Solutions to Problem Set 1

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This is my own solution to Problem Set 1 of 18.014. The first three problems are from Apostol's *Calculus* and the next three problems are from Munkres's Course Note A (1: 18, 20, 43; 9–10).

1. Prove Theorem I.11: If $ab = 0$, then $a = 0$ or $b = 0$.

   Proof. If not, then $a \neq 0$ and $b \neq 0$, which implies

$$\begin{aligned}
1 &= 1 \cdot 1 && \text{by the identity axiom,} \\
&= (aa^{-1})(bb^{-1}) && \text{by the inverse axiom,} \\
&= (a^{-1}a)(bb^{-1}) && \text{by commutativity,} \\
&= a(a(bb^{-1})) && \text{by associativity,} \\
&= a((ab)b^{-1}) && \text{by associativity,} \\
&= a(0 \cdot b^{-1}) && \text{by assumption,} \\
&= a \cdot 0 && \text{by Theorem I.6,} \\
&= 0 && \text{by Theorem I.6.}
\end{aligned}$$

   But we know by Axiom 4 that $1 \neq 0$. Thus, either $a = 0$ or $b = 0$. $\square$

2. Prove Theorem I.25: If $a < c$ and $b < d$, then $a + b < c + d$.

   Proof. We have

$$\begin{aligned}
a + b &< c + b && \text{by Theorem I.18,} \\
&= b + c && \text{by commutativity,} \\
&< d + c && \text{by Theorem I.18,} \\
&= c + d && \text{by commutativity.}
\end{aligned}$$

   $\square$

3. Prove that $||x| - |y|| \leqslant |x - y|$.

   There are two ways to prove this. One is logical and the other is algebraic.

   Proof. If $|x| - |y| \geqslant 0$, then
$$||x| - |y|| = |x| - |y| \leqslant |x - y|.$$
   If $|x| - |y| < 0$, then
$$||x| - |y|| = -(|x| - |y|) = |y| - |x| \leqslant |y - x| = |x - y|.$$
   In all cases, the inequality holds. $\square$

   Proof. We have

$$\begin{aligned}
xy \leqslant |xy| = |x||y| & \iff \\
-2|x||y| \leqslant -2xy & \iff \\
|x|^2 - 2|x||y| + |y|^2 \leqslant x^2 - 2xy + y^2 & \iff \\
(|x| - |y|)^2 \leqslant (x - y)^2 & \iff \\
\sqrt{(|x| - |y|)^2} \leqslant \sqrt{(x - y)^2} & \iff \\
||x| - |y|| \leqslant |x - y|. &
\end{aligned}$$

   $\square$

4. Prove Theorem 6: If a and b are in $\mathbf{P}$, so is ab.

   Proof. We choose a fixed positive integer a and then prove this theorem by induction on b. First, $a \cdot 1$ is in $\mathbf{P}$, since $a \cdot 1 = a$ by the identity axiom and a is in $\mathbf{P}$. Now suppose b is a positive integer such that ab is in $\mathbf{P}$. Then we have $a(b+1) = ab + a$ by distributivity. By the induction hypothesis, ab is a positive integer. And a is also a positive integer. We have already shown that $\mathbf{P}$ is closed under addition. Thus, $ab + a$ is also a positive integer, which means $a(b+1)$ is in $\mathbf{P}$. This proves that $\mathbf{P}$ is also closed under multiplication. □

5. Prove Theorem 12: $a^n b^n = (ab)^n$, where a and b are any real numbers and n is a positive integer.

   Proof. The formula is trivially true for $n = 1$, since $a^1 b^1 = ab = (ab)^n$. Now suppose the formula holds for some n. Then

$$\begin{aligned}
a^{n+1}b^{n+1} &= (a^n a)(b^n b) && \text{by definition of exponent,}\\
&= (a^n b^n)(ab) && \text{by associativity and commutativity,}\\
&= (ab)^n(ab) && \text{by the induction hypothesis,}\\
&= (ab)^{n+1} && \text{by definition of exponent.}
\end{aligned}$$

   This proves the formula holds for any positive integer n. □

6. Let a and h be real numbers; let m be a positive integer. Show by induction that if a and $a+h$ are positive, then

$$(a+h)^m \geqslant a^m + ma^{m-1}h.$$

   Proof. The inequality is true for $m = 1$, since $(a+h)^1 = a + h = a^1 + 1 \cdot a^{1-1}h$. Now suppose it is true for some $m \geqslant 1$, then

$$\begin{aligned}
(a+h)^{m+1} &= (a+h)^m(a+h) && \text{by definition of exponent,}\\
&\geqslant (a^m + ma^{m-1}h)(a+h) && \text{by the induction hypothesis}\\
& && \text{and positivity of } a+h,\\
&= a^{m+1} + ma^m h + a^m h + ma^{m-1}h^2 && \text{by distributivity, associativity,}\\
& && \text{commutativity, and definition of exponent,}\\
&= a^{m+1} + (m+1)a^m h + ma^{m-1}h^2 && \text{by associativity, the identity axiom,}\\
& && \text{and distributivity,}\\
&\geqslant a^{m+1} + (m+1)a^m h && \text{by positivity of } m, a, \text{ and } h^2,\\
& && \text{the closure axiom, and Theorem I.18.}
\end{aligned}$$

□

7.* Let

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{and} \quad G_n = (x_1 x_2 \cdots x_n)^{1/n}$$

   represent the arithmetic and geometric mean respectively for a set of n positive real numbers.

   (a) Prove that $G_n \leqslant A_n$ for $n = 2$.

   Proof. Let a and b be any arbitrary real numbers. Then

$$\begin{aligned}
(a-b)^2 &\geqslant 0 && \text{by Theorem I.20,} && \iff\\
a^2 + b^2 &\geqslant 2ab && \text{by simple algebra.}
\end{aligned}$$

   By Theorem I.35, $x_1^{1/2}$ and $x_2^{1/2}$ are meaningful. Let $a = x_1^{1/2}$ and $b = x_2^{1/2}$. Then

$$\begin{aligned}
\frac{x_1 + x_2}{2} &\geqslant (x_1 x_2)^{1/2} && \text{by simple algebra,} && \iff\\
G_2 &\leqslant A_2 && \text{by definition.}
\end{aligned}$$

□

*Only to be attempted once other problems are completed

(b) Use induction to show $G_n \leqslant A_n$ for any $n = 2^k$ where k is a positive integer.

Proof. The previous part proves that the inequality holds for $k = 1$. Suppose the inequality holds for some $k \geqslant 1$. By the induction hypothesis, we have

$$\frac{1}{2^k}\sum_{i=1}^{2^k} x_i \geqslant \left(\prod_{i=1}^{2^k} x_i\right)^{1/2^k} \tag{1}$$

and

$$\frac{1}{2^k}\sum_{i=2^k+1}^{2^{k+1}} x_i \geqslant \left(\prod_{i=2^k+1}^{2^{k+1}} x_i\right)^{1/2^k}. \tag{2}$$

By adding (1) and (2) and halving the sum, we obtain

$$\frac{1}{2^{k+1}}\sum_{i=1}^{2^{k+1}} x_i \geqslant \frac{1}{2}\left\{\left(\prod_{i=1}^{2^k} x_i\right)^{1/2^k} + \left(\prod_{i=2^k+1}^{2^{k+1}} x_i\right)^{1/2^k}\right\} \geqslant \left(\prod_{i=1}^{2^{k+1}} x_i\right)^{1/2^{k+1}}.$$

The last inequality sign holds by applying the induction hypothesis to the right-hand sides of both (1) and (2). □

(c) Now for any positive integer n, suppose $n < 2^m$ for some integer m. Using the set

$$\{x_1, x_2, \dots, x_n, A_n, A_n, \dots, A_n\}$$

where the A_n appears $2^m - n$ times in the set, show that $G_n \leqslant A_n$.

Proof. Apply to the set the result from the previous problem and we have

$$\begin{aligned} A_n &= \frac{1}{2^m}[(2^m - n)A_n + nA_n] \\ &= \frac{1}{2^m}\left\{(2^m - n)A_n + \sum_{i=1}^{n} x_i\right\} \\ &\geqslant \left\{\left(\prod_{i=1}^{2^m-n} A_n\right)\left(\prod_{i=1}^{n} x_i\right)\right\}^{1/2^m} \\ &= \left(A_n^{2^m-n} G_n^n\right)^{1/2^m} \\ &= \left(\frac{A_n}{G_n}\right)^{\frac{2^m-n}{2^m}} G_n. \end{aligned}$$

Now suppose $A_n < G_n$, then we have

$$\frac{A_n}{G_n} < \left(\frac{A_n}{G_n}\right)^{\frac{2^m-n}{2^m}} < 1,$$

which implies

$$A_n = \frac{A_n}{G_n} \cdot G_n < \left(\frac{A_n}{G_n}\right)^{\frac{2^m-n}{2^m}} G_n \leqslant A_n.$$

This is absurd. Thus, it must be the case that $G_n \leqslant A_n$, which is consistent with the above result. □