Quantum Mechanics

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Problem Set 1

Problem 1. The wave function of this particle is,

$$\psi(x,0) = N\exp\left(-\left(\frac{x-x_0}{2\sigma}\right)^2\right) \tag{1}$$

(1) From the normalization of the wave function,

$$\int_{-\infty}^{\infty}|\psi(x,0)|^2\,dx = N^2\int_{-\infty}^{\infty}\exp\left(-2\left(\frac{x-x_0}{2\sigma}\right)^2\right)dx = 1. \tag{2}$$

Since a range of integration is all space, the translation about x can be ignored. To make a compact form, it needs to change an integral variable.

$$2\left(\frac{x-x_0}{2\sigma}\right)^2 = \left(\frac{x-x_0}{\sqrt{2}\sigma}\right)^2 \longrightarrow t^2,\ dx \longrightarrow \sqrt{2}\sigma dt \tag{3}$$

Then the wave function changes into more comfort form to integrate.

$$N^2\int_{-\infty}^{\infty}\exp\left(-2\left(\frac{x-x_0}{2\sigma}\right)^2\right)dx = \sqrt{2}\sigma N^2\int_{-\infty}^{\infty}e^{-x^2}\,dx \tag{4}$$

To calcualte this integration, we use a idea of double integration,

$$\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty}e^{-r^2}r\,dr\,d\theta \tag{5}$$

$$= \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi. \tag{6}$$

First double integration about coordinate space can be decomposed.

$$\int_{-\infty}^{\infty}e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty}e^{-x^2}\,dx\int_{-\infty}^{\infty}e^{-y^2}\,dy = \left(\int_{-\infty}^{\infty}e^{-x^2}\,dx\right)^2 \tag{7}$$

From this result,

$$\int_{-\infty}^{\infty}e^{-x^2}\,dx = \sqrt{\pi},\quad \sqrt{2\pi}\sigma N^2 = 1. \tag{8}$$

Finally,

$$N = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}}. \tag{9}$$

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(2) First, we have to find $\phi(p,0)$. It is a fourier transformation of ψ.

$$\phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x,0) e^{-\frac{i}{\hbar}px}\, dx = \frac{N}{\sqrt{2\pi\hbar}} \int \exp\left(-\left(\frac{x-x_0}{2\sigma}\right)^2\right) e^{-\frac{i}{\hbar}px}\, dx \tag{10}$$

$$= \frac{N}{\sqrt{2\pi\hbar}} \int \exp\left(-\left(\frac{x-x_0}{2\sigma}\right)^2 - \frac{i}{\hbar}px\right) dx \tag{11}$$

We change a variable for compact form.

$$-\left(\frac{x-x_0}{2\sigma}\right)^2 - \frac{i}{\hbar}px \longrightarrow -t^2 - \frac{i}{\hbar}p\left(2\sigma t - x_0\right),\ dx \longrightarrow 2\sigma dt \tag{12}$$

And transform a exponential of integrated function into the complete square one about x.

$$-x^2 - 2\frac{i}{\hbar}\sigma px = -\left(x + \frac{i}{\hbar}\sigma p\right)^2 - \frac{\sigma^2}{\hbar^2}p^2 \tag{13}$$

In term of integration of x, there is a translation term and it can be negligiable with the same reason in previous. So,

$$\phi(p,0) = \frac{2\sigma N}{\sqrt{2\pi\hbar}} \exp\left(-\frac{\sigma^2}{\hbar^2}p^2 - \frac{i}{\hbar}x_0 p\right) \int\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\frac{2\sigma}{\hbar}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{\sigma^2}{\hbar^2}p^2 - \frac{i}{\hbar}x_0 p\right) \tag{14}$$

Then, $\phi(0,0)$ is,

$$\phi(0,0) = \sqrt{\frac{2\sigma}{\hbar}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \tag{15}$$

(3) Because it is a free particle, the time evolution of $\phi(x,0)$ is $\phi(x,t) = e^{-i\omega t}\phi(p,0)$ and $\omega = \frac{p^2}{2m\hbar}$.

$$\phi(p,t) = \sqrt{\frac{2\sigma}{\hbar}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\alpha\left(p + \frac{ix_0}{2\alpha\hbar}\right)^2 - \frac{x_0^2}{4\alpha\hbar^2}\right), \quad \alpha = \frac{2m\sigma^2 + i\hbar t}{2m\hbar^2} \tag{16}$$

(4) $\psi(x,t)$ is a fourier transformation of $\phi(p,t)$,

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int\int_{-\infty}^{\infty} \phi(p,t) e^{\frac{i}{\hbar}px}\, dp = \frac{1}{\hbar}\sqrt{\frac{\sigma}{\pi}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \int\int_{-\infty}^{\infty} \exp\left(-\alpha\left(p + \frac{ix_0}{2\alpha\hbar}\right)^2 - \frac{x_0^2}{4\alpha\hbar^2} + \frac{i}{\hbar}px\right) dp \tag{17}$$

A exponential of a integrated function can be changed into the complete square form about p.

$$-\alpha\left(p + \frac{ix_0}{2\alpha\hbar}\right)^2 - \frac{x_0^2}{4\alpha\hbar^2} + \frac{i}{\hbar}px = -\alpha\left(p + i\frac{x_0 - x}{2\alpha\hbar}\right)^2 - \frac{(x_0 - x)^2}{4\alpha\hbar^2} \tag{18}$$

So,

$$\psi(x,t) = \frac{1}{\hbar}\sqrt{\frac{\sigma}{\pi}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{(x_0 - x)^2}{4\alpha\hbar^2}\right) \int\int_{-\infty}^{\infty} e^{-\alpha p^2}\, dp \tag{19}$$

$$= \frac{1}{\hbar}\sqrt{\frac{\sigma}{\alpha}} \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{(x_0 - x)^2}{4\alpha\hbar^2}\right) \tag{20}$$

(5) From a (4), $|\psi(x,\, t)|^2$ and $|\psi(0,\, t)|^2$ are,

$$|\psi(x,t)|^2 = \frac{1}{\hbar^2} \frac{\sigma}{\sqrt{\alpha^*\alpha}} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{(x_0 - x)^2}{4\alpha^*\hbar^2} - \frac{(x_0 - x)^2}{4\alpha\hbar^2}\right) \tag{21}$$

$$|\psi(0,t)|^2 = \frac{1}{\hbar^2} \frac{\sigma}{\sqrt{\alpha^*\alpha}} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{{x_0}^2}{4\alpha^*\hbar^2} - \frac{{x_0}^2}{4\alpha\hbar^2}\right) \tag{22}$$

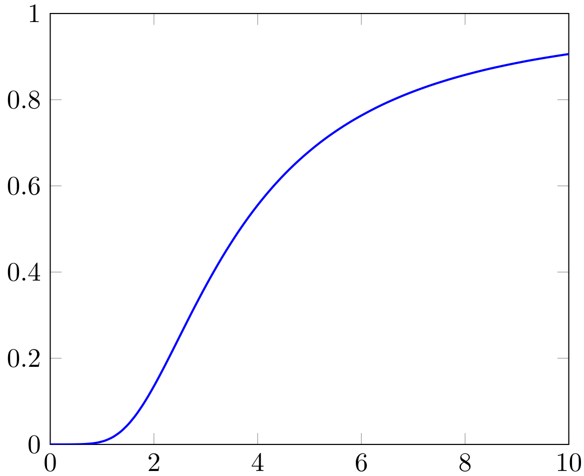
Since coefficients are canceled, there remains only exponential term.

$$\mathcal{S}(t) = \frac{|\psi(x,t)|^2}{|\psi(0,t)|^2} = \exp\left(-\frac{x^2 - 2xx_0}{4\hbar^2}\left(\frac{\alpha^* + \alpha}{|\alpha|^2}\right)\right) \tag{23}$$

We want to know that is the spread increases with time t. Since only α is dependent to time and time is a imaginary part of α, all of what is need to us is in $e^{-\frac{1}{|\alpha|^2}}$.

$$\frac{1}{|\alpha|^2} = \frac{4m^2\hbar^4}{4m^2\sigma^4 + \hbar^2 t^2}, \quad \mathcal{S}(t) = A\exp\left(-\frac{4m^2\hbar^4}{4m^2\sigma^4 + \hbar^2 t^2}\right) \tag{24}$$

A is a constant about t. The form of $\mathcal{S}(t)$ is $\exp\left(-\frac{k_1}{k_2+t^2}\right)$. The derivative of $\mathcal{S}(t)$ is $\frac{2k_1 t}{(k_2+t^2)^2}\mathcal{S}(t)$, being positive in all positive time t. This is a graph of $\exp(-\frac{10}{1+x^2})$ with $4m^2\hbar^2 = 10$, $4\frac{m^2\sigma^4}{\hbar^2} = 1$.



Problem 2. The Hamiltonian of the free particle is,

$$H = \frac{p^2}{2m}. \tag{25}$$

(1) $\langle p_x \rangle$ can be expanded about t.

$$\langle p_x \rangle = \langle p_x \rangle_{t=0} + \frac{1}{1!}\frac{d}{dt}\langle p_x \rangle\bigg|_{t=0} t + \cdots \tag{26}$$

Calculate $\frac{d}{dt}\langle p_x \rangle$ using Hamiltonian,

$$i\hbar\frac{d}{dt}\langle p_x \rangle = \langle [p_x, H] \rangle + i\hbar\left\langle \frac{\partial p_x}{\partial t} \right\rangle = \frac{1}{2m}\langle [p_x, p_x^2] \rangle = 0 \tag{27}$$

So, there remains only $\langle p_x \rangle_{t=0}$,

$$\langle p_x \rangle = \langle p_x \rangle_{t=0} \tag{28}$$

(2) With the same method,

$$\langle x \rangle = \langle x \rangle_{t=0} + \frac{1}{1!}\frac{d}{dt}\langle x \rangle\bigg|_{t=0} t + \frac{1}{2!}\frac{d^2}{dt^2}\langle x \rangle\bigg|_{t=0} t^2 + \cdots \tag{29}$$

We have to calculate two time derivatives.

$$\frac{d}{dt}\langle x\rangle =\frac{1}{2im\hbar}\langle\left[x,p^2\right]\rangle + \left\langle\frac{\partial x}{\partial t}\right\rangle = \frac{1}{m}\langle p_x\rangle \tag{30}$$

$$\frac{d^2}{dt^2}\langle x\rangle =\frac{1}{m}\frac{d}{dt}\langle p_x\rangle = 0 \tag{31}$$

Finally,

$$\langle x\rangle =\langle x\rangle_{t=0} + \frac{\langle p_x\rangle_{t=0}}{2m}t \tag{32}$$

(3) From the definition of the deviation,

$$\left(\Delta p_x\right)^2 = \langle p_x^2\rangle - \langle p_x\rangle^2, \quad \left(\Delta p_x\right)^2_{t=0} = \langle p_x^2\rangle_{t=0} - \langle p_x\rangle^2_{t=0} \tag{33}$$

As we did,

$$\langle p_x^2\rangle = \frac{1}{2m}\langle\left[p^2,p^2\right]\rangle = 0 = \langle p_x^2\rangle_{t=0} \tag{34}$$

And we know that $\langle p_x\rangle^2 = \langle p_x\rangle^2_{t=0}$ by the result of (1). Therefore, $\left(\Delta p_x\right)^2 = \left(\Delta p_x\right)^2_{t=0}$.

(4) From the (3),

$$\frac{d}{dt}\left(\Delta x\right)^2 = \frac{d}{dt}\langle x^2\rangle - \frac{d}{dt}\langle x\rangle^2 \tag{35}$$

We have to calculate two time derivatives.

$$\frac{d}{dt}\langle x^2\rangle =\frac{1}{2im\hbar}\langle\left[x^2,p_x^2\right]\rangle = \frac{2}{m}\langle xp_x\rangle \tag{36}$$

$$\frac{d}{dt}\langle x\rangle =\frac{1}{2im\hbar}\langle\left[x,p_x^2\right]\rangle = \frac{1}{m}\langle p_x\rangle \tag{37}$$

$\langle xp_x\rangle$ can be approximated a series of time t.

$$\langle xp_x\rangle = \langle xp_x\rangle_{t=0} + \frac{1}{1!}\frac{d}{dt}\langle xp_x\rangle\bigg|_{t=0} t + \frac{1}{2!}\frac{d^2}{dt^2}\langle xp_x\rangle\bigg|_{t=0} t^2 + \cdots \tag{38}$$

Most derivative calculations about expectation value are using the Hamiltonian.

$$\frac{d}{dt}\langle xp_x\rangle = \frac{1}{2im\hbar}\langle\left[xp_x,p^2\right]\rangle = \frac{1}{m}\langle p^2\rangle \tag{39}$$

Since $\frac{d}{dt}\langle xp_x\rangle$ is a multiple of $\langle p^2\rangle$, the quadratic and higher term will be vansihed. So,

$$\langle xp_x\rangle = \langle xp_x\rangle_{t=0} + \frac{1}{m}\langle p^2\rangle_{t=0}t \tag{40}$$

Substitute these results in $\frac{d}{dt}\left(\Delta x\right)^2$,

$$\frac{d}{dt}\left(\Delta x\right)^2 = \frac{2}{m}\langle xp_x\rangle - \frac{2}{m}\langle x\rangle\langle p_x\rangle \tag{41}$$

We already know that $\langle x\rangle$ and $\langle p_x\rangle$ can be represented by initial conditions.

$$\frac{d}{dt}\left(\Delta x\right)^2 = \frac{2}{m}\left(\langle xp_x\rangle_{t=0} + \frac{\langle p^2\rangle_{t=0}}{m}t\right) - \frac{2}{m}\langle p_x\rangle_{t=0}\left(\langle x\rangle_{t=0} + \frac{\langle p_x\rangle_{t=0}}{2m}t\right) \tag{42}$$

Problem 3. In this problem, the wave function of a partcle is,

$$\psi(x) = C\exp\left[i\frac{p_0 x}{\hbar} - \frac{\left(x-x_0\right)^2}{2\sigma^2}\right] \tag{43}$$

(1) The normalization constant is calculable from the normalization.

$$C^2 \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx = C^2 \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx = C^2\sigma\sqrt{\pi} \tag{44}$$

The result of the noramlizatoin is must be 1. So,

$$C = \left(\frac{1}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} \tag{45}$$

(2) First, let us find the mean value of x.

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x\psi \, dx = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} x \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx \tag{46}$$

$$\int_{-\infty}^{\infty} x \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx = \int_{-\infty}^{\infty} xe^{-\left(\frac{x}{\sigma}\right)^2} dx + x_0 \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sigma}\right)^2} dx \tag{47}$$

The first term of the right side is a zero, because $xe^{-\left(\frac{x}{\sigma}\right)^2}$ is a even function and this integration is from $-\infty$ to ∞. The calculation of the second term is simple.

$$x_0 \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sigma}\right)^2} dx = x_0 \, \sigma\sqrt{\pi} \tag{48}$$

So, the mean value is a x_0.

$$\langle x\rangle = \frac{1}{\sigma\sqrt{\pi}} \, x_0 \, \sigma\sqrt{\pi} = x_0 \tag{49}$$

The mean value of p is,

$$\langle p\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial\psi}{\partial x} \, dx = \frac{-i\hbar}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{i}{\hbar}p_0 - \frac{x-x_0}{\sigma^2}\right) \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx \tag{50}$$

$$= \frac{-i\hbar}{\sigma\sqrt{\pi}} \left[\frac{i}{\hbar}p_0 \int_{-\infty}^{\infty} \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx - \int_{-\infty}^{\infty} \left(\frac{x-x_0}{\sigma^2}\right) \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx\right] = p_0 \tag{51}$$

Because the second term is a even function about $x = x_0$, it is a zero.

(3) From a (3) in problem 2, we use the definition of the deviation.

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2, \quad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 \tag{52}$$

First we calculate $\langle x^2\rangle$.

$$\langle x^2\rangle = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\left(\frac{x-x_0}{\sigma}\right)^2\right) dx = \frac{1}{\sigma\sqrt{\pi}} \left[\int_{-\infty}^{\infty} x^2 e^{-\left(\frac{x}{\sigma}\right)^2} dx + 2x_0 \int_{-\infty}^{\infty} xe^{-\left(\frac{x}{\sigma}\right)^2} dx + x_0^2 \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sigma}\right)^2} d \tag{53}$$

The middle term of the right side is zero from a (2) and the last term is $x_0^2 \, \sigma\sqrt{\pi}$.

$$\int_{-\infty}^{\infty} x^2 e^{-\left(\frac{x}{\sigma}\right)^2} dx = \sigma^3 \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = -\frac{1}{2}\sigma^3 \left[xe^{-x^2}\right]_{-\infty}^{\infty} + \frac{1}{2}\sigma^3 \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{2}\sigma^3\sqrt{\pi} \tag{54}$$

So,

$$\langle x^2\rangle = \frac{1}{\sigma\sqrt{\pi}} \left[x_0^2 \, \sigma\sqrt{\pi} + \frac{1}{2}\sigma^3\sqrt{\pi}\right] = \frac{1}{2}\sigma^2 + x_0^2 \tag{55}$$

Then $(\Delta x)^2$ is,

$$(\Delta x)^2 = \frac{1}{2}\sigma^2 + x_0^2 - x_0^2 = \frac{1}{2}\sigma^2 \tag{56}$$

the expectation value of p^2 is,

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2 \psi}{\partial x^2}\, dx = -\hbar^2 \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} \left(\psi^* \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} \right) dx \tag{57}$$

There are two terms that seem complicated. But, some of the integration in calculation will be canceled since these are even functions and the integration range is symmetric.

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(\psi^* \frac{\partial \psi}{\partial x} \right) dx = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(\left(\frac{i}{\hbar} p_0 - \frac{x - x_0}{\sigma^2} \right) \exp\left(- \left(\frac{x - x_0}{\sigma} \right)^2 \right) \right) dx = 0 \tag{58}$$

$$\int_{-\infty}^{\infty} \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x}\, dx = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\left(\frac{p_0}{\hbar} \right)^2 + \left(\frac{x - x_0}{\sigma^2} \right)^2 \right) \exp\left(- \left(\frac{x - x_0}{\sigma} \right)^2 \right) dx = \left(\frac{p_0}{\hbar} \right)^2 + \frac{1}{\sigma^2\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx \tag{59}$$

It is a simple gaussian integration.

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = -\frac{1}{2} \left[x e^{-x^2} \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \tag{60}$$

From these results, we can calculate $\langle p^2 \rangle$.

$$\langle p^2 \rangle = p_0^2 + \frac{\hbar^2}{2\sigma^2} \tag{61}$$

Finally, we can calculate $(\Delta p)^2$,

$$(\Delta p)^2 = p_0^2 + \frac{\hbar^2}{2\sigma^2} - p_0^2 = \frac{\hbar^2}{2\sigma^2} \tag{62}$$

Confirm these result does satisfy Heisenberg's uncertainty principle.

$$\Delta x \Delta p = \sqrt{\frac{\hbar^2}{2\sigma^2} \frac{\sigma^2}{2}} = \frac{\hbar}{2} \tag{63}$$

It is in the sense.

Problem 4.

(a) The initial state is,

$$\psi(\boldsymbol{x}, 0) = c_1 \psi_1(\boldsymbol{x}) + c_2 \psi_2(\boldsymbol{x}) \tag{64}$$

Since this particle is in region Ω_1, $c_1 = 1$ and $c_2 = 0$. The time evolutoin of this particle is,

$$\psi(\boldsymbol{x}, t) = c_1 e^{-\frac{i}{\hbar} E_1 t} \psi_1(\boldsymbol{x}) + c_2 e^{-\frac{i}{\hbar} E_2 t} \psi_2(\boldsymbol{x}) = e^{-\frac{i}{\hbar} E_1 t} \psi_1(\boldsymbol{x}) \tag{65}$$

Since the time evolution is dependent to only $\psi_1(\boldsymbol{x})$, it will stay region Ω_1, forever.

(b) The time evolution is,

$$\psi(\boldsymbol{x}, t) = \frac{1}{\sqrt{2}} \left[e^{-\frac{i}{\hbar} E_1 t} \psi_1(\boldsymbol{x}) + e^{-\frac{i}{\hbar} E_2 t} \psi_2(\boldsymbol{x}) \right] \tag{66}$$

Consider the probability density of this particle.

$$|\psi(\boldsymbol{x}, t)|^2 = \frac{1}{2} \left[|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2 + e^{-\frac{i}{\hbar}(E_2 - E_1)t} \psi_1^* \psi_2 + e^{-\frac{i}{\hbar}(E_1 - E_2)t} \psi_1 \psi_2^* \right] \tag{67}$$

Suppose that Ω_3 in $\mathbb{R}^3$ satisfies that $\Omega_1 \cap \Omega_3 = \varnothing$, $\Omega_2 \cap \Omega_3 = \varnothing$ and $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \mathbb{R}$. If $\boldsymbol{x} \notin \Omega_1$, then $\psi_1(\boldsymbol{x}) = 0$. If $\boldsymbol{x} \notin \Omega_2$, then $\psi_2(\boldsymbol{x}) = 0$. And $\psi_1^* \psi_2 = \psi_1 \psi_2^* = 0$ any $\boldsymbol{x} \in \mathbb{R}$ because of following reasons.

(i) $\boldsymbol{x} \in \Omega_1 \Longrightarrow \psi_2 = \psi_2^* = 0$
(ii) $\boldsymbol{x} \in \Omega_2 \Longrightarrow \psi_1 = \psi_1^* = 0$
(iii) $\boldsymbol{x} \in \Omega_3 \Longrightarrow \psi_1 = \psi_2 = 0$

Therefore, $|\psi(\boldsymbol{x},t)|^2 = \frac{1}{2}\left[|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2\right]$. And the probability density is time-independent.

(c) From a (62), since $E_2 - E_1 = \hbar\Omega$,

$$|\psi(\boldsymbol{x},t)|^2 = \frac{1}{2}\left[|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2 + e^{-i\omega t}\psi_1^*\psi_2 + e^{i\omega t}\psi_1\psi_2^*\right] \tag{68}$$

$$= \frac{1}{2}\left[|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2 + e^{-i\omega t}\psi_1^*\psi_2 + \left(e^{-i\omega t}\psi_1^*\psi_2\right)^*\right] \tag{69}$$

Imaginary parts of the last two terms are canceled.

$$|\psi(\boldsymbol{x},t)|^2 = \frac{1}{2}\left[|\psi_1(\boldsymbol{x})|^2 + |\psi_2(\boldsymbol{x})|^2 + \left(\psi_1^*\psi_2 + (\psi_1^*\psi_2)^*\right)\cos\omega t\right] \tag{70}$$

This result is a periodic function about time because the last term is a periodic function of time and other terms are constant about time.

(d) From the continuity equation, We use the integration of this equation because of the right term.

$$\int_{\Omega_2} \frac{\partial\rho}{\partial t}\, dr^3 = \int_{\Omega_2} \nabla\cdot\boldsymbol{J}\, dr^3 \tag{71}$$

The left term can be calculated using (c),

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}|\psi(\boldsymbol{x},t)|^2 = -\omega\psi_1\psi_2\sin\omega t \tag{72}$$

If we integrate this, it will be a zero since ψ_1 and ψ_2 are orthogonal to each other. Consider the right term. This integration is changed into the surface integration following Green's Theorem.

$$\int_{\Omega_2} \nabla\cdot\boldsymbol{J}\, dr^3 = \int_{\Omega_2} \boldsymbol{J}\cdot d\boldsymbol{S} \tag{73}$$

Because wave functions are isotropic, a current has the same value in a different direction. It means that this integration is replaced by the just inner product.

$$\int_{\Omega_2} \boldsymbol{J}\cdot d\boldsymbol{S} = 4\pi R_2^2\boldsymbol{J}\cdot\hat{n} \tag{74}$$

$\hat{n}$ is a vector that is vertical to the surface of a sphere Ω_2. Fianlly,

$$0 = 4\pi R_2^2\boldsymbol{J}\cdot\hat{n} \tag{75}$$

This means that there is no probability current between region Ω_1 and Ω_2.