Solutions to Tutorial 7

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This is my own solutions to the problems from tutorial 7 of 6.041sc.

1. Alice and Bob alternate playing at the casino table. (Alice starts and plays at odd times $i = 1, 3, \ldots$; Bob plays at even times $i = 2, 4, \ldots$.) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$p_G(g) = \begin{cases} 1/3, & g = -2, \\ 1/2, & g = 1, \\ 1/6, & g = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that the net gains at different times are independent. We refer to an outcome of -2 as a "loss."

(a) They keep gambling until the first time where a loss by Bob immediately follows a loss by Alice. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Alice and then one by Bob.)

The probability that any round terminates the gambling is $1/3 \cdot 1/3 = 1/9$. The total number of rounds played is a geometric random variable (denoted by K) with parameter $1/9$, whose the PMF is

$$p_K(k) = \left(\frac{8}{9}\right)^{k-1} \frac{1}{9}.$$

(b) Write down the PMF for Z, defined as the time at which Bob has his third loss.

The random variable Z has the Pascal distribution of order 3 with parameter $1/3$, whose PMF is

$$p_Z(z) = \binom{z-1}{2} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{z-3}.$$

(c) Let N be the number of rounds until each one of them has won at least once. Find $\mathrm{E}[N]$.

We can view the process as a two-step process. The first step is that the gambling continues until any one of them has won. Given the first step, the second step is that the gambling continues until the other player has won. We use W_1 and W_2 to denote the number of rounds during the first and second steps respectively.

At each round, the probablity that Alice wins is

$$\begin{aligned} P(G_i > G_{i+1}) &= p_{G_i}(3)\, P(G_{i+1} < 3) + p_{G_i}(1)\, P(G_{i+1} < 1) \\ &= \frac{1}{6}\left(1 - \frac{1}{6}\right) + \frac{1}{2}\frac{1}{3} = \frac{11}{36}. \end{aligned}$$

By symmetry, the probability that Bob wins at each round is also

$$P(G_i < G_{i+1}) = \frac{11}{36}.$$

This implies that the probability distribution of W_2 is unaffected by the outcome of W_1. The probablity that either Alice or Bob wins at each round is

$$P(G_i > G_{i+1} \text{ or } G_i < G_{i+1}) = P(G_i > G_{i+1}) + P(G_i < G_{i+1}) = \frac{11}{18}.$$

So W_1 and W_2 both have the geometric distribution with parameter $11/18$ and $11/36$ respectively.

Finally, the expectation to be found is

$$\begin{aligned} \mathrm{E}[N] &= \mathrm{E}[W_1 + W_2] \\ &= \mathrm{E}[W_1] + \mathrm{E}[W_2] \\ &= \frac{18}{11} + \frac{36}{11} = \frac{54}{11}. \end{aligned}$$

2. *Sum of a geometric number of independent geometric random variables*

   Let $Y = X_1 + \cdots + X_N$, where the random variable X_i are geometric with parameter p, and N is geometric with parameter q. Assume that the random variables N, X_1, X_2, … are independent. Show that Y is geometric with parameter pq. Hint: Interpret the various random variables in terms of a split Bernoulli process.

3. A train bridge is constructed across a wide river. Trains arrive at the bridge according to a Poisson process of rate $\lambda = 3$ per day.

   (a) If a train arrives on day 0, find the probability that there will be no trains on days 1, 2, and 3.

   Let X be the day of next train arrival, then the probabiltiy to be found is

   $$P(X > 3) = \int_3^{+\infty} 3e^{-3t}\, dt = e^{-3t}\Big|_{+\infty}^{3} = e^{-9}.$$

   (b) Find the probability that the next train to arrive after the first train on day 0, takes more than 3 days to arrive.

   $$P(X > 3) = \int_3^{+\infty} 3e^{-3t}\, dt = e^{-3t}\Big|_{+\infty}^{3} = e^{-9}.$$

   (c) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4th day.

   $$P(X > 2)P(4 \text{ train arrives on 4th day}) = \left(\int_2^{+\infty} 3e^{-3t}\, dt\right)\left(\frac{3^4}{4!}e^{-3}\right) = \frac{3^4}{4!}e^{-9}.$$

   (d) Find the probability that it takes more than 2 days for the 5th train to arrive at the bridge.

   $$\begin{aligned} \int_2^{+\infty} \frac{3^5 t^4}{5!} e^{-3t}\, dt &= \sum_{i=0}^{4} \frac{6^i}{i!} e^{-6} \\ &= (1 + 6 + 18 + 36 + 54)e^{-6} \\ &= 115e^{-6}. \end{aligned}$$