UTM
UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 2
Part 3

Recurrence Relation
&
Recursive Algorithms

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \ldots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$,
  ($a_0, \ a_1, \ldots$)
- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

$$3, 8, 13, 18, 23, \ldots$$

The above sequence shows a pattern, given initial condition, a_1 =3:

$$\underset{a_1}{3},\ \underset{a_2}{3+5},\ \underset{a_3}{8+5},\ \underset{a_4}{13+5},\ \underset{a_5}{18+5}, \ldots$$

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$$a_n = a_{n-1} + 5\ ,\quad n \geq 2$$

$$3, 8, 13, 18, 23, \mathbf{28}, a_{n-1} + 5, \ldots$$

$$a_2 = a_1 + 5\ ,\quad 3 + 5 = 8$$

$$a_3 = a_2 + 5\ ,\quad 8 + 5 = 13$$

$$a_4 = a_3 + 5\ ,\quad 13 + 5 = 18$$

$$\vdots$$

$$a_6 = a_5 + 5\ ,\quad 23 + 5 = 28$$

Example 1

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$$3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$$
$$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$\boldsymbol{a_n = 1 + 2a_{n-1} \ , n \geq 1}$$

First few sequence are:

$$a_1 = 1 + 2\,(1) = 3$$
$$a_2 = 1 + 2(3) = 7$$
$$a_3 = 1 + 2(7) = 15$$

1, 3, 7, 15, 31, 63, …

Example 3

Given initial condition, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$\boldsymbol{a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2}$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$
$$a_3 = 3(9 + 2) = 33$$
$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Exercise

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$, $a_1 = 3$

2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5$, $a_1 = 3$

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$\boldsymbol{a_n} = \boldsymbol{a_{n-1}} + 7,\ \ n \geq 2 \text{ with } \boldsymbol{a_1 = 10}$$

Example 4 -Solution

Number of staff in the first 5 rows:

$\boldsymbol{a_1 = 10}$,

$\boldsymbol{a_2 = a_1} + 7$, $10 + 7 = 17$

$\boldsymbol{a_3 = a_2} + 7$, $17 + 7 = 24$

$\boldsymbol{a_4 = a_3} + 7$, $24 + 7 = 31$

$\boldsymbol{a_5 = a_4} + 7$, $31 + 7 = 38$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:

Look at the differences between terms:

4, 12, 36, 108, …

the difference in the sequence is growing by a factor of 3.

Example 5 -Solution

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, …

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$\boldsymbol{a_n = 3(a_{n-1}) + 2,\ n \geq 1}$**, with initial condition,** $\boldsymbol{a_0 = 1}$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example 6 - Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Example 6 - Solution

Initial condition, $P_0 = 10{,}000$. Then,

$$P_1 = 1.05P_0$$
$$P_2 = 1.05P_1 = (1.05)^2P_0$$
$$P_3 = 1.05P_2 = (1.05)^3P_0$$
$$\ldots$$
$$P_n = 1.05P_{n-1} = (1.05)^nP_0\ ,$$

now we can use this formula to calculate n_{th} term without iteration

Example 6 - Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30}(10,000) = 43,219.42$

After 30 years, the account contains RM 43,219.42.

Example

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Answer:

$a_n = a_{n-1} + 4 \quad , n \geq 2$ with $a_1 = 1$

Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n, that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

example

Factorial problem

- If $n \geq 1$,

$$n! = n\ (n-1)\ ....2.\ 1$$

and $$0! = 1$$

- Notice that, if $n \geq 2$, n factorial can be written,

$$n! = n\ (n-1)(n-2)\ ....2.\ 1$$

$$= n.\ (n-1)!$$

example

- n=5

- 5!

$$5! = 5.\,4.\,3.\,2.\,1 = 5.\,4!$$

$$4! = 4.\,3!$$

$$3! = 3.\,2!$$

Algorithm

- Input: n, integer ≥ 0
- Output: n!
- Factorial (n) {

  if (n=0)

  return 1

  return n*factorial(n-1)

  }

example

- Fibonacci sequence, f_n

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ….

Algorithm

- Input: n
- Output: $f(n)$
- $f(n)$ {

    if (n=1 or n=2)

        return 1

    return $f(n-1) + f(n-2)$

  }

Example

Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

Suppose a_n is the term sequence. The generating rule is $a_n = a_{n-1} + 2$, for $n > 1$. The relevant recursive algorithm can be written as

```
f(n)
{   if (n = 1 )
            return 1
    return f(n – 1) +2
}
```

Use the above recursive algorithm to trace n = 4

Solutions

Trace the output if n = 4

f(4)

n = 4
Because n≠1
return f(3) +2

f(4) =7

Return 5 +2

f(3)

n = 3
Because n≠1
return f(2) +2

f(3) =5

Return 3 +2

f(2)

n = 2
Because n≠1
return f(1) +2

f(2) =3

Return 1 +2

f(1)

n = 1
Because n =1
return 1

f(1) =1

Return 1

Answer = 7

exercise

- Given,

$$s_n = 3 + 6 + 9 + \ldots\ldots + 3n$$

$$s_1 = 3 \qquad s_n = s_{n-1} + 3n \text{ for } n \geq 2,$$

- Write a recursive algorithm to compute S_n, $n \geq 1$.

Algorithm

- Input: n
- Output: $s(n)$
- $s(n)$ {

    if (n=1)

        return 3

    return $s(n\text{-}1) + 3n$

  }

Exercise: Test 1 (2018/2019)

1. Suppose that the number of bacteria in a colony triples every hour.

   a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

   b) If 10 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

2. Find a recurrence relation for the balance B(k) owed at the end of k months on a loan of RM9000 at rate of 8% if a payment of RM150 is made each month.
   [Hint: Express B(k) in terms of B(k-1); the monthly interest is (0.07/12) (B(k-1))]

3. Write a recursive algorithm for computing n^2 where n is a nonnegative integer using the fact that $(n+1)^2 = n^2 + 2n + 1$.