Quantum Mechanics

김현철[1, *]
[1] *Hadron Theory Group, Department of Physics, Inha University, Incheon 22212, Republic of Korea*
(Dated: 2021)

Due date: **April 9, 2022**

Problem Set 2

Problem 1. A constant electric field $\mathcal{E}$ is exerted on a charged linear harmonic oscillator.

(1) Write down the corresponding Schrödinger equation.

(2) Derive the eigenvalues and eigenvectors of the charged linear oscillators under a uniform electric field.

(3) Discuss the change in energy levels and physics. eigenstates.

Hint: Use the operator method.

Problem 2. The generating function $S(x,t)$ for the Hermite polynomial $H_n(x)$ is defined as

$$S(x,t) = e^{x^2-(t-x)^2} = e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{1}$$

(1) Using this generating function, derive the Hermite differential equation.

(2) Derive the following formula from Eq. (1):

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \tag{2}$$

which is called the Rodrigues representation of the Hermite polynomial.

(3) Using Eq. (1), derive the orthogonal relation of the Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n \sqrt{\pi} n! \delta_{nm}. \tag{3}$$

(4) Prove that

$$\left(2x - \frac{d}{dx}\right)^n 1 = H_n(x), \tag{4}$$

(5) Prove

$$\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^{n-1} n! \delta_{m,n-1} + \sqrt{\pi} 2^n (n+1)! \delta_{m,n+1}. \tag{5}$$

*Electronic address: `hchkim@inha.ac.kr`

(6) Prove

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n(x) H_n(x) dx = \sqrt{\pi} 2^n n! \left(n + \frac{1}{2}\right). \tag{6}$$

Problem 3. Given the eigenfunctions and eigenenergies of the SHO,

(1) Compute the kinetic and potential energies at the n^{th} level. Show that the results satisfy the virial theorem.

(2) Show that the n^{th} state of the SHO satisfies

$$\Delta x \Delta p = \left(n + \frac{1}{2}\right) \hbar. \tag{7}$$

Problem 4. If a wavefunction desribes a mixed state of the eigenstates of the SHO given as

$$\psi(x,t) = \frac{1}{\sqrt{2}} [\psi_0(x,t) + \psi_1(x,t)], \tag{8}$$

(1) Investigate how the probability density changes in time.

(2) Prove the following relations

$$\begin{aligned} \langle E \rangle &= \langle H \rangle = \hbar\omega, \\ \langle x \rangle &= \frac{1}{\sqrt{2}\alpha} \cos\omega t, \\ \langle p \rangle &= -\sqrt{\frac{\alpha}{2}} \hbar \sin\omega t, \end{aligned} \tag{9}$$

where $\alpha = \sqrt{m\omega/\hbar}$.

(3) If

$$\psi(x,t) = \frac{1}{\sqrt{2}} [e^{i\delta_0} \psi_0(x,t) + e^{i\delta} \psi_1(x,t)], \tag{10}$$

discuss the effects of the phase factors δ_0 and δ on $\langle x \rangle$ and $\langle p \rangle$.

Problem 5. Derive the wavefunction in momentum space, which corresponds to the eigenfunctions for the SHO in coordinates, $\psi_n(x)$.

Problem 6. At $t = 0$, the wavefunction for a state is described by

$$\psi(x,0) = \sum_n A_n u_n(x) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\alpha^2(x-a)^2/2}. \tag{11}$$

show that after some time t, the probability density changes in time as

$$|\pi(x,t)|^2 = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\alpha^2(x-a\cos\omega t)^2} \tag{12}$$

and discuss the result.

Problem 7. THe Einstein model for a solid assumes that it consists of many SHOs. If the N atoms are similar each other and oscillate similarly in average, the solid can be explained in terms of N SHOs. At a given temperature T, N atoms are in thermal equilibrium. Then, the Boltzmann distribution is given by

$$P_n = \frac{1}{Z} e^{-E_n/kT} \tag{13}$$

with

$$Z = \sum_n e^{-E_n/kT}, \tag{14}$$

where

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega. \tag{15}$$

(1) Derive the mean energy per an SHO

$$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \frac{1}{2}\hbar\omega. \tag{16}$$

(2) If U is the internal energy of the solid, derive the specific heat with constant volume

$$C_V = \frac{\partial U}{\partial T}. \tag{17}$$

Show that when T is large, $C_V = 3R$.

(3) Discuss the physics related to this problem as far as you can.