Solutions to Problem Set 7

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This is my own solutions to the problem set 7 of 6.041sc.

1. Consider a sequence of mutually independent, identically distributed, probabilistic trials. Any particular trial results in either a success (with probability p) or a failure.

   (a) Obtain a simple expression for the probability that the ith success occurs before the jth failure. You may leave your answer in the form of a summation.

   Consider the situation that the ith success happend at the kth trial. If $k \geqslant i + j$, then there is already at least j failures before the ith success. If $k < i + j$, then jth failture must occur after ith success. Thus, the probability that the ith success occurs before the jth failure is the sum of all probabilities of ith success at the kth trial where $k < i + j$, which is

$$\sum_{k<i+j} P(i\text{th success at the } k\text{th trial}) = \sum_{k=i}^{i+j-1} \binom{k-1}{i-1} p^i (1-p)^{k-i}.$$

   (b) Determine the expected value and variance of the number of successes which occur before the jth failure.

   Let X be the number of trials until the jth failure. Then X can be decompose into a sequence of i.i.d. geometric random variables X_1, X_2, …, and X_j with parameter $1 - p$. Let Y be the number of successes before the jth failture. It can be decomposed into a sequence of i.i.d. random variables Y_1, …, and Y_j, where each $Y_i = X_i - 1$. Thus, we have

$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\sum_{i=1}^{j} Y_i] \\
&= \sum_{i=1}^{j} \mathrm{E}[X_i - 1] \\
&= j(\mathrm{E}[X_1] - 1) \\
&= j\left(\frac{1}{1-p} - 1\right) \\
&= \frac{jp}{1-p},
\end{aligned}$$

   and

$$\begin{aligned}
\mathrm{var}(Y) &= \mathrm{var}(\sum_{i=1}^{j} Y_i) \\
&= \sum_{i=1}^{j} \mathrm{var}(X_i - 1) \\
&= j\,\mathrm{var}(X_1) \\
&= \frac{jp}{(1-p)^2}.
\end{aligned}$$

(c) Let L_{17} be described by a Pascal PMF of order 17. Find the numerical values of a and b in the following equation:

$$\sum_{l=42}^{\infty} p_{L_{17}}(l) = \sum_{x=0}^{a} \binom{b}{x} p^x (1-p)^{b-x}.$$

Explain your work.

The LHS can be interpreted as the probability that the 17th success happens after 41th trial. That is equal to the probability of x successes in the first 41 trials for all $x < 17$, which is

$$\sum_{x<17} P(x \text{ successes in the first 41trials}) = \sum_{x=0}^{16} \binom{41}{x} p^x (1-p)^{41-x}.$$

It is easy to see this has the same form of the RHS of the original formula. Thus, $a = 16$ and $b = 41$.

2. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

(a) Determine the probability that Fred gives away his first sample on his third call.

The probability that Fred gives out a sample on any call is $3/4 \times 2/3 = 1/2$. Let X_k be the number of the call on which Fred gives away his kth sample. Thus, the probability to be determined is

$$p_{X_1}(3) = \left(\frac{1}{2}\right)^2 \frac{1}{2} = \frac{1}{8}.$$

(b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

Due to the memorylessness property of Bernoulli process, the probability that Fred will give away his fifth sample on his eleventh call, conditioned upon that he has given away exactly four samples on his first eight calls, is the same as the unconditional probability that he will give away his first sample on the third call. This is the same probability calculated in (a), which is 1/8.

(c) Determine the probability that he gives away his second sample on his fifth call.

$$p_{X_2}(5) = \binom{4}{1} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 = \frac{1}{64}.$$

(d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

$$\begin{aligned} P(X_2 = 5 \mid X_2 \neq 2) &= \frac{P(X_2 = 5, X_2 \neq 2)}{P(X_2 \neq 2)} \\ &= \frac{P(X_2 = 5)}{1 - P(X_2 = 2)} \\ &= \frac{1/64}{1 - (1/2)^2} \\ &= \frac{1}{48}. \end{aligned}$$

(e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

$$
\begin{aligned}
P(X_2 \geqslant 5) &= \sum_{i=5}^{\infty} \binom{i-1}{1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{i-2} \\
&= \sum_{j=0}^{1} \binom{4}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{4-i} \\
&= (1+4)\frac{1}{2^4} = \frac{5}{16}.
\end{aligned}
$$

(f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

Since the events "Door answered" and "A dog lives here" are independent and the outcomes of all calls are independent, we can restrict our attention to only the homes with dogs. Given a home with dogs, the probability of the door being answered is 3/4; and this is the same as a Bernoulli process with parameter 3/4. We can decompose D_m into a sequence of random variables $G_1 - 1$, $G_2 - 1$, …, $G_m - 1$, where each G_i is an i.i.d. geometric random variable. Therefore,

$$
\begin{aligned}
\mathrm{E}[D_m] &= \mathrm{E}[\sum_{i=1}^{m}(G_i - 1)] \\
&= \sum_{i=1}^{m} \mathrm{E}[G_1 - 1] \\
&= m(\mathrm{E}[G_1] - 1) \\
&= \frac{m(1-p)}{p},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{var}(D_m) &= \mathrm{var}(\sum_{i=1}^{m}(G_i - 1)) \\
&= \sum_{i=1}^{m} \mathrm{var}(G_1) \\
&= \frac{m(1-p)}{p^2}.
\end{aligned}
$$

3. Let T_1 and T_2 be exponential random variables with parameter λ, and let S be an exponential random variable with parameter μ. We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1 + T_2, S\}$. *Hint*: See Problem 6.19 in the text.

Since

$$
\begin{aligned}
P(\min\{T_1 + T_2, S\} > t) &= P(T_1 + T_2 > t)\, P(S > t) \\
&= (e^{-\lambda t} + \lambda t e^{-\lambda t}) e^{-\mu t} \\
&= (1 + \lambda t) e^{-(\lambda+\mu)t},
\end{aligned}
$$

therefore

$$
\begin{aligned}
P(\min\{T_1 + T_2, S\} \leqslant t) &= 1 - P(\min\{T_1 + T_2, S\} > t) \\
&= 1 - (1 + \lambda t) e^{-(\lambda+\mu)t}.
\end{aligned}
$$

Take the derivative to get the PDF of $\min\{T_1 + T_2, S\}$,

$$
\begin{aligned}
f_{\min\{T_1+T_2,S\}}(t) &= \frac{d}{dt}(1 - (1 + \lambda t) e^{-(\lambda+\mu)t}) \\
&= -\lambda e^{-(\lambda+\mu)t} + (1 + \lambda t)(\lambda + \mu) e^{-(\lambda+\mu)t} \\
&= (\mu + \lambda(\lambda + \mu)t) e^{-(\lambda+\mu)t}.
\end{aligned}
$$

Integrate to find the expected value of $\min\{T_1 + T_2, S\}$,

$$\begin{aligned}
\mathrm{E}[\min\{T_1 + T_2, S\}] &= \int_0^{+\infty} t f_{\min\{T_1+T_2,S\}}(t)\, dt \\
&= \int_0^{+\infty} (\mu t + \lambda(\lambda+\mu)t^2) e^{-(\lambda+\mu)t}\, dt \\
&= \frac{\mu}{\lambda+\mu} \int_0^{+\infty} t\,(\lambda+\mu) e^{-(\lambda+\mu)t}\, dt \\
&\quad + \lambda \int_0^{+\infty} t^2 (\lambda+\mu) e^{-(\lambda+\mu)t}\, dt \\
&= \frac{\mu}{(\lambda+\mu)^2} + \lambda \left(\frac{1}{(\lambda+\mu)^2} + \left(\frac{1}{\lambda+\mu}\right)^2 \right) \\
&= \frac{\mu + 2\lambda}{(\lambda+\mu)^2}.
\end{aligned}$$

4. A single dot is placed on a very long length of yarn at the textile mill. The yarn is then cut into lengths requested by different customers. The lengths are independent of each other, but all distributed according to the PDF $f_L(\ell)$. Let R be be the length of yarn purchased by that customer whose purchase included the dot. Determine the expected value of R in the following cases:

   (a) $f_L(\ell) = \lambda e^{-\lambda\ell}$ for $\ell \geqslant 0$.
   This problem is equivalent to the random incidence paradox. For this specific PDF f_L, the resulting R is an Erlang distribution of order 2. Thus,

   $$\mathrm{E}[R] = \frac{2}{\lambda}.$$

   (b) $f_L(\ell) = \frac{\lambda^3\ell^2}{2} e^{-\lambda\ell}$ for $\ell \geqslant 0$.
   In this case, L has an Erlang distribution of order 3. It follows that R has an Erlang distribution of order 4. Thus,

   $$\mathrm{E}[R] = \frac{4}{\lambda}.$$

   (c) $f_L(\ell) = \ell e^{\ell}$, for $0 \leqslant \ell \leqslant 1$.
   Section 2.13 *Random Incidence* from *Urban Operations Research* by Larson and Odoni is a very good source for this type of problem. Another source is section 2.8.3 *Random Incidence and Residual Time* from *Fundamentals of Stochastic Networks* by Oliver C. Ibe.
   Both sources uses an argument by saying the $f_R(\ell)\delta$ is proportional to both ℓ and $f_L(\ell)\delta$, that is $f_R(\ell)\delta \propto \ell f_L(\ell)\delta$. Since $f_R(\ell)$ is a PDF and it must integrate to one, then the normalizing constant must be $1/\mathrm{E}[L]$. In this way, we derive that

   $$f_R(\ell) = \frac{\ell f_L(\ell)}{\mathrm{E}[L]}.$$

   Then it is obvious that

   $$\begin{aligned}
   \mathrm{E}[L] &= \int_0^1 \ell^2 e^{\ell}\, d\ell \\
   &= \ell^2 e^{\ell}\Big|_0^1 - 2\int_0^1 \ell e^{\ell}\, d\ell \\
   &= e - 2(\ell e^{\ell} - e^{\ell})\Big|_0^1 \\
   &= e - 2,
   \end{aligned}$$

   and

   $$\begin{aligned}
   \mathrm{E}[R] &= \int_0^1 \frac{\ell^3 e^{\ell}}{\mathrm{E}[L]}\, d\ell \\
   &= \frac{\ell^3 e^{\ell}\Big|_0^1 - 3\,\mathrm{E}[L]}{\mathrm{E}[L]} \\
   &= \frac{2(3-e)}{e-2}.
   \end{aligned}$$

This proportional argument, however, was somewhat elusive to me and I spent one whole day ruminating about this argument. Fortunately, I eventually found a different way to derive f_R that is *intuitive* to understand. First, we can consider the situation where L is a discrete r.v. and treat the continuous version as the limiting case of the discrete version.

5. Consider a Poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t]$ and M be the number of arrivals in $(0, t+s]$, where $t, s \geq 0$.

   (a) Find the conditional PMF of M given N, $p_{M|N}(m \mid n)$, for $m \geqslant n$.
   Due to the memorylessness property of Poisson process, we have

$$p_{M|N}(m \mid n) = p_{N_s}(m-n) = \frac{(\lambda s)^{m-n}}{(m-n)!} e^{-\lambda s}.$$

   (b) Find the joint PMF of N and M, $p_{N,M}(n, m)$.
   By multiplication rule, we have

$$p_{N,M}(n, m) = p_N(n)\, p_{M|N}(m \mid n) = \begin{cases} \dfrac{(\lambda s)^m (t/s)^n}{n!(m-n)!} e^{-\lambda(t+s)}, & 0 \leqslant n \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

   (c) Find the conditional PMF of N given M, $p_{N|M}(n \mid m)$, for $n \leqslant m$, using your answer to part (b).
   By Bayes' rule, we have

$$p_{N|M}(n \mid m) = \frac{p_N(n)\, p_{M|N}(m \mid n)}{p_M(m)}.$$

   For the denominator, by total probability theorem we have

$$\begin{aligned} p_M(m) &= \sum_{n=0}^{\infty} p_N(n)\, p_{M|N}(m \mid n) \\ &= \sum_{n=0}^{m} \frac{(\lambda s)^m (t/s)^n}{n!(m-n)!} e^{-\lambda(t+s)} \\ &= \frac{(\lambda s)^m}{m!} e^{-\lambda(t+s)} \sum_{n=0}^{m} \binom{m}{n} \left(\frac{t}{s}\right)^n \\ &= \left(1 + \frac{t}{s}\right)^m \frac{(\lambda s)^m}{m!} e^{-\lambda(t+s)} \\ &= \frac{[\lambda(t+s)]^m}{m!} e^{-\lambda(t+s)}. \end{aligned}$$

   By looking at the above result, we recognize that p_M can also be derived by direct application of Poission PMF.
   Finally, when $n \leqslant m$ we have

$$\begin{aligned} p_{N|M}(n \mid m) &= \frac{\frac{(\lambda s)^m (t/s)^n}{n!(m-n)!} e^{-\lambda(t+s)}}{\left(1 + \frac{t}{s}\right)^m \frac{(\lambda s)^m}{m!} e^{-\lambda(t+s)}} \\ &= \binom{m}{n} \left(\frac{t}{s}\right)^n \left(1 + \frac{t}{s}\right)^{-m}. \end{aligned}$$

   (d) Rederive your answer to part (c) without using part (b). As a hint, consider what kind of distribution the kth arrival time has if we are given the event $\{M = m\}$, where $k \leqslant m$.
   Some references for the relationship between uniform distribution and Poisson process can be found at *The Uniform Distribution and the Poisson Process* by Ilya Goldsheid, section 1.10 from *Notes on the Poisson Process* by Karl Sigman, theorem 18.5 from *Lecture Notes for Introductory Probability* by Janko Gravner.

The description of the hint is somewhat misleading. It should be read as "consider the joint distribution of m arrival times given $M = m$."

Conditioned upon $M = m$, the joint PDF of m arrival times $X_1, \ldots, X_m$ is

$$
\begin{aligned}
f_{X_1,\ldots,X_m|M}(x_1,\ldots,x_m \mid m) &= \frac{f_{T_1}(x_1) f_{T_2}(x_2 - x_1) \cdots f_{T_n}(x_n - x_{n-1})\, P(T_{n+1} > t + s - x_n)}{P(M = m)} \\
&= \frac{\lambda e^{-\lambda x_1} \lambda e^{-\lambda(x_2 - x_1)} \ldots \lambda e^{-\lambda(x_n - x_{n-1})} e^{-\lambda(t+s-x_n)}}{P(M = m)} \\
&= \frac{\lambda^m e^{-\lambda(t+s)}}{\frac{\lambda^m (t+s)^m}{m!} e^{-\lambda(t+s)}} = \frac{m!}{(t+s)^m}
\end{aligned}
$$

for $0 < x_1 < x_2 < \cdots < x_m < t + s$, where T_k is the kth interarrival time. This is also the joint PDF of the order statistics of some uniform r.v.s $U_1, U_2, \ldots, U_m$. That is

$$
f_{X_1,\ldots,X_m|M}(x_1,\ldots,x_m \mid m) = f_{U_{(1)},\ldots,U_{(m)}}(x_1,\ldots,x_m).
$$

Thus,

$$
\begin{aligned}
p_{N|M}(n \mid m) &= P(X_n < t < X_{n+1} \mid M = m) \\
&= P(U_{(n)} < t < U_{(n+1)}) \\
&= \binom{m}{n} P(U < t)^n P(U > t)^{m-n} \\
&= \binom{m}{n} \left(\frac{t}{t+s}\right)^n \left(\frac{s}{t+s}\right)^{m-n}.
\end{aligned}
$$

(e) Find $\mathrm{E}[NM]$.

By the law of iterated expectation, we have

$$
\begin{aligned}
\mathrm{E}[NM] &= \mathrm{E}[\mathrm{E}[NM \mid N]] \\
&= \mathrm{E}[N\, \mathrm{E}[M \mid N]] \\
&= \mathrm{E}[N(N + \mathrm{E}[M - N \mid N])] \\
&= \mathrm{E}[N(N + \lambda s)] \\
&= \mathrm{E}[N^2] + \lambda s\, \mathrm{E}[N] \\
&= \lambda t(1 + \lambda s).
\end{aligned}
$$

6. The interarrival times for cars passing a checkpoint are independent random variables with PDF

$$
f_T(t) = \begin{cases} 2e^{-2t}, & \text{for } t > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

where the interarrival times are measured in minutes. The successive experimental values of the durations of these interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

(a) Determine the mean and the third moment of the interarrival times.

We can either integrate directly or use moment generating function. By direct integration, the mean is

$$
\begin{aligned}
\mathrm{E}[T] &= \int_0^{+\infty} t \lambda e^{-\lambda t}\, dt = \int_{+\infty}^0 t\, de^{-\lambda t} \\
&= t e^{-\lambda t}\Big|_{+\infty}^0 + \frac{1}{\lambda} \int_{+\infty}^0 de^{-\lambda t} \\
&= \frac{1}{\lambda} = \frac{1}{2}.
\end{aligned}
$$

The kth moment of the interarrival time can be found inductively.

$$\begin{aligned}
\mathrm{E}[T^k] &= \int_0^{+\infty} t^k \lambda e^{-\lambda t}\, dt = \int_{+\infty}^{0} t^k \, de^{-\lambda t} \\
&= t^k e^{-\lambda t}\Big|_{+\infty}^{0} + \frac{k}{\lambda}\int_0^{+\infty} t^{k-1}\lambda e^{-\lambda t}\, dt \\
&= \frac{k}{\lambda}\,\mathrm{E}[T^{k-1}] \\
&= \frac{k}{\lambda}\frac{k-1}{\lambda}\cdots\frac{1}{\lambda} \\
&= \frac{k!}{\lambda^k}.
\end{aligned}$$

Thus, $\mathrm{E}[T^3] = 6/\lambda^3 = 3/4$.
To use moment generating function, we have

$$\begin{aligned}
M_s(T) = \mathrm{E}[e^{sT}] &= \int_0^{+\infty} e^{st}\lambda e^{-\lambda t}\, dt \\
&= \frac{\lambda}{s-\lambda} e^{(s-\lambda)t}\Big|_0^{+\infty} \\
&= \frac{\lambda}{\lambda - s}
\end{aligned}$$

for $s < \lambda$. Thus, we have

$$\begin{aligned}
\mathrm{E}[T] &= \frac{d}{ds} M_s(T)\Big|_{s=0} \\
&= \frac{\lambda}{(\lambda-s)^2}\Big|_{s=0} \\
&= \frac{1}{\lambda} = \frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}[T^3] &= \frac{d^3}{ds^3} M_s(T)\Big|_{s=0} \\
&= \frac{6\lambda}{(\lambda-s)^4}\Big|_{s=0} \\
&= \frac{6}{\lambda^3} = \frac{3}{4}.
\end{aligned}$$

(b) Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.
Due to the memorylessness property of Poisson process, we have

$$p_K(k) = \frac{12^k}{k!} e^{-12}.$$

(c) Determine the PDF and the expected value for the total time required to use up the first dozen computer cards.
This is equivalent to ask for the (3×12)th arrival time. Thus, the PDF and mean for such total time X_{36} are

$$f_{X_{36}}(x) = \frac{2^{36} x^{35}}{35!} e^{-2x}$$

and

$$\mathrm{E}[X_{36}] = \frac{36}{2} = 18.$$

(d) Consider the following two experiments:

i. Pick a card at random from a group of completed cards and note the total time, Y, the card was in service. Find $\mathrm{E}[Y]$ and $\mathrm{var}(Y)$.
It is easy to see that $Y = X_3$. So we have $\mathrm{E}[Y] = 3/2$ and $\mathrm{var}(Y) = 3/4$.

ii. Come to the corner at a random time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time W. Determine $\mathrm{E}[W]$ and $\mathrm{var}(W)$.
Again, this is a random incidence problem. Thus, the service time W is an Erlang distribution of order 4 and we have $\mathrm{E}[W] = 4/2 = 2$ and $\mathrm{var}(W) = 4/4 = 1$.

G1†. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geqslant 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

$$P(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \mid N(G) = k).$$

Since $k = \sum_{i=1}^{n} k_i$ and $G = \bigcup_{1\leqslant i\leqslant n} G_i$, we can see that $\bigcap_{1\leqslant i\leqslant n}\{N(G_i) = k_i\}$ is a subset of $\{N(G) = k\}$. This means $(\bigcap_{1\leqslant i\leqslant n}\{N(G_i) = k_i\}) \cap \{N(G) = k\} = \bigcap_{1\leqslant i\leqslant n}\{N(G_i) = k_i\}$. Thus, also due to the independence property of Poisson process, we have

$$\begin{aligned}
P(N(G_1) = k_1, \ldots, N(G_n) = k_n \mid N(G) = k) &= \frac{P(N(G_1) = k_1, \ldots, N(G_n) = k_n, N(G) = k)}{P(N(G) = k)} \\
&= \frac{P(N(G_1) = k_1, \ldots, N(G_n) = k_n)}{P(N(G) = k)} \\
&= \frac{P(N(G_1) = k_1) \cdots P(N(G_n) = k_n)}{P(N(G) = k)} \\
&= \frac{\frac{(\lambda c_1)^{k_1}}{k_1!} e^{-\lambda c_1} \frac{(\lambda c_2)^{k_2}}{k_2!} e^{-\lambda c_2} \cdots \frac{(\lambda c_n)^{k_n}}{k_n!} e^{-\lambda c_n}}{\frac{(\lambda c)^k}{k!} e^{-\lambda c}} \\
&= \frac{\frac{\lambda^{k_1+k_2+\cdots+k_n} c_1^{k_1} c_2^{k_2} \cdots c_n^{k_n}}{k_1! k_2! \cdots k_n!} e^{-\lambda(c_1+c_2\cdots+c_n)}}{\frac{\lambda^k c^k}{k!} e^{-\lambda c}} \\
&= \binom{k}{k_1, \ldots, k_n} \left(\frac{c_1}{c}\right)^{k_1} \cdots \left(\frac{c_n}{c}\right)^{k_n}.
\end{aligned}$$

†Required for 6.431; optional for 6.041