Problem 1

Question: $\dfrac{d}{dx}\displaystyle\int_{t=0}^{\arcsin x} \ln|\sin t + \cos t|\ dt.$

Answer: $\dfrac{1}{\sqrt{1-x^2}} \ln\left|x + \sqrt{1-x^2}\right|.$

$$
\begin{aligned}
\frac{d}{dx}\int_{t=0}^{\arcsin x} \ln|\sin t + \cos t|\ dt &= \frac{d}{dx}\int_{t=0}^{u} \ln|\sin t + \cos t|\ dt && \langle\, u = \arcsin x \,\rangle \\
&= \ln|\sin u + \cos u| \cdot \frac{du}{dx} && \langle\, \tfrac{d}{dx}\textstyle\int_{t=a}^{u(x)} f(t)\,dt = f(u(x))\frac{du(x)}{dx} \,\rangle \\
&= \frac{\ln|x + \cos(\arcsin x)|}{\sqrt{1-x^2}} && \langle\, u = \arcsin x \text{ and evaluate } \tfrac{du}{dx} \,\rangle \\
&= \frac{\ln|x + \sqrt{1-x^2}|}{\sqrt{1-x^2}} && \langle\, \cos(\arcsin x) = \sqrt{1-x^2} \,\rangle
\end{aligned}
$$

Problem 2

Question: $\dfrac{d}{dx}\displaystyle\int_{t=\sin x}^{\tan x} e^{-t^2}\ dt.$

Answer: $e^{-\tan^2 x} \cdot \sec^2 x - e^{-\sin^2 x} \cdot \cos x.$

$$
\begin{aligned}
\frac{d}{dx}\int_{t=\sin x}^{\tan x} e^{-t^2}\ dt &= \frac{d}{dx}\left(\int_{t=0}^{\tan x} e^{-t^2}\ dt - \int_{t=0}^{\sin x} e^{-t^2}\ dt\right) && \langle\, \text{additivity} \,\rangle \\
&= \frac{d}{dx}\int_{t=0}^{\tan x} e^{-t^2}\ dt - \frac{d}{dx}\int_{t=0}^{\sin x} e^{-t^2}\ dt && \langle\, \text{linearity} \,\rangle \\
&= e^{-\tan^2 x} \cdot \sec^2 x - e^{-\sin^2 x} \cdot \cos x. && \langle\, \tfrac{d}{dx}\textstyle\int_{t=a}^{u(x)} f(t)\,dt = f(u(x))\frac{du(x)}{dx} \,\rangle
\end{aligned}
$$

Problem 3

Question: Which of the following is the leading order term in the Taylor series about $x = 0$ of

$$f(x) = \int_{t=0}^{x} \ln(\cosh t)\, dt$$

Hint: yes, there's more than one way to do this problem... Try using the F.T.I.C. to compute the derivatives.

Answer: $\dfrac{x^3}{3!} = \dfrac{x^3}{6}.$

The constant term of the Taylor series must be zero since $f(0) = \int_{t=0}^{0} x\ln(\cosh t)\, dt$. So if the first derivative at $x = 0$ is nonzero, then $f'(0) \cdot x$ must be the leading order term. And

$$\frac{d}{dx} f(x) = \frac{d}{dx}\int_{t=0}^{x} \ln(\cosh t)\, dt = \ln(\cosh x)\,,$$

so $f'(0) = \ln(\cosh 0) = \ln 1 = 0$. Oops, we have to take the second derivative at $x = 0$,

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\ln(\cosh x) = \frac{\sinh x}{\cosh x}\,,$$

so $f''(0) = \sinh 0/\cosh 0 = 0$. Oops again, let's take the third derivative,

$$\frac{d^3}{dx^3} f(x) = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{1}{\cosh^2 x}\,,$$

so $f'''(0) = 1$. Thus, the leading order term is

$$\frac{x^3}{3!} = \frac{x^3}{6}.$$

Problem 4

Question: We usually use Riemann sums to approximate integrals, but we can go the other way, too, using an antiderivative to approximate a sum. Using only your head (no paper, no calculator), tell me which of the following is the best estimate for

$$\sum_{n=0}^{100} n^3.$$

Answer: 2.5×10^7.

$$\begin{aligned}\sum_{n=0}^{100} n^3 &\approx \int_0^{100} x^3\,dx \\ &= 100^4/4 \\ &= 2.5 \times 10^7\end{aligned}$$