TONGJI UNIVERSITY
SCHOOL OF SOFTWARE ENGINEERING

Assignment #1

Student name: *Jiajie Li (1750655)*

Course: *Computer Vision* – Professor: *Prof. Lin Zhang*
Due date: *April 18th, 2021*

Question 1

1-1 (Math) In our lectures, we mentioned that matrices that can represent isometries can form a group. Specifically, in 3D space, the set comprising matrices $\{\mathbf{M}_i\}$ is actually a group, where $\mathbf{M}_i = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$, $\mathbf{R}_i \in \mathbb{R}^{3\times3}$ is an orthonormal matrix, $\det(\mathbf{R}_i) = 1$, and $\mathbf{t}_i \in \mathbb{R}^{3\times1}$ is a vector. Please prove that the set $\{\mathbf{M}_i\}$ forms a group.

Hint: You need to prove that $\{\mathbf{M}_i\}$ satisfies the four properties of a group, i.e. closure, associativity, existence of identity element, and existence of inverse element for each group element.

Answer. Suppose that the element of group is M_i, and the operation on this group is Matrix Multiply .

(a) **Closure**

$\forall \mathbf{M}_i,\ \mathbf{M}_j \in \{\mathbf{M}_i\}$

$\mathbf{M}_i\mathbf{M}_j = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_j & \mathbf{t}_j \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i\mathbf{R}_j & \mathbf{R}_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix}$

$\mathbf{R}_i, \mathbf{R}_j \in \mathbb{R}^{3\times3} \Rightarrow \mathbf{R}_i\mathbf{R}_j \in \mathbb{R}^{3\times3}$

$\mathbf{R}_i\mathbf{R}_j(\mathbf{R}_i\mathbf{R}_j)^T = \mathbf{R}_i\mathbf{R}_j(\mathbf{R}_j)^T(\mathbf{R}_i)^T = \mathbf{E} \Rightarrow \mathbf{R}_i\mathbf{R}_j$ is an orthonormal matrix

$\det\left(\mathbf{R}_i\mathbf{R}_j\right) = \det(\mathbf{R}_i)\det\left(\mathbf{R}_j\right) = 1$

$\mathbf{t}_i, \mathbf{t}_j \in \mathbb{R}^{3\times1} \Rightarrow \mathbf{R}_i\mathbf{t}_j + \mathbf{t}_i \in \mathbb{R}^{3\times1}$

Hence, $\mathbf{M}_i\mathbf{M}_j \in \{\mathbf{M}_i\}$, Q.E.D.

(b) **Associativity**

$$\begin{aligned}
\mathbf{M}_i\left(\mathbf{M}_j\mathbf{M}_k\right) &= \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_j & \mathbf{t}_j \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_k & \mathbf{t}_k \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_j\mathbf{R}_k & \mathbf{R}_j\mathbf{t}_k + \mathbf{t}_j \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_i\mathbf{R}_j\mathbf{R}_k & \mathbf{R}_i\left(\mathbf{R}_j\mathbf{t}_k + \mathbf{t}_j\right) + \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_i\mathbf{R}_j\mathbf{R}_k & \mathbf{R}_i\mathbf{R}_j\mathbf{t}_k + \mathbf{R}_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}_i\mathbf{R}_j & \mathbf{R}_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_k & \mathbf{t}_k \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \left(\mathbf{M}_i\mathbf{M}_j\right)\mathbf{M}_k
\end{aligned}$$

Hence, $\mathbf{M}_i\left(\mathbf{M}_j\mathbf{M}_k\right) = \left(\mathbf{M}_i\mathbf{M}_j\right)\mathbf{M}_k$, Q.E.D.

(c) **Existence of identity element**

$\forall M_i \in \{M_i\}$

$\exists \mathbf{e} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 1 \end{bmatrix}$, where $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathbf{e}\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} =$

$\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix}\mathbf{e} = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{0} & 1 \end{bmatrix}$

Hence, $\exists \mathbf{e}, \mathbf{e}\mathbf{M}_i = \mathbf{M}_i\mathbf{e} = \mathbf{M}_i$, Q.E.D.

(d) **Existence of inverse element**

$\forall M_i = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \in \{M_i\}$

$\exists M_i^{-1} = \begin{bmatrix} R_i^T & -R_i^T t_i \\ 0 & 1 \end{bmatrix}$

$M_i^{-1}M_i = \begin{bmatrix} R_i^T & -R_i^T t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} = E_{4\times 4}$

$M_iM_i^{-1} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i^T & -R_i^T t_i \\ 0 & 1 \end{bmatrix} = E_{4\times 4}$

Hence, $\forall M_i \in \{M_i\}$, $\exists M_i^{-1}, M_i^{-1}M_i = M_iM_i^{-1} = E$, Q.E.D.

Question 2

1-2 (Math) Gaussian function is

$$G(x, y; \sigma) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

The scale-normalized Laplacian of Gaussian (LOG) is

$$LoG = \sigma^2 \nabla^2 G$$

Please verify that Difference of Gaussian (DOG)

$$DoG = G(x, y; k\sigma) - G(x, y; \sigma)$$

can be a good approximation of LoG.

Answer.

$$\frac{\partial G}{\partial x} = -\frac{x}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$\frac{\partial G}{\partial y} = -\frac{y}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial\left(\frac{\partial G}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\partial G}{\partial y}\right)}{\partial x} = \frac{x^2 - \sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$\frac{\partial^2 G}{\partial y^2} = \frac{\partial\left(\frac{\partial G}{\partial x}\right)}{\partial y} + \frac{\partial\left(\frac{\partial G}{\partial y}\right)}{\partial y} = \frac{y^2 - \sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$LoG = \sigma^2 \nabla^2 G = \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$$\begin{aligned}
\frac{\partial G}{\partial \sigma} &= -\frac{1}{\pi\sigma^3} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) + \frac{1}{2\pi\sigma^5} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)\left(x^2 + y^2\right) \\
&= \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^5} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \\
&= \sigma \nabla^2 G
\end{aligned}$$

Hence,

$$\frac{\partial G}{\partial \sigma} = \sigma \nabla^2 G \tag{1}$$

Performing the caleulus of difference on the equation (1) yields,

$$\sigma \nabla^2 G = \frac{\partial G}{\partial \sigma} \approx \frac{G(x, y, k\sigma) - G(x, y, \sigma)}{k\sigma - \sigma}$$

$$\text{DoG } = G(x, y, k\sigma) - G(x, y, \sigma) \approx (k-1)\sigma^2\nabla^2 G = (k-1) \text{ LoG}$$

As the constant term $(k-1)$ does not affect the location of the extrema. So DoG is an approximation of $\sigma^2\nabla^2 G$, Q.E.D.

Question 3

1-3 (Math) In the lecture, we talked about the least square method to solve an over-determined linear system $\mathbf{Ax} = b, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^{n\times 1}, m > n, \text{rank}(\mathbf{A}) = n$. The closed form solution is $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^T \mathbf{A}^T b$. Try to prove that $\mathbf{A}^T\mathbf{A}$ is non-singular (or in other words, it is invertible).

Answer.

Since $\text{rank}(\mathbf{A}) = n$, according to Schmidt orthogonalization, there must exist an invertible square matrix $\mathbf{P}$ that makes $\mathbf{B} = \mathbf{AP}$, $\mathbf{B}^T\mathbf{B} = \mathbf{E}$.

Hence, B^TB is an invertible matrix, that

$$\mathbf{B}^T\mathbf{B} = (\mathbf{AP})^T(\mathbf{AP}) = \mathbf{P}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{P} = \mathbf{E} \tag{2}$$

is invertible matrix, and all three of the above multiplied square matrices are invertible. Hence, $\mathbf{A}^T\mathbf{A}$ is invertible(non-singular).