§2. 换元积分法

一. 第一换元法

设$\int f(u)\,dx = F(u) + C$且$\varphi(x) \in C^1$, 则$\int f(\varphi(x))\varphi'(x)\,dx = F(\varphi(x)) + C$.

证明：(左边)$' = f(\varphi(x))\varphi'(x) =$ (右边)$'$. □

在实际使用中, 换元法体现了莱布尼茨微分记号的优越性. 将

$$\varphi'(x)\,dx = d\varphi(x)$$

代入原式, 得

$$\int f(\varphi(x))\varphi'(x)\,dx = \int f(\varphi(x))\,d\varphi(x)$$

用u代替$\varphi(x)$

$$\begin{aligned} &= \int f(u)\,du \\ &= F(u) + C \\ &= F(\varphi(x)) + C. \end{aligned}$$

例1：$\displaystyle\int x\sin(x^2)\,dx.$

将$x\,dx = \frac{1}{2}\,dx$代入得

$$\int x\sin(x^2)\,dx = \frac{1}{2}\int \sin(x^2)\,d(x^2)$$

用u代替x^2

$$\begin{aligned} &= \frac{1}{2}\int \sin u\,du \\ &= -\frac{1}{2}\cos(u) + C \\ &= -\frac{1}{2}\cos(x^2) + C. \end{aligned}$$

例2：$\displaystyle\int \cot x\,dx.$

$$\begin{aligned} \text{原式} &= \int \frac{\cos x}{\sin x}\,dx && \langle \cot x = \frac{\cos x}{\sin x}\rangle \\ &= \int \frac{1}{\sin x}\,d(\sin x) && \langle d\sin x = \cos x\,dx\rangle \\ &= \int \frac{du}{u} && \langle u = \sin x\rangle \\ &= \ln|u| + C && \langle d\ln|u| = \frac{1}{u}\,du\rangle \\ &= \ln|\sin x| + C. && \langle u = \sin x\rangle \end{aligned}$$

例3： $\int \frac{dx}{a^2+x^2} \qquad (a \neq 0).$

$$\begin{aligned}
\text{原式} &= \frac{1}{a^2}\int \frac{dx}{1+\left(\frac{x}{a}\right)^2} && \langle \text{因为} a \neq 0, \text{提取因子} \frac{1}{a^2} \rangle \\
&= \frac{1}{a^2}\int \frac{d(au)}{1+u^2} && \langle u = \frac{x}{a} \rangle \\
&= \frac{1}{a}\int \frac{du}{1+u^2} && \langle d(au) = a\,du \rangle \\
&= \frac{1}{a}\arctan u + C && \langle d\arctan u = \frac{1}{1+u^2}\,du \rangle \\
&= \frac{1}{a}\arctan \frac{x}{a} + C. && \langle u = \frac{x}{a} \rangle
\end{aligned}$$

例4： $\int \frac{dx}{\sqrt{a^2-x^2}} \qquad (a > 0).$

$$\begin{aligned}
\text{原式} &= \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} && \langle \text{因为} a > 0, \text{提取因子} \frac{1}{a} \rangle \\
&= \frac{1}{a}\int \frac{d(au)}{\sqrt{1-u^2}} && \langle u = \frac{x}{a} \rangle \\
&= \int \frac{du}{\sqrt{1-u^2}} && \langle d(au) = a\,du \rangle \\
&= \arcsin u + C && \langle d\arcsin u = \frac{1}{\sqrt{1-u^2}}\,du \rangle \\
&= \arcsin \frac{x}{a} + C. && \langle u = \frac{x}{a} \rangle
\end{aligned}$$

二. 第二换元法

设$f(x)$为连续函数, $x = \varphi(t)$连续可导且有反函数, 则

$$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt.$$

若右边的原函数可求得, 记$G(t) = \int f(\varphi(t))\varphi'(t)\,dt$, 则

$$\int f(x)\,dx = G(\varphi^{-1}(x)) + C.$$

例1： $\int \sqrt{a^2-x^2}\,dx.$

假设$a > 0$, 则$-a \leq x \leq a$

$$\begin{aligned}
\text{原式} &= \int a(\sqrt{1-\sin^2 t})\,d(a\sin t) && \langle x = a\sin t,\ t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \rangle \\
&= a^2\int \cos^2 t\,dt && \langle \cos t = \sqrt{1-\sin^2 t},\ d(a\sin t) = a\cos t\,dt \rangle \\
&= a^2\int \frac{1+\cos 2t}{2}\,dt && \langle \cos^2 t = \frac{1+\cos 2t}{2} \rangle \\
&= \frac{a^2}{2}t + \frac{\sin 2t}{4} + C && \langle d\frac{t}{2} = \frac{1}{2}\,dt,\ d\frac{\sin 2t}{4} = \frac{\cos 2t}{2}\,dt \rangle \\
&= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{\sin(2\arcsin\frac{x}{a})}{2} + C. && \langle t = \arcsin\frac{x}{a} \rangle
\end{aligned}$$

例2: $\int \frac{dx}{\sqrt{x^2-a^2}}$.

假设$a>0$, 则$x>a$或$x<-a$

$$
\begin{aligned}
\text{原式} &= \int \frac{dx}{a\sqrt{\sec^2 t-1}} && \langle x=a\sec t,\ t\in(0,\frac{\pi}{2})\rangle \\
&= \int \frac{\cancel{a}\sec t\cancel{\tan t}}{\cancel{a}\cancel{\tan t}}\,dt && \langle d(a\sec t)=a\sec t\tan t\,dt\rangle \\
&= \int \frac{dt}{\cos t} && \langle \sec t=\frac{1}{\cos t}\rangle \\
&= \int \frac{\cos t\,dt}{\cos^2 t} && \langle \text{分子分母同乘以}\cos t\rangle \\
&= \int \frac{d\sin t}{1-\sin^2 t} && \langle d\sin t=\cos t\,dt,\ \cos^2 t=1-\sin^2 t\rangle \\
&= \int \frac{du}{1-u^2} && \langle u=\sin t\rangle \\
&= \frac{1}{2}\left(\int\frac{du}{1-u}+\int\frac{du}{1+u}\right) && \langle \frac{1}{1-u^2}=\frac{1}{2}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)\rangle \\
&= \frac{1}{2}(\ln|1+u|-\ln|1-u|+C) && \langle d\ln|1+u|=\frac{1}{1+u},\ d\ln|1-u|=-\frac{1}{1-u}\rangle \\
&= \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|+C && \langle \text{对数的性质},\ u=\sin t\rangle \\
&= \frac{1}{2}\ln\left|\frac{1+\sqrt{1-\left(\frac{a}{x}\right)^2}}{1-\sqrt{1-\left(\frac{a}{x}\right)^2}}\right|+C && \langle \sin^2 t=1-\left(\frac{a}{x}\right)^2\rangle \\
&= \frac{1}{2}\ln\left|\frac{\left(1+\sqrt{1-\left(\frac{a}{x}\right)^2}\right)^2}{\left(\frac{a}{x}\right)^2}\right|+C && \langle \text{分母有理化}\rangle \\
&= \ln\left(x+\sqrt{x^2-a^2}\right)+C. && \langle \text{对数的性质}\rangle
\end{aligned}
$$

例3: $\int \frac{dx}{x^2\sqrt{x^2+1}}$.

$$
\begin{aligned}
\text{原式} &= \int \frac{d\tan t}{\tan^2 t\sqrt{\tan^2 t+1}} && \langle x=\tan t,\ \text{因为}x\neq 0,\ \text{所以}t\in(-\frac{\pi}{2},\frac{\pi}{2})\setminus\{0\}\rangle \\
&= \int \frac{\sec^{\cancel{2}} t\,dt}{\tan^2 t\cancel{\sec t}} && \langle \sec^2 t-\tan^2 t=1\rangle \\
&= \int \frac{\cos t\,dt}{\sin^2 t} && \langle \sec t=\frac{1}{\cos t},\ \tan t=\frac{\sin t}{\cos t}\rangle \\
&= \int \frac{d\sin t}{\sin^2 t} && \langle d\sin t=\cos t\,dt\rangle \\
&= -\frac{1}{\sin t}+C && \langle \int u^p\,du=\frac{u^{p+1}}{p+1}+C\rangle \\
&= -\frac{\sqrt{1+x^2}}{x}+C. && \frac{1}{\sin t}=\frac{\sqrt{1+x^2}}{x}
\end{aligned}
$$