Chapter 3 Summary

1 Indefinite Integral

Why do we care about indefinite integral? One of the reasons to study indefinite integral is to use it to solve differential equation, which often arises in physics problems.

$$\begin{aligned}
\frac{dv}{dt} &= -g \\
v(t) &= \int -g\,dx \\
&= -gt + C \\
&= -gt + v_0 \\
\frac{dx}{dt} &= -gt + v_0 \\
x(t) &= \int (-gt + v_0)\,dt \\
&= -\frac{1}{2}gt^2 + v_0 t + C \\
&= -\frac{1}{2}gt^2 + v_0 t + x_0.
\end{aligned}$$

v_0 and x_0 are initial speed and initial height, respectively.

The solution to he simple ordinary differential equation (O.D.E.) of the form $\frac{dx}{dt} = f(t)$ is $\int f(t)\,dt$. But what is the solution to the slightly different O.D.E. of the form $\frac{dx}{dt} = f(x)$? But we only focus our attention on the simplest of them, linear O.D.E. , that is the differential equation of the form $\frac{dx}{dt} = ax$, where a is a constant. It's easy to see $x = e^t$ satisfies the equation when $a = 1$, since $\frac{d}{dt}e^t = e^t$. Inspired by this, we can conjecture that

$$x(t) = e^{at} \text{ satisfies } \frac{dx}{dt} = ax.$$

We can verifies this by differentiating e^{at},

$$\frac{d}{dt}x = \frac{d}{dt}e^{at} = ae^{at} = ax.$$

Do we omit something? By observing that

$$\frac{dCx}{dt} = C\frac{dx}{dt} = Cax = aCx,$$

we know that the operator $\frac{d}{dt}$ in the linear O.D.E is homogeneous of degree one and that's also why it's called *linear* O.D.E. Therefore, any scalar multiple of solution to the linear O.D.E. is also a solution to the equation,

$$x(t) = Ce^t = x_0 e^t.$$

Here x_0 means the initial condition.

The above method of solution is sometimes called solution by *ansatz*. We take a reasonable guess which turns out to be correct by verification. However, there exists more principled approaches like solution by series or by integration.

Solution by series

Solution by integration, method of separation.

$$\begin{aligned}
\frac{dx}{dt} &= ax \\
\frac{dx}{x} &= a\,dt \\
\int \frac{dx}{x} &= \int a\,dt \\
\ln x &= at + C \\
x &= e^{at+C} \\
&= e^{C} e^{at} \\
&= Ce^{at} \\
&= x_0 e^{at}.
\end{aligned}$$

2 A Simple O.D.E

What is modeled by $\frac{dx}{dt} = ax$?