CHAPTER 3

COUNTING METHODS (Part 2)

Permutation & Combination

Permutations & Combinations

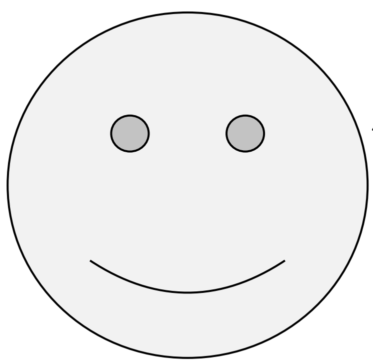
- What's the difference?
- Consider these situations:
  - ***"My fruit salad is a combination of apples, grapes and bananas"***
    - We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.
  - ***"The combination to the safe was 472"***
    - Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more **precise** language:

- If the order **doesn't** matter, it is a **Combination**.
  - Combination means **selection** of things.
  - The word **selection** is used, when the order of things has *no importance*.

- If the order **does** matter, it is a **Permutation**.
  - Permutation means **arrangement** of things.
  - The word **arrangement** is used, if the order of things *is considered*.

To help you to remember, think "**P**ermutation ... **P**osition"

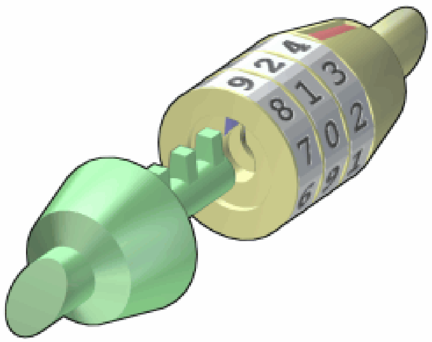


Permutations

There are basically two types of permutation:

- **Repetition is Allowed**: such as the permutation lock (in picture). It could be "333".

- **No Repetition**: for example the first three people in a running race. You can't be first and second.



Permutation – No Repetition

- In this case, you have to **reduce** the number of available choices each time.
- For example, what order could 16 pool balls be in?
  - After choosing, say, number "14" you can't choose it again.

- So, your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, 13, etc.
- And the total permutations would be:

$16 \times 15 \times 14 \times 13 \times \ldots = 20{,}922{,}789{,}888{,}000$

- But maybe you don't want to choose them all, just 3 of them, so that would be only:

$$16 \times 15 \times 14 = 3{,}360$$

- In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

- But how do we write that mathematically?
- Answer: we use the "factorial function"
- The **factorial function** (symbol: **!**) just means to multiply a series of **descending** natural numbers. Examples:
  - $4! = 4 \times 3 \times 2 \times 1 = 24$
  - $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$
  - $1! = 1$

- So, if you wanted to select **all** of the billiard balls, the permutations would be:

16! = 20,922,789,888,000

- A permutation of n distinct elements $x_1, \ldots, x_n$ is an ordering of the n elements $x_1, .. x_n$,
- There are $n!$ permutations of n elements

$$p(n) = n!$$

$$= n.(n-1).(n-2)\ldots2.1$$

Example 1

- There are 6 permutations of three elements
- If the elements are denoted A, B, C, the six permutations are
  - ABC, ACB, BAC, BCA, CAB, CBA
  - 3!= 3.2.1=6

example

- How many permutations of letters *ABCDEF* contain the substring *DEF*?.

DEF	*A*	*B*	*C*

- $4! = 24$

example

- How many permutations of letters *ABCDEF* contain the letters *DEF* together in any order?

DEF	***A***	***B***	***C***

- *DEF* 3! =6
- *DEF, A, B, C* 4! = 24 6.24=144

Permutations - Circular Arrangement

- The permutation in a row or along a line has a beginning and an end.
- But there is nothing like beginning or end or first and last in a **circular permutation**.
- In circular permutations, we consider one of the objects as fixed and the remaining objects are arranged as in linear permutation.

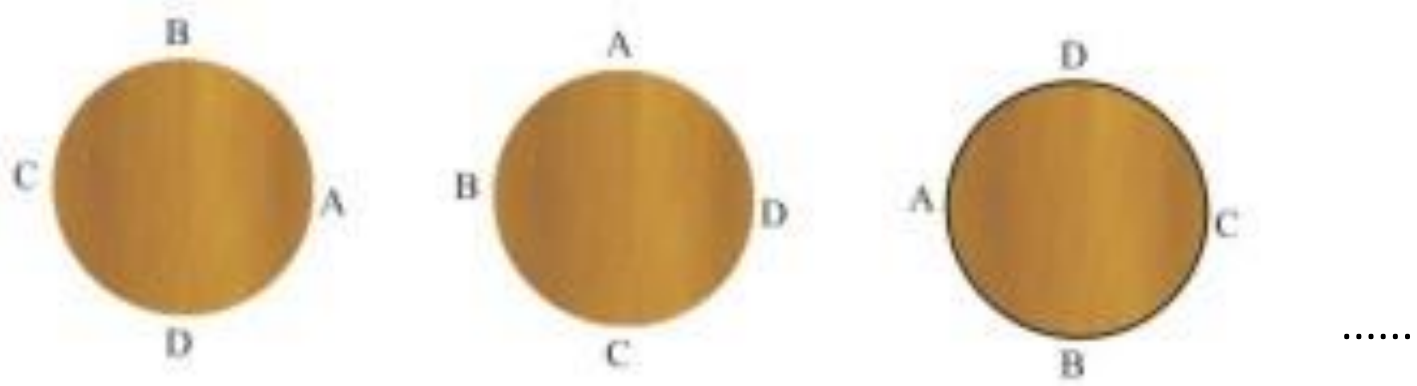
- There are two cases of circular-permutations:

  - If clockwise and anti clockwise orders **are different**, then total number of circular-permutations is given by **$(n$-1)!**

  - If clockwise and anti clockwise orders are taken as **not different**, then total number of circular-permutations is given by

$$\frac{(\boldsymbol{n}-1)!}{2!}$$

- Let’s consider that 4 persons A,B,C, and D are sitting around a round table
- Shifting A, B, C, D, one position in anticlock-wise direction, we get the following arrangements:-



- Thus, we use that if 4 persons are sitting at a round table, then they can be shifted four times.

- But these four arrangements will be the same, because the sequence of A, B, C, D, is same.

- But if A, B, C, D, are sitting in a row, and they are shifted, then the four **linear-arrangement** will be different.

- Hence, if we have ‘4’ things, then for each circular-arrangement, number of linear-arrangements is 4.

- Similarly, if we have ‘***n***’ things, then for each circular-arrangement , number of linear–arrangement is ***n***.

- Hence, the number of circular permutations is

$$P_n = (n-1)!$$

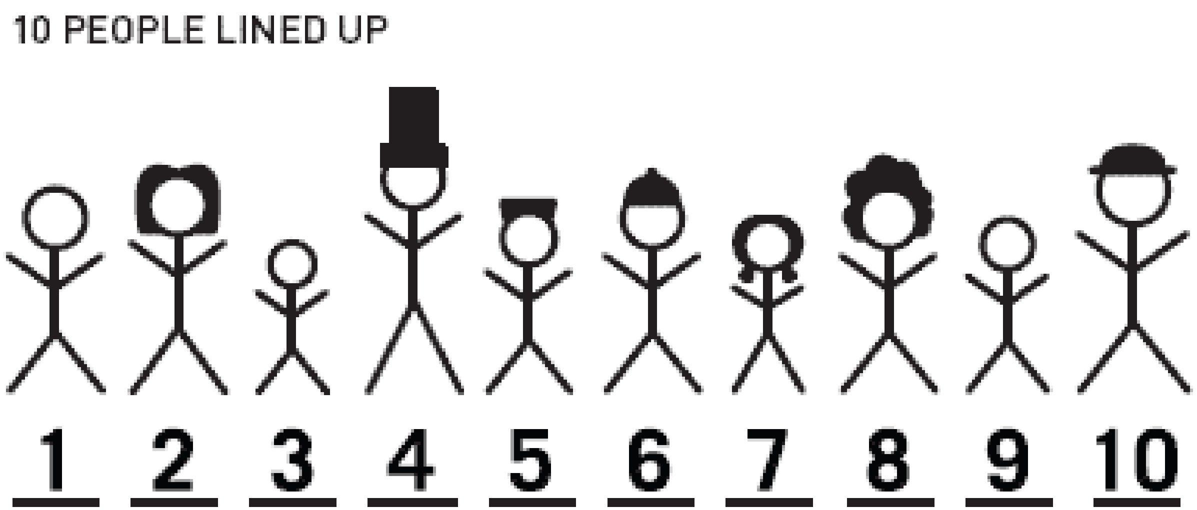
- The number is $(n-1)!$ instead of the usual factorial, $n!$ since all cyclic permutations of objects are equivalent because the circle can be rotated.

- Thus, the number of permutations of 4 objects in a row = 4!

- Where as the number of circular permutations of 4 objects is (4-1)! = 3!

Example 1

- Suppose we are expecting ten people for dinner.
- How many ways can we seat them around a circular table?

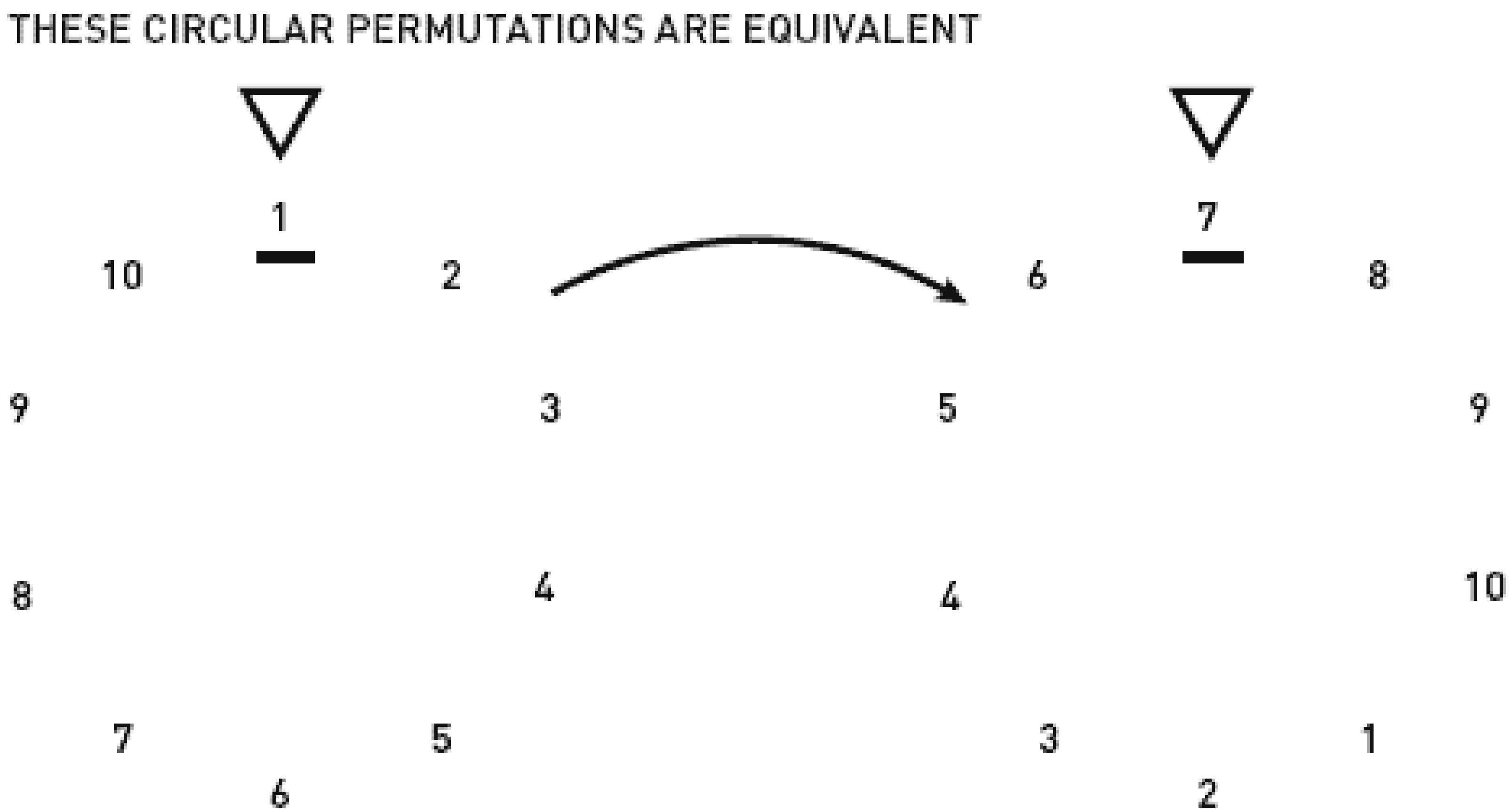


Example 1 - Solution

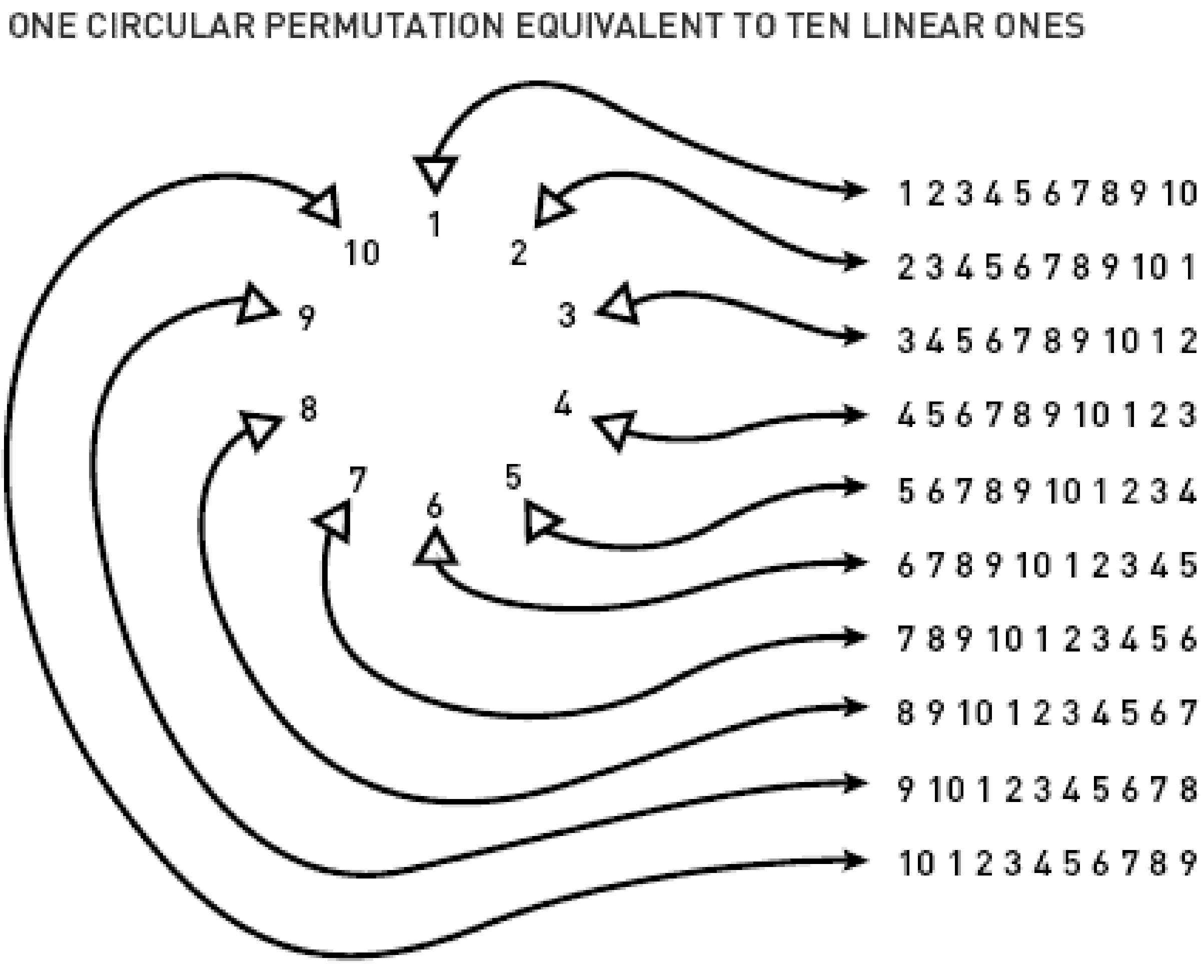
- First, let's think about how many ways we can line them up.
- As we indicated above, there will be 10! ways to line up ten guests
- 10 for the first position, 9 for the second, 8 for the third, and so on.

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

How does this change if they are seated around a circular table?



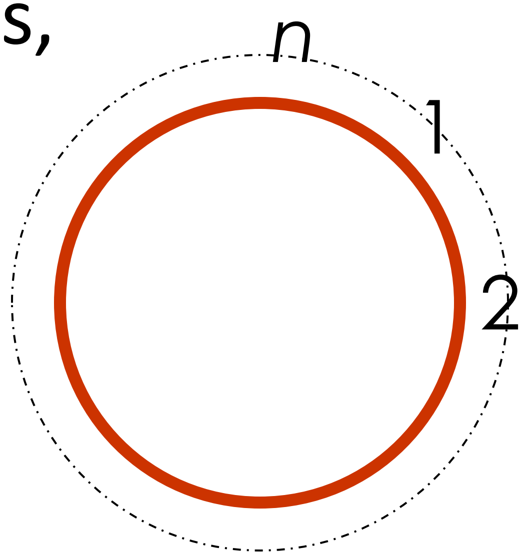
- Notice that every **circular arrangement** corresponds to ten different linear arrangements.



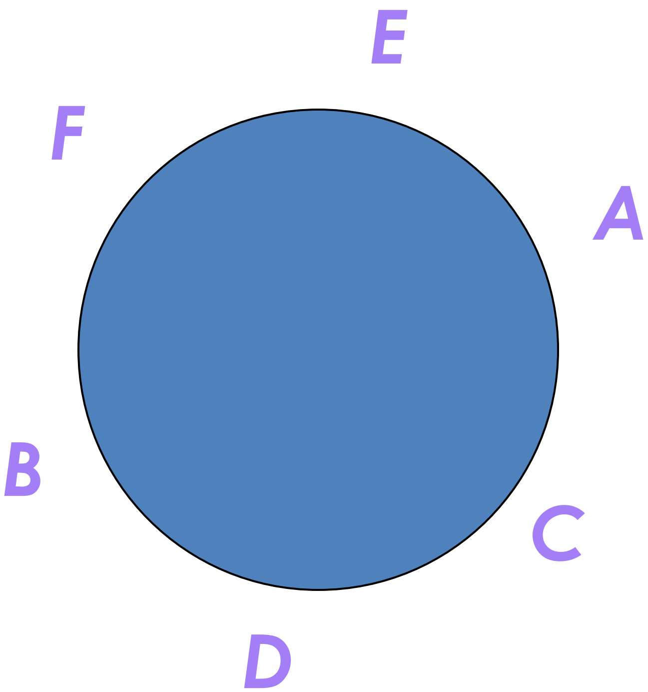
- Using our reasoning from before, we can see that the number of circular arrangements is equal to the number of linear arrangements, 10!, divided by ten to compensate for the fact that each circular permutation corresponds to ten different linear ones.
- This gives as the number of ways to arrange ten guests around a table.
- We can generalize this to say that ***n*** elements can be arranged in **(*n*-1)!** ways around a circle.

Permutations (circular arrangement)

- Permutations of n elements,
  - P(n) = (n-1)!



example



$(n-1)! = (6-1)! = 5! = 120$

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More Examples

- How many circular permutation can you form with 10 objects

  $(n-1)! = (10-1)!=9!=362880$

- How many ways can four boys and two girls be seated at a round table?

  $(n-1)!=(6-1)!=5!=120$

Exercise

- In how many ways can five people A, B, C, D, and E be seated around a circular table if

  a) A and B must sit next to each other

  b) A and B must not sit next to each other

  c) A and B must be together and CD must be together

Solution

a) AB, C, D, E = (4-1)!=3!

AB switch places ; 3!.2!=12

b) (5-1)! ; consider five people = 4!

subtract with (a); 4!-12=12

c) 3 groups: AB, CD and E: 2!2!2!=8

r-permutations
(permutations without repetition)

- An r-permutations of n (distinct) elements $x_1,.., x_n$ is an ordering of an r-element subset of $\{x_1,.., x_n\}$.

- The number of r-permutations of a set of n distinct elements is,

$$P(n,r) = {}^nP_r = {}_nP_r = \frac{n!}{(n-r)!}$$

Example 1

- Back to our billiard balls example.
- If you wanted to select just 3, then you have to stop the multiplying after 14. How do you do that? There is a neat trick ... you divide by **13!** ...

$$\frac{16\times15\times14\times13\times12\times\ldots}{13\times12\times\ldots}=16\times15\times14=3{,}360$$

- Do you see? **16! / 13! = 16 × 15 × 14**

- Our "order of 3 out of 16 pool balls example" would be:

$$P(16,3) = \frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20,922,789,888,000}{6,227,020,800} = 3,360$$

- which is just the same as:

$$16 \times 15 \times 14 = 3,360$$

Example 2

- How many ways can first and second place be awarded to 10 people?

$$P(10,2) = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3,628,800}{40,320} = 90$$

- which is just the same as:

$$10 \times 9 = 90$$

example

- 2- permutations of a, b, c are,
  - *ab, ac, ba, bc, ca, cb*

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3.2 = 6$$

example

- In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

$$P(10,4)=\frac{10!}{(10-4)!}=\frac{10!}{6!}=10.9.8.7=5040$$

example

- How many dance pairs, (dance pairs means a pair (W,M), where W stands for a women and M for man), can be formed from a group of 6 women and 10 men?

$$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 10.9.8.7.6.5 = 151200$$

example

- How many numbers between 20000 and 50000 can be formed with the digits 1,2,3,4,5,6 such that no digits are repeated in any of the numbers so formed?

example

20000 – 50000 { 1,2,3,4,5,6 }

$P(6,5) = 6.5.4.3.2 = 720$ _ _ _ _ _

$P(5,4) = 5.4.3.2 = 120$ 1 _ _ _ _

$P(5,4) = 5.4.3.2 = 120$ 5 _ _ _ _

$P(5,4) = 5.4.3.2 = 120$ 6 _ _ _ _

720-120-120-120=360

Or

3. 5. 4. 3. 2 = 360

example

- In how many ways can six boys and five girls stand in a line so that no two girls are next to each other?
  - $G\ B_1G\ B_2G\ B_3G\ B_4G\ B_5G\ B_6G$

$$P(7,5) = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520 \qquad P(6,6) = 6! = 720$$

$P(7,5).\ P(6,6) = 1814400$

r-Permutations

(Permutations with repetitions allowed)

- r-permutations of a set of n distinct elements if repetitions are allowed.

$$P(n,r) = n^r$$

- In other words:
  - There are ***n*** possibilities for the first choice,
  - THEN there are ***n*** possibilities for the second choice,
  - and so on, multiplying each time.
- Which is easier to write down using an exponent of ***r***:

$$\boldsymbol{n \times n \times \ldots (r \text{ times}) = n^r}$$

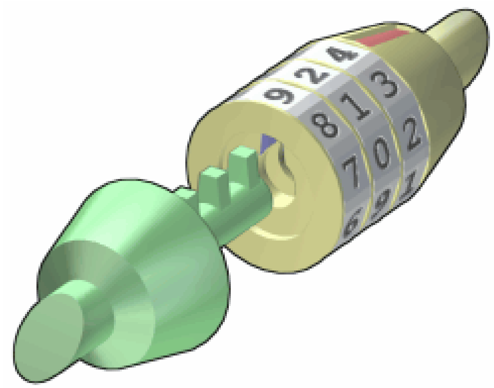
Example 1

- In the lock, there are 10 numbers to choose from (0,1,..9) and you choose 3 of them:

10 × 10 × ... (3 times)

= 10^3

= 1,000 permutations



Example 2

- How many five-letters word can be formed from the letters *A-Z*?

  - $P(26,5) = 26^5$

Permutations

(***n*** objects, ***k*** different types)

- A collection of n objects of k different types.

- The total number of different arrangements of these n objects is,

$$P(n) = \frac{n!}{(n_1!n_2!\ldots n_k!)}$$

example

- Find the number of different ways the letters of the word ASSESSMENT can be arranged?

- 10 letters, A (1), S(4), E(2), M(1), N(1), T(1)

$$P(10) = \frac{10!}{(1!\ 4!\ 2!\ 1!\ 1!\ 1!)} = 75600$$

Exercise 1

- In how many ways can 10 distinct books be divided among 3 students if Khairin gets 4 books and Nurina and Sarah each get 3 books.

Exercise 2

1. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if
   i) 4 letters are used at a time ,
   ii) all letters are used at a time
   iii) All letters are used but first letter is vowel

2. One hundred twelve people bought raffle tickets to enter a random drawing for three prizes. How many ways can three names be drawn for first prize, second prize, and third prize?
3. In how many ways can the letter of the word 'JUDGE' be arranged such that the vowels always come together?

Combinations

- There are also two types of combinations (remember the order does **not** matter now):
- **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
- **No Repetition**: such as combination of subjects to enroll (Maths., English, Music)

Combinations (without Repetition)

Given a set $X=\{x_1,.., x_n\}$ containing n (distinct) elements.

An r-combination of X is an unordered selection of r-elements of X.

The number of r-combinations of a set of n distinct elements is,

$$C(n,r)={}^{n}C_{r}={}_{n}C_{r}=\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

Example 1

- Back to our pool ball example, let us say that you just want to know which 3 pool balls were chosen, not the order.
- We already know that 3 out of 16 gave us 3,360 permutations.
- But many of those will be the same to us now, because we don't care what order!

- For example, let us say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

- So, the permutations will have 6 times as many possibilities.

- In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it.

- The answer is:

$$\mathbf{3! = 3 \times 2 \times 1 = 6}$$

- So, all we need to do is adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in the order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

- So, our pool ball example (now without order) is: $$\frac{16!}{3!(16-3)!} = \frac{16!}{3! \times 13!} = \frac{20,922,789,888,000}{6 \times 6,227,020,800} = 560$$

- Or you could do it this way:

$$\frac{16 \times 15 \times 14}{3 \times 2 \times 1} = \frac{3360}{6} = 560$$

example

- In how many ways can we select a committee of three from a group of 10 distinct persons?

$$C(10,3) = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120$$

example

- In how many ways can we select a committee of 2 women and 3 men from a group of 5 distinct women and 6 distinct men?

$$C(5,2) = \frac{5!}{2!3!} = 10$$

$$C(6,3) = \frac{6!}{3!3!} = 20$$

10.20 = 200

example

- How many 8-bit strings contain exactly four 1’s ?

$$C(8,4) = \frac{8!}{4!4!} = 70$$

example

- A committee of six is to be made from 4 students and 8 lecturers. In how many ways can this be done:

  - If the committee contains exactly 3 students?

  - If the committee contains at least 3 students?

example

- If the committee contains exactly 3 students?
- Select 3 students

$$C(4,3) = \frac{4!}{3!\ 1!} = 4$$

$$C(8,3) = \frac{8!}{3!\ 5!} = 56$$

- Select 3 lecturers
- 4.56 = 224

example

- If the committee contains at least 3 students?

- We have to consider 2 cases
  - Case 1: 3 students and 3 lecturers
  - Case 2: 4 students and 2 lecturers

example

- Case 1: 3 students and 3 lecturers
  - 224 ways

- Case 2: 4 students and 2 lecturers

$$C(4,4) = 1 \qquad C(8,2) = \frac{8!}{2!6!} = 28$$

  - 1.28 = 28

- Case 1+ case 2 = 224 + 28 = 252

example

- A student is required to answer 7 out of 12 questions, which are divided into two groups, each containing 6 questions.

- The student is not permitted to answer more than 5 questions from either group.

- In how many different ways can the student choose the 7 questions?

example

Number of question from group A		Number of question from group B
5		2
4	$C(6,5).C(6,2)=90$	3
3	$C(6,4).C(6,3)=300$	4
2	$C(6,3).C(6,4)=300$	5
	$C(6,2).C(6,5)=90$	

$$\boxed{90+300+300+90 = 780}$$

More exercises

1. Ahmad bought a machine to make fresh juice. He has five different fruits: strawberry, oranges, apples, pineapples and lemons. If he only use two fruits, how many different juice drinks can Ahmad make?

2. There are 25 people who work in an office together. Five of these people are selected to attend five different conferences. The first person selected will go to a conference in Hawaii, the second will go to New York, the third will go to San Diego, the fourth will go to Atlanta and the fifth will go to Nashville. How many such selection are possible

3. In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Combinations
(Repetition Allowed)

The number of r-combinations of n objects with repetitions allowed is,

$$C(n+r-1,r)=\binom{n+r-1}{r}=\frac{(n+r-1)!}{r!(n-1)!}$$

n is the number of things/objects to choose from, and you choose ***r*** of them (repetition allowed, order doesn't matter)

Combinations with Repetition

- Let us say there are five flavors of icecream: **banana, chocolate, lemon, strawberry and vanilla**.
- You can have three scoops.
- How many variations will there

Combinations with Repetition

- Let's use letters for the flavors:

  {b, c, l, s, v}

- Example selections would be
  - {c, c, c} (3 scoops of chocolate)
  - {b, l, v} (one each of banana, lemon and vanilla)
  - {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are ***n*=5** things to choose from, and you choose ***r*=3** of them. Order does not matter, and you **can** repeat!)

Combinations with Repetition

$$C(n+r-1, r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

- where *n* is the number of things to choose from, and you choose *r* of them
(Repetition allowed, order doesn't matter)

Combinations with Repetition

- So, what about our ice-cream example, what is the answer?

$$C(n+r-1,r)=\binom{n+r-1}{r}=\frac{(n+r-1)!}{r!(n-1)!}$$

$$C(5+3-1,3)=\binom{5+3-1}{3}=\frac{(5+3-1)!}{3!(5-1)!}=\frac{7!}{3!\times 4!}=\frac{5040}{6\times 24}=35$$

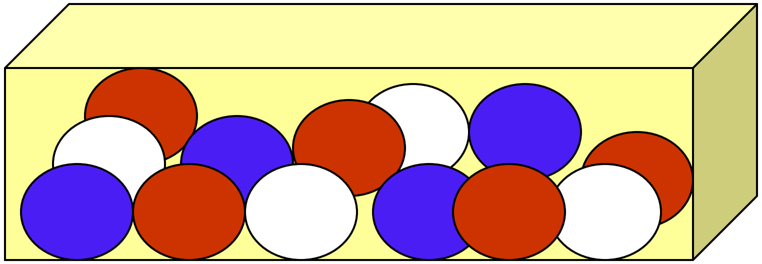
example

- A bakery makes 4 different varieties of donuts. Khairin wants to buy 6 donuts. How many different ways can he do it?

$$C(4+6-1,6)=\frac{(4+6-1)!}{6!(4-1)!}=\frac{9!}{6!3!}=84$$

example

- There is a box containing identical white, red and blue balls.



- In how many ways can we select 4 balls?

$$C(3+4-1,4)=\frac{(3+4-1)!}{4!(3-1)!}=\frac{6!}{4!2!}=15$$

Summary

Which formula to use?

	Order Matters (Permutations)	Order Does Not Matter (Combinations)
Repetition is allowed	$P_n = n^r$	$C(n+r-1,r)=\frac{(n+r-1)!}{r!(n-1)!}$
Repetition is not allowed	$P(n,r)=\frac{n!}{(n-r)!}$	$C(n,r)=\frac{n!}{r!(n-r)!}$

exercise

- There is a shipment of 50 microprocessors of which four are defective.

  – In how many ways can we select a set of 4 microprocessors?

  – In how many ways can we select a set of 4 microprocessor containing at least 1 defective microprocessor?