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INSPIRING CREATIVE AND INNOVATIVE MINDS

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Chapter 2 (Part 1)

Relations

Relations

- A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$.
- If $(x, y) \in R$, we write

$$x R y \quad (x \text{ is related to } y)$$

(Binary) relation from X to Y , where $x \in X$, $y \in Y$,
 $(x, y) \in X \times Y$ and $R \subseteq X \times Y$

$$x R y \Leftrightarrow (x, y) \in R$$



example

- $A = \{ 1, 2, 3, 4 \} , \quad B = \{ p, q, r \}$
- $R = \{ (1, q), (2, r), (3, q), (4, p) \}$
- $R \subseteq A \times B$
- R is the relation from A to B
 $1 R q \qquad 3 \not R p$

example

$$aRb \leftrightarrow a-b \in \mathbb{Z}^{\text{even}}$$

- *Finite set:* $A = \{ 1, 2 \}$, $B = \{ 1, 2, 3 \}$

$$R = \{ (1, 1), (2, 2), (1, 3) \}$$

- *Infinite set:* $A = \mathbb{Z}$ and $B = \mathbb{Z}$

$$R = \{ \dots(-3, -1), (-2, 2), (1, 3), \dots \}$$

(note: \mathbb{Z} is set of integers)



example

- $A = \{ \text{New Delhi, Ottawa, London, Paris, Washington} \}$
- $B = \{ \text{Canada, England, India, France, United States} \}$
- Let $x \in A$, $y \in B$. Define the relation between x and y by “ x is the capital of y ”



- $R = \{ \text{(New Delhi, India),}$
 (Ottawa, Canada),
 $\text{(London, England),}$
 (Paris, France),
 $\text{(Washington, United States) } \}$

- “less than” relation from $A=\{0, 1, 2\}$ to $B=\{1, 2, 3\}$

- Traditional notation:

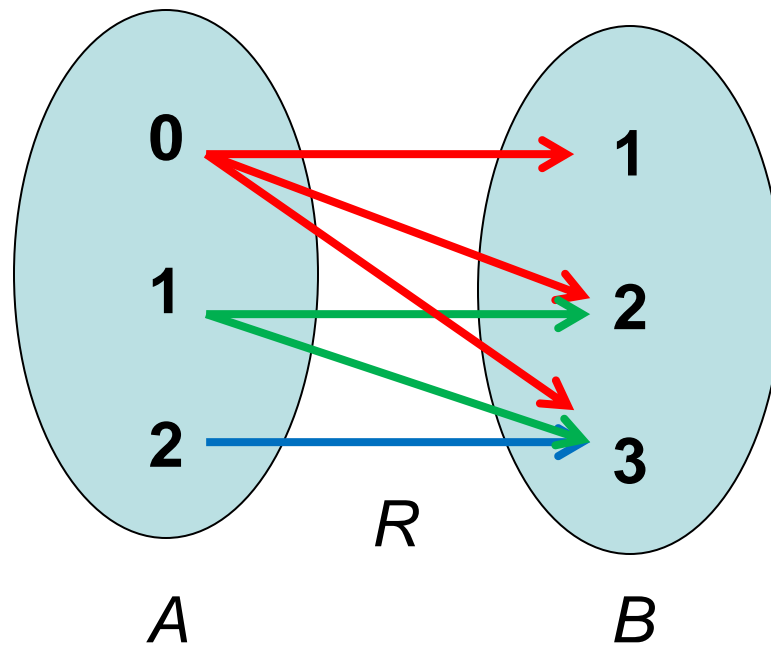
$$0<1, 0<2, 0<3, 1<2, 1<3, 2<3$$

- Set notation

$$A \times B = \{ (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) \}$$

$$R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$$

■ Arrow diagrams





Domain and Range

- Let R , a relation from A to B .
- The set, $\{ a \in A \mid (a,b) \in R \text{ for some } b \in B \}$ is called the **domain** of R .
- The set, $\{ b \in B \mid (a,b) \in R \text{ for some } a \in A \}$ is called the **range** of R .
- In case $A=B$, we call R a(binary) relation on A .

example

- Let R be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, and $x, y \in X$.
- Then,
 $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) \}$
- The domain and range of R are both equal to X .

example

- Let $X = \{ 2, 3, 4 \}$ and $Y = \{ 3, 4, 5, 6, 7 \}$
If we define a relation R from X to Y by,
 $(x,y) \in R$ if x divides y (with zero remainder)

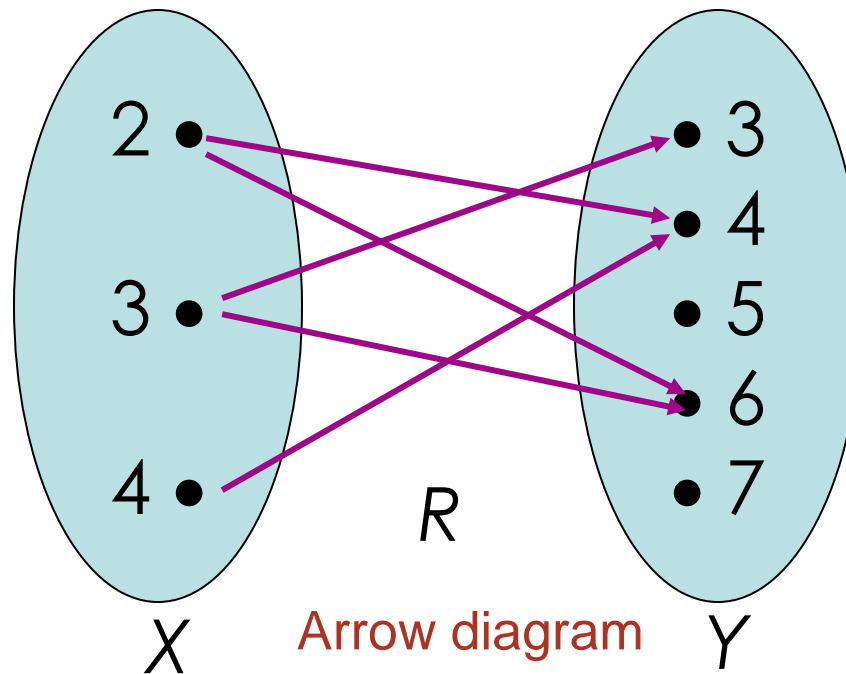
- We obtain,
 $R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$

The domain of R is $\{2,3,4\}$

The range of R is $\{3,4,6\}$

example

$$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$$





Exercise

■ Write the relation R as $(x,y) \in R$

(a) The relation R on $\{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x^2 \geq y$.

(b) The relation R on $\{1,2,3,4,5\}$ defined by $(x,y) \in R$ if 3 divides $x-y$.

Exercise

■ Find range and domain for:

(a) The relation R on $\{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x^2 \geq y$.

(b) The relation $R = \{ (1, 2), (2, 1), (3, 3), (1, 1), (2, 2) \}$ on $X = \{1, 2, 3\}$

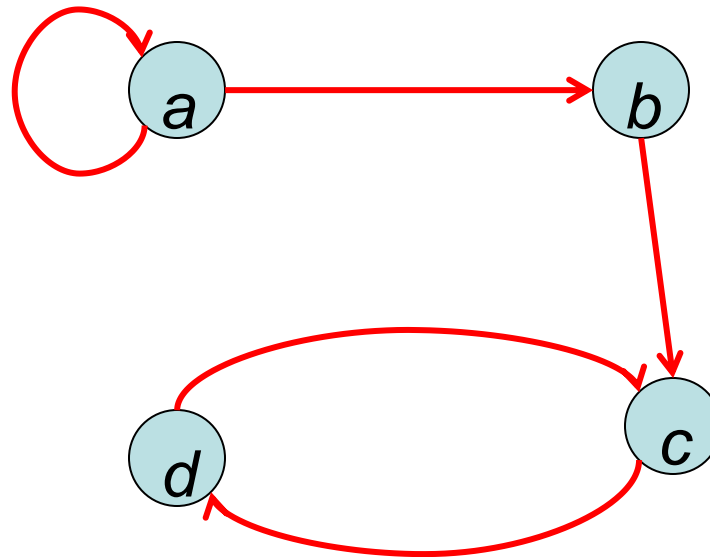


Digraph

- An informative way to picture a relation on a set is to draw its digraph.
- Let R be a relation on a finite set A .
- Draw dots (**vertices**) to represent the elements of A .
- If the element $(a,b) \in R$, draw an arrow (called a **directed edge**) from a to b .

example

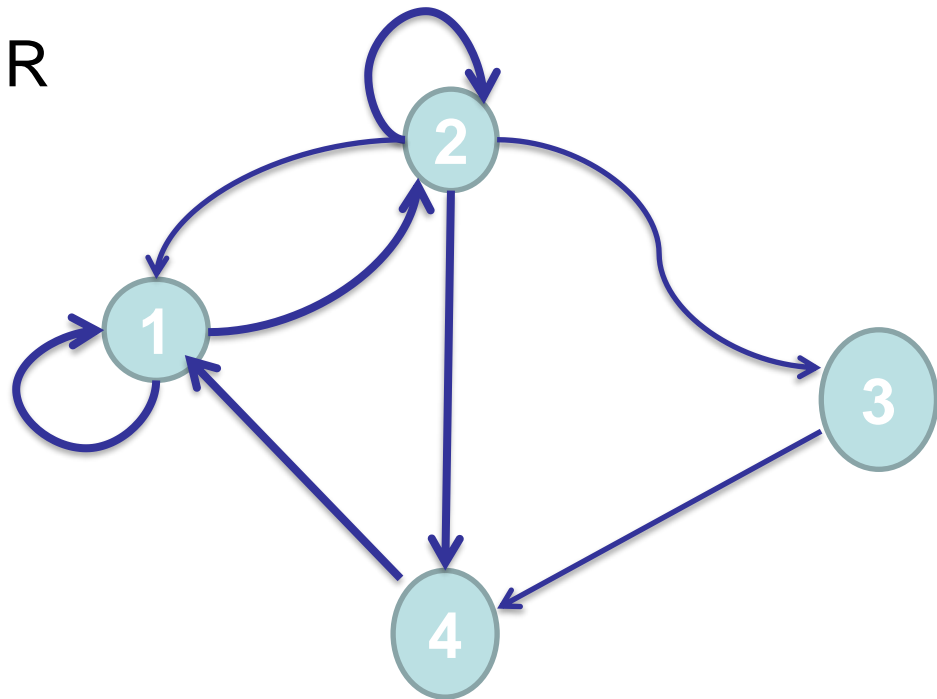
- The relation R on $A = \{a, b, c, d\}$,
 $R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



example

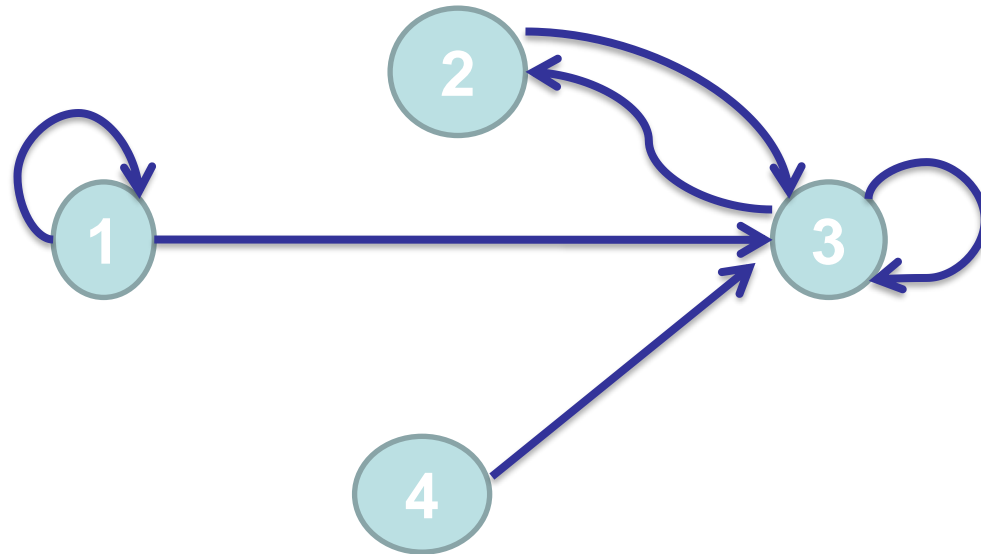
- Let, $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$

- Draw the digraph of R



example

Find the relation determined by digraph below.



- Since $a R b$ if and only if there is an edge from a to b , so
 $R = \{ (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) \}$



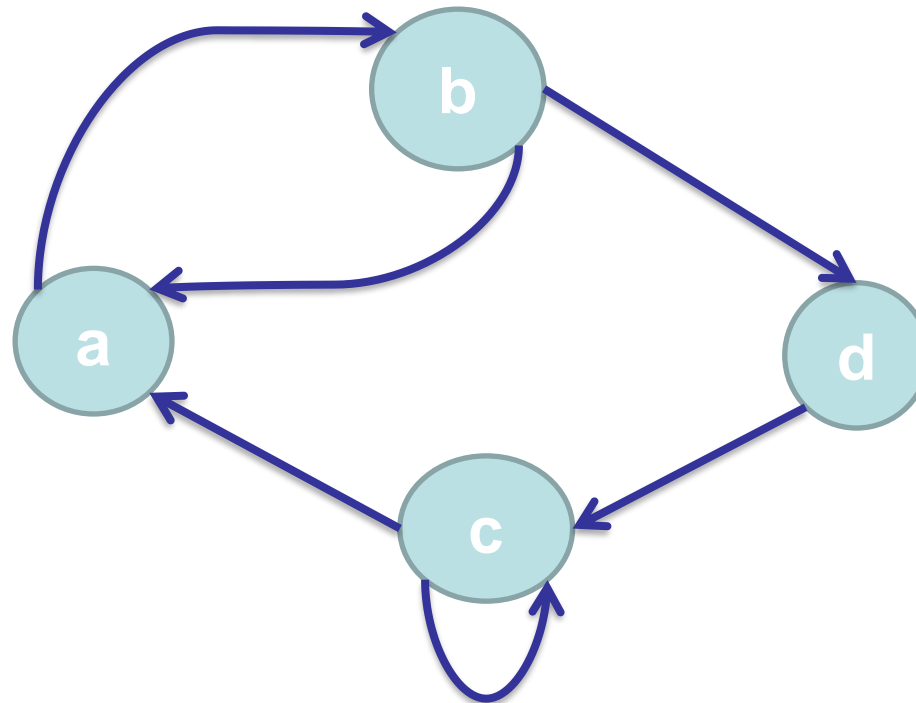
Draw the diagraph of the relation:

(a) $R = \{ (a,c), (b, d), (a,b), (c,d) \}$

(b) The relation R on $\{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x^2 \geq y$

exercise

Write the relation as a set of ordered pair.





Matrices of Relations

- A matrix is a convenient way to represent a relation R from A to B .
- Label **the rows** with the **elements of A** (in some arbitrary order)
- Label **the columns** with the **elements of B** (in some arbitrary order)

Matrices of Relations

- Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_p\}$ be finite nonempty sets.
- Let R be a relation from A into B .
- Let $M_R = [m_{ij}]_{n \times p}$ be the Boolean $n \times p$ matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$



Matrices of Relations

$$M_R = \begin{bmatrix} m_{11} & m_{12} & \dots & \dots & m_{1p} \\ m_{21} & m_{22} & \dots & \dots & m_{2p} \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ m_{n1} & m_{n2} & \dots & \dots & m_{np} \end{bmatrix}$$



example

- Let $A = \{1, 3, 5\}$ and $B = \{1, 2\}$
- Let R be a relation from A to B and $R = \{(1,1), (3,2), (5,1)\}$
- Then the matrix represent R is

$$\begin{array}{c} \\ \\ \end{array} \begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right] \end{array}$$

example

- The relation,

$$R = \{ (1,b), (1,d), (2,c), (3,c), (3,b), (4,a) \}$$

from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	0	1	0	1
2	0	0	1	0
3	0	1	1	0
4	1	0	0	0

or

	<i>d</i>	<i>b</i>	<i>a</i>	<i>c</i>
2	0	0	0	1
3	0	1	0	1
4	0	0	1	0
1	1	1	0	0

example

- The matrix of the relation R from $\{2, 3, 4\}$ to $\{5, 6, 7, 8\}$ defined by

$x R y$ if x divides y

	5	6	7	8
2	0	1	0	1
3	0	1	0	0
4	0	0	0	1

example

- Let $A = \{ a, b, c, d \}$
- Let R be a relation on A .
- $R = \{ (a,a), (b,b), (c,c), (d,d), (b,c), (c,b) \}$

	a	b	c	d
a	1	0	0	0
b	0	1	1	0
c	0	1	1	0
d	0	0	0	1



exercise

- Let $A = \{ 1, 2, 3, 4 \}$ and R be a relation on A .
 $R = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$

(a) What is R (represent)?

(b) What is matrix representation of R ?



In Degree and Out Degree

- If R is a relation on a set A and $a \in A$, then the **in-degree** of a (relative to relation R) is the number of $b \in A$ such that $(b, a) \in R$.
- The **out degree** of a is the number of $b \in A$ such that $(a, b) \in R$



In Degree and Out Degree

- Meaning that, in terms of the digraph of R , is that the in-degree of a vertex is
“the number of edges terminating at the vertex”
- The out-degree of a vertex is
“ the number of edges leaving the vertex”

Example

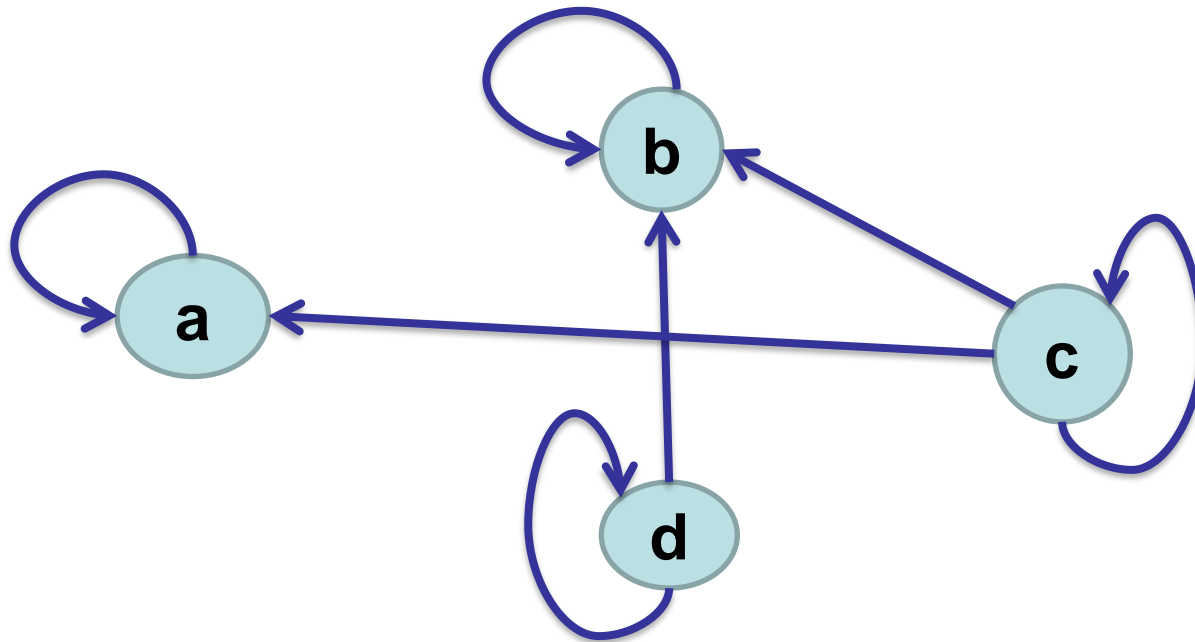
- Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Construct the digraph of R , and list in-degrees and out-degrees of all vertices.

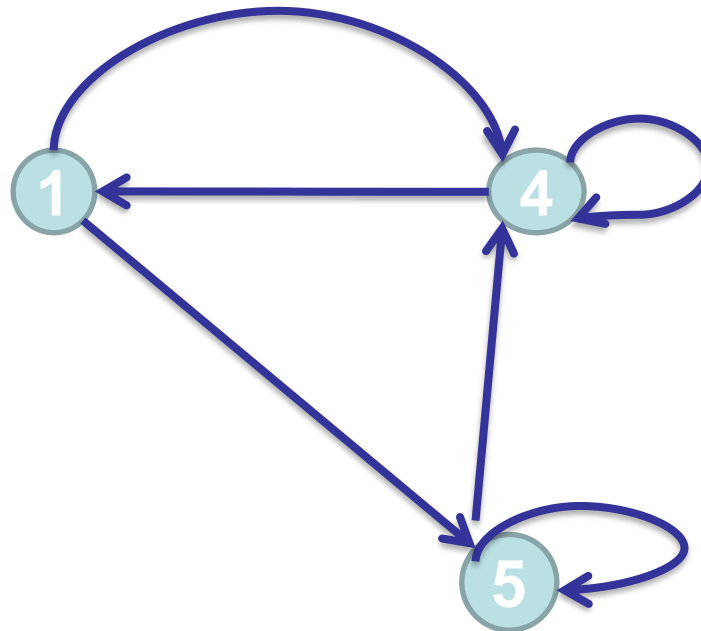
Example

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



exercise

- Let $A = \{1, 4, 5\}$ and let R be given by the digraph shown below.
- Find M_R and R



example

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- An airline services the five cities c_1, c_2, c_3, c_4 and c_5 .
- Table below gives the cost (in dollars) of going from c_i to c_j . Thus the cost of going from c_1 to c_3 is \$100, while the cost of going from c_4 to c_2 is \$200

To from	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

example

- If the relation R on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\} : c_i R c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to \$180.
- Find R .

$$R = \{(c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_4), (c_3, c_1), (c_3, c_2), (c_4, c_3), (c_4, c_5), (c_4, c_5), (c_5, c_2), (c_5, c_4)\}$$



exercise

- From the previous example, find the matrices of relations for R .

$$R = \{(c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_4), (c_3, c_1), (c_3, c_2), (c_4, c_3), (c_4, c_5), (c_4, c_5), (c_5, c_2), (c_5, c_4)\}$$



Reflexive Relations

- A relation R on a set X is called **reflexive** if $(x,x) \in R$ for every $x \in X$.
 - That is, if xRx for all $x \in X$.
- (R is reflexive if every element $x \in X$ is related to itself)



example

- The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$ is reflexive because for each element $x \in X$, $(x, x) \in R$
- $(1, 1), (2, 2), (3, 3), (4, 4)$ are each in R .



- The relation,

$$R = \{ (a,a), (b,c), (c,b), (d,d) \}$$

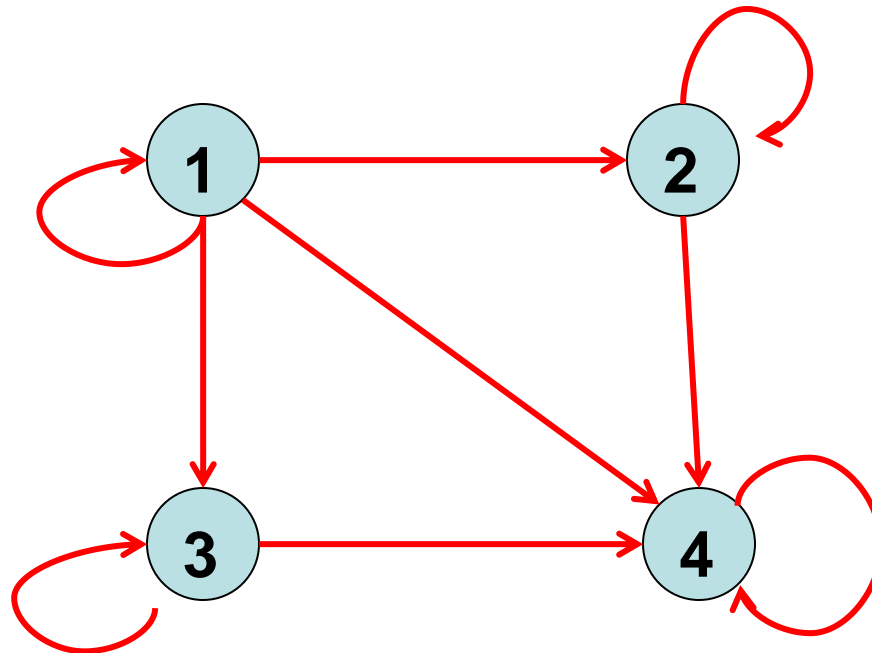
on $X = \{a, b, c, d\}$

is not reflexive.

- For example, $b \in X$, but $(b,b) \notin R$

Reflexive Relations

- The digraph of a reflexive relation has a loop at every vertex.
- example





■ Irreflexive

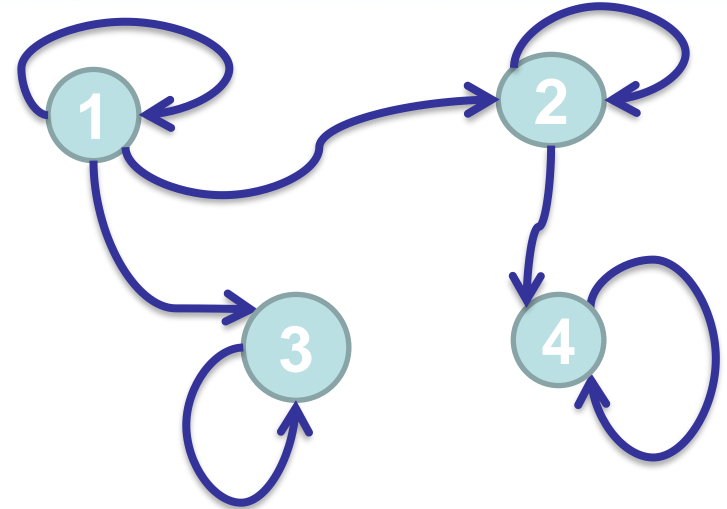
- A relation R on a set A is **irreflexive** if $x \not R x$ or $(x, x) \notin R; \forall x: x \in X$

■ Not Reflexive

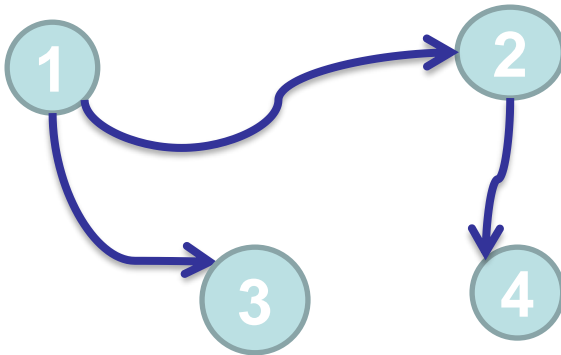
- A Relation R is **not reflexive** if at least one pair of $(x, x) \notin R, \forall x: x \in X$

example

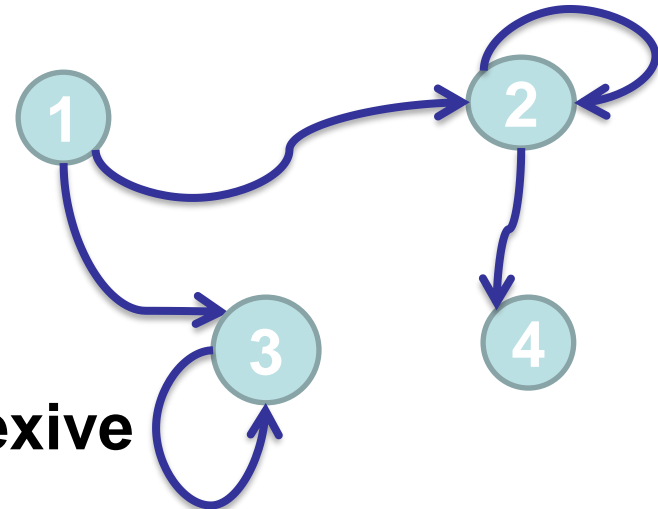
Reflexive



Irreflexive



Not reflexive





- Consider the following relations on the set $\{1, 2, 3\}$

$$R_1 = \{ (1,1), (1,2), (2,1), (2,2), (3,1), (3,3) \}$$

$$R_2 = \{ (1,1), (1,3), (2,2), (3,1) \}$$

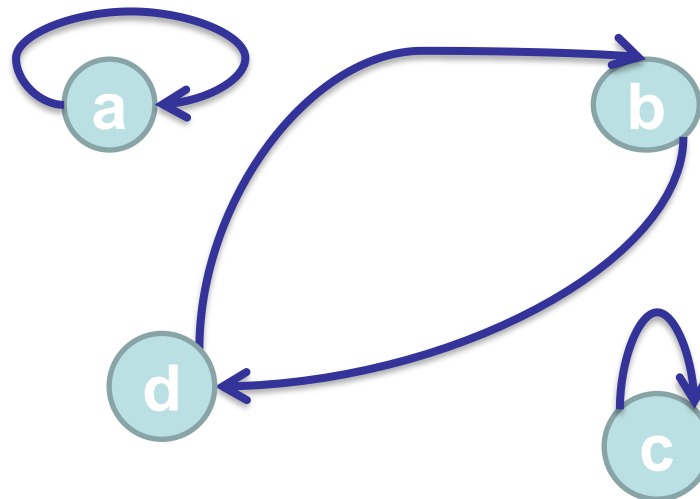
$$R_3 = \{ (2,3) \}$$

$$R_4 = \{ (1,1) \}$$

Which of them are reflexive?

exercise

- (i) Let R be the relation on $X=\{1,2,3,4\}$ defined by $(x,y)\in R$ if $x\leq y$, $x,y\in X$. Determine whether R is a reflexive relation.
- (ii) The relation R on $X=\{a,b,c,d\}$ given by the below diagraph. Is R a reflexive relation?

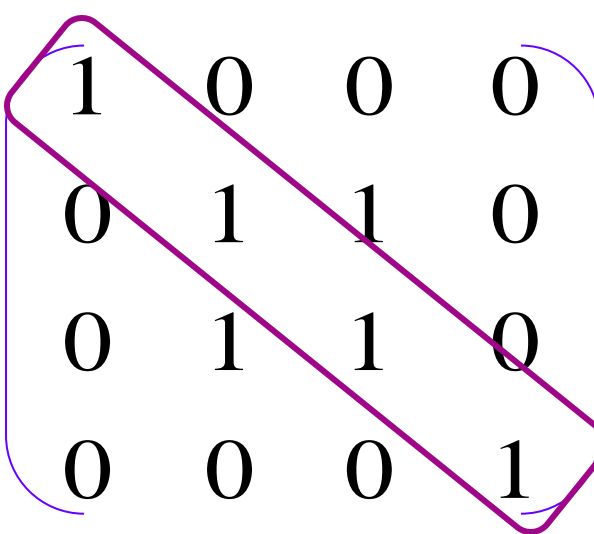


Reflexive Relations

- The relation R is reflexive if and only if the matrix of relation has 1's on the main diagonal.

- example

	a	b	c	d
a	1	0	0	0
b	0	1	1	0
c	0	1	1	0
d	0	0	0	1



Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

■ example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

example

- The relation R is not reflexive.

	a	b	c	d
a	1	0	0	0
b	0	0	1	0
c	0	1	1	0
d	0	0	0	1

$b \in X$
 $(b, b) \notin R$

exercise

Let $A = \{ 1,2,3,4 \}$. Construct the matrix of relation of R . Then, determine whether the relation is reflexive, not reflexive or irreflexive.

- (i) $R = \{ (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4) \}$
- (ii) $R = \{ (1,3), (1,1), (3,1), (1,2), (3,3), (4,4) \}$
- (iii) $R = \{ (1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4) \}$
- (iv) $R = \{ (1,2), (1,3), (3,2), (1,4), (4,2), (3,4) \}$

Symmetric Relations

- A relation R on a set X is called symmetric if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

$$\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R$$

- Let M be the matrix of relation R .
The relation R is symmetric if and only if for all i and j , the ij th entry of M is equal to the ji th entry of M .

Symmetric Relations

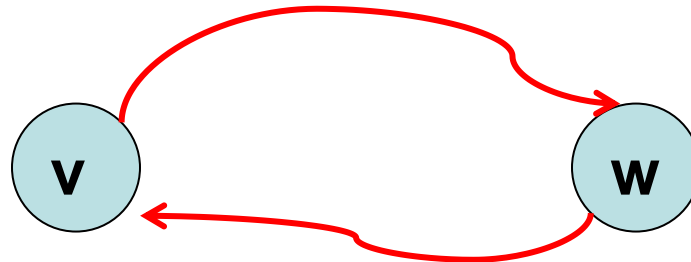
- The matrix of relation M_R is symmetric if $M_R = M_R^T$

- Example

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} = M_R^T$$

Symmetric Relations

- The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w , there is also a directed edge from w to v .



example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$(b,c) \in R$
 $(c,b) \in R$

	a	b	c	d
a	1	0	0	0
b	0	0	1	0
c	0	1	0	0
d	0	0	0	1

symmetric

example

- The relation R on $X = \{ 1, 2, 3, 4 \}$, defined by
 $(x, y) \in R$ if $x \leq y, x, y \in X$

$(2, 3) \in R$
 $(3, 2) \notin R$

	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

not symmetric

Antisymmetric Relations

- A relation, R on a set X is called antisymmetric, if for all $x, y \in X$, if $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$.
- A relation R on set X is antisymmetric if $x \neq y$, whenever xRy , then $y \not Rx$. In other word if whenever xRy , then yRx then it implies that $x=y$

$$\forall x, y \in A, (x, y) \in R \wedge x \neq y \rightarrow (y, x) \notin R$$

Or

$$\forall x, y \in A, (x, y) \in R \wedge (y, x) \in R \rightarrow x = y$$



Antisymmetric Relations

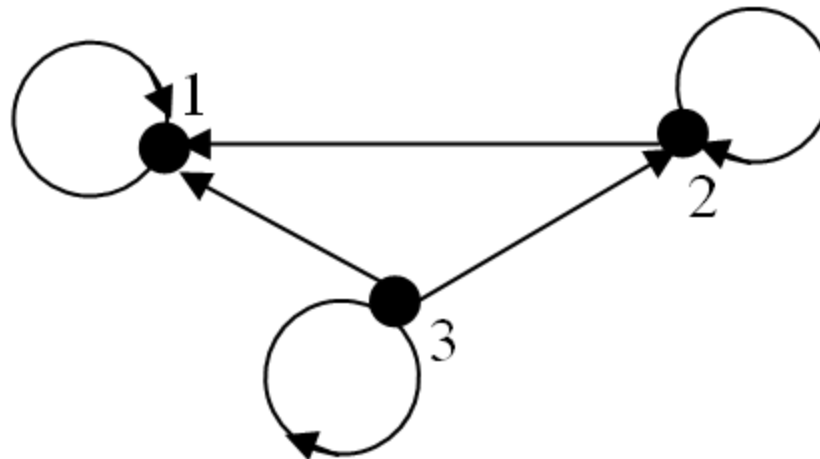
- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation R satisfies the property that if $i \neq j$, then $m_{ij} = 0$ or $m_{ji} = 0$.
- If R is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i .
- At least one directed relation and one way

Example

- Let R be a relation on $A = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$



example

- The relation R on $X = \{ 1, 2, 3, 4 \}$ defined by,
 $(x,y) \in R$ if $x \leq y, x,y \in X$

$$(1,2) \in R$$

$$(2,1) \notin R$$

	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

antisymmetric

example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\begin{aligned} (b,c) &\in R \\ (c,b) &\in R \end{aligned}$$

	a	b	c	d
a	1	0	0	0
b	0	0	1	0
c	0	1	0	0
d	0	0	0	1

not antisymmetric

example

- The relation
 $R = \{ (a,a), (b,b), (c,c) \}$
on $X = \{ a, b, c \}$
- R has no members of the form
 (x,y) with $x \neq y$, then R is
antisymmetric

	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

example

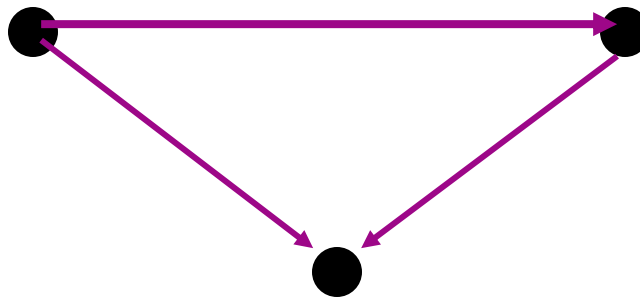
- Antisymmetric
- Reflexive
- Symmetric

	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

- “Antisymmetric” is not the same as “not symmetric”

Antisymmetric Relations

- The digraph of an antisymmetric relation has at most one directed edge between each pair of vertices.
- Example





Asymmetric

A relation R on set A is asymmetric if whenever aRb , then $b \not R a$.

$$\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \notin R$$

In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.

Asymmetric

- The matrix $M_R = [m_{ij}]$ of an asymmetric relation R satisfies the property that
If $m_{ij} = 1$ then $m_{ji} = 0$
 $m_{ii} = 0$ for all i (the main diagonal of matrix M_R consists entirely of 0's or otherwise)
- If R is asymmetric relation, then the digraph of R cannot simultaneously have an edge from vertex i to vertex j and an edge from vertex j to vertex i
- All edges are “one way street”

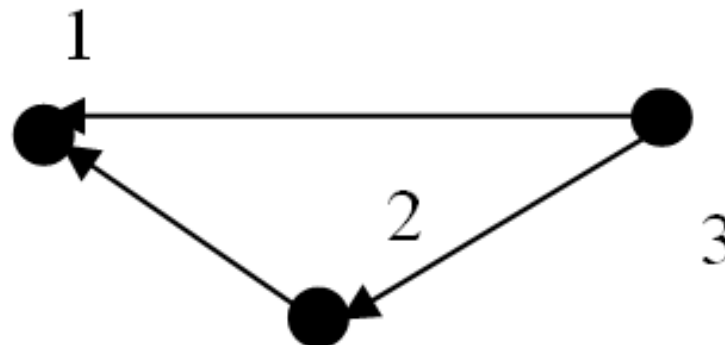
Example

- Let R be the relation on $A = \{1, 2, 3\}$ defined by $(a, b) \in R$ if $a > b$, $a, b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$





Not Symmetric

- Let R be a relation on a set A .
- Then R is called ***not symmetric***, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Not Symmetric and Not Antisymmetric

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- Let R be a relation on a set A . Then R is called ***not symmetric*** and ***not antisymmetric***, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

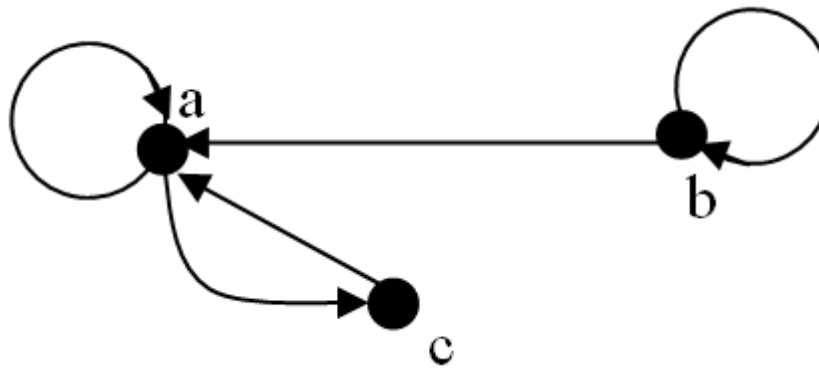
AND

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Example

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$(a, c), (c, a) \in R$ and also $(b, a) \in R$ but $(a, b) \notin R$





Example

1. Let $A=\mathbb{Z}$, the set of integers and let $R=\{(a,b)\in A\times A \mid a<b\}$.
So that R is the relation “less than”.

Is R symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

Determine whether R symmetric, asymmetric or antisymmetric.

Question 1

- **Symmetric** : If $a < b$, then it is not true that $b < a$, so R is not symmetric
- **Assymmetric** : If $a < b$ then $b > a$ (b is greater than a), so R is assymmetric
- **Antisymmetric** : If $a \neq b$, then either $a > b$ or $b > a$, so R is antisymmetric

Question 2

- R is not symmetric since $(1,2) \in R$, but $(2,1) \notin R$
- R is not asymmetric, since $(2,2) \in R$
- R is antisymmetric, since $a \neq b$, either $(a,b) \notin R$ or $(b,a) \notin R$

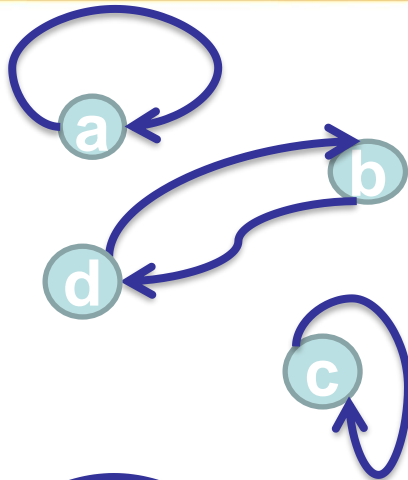


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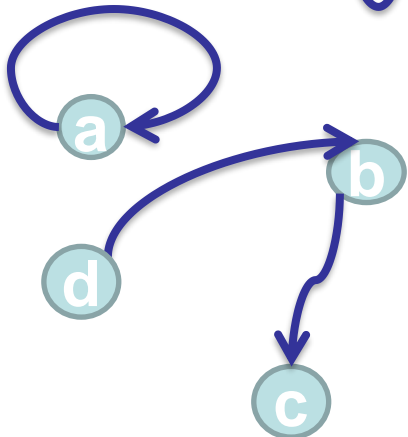
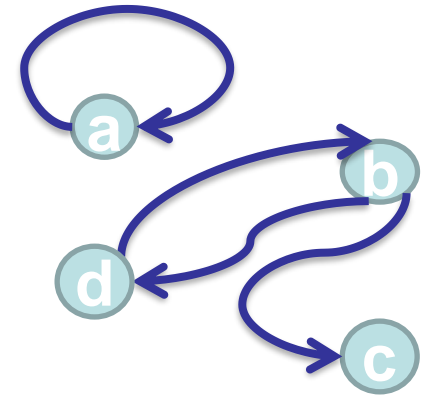
Example

Summary on Symmetric



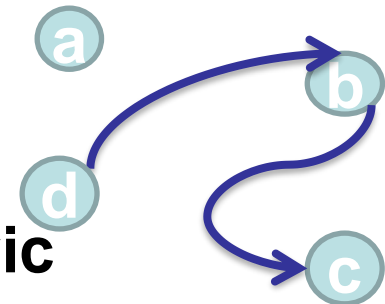
Symmetric

Not Symmetric



Antisymmetric

Asymmetric





Exercise

Let $A = \{1, 2, 3, 4\}$. Construct the matrix of relation of R . Then, determine whether the relation is symmetric, asymmetric, not symmetric or antisymmetric.

- (i) $R = \{ (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4) \}$
- (ii) $R = \{ (1,3), (1,1), (3,1), (1,2), (3,3), (4,4) \}$
- (iii) $R = \{ (1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4) \}$

Transitive Relations

- A relation R on a set X is called transitive if for all $x, y, z \in X$,
if (x, y) and $(y, z) \in R$ then $(x, z) \in R$
- It is often convenient to say what it means for a relation to be not transitive.
- A relation R on X is **not transitive** if there exists x, y , and z in X so that xRy and yRz , but $x \not R z$. If such x, y , and z do not exist, then R is transitive.



Transitive Relations

- Let M_R be the matrix of relation R .
The relation R is transitive if,

$$M_R \otimes M_R = M_R.$$

\otimes Boolean product.



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Boolean Algebra

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

example

- The relation R on $X = \{ 1, 2, 3, 4 \}$ defined by
 $(x, y) \in R$ if $x \leq y, x, y \in X$

	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

example

■ $M_R \otimes M_R = M_R$

transitive

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$ is not transitive.
- (b,c) and $(c,b) \in R$, but $(b,b) \notin R$.



example

- Consider the following relations on the set $\{1, 2, 3\}$

$$R1 = \{ (1,1), (1,2), (2,3) \}$$

$$R2 = \{ (1,2), (2,3), (1,3) \}$$

- Which of them is transitive?

example

Let R be a relation on $A=\{1,2,3\}$ is defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$. Find R . Is R a transitive relation?

Solution:

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

R is a transitive relation because

$$(1,2) \text{ and } (2,2)\in R, (1,2)\in R$$

$$(1,2) \text{ and } (2,3)\in R, (1,3)\in R$$

$$(1,3) \text{ and } (3,3)\in R, (1,3)\in R$$

$$(2,2) \text{ and } (2,3)\in R, (2,3)\in R$$

$$(2,3) \text{ and } (3,3)\in R, (2,3)\in R$$



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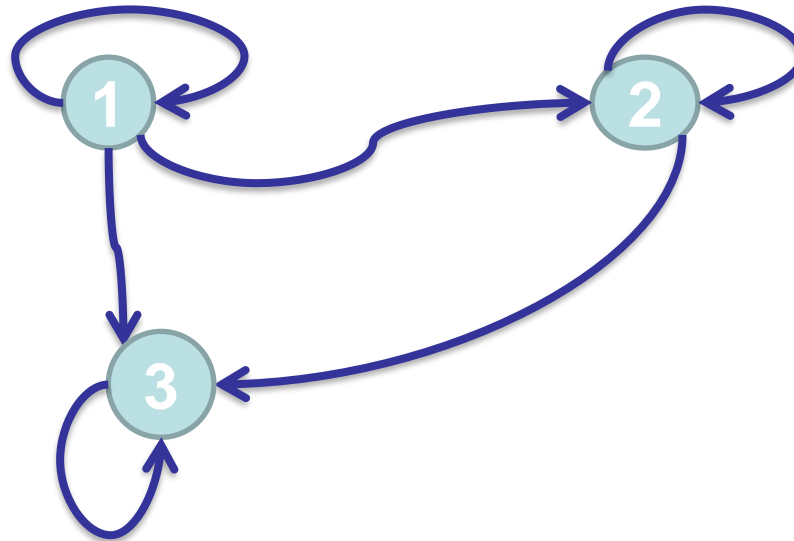
Transitive Relations

In the digraph of R , R is a transitive relation if and only if there is a directed edge from one vertex a to another vertex b , and if there exists a directed edge from vertex b to vertex c , then there must exist a directed edge from a to c

example

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

The diagraph:



example

The relation R on $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$, is a transitive. The matrix of relation M_R ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that, $(1,2)$ and $(2,3)\in R$, $(1,3)\in R$

example

The relation R on $A=\{1,2,3\}$ defined by $(a,b) \in R$ if $a \leq b$, $a,b \in A$, is a transitive. The matrix of relation M_R ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that, $(1,2)$ and $(2,3) \in R$, $(1,3) \in R$

Example

The relation R on $A=\{a,b,c,d\}$ IS $r=\{(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)\}$ is not transitive. The matrix of relation M_R ,

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & \textcircled{0} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$M_R \otimes M_R \neq M_R$

The product of boolean,

$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \otimes \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & \textcircled{1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Note that, (a,c) and $(c,b) \in R$, $(a,b) \notin R$



Equivalence Relations

- A relation R that is reflexive, symmetric and transitive on a set X is called an equivalence relation on X .

example

Let $R=\{(1,1), (1,3), (2,2), (3,1), (3,3)\}$ on $\{1,2,3\}$, the matrix of the relation M_R ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

example

Let $R=\{(1,1), (1,3), (2,2), (3,1), (3,3)\}$ on $\{1,2,3\}$, the matrix of the relation M_R ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

example

The transpose matrix M_R , M_R^T is equal to M_R , so R is symmetric

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix} \quad M_R^T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of Boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

example

- The relation, $R = \{ (1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5) \}$
on $\{ 1,2,3,4,5 \}$

- Reflexive?
- Symmetric?
- Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

example

■ Reflexive?

1	0	1	0	1
0	1	0	1	0
1	0	1	0	1
0	1	0	1	0
1	0	1	0	1

Reflexive ✓

example

■ Symmetric?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric ✓

example

■ Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$



example

- Reflexive
- Symmetric
- Transitive



Equivalence relation

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

example

- The relation R on $X = \{ 1, 2, 3, 4 \}$, defined by
 $(x, y) \in R$ if $x \leq y, x, y \in X$
- Not symmetric
 - $(2, 3) \in R$ but $(3, 2) \notin R$
- R is not equivalence relation on X .



Partial Orders

- A relation, R on a set X is called a **partial order** if R is reflexive, antisymmetric, and transitive.



example

Let R be a relation on a set $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b, a,b\in R$.

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

R is reflexive, antisymmetric and transitive.

So R is a partial order relation.



- The relation R defined on the positive integers by
 $(x,y) \in R$ if x divides y (evenly)

is reflexive, antisymmetric and transitive

- R is a partial order.



example

- The relation R defined on the set of integers by

$$(x,y) \in R \text{ if } x \leq y$$

is reflexive, antisymmetric, and transitive.

- R is a partial order.



The relation R on the set $\{1,2,3,4,5\}$ defined by the rule $(x,y) \in R$ if $x+y \leq 6$

- (i) List the elements of R
- (ii) Find the domain of R
- (iii) Find the range of R
- (iv) Is the relation of R reflexive, symmetric, asymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?



exercise

The relation R on the set $\{1,2,3,4,5\}$ defined by the rule $(x,y) \in R$ if 3 divides $x-y$

- (i) List the elements of R
- (ii) Find the domain of R
- (iii) Find the range of R
- (iv) Is the relation of R reflexive, symmetric, asymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?



exercise

The relation R on the set $\{1,2,3,4,5\}$ defined by the rule $(x,y) \in R$ if $x=y-1$

- (i) List the elements of R
- (ii) Find the domain of R
- (iii) Find the range of R
- (iv) Is the relation of R reflexive, symmetric, asymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?