Chapter 1

SET THEORY

[Part 1: Set & Subset]

Introduction

Why are we studying sets

- The concept of set is basic to all of mathematics and mathematical applications.
- Serves as a basis of description of higher concept and mathematical reasoning
- Set is fundamental in many areas of Computer Science.

Set

- A set is a **well-defined collection of distinct objects.**
- These objects are called **members** or **elements** of the set.
- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

- A is a set of all positive integers less than 10,

  $A=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- B is a set of first 5 positive odd integers,

  $B=\{1, 3, 5, 7, 9\}$

- C is a set of vowels, $C=\{a, e, i, o, u\}$

Defining Sets

This can be done by:

- Listing ALL elements of the set within braces.
- Listing enough elements to show the pattern then an ellipsis.
- Use set builder notation to define "rules" for determining membership in the set

Example

1. Listing ALL elements. $A = \{1,\ 2,\ 3,\ 4\}$ explicitly

2. Demonstrating a pattern. $\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$ implicitly

3. Using set builder notation. $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$

implicitly

Sets

A set is determined by its elements and not by any particular order in which the element might be listed.

Example, $A=\{1, 2, 3, 4\}$,

A might just as well be specified as

$\{2, 3, 4, 1\}$ or $\{4, 1, 3, 2\}$

Sets

The elements making up a set are assumed to be **distinct**, we may have duplicates in our list, only one occurrence of each element is in the set.

Example

$\{a, b, c, a, c\} \longrightarrow \{a, b, c\}$

$\{1, 3, 3, 5, 1\} \longrightarrow \{1, 3, 5\}$

Sets

- Use uppercase letters *A*, *B*, *C* … to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

$X = \{ a, b, c, d, e \}$, $b \in X$ and $m \notin X$

$A = \{\{1\}, \{2\}, 3, 4\}$, $\{2\} \in A$ and $1 \notin A$

Sets

- Use uppercase letters A, B, C … to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

$X=\{ a, b, c, d, e \}$, $b \in X$ and $m \notin X$

$A=\{\{1\}, \{2\}, 3, 4\}$, $\{2\} \in A$ and $1 \notin A$

Sets

- If a set is a large finite set or an infinite set, we can describe it by **listing a property necessary for memberships**

- Let S be a set, the notation,

$$A = \{x \mid x \in S, P(x)\} \text{ or } A = \{x \in S \mid P(x)\}$$

means that A is the set of all elements x of S such that x satisfies the property P.

Example

- Let $A=\{1, 2, 3, 4, 5, 6\}$, we can also write A as,

  $A=\{x \mid x \in Z,\ 0< x <7\}$ if Z denotes the set of integers.

- Let $B=\{x \mid x \in Z, x>0\}$, $B=\{1, 2, 3, 4, \ldots\}$

Example

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of positive integers: $\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$

The set of Rational Numbers (fractions): $1/2, 2/3, 5/7, etc \in \mathbb{Q}$

More formally: $\mathbb{Q} = \left\{ a/b \mid a, b \in \mathbb{R}, b \neq 0 \right\}$

The set of Irrational Numbers: $\sqrt{2}, \pi, \text{or } e$ are irrational

The Real numbers $= \mathbb{R} =$ the union of the rational numbers with the irrational numbers

Premier Digital Tech University™

Some Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x \mid x \text{ is an odd positive integer}\}$ represents the set $\{1, 3, 5, 7, 9, \ldots\}$

$\{x \mid x \text{ is an odd positive integer}\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " $|$ ", stands for "such that"

Other "short-hand" notation used in working with sets

"$\forall$" stands for "for every"

"$\exists$" stands for "there exists"

"$\bigcup$" stands for "union"

"$\bigcap$" stands for "intersection"

"$\subseteq$" stands for "is a subset of"

"$\subset$" stands for "is a (proper) subset of"

"$\not\subset$" stands for "is a not a (proper) subset of"

"$\varnothing$" stands for the "empty set"

"$\in$" stands for "is an element of"

"$\notin$" stands for "is not an element of"

"$\times$" stands for "cartesian cross product"

"$=$" stands for "is equal to"

Subset

If every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.

$A=B$, if $A \subseteq B$ and $B \subseteq A$

The empty set ($\varnothing$) is a subset of every set.

Example $A=\{1, 2, 3\}$

Subset of A,

$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

Note: A is a subset of A

Exercise

Answer true or false

a) $\{x\} \subseteq \{x\}$

b) $\{x\} \in \{x\}$

c) $\{x\} \in \{x,\{x\}\}$

d) $\{x\} \subseteq \{x,\{x\}\}$

e) $\{\{x\}\} \subseteq \{x,\{x\}\}$

f) $x \in \{x,\{x\}\}$

Proper Subset

If $A \subseteq B$ and B contains an element that is not in A, then we say "A is a **proper subset** of B": $A \subset B$ or $B \supset A$.

Formally: $A \subseteq B$ means $\forall x\ [x \in A \rightarrow x \in B]$.

For all sets: $A \subseteq A$.

Note: If *A* is a subset of *B* and *A* does not equal B, we say that *A* is a proper subset of *B* ($A \subseteq B$ and $A \neq B$ ($B \not\subseteq A$))

Example

- Let, $A=\{1, 2, 3\}$

  Proper subset of A,
  $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

- Let, $B=\{1, 2, 3, 4, 5, 6\}$

  A is proper subset of *B.*

Example

A={a,b,c,d,e,f,g,h}
B={b,d,e}
C={a,b,c,d,e}
D={r,s,d,e}

Proper subset of *A* ??

Premier Digital Tech University™

Empty Sets

The **empty set** $\varnothing$ or {} but not $\{\varnothing\}$
is the set without elements.

Note:

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$
  $A \cap A' = \varnothing$, $A \cup A' = U$
  $U' = \varnothing$, $\varnothing' = U$

Example

$\varnothing = \{x \mid x$ is a real number and $x^2 = -3\}$

$\varnothing = \{x \mid x$ is positive integer and $x^3 < 0\}$

Premier Digital Tech University™

Equal Sets

The sets A and B are **equal** (A=B) if and only if each element of A is an element of B and vice versa.

Formally: A=B means $\forall x\ [x \in A \leftrightarrow x \in B]$.

Example

$A=\{a, b, c\}$, $B=\{b, c, a\}$, $A=B$

$C=\{1, 2, 3, 4\}$

$D=\{x \mid x$ is a positive integer and $2x < 10\}$,

$C=D$

Exercise

Determine whether each pair of sets is equal

a) $\{1, 2, 2, 3\}, \{1, 3, 2\}$

b) $\{ x \mid x^2 + x = 2\}, \{1, -2\}$

c) $\{x \mid x$ is a real number and $0 < x \leq 2\}, \{1,2\}$

Equivalent Sets

Two sets, A and B, are equivalent if there exists a one-to-one correspondence between them.

When we say sets "have the same size", we mean that they are equivalent.

Example

Set A= {A, B, C, D, E} and **Set** B={1, 2, 3, 4, 5}

Note:

- An equivalent set is simply a set with an **equal number of elements**.
- The sets do not have to have the same exact elements, just the same number of elements.

Finite Sets

A set A is finite

if it is empty

or

if there is a natural number n such that set A is equivalent to

$\{1, 2, 3, \ldots n\}$.

Example

$A = \{1, 2, 3, 4\}$

$B = \{x \mid x \text{ is an integer, } 1 \leq x \leq 4\}$

Note:

There exists a nonnegative integer n such that *A* has *n* elements (*A* is called a finite set with *n* elements)

Infinite Sets

- An infinite set is a set whose **elements can not be counted**.
- An infinite set is one that has **no last element**

Are all infinite sets equivalent?

An infinite set is a set that can be placed into a **one-to-one correspondence** with a proper subset of itself.

Example

Infinite sets

$Z = \{x \mid x \text{ is an integer}\}$
or $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$S = \{x \mid x \text{ is a real number and } 1 \leq x \leq 4\}$
$D = \{x \mid x \text{ is an integer, } x > 0\}$

Finite Sets

$C = \{5, 6, 7, 8, 9, 10\}$

$B = \{x \mid x \text{ is an integer, } 10 < x < 20\}$

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set U.
- This set U is called a universal set or a universe.
- The set U must be explicitly given or inferred from the context

Universal Set

Typically we consider a set A
a part of a **universal set** $\mathcal{U}$,
which consists of all possible elements.

To be entirely correct we should say

$$\forall x \in \mathcal{U}\ [x \in A \leftrightarrow x \in B]$$

instead of

$$\forall x\ [x \in A \leftrightarrow x \in B] \text{ for } A=B.$$

Note that $\{ x \mid 0<x<5\}$ is can be ambiguous.
Compare $\{ x \mid 0<x<5, x \in N \}$ with $\{ x \mid 0<x<5, x \in Q \}$

Example

- The sets $A=\{1,2,3\}$, $B=\{2,4,6,8\}$ and $C=\{5,7\}$
- One may choose $U=\{1,2,3,4,5,6,7,8\}$ as a universal set.
- Any superset of U can also be considered a universal set for these sets A, B, and C.

For example, $U=\{x \mid x \text{ is a positive integer}\}$

Cardinality of Set

- Let S be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|S|=n$ and say that the cardinality (or the number of elements) of S is n.

Example

$A = \{1, 2, 3\}$, $|A|=3$

$B = \{a,b,c,d,e,f,g\}$, $|B|=7$

Exercise

If M is finite, determine the $|M|$

a) $M = \{1, 2, 3, 4\}$
b) $M = \{4, 4, 4\}$
c) $M = \{\}$
d) $M = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

Power Set

- The set of all subsets of a set A, denoted $P(A)$, is called the power set of A.

$$P(A) = \{X \mid X \subseteq A\}$$

If $|A| = n$, then $|P(A)| = 2^n$

Example $A = \{1,2,3\}$

The power set of A,

$P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Notice that $|A| = 3$, and $|P(A)| = 2^3 = 8$

Exercise

List the member of $P(\{a, b, c, d\})$.
Which are proper subset of $\{a, b, c, d\}$?

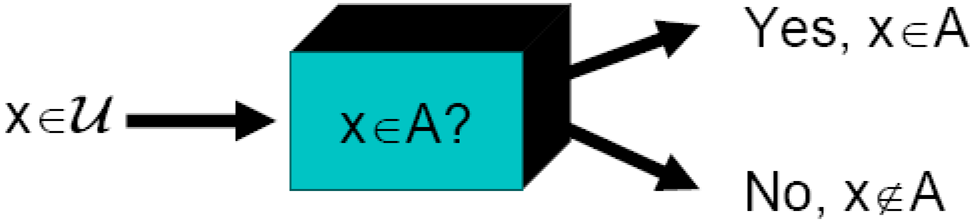
How to Think of Sets

The elements of a set do not have an ordering, hence $\{a,b,c\} = \{b,c,a\}$

The elements of a set do not have multitudes, hence $\{a,a,a\} = \{a,a\} = \{a\}$

All that matters is: "Is x an element of A or not?"

The size of A is thus the number of *different* elements



Thank You

UTM
UNIVERSITI TEKNOLOGI MALAYSIA

Premier Digital Tech University™

innovative ● entrepreneurial ● global