§3. 分部积分法

若$u(x)$, $v(x)$连续可导, 则$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$, 就有

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

或者

$$\underbrace{\int u(x)\,dv(x)}_{\text{难}} = u(x)v(x) - \underbrace{\int v(x)\,du(x)}_{\text{易}}.$$

这种方法适用于函数本身比较难, 但是其导函数比较简单. 这样的函数$u(x)$一般有

$$\begin{array}{cccl} \ln x, & \arctan x, & \arcsin x & \text{函数复杂, 导数简单,} \\ e^x, & \sin x, & \cos x & \text{函数导数, 难度相同.} \end{array}$$

例1: $\int \ln x\,dx.$

$$\begin{aligned} \int \underbrace{\ln x}_{u(x)}\, \underbrace{dx}_{dv(x)} &= x\ln x - \int \underbrace{x}_{v(x)}\, \underbrace{d\ln x}_{du(x)} && \langle\text{分部积分法}\rangle \\ &= x\ln x - \int dx && \langle d\ln x = \frac{1}{x}\,dx\rangle \\ &= x\ln x - x + C. && \langle dx = dx\rangle \end{aligned}$$

例2: $\int x\arctan x\,dx.$

$$\begin{aligned} \int x\,\underbrace{\arctan x}_{u(x)}\,dx &= \int \arctan x\, d\frac{x^2}{2} && \langle d\frac{x^2}{2} = x\,dx\rangle \\ &= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\,d(\arctan x) && \langle\text{分部积分法}\rangle \\ &= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2(1+x^2)}\,dx && \langle d(\arctan x) = \frac{1}{1+x^2}\rangle \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)\,dx && \langle \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}\rangle \\ &= \frac{1}{2}\left(x^2\arctan x - \int dx + \int \frac{dx}{1+x^2}\right) && \langle\text{积分的加法法则}\rangle \\ &= \frac{1}{2}[(x^2+1)\arctan x - x] + C. && \langle dx = dx,\ d(\arctan x) = \frac{1}{1+x^2}\rangle \end{aligned}$$

例3: $\int x^2 e^x\,dx.$

$$\begin{aligned} \text{原式} &= \int x^2\,de^x \\ &= x^2e^x - \int e^x\,d(x^2) && \langle\text{分部积分法}\rangle \\ &= x^2e^x - 2\int x\,de^x \\ &= x^2e^x - 2\left(xe^x - \int e^x\,dx\right) && \langle\text{分部积分法}\rangle \\ &= (x^2 - 2x + 2)e^x + C. \end{aligned}$$

例4： $\int x\sin(2x)\,dx.$

$$\begin{aligned}
\text{原式} &= \frac{1}{2}\int x\,d[-\cos(2x)] \\
&= \frac{1}{2}\left(\int \cos(2x)\,dx - x\cos(2x)\right) && \langle\text{分部积分法}\rangle \\
&= \frac{1}{2}\left(\frac{1}{2}\sin(2x) - x\cos(2x)\right) + C.
\end{aligned}$$

例5： $\int e^x \sin x\,dx.$

$$\begin{aligned}
-\int e^x\,d\cos x = \text{原式} &= \int \sin x\,de^x \\
-e^x\cos x + \int \cos x\,de^x = \text{原式} &= e^x\sin x - \int e^x\,d\sin x && \langle\text{分部积分法}\rangle
\end{aligned}$$

因为$\int \cos x\,de^x = \int e^x\cos x\,dx = \int e^x\,d\sin x$, 所以

$$\begin{aligned}
\int e^x\cos x\,dx &= \frac{\sin x + \cos x}{2}e^x + C \\
\text{原式} &= e^x\sin x - \frac{\sin x + \cos x}{2}e^x + C \\
&= \frac{\sin x - \cos x}{2}e^x + C.
\end{aligned}$$