3. $$\bigcup_{n=1}^{\infty}\left(1+\frac{1}{n}, 2-\frac{1}{n}\right)=(1,2)$$

Proof

Let $A=\bigcup_{n=1}^{\infty}\left(1+\frac{1}{n}, 2-\frac{1}{n}\right)$ and $B=(1,2)$.

For any arbitrary x in B, we can always find n_0 such that x is also in $(1+\frac{1}{n_0}, 2-\frac{1}{n_0})$.

Just let $n_0=\left\lfloor\frac{1}{\min\{x-1,2-x\}}\right\rfloor+1$.

Therefore, x is in A and $B\subset A$.

$\left(1+\frac{1}{n}, 2-\frac{1}{n}\right)\subset B$ for all $n\geq 1$

Therefore, $A\subset B$.

Combine the fact $A\subset B$ and $B\subset A$, we know $A=B$. ■

4. $$\bigcap_{n=1}^{\infty}\left(2-\frac{1}{n}, 3-\frac{1}{n}\right]=\{2\}$$

Proof

Let $A=\bigcap_{n=1}^{\infty}\left(2-\frac{1}{n}, 3-\frac{1}{n}\right]$.

$2-\frac{1}{n}<2$ and $3-\frac{1}{n}\geq 2$ for all $n\geq 1$

2 is in $\left(2-\frac{1}{n}, 3-\frac{1}{n}\right]$ for all $n\geq 1$

Therefore, $2\in A$ and $\{2\}\subset A$.

For any positive ε, we can always find n_0 and n_1 such that $2-\varepsilon$ is not in $\left(2-\frac{1}{n_0}, 3-\frac{1}{n_0}\right]$ and $2+\varepsilon$ is not in $\left(2-\frac{1}{n_1}, 3-\frac{1}{n_1}\right]$.

Just let $n_0=\left\lfloor\frac{1}{\varepsilon}\right\rfloor+1$ and $n_1=1$.

Therefore, 2 is unique to A and $A=\{2\}$. ■

7. $M_1 = \inf(A \cup B)$, $M_2 = \min\{\inf A, \inf B\} \quad \vdash M_1 = M_2$
$N_1 = \sup(A \cup B)$, $N_2 = \max\{\sup A, \sup B\} \vdash N_1 = N_2$

Proof

Let $m_1 = \inf A$ and $m_2 = \inf B$. And suppose $m_1 \le m_2$.

$\forall x \in A \ \ (m_1 \le x)$

$\forall x \in B \ \ (m_2 \le x)$

$x \in A \Rightarrow m_1 \le x$

$x \in B \Rightarrow m_1 \le m_2 < x$

$\forall x \in A \cup B \ \ (m_1 \le x)$ (1)

$\forall \varepsilon > 0 \ \exists x \in A \ \ (x < m_1 + \varepsilon)$ (2)

Let $f(\varepsilon)$ be a skolem function for the existential quantifier in $\exists x \in A$ in (2).

Let $x_0 = f(\varepsilon)$.

$x_0 \in A \cup B$

$\forall \varepsilon > 0 \ \exists x_\varepsilon \in A \cup B \ \ (x_\varepsilon < m_1 + \varepsilon)$ (3)

Combine (2) and (3), we get $M_1 = \inf(A \cup B) = m_1 = \min\{\inf A, \inf B\} = M_2$.

Similarly, we can use the same approach to prove $N_1 = N_2$. ■

8. $M_1 = \inf(A \cap B)$, $M_2 = \max\{\inf A, \inf B\} \quad \vdash M_1 \ge M_2$
$N_1 = \sup(A \cap B)$, $N_2 = \min\{\sup A, \sup B\} \vdash N_1 \le N_2$

Proof

Let $m_1 = \inf A$ and $m_2 = \inf B$. And suppose $m_1 \le m_2$ and $M_1 < M_2$.

$M_2 = \max\{m_1, m_2\} = m_2$

$\forall x \in A \ \ (m_1 \le x)$

$\forall x \in B \ \ (m_2 \le x)$

$x \in A \wedge x \in B \Rightarrow m_1 \le m_2 \le x$

$\forall x \in A \cap B \ \ (m_1 \le m_2 < x)$

$M_2 = m_2$ is a lower bound for $A \cap B$.

$\forall \varepsilon > 0 \ \exists x \in A \cap B \ \ (x < M_1 + \varepsilon < M_2 + \varepsilon)$

M_2 is a greatest lower bound for $A \cap B$.

$M_1 = M_2$ because infimum is unique.

This contradicts with the supposition $M_1 < M_2$.

It is impossible for M_1 to be less than M_2.

Let $A = (3, 5)$ and $B = (4, 6)$.

$M_1 = M_2 = 4$

Let $A = (3, 5) \cup (6, 7)$ and $B = [5, 8]$.

$6 = M_1 > M_2 = 5$

Therefore, $M_1 \geq M_2$.

Similarly, we can do the same thing for N_1 and N_2 ■

10. $A, B \subset \mathbb{R}^+$, both sets are bounded.
$P = \{\, x \mid x = a + b, a \in A, b \in B \,\}$
$Q = \{\, x \mid x = a - b, a \in A, b \in B \,\}$

$\sup P = \sup A + \sup B$ True

$\inf P = \inf A + \inf B$ True

$\sup Q = \sup A - \sup B$ False

$\inf Q = \inf A - \inf B$ False

Proof

Let $m_1 = \sup A$ and $m_2 = \sup B$ and $M = m_1 + m_2$.

$\forall x_1 \in A\ \forall x_2 \in B\ \ (x_1 \leq m_1 \wedge x_2 \leq m_2)$

$\forall x_1 \in A\ \forall x_2 \in B\ \ (x_1 + x_2 \leq m_1 + m_2)$

$\forall x \in P\ \ (x \leq M = m_1 + m_2)$ (1)

$\forall \varepsilon > 0\ \exists x_1 \in A, x_2 \in B\ \ (x_1 > m_1 - \varepsilon \wedge x_2 > m_2 - \varepsilon)$

$\forall \varepsilon > 0\ \exists x_1 \in A, x_2 \in B\ \ (x_1 + x_2 > m_1 + m_2 - 2\varepsilon)$

ε is arbitrary, so we have

$\forall \varepsilon > 0\ \exists x \in P\ \ (x > M - \varepsilon = m_1 + m_2 - \varepsilon)$ (2)

Combine (1) and (2), M is a least upper bound for P.

$\sup P = \sup A + \sup B$ because supremum is unique.

Similarly, $\inf P = \inf A + \inf B$.

Let $A = (8, 9)$ and $B = (1,2)$.

$8 = \sup Q \neq \sup A - \sup B = 7$

$6 = \inf Q \neq \inf A - \inf B = 7$ ■