Solutions to Recitation 5

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This is my own solution to the Recitation 5 of 18.014, which includes Exercise 9abc, 11, 1ab, and 4a from Apostol's *Calculus* (1: 57, 63).

1. This exercise develops some fundamental properties of polynomials. Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree n. Prove each of the following:

   (a) If $n \geqslant 1$ and $f(0) = 0$, then $f(x) = x\,g(x)$, where g is a polynomial of degree $n-1$.

   Proof. Since $n \geqslant 1$, we can rewrite f as

$$f(x) = c_0 + \sum_{k=1}^{n} c_k x^k = c_0 + x \sum_{k=1}^{n} c_k x^{k-1}.$$

   But $f(0) = 0$, which implies $c_0 = 0$. Then let $g(x) = \sum_{k=1}^{n} c_k x^{k-1}$ and we obtain

$$f(x) = x \sum_{k=1}^{n} c_k x^{k-1} = x\,g(x). \qquad \square$$

   (b) For each real a, the function p given by $p(x) = f(x+a)$ is a polynomial of degree n.

   Proof. We have

$$\begin{aligned} f(x+a) &= \sum_{k=0}^{n} c_k (x+a)^k \\ &= \sum_{k=0}^{n} \left\{ c_k \sum_{i=0}^{k} \binom{k}{i} a^{k-i} x^i \right\} \\ &= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} c_k a^{k-i} x^i \\ &= \sum_{k=0}^{n} \left\{ \sum_{i=0}^{n-k} \binom{k+i}{k} c_{k+i} a^i \right\} x^k. \qquad \square \end{aligned}$$

   The above proof says too much, which not only shows the new polynomial is of the same degree but also provide a way to compute the coefficients for it. A less combinatorial proof is as follows.

   Proof. The statement clearly holds for $n = 0$ since $f(x+a) = f(x) = c_0$. Suppose the statement holds for some $n \geqslant 0$ and f is a polynomial of degree $n+1$. We decompose $f(x+a)$ into

$$f(x+a) = \sum_{k=0}^{n+1} c_k (x+a)^k = \left(\sum_{k=0}^{n} c_k (x+a)^k \right) + c_{n+1} (x+a)^{n+1}.$$

   Let $g(x) = \sum_{k=0}^{n} c_k x^k$. Since g is a polynomial of degree n, then there must be a sequence of coefficients d_k such that $g(x+a) = \sum_{k=0}^{n} d_k x^k$. Thus

$$\begin{aligned} f(x+a) &= g(x+a) + c_{n+1}(x+a)^{n+1} \\ &= \sum_{k=0}^{n} d_k x^k + c_{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} x^k \\ &= \left(\sum_{k=0}^{n} d_k x^k \right) + \left\{ \sum_{k=0}^{n} \binom{n+1}{k} c_{n+1} a^{n+1-k} x^k \right\} + c_{n+1} x^{n+1} \\ &= \left\{ \sum_{k=0}^{n} \left[d_k + \binom{n+1}{k} c_{n+1} a^{n+1-k} \right] x^k \right\} + c_{n+1} x^{n+1}. \end{aligned}$$

Let

$$e_k = \begin{cases} d_k + \binom{n+1}{k} c_{n+1} a^{n+1-k}, & 0 \leqslant k \leqslant n, \\ c_{n+1}, & k = n+1. \end{cases}$$

Then we could rewrite $f(x+a)$ as

$$f(x+a) = \left(\sum_{k=0}^{n} e_k x^k \right) + e_{n+1} x^{n+1} = \sum_{k=0}^{n+1} e_k x^k.$$

□

(c) If $n \geqslant 1$ and $f(a) = 0$ for some real a, then $f(x) = (x-a)\,h(x)$, where h is a polynomial of degree $n-1$.

Proof. Let $g(x) = f(x+a)$. By the result from (b), g is a polynomial of degree n. Since $n \geqslant 1$ and $g(0) = f(a) = 0$, by the result from (a), $g(x) = x\,p(x)$, where $p(x)$ is a polynomial of degree $n-1$. Let $h(x) = p(x-a)$. Again by the result from (b), h is also a polynomial of degree $n-1$. Then $f(x) = g(x-a) = (x-a)\,p(x-a) = (x-a)\,h(x)$. □

2. In each case, find all polynomials p of degree $\leqslant 2$ which satisfy the given conditions for all real x.

(a) $p(x) = p(1-x)$.
The coefficient c_0 can be any real number regardless of the degree.
When the degree is zero, the condition clearly holds for all constant polynonimals.
When the degree is 1 ($c_1 \neq 0$), we have

$$\begin{gathered} c_1 x = c_1(1-x), \text{ or} \\ c_1(2x-1) = 0. \end{gathered}$$

Since c_1 is not zero, it must be the case that $x = 1/2$. This means there is no polynonimal of degree 1 which satisfies the above condition for all real x.
When the degree is 2 ($c_2 \neq 0$), we have

$$c_1 x + c_2 x^2 = c_1(1-x) + c_2(1-x)^2,$$

which simplies to

$$(c_1 + c_2)(2x-1) = 0.$$

By the same reasoning in the case of degree 1, it must be the case that $c_1 + c_2 = 0$.

(b) $p(x) = p(1+x)$.
As in the previous exercise, the coefficient c_0 can be any real number regardless of the degree.
By the same reasoning, it's easy to show that there is no polynonimal of degree 1 which satisfies the above condition for all real x.
When the degree is 2 ($c_2 \neq 0$), we have

$$c_1 x + c_2 x^2 = c_1(1+x) + c_2(1+x)^2,$$

which simplies to

$$c_1 + c_2(2x+1) = 0.$$

Since c_2 is not zero, no matter what value c_1 is, we can always find a real number x such that $c_1 + c_2(2x+1) \neq 0$. This means there is no polynonimal of degree 2 which satisfies the above condition for all real x either.

(c) $p(2x) = 2p(x)$.
The coefficient c_0 must be zero regardless of the degree for the above condition to hold for all real x, since

$$p(2 \cdot 0) = 2p(0), \text{ or}$$
$$p(0) = c_0 = 0.$$

When the degree is zero, the only polynonimal satisfying the stated condition for all real x is $p(x) = 0$.
When the degree is 1, we have

$$c_0 + c_1(2x) = 2(c_0 + c_1 x)$$

and since $c_0 = 0$, the above equality can be simplified into

$$2c_1 x = 2c_1 x,$$

which is a tautology. This means c_1 can take any value except zero as long as c_0 is zero.
When the degree is 2, we have

$$c_2(2x)^2 = 2c_2 x^2, \text{ or}$$
$$2c_2 x^2 = 0.$$

Since $c_2 \neq 0$, the last equality is only satisfied when $x = 0$. This means there is no polynonimal of degree 2 satisfying the stated condition for all real x.

(d) $p(3x) = p(x + 3)$.
As before, the coefficient c_0 can be any real number regardless of the degree.
When the degree is zero, the stated condition holds for all constant polynonimals.
When the degree is 1 ($c_1 \neq 0$), we have

$$c_1(3x) = c_1(x + 3), \text{ or}$$
$$x = \frac{3}{2}.$$

This means there is no polynonimal of degree 1 satisfying the stated condition for all real x.
When the degree is 2, we have

$$c_1(3x) + c_2(3x)^2 = c_1(x + 3) + c_2(x + 3)^2, \text{ or}$$
$$[c_1 + c_2(4x + 3)](2x - 3) = 0.$$

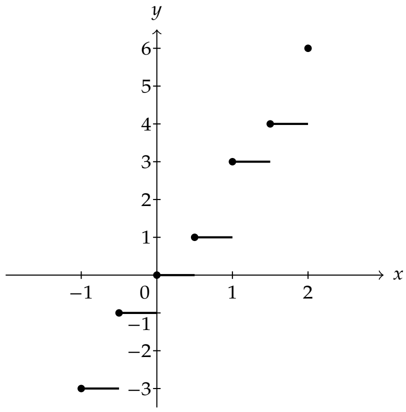
Solve the last equation and we obtain

$$x = -\frac{1}{4}\left(\frac{c_1}{c_2} + 3\right) \quad \text{or} \quad x = \frac{3}{2}.$$

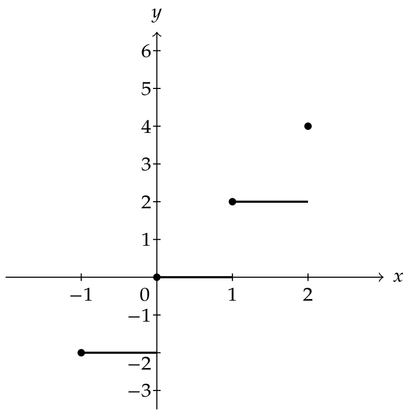
This means there is no polynonimal of degree 2 satisfying the stated condition for all real x.

3. Let $f(x) = [x]$ and let $g(x) = [2x]$ for all real x. In each case, draw the graph of the function h defined over the interval $[-1, 2]$ by the formula given.

(a) $h(x) = f(x) + g(x)$.



(b) $h(x) = f(x) + g(x/2)$.



4. Prove that $[x+n] = [x] + n$ for every integer n.

Proof. By definition of greatest-integer function, we have

$$[x] \leqslant x < [x] + 1 \tag{1}$$

and

$$[x+n] \leqslant x + n < [x+n] + 1. \tag{2}$$

We multiply (1) by -1, add the result to (2), and obtain

$$[x+n] - [x] - 1 < n < [x+n] - [x] + 1,$$

which is the same as

$$n - 1 < [x+n] - [x] < n + 1.$$

The only integer lies between $n-1$ and $n+1$ is n and this is to say $[x+n] - [x] = n$. □