Solutions to Problem Set 4

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This is my own solution to Problem Set 4 of 18.014. Problem 1 and 6 are taken from Apostol's *Calculus* and Problem 3 is taken from Munkres's Course Note F (1: 83, 94;).

1. (a) $\lim_{x\to 0} \dfrac{\sin 5x}{\sin x} = 5.$

   (b) $\lim_{x\to 0} \dfrac{\sin 5x - \sin 3x}{x} = 2.$

   (c) $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \dfrac{1}{2}.$

2. Let $A(x) = \int_{-2}^{x} f(t)\,dt$ where $f(t) = -1$ if $t < 0$ and $f(t) = 1$ if $t \geqslant 0$. Graph $y = A(x)$ for $x \in [-2, 2]$. Using ϵ–δ language, show that $\lim_{x\to 0} A(x)$ exists and find its value.

3. Let $f(x)$ be defined for all x, and continuous except for $x = -1$ and $x = 3$. Let

$$g(x) = \begin{cases} x^2+1, & \text{for } x > 0, \\ x-3, & \text{for } x \leqslant 0. \end{cases}$$

   For what values of x can you be sure that $f(g(x))$ is continuous? Explain.

4. Suppose that g, h are two continuous functions on $[a, b]$. Suppose there exists $c \in (a, b)$ such that $g(c) = h(c)$. Define $f(x)$ such that $f(x) = g(x)$ for $x < c$ and $f(x) = h(x)$ for $x \geqslant c$. Prove that f is continuous on $[a, b]$.

5. Let $f(x) = \sin(1/x)$ for $x \in \mathbf{R}$, $x \neq 0$. Show that for any $a \in \mathbf{R}$, the function $g(x)$ defined by

$$g(x) = \begin{cases} f(x), & x \neq 0, \\ a, & x = 0, \end{cases}$$

   is not continuous at $x = 0$.

6. Given a real-valued function f which is continuous on the closed inverval $[0, 1]$. Assume that $0 \leqslant f(x) \leqslant 1$ for each x in $[0, 1]$. Prove that there is at least one point c in $[0, 1]$ for which $f(c) = c$. Such a point is called a *fixed point* of f. The result of this exercise is a special case of *Brouwer's fixed-point theorem*.

7.* Let f be a bounded function that is integrable on $[a, b]$. Prove that there exists $c \in \mathbf{R}$ with $a \leqslant c \leqslant b$ such that $\int_a^b f(x)\,dx = 2\int_a^c f(x)\,dx$.