Problem 1

Question: $\int \dfrac{5+x}{x^2+x-6}\,dx.$

Answer: $\dfrac{7}{5}\ln|x-2| - \dfrac{2}{5}\ln|x+3| + C.$

$$\int \frac{5+x}{x^2+x-6}\,dx = \int \frac{5+x}{(x-2)(x+3)}\,dx$$
$$= \int \left(\frac{A}{x-2} + \frac{B}{x+3} \right)\,dx$$
$$\Downarrow$$
$$A(x+3) + B(x-2) = 5+x$$
$$\Downarrow \text{ direct method}$$
$$A = \frac{7}{5} \quad B = -\frac{2}{5}$$
$$\Downarrow$$
$$\int \frac{5+x}{x^2+x-6}\,dx = \frac{1}{5}\int \left(\frac{7}{x-2} - \frac{2}{x+3} \right)\,dx$$
$$= \frac{1}{5}\left(7\int \frac{dx}{x-2} - 2\int \frac{dx}{x+3} \right)$$
$$= \frac{1}{5}\left(7\ln|x-2| - 2\ln|x+3| + C \right)$$
$$= \frac{7}{5}\ln|x-2| - \frac{2}{5}\ln|x+3| + C.$$

Problem 2

Question: $\int \dfrac{2x+3}{6x^2+5x+1}\,dx.$

Answer: $\dfrac{7}{3}\ln|3x+1| - 2\ln|2x+1| + C.$

$$\int \frac{2x+3}{6x^2+5x+1}\,dx = \int \frac{2x+3}{(2x+1)(3x+1)}\,dx$$
$$= \int \left(\frac{A}{2x+1} + \frac{B}{3x+1} \right)\,dx$$
$$\Downarrow$$
$$A(3x+1) + B(2x+1) = 2x+3$$
$$\Downarrow \text{ direct method}$$
$$A = -4 \quad B = 7$$
$$\Downarrow$$
$$\int \frac{2x+3}{6x^2+5x+1}\,dx = 7\int \frac{dx}{3x+1} - 4\int \frac{dx}{2x+1}$$
$$= \frac{7}{3}\ln|3x+1| - 2\ln|2x+1| + C.$$

Problem 3

Question: $\int \dfrac{x^2-x+5}{(x-2)(x-1)(x+3)}\,dx.$

Answer: $\dfrac{7}{5}\ln|x-2| - \dfrac{5}{4}\ln|x-1| + \dfrac{17}{20}\ln|x+3| + C.$

$$\int \frac{x^2 - x + 5}{(x-2)(x-1)(x+3)} \, dx = \int \left(\frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x+3} \right) dx$$

$$\Downarrow$$

$$A(x-1)(x+3) + B(x-2)(x+3) + C(x-2)(x-1) = x^2 - x + 5$$

$$\Downarrow \text{ direct method}$$

$$A = \frac{7}{5} \quad B = -\frac{5}{4} \quad C = \frac{17}{20}$$

$$\Downarrow$$

$$\int \frac{x^2 - x + 5}{(x-2)(x-1)(x+3)} \, dx = \frac{7}{5} \int \frac{dx}{x-2} - \frac{5}{4} \int \frac{dx}{x-1} + \frac{17}{20} \int \frac{dx}{x+3}$$

$$= \frac{7}{5} \ln|x-2| - \frac{5}{4} \ln|x-1| + \frac{17}{20} \ln|x+3| + C.$$

Problem 4

Question: $\int \dfrac{2x-1}{x^3-x} \, dx$.

Answer: $\ln|x| + \dfrac{1}{2} \ln|x-1| - \dfrac{3}{2} \ln|x+1| + C$.

$$\int \frac{2x-1}{x^3-x} \, dx = \int \frac{2x-1}{x(x-1)(x+1)} \, dx$$

$$= \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \right) dx$$

$$\Downarrow$$

$$A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 2x - 1$$

$$\Downarrow \text{ direct method}$$

$$A = 1 \quad B = \frac{1}{2} \quad C = -\frac{3}{2}$$

$$\Downarrow$$

$$\int \frac{2x-1}{x^3-x} \, dx = \ln|x| + \frac{1}{2} \ln|x-1| - \frac{3}{2} \ln|x+1| + C.$$

Problem 5

Question: $\int \dfrac{x^2-3}{x^2-4} \, dx$.

Hint: start by performing long division of the numerator by the denominator.

Answer: $x + \dfrac{1}{4} \ln \left| \dfrac{x-2}{x+2} \right| + C$.

$$\int \frac{x^2-3}{x^2-4} \, dx = \int \left(1 + \frac{1}{4(x-2)} - \frac{1}{4(x+2)} \right) dx$$

$$= x + \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$$

$$= x + \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C.$$

Problem 6

Question: $\displaystyle\int \frac{dx}{x^2 - 4x + 8}$.

Answer: $\displaystyle\frac{1}{2}\arctan\frac{x-2}{2} + C$.

$$
\begin{aligned}
\int \frac{dx}{x^2 - 4x + 8} &= \int \frac{dx}{(x-2)^2 + 4} \\
&= \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} && \langle\,\text{Let } x = 2\tan\theta + 2\,\rangle \\
&= \frac{1}{2}\theta + C \\
&= \frac{1}{2}\arctan\frac{x-2}{2} + C. && \langle\,\text{substitute back with } \theta = \arctan\frac{x-2}{2}\,\rangle
\end{aligned}
$$