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Homework 1

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1 Section 1

(Theorem 2.5.4.)

Theorem 2.5.4. Hewitt-Savage $0-1$ law. If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{E}$ then $P(A) \in \{0,1\}$.

Proof Let $A \in \mathcal{E}$. As in the proof of Kolmogorov's 0-1 law, we will show A is independent of itself, i.e., $P(A)=P(A \cap A)=P(A)P(A)$ so $P(A) \in \{0,1\}$. Let $A_n \in \sigma\left(X_1, \ldots, X_n\right)$ so that

(a)
$$P\left(A_n \Delta A\right) \rightarrow 0$$

Here $A \Delta B=(A-B) \cup(B-A)$ is the symmetric difference. The existence of the A_n 's is proved in part ii of Lemma A.2.1. A_n can be written as $\left\{\omega:\left(\omega_1, \ldots, \omega_n\right) \in B_n\right\}$ with $B_n \in \mathcal{S}^n$. Let

$$\pi(j)= \begin{cases}j+n & \text { if } 1 \leq j \leq n \\ j-n & \text { if } n+1 \leq j \leq 2 n \\ j & \text { if } j \geq 2 n+1\end{cases}$$

Observing that π^2 is the identity (so we don't have to worry about whether to write π or π^{-1}) and the coordinates are i.i.d. (so the permuted coordinates are) gives

(b) $\quad P\left(\omega: \omega \in A_n \Delta A\right)=P\left(\omega: \pi \omega \in A_n \Delta A\right)$

Now $\{\omega: \pi \omega \in A\}=\{\omega: \omega \in A\}$, since A is permutable, and

$$\left\{\omega: \pi \omega \in A_n\right\}=\left\{\omega:\left(\omega_{n+1}, \ldots, \omega_{2 n}\right) \in B_n\right\}$$

If we use $A_n^{\prime}$ to denote the last event then we have

(c) $\quad\left\{\omega: \pi \omega \in A_n \Delta A\right\}=\left\{\omega: \omega \in A_n^{\prime} \Delta A\right\}$ Combining (b) and (c) gives

(d) $\quad P\left(A_n \Delta A\right)=P\left(A_n^{\prime} \Delta A\right)$ It is easy to see that

$$|P(B)-P(C)| \leq\mid P(B \Delta C \mid$$

so (d) implies $P\left(A_n\right), P\left(A_n^{\prime}\right) \rightarrow P(A)$. Now $A-C \subset(A-B) \cup(B-C)$ and with a similar inequality for $C-A$ implies $A \Delta C \subset(A \Delta B) \cup(B \Delta C)$. The last inequality, (d), and (a) imply

$$P\left(A_n \Delta A_n^{\prime}\right) \leq P\left(A_n \Delta A\right)+P\left(A \Delta A_n^{\prime}\right) \rightarrow 0$$

The last result implies

$$\begin{aligned} 0 &\leq P\left(A_n\right)-P\left(A_n \cap A_n'\right) \\ &\leq P\left(A_n \cup A_n'\right)-P\left(A_n \cap A_n'\right)=P\left(A_n \Delta A_n'\right) \rightarrow 0 \end{aligned}$$

so $P\left(A_n \cap A_n'\right) \rightarrow P(A)$. But A_n and A_n' are independent, so

$$P\left(A_n \cap A_n'\right)=P\left(A_n\right) P\left(A_n'\right) \rightarrow P(A)^2$$

This shows $P(A)=P(A)^2$, and proves Theorem 2.5.4. $\square$