Solutions to Problem Set 3

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This is my own solution to Problem Set 3 of 18.014. The first two problems are from Apostol's *Calculus* (1: 83, 94).

1. Find all values of c for which

   (a) $\int_0^c x(1-x)\,dx = 0$,

   (b) $\int_0^c |x(1-x)|\,dx = 0$.

   (a) We have

   $$\int_0^c x(1-x)\,dx = \int_0^c (x-x^2)\,dx = \frac{c^2}{2} - \frac{c^3}{3} = c^2\left(\frac{1}{2} - \frac{c}{3}\right) = 0.$$

   Solve the above equation and we obtain either $c = 0$ or $c = 3/2$.

   (b) We have

   $$\begin{aligned}\int_0^c |x(1-x)|\,dx &= \begin{cases} \int_0^c x(1-x)\,dx, & 0 \leqslant c \leqslant 1, \\ \int_0^1 x(1-x)\,dx + \int_1^c x(x-1)\,dx, & c > 1, \\ \int_0^c x(x-1)\,dx, & c < 0, \end{cases} \\ &= \begin{cases} c^2\left(\frac{1}{2} - \frac{c}{3}\right), & 0 \leqslant c \leqslant 1, \\ \frac{1}{6} + \frac{c^3-1}{3} - \frac{c^2-1}{2}, & c > 1 \\ c^2\left(\frac{c}{3} - \frac{1}{2}\right), & c < 0, \end{cases} \\ &= 0.\end{aligned}$$

   The only solution for the above equation is $c = 0$.

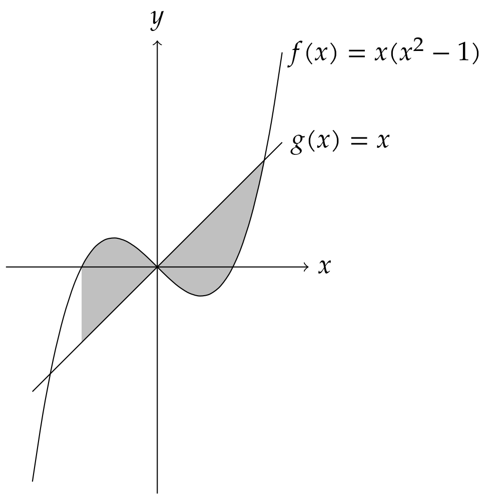
   Another way to solve the problem is to look at the geometrical interpretation of this integral.

2. Let $f(x) = x(x^2-1)$, $g(x) = x$, $a = -1$, and $b = \sqrt{2}$. Compute the area of the region S between the graphs of f and g over the interval $[a, b]$. Make a sketch of the two graphs and indicate S by shading.

   We have

   $$\begin{aligned} a(S) &= \int_{-1}^0 (f(x) - g(x))\,dx + \int_0^{\sqrt{2}} (g(x) - f(x))\,dx \\ &= \int_{-1}^0 (x^3 - 2x)\,dx + \int_0^{\sqrt{2}} (2x - x^3)\,dx \\ &= \int_0^1 (2x - x^3)\,dx + \int_0^{\sqrt{2}} (2x - x^3)\,dx \\ &= 1 - \frac{1}{4} + 2 - 1 = \frac{7}{4}. \end{aligned}$$

   The sketch is as follows.



3. For step functions s and t defined on $[a, b]$, prove the Cauchy-Schwarz inequality

$$\left(\int_a^b s \cdot t\right)^2 \leqslant \int_a^b s^2 \cdot \int_a^b t^2.$$

Show that the equality if and only if $s = ct$ for some $c \in \mathbf{R}$.

Proof. Let P_1 and P_2 be partitions for s and t respectively. Let $P = \{x_0, x_1, \dots, x_n\}$ be the common refinement of P_1 and P_2. Denote $s(x) = s_k$ and $t(x) = t_k$ for all x on every open interval (x_{k-1}, x_k). Then we have

$$\begin{aligned}
\left(\int_a^b s \cdot t\right)^2 &= \left\{\sum_{k=1}^n s_k t_k (x_k - x_{k-1})\right\}^2 \\
&= \left\{\sum_{k=1}^n (s_k\sqrt{x_k - x_{k-1}})(t_k\sqrt{x_k - x_{k-1}})\right\}^2 \\
&\leqslant \left\{\sum_{k=1}^n s_k^2 (x_k - x_{k-1})\right\}\left\{\sum_{k=1}^n t_k^2 (x_k - x_{k-1})\right\} \\
&= \int_a^b s^2 \cdot \int_a^b t^2.
\end{aligned}$$

The equality holds if and only if there exists a constant c such that

$$s_k\sqrt{x_k - x_{k-1}} = ct_k\sqrt{x_k - x_{k-1}}$$

for each k. Since $x_k - x_{k-1} > 0$, we can cancel out the $\sqrt{x_k - x_{k-1}}$ factor from both sides of the above equation and obtain $s_k = ct_k$ for each k. □

4.* Let $B = \{x \in [0, 1] \mid x = m/2^n \text{ for some } m, n \in \mathbf{Z}\}$. Prove that the function

$$f(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases}$$

is not integrable on $[0, 1]$ by our definition of integrability.

Lemma. *If $0 \leqslant \alpha < \beta \leqslant 1$, then we can always find a third number γ such that $\gamma \in B$ and $\alpha < \gamma < \beta$.*

Proof. Let s and t be two arbitrary step functions satisfying $s \leqslant f \leqslant t$ on $[0, 1]$. We can always find a common partition $P = \{x_0, x_1, \dots, x_n\}$ for both s and t such that $s(x) = s_k$ and $t(x) = t_k$ on every open interval (x_{k-1}, x_k). The constant t_k cannot be less than 1. If so, we can always find a $\alpha \in B$ such that $x_{k-1} < \alpha < x_k$. This means $t(\alpha) < f(\alpha)$ for this α, which contradicts our choice of t. By a similar argument, the constant s_k cannot be greater than 0. This is to say, $s_k \leqslant 0$ and $t_k \geqslant 1$ for each k. Then we have

$$\begin{aligned}\int_0^1 t - \int_0^1 s &= \int_0^1 (t-s) \\ &= \sum_{k=1}^{n} (t_k - s_k)(x_k - x_{k-1}) \\ &\geqslant \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 1.\end{aligned}$$

Choose an ϵ where $0 < \epsilon < 1$. Due to the arbitrariness of s and t, the Riemann condition is broken here. Thus, the function f is not integrable on $[0, 1]$. □

Proof of the lemma. It is easy to show that $n < 2^n$ for every positive integer n. Let $x = \beta - \alpha$. By the Archimedean property, we can always find an n such that $1 < nx < 2^n x$. This means $1/2^n < x$. We apply the Archimedean property again along with the well-ordering principle and find a smallest positive integer m such that $\alpha < m/2^n$. Suppose $m/2^n \geqslant \beta$. Then we have $\alpha < \beta - 1/2^n \leqslant m/2^n - 1/2^n = (m-1)/2^n$. We find another integer $m - 1$ which is smaller than m and this contradicts our choice of m. Then it must be the case $m/2^n < \beta$. Let $\gamma = m/2^n$. Then we have $\alpha < \gamma < \beta$. □