Chapter 4 Summary

1 Arclength

1.1 Principles

Rotate the curve $y = f(x)$ about x-axis. dL denotes the length element, and

$$\begin{aligned} dL &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}$$

and the total length is

$$\begin{aligned} L &= \int dL \\ &= \int \sqrt{1 + \frac{dy}{dx}}\, dx. \end{aligned}$$

The parametric version of the above formulas would be

$$\begin{aligned} dL &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{\left(\frac{dx}{dt}\, dt\right)^2 + \left(\frac{dy}{dt}\, dt\right)^2} \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned}$$

and the total length is

$$\begin{aligned} L &= \int dL \\ &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \end{aligned}$$

1.2 Examples

The circumference of a circle is given by the following derivation.

$$dL = 2\sqrt{1 + (x/y)^2}\, dx$$

$$
\begin{aligned}
&= 2\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx \\
&= \frac{2r}{\sqrt{r^2 - x^2}}\,dx
\end{aligned}
$$

and the circumference is

$$
\begin{aligned}
L &= \int_{-r}^{+r} \frac{2r}{\sqrt{r^2 - x^2}}\,dx \\
&= 2r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\
&= 2\pi r.
\end{aligned}
$$

The parametric version is similar.

$$
\begin{aligned}
dL &= \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\,dt \\
&= r\,dt \\
L &= \int_0^{2\pi} r\,dt \\
&= 2\pi r.
\end{aligned}
$$

2 Surface Area

2.1 Principles

The surface area element of a cone's side is given by

$$
dS = \frac{1}{2} r\,d\theta \cdot L
$$

and the surface area is

$$
\begin{aligned}
S &= \int_0^{2\pi} \frac{1}{2} r L\,d\theta \\
&= \pi r L.
\end{aligned}
$$

The surface area of a revolution around x-axis is thus

$$
\begin{aligned}
S &= \int dS \\
&= \int \pi(r\,dL + L\,dr) \\
&= \int 2\pi r\,dL \\
&= \int 2\pi y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.
\end{aligned}
$$

Explanation $dS = \pi(r\,dL + L\,dr)$ is an application of dffirential operator, $r\,dL = L\,dr$ is due to similar triangle.

2.2 Examples

2.2.1 Ball

The surface area of a ball is

$$
\begin{aligned}
S &= \int_{-r}^{+r} 2\pi y \sqrt{1 + \frac{x^2}{y^2}}\, dx \\
&= 2\pi r \int_{-r}^{+r} dx \\
&= 4\pi r^2.
\end{aligned}
$$

2.2.2 Parabola

The surface area of revolving the parabola $y = ax^2$ around its symmetry axis is

$$
\begin{aligned}
S &= \int_0^h 2\pi x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= 2\pi \int_0^h \sqrt{\frac{y}{a}} \sqrt{1 + \frac{1}{4ay}}\, dy \\
&= \frac{2\pi}{\sqrt{a}} \int_0^h \sqrt{y + \frac{1}{4a}}\, dy \\
&= \frac{4\pi}{3\sqrt{a}} \left(y + \frac{1}{4a}\right)^{3/2} \Bigg|_0^h \\
&= \frac{\pi}{6a^2} [(4ah + 1)^{3/2} - 1].
\end{aligned}
$$

3 Work

3.1 Principles

$$
\begin{aligned}
\text{Work} &= \text{Force} \times \text{Distance} \\
W &= \int dW.
\end{aligned}
$$

If the force is constant, it would be easy to calculate the work.

3.2 Examples

3.2.1 Springs

$$
\begin{aligned}
&\text{Linear} \quad F(x) = \kappa x \\
&\text{Hard} \qquad\quad\;\; = \kappa x + O(x^2) \\
&\text{Soft} \qquad\quad\;\;\, = \kappa x - O(x^2)
\end{aligned}
$$

And the total work is the sum of all instantaneous work, which is the product of force and instantaneous displacement:

$$
\begin{aligned}
dW &= F(x)\,dx \\
W &= \int dW \\
&= \int F(x)\,dx.
\end{aligned}
$$

3.2.2 Pumps

Let ρ denote the weight density. $A(x)$ is the cross-sectional area. Then

$$
\begin{aligned}
dW &= F(x) \cdot (h-x) \\
&= \rho\,dV \cdot (h-x) \\
&= \rho A(x)(h-x)\,dx \\
W &= \int \rho A(x)(h-x)\,dx.
\end{aligned}
$$

3.2.3 Holes

Divide the work evenly for two workers digging the earth. Let D be the depth, other symbols ditto. The work done is given by

$$
\begin{aligned}
dW &= \rho A\,dx \cdot x \\
W(x) &= \int_0^x \rho A x\,dx.
\end{aligned}
$$

Assume A is constant, then

$$
\begin{aligned}
W(D) &= \rho A \left.\frac{x^2}{2}\right|_0^D \\
&= \frac{\rho A D^2}{2}
\end{aligned}
$$

and divide the work equally,

$$
\begin{aligned}
2W(\tilde{D}) &= W(D) \\
\tilde{D} &= \frac{D}{\sqrt{2}} \approx 0.707D.
\end{aligned}
$$

3.2.4 Swimming Pool

Your swimming pool is 3 m deep, 10 m long and 6 m wide. If the pool is initially full, how much work is required to drain two thirds of the water in the pool (that is, until the water is only 1 m deep)? Assume that the density of water is

$1\,000\,\mathrm{kg/m^3}$, and that the acceleration of gravity is $g = 10\,\mathrm{m/s^2}$.

$$\begin{aligned}
dW &= F(x) \cdot x \\
&= \rho g \, dV \cdot x \\
&= \rho g A x \, dx \\
W &= \int_0^2 \rho g A x \, dx \\
&= \rho g A \left. \frac{x^2}{2} \right|_0^2 \\
&= 2\rho g A \\
&= 1.2 \times 10^6 \,\mathrm{J}.
\end{aligned}$$

4 Element

4.1 Principles

Steps

1. Determine the differential element du.
2. Integrate to compute $u = \int du$.

4.2 Examples

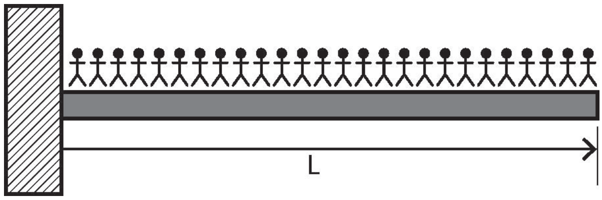
4.2.1 Torque

$$\text{Torque} = \text{(Perpendicular) Force} \times \text{Distance}$$

$$\begin{aligned}
d\tau &= x \, dF \\
dF &= g \, dM \\
dM &= \rho(x) \, dx \\
\tau = \int d\tau &= \int x g \rho(x) \, dx
\end{aligned}$$

Cantilever Beam Consider a cantilever beam of length L. Suppose that N people, each of mass m_0, stand on it equally spaced, so that their combined weight is supported uniformly along the beam. If $L = 20\,\mathrm{m}$, and $m_0 = 75\,\mathrm{kg}$, at the point of attachment, the beam can withstand a maximum torque of $\tau_{\max} = 1.5 \cdot 10^6\,\mathrm{N{\cdot}m}$, what is the maximum number of people that can stand on it? Assume the acceleration of gravity to be $g = 10\,\mathrm{m/s^2}$.

The question in hand is discrete in nature but we can still approximate the solution continuously. Since people stand on it *equally* spaced, the linear density



ρ can be approximated by Nm_0/L, where Nm_0 is total mass and L is the length of the beam. So the total torque is

$$\begin{aligned}\tau &= 37.5N \int_0^{20} x\,dx \\ &= 7\,500N \text{ N·m}.\end{aligned}$$

It is required that $\tau \leqslant \tau_{\max}$, so

$$\begin{aligned}7\,500N \text{ N·m} &\leqslant 1.5 \cdot 10^6 \text{ N·m} \\ N &\leqslant 200.\end{aligned}$$

At maximum, 200 people can stand on it.

4.2.2 Force on the side of a tank

The pressure is defined by the force under the area, thus

$$P = \frac{F}{A} = \frac{\rho V}{A} = \rho x$$

and the differential version would be

$$dF = P\,dA.$$

Integrating this will give the total force on the side of a tank is

$$\begin{aligned}F &= \int dF \\ &= \int P\,dA.\end{aligned}$$

4.2.3 Present Value

$$\begin{aligned}P(t) &= P_0\,e^{rt} \\ P_0 &= P(t)\,e^{-rt}.\end{aligned}$$

$I(t)$ denotes the income stream and PV is the present value of this income stream.

$$\begin{aligned} dPV &= e^{-rt}\, dI \\ &= e^{-rt} I(t)\, dt \\ PV &= \int_0^{\infty} dPV \\ &= \int_0^{\infty} e^{-rt} I(t)\, dt. \end{aligned}$$

A Complicated Example Consider two potential income streams, each valued based on an assumption of a constant return on investment at rate $r > 0$. The first, I_1, starts off slow, then peaks, and then decreases. The second, I_2, starts off high, then decreases. Both oscillate eventually with the same period. The specific formulas are

$$I_1(t) = I_0 + A \sin \frac{\pi t}{P} \quad \text{and} \quad I_2(t) = I_0 + A \cos \frac{\pi t}{P}.$$

Here, $I_0 > 0$ is a constant (the baseline income), $A > 0$ is a constant (the amplitude of fluctuation) and $P > 0$ is a constant (the half-period). Assume $\pi > Pr$. Which income stream has the greater present value over the time interval $[0, P]$? Which has the greater present value over the time interval $[0, +\infty)$?

$$\begin{aligned} PV_1(t) &= \int_0^t e^{-rt}(I_0 + A \sin \frac{\pi t}{P})\, dt \\ &= \frac{I_0}{r}(1 - e^{-rt}) + A \int_0^t e^{-rt} \sin \frac{\pi t}{P}\, dt \end{aligned}$$

and

$$\begin{aligned} PV_2(t) &= \int_0^t e^{-rt}(I_0 + A \cos \frac{\pi t}{P})\, dt \\ &= \frac{I_0}{r}(1 - e^{-rt}) + A \int_0^t e^{-rt} \cos \frac{\pi t}{P}\, dt. \end{aligned}$$

Integration by parts will give the following system of equations,

$$\begin{cases} \displaystyle\int e^{-rt} \sin \frac{\pi t}{P}\, dt = -\frac{1}{r} e^{-rt} \sin \frac{\pi t}{P} + \frac{\pi}{Pr} \int e^{-rt} \cos \frac{\pi t}{P}\, dt \\ \displaystyle\int e^{-rt} \cos \frac{\pi t}{P}\, dt = -\frac{1}{r} e^{-rt} \cos \frac{\pi t}{P} - \frac{\pi}{Pr} \int e^{-rt} \sin \frac{\pi t}{P}\, dt. \end{cases}$$

We solve this system of equations with the indefinite integrals being the two unknowns and obtain the antiderivatives:

$$\begin{aligned} \int e^{-rt} \sin \frac{\pi t}{P}\, dt &= -\frac{P^2 r}{P^2 r^2 + \pi^2} e^{-rt}(\sin \frac{\pi t}{P} + \frac{\pi}{Pr} \cos \frac{\pi t}{P}) + C, \\ \int e^{-rt} \cos \frac{\pi t}{P}\, dt &= \frac{P^2 r}{P^2 r^2 + \pi^2} e^{-rt}(\frac{\pi}{Pr} \sin \frac{\pi t}{P} - \cos \frac{\pi t}{P}) + C. \end{aligned}$$

Do not be scared by these horrifying formulas. Only the definite integrals $PV(P)$ and $PV(\infty)$ are of interest to us. There is a lot of canceling and factoring in the algebraic process. The final results are

$$
\begin{aligned}
PV_1(P) &= \frac{I_0}{r}(1 - e^{-rP}) + \frac{AP\pi}{P^2r^2 + \pi^2}(1 + e^{-rP}), \\
PV_1(\infty) &= \frac{I_0}{r} + \frac{AP\pi}{P^2r^2 + \pi^2}, \\
PV_2(P) &= \frac{I_0}{r}(1 - e^{-rP}) + \frac{AP^2r}{P^2r^2 + \pi^2}(1 + e^{-rP}),
\end{aligned}
$$

and

$$
PV_2(\infty) = \frac{I_0}{r} + \frac{AP^2r}{P^2r^2 + \pi^2}.
$$

Since $\pi > Pr$, the income stream $I_1(t)$ has the greater present value over both $[0, P]$ and $[0, +\infty)$.

Save Money for College We have learned about present value of an income stream $I(t)$; one may also reverse the derivation to determine the future value of the income at a time $t = T$. The future value element of $I(t)$ is

$$
dFV = e^{r(T-t)} I(t)\, dt,
$$

assuming a continuous compounding at fixed interest rate r.

If you save for a child's college at a rate of \$5 000/year starting at the child's birth, how much money will be available when she is 20? Assume a fixed return 5% on investments.

$$
\begin{aligned}
FV &= \int_0^T dFV \\
&= \int_0^T e^{r(T-t)} I(t)\, dt.
\end{aligned}
$$

Since $T = 20$, $r = 0.05$, and $I(t) = 5\,000$, then

$$
\begin{aligned}
FV &= 5\,000 \int_0^{20} e^{0.05(20-t)}\, dt \\
&= -100\,000\, e^{1-0.05t} \Big|_0^{20} \\
&= 100\,000\,(e - 1)
\end{aligned}
$$

5 Average

5.1 Principles

5.1.1 Mean of a Function

Definition The average of f, denoted by $\bar{f}$, over the interval $[a, b]$ satisfies

$$\int_a^b f - \bar{f}\, dx = 0.$$

Some algebra will turn the above formula into the following:

$$\begin{aligned}
\int_a^b f - \bar{f}\, dx &= 0 \\
\int_a^b f\, dx &= \int_a^b \bar{f}\, dx \\
\bar{f} = \frac{\displaystyle\int_a^b f\, dx}{\displaystyle\int_a^b dx} &= \frac{1}{b-a} \int_a^b f\, dx.
\end{aligned}$$

We'd better view the average as the ratio of the integral of f and the integral of 1, since it is more flexible and can be extended to infinite and discrete situation, where the domain of integration is not a line segment. In general, we write

$$\bar{f} = \frac{\displaystyle\int_D f\, dx}{\displaystyle\int_D dx},$$

where D is the domain of integration. When D is discrete, we have

$$\bar{f} = \frac{1}{n} \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} f_i \Big/ \sum_{i=1}^{n} 1.$$

5.1.2 Root Mean Square

Definition The root mean square or quadratic mean of a function is

$$f_{RMS} = \sqrt{\overline{f^2}}.$$

5.2 Examples

What is the average over $0 \leqslant x \leqslant T$?

5.2.1 Monomial

$$\frac{1}{T} \int_0^T x^n\, dx = \frac{T^n}{n+1}$$

5.2.2 Exponential

$$\frac{1}{T}\int_0^T e^x\,dx = \frac{e^T - 1}{T}$$

5.2.3 Logarithm

$$\begin{aligned}\frac{1}{T}\int_1^T \ln x\,dx &= \frac{1}{T}(x\ln x - x)\Big|_1^T \\ &= \ln T - 1 + \frac{1}{T}\end{aligned}$$

5.2.4 Density of the Earth

$$\bar{\rho} \neq \frac{\displaystyle\int \rho(r)\,dr}{\displaystyle\int dr}$$

$$\bar{\rho} = \frac{\displaystyle\int \rho\,dV}{\displaystyle\int dV} = \frac{\displaystyle\int dM}{\displaystyle\int dV} = \frac{\displaystyle\int_0^R \rho\,4\pi r^2\,dr}{\displaystyle\int_0^R 4\pi r^2\,dr}$$

5.2.5 Blood Flow

Poiseuilless's Law states

$$v(r) = \frac{P}{4\mu l}(R^2 - r^2),$$

where P is the pressure, μ the viscosity, and l the length. Thus

$$\begin{aligned}\bar{v} &= \frac{\displaystyle\int v\,dA}{\displaystyle\int dA} = \frac{\displaystyle\int_0^R v\,2\pi r\,dr}{\displaystyle\int_0^R 2\pi r\,dr} \\ &= \frac{1}{\pi R^2}\int_0^R \frac{P}{4\mu l}(R^2 - r^2)\,2\pi r\,dr \\ &= \frac{P}{2R^2\mu l}\left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^R \\ &= \frac{1}{2}\frac{P}{4\mu l}R^2 = \frac{1}{2}v_{\max}.\end{aligned}$$

5.2.6 Voltage

Alternating current gives a sinusoid. Its formula is

$$V(t) = V_P \sin(\omega t),$$

where V_P is the amplitude or peak voltage and ω is the frequency. The period $T = 2\pi/\omega$. However, the average voltage is

$$\bar{V} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} V_P \sin(\omega t)\, dt = 0,$$

which is totally useless. Instead, the root mean square is more useful here. It is

$$\begin{aligned} V_{RMS} &= \sqrt{\frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} V_P^2 \sin^2(\omega t)\, dt} \\ &= \sqrt{\frac{V_P^2}{4\pi} \left(\omega t - \frac{\sin 2\omega t}{2} \right) \Bigg|_0^{\frac{2\pi}{\omega}}} \\ &= \frac{V_P}{\sqrt{2}}. \end{aligned}$$