Solutions to Practice Exam 1

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This is my own solution to Practice Exam 1 of 18.014.

1. Compute $\int_{99}^{103} (2x - 198)^2 [\sqrt{x - 99}]\, dx$ where here $[x]$ is defined to be the largest integer $\leqslant x$.

   We have

$$\begin{aligned}\int_{99}^{103} (2x - 198)^2 [\sqrt{x - 99}]\, dx &= \int_0^4 (2x)^2 [\sqrt{x}]\, dx \\ &= \int_0^1 0\, dx + 4 \int_1^4 x^2\, dx \\ &= 4 \cdot \frac{4^3 - 1^3}{3} \\ &= 84.\end{aligned}$$

2. Let S be a square pyramid with base area r^2 and height h. Using Cavalieri's Theorem, determine the volume of the pyramid.

   By elementary geometry, we have

$$a_S(x) = \frac{(h - x)^2 r^2}{h^2}.$$

   Then

$$\begin{aligned}v(S) &= \int_0^h a_S(x)\, dx \\ &= \int_0^h \frac{(h - x)^2 r^2}{h^2}\, dx \\ &= \frac{r^2}{h^2} \int_0^h (h - x)^2\, dx \\ &= \frac{r^2}{h^2} \int_{-h}^0 x^2\, dx \\ &= \frac{r^2}{h^2} \cdot \frac{h^3}{3} = \frac{r^2 h}{3}.\end{aligned}$$

3. Let f be an integrable function on $[0, 1]$. Prove that $|f|$ is integrable on $[0, 1]$.

   Proof. Since f is integrable on $[0, 1]$, for any $\epsilon > 0$, we can always find step functions s and t such that $s \leqslant f \leqslant t$ and

$$\int_0^1 t - \int_0^1 s < \epsilon.$$

   Let $P = \{x_0, x_1, \dots, x_n\}$ be a common partition for s and t. Denote $s(x) = s_k$ and $t(x) = t_k$ on every open interval (x_{k-1}, x_k). We construct two other step functions

$$s'(x) = \begin{cases} s(x), & s(x) \geqslant 0, \\ -t(x), & t(x) \leqslant 0, \end{cases} \quad \text{and} \quad t'(x) = \begin{cases} t(x), & s(x) \geqslant 0, \\ -s(x), & t(x) \leqslant 0. \end{cases}$$

   It is easy to verify that $s' \leqslant |f| \leqslant t'$ and

$$\int_0^1 t' - \int_0^1 s' = \int_0^1 t - \int_0^1 s < \epsilon.$$

   This means that $|f|$ also satisfies the Riemann condition and is thus integrable. □

4. The well-ordering principle states that every non-empty subset of the natural numbers has a least element. Prove the well-ordering principle implies the principle of mathematical induction.

   Proof. Let T be the set of all positive integers not in S. Suppose T is nonempty. By the well-ordering principle, T has a smallest member. Let m denote such a member. The fact that $m \notin S$ implies $m \neq 1$. It cannot be the case that $m < 1$ since 1 is the smallest positive integer. Thus, $m > 1$. Then $m - 1$ is another integer since $\mathbf{Z}$ is closed under subtraction. And it is positive since $m > 1$ implies $m - 1 > 0$. It cannot be in S. If so, then m will be in S, which contradicts our definition of T. This means $m - 1$ is another positive integer not in S and is thus in T. Notice that $m - 1$ is smaller than m. This contradicts the well-ordering principle. Thus, T must be empty. This is to say that S contains all positive integers. $\square$

5. Suppose $\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x) = A$. Prove $\lim_{x\to p} f(x) = A$.

   Proof. For any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - A| < \epsilon \quad \text{whenener} \quad 0 < x - p < \delta_1$$

   and

$$|f(x) - A| < \epsilon \quad \text{whenener} \quad 0 < p - x < \delta_2.$$

   Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|f(x) - A| < \epsilon \quad \text{whenener} \quad 0 < |x - p| < \delta. \qquad \square$$