UNIVERSITI TEKNOLOGI MALAYSIA

INSPIRING CREATIVE AND INNOVATIVE MINDS

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Chapter 2

(Part 2)

Functions

Functions

- Let X and Y be nonempty sets
- A function f from X to Y is a relation from X to Y having the properties.
  - The domain of f is X
  - If (x,y), $(x,y') \in f$, then $y=y'$

(e.g. f(1)=b, f(2)=b is a function, but f(1)=a, f(1)=b is NOT a function)

Functions

- A function from X to Y is denoted, $f : X \to Y$
- The domain of f is the set X.
- The set Y is called the codomain or target of f.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

example

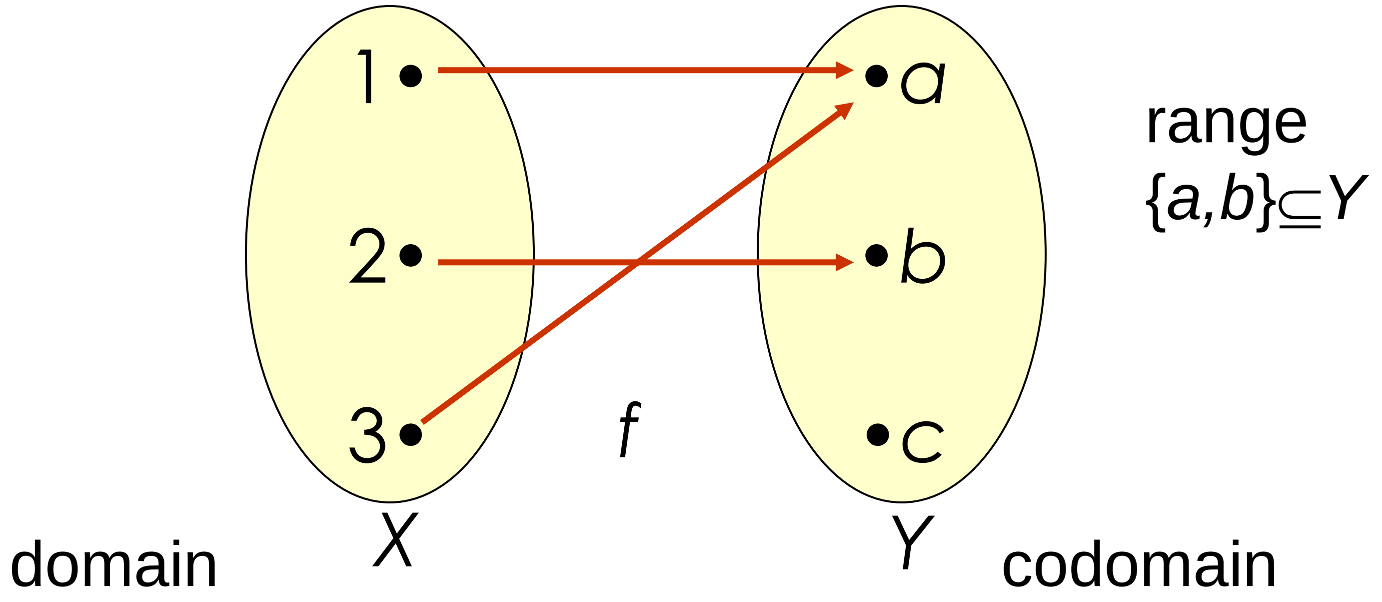
- The relation, $f = \{ (1,a), (2,b), (3,a) \}$
  from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$ is a function from X to Y.

- The domain of f is X

- The range of f is $\{a, b\}$

example

- $f = \{ (1,a), (2,b), (3,a) \}$



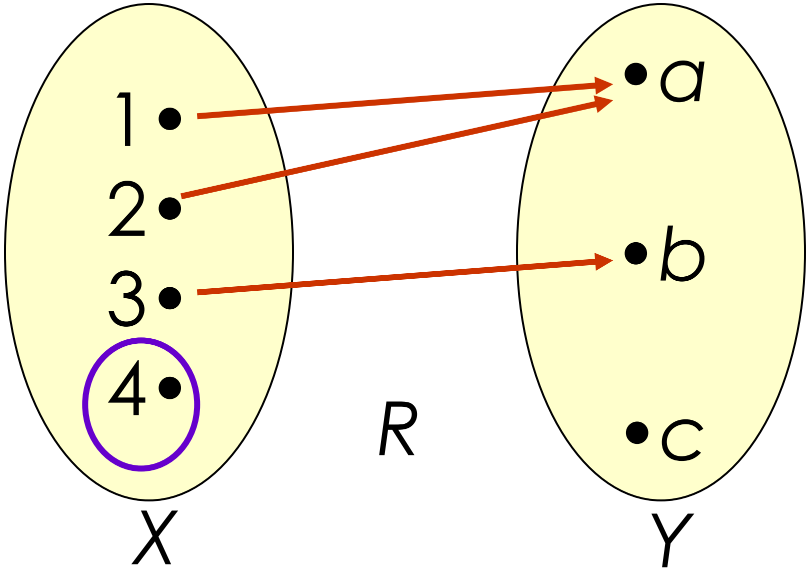
example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$
  from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y.

- The domain of R, $\{ 1,2,3 \}$ is not equal to X.

example

$R = \{(1,a), (2,a), (3,b)\}$



There is no arrow from 4

example

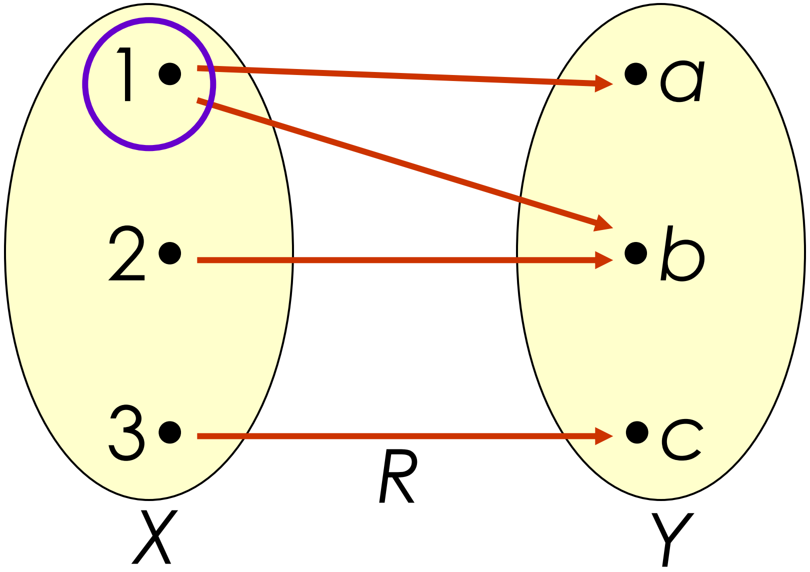
- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$
  from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y

- (1,a) and (1,b) in R but $a \neq b$.

example

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



example

- For the function, $f = \{(1,a), (2,b), (3,a)\}$
- We may write

  $f(1)=a$, $f(2)=b$, $f(3)=a$
- Notation $f(x)$ is used to define a function

example

- $f(x) = x^2$

  $f(2) = 4$, $f(-3.5) = 12.25$, $f(0) = 0$

  $f = \{(x, x^2) \mid x \text{ is a real number}\}$

One-to-one

- A function f from X to Y, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.

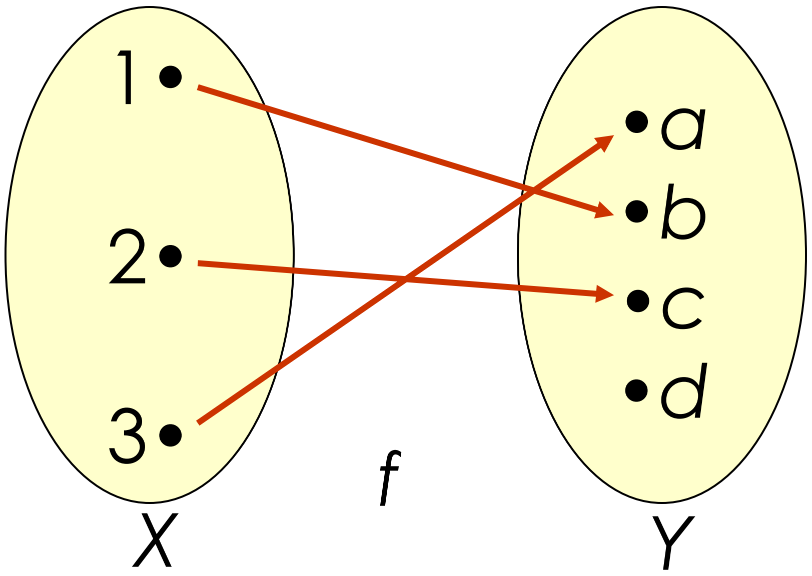
- For all x_1, x_2 , if $f(x_1) = f(x_2)$, then $x_1=x_2$.

  $\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$

example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c, d \}$ is one-to-one.

Each element in *Y* has at most one arrow pointing to it



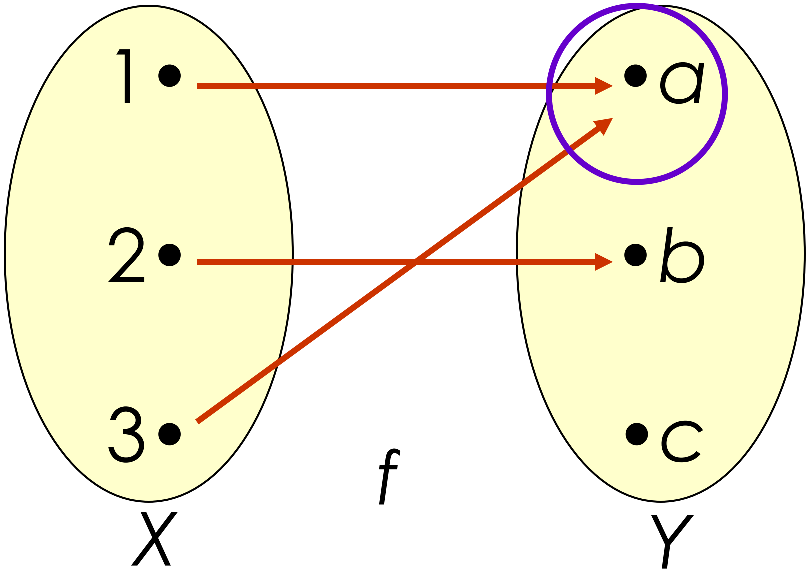
example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$ is NOT one-to-one.

- $f(1) = a = f(3)$

example

$f = \{\ (1,a),\ (2,b),\ (3,a)\ \}$



a has 2 arrows pointing to it

example

Show that the function,

$$f\,(n) = 2n+1$$

on the set of positive integers is one-to-one.

Example

- For all positive integer, $n_1$and n_2 if $f\,(n_1) = f\,(n_2)$, then $n_1 = n_2$.

- Let, $f\,(n_1) = f\,(n_2)$, $f\,(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ (-1)

  $2n_1 = 2n_2$ $(\div 2)$

  $n_1 = n_2$

- This shows that f is one-to-one.

example

- Show that the function,

  $$f\,(n) = 2^n - n^2$$

  from the set of positive integers to the set of integers is NOT one-to-one.

example

- Need to find 2 positive integers, n_1 and n_2
  $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.

- Trial and error,

  $$f(2) = f(4)$$

- f is not one-to-one.

Onto

- If f is a function from X to Y and the range of f is Y, f is said to be onto Y

  (or an onto function or a surjective function)

- For every $y \in Y$, there exists at least one $x \in X$ such that $f(x)=y$

$$\forall y \in Y \ \exists x \in X \ (f(x)=y)$$

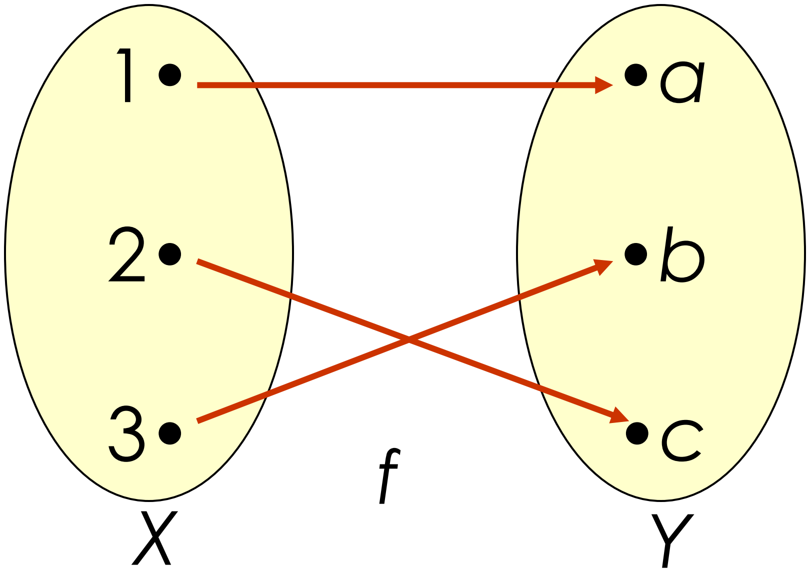
example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$
  from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$
  is one-to-one and onto Y.

example

- $f = \{ (1,a), (2,c), (3,b) \}$

One-to-one
Each element in Y has at most one arrow



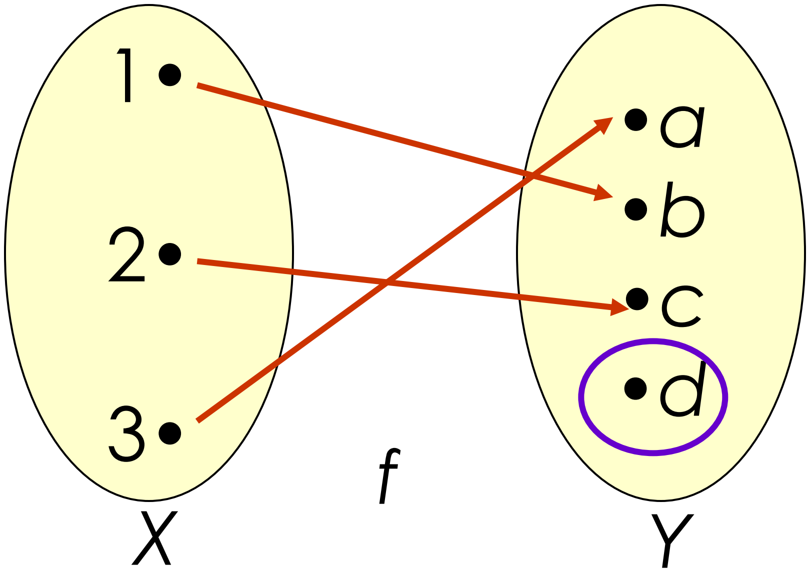
Onto
Each element in Y has at least one arrow pointing to it

example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto $Y = \{a, b, c, d\}$
- It is onto $\{a, b, c\}$

example

$f = \{ (1,b), (3,a), (2,c) \}$



not onto
no arrow pointing to *d*

Bijection

- *f* is called one-to-one correspondence (or bijective or bijection) if *f* is both one-to-one and onto.

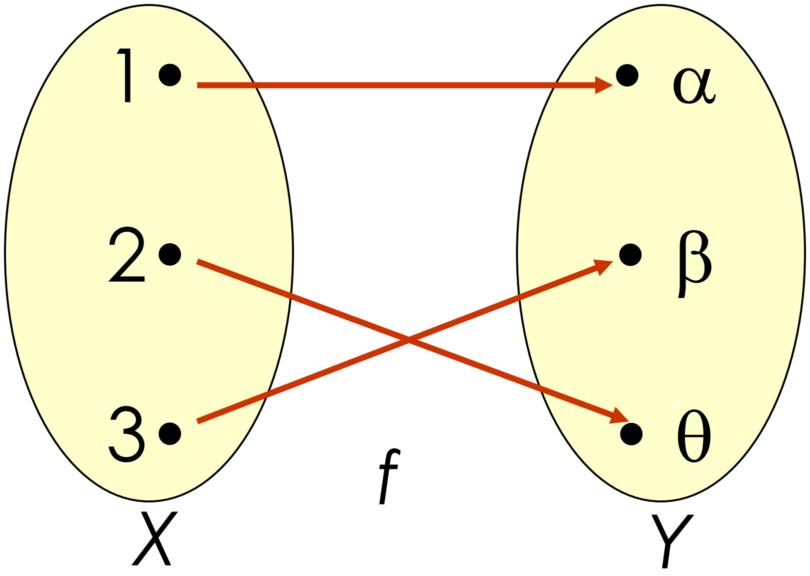
example

- The function, $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{\alpha, \beta, \theta \}$ is one-to-one and onto Y.

- The function *f* is a bijection

example

$f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$



One-to-one and onto *Y*
-**bijection**

exercise

Determine which of the relations *f* are functions from the set *X* to the set *Y*.

a) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and
$f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

b) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and
$f = \{ (-2,-3), (1,4), (2,5)\}$

c) $X = Y = \{ -3, -1, 0, 2 \}$ and
$f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

In case any of these relations are functions, determine if they are one-to-one, onto *Y*, and/or bijection.

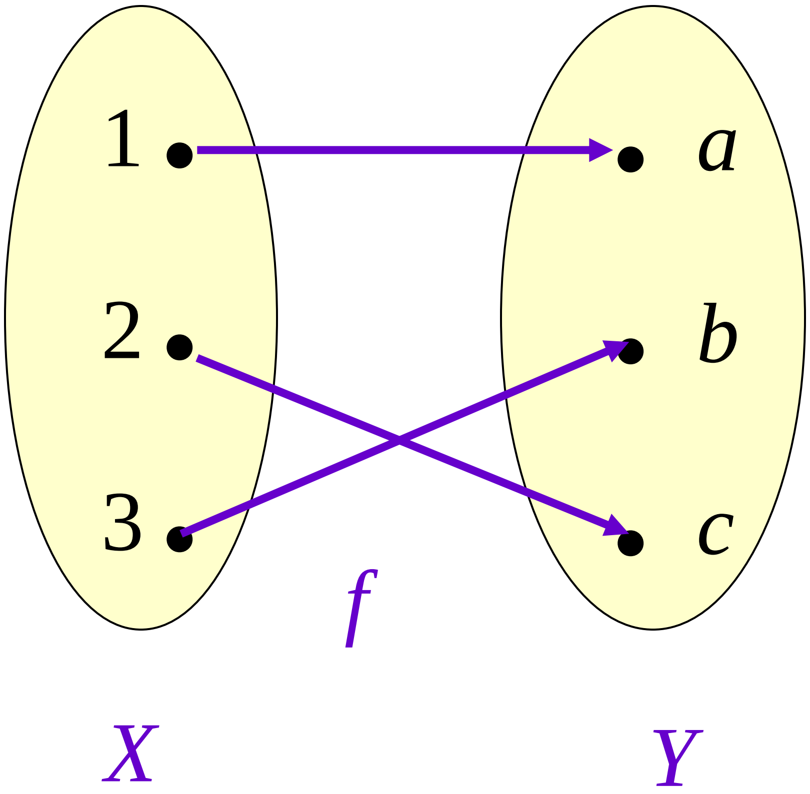
Inverse function

- Let $f: X \to Y$ be a function.

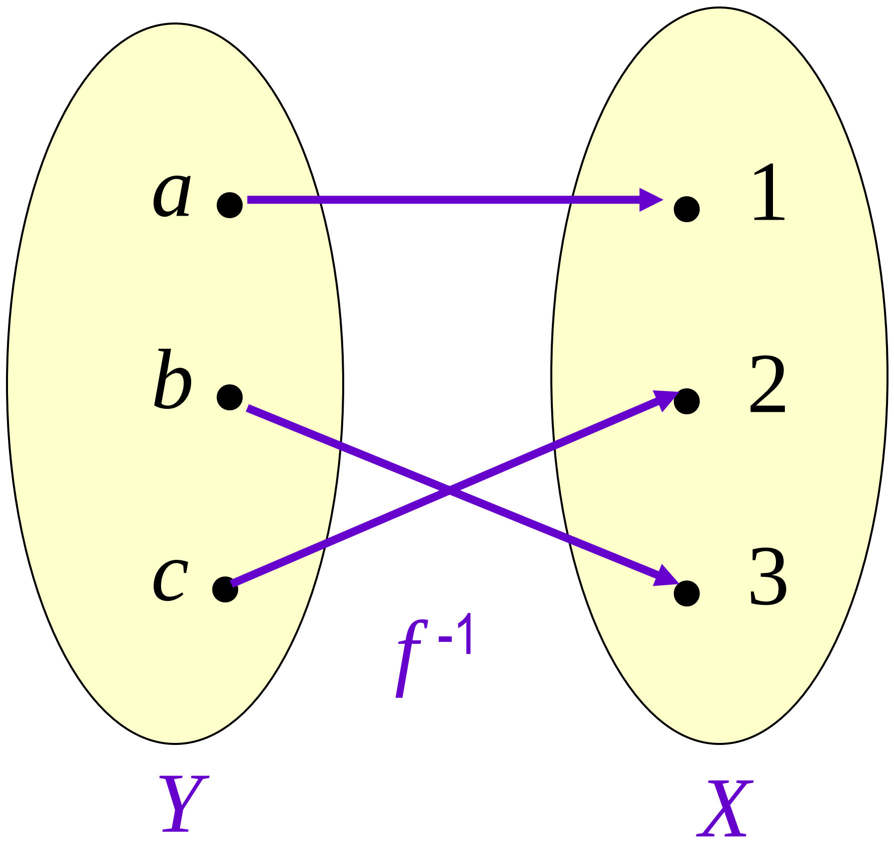
- The inverse relation $f^{-1} \subseteq Y \times X$ is a function from Y to X, if and only if f is both one-to-one and onto Y.

example

- $f = \{(1,a),(2,c),(3,b)\}$
- $f^{-1} = \{(a,1),(c,2),(b,3)\}$



example



Example

- The function, $f(x) = 9x + 5$ for all $x \in R$
  (R is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, f^{-1} exists.
- Let $(y, x) \in f^{-1}$, $f^{-1}(y) = x$
  $(x,y) \in f$, $y = 9x + 5$
  $x = (y-5)/9$
  $f^{-1}(y)=(y-5)/9$

exercise

- Find each inverse function.

  a) $f\,(x) = 4x + 2,\ \ x \in R$

  b) $f\,(x) = 3 + (1/x),\ \ x \in R$

Solution

- $f(x) = 4x + 2$

$$y = 4x + 2$$
$$x = 4y + 2$$
$$4y = x - 2y$$
$$= \frac{x-2}{4} = f^{-1}(x) = \frac{x-2}{4}$$

Exercise

Given $f(x) = x^3$+1 find f^{-1}(9)

Composition

- Suppose that g is a function from X to Y and f is a function from Y to Z.

- The composition of f with g,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

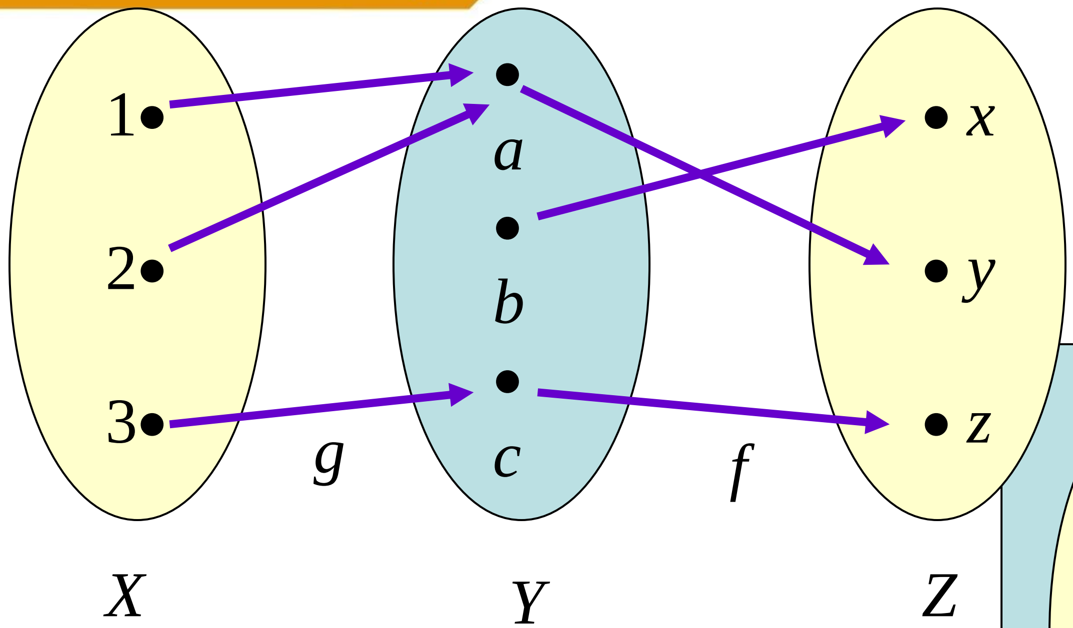
from X to Z

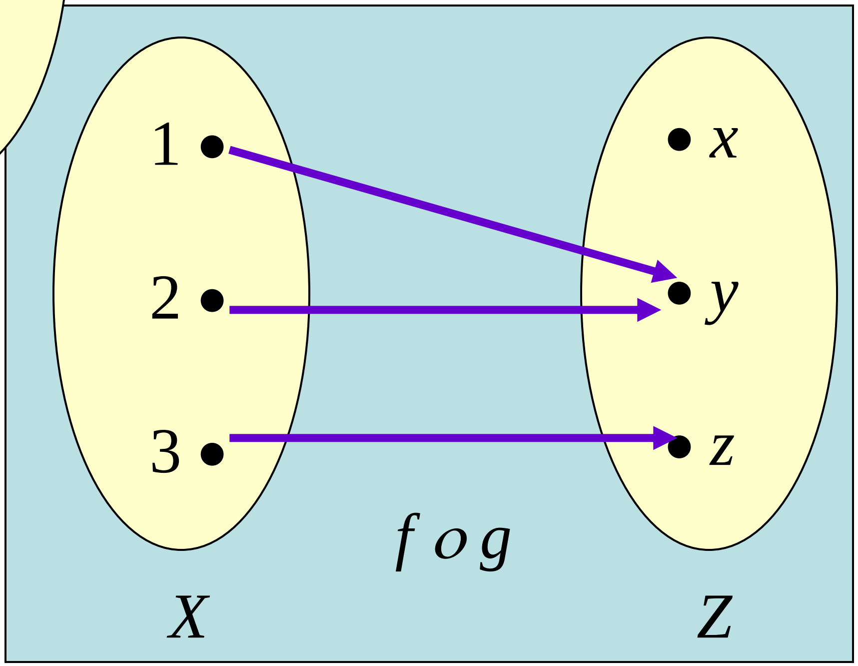
example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$
  a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ and
  $$f = \{ (a,y), (b,x), (c,z) \}$$
  a function from Y to $Z = \{ x, y, z \}$

- The composition function from X to Z is the function
  $$f \circ g = \{ (1,y), (2,y), (3,z) \}$$

example





example

- $f(x) = \log_3 x$ and $g(x) = x^4$

  $f(g(x)) = \log_3(x^4)$

  $g(f(x)) = (\log_3 x)^4$

Note: $f \circ g \neq g \circ f$

example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

example

- Composition sometimes allows us to decompose complicated functions into simpler functions.

- example

$$f(x) = \sqrt{\sin 2x}$$

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$

Exercise

Find $(g \circ f)(x)$ if $f(x) = \dfrac{5}{x+4}$ and $g(x) = \dfrac{3x}{2x-1}$

Let $f(x) = 5x^2 + 3x - 1$ and $g(x) = 2x - 7$.

(a) $(f \circ g)(2)$

(b) $(g \circ f)(0)$

exercise

- Let f dan g be functions from the positive integers to the positive integers defined by the equations,

  $f(n) = n^2$, $g(n) = 2^n$

- Find the compositions

  a) $f \circ f$

  b) $g \circ g$

  c) $f \circ g$

  d) $g \circ f$