Solutions to Recitation 15

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This is my own solutions to the problems from recitation 15 of 6.041SC.

1. Beginning at time $t = 0$, we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

$$\text{for type-A bulbs:} \quad f_X(x) = \begin{cases} e^{-x}, & x \geqslant 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{for type-B bulbs:} \quad f_X(x) = \begin{cases} 3e^{-3x}, & x \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the expected time until the first failure.

Let Y be the random variable such that $Y = 0$ whenever type-A bulb is chosen and $Y = 1$ whenever type-B bulb is chosen. The expected time until the first failure actually is $\mathrm{E}[X]$. By the law of iterated expectation, we have

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[\mathrm{E}[X \mid Y]] \\ &= p_Y(0)\,\mathrm{E}[X \mid Y = 0] + p_Y(1)\,\mathrm{E}[X \mid Y = 1] \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

(b) Find the probability that there are no bulb failures before time t.

This probability is $P(X > t)$. By total probability theorem, we have

$$\begin{aligned} P(X > t) &= P(Y = 0)P(X > t \mid Y = 0) + P(Y = 1)P(X > t \mid Y = 1) \\ &= \frac{1}{2}\int_t^{+\infty} e^{-x}\,dx + \frac{1}{2}\int_t^{+\infty} 3e^{-3x}\,dx \\ &= \frac{1}{2}e^{-x}\Big|_{+\infty}^{t} + \frac{1}{2}e^{-3x}\Big|_{+\infty}^{t} \\ &= \frac{1}{2}(e^{-t} + e^{-3t}). \end{aligned}$$

(c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

By Bayes' theorem, we have

$$\begin{aligned} P(Y = 0 \mid X > t) &= \frac{P(Y = 0)P(X > t \mid Y = 0)}{P(X > t)} \\ &= \frac{e^{-t}/2}{(e^{-t} + e^{-3t})/2} \\ &= \frac{1}{1 + e^{-2t}}. \end{aligned}$$

(d) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

Let X_1, X_2 be the lifetime of first two type-B bulbs and X_3 be the lifetime of first type-A bulb. The probability to be determined is actually $P(X_1 + X_2 > X_3) = P(X_1 + X_2 - X_3 > 0)$. Let $S = X_1 + X_2$ and $Z = S - X_3$. By Erlang distribution, when $s \geqslant 0$ we have

$$f_S(s) = \frac{3^2 s e^{-3s}}{(2-1)!} = 9se^{-3s}.$$

By convolution formula, we have

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_S(s) f_{-X_3}(z-s)\, ds \\
&= \int_{-\infty}^{+\infty} f_S(s) f_{X_3}(s-z)\, ds \\
&= \int_{\max(0,z)}^{+\infty} 9se^{-3s} e^{-(s-z)}\, ds \\
&= 9e^z \int_{\max(0,z)}^{+\infty} se^{-4s}\, ds \\
&= \frac{9}{4} e^z \left(s + \frac{1}{4}\right) e^{-4s} \Big|_{+\infty}^{\max(0,z)} \\
&= \begin{cases} \frac{9}{16} e^z, & z \leqslant 0, \\ \frac{9}{4}\left(z + \frac{1}{4}\right) e^{-3z}, & z > 0. \end{cases}
\end{aligned}$$

Thus the probability to be determined is

$$\begin{aligned}
P(Z > 0) &= 1 - P(Z \leqslant 0) \\
&= 1 - \int_{-\infty}^{0} \frac{9}{16} e^z\, dz \\
&= 1 - \frac{9}{16} = \frac{7}{16}.
\end{aligned}$$

(e) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

Let K be the number of type-B bulbs occurred in the process and X be the total period of illumination provided by type-B bulbs. Fix $K = k$, we have

$$\mathrm{E}[X \mid K = k] = \frac{k}{3}$$

and

$$\mathrm{var}(X \mid K = k) = \frac{k}{9}.$$

Therefore,

$$\mathrm{E}[X \mid K] = \frac{K}{3}$$

and

$$\mathrm{var}(X \mid K) = \frac{K}{9}.$$

By law of iterated expectation and law of total variance, we have

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[\mathrm{E}[X \mid K]] \\
&= \mathrm{E}[K/3] \\
&= \frac{1}{3}\,\mathrm{E}[K] \\
&= \frac{1}{3}\cdot 12 \cdot \frac{1}{2} = 2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}[\mathrm{var}(X \mid K)] + \mathrm{var}(\mathrm{E}[X \mid K]) \\
&= \mathrm{E}[K/9] + \mathrm{var}(K/3) \\
&= \frac{1}{9}\,\mathrm{E}[K] + \frac{1}{9}\,\mathrm{var}(K) \\
&= \frac{1}{9}(6+3) = 1.
\end{aligned}$$

(f) Given that there are no failures until time t, find the expected value of the time until the first failure.

The entire process is not memoryless; however, condtioned upon chosen type of bulbs, the conditional probability of bulb lifetime is memeryless. Using notation from (a)–(c), we have

$$\begin{aligned}
\mathrm{E}[X \mid X > t] &= t + \mathrm{E}[X - t \mid X > t] \\
&= t + P(Y = 0 \mid X > t)\,\mathrm{E}[X - t \mid X > t, Y = 0] \\
&\quad + P(Y = 1 \mid X > t)\,\mathrm{E}[X - t \mid X > t, Y = 1] \\
&= t + \frac{1}{1 + e^{-2t}} + \frac{1}{3(1 + e^{2t})}.
\end{aligned}$$

2. A service station handles jobs of two types, A and B. (Multiple jobs can be processed simultaneously.) Arrivals of the two job types are independent Poisson processes with parameters $\lambda_A = 3$ and $\lambda_B = 4$ per minute, respectively. Type A jobs stay in the service station for exactly one minute. Each type B job stays in the service station for a random but integer amount of time which is geometrically distributed, with mean equal to 2, and independent of everything else. The service station started operating at some time in the remote past.

(a) What is the mean, variance, and PMF of the total number of jobs that arrive within a given three-minute interval?

Let K be the total number of job arrivals within a given three-minute interval. The merging of two Poisson process is another Poisson process with parameter $\lambda_A + \lambda_B$, thus we have

$$p_K(k) = \frac{(3(3+4))^k}{k!} e^{-3(3+4)} = \frac{21^k}{k!} e^{-21},$$

$$\mathrm{E}[K] = 21,$$

and

$$\mathrm{var}(K) = 21.$$

(b) We are told that during a 10-minute interval, exactly 10 new jobs arrived. What is the probability that exactly 3 of them are of type A?

Since the number of type A and type B job arrivals during a 10-minute interval is indepent from each other. Let $K_{\lambda\tau}$ denoate the Poisson random variable with parameter $\lambda\tau$, the probability to be determined is

$$
\begin{aligned}
P(K_{3\times 10} = 3 \mid K_{(3+4)\times 10} = 10) &= \frac{P(K_{30} = 3)\, P(K_{40} = 7)}{P(K_{70} = 10)} \\
&= \frac{\frac{30^3}{3!}e^{-30}\, \frac{40^7}{7!}e^{-40}}{\frac{70^{10}}{10!}e^{-70}} \\
&= \binom{10}{3}\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^7 \approx 18.79\%.
\end{aligned}
$$

(c) At time 0, no job is present in the service station. What is the PMF of the number of type B jobs that arrive in the future, but before the first type A arrival?

Let K denote the number of type B job arrival before the first type A job arrival and X denote the time of first type A arrival. Then, we have

$$
P(K = k \mid X = t) = \frac{(4t)^k}{k!}e^{-4t}
$$

and

$$
\begin{aligned}
P(K) &= \int_0^{+\infty} f_X(t) P(K = k \mid X = t)\, dt \\
&= \frac{3 \times 4^k}{k!} \int_0^{+\infty} t^k\, e^{-7t}\, dt.
\end{aligned}
$$

3. Let X, Y, and Z be independent exponential random variables with parameters λ, μ, and ν, respectively. Find $P(X < Y < Z)$.

$$
\begin{aligned}
P(X < Y < Z) &= \iiint_{\{(x,y,z) \mid x<y<z\}} f_X(x) f_Y(y) f_Z(z)\, dx\, dy\, dz \\
&= \int_0^{+\infty}\int_x^{+\infty}\int_y^{+\infty} f_X(x) f_Y(y) f_Z(z)\, dz\, dy\, dx \\
&= \int_0^{+\infty} f_X(x) \int_x^{+\infty} f_Y(y) \int_y^{+\infty} f_Z(z)\, dz\, dy\, dx \\
&= \int_0^{+\infty} \lambda e^{-\lambda x} \int_x^{+\infty} \mu e^{-\mu y} \int_y^{+\infty} \nu e^{-\nu z}\, dz\, dy\, dx \\
&= \int_0^{+\infty} \lambda e^{-\lambda x} \int_x^{+\infty} \mu e^{-(\mu+\nu) y}\, dy\, dx \\
&= \frac{\lambda\mu}{\mu+\nu} \int_0^{+\infty} e^{-(\lambda+\mu+\nu)x}\, dx \\
&= \frac{\lambda\mu}{(\mu+\nu)(\lambda+\mu+\nu)}.
\end{aligned}
$$