Which questions should we spend more time on? Any questions from past tutorials please feel free to clarify =)

EE2211 Introduction to Machine Learning

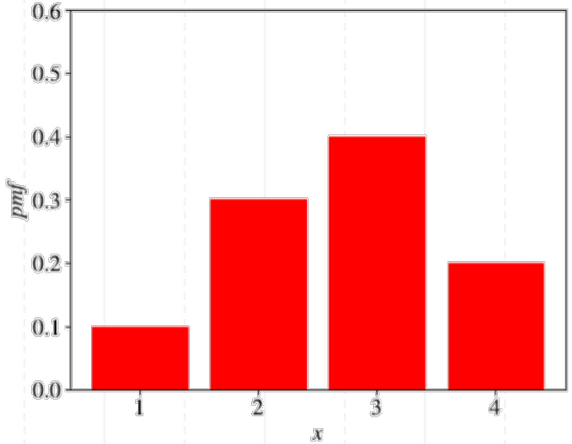
T14 & T22, Chua Dingjuan elechuad@nus.edu.sg
Slides @ tiny.cc/ee2211tut

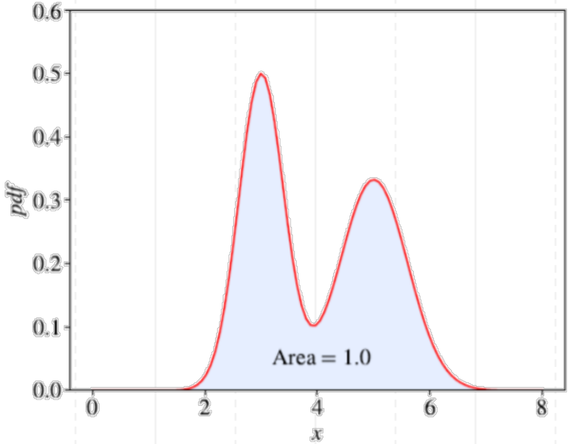
Tutorial 3

Let’s start with a brief summary

Brief Summary of Key Points

- PMF vs PDF (mass vs density)





- Python
  - Dictionary
  - `Scipy stats.norm.cdf`

- Bayes' Rule

$$\Pr(y|x) = \frac{\Pr(y)\Pr(x|y)}{\Pr(x)} = \frac{\Pr(y)\Pr(x|y)}{\Sigma_y \Pr(y)\Pr(x|y)}$$

Discussion of Solutions 3

Q1, 2, 3, 10, **4, 5,** 6(self reading!), 7-13

Question1

The random variable N has probability mass function (PMF) $P_N(n) = \begin{cases} c\left(1/2\right)^n, & n = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$

$P_N(0) = c$, $P_N(1) = \frac{1}{2}c$, $P_N(2) = \frac{1}{4}c$

(a) What is the value of the constant c?

(b) What is $\Pr[N \leq 1]$?

Question1

The random variable N has probability mass function (PMF) $P_N(n) = \begin{cases} c(1/2)^n, & n = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$

(a) What is the value of the constant c?

(b) What is Pr[N ≤ 1]?

$P_N(0) = c, \; P_N(1) = \frac{1}{2}c, \; P_N(2) = \frac{1}{4}c$

Solution

(a) Sum of all probabilities = 1 (PMF sum up to one)

$$\sum_{n=0}^{2} P_X(n) = c + \frac{c}{2} + \frac{c}{4} = 1 \quad \text{implying } c = \frac{4}{7}$$

(b) Pr[N ≤ 1] = $P_N(n = 0) + P_N(n = 1) = c + \frac{c}{2} = \frac{6}{7}$

Question2

The random variable X has probability density function (PDF)

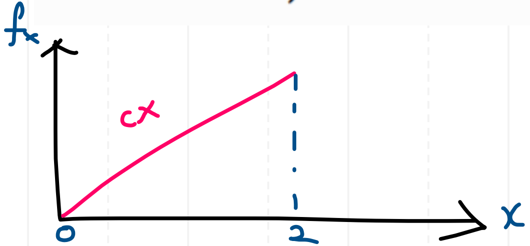
$$f_X(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $\Pr[0 \le X \le 1]$,

(c) $\Pr[-1/2 \le X \le 1/2]$.



Question2

The random variable X has probability density function (PDF)

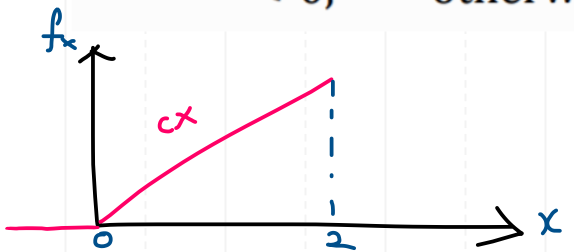
$$f_X(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $\Pr[0 \le X \le 1]$,

(c) $\Pr[-1/2 \le X \le 1/2]$.



Solution

(a) Sum of all probabilities = 1 (PDF area under the curve sum up to one)

$$\int_0^2 cx\,dx = c\left[\frac{x^2}{2}\right]_0^2 = c\left[\frac{4}{2}\right] = 2c = 1$$

$$c = 1/2.$$

Question2

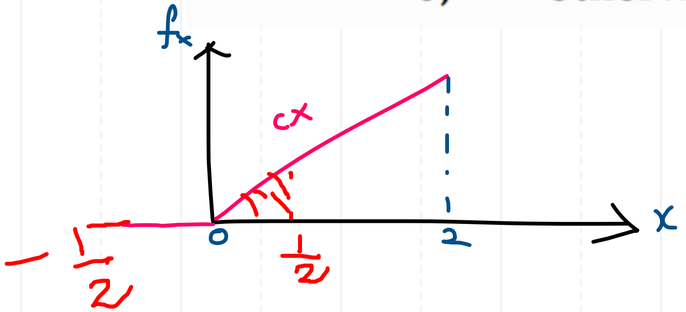
The random variable X has probability density function (PDF) $f_X(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

Use the PDF to find

(a) the constant c,

(b) $\Pr[0 \le X \le 1]$,

(c) $\Pr[-1/2 \le X \le 1/2]$.



Solution

(b) $\Pr[0 \le X \le 1]$

$$Pr[0 \le X \le 1] = \int_0^1 x/2\, dx = 1/4$$

(c) $\Pr[-1/2 \le X \le 1/2]$

$$Pr[-1/2 \le X \le 1/2] = \int_0^{1/2} x/2\, dx = 1/16$$

Question3

Let A = {resistor is within 50Ω of the nominal value}.
The probability that a resistor is from machine B is Pr[B] = 0.3. The probability that a resistor is acceptable, i.e., within 50 Ω of the nominal value, is Pr[A] = 0.78. Given that a resistor is from machine B, the conditional probability that it is acceptable is Pr[A|B] = 0.6.
What is the probability that an acceptable resistor comes from machine B?

Solution

Pr[A] = resistor is ±50Ω of nominal value = 0.78

Pr[B] = resistor from machine B = 0.3

Pr[A|B] = P (resistor is ±50Ω of nominal value, given that it comes from B)
= 0.6

acceptable

Pr[B|A] = P (resistor is from B, given that it is ±50Ω of nominal value) = ?

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{Pr(A)}$$

$$= (0.6)(0.3)/(0.78) \approx 0.23$$

Question4

Consider tossing a fair six-sided die. There are only six outcomes possible, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose we toss two dice and assume that each throw is independent.

outcome of one does not influence the other

(a) What is the probability that the sum of the dice equals seven?

i. List out all pairs of possible outcomes together with their sums from the two throws.

(hint: enumerate all the items in range(1,7))

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

(hint: use dictionary from collections import defaultdict to collect all of the (a, b) pairs that sum to each of the possible values from two to twelve)

(b) What is the probability that half the product of three dice will exceed their sum?

Question4 Solution

(a) What is the probability that the sum of the dice equals seven?

Python Dictionary

CS1010E

- You search for the key in the dictionary
- Then look for its value

Key → Value

- Each key has a correspondent value

key : value pair

```
>>> students = {'A100000X':'John', 'A123456X':'Peter',
'A999999X':'Paul'}
>>> students['A123456X']
'Peter'
```

Tuples use '(' and ')'
Lists use '[' and ']'
Dictionary use '{' and '}'

Question4 Solution

(a) What is the probability that the sum of the dice equals seven?

i. List out all pairs of possible outcomes together with their sums from the two throws.

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

```
#(i)
d={(i,j):i+j for i in range(1,7) for j in range(1,7)}


#(ii) collect all of the (a, b) pairs that sum to
#each of the possible values from two to twelve

from collections import defaultdict
dinv = defaultdict(list)

for i,j in d.items(): dinv[j].append(i)

#      Compute the probability measured for each sum
#      including the sum equals seven

X={i:len(j)/36. for i,j in dinv.items() }

print(X)
```

Question4 Solution

(a) What is the probability that the sum of the dice equals seven?

i. List out all pairs of possible outcomes together with their sums from the two throws.

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

tuple

```
#(i)
d={(i,j):i+j for i in range(1,7) for j in range(1,7)}
```

```
>>> print([i for i in range(1,7) for j in range (1,7)])
[1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 5, 5, 5
, 5, 5, 5, 6, 6, 6, 6, 6, 6]
>>> print([j for i in range(1,7) for j in range (1,7)])
[1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 1, 2, 3
, 4, 5, 6, 1, 2, 3, 4, 5, 6]
```

```
>>> d={(i,j):i+j for i in range(1,7) for j in range(1,7)}

>>> d
{(1, 1): 2, (1, 2): 3, (1, 3): 4, (1, 4): 5, (1, 5): 6, (1, 6): 7, (2, 1): 3, (2,
2): 4, (2, 3): 5, (2, 4): 6, (2, 5): 7, (2, 6): 8, (3, 1): 4, (3, 2): 5, (3, 3):
6, (3, 4): 7, (3, 5): 8, (3, 6): 9, (4, 1): 5, (4, 2): 6, (4, 3): 7, (4, 4): 8, (
4, 5): 9, (4, 6): 10, (5, 1): 6, (5, 2): 7, (5, 3): 8, (5, 4): 9, (5, 5): 10, (5,
6): 11, (6, 1): 7, (6, 2): 8, (6, 3): 9, (6, 4): 10, (6, 5): 11, (6, 6): 12}
```

Question4 Solution

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

```
#(i)
d={(i,j):i+j for i in range(1,7) for j in range(1,7)}


#(ii) collect all of the (a, b) pairs that sum to
#each of the possible values from two to twelve

from collections import defaultdict
dinv = defaultdict(list)

for i,j in d.items(): dinv[j].append(i)

#     Compute the probability measured for each sum
#     including the sum equals seven

X={i:len(j)/36. for i,j in dinv.items() }

print(X)
```

The main difference between `defaultdict` and `dict` is that when you try to access or modify a `key` that's not present in the dictionary, a default `value` is automatically given to that `key`.

```
>>> from collections import defaultdict
>>> dinv = defaultdict(list)
>>> dinv
defaultdict(<class 'list'>, {})
```

Call `list()` to create a new empty list

`dinv ['key'] . append( value )`

dinv ['key'] . append(value)

(1, 1)

(1, 2)

Question4 Solution

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

```
#(i)
d={(i,j):i+j for i in range(1,7) for j in range(1,7)}


#(ii) collect all of the (a, b) pairs that sum to
#each of the possible values from two to twelve

from collections import defaultdict
dinv = defaultdict(list)

for i,j in d.items(): dinv[j].append(i)

#     Compute the probability measured for each sum
#     including the sum equals seven
```

sum values

dice pair values

sum

dice pair

```
>>> d.items()
dict_items([((1, 1), 2), ((1, 2), 3),
, 6), 7), ((2, 1), 3), ((2, 2), 4),
```

```
>>> for i,j in d.items(): print (i)

(1, 1)
(1, 2)
(1, 3)
(1, 4)
(1, 5)
(1, 6)
(2, 1)
```

```
>>> for i,j in d.items(): print (j)

2
3
4
5
6
7
3
```

```
defaultdict(<class 'list'>, {2: [(1, 1)], 3: [(1, 2), (2, 1)], 4: [(1, 3), (2, 2), (
3, 1)], 5: [(1, 4), (2, 3), (3, 2), (4, 1)], 6: [(1, 5), (2, 4), (3, 3), (4, 2), (5,
1)], 7: [(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)], 8: [(2, 6), (3, 5), (4, 4)
, (5, 3), (6, 2)], 9: [(3, 6), (4, 5), (5, 4), (6, 3)], 10: [(4, 6), (5, 5), (6, 4)]
, 11: [(5, 6), (6, 5)], 12: [(6, 6)]})
```

Question4 Solution

ii. Collect all of the (a, b) pairs that sum to each of the possible values from two to twelve (including the sum equals seven).

```
#(i)
d={(i,j):i+j for i in range(1,7) for j in range(1,7)}


#(ii) collect all of the (a, b) pairs that sum to
#each of the possible values from two to twelve

from collections import defaultdict
dinv = defaultdict(list)

for i,j in d.items(): di

#     Compute the probab
#     including the sum equals seven

X={i:len(j)/36. for i,j in dinv.items() }

print(X)
```

```
defaultdict(<class 'list'>, {2: [(1, 1)], 3: [(1, 2), (2, 1)], 4: [(1, 3), (2, 2), (3, 1)], 5: [(1, 4), (2, 3), (3, 2), (4, 1)], 6: [(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)], 7: [(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)], 8: [(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)], 9: [(3, 6), (4, 5), (5, 4), (6, 3)], 10: [(4, 6), (5, 5), (6, 4)], 11: [(5, 6), (6, 5)], 12: [(6, 6)]})
```

sum

dice pair values

```
{2: 0.027777777777777776, 3: 0.05555555555555555, 4: 0.08333333333333333, 5: 0.1111111111111111, 6: 0.1388888888888889, 7: 0.16666666666666666, 8: 0.1388888888888889, 9: 0.1111111111111111, 10: 0.08333333333333333, 11: 0.05555555555555555, 12: 0.027777777777777776}
```

Question4

Consider tossing a fair six-sided die. There are only six outcomes possible, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose we toss two dice and assume that each throw is independent.

(b) What is the probability that half the product of three dice will exceed their sum?

```
# (b) What is the probability that half the product of three dice will exceed their sum?

d={(i,j,k):((i*j*k)/2>i+j+k) for i in range(1,7)
                                for j in range(1,7)
                                   for k in range(1,7)}

dinv = defaultdict(list)
for i,j in d.items(): dinv[j].append(i)
X={i:len(j)/6.0**3 for i,j in dinv.items() }
print(X)
```

Question4

Consider tossing a fair six-sided die. There are only six outcomes possible, Ω = {1, 2, 3, 4, 5, 6}. Suppose we toss two dice and assume that each throw is independent.

(b) What is the probability that half the product of three dice will exceed their sum?

```
# (b) What is the probability that half the product of three dice will exceed their sum?

d={(i,j,k):((i*j*k)/2>i+j+k) for i in range(1,7)
                                for j in range(1,7)
                                    for k in range(1,7)}
```

```
>>> d={(i,j,k):((i*j*k)/2>i+j+k) for i in range(1,7)
                                for j in range(1,7)
                                    for k in range(1,7)}
>>> d
{(1, 1, 1): False, (1, 1, 2): False, (1, 1, 3): False, (1, 1, 4): False, (1, 1, 5): False, (1, 1, 6): False, (1, 2, 1): False, (1, 2, 2): False,
(1, 2, 3): False, (1, 2, 4): False, (1, 2, 5): False, (1, 2, 6): False, (1, 3, 1): False, (1, 3, 2): False, (1, 3, 3): False, (1, 3, 4): False,
(1, 3, 5): False, (1, 3, 6): False, (1, 4, 1): False, (1, 4, 2): False, (1, 4, 3): False, (1, 4, 4): False, (1, 4, 5): False, (1, 4, 6): True, (
1, 5, 1): False, (1, 5, 2): False, (1, 5, 3): False, (1, 5, 4): False, (1, 5, 5): True, (1, 5, 6): True, (1, 6, 1): False, (1, 6, 2): False, (1,
```

Question4

Consider tossing a fair six-sided die. There are only six outcomes possible, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose we toss two dice and assume that each throw is independent.

(b) What is the probability that half the product of three dice will exceed their sum?

```
# (b) What is the probability that half the product of three dice will exceed their sum?

d={(i,j,k):((i*j*k)/2>i+j+k) for i in range(1,7)
                              for j in range(1,7)
                                 for k in range(1,7)}

dinv = defaultdict(list)
for i,j in d.items(): dinv[j].append(i)
```

```
>>> dinv = defaultdict(list)
>>> for i,j in d.items(): dinv[j].append(i)

>>> dinv
defaultdict(<class 'list'>, {False: [(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 1, 5), (1, 1, 6), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 2, 4
), (1, 2, 5), (1, 2, 6), (1, 3, 1), (1, 3, 2), (1, 3, 3), (1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 1), (1, 4, 2), (1, 4, 3), (1, 4, 4), (1, 4, 5)
, (1, 5, 1), (1, 5, 2), (1, 5, 3), (1, 5, 4), (1, 6, 1), (1, 6, 2), (1, 6, 3), (2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 1, 4), (2, 1, 5), (2, 1, 6),
(2, 2, 1), (2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 3, 1), (2, 3, 2), (2, 4, 1), (2, 4, 2), (2, 5, 1), (2, 6, 1), (3, 1, 1), (3, 1, 2), (3, 1, 3), (
3, 1, 4), (3, 1, 5), (3, 1, 6), (3, 2, 1), (3, 2, 2), (3, 3, 1), (3, 4, 1), (3, 5, 1), (3, 6, 1), (4, 1, 1), (4, 1, 2), (4, 1, 3), (4, 1, 4), (4
, 1, 5), (4, 2, 1), (4, 2, 2), (4, 3, 1), (4, 4, 1), (4, 5, 1), (5, 1, 1), (5, 1, 2), (5, 1, 3), (5, 1, 4), (5, 2, 1), (5, 3, 1), (5, 4, 1), (6,
1, 1), (6, 1, 2), (6, 1, 3), (6, 2, 1), (6, 3, 1)], True: [(1, 4, 6), (1, 5, 5), (1, 5, 6), (1, 6, 4), (1, 6, 5), (1, 6, 6), (2, 2, 5), (2, 2, 6
), (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 3), (2, 4, 4), (2, 4, 5), (2, 4, 6), (2, 5, 2), (2, 5, 3), (2, 5, 4), (2, 5, 5), (2, 5, 6)
, (2, 6, 2), (2, 6, 3), (2, 6, 4), (2, 6, 5), (2, 6, 6), (3, 2, 3), (3, 2, 4), (3, 2, 5), (3, 2, 6), (3, 3, 2), (3, 3, 3), (3, 3, 4), (3, 3, 5),
(3, 3, 6), (3, 4, 2), (3, 4, 3), (3, 4, 4), (3, 4, 5), (3, 4, 6), (3, 5, 2), (3, 5, 3), (3, 5, 4), (3, 5, 5), (3, 5, 6), (3, 6, 2), (3, 6, 3), (
```

Question4

Consider tossing a fair six-sided die. There are only six outcomes possible, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose we toss two dice and assume that each throw is independent.

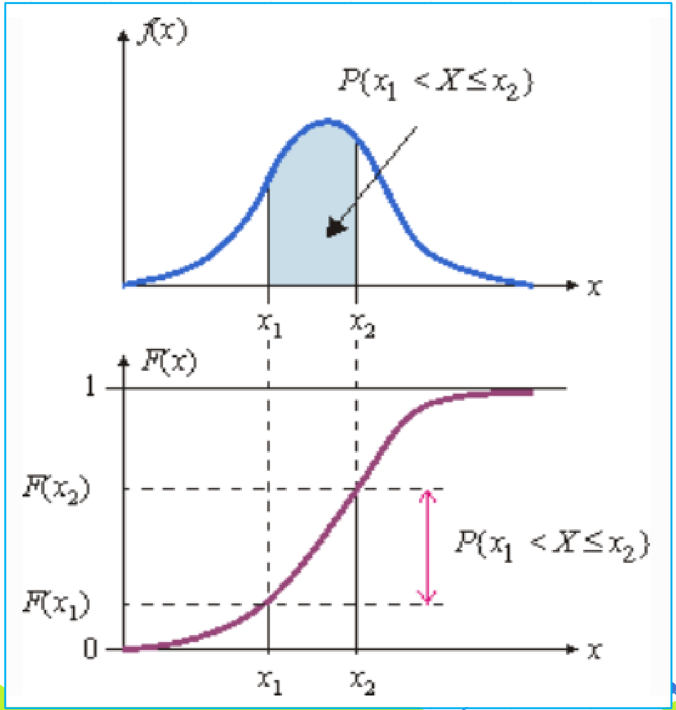
(b) What is the probability that half the product of three dice will exceed their sum?

```
# (b) What is the probability that half the product of three dice will exceed their sum?

d={(i,j,k):((i*j*k)/2>i+j+k) for i in range(1,7)
                                for j in range(1,7)
                                    for k in range(1,7)}

dinv = defaultdict(list)
for i,j in d.items(): dinv[j].append(i)
X={i:len(j)/6.0**3 for i,j in dinv.items() }
print(X)
```

```
>>> X
{False: 0.37037037037037035, True: 0.6296296296296297}
```

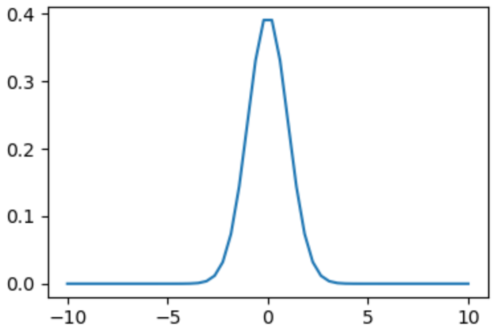
Question5

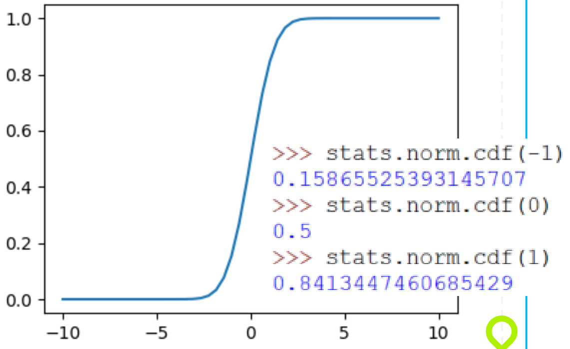
Assuming a normal (Gaussian) distribution with mean 30 Ω and standard deviation of 1.8 Ω, determine the probability that a resistor coming off the production line will be within the range of 28 Ω to 33 Ω. (Hint: use `stats.norm.cdf` function from `scipy import stats`)



`stats.norm.cdf:`

- Creates a normal variable
- Returns the cumulative distribution function value
- `stats.norm.cdf(cdfvalue, mu, sigma)`





Question5

Assuming a normal (Gaussian) distribution with mean 30 Ω and standard deviation of 1.8 Ω, determine the probability that a resistor coming off the production line will be within the range of 28 Ω to 33 Ω. (Hint: use `stats.norm.cdf` function from `scipy import stats`)

```
mu = 30 # mean = 30Ω
sigma = 1.8 # standard deviation = 1.8Ω
x1 = 28 # lower bound = 28Ω
x2 = 33 # upper bound = 33Ω

## calculate probabilities
# probability from Z=0 to lower bound
p_lower = stats.norm.cdf(x1,mu,sigma)

# probability from Z=0 to upper bound
p_upper = stats.norm.cdf(x2,mu,sigma)

# probability of the interval
Prob = (p_upper) - (p_lower)
```

$= 28$

$= 33$

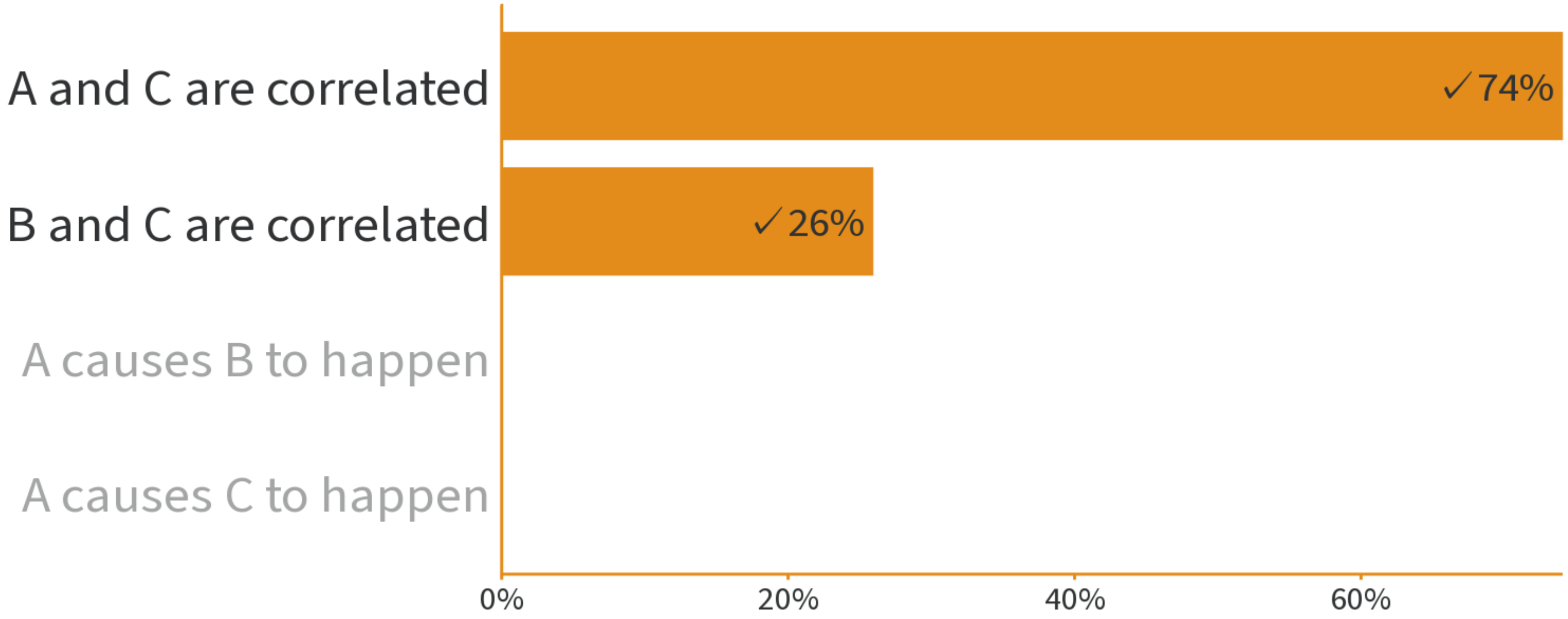
```
>>> p_lower
0.13326026290250537

>>> p_upper
0.9522096477271853

>>> Prob
0.8189493848246799
```

$P(R \leq 28)$

$P(R \leq 33)$

$P(28 \leq R \leq 33)$

Q7 If A and B are correlated, but they're actually caused by C, which of the following statements are correct?



NUS National University of Singapore

Q9 We toss a coin and observe which side is facing up. Which of the following statements represent valid probability assignments for observing head P['H'] and tail P['T']?

P['H']=0.2, P['T']=0.9

P['H']=0.0, P['T']=1.0 (

P['H']=-0.1, P['T']=1.1

P['H']=P['T']=0.5

Question9

Are the two vectors $\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}$ linearly dependent?

Question10

The rank of the matrix $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ is :

Question11

The rank of the matrix $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ is :

Question9

Are the two vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$ linearly dependent?

No. They are not multiples of each other. $\begin{bmatrix}4\\5\\6\end{bmatrix} \neq n\begin{bmatrix}1\\2\\3\end{bmatrix}$

Question10

The rank of the matrix $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ is :

Using row-echelon form : $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 3\\0 & -2\end{bmatrix}$

row2 = row2-2*row1

# of non-zero rows / columns = 2

Therefore rank is 2.

Question11

The rank of the matrix $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ is :

Using row-echelon form, rank is 2.

$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & -6 & -12\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix}$

row2 = row2 - 4*row1
row3 = row3 - 7*row1

row3 = row3 - 2*row2