Solutions to Problem Set 2

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This is my own solution to Problem Set 2 of 18.014. The first four problems are from Apostol's *Calculus* (1: 57, 60, 70).

1. (a) Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be either a zero polynomial or a polynomial of degree at most n. If $f(x) = 0$ for $n+1$ distinct real values of x, then f is a zero polynomial.

   Proof. Let x_k denote the zeros for f. The statement clearly holds for $n = 0$. If f is of degree 0, then $f(x_0) = c_0 = 0$, which contradicts the assumption $c_0 \neq 0$. Thus, f must be a zero polynomial. Suppose the statement holds for some $n \geqslant 0$. We are going to show that the statement also holds for $n+1$. Assume that f is either a zero polynomial or a polynomial of degree at most $n+1$ and have $n+2$ distinct zeros. The polynomial f can be factored into $(x - x_0)\,h(x)$, where h is either a zero polynomial or a polynomial of degree at most n. Our induction basis implies the degree of f cannot be 0. If the degree of f is at least one, then this is so by the result we proved in Recitation 5. If f is a zero polynomial, then $f(x) = 0 = (x - x_0) \cdot 0$. In both cases, the stated decomposition holds. Other than x_0, f still has $n+1$ distinct zeros. By Theorem I.11, h must have $n+1$ distinct zeros. By our induction hypothesis, h is a zero polynomial. This means f is also a zero polynomial. □

   An alternative view, which is less wordy, to look at the problem is as follows.

   Proof. Suppose the degree of the polynomial f is well-defined. Then let m denote the degree of f. By the result from Recitation 5, we have

$$
\begin{aligned}
f(x) &= h_0(x) \\
&= (x - x_0)\,h_1(x) \\
&= (x - x_0)(x - x_1)\,h_2(x) \\
&= \left\{ \prod_{k=0}^{m-1} (x - x_k) \right\} h_m(x),
\end{aligned}
$$

   where each h_i is a polynomial of degree $m - i$. Since $0 \leqslant m \leqslant n$ and f has $n+1$ distinct zeros, there is at least one x_m different from $x_0, x_1, \ldots, x_{m-1}$ such that

$$
f(x_m) = \left\{ \prod_{k=0}^{m-1} (x_m - x_k) \right\} h_m(x_m) = 0.
$$

   The product in the braces is not zero, then by Theorem I.11 $h_m(x_m) = 0$. This is impossible since h_m is of degree 0. Thus, f must be a zero polynomial. □

   (b) Let $f(x) = \sum_{k=0}^{n} c_k x^k$ and $g(x) = \sum_{k=0}^{m} b_k x^k$ be polynomials of degree n and m respectively and $m \geqslant n$. Prove that if $g(x) = f(x)$ for $m+1$ distinct real values of x, then $m = n$, $b_k = c_k$ for each k, and $g(x) = f(x)$ for all real x.

   Proof. Let $h(x) = g(x) - f(x)$. Then it is easy to see that h is either a zero polynomial or a polynomial of degree at most m. The polynomial h has $m+1$ distinct zeros since $g(x) = f(x)$ for

$m+1$ distinct real values of x. By the result from the previous part, h is thus a zero polynomial. We have

$$\begin{aligned} h(x) &= g(x) - f(x) \\ &= \sum_{k=0}^{m} b_k x^k - \sum_{k=0}^{n} c_k x^k \\ &= \sum_{k=0}^{n} b_k x^k + \sum_{k=n+1}^{m} b_k x^k - \sum_{k=0}^{n} c_k x^k \\ &= \sum_{k=n+1}^{m} b_k x^k + \sum_{k=0}^{n} (b_k x^k - c_k x^k) \\ &= \sum_{k=n+1}^{m} b_k x^k + \sum_{k=0}^{n} (b_k - c_k) x^k \\ &= 0. \end{aligned}$$

If $m > n$, then the above won't hold since $b_m \neq 0$. Thus, $m = n$ and $b_k - c_k = 0$ for each k. □

2. Let $A = \{1, 2, 3, 4, 5\}$, and let $\mathcal{M}$ denote the class of all subsets of A. (There are 32 altogether, counting A itself and the empty set $\emptyset$.) For each set S in $\mathcal{M}$, let $n(S)$ denote the number of distinct elements in S. If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that the set function n satisfies the first three axioms for area.

We have $S \cup T = \{1, 2, 3, 4, 5\}$, $S \cap T = \{3, 4\}$, $S - T = \{1, 2\}$, and $T - S = \{5\}$. Thus, $n(S \cup T) = 5$, $n(S \cap T) = 2$, $n(S - T) = 2$ and $n(T - S) = 1$.

Proof. The empty set $\emptyset$ has the least number of distinct elements, namely 0. Any nonempty set S has at least one element and hence $n(S) > 0$. Thus, $n(S) \geqslant 0$ for each set $S \in \mathcal{M}$. This proves the nonnegative property.

Now suppose S and T are both in $\mathcal{M}$. For any $x \in S \cup T$, either $x \in S$ or $x \in T$. If $x \in S$, then $x \in A$ since $S \subseteq A$. Similarly, if $x \in T$, then $x \in A$ since $T \subseteq A$. Thus, $x \in A$. This means $S \cup T$ is a subset of A and thus is in $\mathcal{M}$. In the same fashion, we can show that $S \cap T$ is also in $\mathcal{M}$. It is easy to see that the sets $S \cap T$, $S - T$, $T - S$ are disjoint from each other and

$$\begin{aligned} S &= (S - T) \cup (S \cap T), \\ T &= (T - S) \cup (S \cap T), \\ S \cup T &= (S - T) \cup (S \cap T) \cup (T - S). \end{aligned}$$

Thus, we have

$$\begin{aligned} n(S) &= n(S - T) + n(S \cap T), && (1) \\ n(T) &= n(T - S) + n(S \cap T), && (2) \\ n(S \cup T) &= n(S - T) + n(S \cap T) + n(T - S). && (3) \end{aligned}$$

Substitute (1) and (2) into (3) and we obtain

$$n(S \cup T) = n(S) + n(T) - n(S \cap T).$$

This proves the additive property.

With additional condition $S \subseteq T$, we have $S \cup T = T$, $S \cap T = S$, and $S - T = \emptyset$. Substitute into (3) and we get

$$\begin{aligned} n(T) &= n(\emptyset) + n(S) + n(T - S), \text{ or} \\ n(T - S) &= n(T) - n(S). \end{aligned}$$

This proves the difference property. □

3. (a) Compute $\int_0^9 [\sqrt{t}]\, dt$.

   We can find a partition $\{0, 1, 4, 9\}$ for the step function $[\sqrt{t}]$ and then

$$\int_0^9 [\sqrt{t}]\, dt = 0 \times 1 + 1 \times (4 - 1) + 2 \times (9 - 4) = 13.$$

(b) If n is a positive integer, prove that $\int_0^{n^2} [\sqrt{t}]\, dt = n(n-1)(4n+1)/6$.

Proof. We can find a partition $\{0, 1, 4, \dots, n^2\}$ for the step function $[\sqrt{t}]$ and then

$$\begin{aligned}
\int_0^{n^2} [\sqrt{t}]\, dt &= \sum_{k=1}^{n} (k-1)\{k^2 - (k-1)^2\} \\
&= \sum_{k=1}^{n} (k-1)(2k-1) \\
&= \sum_{k=1}^{n} (2k^2 - 3k + 1) \\
&= 2\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= \frac{n(n+1)(2n+1)}{3} - \frac{3n(n+1)}{2} + n \\
&= \frac{n(n-1)(4n+1)}{6}.
\end{aligned}$$

□

4. If, instead of defining integrals of step functions by using formula (1.3), we used the definition

$$\int_a^b s(x)\, dx = \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}),$$

a new and different theory of integration would result. Which of the following properties would remain valid in this new theory?

(a) $\displaystyle\int_a^b s + \int_b^c s = \int_a^c s.$
This is valid.

Proof. Let $P_1 = \{x_0, x_1, \dots, x_n\}$ be a partition for s on $[a, b]$ and $P_2 = \{x_0', x_1', \dots, x_m'\}$ be a partition for s on $[b, c]$. Then $s(x) = s_k$ on every open interval (x_{k-1}, x_k) and $s(x) = s_j'$ on every open interval (x_{j-1}', x_j'), where $0 \leqslant k \leqslant n$, $0 \leqslant j \leqslant m$, and s_k and s_j' are all constants. Notice that $x_n = x_0' = b$. We now can form a new partition $P = \{x_0, x_1, \dots, x_n, x_1', \dots, x_m'\}$. We set $x_k = x_{k-n}'$ and $s_k = s_{k-n}'$ when $k > n$. Then

$$\begin{aligned}
\int_a^b s + \int_b^c s &= \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}) + \sum_{j=1}^{m} s_j'^3 (x_j' - x_{j-1}') \\
&= \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}) + \sum_{k=n+1}^{n+m} s_k^3 (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n+m} s_k^3 (x_k - x_{k-1}) \\
&= \int_a^c s.
\end{aligned}$$

□

(b) $\displaystyle\int_a^b c \cdot s = c \int_a^b s.$
This is invalid. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition for s on $[a, b]$ and $s(x) = s_k$ on every open interval (x_k, x_{k-1}). Then

$$\int_a^b c \cdot s = \sum_{k=1}^{n} (cs_k)^3 (x_k - x_{k-1}) = \sum_{k=1}^{n} c^3 s_k^3 (x_k - x_{k-1}) \neq c \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}) = c \int_a^b s.$$

However, we do have

$$\int_a^b c \cdot s = c^3 \int_a^b s.$$

5. Prove, using properties of the integral, that for $a, b > 0$

$$\int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}.$$

Define a function $f(w) = \int_1^w \frac{1}{x}\,dx$, for $w \in \mathbf{R}^+$. Rewrite the equation above in terms of the function f. Give an example of a function that has the same property as the one displayed here by f.

Proof. We have

$$\begin{aligned}
\int_1^b \frac{dx}{x} &= \frac{1}{a}\int_a^{ab} \frac{dx}{x/a} && \text{by Theorem 1.19,} \\
&= \int_a^{ab} \frac{dx}{x} && \text{by homogeneity and cancellation,} \\
&= \int_a^1 \frac{dx}{x} + \int_1^{ab} \frac{dx}{x} && \text{by Theorem 1.17,} \\
&= -\int_1^a \frac{dx}{x} + \int_1^{ab} \frac{dx}{x} && \text{by our convention.}
\end{aligned}$$

Move the first term of the RHS to the LHS and we obtain the equation. □

We can rewrite the equation in terms of the function f as

$$f(a) + f(b) = f(ab).$$

The logarithm functions observe the stated property

$$\log a + \log b = \log ab.$$

6. Suppose we define $\int_a^b s(x)\,dx = \sum s_k (x_{k-1} - x_k)^2$ for a step function $s(x)$ with partition $P = \{x_0, x_1, \dots, x_n\}$. Is this integral well-defined? That is, will the value of the integral be independent of the choice of partition? (If well-defined, prove it. If not well-defined, provide a counterexample.)

Clearly, this integral is not well-defined since the factor $(x_{k-1} - x_k)^2$ does not observe telescoping property under refinement. A counterexample will be as follows. Let us define $s(x) = 1$ on $[0, 2]$. Then both $P_1 = \{0, 2\}$ and $P_2 = \{0, 1, 2\}$ are partitions for s, with P_2 being a refinement of P_1. However, we have

$$\int_0^2 s = (2-0)^2 = 4 \neq 2 = (1-0)^2 + (2-1)^2 = \int_0^2 s.$$

7.* Define the function (where n is in the positive integers)

$$f(x) = \begin{cases} x, & x = \frac{1}{n^2}, \\ 0, & x \neq \frac{1}{n^2}. \end{cases}$$

Prove that f is integrable on $[0, 1]$ and that $\int_0^1 f(x)\,dx = 0$.

Proof. For any $\epsilon > 0$, by the Archimedean property of real numbers we can always find a positive integer n_0 such that $1/n_0^2 < \epsilon$. Then we can define two step functions s and t on $[0, 1]$ as follows. Let

$$s(x) = \begin{cases} x, & x = \frac{1}{n^2} \text{ and } x > \frac{1}{n_0^2}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad t(x) = \begin{cases} x, & x = \frac{1}{n^2} \text{ and } x > \frac{1}{n_0^2}, \\ \frac{1}{n_0^2}, & 0 \leqslant x \leqslant \frac{1}{n_0^2}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$\int_0^1 s = 0 \quad \text{and} \quad \int_0^1 t = \frac{1}{n_0^4}.$$

Therefore, we have

$$\int_0^1 t - \int_0^1 s = \frac{1}{n_0^4} \leqslant \frac{1}{n_0^2} < \epsilon$$

and

$$\int_0^1 s \leqslant 0 \leqslant \int_0^1 t.$$

By Theorem 2 from Course Note C, the function f is integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$. $\square$