Solutions to Recitation 17

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This is my own solutions to the problems from recitation 17 of 6.041SC.

1. Iwana Passe is taking a multiple-choice exam. You may assume that the number of questions is infinite. *Simultaneously, but independently*, her conscious and subconscious faculties are generating answers for her, each in a Poisson manner. (Her conscious and subconscious are always working on different questions.) Conscious responses are generated at the rate λ_c responses per minute. Subconscious responses are generated at the rate λ_s responses per minute. Assume $\lambda_c \neq \lambda_s$. Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct. Iwana responds only once to each question, and you can assume that her time for recording these conscious and subconscious responses is negligible.

   (a) Determine $p_K(k)$, the probability mass function for the number of conscious responses Iwana makes in an interval of T minutes.

$$p_K(k) = \frac{(\lambda_c T)^k}{k!} e^{-\lambda_c T}.$$

   (b) If we pick any question to which Iwana has responded, what is the probability that her answer to that question:

      i. represents a conscious response?

$$\frac{\lambda_c}{\lambda_c + \lambda_s}.$$

      ii. represents a conscious correct response?

$$\frac{\lambda_c p_c}{\lambda_c + \lambda_s}.$$

   (c) If we pick an interval of T minutes, what is the probability that in that interval Iwana will make exactly r conscious responses and s subconscious responses?

$$\frac{\lambda_c^r \lambda_s^s T^{r+s}}{r!s!} e^{-(\lambda_c + \lambda_s)T}.$$

   (d) Determine the probability density function for random variable X, where X is the time from the start of the exam until Iwana makes her first conscious response which is preceded by at least one subconscious response.

$$\frac{\lambda_c \lambda_s}{\lambda_c - \lambda_s}(e^{-\lambda_s x} - e^{-\lambda_c x}).$$

2. Shem, a local policeman, drives from intersection to intersection in times that are independent and all exponentially distributed with parameter λ. At each intersection he observes (and reports) a car accident with probability p. (This activity does not slow his driving at all.) Independently of all else, Shem receives extremely brief radio calls in a Poisson manner with an average rate of μ calls per hour.

   (a) Determine the PMF for N, the number of intersections Shem visits up to and including the one where he reports his first accident.

$$p_N(n) = (1-p)^{n-1}p.$$

   (b) Determine the PDF for Q, the length of time Shem drives between reporting accidents.

$$f_Q(q) = \lambda p e^{-\lambda p q}.$$

   (c) What is the PMF for M, the number of accidents which Shem reports in two hours?

$$p_M(m) = \frac{(2\lambda p)^m}{m!} e^{-2\lambda}.$$

   (d) What is the PMF for K, the number of accidents Shem reports between his receipt of two successive radio calls?

$$p_K(k) = \left(\frac{\lambda p}{\mu + \lambda p}\right)^k \frac{\mu}{\mu + \lambda p}.$$

   (e) We observe Shem at a random instant long after his shift has begun. Let W be the total time from Shem's last radio call until his next radio call. What is the PDF of W?

$$f_W(w) = \mu^2 w e^{-\mu w}.$$

3. *Random incidence in an Erlang arrival process*

   Consider an arrival process in which the interarrival times are independent Erlang random variables of order 2, with mean $2/\lambda$. Assume that the arrival process has been ongoing for a very long time. An external observer arrives at a given time t. Find the PDF of the length of the interarrival interval that contains t.

   Let X denote the length of the interarrival interval that contains t. The random variable X can be decomposed into three component X_1, X_2, and X_3, each of which has an exponential distribution with parameter λ. Thus, by convolution, we have

$$\begin{aligned} f_{X_1+X_2}(x) &= \int_0^x f_{X_1}(t) f_{X_2}(x-t)\,dt \\ &= \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)}\,dt \\ &= \lambda^2 e^{-\lambda x} \int_0^x dt \\ &= \lambda^2 x e^{-\lambda x} \end{aligned}$$

and

$$
\begin{aligned}
f_X(x) &= \int_0^x f_{X_1+X_2}(t) f_{X_3}(x-t)\, dt \\
&= \int_0^x \lambda^2 t e^{-\lambda t}\, \lambda e^{-\lambda(x-t)}\, dt \\
&= \lambda^3 e^{-\lambda x} \int_0^x t\, dt \\
&= \frac{\lambda^3 x^2}{2} e^{-\lambda x}.
\end{aligned}
$$

Hence, the PDF of X is an Erlang distribution of order 3 with parameter λ.