Solutions to Tutorial 8

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This is my own solutions to the problems from tutorial 8 of 6.041SC.

1. Type A, B, and C items are placed in a common buffer, each type arriving as part of an independent Poisson process with average arrival rates, respectively, of a, b, and c items per minute. For the first four parts of this problem, assume the buffer is discharged immediately whenever it contains a total of ten items.

   (a) What is the probability that, of the first ten items to arrive at the buffer, only the first and one other are type A?

$$\frac{a}{a+b+c}\binom{9}{1}\frac{a}{a+b+c}\left(\frac{b+c}{a+b+c}\right)^8 = \frac{9a^2(b+c)^8}{(a+b+c)^{10}}.$$

   (b) What is the probability that any particular discharge of the buffer contains five times as many type A items as type B items?

$$\binom{10}{5,1,4}\left(\frac{a}{a+b+c}\right)^5\left(\frac{b}{a+b+c}\right)\left(\frac{c}{a+b+c}\right)^4 + \left(\frac{c}{a+b+c}\right)^{10} = \frac{1260\,a^5bc^4 + c^{10}}{(a+b+c)^{10}}.$$

   (c) Determine the PDF, expectation, and variance for the total time between consecutive discharges of the buffer.
   Let T be the time between consecutive discharges of the buffer. Then,

$$f_T(t) = \frac{(a+b+c)^{10}t^9}{9!}e^{-(a+b+c)t},$$
$$E[T] = \frac{10}{a+b+c},$$

   and

$$\mathrm{var}(T) = \frac{10}{(a+b+c)^2}.$$

   (d) Determine the probability that exactly two of each of the three item types arrive at the buffer input during any particular five minute interval.

$$\left(\frac{(5a)^2}{2}e^{-5a}\right)\left(\frac{(5c)^2}{2}e^{-5c}\right)\left(\frac{(5c)^2}{2}e^{-5c}\right) = \frac{(125abc)^2}{8}e^{-5(a+b+c)}.$$

2. A store opens at $t = 0$ and *potential* customers arrive in a Poisson manner at an average arrival rate of λ potential customers per hour. As long as the store is open, and independently of all other events, each particular potential customer becomes an *actual* customer with probability p. The store closes as soon as ten actual customers have arrived.

   (a) What is the probability that exactly three of the first five potential customers become actual customers?

$$\binom{5}{3}p^3(1-p)^2.$$

(b) What is the probability that the fifth potential customer to arrive becomes the third actual customer?

$$\binom{4}{2} p^3 (1-p)^2.$$

(c) What is the PDF and expected value for L, the duration of the interval from store opening to store closing?

$$f_L(t) = \frac{(\lambda p)^{10} t^9}{9!} e^{-\lambda p t},$$

and

$$E[L] = \frac{10}{\lambda p}.$$

(d) Given only that exactly three of the first five potential customers became actual customers, what is the conditional expected value of the *total* time the store is open?

$$\frac{5}{\lambda} + \frac{7}{\lambda p}.$$

(e) Considering only customers arriving between $t = 0$ and the closing of the store, what is the probability that no two actual customers arrive within τ time units of each other?

$$e^{-9\lambda p \tau}.$$

3. Consider a Poisson process with parameter λ, and an independent random variable T, which is exponential with parameter ν. Find the PMF of the number of Poisson arrivals during the time interval $[0, T]$.

Let K be the number of Poisson arrivals. Then,

$$\begin{aligned}
p_K(k) &= \int_0^{+\infty} f_T(t)\, p_{K|T}(k \mid t)\, dt \\
&= \int_0^{+\infty} \nu e^{-\nu t}\, \frac{(\lambda t)^k}{k!} e^{-\lambda t}\, dt \\
&= \frac{\nu \lambda^k}{k!} \int_0^{+\infty} t^k e^{-(\lambda+\nu)t}\, dt \\
&= \frac{\nu \lambda^k}{k!} \left(\sum_{k=0}^{k} \frac{k!\, t^i}{i! (\lambda+\nu)^{k+1-i}} e^{-(\lambda+\nu)t} \right) \Bigg|_{+\infty}^{0} \\
&= \frac{\nu \lambda^k}{(\lambda+\nu)^{k+1}}.
\end{aligned}$$