

Solutions to Problem Set 1

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This is my own solution to the problem set 1 of [18.014](#). The first three problems are from Apostol's *Calculus* and the next three problems are from Munkres's Course Note A (1: 18, 20, 43; 9–10).

1. Prove Theorem I.11: If $ab = 0$, then $a = 0$ or $b = 0$.

Proof. If not, then $a \neq 0$ and $b \neq 0$, which implies

$$\begin{aligned}
 1 &= 1 \cdot 1 && \text{by the identity axiom,} \\
 &= (aa^{-1})(bb^{-1}) && \text{by the inverse axiom,} \\
 &= (a^{-1}a)(bb^{-1}) && \text{by commutativity,} \\
 &= a(bb^{-1}) && \text{by associativity,} \\
 &= a(ab)b^{-1} && \text{by associativity,} \\
 &= a(0 \cdot b^{-1}) && \text{by assumption,} \\
 &= a \cdot 0 && \text{by Theorem I.6,} \\
 &= 0 && \text{by Theorem I.6.}
 \end{aligned}$$

But we know by Axiom 4 that $1 \neq 0$. Thus, either $a = 0$ or $b = 0$. □

2. Prove Theorem I.25: If $a < c$ and $b < d$, then $a + b < c + d$.

Proof. We have

$$\begin{aligned}
 a + b &< c + b && \text{by Theorem I.18,} \\
 &= b + c && \text{by commutativity,} \\
 &< d + c && \text{by Theorem I.18,} \\
 &= c + d && \text{by commutativity.}
 \end{aligned}$$
□

3. Prove that $||x| - |y|| \leq |x - y|$.

There are two ways to prove this. One is logical and the other is algebraic.

Proof. If $|x| - |y| \geq 0$, then

$$||x| - |y|| = |x| - |y| \leq |x - y|.$$

If $|x| - |y| < 0$, then

$$||x| - |y|| = -(|x| - |y|) = |y| - |x| \leq |y - x| = |x - y|.$$

In all cases, the inequality holds. □

Proof. We have

$$\begin{aligned}
 xy &\leq |xy| = |x||y| && \iff \\
 -2|x||y| &\leq -2xy && \iff \\
 |x|^2 - 2|x||y| + |y|^2 &\leq x^2 - 2xy + y^2 && \iff \\
 (|x| - |y|)^2 &\leq (x - y)^2 && \iff \\
 \sqrt{(|x| - |y|)^2} &\leq \sqrt{(x - y)^2} && \iff \\
 ||x| - |y|| &\leq |x - y|. && \iff
 \end{aligned}$$
□

4. Prove Theorem 6: If a and b are in \mathbf{P} , so is ab .

Proof. We choose a fixed positive integer a and then prove this theorem by induction on b . First, $a \cdot 1$ is in \mathbf{P} , since $a \cdot 1 = a$ by the identity axiom and a is in \mathbf{P} . Now suppose b is a positive integer such that ab is in \mathbf{P} . Then we have $a(b+1) = ab + a$ by distributivity. By the induction hypothesis, ab is a positive integer. And a is also a positive integer. We have already shown that \mathbf{P} is closed under addition. Thus, $ab + a$ is also a positive integer, which means $a(b+1)$ is in \mathbf{P} . This proves that \mathbf{P} is also closed under multiplication. \square

5. Prove Theorem 12: $a^n b^n = (ab)^n$, where a and b are any real numbers and n is a positive integer.

Proof. The formula is trivially true for $n = 1$, since $a^1 b^1 = ab = (ab)^1$. Now suppose the formula holds for some n . Then

$$\begin{aligned} a^{n+1} b^{n+1} &= (a^n a) (b^n b) && \text{by definition of exponent,} \\ &= (a^n b^n) (ab) && \text{by associativity and commutativity,} \\ &= (ab)^n (ab) && \text{by the induction hypothesis,} \\ &= (ab)^{n+1} && \text{by definition of exponent.} \end{aligned}$$

This proves the formula holds for any positive integer n . \square

6. Let a and h be real numbers; let m be a positive integer. Show by induction that if a and $a+h$ are positive, then

$$(a+h)^m \geq a^m + ma^{m-1}h.$$

Proof. The inequality is true for $m = 1$, since $(a+h)^1 = a+h = a^1 + 1 \cdot a^{1-1}h$. Now suppose it is true for some $m \geq 1$, then

$$\begin{aligned} (a+h)^{m+1} &= (a+h)^m (a+h) && \text{by definition of exponent,} \\ &\geq (a^m + ma^{m-1}h)(a+h) && \text{by the induction hypothesis} \\ &&& \text{and positivity of } a+h, \\ &= a^{m+1} + ma^m h + a^m h + ma^{m-1}h^2 && \text{by distributivity, associativity,} \\ &&& \text{commutativity, and definition of exponent,} \\ &= a^{m+1} + (m+1)a^m h + ma^{m-1}h^2 && \text{by associativity, the identity axiom,} \\ &&& \text{and distributivity,} \\ &\geq a^{m+1} + (m+1)a^m h && \text{by positivity of } m, a, \text{ and } h^2, \\ &&& \text{the closure axiom, and Theorem I.18.} \end{aligned} \quad \square$$

- 7.* Let

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{and} \quad G_n = (x_1 x_2 \cdots x_n)^{1/n}$$

represent the arithmetic and geometric mean respectively for a set of n positive real numbers.

- (a) Prove that $G_n \leq A_n$ for $n = 2$.

Proof. Let a and b be any arbitrary real numbers. Then

$$\begin{aligned} (a-b)^2 &\geq 0 && \text{by Theorem I.20,} && \Leftrightarrow \\ a^2 + b^2 &\geq 2ab && \text{by simple algebra.} \end{aligned}$$

By Theorem I.35, $x_1^{1/2}$ and $x_2^{1/2}$ are meaningful. Let $a = x_1^{1/2}$ and $b = x_2^{1/2}$. Then

$$\begin{aligned} \frac{x_1 + x_2}{2} &\geq (x_1 x_2)^{1/2} && \text{by simple algebra,} && \Leftrightarrow \\ G_2 &\leq A_2 && \text{by definition.} \end{aligned} \quad \square$$

*Only to be attempted once other problems are completed

(b) Use induction to show $G_n \leq A_n$ for any $n = 2^k$ where k is a positive integer.

Proof. The previous part proves that the inequality holds for $k = 1$. Suppose the inequality holds for some $k \geq 1$. By the induction hypothesis, we have

$$\frac{1}{2^k} \sum_{i=1}^{2^k} x_i \geq \left(\prod_{i=1}^{2^k} x_i \right)^{1/2^k} \quad (1)$$

and

$$\frac{1}{2^k} \sum_{i=2^{k+1}}^{2^{k+1}} x_i \geq \left(\prod_{i=2^{k+1}}^{2^{k+1}} x_i \right)^{1/2^k} \quad (2)$$

By adding (1) and (2) and halving the sum, we obtain

$$\frac{1}{2^{k+1}} \sum_{i=1}^{2^{k+1}} x_i \geq \frac{1}{2} \left\{ \left(\prod_{i=1}^{2^k} x_i \right)^{1/2^k} + \left(\prod_{i=2^{k+1}}^{2^{k+1}} x_i \right)^{1/2^k} \right\} \geq \left(\prod_{i=1}^{2^{k+1}} x_i \right)^{1/2^{k+1}}$$

The last inequality sign holds by applying the induction hypothesis to the right-hand sides of both (1) and (2). \square

(c) Now for any positive integer n , suppose $n < 2^m$ for some integer m . Using the set

$$\{x_1, x_2, \dots, x_n, A_n, A_n, \dots, A_n\}$$

where the A_n appears $2^m - n$ times in the set, show that $G_n \leq A_n$.

Proof. Apply to the set the result from the previous problem and we have

$$\begin{aligned} A_n &= \frac{1}{2^m} [(2^m - n)A_n + nA_n] \\ &= \frac{1}{2^m} \left\{ (2^m - n)A_n + \sum_{i=1}^n x_i \right\} \\ &\geq \left\{ \left(\prod_{i=1}^{2^m-n} A_n \right) \left(\prod_{i=1}^n x_i \right) \right\}^{1/2^m} \\ &= (A_n^{2^m-n} G_n^n)^{1/2^m} \\ &= \left(\frac{A_n}{G_n} \right)^{\frac{2^m-n}{2^m}} G_n. \end{aligned}$$

Now suppose $A_n < G_n$, then we have

$$\frac{A_n}{G_n} < \left(\frac{A_n}{G_n} \right)^{\frac{2^m-n}{2^m}} < 1,$$

which implies

$$A_n = \frac{A_n}{G_n} \cdot G_n < \left(\frac{A_n}{G_n} \right)^{\frac{2^m-n}{2^m}} G_n \leq A_n.$$

This is absurd. Thus, it must be the case that $G_n \leq A_n$, which is consistent with the above result. \square