TONGJI UNIVERSITY
SCHOOL OF SOFTWARE ENGINEERING

Assignment #2

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Course: *Computer Vision* – Professor: *Prof. Lin Zhang*
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Question 1

2-1 (Math) In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous coordinate of the infinity point of this line?

Answer.

As defined, all infinity points on plane comprise an infinity line, and homogeneous coordinate of the infinity line is $(0, 0, 1)^T$.

Hence, the intersection of line $x - 3y + 4 = 0$ and the infinity line is the infinity point of line $x - 3y + 4 = 0$.

Convert the line in the Cartesian coordinate system to the homogeneous coordinate system by replacing x, y by the homogeneous coordinates $x/w, y/w$ respectively, and rewrite the equation of the line.

$$\frac{x}{z} - 3\frac{y}{z} + 4 = 0 \Rightarrow x - 3y + 4z = 0 \tag{1}$$

Hence, the homogeneous coordinates of these two lines are $(0, 0, 1)^T$ and $(1, -3, 4)^T$, On the projective plane, the intersection of two lines $\mathbf{l}, \mathbf{l}'$ is the point $x = \mathbf{l} \times \mathbf{l}'$

$$(0,0,1)^T \times (1,-3,4)^T = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & -3 & 4 \end{vmatrix} = 3i + j = (3,1,0)^T \tag{2}$$

Since the infinity point actually represents a direction, it is more safe to represent it as a norm vector, the norm vector of $(3, 1, 0)^T$ is $(\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, 0)^T$

Question 2

2-2 (Math) A, B, C and D are four points in 3D Euclidean space, their coordinates are (x_i, y_i, z_i), $i = 1, 2, 3, 4$, respectively. Please prove that:

$$\text{These four points are coplanar} \Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Answer.

Since the determination of four points co-planarity can be transformed into the determination of three vectors co-planarity. Hence,

$$\begin{aligned} & \text{These four points are coplanar} \\ & \Leftrightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD} \text{ are coplanar} \\ & \Leftrightarrow (\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = 0 \\ & \Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0 \end{aligned} \tag{3}$$

Since,

$$\begin{aligned} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} &= \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 &\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0 \\ &\Leftrightarrow \text{These four points are coplanar} \end{aligned} \tag{5}$$

Q.E.D.

Question 3

2-3 (Math) On the normalized retinal plane, suppose that $\mathbf{p}_n$ is an ideal point of projection without considering distortion. If distortion is considered, $\mathbf{p}_n = (x, y)^T$ is mapped to $\mathbf{p}_a = (x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is,

$$\begin{cases} x_d = x\left(1 + k_1 r^2 + k_2 r^4\right) + 2\rho_1 xy + \rho_2\left(r^2 + 2x^2\right) + xk_3 r^6 \\ y_d = y\left(1 + k_1 r^2 + k_2 r^4\right) + 2\rho_2 xy + \rho_1\left(r^2 + 2y^2\right) + yk_3 r^6 \end{cases}$$

where $r^2 = x^2 + y^2$ For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of $\mathbf{p}_d$ w.r.t $\mathbf{p}_n$, i.e., $\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$
It should be noted that in this question $\mathbf{p}_d$ is the function of $\mathbf{p}_n$ and all the other parameters can be regarded as constants.

Answer.

$$\begin{aligned} r^2 = x^2 + y^2, r^4 = x^4 + 2x^2y^2 + y^4, r^6 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 \\ x_d =& x\left(1 + k_1 r^2 + k_2 r^4\right) + 2\rho_1 xy + \rho_2\left(r^2 + 2x^2\right) + xk_3 r^6 \\ =& k_3x^7 + 3k_3x^5y^2 + k_2x^5 + 3k_3x^3y^4 + 2k_2x^3y^2 + k_1x^3 \\ &+ 3\rho_2x^2 + k_3xy^6 + k_2xy^4 + k_1xy^2 + 2\rho_1xy + x + \rho_2y^2 \\ y_d =& y\left(1 + k_1 r^2 + k_2 r^4\right) + 2\rho_2 xy + \rho_1\left(r^2 + 2y^2\right) + yk_3 r^6 \\ =& k_3y^7 + 3k_3y^5x^2 + k_2y^5 + 3k_3y^3x^4 + 2k_2y^3x^2 + k_1y^3 \\ &+ 3\rho_1y^2 + k_3yx^6 + k_2yx^4 + k_1yx^2 + 2\rho_2yx + y + \rho_1x^2 \end{aligned} \tag{6}$$

Hence,

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial x_d}{\partial x} &= 7k_3x^6 + 15k_3x^4y^2 + 5k_2x^4 + 9k_3x^2y^4 + 6k_2x^2y^2 + 3k_1x^2 \\ &\quad + 6\rho_2x + k_3y^6 + k_2y^6 + k_2y^6 + k_1y^2 + 2\rho_1y + 1 \\ \frac{\partial x_d}{\partial y} &= 6k_3x^5y + 12k_3x^3y^3 + 4k_2x^3y + 6k_3xy^5 + 4k_2xy^3 + 2k_1xy + 2\rho_1x + 2\rho_2y \\ \frac{\partial y_d}{\partial y} &= 7k_3y^6 + 15k_3y^4x^2 + 5k_2y^4 + 9k_3y^2x^4 + 6k_2y^2x^2 + 3k_1y^2 \\ &\quad + 6\rho_1y + k_3x^6 + k_2x^6 + k_2x^6 + k_1x^2 + 2\rho_2x + 1 \\ \frac{\partial y_d}{\partial x} &= 6k_3y^5x + 12k_3y^3x^3 + 4k_2y^3x + 6k_3yx^5 + 4k_2yx^3 + 2k_1yx + 2\rho_2y + 2\rho_1x \end{aligned}$$

Question 4

2-4 (Math) In our lecture, we mentioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix (represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix.

Suppose that

$\mathbf{r} = \theta\mathbf{n} \in \mathbb{R}^{3\times 1}$, where $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a 3D unit vector and θ is a real number denoting the rotation angle.

With Rodrigues formula, $\mathbf{r}$ can be converted to its rotation matrix form, $\mathbf{R} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta\mathbf{n}^\wedge$

and obviously $\mathbf{R} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote u by the vectorized form of $\mathbf{R}$, i.e.,
$\mathbf{u}\left(R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33}\right)^T$

Please give the concrete form of Jacobian matrix of $\mathbf{u}$ w.r.t $\mathbf{r}$, i.e., $\frac{d\mathbf{u}}{d\mathbf{r}^T} \in \mathbb{R}^{9\times 3}$
In order to make it easy to check your result, please follow the following notation requirements, $\alpha\sin\theta, \beta\cos\theta, \gamma 1-\cos\theta$

Answer.
Let $\alpha = \sin\theta, \beta = \cos\theta, \gamma = (1-\cos\theta)$, therefore

$$\mathbf{R} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta\mathbf{n}^\wedge = \beta\mathbf{I} + \gamma\mathbf{n}\mathbf{n}^T + \alpha\mathbf{n}^\wedge$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{n}\mathbf{n}^T = \begin{bmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{bmatrix}$$

$$\mathbf{n}^\wedge = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

Hence,

$$\mathbf{R} = \begin{bmatrix} \beta + \gamma n_1^2 & \gamma n_1 n_2 - \alpha n_3 & \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_1 n_2 + \alpha n_3 & \beta + \gamma n_2^2 & \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_1 n_3 - \alpha n_2 & \gamma n_2 n_3 + \alpha n_1 & \beta + \gamma n_3^2 \end{bmatrix} \tag{7}$$

$$
\begin{aligned}
\frac{dR_{11}}{dr^T} &= \frac{d\left(\beta+\gamma n_1^2\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} 2\gamma n_1 & 0 & 0 \end{array}\right] \\
\frac{dR_{12}}{dr^T} &= \frac{d\left(\gamma n_1 n_2-\alpha n_3\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} \gamma n_2 & \gamma n_1 & -\alpha \end{array}\right] \\
\frac{dR_{13}}{dr^T} &= \frac{d\left(\gamma n_1 n_3+\alpha n_2\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} \gamma n_3 & \alpha & \gamma n_1 \end{array}\right] \\
\frac{dR_{21}}{dr^T} &= \frac{d\left(\gamma n_1 n_2+\alpha n_3\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} \gamma n_2 & \gamma n_1 & \alpha \end{array}\right] \\
\frac{dR_{22}}{dr^T} &= \frac{d\left(\beta+\gamma n_2^2\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} 0 & 2\gamma n_2 & 0 \end{array}\right] \\
\frac{dR_{23}}{dr^T} &= \frac{d\left(\gamma n_2 n_3-\alpha n_1\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} -\alpha & \gamma n_3 & \gamma n_2 \end{array}\right] \\
\frac{dR_{31}}{dr^T} &= \frac{d\left(\gamma n_1 n_3-\alpha n_2\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} \gamma n_3 & -\alpha & \gamma n_1 \end{array}\right] \\
\frac{dR_{32}}{dr^T} &= \frac{d\left(\gamma n_2 n_3+\alpha n_1\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} \alpha & \gamma n_3 & \gamma n_2 \end{array}\right] \\
\frac{dR_{33}}{dr^T} &= \frac{d\left(\beta+\gamma n_3^3\right)}{d(\theta n)^T} = \frac{1}{\theta}\left[\begin{array}{ccc} 0 & 0 & 2\gamma n_3 \end{array}\right]
\end{aligned} \tag{8}
$$

Thus,

$$
\begin{aligned}
\frac{d\mathbf{u}}{d\mathbf{r}^T} &= \frac{1}{\theta}\begin{bmatrix}
2\gamma n_1 & 0 & 0 \\
\gamma n_2 & \gamma n_1 & -\alpha \\
\gamma n_3 & \alpha & \gamma n_1 \\
\gamma n_2 & \gamma n_1 & \alpha \\
0 & 2\gamma n_2 & 0 \\
-\alpha & \gamma n_3 & \gamma n_2 \\
\gamma n_3 & \alpha & \gamma n_1 \\
\gamma n_3 & \alpha & \gamma n_1 \\
0 & 0 & 2\gamma n_3
\end{bmatrix} \\
&= \frac{1}{\theta}\begin{bmatrix}
(2-2\cos\theta)n_1 & 0 & 0 \\
(1-\cos\theta)n_2 & (1-\cos\theta)n_1 & -\sin\theta \\
(1-\cos\theta)n_3 & \sin\theta & (1-\cos\theta)n_1 \\
(1-\cos\theta)n_2 & (1-\cos\theta)n_1 & \sin\theta \\
0 & (2-2\cos\theta)n_2 & 0 \\
-\sin\theta & (1-\cos\theta)n_3 & (1-\cos\theta)n_2 \\
(1-\cos\theta)n_3 & \sin\theta & (1-\cos\theta)n_1 \\
(1-\cos\theta)n_3 & \sin\theta & (1-\cos\theta)n_1 \\
0 & 0 & (2-2\cos\theta)n_3
\end{bmatrix}
\end{aligned} \tag{9}
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