Quantum Mechanics

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(Dated: 2021)

Due date: **March 2. 2022**

Problem Set 1

Problem 1. The wave function for a free particle is given by

$$\psi(x,\,0) = N\exp\left(i\frac{p_0 x}{\hbar} - \frac{(x-x_0)^2}{4\sigma^2}\right),$$

where $\sigma \in \mathbb{R}$ is a contant and N is a normalization constant.

(1) Derive the normalization constant N.

(2) Derive the wave function $\phi(0,\,0)$ in momentum space.

(3) Find $\phi(p,\,t)$.

(4) Find $\psi(x\,t)$.

(5) Show that the the spread in the spatial probability distribution increases with time t. Note that the spread is defined as

$$\mathcal{S}(t) = \frac{|\psi(x,\,t)|^2}{|\psi(0,t)|^2}.$$

Solution

(1) From the normalization of the wave function,

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2\,dx = N^2\int_{-\infty}^{\infty}\exp\left(-2\left(\frac{x-x_0}{2\sigma}\right)^2\right)\,dx = 1. \tag{1}$$

Since a range of integration is all space, the translation about x can be ignored. To make a compact form, it needs to change an integral variable.

$$t \equiv \left(\frac{x-x_0}{\sqrt{2}\sigma}\right),\quad dt = \frac{1}{\sqrt{2}\sigma}dx \tag{2}$$

Then the wave function changes into more comfort form to integrate.

$$N^2\int_{-\infty}^{\infty}\exp\left(-2\left(\frac{x-x_0}{2\sigma}\right)^2\right)\,dx = \sqrt{2}\sigma N^2\int_{-\infty}^{\infty} e^{-t^2}\,dt \tag{3}$$

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To calcualte this integration, we use a idea of double integration,

$$\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta \tag{4}$$

$$= \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi. \tag{5}$$

First double integration about coordinate space can be decomposed.

$$\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty} e^{-x^2}\,dx\int_{-\infty}^{\infty} e^{-y^2}\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 \tag{6}$$

From this result,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}, \quad \sqrt{2\pi}\sigma N^2 = 1. \tag{7}$$

Finally we obtain the normalization constant,

$$N = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}}. \tag{8}$$

(2) We will find $\phi(p,0)$ first. $\phi(p,0)$ is the Fourier transform of $\psi(x,0)$.

$$\phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}}\int \psi(x,0)e^{-\frac{i}{\hbar}px}\,dx = \frac{N}{\sqrt{2\pi\hbar}}\int \exp\left(-\left(\frac{x-x_0}{2\sigma}\right)^2\right)e^{-\frac{i}{\hbar}px}\,dx \tag{9}$$

$$= \frac{N}{\sqrt{2\pi\hbar}}\int \exp\left(-\left(\frac{x-x_0}{2\sigma}\right)^2 - \frac{i}{\hbar}px\right)\,dx \tag{10}$$

To make it compact form, let us erase the translation term and change the variable.

$$u \equiv \frac{x-x_0}{2\sigma}, \quad du = \frac{1}{2\sigma}dx \tag{11}$$

Then, a $\phi(p,0)$ is,

$$\phi(p,0) = \frac{2\sigma N}{\sqrt{2\pi\hbar}}\int \exp\left(-u^2 - \frac{i}{\hbar}p(2\sigma u + x_0)\right)\,du \tag{12}$$

$$= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}px_0}\int \exp\left(-u^2 - 2\frac{i}{\hbar}\sigma pu\right)\,du. \tag{13}$$

And, a exponential of integrated function can be expressed in terms of complete square form about u.

$$-u^2 - 2\frac{i}{\hbar}\sigma pu = -\left(u + \frac{i}{\hbar}\sigma p\right)^2 - \frac{\sigma^2}{\hbar^2}p^2 \tag{14}$$

$\frac{i}{\hbar}\sigma p$ is the translation term that can be ignored since the integration range is from $-\infty$ to ∞,

$$\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}px_0}\int \exp\left(-u^2 - 2\frac{i}{\hbar}\sigma pu\right)\,du \tag{15}$$

$$= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}px_0}\int \exp\left(-\left(u + \frac{i}{\hbar}\sigma p\right)^2 - \frac{\sigma^2}{\hbar^2}p^2\right)\,du = \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{i}{\hbar}px_0}e^{-\frac{\sigma^2}{\hbar^2}p^2}\int e^{-u^2}\,du \tag{16}$$

(3)

(4)

(5)

Problem 2. The Hamiltonian for a free particle is given by

$$H = \frac{p^2}{2m}.$$

(1) Show

$$\langle p_x \rangle = \langle p_x \rangle_{t=0}.$$

(2) Show

$$\langle x \rangle = \frac{\langle p_x \rangle_{t=0}}{m} t + \langle x \rangle_{t=0}.$$

(3) Show

$$(\Delta p_x)^2 = (\Delta p_x)^2_{t=0}.$$

(4) Find $d(\Delta x)^2/dt$ as a function of time and initial conditions.

Problem 3. The state of a particle is described by the following wavefunction:

$$\psi(x) = C \exp\left[i\frac{p_0 x}{\hbar} - \frac{(x-x_0)^2}{2\sigma^2}\right]$$

where p_0, x_0, and a are real parameters.

(1) Find the normalization constant C.

(2) Find the mean values of x and p.

(3) Find the standard deviations Δx and Δp.

Problem 4*. Consider a particle and two normalized energy eigenfunctions $\psi_1(\boldsymbol{x})$ and $\psi_2(\boldsymbol{x})$ corresponding to the eigenvalues $E_1 \neq E_2$. Assume that the eigenfunctions vanish outside the two non-overlapping regions Ω_1 and Ω_2, respectively.

(1) (a) Show that, if the particle is initially in region Ω_1 then it will stay there forever.

(b) If, initially, the particle is in the state with wave function

$$\psi(\boldsymbol{x},\, 0) = \frac{1}{\sqrt{2}}[\psi_1(\boldsymbol{x}) + \psi_2(\boldsymbol{x})]$$

show that the probability density $|\psi(\boldsymbol{x}, t)|^2$ is independent of time.

(c) Now assume that the two regions Ω_1 and Ω_2 overlap partially. Starting with the initial wave function of case (b), show that the probability density is a periodic function of time. ($E_2 - E_1 = \hbar\omega$).

(d) Starting with the same initial wave function and assuming that the two eigenfunctions are real and isotropic, take the two partially overlapping regions Ω_1 and Ω_2 to be two concentric spheres of radii $R_1 > R_2$. Compute the probability current that flows through Ω_1.

(Problem 4 is a bit difficult. To solve (3), introduce phase factors for $\psi_1(\boldsymbol{x})$ and $\psi_2(\boldsymbol{x})$ and consider the interference term when one computes the probability density. To solve (4), consider the current density and continuity equation.