

Quantum Mechanics

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Problem Set 1

Problem 1. The wave function for a free particle is given by

$$\psi(x, 0) = N \exp \left(i \frac{p_0 x}{\hbar} - \frac{(x - x_0)^2}{4\sigma^2} \right),$$

where $\sigma \in \mathbb{R}$ is a constant and N is a normalization constant.

- (1) Derive the normalization constant N .
- (2) Derive the wave function $\phi(0, 0)$ in momentum space.
- (3) Find $\phi(p, t)$.
- (4) Find $\psi(x, t)$.
- (5) Show that the spread in the spatial probability distribution increases with time t . Note that the spread is defined as

$$\mathcal{S}(t) = \frac{|\psi(x, t)|^2}{|\psi(0, t)|^2}.$$

Solution

- (1) From the normalization of the wave function,

$$\int_{-\infty}^{\infty} |\psi(x, 0)|^2 dx = N^2 \int_{-\infty}^{\infty} \exp \left(-2 \left(\frac{x - x_0}{2\sigma} \right)^2 \right) dx = 1. \quad (1)$$

Since a range of integration is all space, the translation about x can be ignored. To make a compact form, it needs to change an integral variable.

$$t \equiv \left(\frac{x - x_0}{\sqrt{2}\sigma} \right)^2, \quad dt = \frac{1}{\sqrt{2}\sigma} dx \quad (2)$$

Then the wave function changes into more comfort form to integrate.

$$N^2 \int_{-\infty}^{\infty} \exp \left(-2 \left(\frac{x - x_0}{2\sigma} \right)^2 \right) dx = \sqrt{2}\sigma N^2 \int_{-\infty}^{\infty} e^{-t^2} dt \quad (3)$$

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To calculate this integration, we use an idea of double integration,

$$\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \quad (4)$$

$$= \int_0^{2\pi} \frac{1}{2} d\theta = \pi. \quad (5)$$

First double integration about coordinate space can be decomposed.

$$\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 \quad (6)$$

From this result,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \sqrt{2\pi}\sigma N^2 = 1. \quad (7)$$

Finally we obtain the normalization constant,

$$N = \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{1}{4}}. \quad (8)$$

(2)

(3)

(4)

(5)

Problem 2. The Hamiltonian for a free particle is given by

$$H = \frac{p^2}{2m}.$$

(1) Show

$$\langle p_x \rangle = \langle p_x \rangle_{t=0}.$$

(2) Show

$$\langle x \rangle = \frac{\langle p_x \rangle_{t=0}}{m} t + \langle x \rangle_{t=0}.$$

(3) Show

$$(\Delta p_x)^2 = (\Delta p_x)_{t=0}^2.$$

(4) Find $d(\Delta x)^2/dt$ as a function of time and initial conditions.

Problem 3. The state of a particle is described by the following wavefunction:

$$\psi(x) = C \exp \left[i \frac{p_0 x}{\hbar} - \frac{(x - x_0)^2}{2\sigma^2} \right]$$

where p_0 , x_0 , and σ are real parameters.

- (1) Find the normalization constant C .
- (2) Find the mean values of x and p .
- (3) Find the standard deviations Δx and Δp .

Problem 4*. Consider a particle and two normalized energy eigenfunctions $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ corresponding to the eigenvalues $E_1 \neq E_2$. Assume that the eigenfunctions vanish outside the two non-overlapping regions Ω_1 and Ω_2 , respectively.

- (1) (a) Show that, if the particle is initially in region Ω_1 then it will stay there forever.
- (b) If, initially, the particle is in the state with wave function

$$\psi(\mathbf{x}, 0) = \frac{1}{\sqrt{2}} [\psi_1(\mathbf{x}) + \psi_2(\mathbf{x})]$$

show that the probability density $|\psi(\mathbf{x}, t)|^2$ is independent of time.

- (c) Now assume that the two regions Ω_1 and Ω_2 overlap partially. Starting with the initial wave function of case (b), show that the probability density is a periodic function of time. ($E_2 - E_1 = \hbar\omega$).
- (d) Starting with the same initial wave function and assuming that the two eigenfunctions are real and isotropic, take the two partially overlapping regions Ω_1 and Ω_2 to be two concentric spheres of radii $R_1 > R_2$. Compute the probability current that flows through Ω_1 .

(Problem 4 is a bit difficult. To solve (3), introduce phase factors for $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ and consider the interference term when one computes the probability density. To solve (4), consider the current density and continuity equation.