+1)/(1)}(x;t)$, we must also have

$$\psi(h_k(x)) = \psi(\widetilde{g}_{(k+1)/(1)}(x;t)) = \widetilde{g}_{(1^{k+1})/(1)}(x;t) = e_k[X + T_k - T_1].$$

Since $e_k[X + T_{k-1}] \neq e_k[X + T_k - T_1]$, the map ψ cannot exist.

4. A proof of the Jacobi–Trudi formula

In this section we prove the Jacobi–Trudi formula for $\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}$ in Theorem 3.2. The basic idea of the proof is to show that both sides of the equation satisfy the same recurrence relation. We first introduce several definitions.

A *diagram* is just a (finite) set of pairs (i,j) of positive integers. Similarly to Young diagrams we also visualize a diagram ρ as an array of squares where a square is placed in row i and column j for each $(i,j) \in \rho$. For a diagram ρ, define ρ_i to be the number of j's such that $(i,j) \in \rho$. The kth *row* (resp. *column*) of ρ is the set of cells $(i,j) \in \rho$ with $i = k$ (resp. $j = k$). For two diagrams σ and ρ with $\rho \subseteq \sigma$, denote by $\sigma - \rho$ their set-theoretic difference, which is also a diagram. If a diagram ρ is a Young diagram with at most n rows, it is identified with the partition $(\rho_1, \ldots, \rho_n)$ as before. If R is an RPP of shape λ/μ and $\rho \subseteq \lambda/\mu$ is a diagram, the restriction of R to ρ is denoted by $R|_\rho$. We extend the definition of an RPP of shape λ/μ to an RPP of shape ρ for any diagram ρ as follows. A *reverse plane partition (RPP)* of shape ρ is a filling R of ρ with positive integers such that $R(i,j) \le R(i',j')$ for all $(i,j),(i',j') \in \rho$ with $i \le i'$ and $j \le j'$. The notation used for RPPs of shape λ/μ will be extended to RPPs of shape ρ in the obvious way. For example, RPP_ρ is the set of RPPs of shape ρ and $\mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$ is the set of elements $R \in \mathrm{RPP}_\rho$ with the additional condition that $\alpha_i + 1 \le R(i,j) \le \beta_i$ for all $(i,j) \in \rho$.

Let $\mu, \lambda \in \mathrm{Par}_n$ with $\mu \subseteq \lambda$. We define a total order $\prec$ on the cells in λ/μ as follows: $(i,j) \prec (i',j')$ if and only if $j > j'$ or $j = j'$ and $i < i'$. Note that by definition $(i,j) \prec (i',j')$ implies $(i,j) \ne (i',j')$. Denote by $(\lambda/\mu)^{(m)}$ the set of the first m cells in λ/μ in the total order $\prec$. Note that $(\lambda/\mu)^{(m)}$ is not necessarily a skew shape, see Figure 3.

Definition 4.1. Let $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \lambda \in \mathrm{Par}_n$ with $\mu \subseteq \lambda$. Let $\rho = (\lambda/\mu)^{(m)}$ for some $0 \le m \le |\lambda/\mu|$ and let $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$. Then we define

$$\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0) = \{R \in \mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)} : R|_\rho = R_0\},$$
$$C(\rho) = \{1 \le i \le n : \rho_i > 0\},$$
$$B(R_0,\beta) = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n),$$

where $\widetilde{\beta}_i$ is defined by

$$\widetilde{\beta}_i = \begin{cases} R_0(i, \lambda_i - \rho_i + 1), & \text{if } i \in C(\rho), \\ \beta_i, & \text{if } i \notin C(\rho). \end{cases}$$

Note that if $i \in C(\rho)$, then $(i, \lambda_i - \rho_i + 1)$ is the leftmost cell in the ith row of ρ.

One may consider an element in $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0)$ as an RPP in $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}$ that can be obtained from R_0 by filling the remaining cells in $(\lambda/\mu) - \rho$. The motivation for introducing $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0)$ is to construct an RPP of shape λ/μ by filling the cells in λ/μ one at a time with respect to the order of the cells given by $\prec$. This will allow us to find a recurrence relation for a generating function for restricted RPPs.
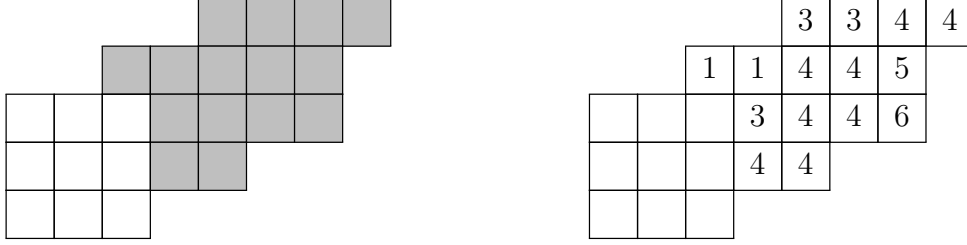
FIGURE 3. The left diagram shows λ/μ and $\rho = (\lambda/\mu)^{(m)}$, where $\lambda = (8,7,7,5,3)$, $\mu = (4,2)$, $m = 15$, and the cells in ρ are the gray cells. The right diagram shows an RPP of shape ρ.

Note that each element $\widetilde{\beta}_i$ in $B(R_0, \beta)$ acts as an upper bound for the remaining entries in row i for an RPP in $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0)$. For example, if R_0 is the RPP shown in Figure 3, and $\alpha = (0,0,1,1,2)$ and $\beta = (5,5,6,7,7)$, then $C(\rho) = \{1,2,3,4\}$ and $B(R_0, \beta) = (3,1,3,4,7)$. Note also that if the ith row of $(\lambda/\mu) - \rho$ is empty, then the lower and upper bounds for the entries in row i are irrelevant.

For $R \in \mathrm{RPP}_{\lambda/\mu}$, we define

$$\mathrm{wt}(R) = \prod_{i \geq 1} x_i^{a_i(R)} t_i^{b_i(R)},$$

where $a_i(R)$ is the number of columns containing an i and $b_i(R)$ is the number of cells (i,j) such that $(i,j),(i+1,j) \in \lambda/\mu$ and $R(i,j) = R(i+1,j)$. We also define

$$\overline{\mathrm{wt}}(R) = \prod_{j \geq 1} x_j^{\overline{a}_j(R)} t_j^{\overline{b}_j(R)},$$

where $\overline{a}_j(R)$ is the number of rows containing a j and $\overline{b}_j(R)$ is the number of cells (i,j) such that $(i,j),(i,j+1) \in \lambda/\mu$ and $R(i,j) = R(i,j+1)$.

Note that an RPP R and its transpose R' satisfy $\overline{\mathrm{wt}}(R) = \mathrm{wt}(R')$.

Definition 4.2. For any $\alpha, \beta \in \mathbb{N}^n$, $\mu, \lambda \in \mathrm{Par}_n$ with $\mu \subseteq \lambda$, and a fixed RPP R_0 of shape $\rho = (\lambda/\mu)^{(m)}$, define

$$\mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{R \in \mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0)} \mathrm{wt}(R),$$

$$\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{R \in \mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(R_0)} \overline{\mathrm{wt}}(R).$$

Note that, by definition,

$$\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}(x;t) = \mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(\emptyset),$$

$$\widetilde{g}_{\lambda/\mu}^{\mathrm{col}(\alpha,\beta)}(x;t) = \overline{\mathcal{R}}_{\lambda',\mu'}^{\alpha,\beta}(\emptyset),$$

where $\emptyset$ is the unique filling of the empty diagram and we define $\mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = 0$ if $\mu \not\subseteq \lambda$. In order to avoid using transposes in the proof, instead of the latter equation above we will consider

$$\widetilde{g}_{\lambda'/\mu'}^{\mathrm{col}(\alpha,\beta)}(x;t) = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset).$$

Definitions 4.1 and 4.2 will also be used in the next section. We need one more definition for this section.

Definition 4.3. For $\alpha, \beta, \mu, \lambda \in \mathbb{N}^n, C \subseteq [n]$, and $1 \leq i, j \leq n$, define

$$e_{\lambda,\mu}^{\alpha,\beta}(i,j) = e_{\lambda_i-i-\mu_j+j}[X_{(\alpha_j,\beta_i]} + T_{\lambda_i-1} - T_{\mu_j}],$$

$$\overline{e}_{\lambda,\mu}^{\alpha,\beta}(i,j) = e_{\lambda_i-i-\mu_j+j}[X_{(\alpha_j,\beta_i-1]} + T_{\lambda_i} - T_{\mu_j}],$$

$$e_{\lambda,\mu}^{\alpha,\beta}(C;i,j) = \begin{cases} e_{\lambda,\mu}^{\alpha,\beta}(i,j), & \text{if } i \notin C, \\ \overline{e}_{\lambda,\mu}^{\alpha,\beta}(i,j), & \text{if } i \in C, \end{cases}$$

$$E_{\lambda,\mu}^{\alpha,\beta}(C) = \det(e_{\lambda,\mu}^{\alpha,\beta}(C;i,j))_{1 \leq i,j \leq n},$$

$$E_{\lambda,\mu}^{\alpha,\beta} = \det(e_{\lambda,\mu}^{\alpha,\beta}(i,j))_{1 \leq i,j \leq n}.$$

If $\mu \subseteq \lambda$, $\rho = (\lambda/\mu)^{(m)}$, and $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$, we define

$$(4.1) \qquad \mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \overline{\mathrm{wt}}(R_0) E_{\lambda-\rho,\mu}^{\alpha,B(R_0,\beta)}(C(\rho)).$$

Note that

$$E_{\lambda,\mu}^{\alpha,\beta} = E_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = \mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(\emptyset),$$

where the second $\emptyset$ stands for the unique filling of the empty diagram $(\lambda/\mu)^{(0)}$.

Using the notation above Theorem 3.2 can be stated as

$$E_{\lambda,\mu}^{\alpha,\beta} = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset).$$

Our strategy is to show that both sides of the above equation satisfy the same recurrence relation.

We will frequently use the following lemmas, which can easily be proved using elementary linear algebra.

Lemma 4.4. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a matrix. If there is an integer $1 \leq k \leq n$ such that $a_{i,j} = 0$ for all $k \leq i \leq n$ and $1 \leq j \leq k$, then $\det(A) = 0$. Similarly, if there is an integer $1 \leq k \leq n$ such that $a_{i,j} = 0$ for all $1 \leq i \leq k$ and $k \leq j \leq n$, then $\det(A) = 0$.*

Lemma 4.5. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a matrix. If there is an integer $0 \leq k \leq n$ such that $a_{i,j} = 0$ for all $k+1 \leq i \leq n$ and $1 \leq j \leq k$, then $\det(A) = \det(a_{i,j})_{1 \leq i,j \leq k} \det(a_{i,j})_{k+1 \leq i,j \leq n}$. Similarly, if there is an integer $0 \leq k \leq n$ such that $a_{i,j} = 0$ for all $1 \leq i \leq k$ and $k+1 \leq j \leq n$, then $\det(A) = \det(a_{i,j})_{1 \leq i,j \leq k} \det(a_{i,j})_{k+1 \leq i,j \leq n}$.*

Lemma 4.6. *Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a matrix. Assume that there is an integer $1 \leq k \leq n$ such that $a_{i,j} = \chi(i = j = k)$ for all $k \leq i \leq n$ and $1 \leq j \leq k$. Then, $\det(A) = \det(a_{i,j})_{1 \leq i,j \leq k-1} \det(a_{i,j})_{k+1 \leq i,j \leq n}$. Furthermore, each nonzero term in the expansion of $\det(A)$ must contain the (k,k) entry (which is 1).*

4.1. **Technical lemmas.** In this subsection we give a list of lemmas that will be used to prove Theorem 3.2.

From now on, once n is given, let $\epsilon_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the sequence of $n-1$ zeros and one 1, where the unique 1 is at position k. For a statement p, we define $\chi(p) = 1$ if p is true and $\chi(p) = 0$ otherwise.

Let QPar_n denote the set of $\alpha \in \mathbb{N}^n$ such that $\alpha_i \leq \alpha_{i+1} + 1$ for all $i \in [n-1]$. Note that if $\alpha \in \mathrm{QPar}_n$ and $1 \leq i \leq j \leq n$, then $\alpha_i \leq \alpha_j + j - i$.

We note a simple but crucial fact: If $k \in \mathbb{Z}$, and if A is a sum of fewer than k variables, then

$$(4.2) \qquad\qquad\qquad\qquad e_k[A] = 0.$$

(This is because $e_k[A] = e_k(a_1, a_2, \ldots, a_m)$ whenever $A = a_1 + a_2 + \cdots + a_m$ is a sum of variables.)

Now we give a list of lemmas.

Lemma 4.7. *Let Z be a formal power series with integer coefficients and let z be any (single) variable. Then, for any integer k,*

$$e_k[Z] = e_k[Z - z] + z e_{k-1}[Z - z].$$

Proof. Since $e_k(x) = 0$ for $k < 0$ and $e_0(x) = 1$, the equation is clear when $k \leq 0$. Now we assume $k \geq 1$. By (2.4), for any $m \geq 0$, we have

$$e_m[Z - z] = \sum_{i=0}^{m} e_{m-i}[Z](-1)^i h_i(z) = \sum_{i=0}^{m} e_{m-i}[Z](-z)^i.$$

Thus

$$e_k[Z - z] + z e_{k-1}[Z - z] = \sum_{i=0}^{k} e_{k-i}[Z](-z)^i - \sum_{i=0}^{k-1} e_{k-i-1}[Z](-z)^{i+1} = e_k[Z],$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.8. *If $i \geq j \geq 0$ and $k > i - j$, then*

$$e_k[T_i - T_j] = 0.$$

Proof. By definition we have $e_k[T_i - T_j] = e_k[t_{j+1} + \cdots + t_i] = e_k(t_{j+1}, \ldots, t_i)$, which is equal to 0 because $k > i - j$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.9. *Let $\alpha, \beta \in \mathbb{N}^n$, $\mu \in \mathrm{Par}_n$, and $C \subseteq [n]$. Then*

$$E_{\mu,\mu}^{\alpha,\beta}(C) = 1.$$

Proof. If $i \geq j$, we have $\mu_i - i - \mu_j + j \leq 0$, where the equality holds if and only if $i = j$. This shows that for all $1 \leq j \leq i \leq n$, we have $e_{\mu,\mu}^{\alpha,\beta}(C; i, j) = \chi(i = j)$ since $e_{\mu_i - i - \mu_j + j}[Y - Z] = \chi(i = j)$ for any sums Y and Z of variables. Therefore the matrix $(e_{\mu,\mu}^{\alpha,\beta}(C; i, j))_{1 \leq i,j \leq n}$ is upper uni-triangular and $E_{\mu,\mu}^{\alpha,\beta}(C) = \det(e_{\mu,\mu}^{\alpha,\beta}(C; i, j))_{1 \leq i,j \leq n} = 1$. $\quad\square$

Lemma 4.10. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\lambda, \mu \in \mathrm{Par}_n$ with $\mu \not\subseteq \lambda$. Then for any subset $C \subseteq [n]$,*

$$E_{\lambda,\mu}^{\alpha,\beta}(C) = 0.$$

Proof. Since $\mu \not\subseteq \lambda$, there is an integer $1 \le k \le n$ such that $\mu_k > \lambda_k$. Then for all $k \le i \le n$ and $1 \le j \le k$, we have $\lambda_i - i - \mu_j + j \le \lambda_k - k - \mu_k + k < 0$, and therefore $e_{\lambda,\mu}^{\alpha,\beta}(C; i, j) = 0$ since $e_m(x) = 0$ for $m < 0$. By Lemma 4.4 this shows that $E_{\lambda,\mu}^{\alpha,\beta}(C) = \det(e_{\lambda,\mu}^{\alpha,\beta}(C; i, j))_{1 \le i,j \le n} = 0$. $\qquad\square$

Lemma 4.11. *Let $\alpha, \beta \in \mathrm{QPar}_n$ and $\lambda, \mu \in \mathrm{Par}_n$. Suppose that $\alpha_k \ge \beta_k$ and $\mu_k < \lambda_k$ for some $1 \le k \le n$. Then*

$$E_{\lambda,\mu}^{\alpha,\beta} = 0.$$

Proof. By Lemma 4.4 it is enough to show that $e_{\lambda,\mu}^{\alpha,\beta}(i, j) = 0$ assuming $1 \le i \le k$ and $k \le j \le n$. Since $\alpha, \beta \in \mathrm{QPar}_n$, we have $\alpha_j + j - k \ge \alpha_k \ge \beta_k \ge \beta_i - k + i$. Thus $\beta_i - \alpha_j \le j - i$ and $X_{(\alpha_j,\beta_i]}$ is a sum of at most $j - i$ variables. Furthermore, $T_{\lambda_i-1} - T_{\mu_j}$ is a sum of $\lambda_i - 1 - \mu_j$ variables (since $\lambda_i \ge \lambda_k > \mu_k \ge \mu_j$). Using these two facts, we obtain

$$e_{\lambda,\mu}^{\alpha,\beta}(i, j) = e_{\lambda_i-i-\mu_j+j}[X_{(\alpha_j,\beta_i]} + T_{\lambda_i-1} - T_{\mu_j}] = 0,$$

because $X_{(\alpha_j,\beta_i]} + T_{\lambda_i-1} - T_{\mu_j}$ is a sum of at most $\lambda_i - i - \mu_j + j - 1$ variables (and because of (4.2)). This completes the proof. $\qquad\square$

Lemma 4.12. *Let $\alpha \in \mathrm{QPar}_n$, $\beta \in \mathbb{N}^n$, $\mu \in \mathrm{Par}_n$, and $\lambda \in \mathbb{N}^n$. Suppose that $k \in [n]$ is an integer satisfying the following conditions:*

    (1) $\mu_k < \lambda_k$,
    (2) $\alpha_k < \beta_k$, and
    (3) if $1 \le j < k$ and $\mu_j < \lambda_k$, then $\alpha_j < \beta_k$.

Then for any $C \subseteq [n]$, we have

$$E_{\lambda,\mu}^{\alpha,\beta}(C) = E_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}(C \setminus \{k\}) + y E_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}(C \cup \{k\}),$$

where

$$y = \begin{cases} x_{\beta_k}, & \text{if } k \notin C, \\ t_{\lambda_k}, & \text{if } k \in C. \end{cases}$$

Proof. We consider the two cases $k \notin C$ and $k \in C$.

    Case 1: $k \notin C$. We claim that, for all $1 \le j \le n$,

(4.3) $$e_{\lambda,\mu}^{\alpha,\beta}(k, j) = e_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}(k, j) + x_{\beta_k} \overline{e}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}(k, j).$$

By definition we can rewrite (4.3) as

(4.4) $$e_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k]} + T_{\lambda_k-1} - T_{\mu_j}] = e_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k-1} - T_{\mu_j}]$$
$$+ x_{\beta_k} e_{\lambda_k-k-\mu_j+j-1}[X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k-1} - T_{\mu_j}].$$

If $\alpha_j < \beta_k$, (4.4) follows from Lemma 4.7 with $Z = X_{(\alpha_j,\beta_k]} + T_{\lambda_k-1} - T_{\mu_j}$ and $z = x_{\beta_k}$.

Suppose now that $\alpha_j \geq \beta_k$. Then $X_{(\alpha_j,\beta_k]} = 0$ and $X_{(\alpha_j,\beta_k-1]} = 0$. Therefore in order to prove the claim (4.4), it suffices to show

$$(4.5) \qquad\qquad e_{\lambda_k-k-\mu_j+j-1}[T_{\lambda_k-1} - T_{\mu_j}] = 0,$$

because then both sides of (4.4) are equal to $e_{\lambda_k-k-\mu_j+j}[T_{\lambda_k-1} - T_{\mu_j}]$. We will prove (4.5) by considering the two cases $j \leq k$ and $k < j$.

First we assume $j \leq k$. Since $\alpha_j \geq \beta_k$ and $\alpha_k < \beta_k$, we have $j \neq k$, thus $j < k$. Thus, by condition (3), if $\mu_j < \lambda_k$, then $\alpha_j < \beta_k$, which contradicts the assumption $\alpha_j \geq \beta_k$. Thus we must have $\lambda_k \leq \mu_j$. Since $j \leq k$ and $\lambda_k \leq \mu_j$, we have $\lambda_k - k - \mu_j + j - 1 \leq -1$, which shows (4.5).

Now we assume $k < j$. Then $\lambda_k > \mu_k \geq \mu_j$. Since $\lambda_k - k - \mu_j + j - 1 > \lambda_k - \mu_j - 1$ and $\lambda_k - 1 \geq \mu_j$, we obtain (4.5) by Lemma 4.8. This establishes the claim (4.3).

Using (4.3) and the linearity of the determinant in its kth row, we obtain the identity in the lemma in this case.

Case 2: $k \in C$. We claim that, for all $1 \leq j \leq n$,

$$(4.6) \qquad\qquad \bar{e}^{\alpha,\beta}_{\lambda,\mu}(k,j) = e^{\alpha,\beta-\epsilon_k}_{\lambda,\mu}(k,j) + t_{\lambda_k}\bar{e}^{\alpha,\beta}_{\lambda-\epsilon_k,\mu}(k,j).$$

By Lemma 4.7 with $Z = X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k} - T_{\mu_j}$ and $z = t_{\lambda_k}$, we have

$$e_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k} - T_{\mu_j}] = e_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k-1} - T_{\mu_j}]$$
$$+ t_{\lambda_k}e_{\lambda_k-k-\mu_j+j-1}[X_{(\alpha_j,\beta_k-1]} + T_{\lambda_k-1} - T_{\mu_j}],$$

which is exactly (4.6). Using (4.6) and the linearity of the determinant in its kth row, we obtain the identity in the lemma in this case, which completes the proof. $\square$

Lemma 4.13. *Suppose that $\alpha, \beta, \mu, \lambda \in \mathbb{N}^n$, $C \subseteq [n]$, and $r \in [n]$ satisfy $r \notin C$, $r-1 \in C$, $\lambda_r = \lambda_{r-1} + 1$, and $\beta_r = \beta_{r-1} - 1$. Then $E^{\alpha,\beta}_{\lambda,\mu}(C) = 0$.*

Proof. We compare rows r and $r-1$ of the matrix in the definition of

$$E^{\alpha,\beta}_{\lambda,\mu}(C) = \det\left(e^{\alpha,\beta}_{\lambda,\mu}(C;i,j)\right)_{1\leq i,j\leq n}.$$

Since $r \notin C$, the (r,j) entry of the matrix is

$$e^{\alpha,\beta}_{\lambda,\mu}(C;r,j) = e^{\alpha,\beta}_{\lambda,\mu}(r,j) = e_{\lambda_r-r-\mu_j+j}[X_{(\alpha_j,\beta_r]} + T_{\lambda_r-1} - T_{\mu_j}].$$

Since $r-1 \in C$, the $(r-1,j)$ entry of the matrix is

$$e^{\alpha,\beta}_{\lambda,\mu}(C;r-1,j) = \bar{e}^{\alpha,\beta}_{\lambda,\mu}(r-1,j) = e_{\lambda_{r-1}-(r-1)-\mu_j+j}[X_{(\alpha_j,\beta_{r-1}-1]} + T_{\lambda_{r-1}} - T_{\mu_j}].$$

Since $\lambda_r = \lambda_{r-1} + 1$ and $\beta_r = \beta_{r-1} - 1$, the right hand sides of the above two equations are equal. Therefore rows $r-1$ and r of the matrix are identical, which implies $E^{\alpha,\beta}_{\lambda,\mu}(C) = 0$. $\square$

4.2. **Proof of Theorem 3.2.** We first show that $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ and $\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ satisfy the same recurrence relation under certain conditions.

Proposition 4.14. *Let* $\alpha, \beta \in \mathrm{QPar}_n$ *and* $\lambda, \mu \in \mathrm{Par}_n$ *with* $\alpha < \beta$ *and* $\mu < \lambda$. *Fix* $(r, c) \in \lambda/\mu$ *and* $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$, *where* ρ *is the set of cells* $(i, j) \in \lambda/\mu$ *with* $(i, j) \prec (r, c)$. *Let* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n) = B(R_0, \beta)$. *Then*

$$(4.7) \qquad \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{k=a}^{\widetilde{\beta}_r} \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}),$$

$$(4.8) \qquad \mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{k=a}^{\widetilde{\beta}_r} \mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}),$$

where $R_0 \cup \{k\}$ *is the RPP obtained from* R_0 *by adding the cell* (r, c) *with entry* k, *and*

$$a = \begin{cases} \widetilde{\beta}_{r-1}, & \text{if } (r-1, c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \alpha_r + 1, & \text{otherwise.} \end{cases}$$

Proof. Observe that in order to construct an RPP R in $\mathrm{RPP}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}$ such that $R|_\rho = R_0$ we must fill the (r, c) cell of R with one of the integers $a, a+1, \ldots, \widetilde{\beta}_r$. Therefore the first identity (4.7) is immediate from the definition of $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0)$.

It remains to prove the second identity (4.8). From the equality (4.14) below, we have $\alpha_r < \widetilde{\beta}_r$.

Let $C = C(\rho) = \{1 \leq i \leq n : \rho_i > 0\}$ and

$$(4.9) \qquad s = \begin{cases} \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1, & \text{if } (r-1, c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \widetilde{\beta}_r - \alpha_r, & \text{otherwise.} \end{cases}$$

Lemma 4.15 below (specifically, (4.15)) shows that $s \geq 0$. Since $s = \widetilde{\beta}_r - a + 1$, this shows that s is the number of integers k satisfying $a \leq k \leq \widetilde{\beta}_r$. We will consider the two cases $s = 0$ and $s \geq 1$.

We first consider the case $s = 0$. In this case, $\widetilde{\beta}_r = a - 1$, so that the right hand side of (4.8) is zero. Recall from (4.1) that the left hand side of (4.8) is

$$\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \overline{\mathrm{wt}}(R_0) E_{\lambda-\rho,\mu}^{\alpha,B(R_0,\beta)}(C(\rho)) = \overline{\mathrm{wt}}(R_0) E_{\lambda-\rho,\mu}^{\alpha,\widetilde{\beta}}(C),$$

since $\widetilde{\beta} = B(R_0, \beta)$ and $C = C(\rho)$. Hence, to prove (4.8), it suffices to show that

$$(4.10) \qquad E_{\lambda-\rho,\mu}^{\alpha,\widetilde{\beta}}(C) = 0.$$

From (4.16) below, we get $r \notin C(\rho) = C$, whence $C = C \setminus \{r\}$. Therefore (4.10) follows from Lemma 4.16. This shows (4.8) for the case $s = 0$.

It remains to prove (4.8) for the case $s \geq 1$. Let

$$y = \begin{cases} x_{\widetilde{\beta}_r}, & \text{if } r \notin C, \\ t_{\lambda_r - \rho_r}, & \text{if } r \in C. \end{cases}$$

Since $(r, c) \in (\lambda/\mu) - \rho$, we have $\mu_r < (\lambda - \rho)_r$. We also know that $\alpha_r < \widetilde{\beta}_r$. By (4.19) below, if $j < r$ and $\mu_j < (\lambda - \rho)_r$, then $\alpha_j < \widetilde{\beta}_r - (s-1) \leq \widetilde{\beta}_r$ (since $s \geq 1$). Therefore by Lemma 4.12 we have

$$(4.11) \qquad E^{\alpha, \widetilde{\beta}}_{\lambda - \rho, \mu}(C) = y E^{\alpha, \widetilde{\beta}}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}) + E^{\alpha, \widetilde{\beta} - \epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}).$$

For $1 \leq i \leq s-1$, by (4.18) below, we have $\alpha_r < \widetilde{\beta}_r - (s-1) \leq \widetilde{\beta}_r - i = (\widetilde{\beta} - i\epsilon_r)_r$ (whence $\widetilde{\beta} - i\epsilon_r \in \mathbb{N}^n$), and furtheremore, by (4.19) below, we have $\alpha_j < \widetilde{\beta}_r - (s-1) \leq (\widetilde{\beta} - i\epsilon_r)_r$ for each $j < r$ satisfying $\mu_j < (\lambda - \rho)_r$. Therefore we can apply Lemma 4.12 repeatedly to $E^{\alpha, \widetilde{\beta} - i\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\})$, for $i = 1, 2, \ldots, s-1$, to get

$$E^{\alpha, \widetilde{\beta} - \epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) = E^{\alpha, \widetilde{\beta} - 2\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) + x_{\widetilde{\beta}_r - 1} E^{\alpha, \widetilde{\beta} - \epsilon_r}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}),$$

$$E^{\alpha, \widetilde{\beta} - 2\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) = E^{\alpha, \widetilde{\beta} - 3\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) + x_{\widetilde{\beta}_r - 2} E^{\alpha, \widetilde{\beta} - 2\epsilon_r}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}),$$

$$\vdots$$

$$E^{\alpha, \widetilde{\beta} - (s-1)\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) = E^{\alpha, \widetilde{\beta} - s\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) + x_{\widetilde{\beta}_r - (s-1)} E^{\alpha, \widetilde{\beta} - (s-1)\epsilon_r}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}).$$

Since $s \geq 1$, combining (4.11) and the above equations yields

$$(4.12) \qquad E^{\alpha, \widetilde{\beta}}_{\lambda - \rho, \mu}(C) = y E^{\alpha, \widetilde{\beta}}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}) + E^{\alpha, \widetilde{\beta} - s\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\})$$

$$+ \sum_{i=1}^{s-1} x_{\widetilde{\beta}_r - i} E^{\alpha, \widetilde{\beta} - i\epsilon_r}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}).$$

The second summand of the right hand side of (4.12) vanishes:

$$(4.13) \qquad E^{\alpha, \widetilde{\beta} - s\epsilon_r}_{\lambda - \rho, \mu}(C \setminus \{r\}) = 0,$$

because of Lemma 4.16 below.

In view of $C = C(\rho)$ and $\widetilde{\beta} = B(R_0, \beta)$, we can rewrite (4.1) as

$$\mathcal{E}^{\alpha, \beta}_{\lambda, \mu}(R_0) = \overline{\mathrm{wt}}(R_0) E^{\alpha, \widetilde{\beta}}_{\lambda - \rho, \mu}(C).$$

Thus, multiplying both sides of (4.12) by $\overline{\mathrm{wt}}(R_0)$ and using (4.13) we obtain

$$\mathcal{E}^{\alpha, \beta}_{\lambda, \mu}(R_0) = y \overline{\mathrm{wt}}(R_0) E^{\alpha, \widetilde{\beta}}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}) + \sum_{i=1}^{s-1} x_{\widetilde{\beta}_r - i} \overline{\mathrm{wt}}(R_0) E^{\alpha, \widetilde{\beta} - i\epsilon_r}_{\lambda - \rho - \epsilon_r, \mu}(C \cup \{r\}).$$

One can easily check that $y\overline{\text{wt}}(R_0) = \overline{\text{wt}}(R_0 \cup \{\widetilde{\beta}_r\})$ and $x_{\widetilde{\beta}_r - i}\overline{\text{wt}}(R_0) = \overline{\text{wt}}(R_0 \cup \{\widetilde{\beta}_r - i\})$ for $1 \leq i \leq s-1$. Hence we can rewrite the above equation as

$$\mathcal{E}^{\alpha,\beta}_{\lambda,\mu}(R_0) = \sum_{i=0}^{s-1} \overline{\text{wt}}(R_0 \cup \{\widetilde{\beta}_r - i\}) E^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}(C \cup \{r\}) = \sum_{i=0}^{s-1} \mathcal{E}^{\alpha,\beta}_{\lambda,\mu}(R_0 \cup \{\widetilde{\beta}_r - i\}),$$

where the last equality follows from $\widetilde{\beta} - i\epsilon_r = B(R_0 \cup \{\widetilde{\beta}_r - i\}, \beta)$. Since $s = \widetilde{\beta}_r - a + 1$, this is equivalent to (4.8) and the proof is completed. $\qquad\square$

The following lemma proves two statements used in the proof of Proposition 4.14.

Lemma 4.15. *Following the notation in Proposition 4.14 and letting*

$$s = \begin{cases} \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1, & \text{if } (r-1,c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \widetilde{\beta}_r - \alpha_r, & \text{otherwise}, \end{cases}$$

we have

(4.14) $$\alpha_r < \widetilde{\beta}_r,$$

(4.15) $$s \geq 0,$$

(4.16) $$\text{if } s = 0, \text{ then } r \notin C(\rho),$$

(4.17) $$\text{if } \beta_{r-1} \leq \beta_r, \text{ then } s \geq 1,$$

(4.18) $$\alpha_r < \widetilde{\beta}_r - (s-1), \text{ and}$$

(4.19) $$\text{if } j < r \text{ and } \mu_j < (\lambda - \rho)_r, \text{ then } \alpha_j < \widetilde{\beta}_r - (s-1).$$

Proof. The first statement (4.14) is easiest to prove: If $r \notin C(\rho)$, then $\widetilde{\beta}_r = \beta_r \geq \alpha_r + 1$ (since $\alpha < \beta$); if $r \in C(\rho)$, then $\widetilde{\beta}_r = R_0(r, \lambda_r - \rho_r + 1) \geq \alpha_r + 1$ (since $R_0 \in \text{RPP}^{\text{row}(\alpha,\beta)}_\rho$). In either case we have $\widetilde{\beta}_r \geq \alpha_r + 1$, and thus (4.14) holds.

We will prove the next two statements (4.15) and (4.16) by considering the two cases in the definition of s.

Case 1: $(r-1,c) \in \rho$ and $\widetilde{\beta}_{r-1} \geq \alpha_r + 1$. Then $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1$. Note that $\beta_{r-1} \leq \beta_r + 1$ since $\beta \in \text{QPar}_n$. Observe that $r - 1 \in C(\rho)$ and $(r-1,c)$ is the leftmost cell in the $(r-1)$st row of ρ. Thus $\widetilde{\beta}_{r-1} = R_0(r-1,c)$. If $r \notin C(\rho)$, we have

(4.20) $$\widetilde{\beta}_{r-1} = R_0(r-1,c) \leq \beta_{r-1} \leq \beta_r + 1 = \widetilde{\beta}_r + 1$$

by the definition of $\widetilde{\beta}_r$, and thus $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1 \geq 0$. If $r \in C(\rho)$, then $(r, c+1)$ is the leftmost cell of the rth row of ρ and therefore

(4.21) $$\widetilde{\beta}_{r-1} = R_0(r-1,c) \leq R_0(r,c+1) = \widetilde{\beta}_r$$

and thus $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1 \geq 1 \geq 0$. Thus we always have $s \geq 0$. This means that (4.15) holds.

If $r \in C(\rho)$, then $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1 \geq 1$ by (4.21), and thus $s \neq 0$. Taking the contrapositive yields (4.16). So we have proved both (4.15) and (4.16) in this case.

Case 2: $(r-1, c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1$. Then, $s = \widetilde{\beta}_r - \alpha_r$. From (4.14), we have $\widetilde{\beta}_r \geq \alpha_r + 1$. Therefore $s = \widetilde{\beta}_r - \alpha_r \geq 1$ (so $s = 0$ cannot happen in this case), which completes the proof of the two statements (4.15) and (4.16).

The fourth statement (4.17) is proved just as we proved (4.15), except that $\beta_{r-1} \leq \beta_r + 1 = \widetilde{\beta}_r + 1$ is replaced by $\beta_{r-1} \leq \beta_r = \widetilde{\beta}_r$ in (4.20).

Now we prove the fifth statement (4.18). By definition of s we have

$$(4.22) \qquad \widetilde{\beta}_r - (s-1) = \begin{cases} \widetilde{\beta}_{r-1}, & \text{if } (r-1, c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \alpha_r + 1, & \text{otherwise}, \end{cases}$$

which implies $\widetilde{\beta}_r - (s-1) \geq \alpha_r + 1$. This shows the fifth statement (4.18).

For the last statement (4.19) suppose that $j < r$ and $\mu_j < (\lambda - \rho)_r$. We have $\mu_j < (\lambda - \rho)_r = c$ by the definition of ρ, thus $(j, c) \notin \mu$. In view of $j < r$, this leads to $(j, c) \in \lambda/\mu$ and consequently $(j, c) \in \rho$. Hence, $(r-1, c) \in \rho$ (since $j \leq r - 1 < r$), and thus (as in Case 1 above) $\widetilde{\beta}_{r-1} = R_0(r-1, c)$. Now

$$\alpha_j < \alpha_j + 1 \leq R_0(j, c) \leq R_0(r-1, c) = \widetilde{\beta}_{r-1} \leq \widetilde{\beta}_r - (s-1),$$

where the last inequality follows from (4.22) because $(r-1, c) \in \rho$. This shows (4.19). $\square$

Now we prove the identity (4.13) used in the proof of Proposition 4.14.

Lemma 4.16. *Following the notation in Proposition 4.14 we have*

$$(4.23) \qquad\qquad E_{\lambda-\rho,\mu}^{\alpha,\widetilde{\beta}-s\epsilon_r}(C \setminus \{r\}) = 0,$$

where $C = C(\rho)$ and

$$s = \begin{cases} \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1, & \text{if } (r-1, c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \widetilde{\beta}_r - \alpha_r, & \text{otherwise}. \end{cases}$$

Proof. Clearly, $\lambda - \rho \in \mathbb{N}^n$. Also, (4.18) shows that $\widetilde{\beta}_r - s \geq \alpha_r \geq 0$, whence $\widetilde{\beta} - s\epsilon_r \in \mathbb{N}^n$. We consider the two cases in the definition of s separately.

Case 1: $(r-1, c) \in \rho$ and $\widetilde{\beta}_{r-1} \geq \alpha_r + 1$. Then $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1$. In order to prove (4.23) it suffices to check that $\alpha, \widetilde{\beta} - s\epsilon_r, \lambda - \rho, \mu, C \setminus \{r\}$, and r satisfy the conditions for $\alpha, \beta, \lambda, \mu, C$, and r in Lemma 4.13. Since $(r-1, c) \in \rho$ and ρ is the set of all cells $(i, j) \prec (r, c)$ we have $(\lambda - \rho)_r = (\lambda - \rho)_{r-1} + 1$. The fact $(r-1, c) \in \rho$ also implies $r - 1 \in C \setminus \{r\}$, and clearly $r \notin C \setminus \{r\}$. Since $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1$, we have $(\widetilde{\beta} - s\epsilon_r)_r = (\widetilde{\beta} - s\epsilon_r)_{r-1} - 1$. Therefore the conditions in Lemma 4.13 hold, and (4.23) is proved in this case.

Case 2: $(r-1, c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1$. Then $s = \widetilde{\beta}_r - \alpha_r$. Let d be the integer such that $(d, c) \in \lambda/\mu$ and $(d-1, c) \notin \lambda/\mu$. In other words, (d, c) is the topmost cell in the cth column of λ/μ, see Figure 4. Then $1 \leq d \leq r$. Let $\kappa = \widetilde{\beta} - s\epsilon_r$ and $\sigma = \lambda - \rho$, so that

$$E_{\lambda-\rho,\mu}^{\alpha,\widetilde{\beta}-s\epsilon_r}(C \setminus \{r\}) = E_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}) = \det(e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j))_{1 \leq i, j \leq n}.$$

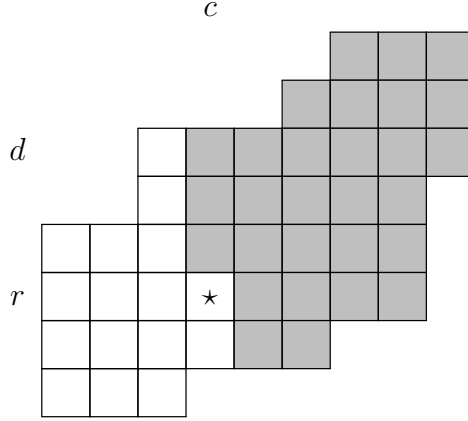FIGURE 4. An example of λ/μ and ρ, where the cells in ρ are the gray cells. The (r, c) cell is marked with a star and the row indices d and r, and the column index c are shown.

We claim that

$$(4.24) \qquad E_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}) = \det(e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j))_{1 \leq i,j \leq d-1} \det(e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j))_{d \leq i,j \leq n}.$$

To show (4.24), by Lemma 4.5 it suffices to show that for all $d \leq i \leq n$ and $1 \leq j \leq d-1$, we have

$$(4.25) \qquad e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j) = 0.$$

By the definitions of ρ and d (see Figure 4), if $d \leq i \leq n$ and $1 \leq j \leq d-1$, we have

$$\sigma_i = (\lambda - \rho)_i \leq c \leq \mu_j$$

(since $(j, c) \in \mu$) and therefore

$$(4.26) \qquad \sigma_i - i - \mu_j + j < \sigma_i - \mu_j \leq 0.$$

Then (4.25) follows from (4.26) and the claim (4.24) is proved.

By (4.24), to show (4.23) it suffices to show that

$$(4.27) \qquad \det(e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j))_{d \leq i,j \leq n} = 0.$$

By Lemma 4.4, to show (4.27) it is enough to show (4.25) for all $d \leq i \leq r$ and $r \leq j \leq n$.

By Definition 4.3 we have

$$e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j) = \begin{cases} e_{\sigma_i - i - \mu_j + j}[X_{(\alpha_j, \kappa_i - 1]} + T_{\sigma_i} - T_{\mu_j}], & \text{if } i \in C \setminus \{r\}, \\ e_{\sigma_i - i - \mu_j + j}[X_{(\alpha_j, \kappa_i]} + T_{\sigma_i - 1} - T_{\mu_j}], & \text{if } i \notin C \setminus \{r\}. \end{cases}$$

Suppose $i = r$ and $r \leq j \leq n$. Then the above equation becomes

$$(4.28) \qquad e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; r, j) = e_{\sigma_r - r - \mu_j + j}[X_{(\alpha_j, \kappa_r]} + T_{\sigma_r - 1} - T_{\mu_j}].$$

Since

$$\sigma_r = (\lambda - \rho)_r > \mu_r \geq \mu_j,$$

the sum $X_{(\alpha_j,\kappa_r]} + T_{\sigma_r-1} - T_{\mu_j}$ is a sum of $\max\{0, \kappa_r - \alpha_j\} + \sigma_r - 1 - \mu_j$ variables. Since

$$\kappa_r = (\widetilde{\beta} - s\epsilon_r)_r = \widetilde{\beta}_r - s = \alpha_r \leq \alpha_j + j - r$$

(a consequence of $\alpha \in \mathrm{QPar}_n$ and $r \leq j$) and therefore $\kappa_r - \alpha_j \leq j - r$ and consequently $\max\{0, \kappa_r - \alpha_j\} \leq \max\{0, j - r\} = j - r$ (since $r \leq j$), we have

$$\max\{0, \kappa_r - \alpha_j\} + \sigma_r - 1 - \mu_j \leq j - r + \sigma_r - 1 - \mu_j < \sigma_r - r - \mu_j + j.$$

Thus, (4.2) shows that the right hand side of (4.28) is zero, and (4.25) holds for $i = r$ and $r \leq j \leq n$.

It remains to prove (4.25) for $d \leq i \leq r - 1$ and $r \leq j \leq n$. Recall the assumption that $(r - 1, c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1$. If $(r - 1, c) \notin \rho$, then $r = d$ and there is no integer i with $d \leq i \leq r - 1$. Therefore we may assume $(r - 1, c) \in \rho$ and $\widetilde{\beta}_{r-1} < \alpha_r + 1$. Since $d \leq i \leq r - 1$, we have $(i, c) \in \rho$, which is the leftmost cell in the ith row of ρ. Thus $\widetilde{\beta}_i = R_0(i, c)$ and considering the case $i = r - 1$ we also have $\widetilde{\beta}_{r-1} = R_0(r - 1, c)$. Then we obtain

$$\kappa_i = (\widetilde{\beta} - s\epsilon_r)_i = \widetilde{\beta}_i = R_0(i, c) \leq R_0(r - 1, c) = \widetilde{\beta}_{r-1} < \alpha_r + 1 \leq \alpha_j + j - r + 1$$

(a consequence of $\alpha \in \mathrm{QPar}_n$ and $r \leq j$), which shows

$$\kappa_i - 1 - \alpha_j < j - r < j - i.$$

Combined with $0 < j - i$ (which is because $i \leq r - 1 < r \leq j$), this yields

(4.29) $\max\{0, \kappa_i - 1 - \alpha_j\} < j - i.$

Since $(i, c) \in \rho$, we have $i \in C(\rho) = C$. Thus $i \in C \setminus \{r\}$, and consequently

(4.30) $e_{\sigma,\mu}^{\alpha,\kappa}(C \setminus \{r\}; i, j) = e_{\sigma_i-i-\mu_j+j}[X_{(\alpha_j,\kappa_i-1]} + T_{\sigma_i} - T_{\mu_j}].$

On the other hand, since $d \leq i \leq r - 1$ and $j \geq r$, we have

$$\sigma_i = (\lambda - \rho)_i = c - 1 \qquad \text{(since } (i, c) \in \rho \text{ and } (i, c - 1) \notin \rho\text{)}$$
$$\geq \mu_r \qquad \text{(since } (r, c) \in \lambda/\mu\text{)}$$
$$\geq \mu_j.$$

Therefore $X_{(\alpha_j,\kappa_i-1]} + T_{\sigma_i} - T_{\mu_j}$ is a sum of $\max\{0, \kappa_i - 1 - \alpha_j\} + \sigma_i - \mu_j$ variables. By (4.29), we have

$$\max\{0, \kappa_i - 1 - \alpha_j\} + \sigma_i - \mu_j < j - i + \sigma_i - \mu_j = \sigma_i - i - \mu_j + j,$$

which implies (via (4.2)) that the right hand side of (4.30) is zero. Thus we obtain (4.25) and the proof is completed. $\qquad\square$

The parallel recurrence relations for $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ and $\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ in Proposition 4.14 can be used to conclude that they are equal.

Proposition 4.17. *Let $\alpha, \beta \in \mathrm{QPar}_n$, $\lambda, \mu \in \mathrm{Par}_n$ with $\alpha < \beta$ and $\mu < \lambda$. Let $\rho = (\lambda/\mu)^{(m)}$ for some $0 \leq m \leq |\lambda/\mu|$ and let $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$. Then*

$$\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0).$$

Proof. We use induction on $N = |\lambda/\mu| - |\rho|$. For the base case, suppose $N = 0$ so that $\rho = \lambda/\mu$. Then clearly $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \overline{\mathrm{wt}}(R_0)$ and, by Lemma 4.9,

$$\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \overline{\mathrm{wt}}(R_0) E_{\mu,\mu}^{\alpha,B(R_0,\beta)}(C(\rho)) = \overline{\mathrm{wt}}(R_0).$$

For the inductive step let $0 < N \leq |\lambda/\mu|$ and assume the assertion for $N - 1$. Since $\rho \neq \lambda/\mu$, we can find $(r, c) \in \lambda/\mu$ such that $\rho = \{(i, j) \in \lambda/\mu : (i, j) \prec (r, c)\}$. By Proposition 4.14 and the induction hypothesis, we obtain

$$\mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{k=a}^{\widetilde{\beta}_r} \mathcal{E}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}) = \sum_{k=a}^{\widetilde{\beta}_r} \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}) = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(R_0),$$

where a and $\widetilde{\beta}$ are given as in Proposition 4.14. Hence the assertion still holds for N and the proof follows by induction. $\qquad\square$

Now we are ready to prove Theorem 3.2, which can be restated as follows.

Theorem 4.18. *Let* $\alpha, \beta \in \mathbb{N}^n$ *and* $\mu, \lambda \in \mathrm{Par}_n$. *If* $\alpha_i \leq \alpha_{i+1} + 1$ *and* $\beta_i \leq \beta_{i+1} + 1$ *whenever* $\mu_i < \lambda_{i+1}$, *then*

$$(4.31) \qquad\qquad E_{\lambda,\mu}^{\alpha,\beta} = \overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset).$$

Proof. We will successively reduce the cases so that we eventually have the assumptions $\alpha, \beta \in \mathrm{QPar}_n$, $\alpha < \beta$ and $\mu < \lambda$ in Proposition 4.17. For a diagram σ, we denote by $\delta(\sigma)$ the diagram obtained by translating σ down by one row, so that $\delta^k(\sigma) = \{(i+k, j) : (i, j) \in \sigma\}$ for all $k \geq 0$. Note that there is a canonical bijection between the RPPs R of shape σ and the RPPs R' of shape $\delta^k(\sigma)$, and that this bijection satisfies $\overline{\mathrm{wt}}(R') = \overline{\mathrm{wt}}(R)$.

If $\mu \not\subseteq \lambda$, both sides of the equation (4.31) are zero by Lemma 4.10 and the definition of $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset)$. Hence we may assume $\mu \subseteq \lambda$. Thus, either $\lambda_k = \mu_k$ for some $1 \leq k \leq n$, or $\mu < \lambda$.

Suppose that $\lambda_k = \mu_k$ for some $1 \leq k \leq n$. Then, for $k \leq i \leq n$ and $1 \leq j \leq k$, since $\lambda_i - i - \mu_j + j \leq \lambda_k - k - \mu_k + k = 0$ and the equality holds if and only if $i = j = k$, we have

$$e_{\lambda,\mu}^{\alpha,\beta}(i, j) = e_{\lambda_i - i - \mu_j + j}[X_{(\alpha_j,\beta_i]} + T_{\lambda_i - 1} - T_{\mu_j}] = \chi(i = j = k).$$

By Lemma 4.6 this shows that

$$E_{\lambda,\mu}^{\alpha,\beta} = E_{\lambda^{(1)},\mu^{(1)}}^{\alpha^{(1)},\beta^{(1)}} E_{\lambda^{(2)},\mu^{(2)}}^{\alpha^{(2)},\beta^{(2)}},$$

where $\gamma^{(1)} = (\gamma_1, \ldots, \gamma_{k-1})$ and $\gamma^{(2)} = (\gamma_{k+1}, \ldots, \gamma_n)$ for each $\gamma \in \{\alpha, \beta, \lambda, \mu\}$. Since the skew shape λ/μ is the disjoint union of $\lambda^{(1)}/\mu^{(1)}$ and $\delta^k(\lambda^{(2)}/\mu^{(2)})$, the definition of $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(C)$ immediately gives

$$\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = \overline{\mathcal{R}}_{\lambda^{(1)},\mu^{(1)}}^{\alpha,\beta}(\emptyset) \overline{\mathcal{R}}_{\delta^k(\lambda^{(2)}),\delta^k(\mu^{(2)})}^{\alpha,\beta}(\emptyset) = \overline{\mathcal{R}}_{\lambda^{(1)},\mu^{(1)}}^{\alpha^{(1)},\beta^{(1)}}(\emptyset) \overline{\mathcal{R}}_{\lambda^{(2)},\mu^{(2)}}^{\alpha^{(2)},\beta^{(2)}}(\emptyset).$$

Hence, by induction, it suffices to consider the case $\mu < \lambda$.

Suppose that there is an integer $k \in [n-1]$ such that $\mu_k \geq \lambda_{k+1}$. Then we have

$$\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = \overline{\mathcal{R}}_{\lambda^{(1)},\mu^{(1)}}^{\alpha,\beta}(\emptyset)\overline{\mathcal{R}}_{\delta^k(\lambda^{(2)}),\delta^k(\mu^{(2)})}^{\alpha,\beta}(\emptyset) = \overline{\mathcal{R}}_{\lambda^{(1)},\mu^{(1)}}^{\alpha^{(1)},\beta^{(1)}}(\emptyset)\overline{\mathcal{R}}_{\lambda^{(2)},\mu^{(2)}}^{\alpha^{(2)},\beta^{(2)}}(\emptyset).$$

since the skew shape λ/μ is the disjoint union of $\lambda^{(1)}/\mu^{(1)}$ and $\delta^k(\lambda^{(2)}/\mu^{(2)})$, where $\gamma^{(1)} = (\gamma_1, \ldots, \gamma_k)$ and $\gamma^{(2)} = (\gamma_{k+1}, \ldots, \gamma_n)$ for each $\gamma \in \{\alpha, \beta, \lambda, \mu\}$. For all $k+1 \leq i \leq n$ and $1 \leq j \leq k$, we have $e_{\lambda,\mu}^{\alpha,\beta}(i,j) = 0$ because

$$\lambda_i - i - \mu_j + j \leq \lambda_{k+1} - (k+1) - \mu_k + k < 0.$$

By Lemma 4.5, this shows that

$$E_{\lambda,\mu}^{\alpha,\beta} = E_{\lambda^{(1)},\mu^{(1)}}^{\alpha^{(1)},\beta^{(1)}} E_{\lambda^{(2)},\mu^{(2)}}^{\alpha^{(2)},\beta^{(2)}}.$$

Thus, by induction, we may assume that $\mu_k < \lambda_{k+1}$ for all $k \in [n-1]$. In this case by assumption we have $\alpha, \beta \in \mathrm{QPar}_n$.

Suppose now that $\alpha_k \geq \beta_k$ for some $1 \leq k \leq n$. Then by Lemma 4.11 we have $E_{\lambda,\mu}^{\alpha,\beta} = 0$. Again, by definition, $\overline{\mathcal{R}}_{\lambda,\mu}^{\alpha,\beta}(\emptyset) = 0$.

The remaining case is that $\mu < \lambda$ and $\alpha < \beta$. This is done in Proposition 4.17 with $R_0 = \emptyset$ and the proof is completed. $\square$

Finally we show that Theorems 3.1 and 3.2 are equivalent.

Proposition 4.19. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \lambda \in \mathrm{Par}_n$. Then*

$$\det\left(e_{\lambda_i-\mu_j-i+j}(x_{\alpha_j+1}, \ldots, x_{\beta_i}, t_{\mu_j+1}, \ldots, t_{\lambda_i-1})\right)_{1 \leq i,j \leq n}$$
$$= \det\left(e_{\lambda_i-\mu_j-i+j}[X_{(\alpha_j,\beta_i]} + T_{\lambda_i-1} - T_{\mu_j}]\right)_{1 \leq i,j \leq n}.$$

Proof. If $\mu \not\subseteq \lambda$, both sides of the equation are zero by the same argument as in the proof of Lemma 4.10. Hence we may assume that $\mu \subseteq \lambda$. Let A and B be the matrices in the left hand side and in the right hand side respectively. We investigate the contribution of the (i,j)-entries $A_{i,j}$ and $B_{i,j}$ in the determinants when $A_{i,j} \neq B_{i,j}$.

Suppose $A_{i,j} \neq B_{i,j}$. Since $\mu_j < \lambda_i$ implies $A_{i,j} = B_{i,j}$, we must have $\lambda_i \leq \mu_j$. Note that $\lambda_i \leq \mu_j \leq \lambda_j$. If $\lambda_i < \lambda_j$, then $i > j$ and $\lambda_i - \mu_j - i + j < 0$, which implies $A_{i,j} = B_{i,j} = 0$, a contradiction. Thus we must have $\lambda_i = \lambda_j$. Since $\lambda_i \leq \mu_j \leq \lambda_j$, we also have $\mu_j = \lambda_j$. We now use an argument in the proof of Theorem 4.18. For $j \leq r \leq n$ and $1 \leq s \leq j$, since $\lambda_r - \mu_s - r + s \leq \lambda_j - \mu_j - j + j = 0$ and the equality holds if and only if $r = s = j$, we have

$$A_{r,s} = B_{r,s} = \chi(r = s = j).$$

Therefore, by the second claim of Lemma 4.6, each nonzero term in the expansion of $\det(A)$ and $\det(B)$ must contain the (j,j) entry, which is 1 for both matrices. Thus if $A_{i,j} \neq B_{i,j}$, these entries $A_{i,j}$ and $B_{i,j}$ do not contribute to the determinants, which implies $\det(A) = \det(B)$. $\square$

Theorem 3.1 now follows from Theorem 3.2 and Proposition 4.19.

5. A proof of the Jacobi–Trudi formula

In this section we prove the Jacobi–Trudi formula for $\widetilde{g}_{\lambda/\mu}^{\mathrm{row}(\alpha,\beta)}$ in Theorem 3.3. The proof is similar to (but not exactly the same as) that in the previous section.

We use the notation in Definitions 4.1 and 4.2 from the previous section. The notation below will also be used throughout this section.

Definition 5.1. For $\alpha, \beta, \mu, \lambda \in \mathbb{N}^n, C \subseteq [n]$, and $1 \leq i, j \leq n$, define

$$h_{\lambda,\mu}^{\alpha,\beta}(i,j) = h_{\lambda_i - i - \mu_j + j}[X_{(\alpha_j, \beta_i]} + T_{i-1} - T_{j-1}],$$

$$\overline{h}_{\lambda,\mu}^{\alpha,\beta}(i,j) = \chi(\alpha_j < \beta_i) h_{\lambda,\mu}^{\alpha,\beta}(i,j),$$

$$h_{\lambda,\mu}^{\alpha,\beta}(C; i, j) = \begin{cases} h_{\lambda,\mu}^{\alpha,\beta}(i,j), & \text{if } i \notin C, \\ \overline{h}_{\lambda,\mu}^{\alpha,\beta}(i,j), & \text{if } i \in C, \end{cases}$$

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = \det\left(h_{\lambda,\mu}^{\alpha,\beta}(C; i, j)\right)_{1 \leq i,j \leq n},$$

$$H_{\lambda,\mu}^{\alpha,\beta} = H_{\lambda,\mu}^{\alpha,\beta}(\emptyset),$$

$$\overline{H}_{\lambda,\mu}^{\alpha,\beta} = H_{\lambda,\mu}^{\alpha,\beta}([n]).$$

If $\mu \subseteq \lambda$, $\rho = (\lambda/\mu)^{(m)}$, and $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$, we define

$$(5.1) \qquad\qquad \mathcal{H}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \mathrm{wt}(R_0) \overline{H}_{\lambda-\rho,\mu}^{\alpha,B(R_0,\beta)}.$$

Note that in the definition of $\mathcal{H}_{\lambda,\mu}^{\alpha,\beta}(R_0)$, we used $\overline{H}$ instead of H. Using the notation above, Theorem 3.3 can be rewritten as

$$H_{\lambda,\mu}^{\alpha,\beta} = \mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(\emptyset).$$

We will show that both sides of the above equation satisfy the same recurrence relation.

5.1. **Technical lemmas.** In this subsection we give a list of lemmas that will be used to prove Theorem 3.3.

Let RPar_n denote the set of $\alpha \in \mathbb{N}^n$ such that $\alpha_i \leq \alpha_{i+1}$ for all $i \in [n-1]$. Note that if $\alpha \in \mathrm{RPar}_n$ and $1 \leq i \leq j \leq n$, then $\alpha_i \leq \alpha_j$. Clearly, $\mathrm{RPar}_n \subseteq \mathrm{QPar}_n$, where QPar_n is defined as in Section 4.

Lemma 5.2. *Let Z be any formal power series with integer coefficients and let z be a (single) variable. Then, for any integer k,*

$$h_k[Z] = h_k[Z - z] + z h_{k-1}[Z].$$

Proof. Since $h_k(x) = 0$ for $k < 0$ and $h_0(x) = 1$, the equation holds for $k \leq 0$. For $k \geq 1$, by (2.3), we have

$$h_k[Z - z] = \sum_{i=0}^k (-1)^i e_i(z) h_{k-i}[Z] = h_k[Z] - z h_{k-1}[Z],$$

which is equivalent to the equation in the lemma. $\qquad\square$

Lemma 5.3. *Let i, j, k be positive integers such that $j \geq i$ and $k > j - i$. Then*

$$h_k[T_{i-1} - T_{j-1}] = 0.$$

Proof. By the property (2.1) of the plethystic substitution, we have

$$h_k[T_{i-1} - T_{j-1}] = h_k[-t_i - t_{i+1} - \cdots - t_{j-1}] = (-1)^k e_k(t_i, t_{i+1}, \ldots, t_{j-1}) = 0,$$

as desired. □

Lemma 5.4. *Let $\alpha, \beta \in \mathbb{N}^n$, $\mu \in \mathrm{Par}_n$, and $C \subseteq [n]$. If $\alpha < \beta$, then*

$$H_{\mu,\mu}^{\alpha,\beta}(C) = 1.$$

Proof. This can be proved by the same argument as in the proof of Lemma 4.9. Note that we need the condition $\alpha < \beta$ to ensure that the (i, i) entry

$$h_{\mu,\mu}^{\alpha,\beta}(C; i, i) = \chi(\alpha_i < \beta_i) h_{\mu,\mu}^{\alpha,\beta}(i, i)$$

is 1 for $i \in C$. □

Lemma 5.5. *Let $\alpha, \beta \in \mathbb{N}^n$ and $\lambda, \mu \in \mathrm{Par}_n$. If $\mu \not\subseteq \lambda$, then for any $C \subseteq [n]$,*

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = 0.$$

Proof. This can be proved by the same argument as in the proof of Lemma 4.10. □

Lemma 5.6. *Let $\alpha, \beta \in \mathrm{RPar}_n$ and $\mu, \lambda \in \mathrm{Par}_n$. Suppose that $\alpha_k \geq \beta_k$ and $\mu_k < \lambda_k$ for some $1 \leq k \leq n$. Then for any $C \subseteq [n]$,*

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = 0.$$

Proof. Since $h_{\lambda,\mu}^{\alpha,\beta}(C; i, j)$ is a multiple of $h_{\lambda,\mu}^{\alpha,\beta}(i, j)$, by Lemma 4.4, it suffices to show $h_{\lambda,\mu}^{\alpha,\beta}(i, j) = 0$ assuming $1 \leq i \leq k$ and $k \leq j \leq n$. From $\alpha, \beta \in \mathrm{RPar}_n$, we obtain $\alpha_j \geq \alpha_k \geq \beta_k \geq \beta_i$, so $X_{(\alpha_j, \beta_i]} = 0$. Because of this, and of

$$\lambda_i - i - \mu_j + j \geq \lambda_k - \mu_k - i + j > j - i,$$

by Lemma 5.3,

$$h_{\lambda,\mu}^{\alpha,\beta}(i, j) = h_{\lambda_i - i - \mu_j + j}[X_{(\alpha_j, \beta_i]} + T_{i-1} - T_{j-1}] = h_{\lambda_i - i - \mu_j + j}[T_{i-1} - T_{j-1}] = 0,$$

as desired. □

Lemma 5.7. *Let $\alpha \in \mathrm{RPar}_n$, $\beta \in \mathbb{N}^n$, $\mu \in \mathrm{Par}_n$, and $\lambda \in \mathbb{N}^n$. Suppose that there is an integer $1 \leq k \leq n$ such that $\alpha_k < \beta_k$ and $\mu_k < \lambda_k$. Then for any subset $C \subseteq [n]$ with $k \in C$, we have*

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = H_{\lambda,\mu}^{\alpha,\beta}(C \setminus \{k\}).$$

Proof. It is sufficient to show that for all $1 \leq j \leq n$,

(5.2) $$h_{\lambda,\mu}^{\alpha,\beta}(C; k, j) = h_{\lambda,\mu}^{\alpha,\beta}(C \setminus \{k\}; k, j).$$

For a contradiction suppose that (5.2) does not hold for some $1 \leq j \leq n$. Since

$$h_{\lambda,\mu}^{\alpha,\beta}(C; k, j) = \chi(\alpha_j < \beta_k) h_{\lambda,\mu}^{\alpha,\beta}(k, j),$$

$$h_{\lambda,\mu}^{\alpha,\beta}(C \setminus \{k\}; k, j) = h_{\lambda,\mu}^{\alpha,\beta}(k, j),$$

we must have $\chi(\alpha_j < \beta_k) = 0$, or equivalently, $\beta_k \leq \alpha_j$. Then by assumption we have $\alpha_k < \beta_k \leq \alpha_j$, which implies $k < j$ (since $\alpha \in \mathrm{RPar}_n$). Thus

$$\lambda_k - k - \mu_j + j \geq \lambda_k - k - \mu_k + j > j - k.$$

By Lemma 5.3,

$$h_{\lambda,\mu}^{\alpha,\beta}(k, j) = h_{\lambda_k - k - \mu_j + j}[X_{(\alpha_j, \beta_k]} + T_{k-1} - T_{j-1}] = h_{\lambda_k - k - \mu_j + j}[T_{k-1} - T_{j-1}] = 0.$$

But this implies that both sides of (5.2) are zero, a contradiction. Therefore (5.2) is true for all $1 \leq j \leq n$, which completes the proof. $\qquad\square$

Lemma 5.8. *Let $\alpha \in \mathrm{RPar}_n$, $\beta \in \mathbb{N}^n$, and $\mu, \lambda \in \mathrm{Par}_n$ with $\alpha < \beta$ and $\mu < \lambda$. Then, for any $C \subseteq [n]$,*

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = H_{\lambda,\mu}^{\alpha,\beta}.$$

Proof. We claim that, if $k \in C$, then

$$H_{\lambda,\mu}^{\alpha,\beta}(C) = H_{\lambda,\mu}^{\alpha,\beta}(C \setminus \{k\}).$$

By assumption we have $\alpha_k < \beta_k$ and $\mu_k < \lambda_k$. Thus the claim follows from Lemma 5.7. Applying the claim iteratively yields the desired result. $\qquad\square$

Lemma 5.9. *Let $\alpha \in \mathrm{RPar}_n$, $\beta \in \mathbb{N}^n$, $\mu \in \mathrm{Par}_n$, and $\lambda \in \mathbb{N}^n$. Suppose that k is an integer satisfying $\alpha_k < \beta_k$ and $\mu_k < \lambda_k$. Then*

$$\overline{H}_{\lambda,\mu}^{\alpha,\beta} = \begin{cases} \overline{H}_{\lambda,\mu}^{\alpha,\beta-\epsilon_k} + x_{\beta_k} \overline{H}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}, & \text{if } \alpha_k + 1 < \beta_k, \\ H_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}([n] \setminus \{k\}) + x_{\beta_k} \overline{H}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}, & \text{if } \alpha_k + 1 = \beta_k. \end{cases}$$

Proof. We first claim that, for all $1 \leq j \leq n$,

(5.3) $$\qquad h_{\lambda,\mu}^{\alpha,\beta}(k, j) = h_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}(k, j) + x_{\beta_k} \overline{h}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}(k, j).$$

The claim is restated as

$$h_{\lambda_k - k - \mu_j + j}[X_{(\alpha_j, \beta_k]} + T_{k-1} - T_{j-1}] = h_{\lambda_k - k - \mu_j + j}[X_{(\alpha_j, \beta_k - 1]} + T_{k-1} - T_{j-1}]$$
$$+ x_{\beta_k} \chi(\alpha_j < \beta_k) h_{\lambda_k - k - \mu_j + j - 1}[X_{(\alpha_j, \beta_k]} + T_{k-1} - T_{j-1}].$$

If $\alpha_j \geq \beta_k$, both sides of the above equation are equal to $h_{\lambda_k - k - \mu_j + j}[T_{k-1} - T_{j-1}]$. If $\alpha_j < \beta_k$, the equation follows from Lemma 5.2 with $Z = X_{(\alpha_j, \beta_k]} + T_{k-1} - T_{j-1}$ and $z = x_{\beta_k}$. This establishes (5.3).

By Lemma 5.7 and (5.3),

(5.4) $$\qquad \overline{H}_{\lambda,\mu}^{\alpha,\beta} = H_{\lambda,\mu}^{\alpha,\beta}([n]) = H_{\lambda,\mu}^{\alpha,\beta}([n] \setminus \{k\}) = H_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}([n] \setminus \{k\}) + x_{\beta_k} \overline{H}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta},$$

where the last equality follows from the linearity of the determinant in its kth row. This shows the lemma for the case $\alpha_k + 1 = \beta_k$. If $\alpha_k + 1 < \beta_k$, Lemma 5.7 gives

$$H_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}([n] \setminus \{k\}) = H_{\lambda,\mu}^{\alpha,\beta-\epsilon_k}([n]) = \overline{H}_{\lambda,\mu}^{\alpha,\beta-\epsilon_k},$$

which together with (5.4) finishes the proof. $\qquad\square$

Lemma 5.10. *Let* $\alpha, \beta, \mu, \lambda \in \mathbb{N}^n$, *and* $2 \le k \le n$ *with* $\beta_k = \beta_{k-1}$ *and* $\lambda_k = \lambda_{k-1} + 1$. *Then*

$$\overline{H}_{\lambda,\mu}^{\alpha,\beta} = t_{k-1} \overline{H}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}.$$

Proof. We claim that, for all $1 \le j \le n$,

(5.5) $$\overline{h}_{\lambda,\mu}^{\alpha,\beta}(k,j) = \overline{h}_{\lambda,\mu}^{\alpha,\beta}(k-1,j) + t_{k-1}\overline{h}_{\lambda-\epsilon_k,\mu}^{\alpha,\beta}(k,j).$$

To prove the claim, since $\chi(\alpha_j < \beta_k) = \chi(\alpha_j < \beta_{k-1})$, it suffices to show that

$$h_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k]} + T_{k-1} - T_{j-1}] = h_{\lambda_{k-1}-(k-1)-\mu_j+j}[X_{(\alpha_j,\beta_{k-1}]} + T_{k-2} - T_{j-1}]$$
$$+ t_{k-1}h_{\lambda_k-k-\mu_j+j-1}[X_{(\alpha_j,\beta_k]} + T_{k-1} - T_{j-1}],$$

which is, by assumption, the same as

$$h_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k]} + T_{k-1} - T_{j-1}] = h_{\lambda_k-k-\mu_j+j}[X_{(\alpha_j,\beta_k]} + T_{k-2} - T_{j-1}]$$
$$+ t_{k-1}h_{\lambda_k-k-\mu_j+j-1}[X_{(\alpha_j,\beta_k]} + T_{k-1} - T_{j-1}].$$

This follows from Lemma 5.2 with $Z = X_{(\alpha_j,\beta_k]} + T_{k-1} - T_{j-1}$ and $z = t_{k-1}$.

Using (5.5) and subtracting the $(k-1)$st row from the kth row of the matrix for the determinant $\overline{H}_{\lambda,\mu}^{\alpha,\beta}$ we obtain the lemma. $\qquad\square$

5.2. **Proof of Theorem 3.3.** We first show that $\mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ and $\mathcal{H}_{\lambda,\mu}^{\alpha,\beta}(R_0)$ satisfy the same recurrence relation under certain conditions.

Proposition 5.11. *Let* $\alpha, \beta \in \mathrm{RPar}_n$, $\lambda, \mu \in \mathrm{Par}_n$ *with* $\alpha < \beta$ *and* $\mu < \lambda$. *Fix* $(r,c) \in \lambda/\mu$ *and* $R_0 \in \mathrm{RPP}_\rho^{\mathrm{row}(\alpha,\beta)}$, *where* ρ *is the set of cells* $(i,j) \in \lambda/\mu$ *with* $(i,j) \prec (r,c)$. *Let* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n) = B(R_0, \beta)$. *Then*

(5.6) $$\mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{k=a}^{\widetilde{\beta}_r} \mathcal{R}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}),$$

(5.7) $$\mathcal{H}_{\lambda,\mu}^{\alpha,\beta}(R_0) = \sum_{k=a}^{\widetilde{\beta}_r} \mathcal{H}_{\lambda,\mu}^{\alpha,\beta}(R_0 \cup \{k\}),$$

where $R_0 \cup \{k\}$ *is the RPP obtained from* R_0 *by adding the cell* (r,c) *with entry* k, *and*

$$a = \begin{cases} \widetilde{\beta}_{r-1}, & \text{if } (r-1,c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \ge \alpha_r + 1, \\ \alpha_r + 1, & \text{otherwise.} \end{cases}$$

Proof. The first identity (5.6) is immediate from the definition of $\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(R_0)$. For the second identity (5.7), let $C = C(\rho) = \{1 \leq i \leq n : \rho_i > 0\}$ and

$$s = \begin{cases} \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1, & \text{if } (r-1,c) \in \rho \text{ and } \widetilde{\beta}_{r-1} \geq \alpha_r + 1, \\ \widetilde{\beta}_r - \alpha_r, & \text{otherwise.} \end{cases}$$

Note that $\beta \in \mathrm{RPar}_n \subseteq \mathrm{QPar}_n$ and $\beta_{r-1} \leq \beta_r$; thus, (4.17) yields $s \geq 1$. Furthermore, (4.18) yields $\widetilde{\beta}_r - s \geq \alpha_r \geq 0$, thus $\widetilde{\beta} - s\epsilon_r \in \mathbb{N}^n$. Finally, $\lambda - \rho \in \mathbb{N}^n$. We consider the two cases in the definition of s.

Case 1: $(r-1,c) \in \rho$ and $\widetilde{\beta}_{r-1} \geq \alpha_r + 1$. Then $s = \widetilde{\beta}_r - \widetilde{\beta}_{r-1} + 1$. Therefore, for $0 \leq i \leq s-2$, we have

$$(\widetilde{\beta} - i\epsilon_r)_r = \widetilde{\beta}_r - i > \widetilde{\beta}_r - (s-1) = \widetilde{\beta}_{r-1} \geq \alpha_r + 1.$$

Note that since $(r,c) \in (\lambda/\mu) - \rho$, we have $(\lambda - \rho)_r > \mu_r$. Thus we can apply Lemma 5.9 repeatedly to $\overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho,\mu}$, for $0 \leq i \leq s-2$:

$$\overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_r} \overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho-\epsilon_r,\mu},$$

$$\overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-2\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_{r}-1} \overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu},$$

$$\vdots$$

$$\overline{H}^{\alpha,\widetilde{\beta}-(s-2)\epsilon_r}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_r-(s-2)} \overline{H}^{\alpha,\widetilde{\beta}-(s-2)\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

Combining the above equations gives

$$(5.8) \qquad \overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho,\mu} + \sum_{i=0}^{s-2} x_{\widetilde{\beta}_r-i} \overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

Note that if $s = 1$, then (5.8) is trivial. Since $(r-1,c) \in \rho$, by the construction of ρ we have $(\lambda - \rho)_r = (\lambda - \rho)_{r-1} + 1$. Since

$$(\widetilde{\beta} - (s-1)\epsilon_r)_r = \widetilde{\beta}_r - (s-1) = \widetilde{\beta}_{r-1} = (\widetilde{\beta} - (s-1)\epsilon_r)_{r-1},$$

Lemma 5.10 gives

$$(5.9) \qquad \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho,\mu} = t_{r-1} \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

By (5.8) and (5.9) we obtain

$$(5.10) \qquad \overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu} = t_{r-1} \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu} + \sum_{i=0}^{s-2} x_{\widetilde{\beta}_r-i} \overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

Since $\widetilde{\beta} = B(R_0, \beta)$, we can rewrite (5.1) as

$$\mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0) = \mathrm{wt}(R_0) \overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu}.$$

Since

$$t_{r-1}\,\mathrm{wt}(R_0) = \mathrm{wt}(R_0 \cup \{\widetilde{\beta}_{r-1}\}) = \mathrm{wt}(R_0 \cup \{\widetilde{\beta}_r - (s-1)\}),$$

$$x_{\widetilde{\beta}_r - i}\,\mathrm{wt}(R_0) = \mathrm{wt}(R_0 \cup \{\widetilde{\beta}_r - i\}) \qquad \text{for } 0 \le i \le s-2,$$

multiplying both sides of (5.10) by $\mathrm{wt}(R_0)$ gives

$$\mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0) = \sum_{i=0}^{s-1} \mathrm{wt}(R_0 \cup \{\widetilde{\beta}_r - i\})\overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu} = \sum_{i=0}^{s-1} \mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0 \cup \{\widetilde{\beta}_r - i\}),$$

where the last equality follows from $\widetilde{\beta} - i\epsilon_r = B(R_0 \cup \{\widetilde{\beta}_r - i\}, \beta)$. The above equation is the same as (5.7).

Case 2: $(r-1,c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1$. Then $s = \widetilde{\beta}_r - \alpha_r$. As in Case 1 we have $(\lambda - \rho)_r > \mu_r$, and, for $0 \le i \le s-1$,

$$(\widetilde{\beta} - i\epsilon_r)_r = \widetilde{\beta}_r - i \ge \widetilde{\beta}_r - (s-1) = \alpha_r + 1,$$

where the equality holds if and only if $i = s-1$. Thus we can apply Lemma 5.9 repeatedly to $\overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho,\mu}$, for $0 \le i \le s-1$:

$$\overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_r}\overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho-\epsilon_r,\mu},$$

$$\overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-2\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_r-1}\overline{H}^{\alpha,\widetilde{\beta}-\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu},$$

$$\vdots$$

$$\overline{H}^{\alpha,\widetilde{\beta}-(s-2)\epsilon_r}_{\lambda-\rho,\mu} = \overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho,\mu} + x_{\widetilde{\beta}_r-(s-2)}\overline{H}^{\alpha,\widetilde{\beta}-(s-2)\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu},$$

$$\overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho,\mu} = H^{\alpha,\widetilde{\beta}-s\epsilon_r}_{\lambda-\rho,\mu}([n] \setminus \{r\}) + x_{\widetilde{\beta}_r-(s-1)}\overline{H}^{\alpha,\widetilde{\beta}-(s-1)\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

Lemma 5.12 below shows $H^{\alpha,\widetilde{\beta}-s\epsilon_r}_{\lambda-\rho,\mu}([n] \setminus \{r\}) = 0$. Thus, combining the above equations, we obtain

$$\overline{H}^{\alpha,\widetilde{\beta}}_{\lambda-\rho,\mu} = \sum_{i=0}^{s-1} x_{\widetilde{\beta}_r-i}\overline{H}^{\alpha,\widetilde{\beta}-i\epsilon_r}_{\lambda-\rho-\epsilon_r,\mu}.$$

Observe that, for $0 \le i \le s-1$, we have $x_{\widetilde{\beta}_r-i}\,\mathrm{wt}(R_0) = \mathrm{wt}(R_0 \cup \{\widetilde{\beta}_r - i\})$ (because we have $(r-1,c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1 = \widetilde{\beta}_r - (s-1) \le \widetilde{\beta}_r - i$) and $\widetilde{\beta} - i\epsilon_r = B(R_0 \cup \{\widetilde{\beta}_r - i\}, \beta)$. Thus, similarly to the argument in Case 1, multiplying both sides of the above equation by $\mathrm{wt}(R_0)$ gives

$$\mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0) = \sum_{i=0}^{s-1} \mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0 \cup \{\widetilde{\beta}_r - i\}),$$

which is the same as (5.7). This completes the proof.   □

We now prove a statement used in the proof of Proposition 5.11.

Lemma 5.12. *Following the notation in Proposition 5.11, suppose $(r-1,c) \notin \rho$ or $\widetilde{\beta}_{r-1} < \alpha_r + 1$, and let $s = \widetilde{\beta}_r - \alpha_r$. Then we have*

$$(5.11) \qquad H^{\alpha,\widetilde{\beta}-s\epsilon_r}_{\lambda-\rho,\mu}([n] \setminus \{r\}) = 0.$$

Proof. We proceed similarly as in the proof of Lemma 4.16. Let d be the integer such that $(d,c) \in \lambda/\mu$ and $(d-1,c) \notin \lambda/\mu$, see Figure 4. Then $1 \le d \le r$. Let $\kappa = \widetilde{\beta} - s\epsilon_r$ and $\sigma = \lambda - \rho$, so that

$$H^{\alpha,\widetilde{\beta}-s\epsilon_r}_{\lambda-\rho,\mu}([n] \setminus \{r\}) = H^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}) = \det(h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j))_{1 \le i,j \le n}.$$

By the same argument as in the proof of (4.24) we have

$$H^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}) = \det(h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j))_{1 \le i,j \le d-1} \det(h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j))_{d \le i,j \le n}.$$

Therefore, it suffices to show that

$$\det(h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j))_{d \le i,j \le n} = 0.$$

By Lemma 4.4, in order to show the above equation it is enough to show that, for all $d \le i \le r$ and $r \le j \le n$,

$$(5.12) \qquad h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j) = 0.$$

By Definition 5.1 we have

$$h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j) = \begin{cases} h_{\sigma_i-i-\mu_j+j}[X_{(\alpha_j,\kappa_i]} + T_{i-1} - T_{j-1}], & \text{if } i = r, \\ \chi(\alpha_j < \kappa_i) h_{\sigma_i-i-\mu_j+j}[X_{(\alpha_j,\kappa_i]} + T_{i-1} - T_{j-1}], & \text{if } i \ne r. \end{cases}$$

Suppose $i = r$ and $r \le j \le n$. Then the above equation becomes

$$h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; r, j) = h_{\sigma_r-r-\mu_j+j}[X_{(\alpha_j,\kappa_r]} + T_{r-1} - T_{j-1}].$$

Since $\kappa_r = (\widetilde{\beta} - s\epsilon_r)_r = \widetilde{\beta}_r - s = \alpha_r \le \alpha_j$, this equation can be further simplified to

$$(5.13) \qquad h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; r, j) = h_{\sigma_r-r-\mu_j+j}[T_{r-1} - T_{j-1}].$$

Since $(r,c) \in \rho$, we have

$$\sigma_r = (\lambda - \rho)_r > \mu_r \ge \mu_j,$$

which implies

$$\sigma_r - r - \mu_j + j > j - r.$$

Thus, Lemma 5.3 gives

$$(5.14) \qquad h_{\sigma_r-r-\mu_j+j}[T_{r-1} - T_{j-1}] = 0.$$

By (5.13) and (5.14) we obtain (5.12) for $i = r$ and $r \le j \le n$.

It remains to prove (5.12) for $d \le i \le r - 1$ and $r \le j \le n$. In this case,

$$h^{\alpha,\kappa}_{\sigma,\mu}([n] \setminus \{r\}; i, j) = \chi(\alpha_j < \kappa_i) h_{\sigma_i-i-\mu_j+j}[X_{(\alpha_j,\kappa_i]} + T_{i-1} - T_{j-1}].$$

Thus it suffices to show that $\kappa_i \leq \alpha_j$. Since $d \leq r - 1 < r$, we have $(r-1, c) \in \rho$. Then by assumption in this lemma, $\widetilde{\beta}_{r-1} < \alpha_r + 1$. Since $d \leq i \leq r - 1$, we have $(i, c) \in \rho$; in view of $(i, c-1) \notin \rho$, this leads to $\widetilde{\beta}_i = R_0(i, c)$, and similarly $\widetilde{\beta}_{r-1} = R_0(r-1, c)$. Therefore

$$\kappa_i = (\widetilde{\beta} - s\epsilon_r)_i = \widetilde{\beta}_i = R_0(i, c) \leq R_0(r-1, c) = \widetilde{\beta}_{r-1} < \alpha_r + 1 \leq \alpha_j + 1$$

(since $\alpha \in \mathrm{RPar}_n$), which shows $\kappa_i \leq \alpha_j$, as desired. $\qquad\square$

The fact that $\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(R_0)$ and $\mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0)$ satisfy the same recurrence relation can be used to show that they are equal.

Proposition 5.13. *Let* $\alpha, \beta \in \mathrm{RPar}_n$, $\lambda, \mu \in \mathrm{Par}_n$ *with* $\alpha < \beta$ *and* $\mu < \lambda$. *Let* $\rho = (\lambda/\mu)^{(m)}$ *for some* $0 \leq m \leq |\lambda/\mu|$ *and let* $R_0 \in \mathrm{RPP}^{\mathrm{row}(\alpha,\beta)}_{\rho}$. *Then*

$$\mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(R_0) = \mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(R_0).$$

Proof. This can be proved by the same argument as in the proof of Proposition 4.17 where we use Lemma 5.4 and Proposition 5.11 in place of Lemma 4.9 and Proposition 4.14, respectively. $\qquad\square$

Now we are ready to prove Theorem 3.3, which can be restated as follows.

Theorem 5.14. *Let* $\alpha, \beta \in \mathbb{N}^n$ *and* $\mu, \lambda \in \mathrm{Par}_n$. *If* $\alpha_i \leq \alpha_{i+1}$ *and* $\beta_i \leq \beta_{i+1}$ *whenever* $\mu_i < \lambda_{i+1}$, *then*

$$H^{\alpha,\beta}_{\lambda,\mu} = \mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset).$$

Proof. As we did in the proof of Theorem 4.18 we will successively reduce the cases so that we eventually have the assumptions $\alpha, \beta \in \mathrm{RPar}_n$, $\alpha < \beta$ and $\mu < \lambda$ in Proposition 5.13. For a diagram σ, we denote by $\delta(\sigma)$ the diagram obtained by translating σ down by one row, so that $\delta^k(\sigma) = \{(i + k, j) : (i, j) \in \sigma\}$ for all $k \geq 0$. Let ϕ be the shifting operator on $\mathbb{Q}[[x_1, x_2, \ldots, t_1, t_2, \ldots]]$ replacing each variable t_i by t_{i+1}. Then ϕ^k is an algebra homomorphism and it sends $T_{i-1} - T_{j-1}$ to $T_{i+k-1} - T_{j+k-1}$ for all positive integers i, j, k. Note that there is a canonical bijection between the RPPs R of shape σ and the RPPs R' of shape $\delta^k(\sigma)$, and that this bijection satisfies $\mathrm{wt}(R') = \phi^k(\mathrm{wt}(R))$.

If $\mu \not\subseteq \lambda$, both sides are zero by Lemma 5.5 and the definition of $\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset)$. Hence we may assume $\mu \subseteq \lambda$. Thus, either $\lambda_k = \mu_k$ for some $1 \leq k \leq n$, or $\mu < \lambda$.

Suppose that $\lambda_k = \mu_k$ for some $1 \leq k \leq n$. Then for $k \leq i \leq n$ and $1 \leq j \leq k$, we have $\lambda_i - i - \mu_j + j \leq \lambda_k - k - \mu_k + k = 0$, where the equality holds if and only if $i = j = k$. Thus

$$h^{\alpha,\beta}_{\lambda,\mu}(i, j) = h_{\lambda_i - i - \mu_j + j}[X_{(\alpha_j, \beta_i]} + T_{i-1} - T_{j-1}] = \chi(i = j = k).$$

By Lemma 4.6, this implies

$$H^{\alpha,\beta}_{\lambda,\mu} = H^{\alpha^{(1)},\beta^{(1)}}_{\lambda^{(1)},\mu^{(1)}} \phi^k \left(H^{\alpha^{(2)},\beta^{(2)}}_{\lambda^{(2)},\mu^{(2)}} \right),$$

where $\gamma^{(1)} = (\gamma_1, \ldots, \gamma_{k-1})$ and $\gamma^{(2)} = (\gamma_{k+1}, \ldots, \gamma_n)$ for each $\gamma \in \{\alpha, \beta, \lambda, \mu\}$. The definition of $\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset)$ immediately gives

$$\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset) = \mathcal{R}^{\alpha,\beta}_{\lambda^{(1)},\mu^{(1)}}(\emptyset) \mathcal{R}^{\alpha,\beta}_{\delta^k(\lambda^{(2)}),\delta^k(\mu^{(2)})}(\emptyset) = \mathcal{R}^{\alpha^{(1)},\beta^{(1)}}_{\lambda^{(1)},\mu^{(1)}}(\emptyset) \phi^k \left(\mathcal{R}^{\alpha^{(2)},\beta^{(2)}}_{\lambda^{(2)},\mu^{(2)}}(\emptyset) \right)$$

because λ/μ is the disjoint union of $\lambda^{(1)}/\mu^{(1)}$ and $\delta^k(\lambda^{(2)}/\mu^{(2)})$. Hence, by induction, it suffices to consider the case $\mu < \lambda$.

Suppose that there is an integer $k \in [n-1]$ such that $\mu_k \geq \lambda_{k+1}$. Then we have

$$\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset) = \mathcal{R}^{\alpha,\beta}_{\lambda^{(1)},\mu^{(1)}}(\emptyset)\mathcal{R}^{\alpha,\beta}_{\delta^k(\lambda^{(2)}),\delta^k(\mu^{(2)})}(\emptyset) = \mathcal{R}^{\alpha^{(1)},\beta^{(1)}}_{\lambda^{(1)},\mu^{(1)}}(\emptyset)\phi^k\left(\mathcal{R}^{\alpha^{(2)},\beta^{(2)}}_{\lambda^{(2)},\mu^{(2)}}(\emptyset)\right),$$

where $\gamma^{(1)} = (\gamma_1, \ldots, \gamma_k)$ and $\gamma^{(2)} = (\gamma_{k+1}, \ldots, \gamma_n)$ for each $\gamma \in \{\alpha, \beta, \lambda, \mu\}$, because λ/μ is the disjoint union of $\lambda^{(1)}/\mu^{(1)}$ and $\delta^k(\lambda^{(2)}/\mu^{(2)})$. For all $k+1 \leq i \leq n$ and $1 \leq j \leq k$, we have $h^{\alpha,\beta}_{\lambda,\mu}(i,j) = 0$ because

$$\lambda_i - i - \mu_j + j \leq \lambda_{k+1} - (k+1) - \mu_k + k < 0.$$

By Lemma 4.5, this implies

$$H^{\alpha,\beta}_{\lambda,\mu} = H^{\alpha^{(1)},\beta^{(1)}}_{\lambda^{(1)},\mu^{(1)}}\phi^k\left(H^{\alpha^{(2)},\beta^{(2)}}_{\lambda^{(2)},\mu^{(2)}}\right).$$

Thus, by induction, we may assume $\mu_k < \lambda_{k+1}$ for all $k \in [n-1]$. In this case by assumption we have $\alpha, \beta \in \mathrm{RPar}_n$.

Suppose that $\alpha_k \geq \beta_k$ for some $1 \leq k \leq n$. Then by Lemma 5.6 with $C = \emptyset$ we have $H^{\alpha,\beta}_{\lambda,\mu} = 0$. Again, by definition, $\mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset) = 0$.

The remaining case is that $\alpha, \beta \in \mathrm{RPar}_n$, $\mu < \lambda$, and $\alpha < \beta$. In this case, by Lemma 5.8 and Proposition 5.13,

$$H^{\alpha,\beta}_{\lambda,\mu} = H^{\alpha,\beta}_{\lambda,\mu}([n]) = \overline{H}^{\alpha,\beta}_{\lambda,\mu} = \mathcal{H}^{\alpha,\beta}_{\lambda,\mu}(\emptyset) = \mathcal{R}^{\alpha,\beta}_{\lambda,\mu}(\emptyset),$$

which completes the proof. $\qquad\square$

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