Solutions to Recitation 7

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This is my own solution to the Recitation 7 from 18.014, which includes Exercise 4, 10, and 15 from Apostol's *Calculus* (1: 70–71).

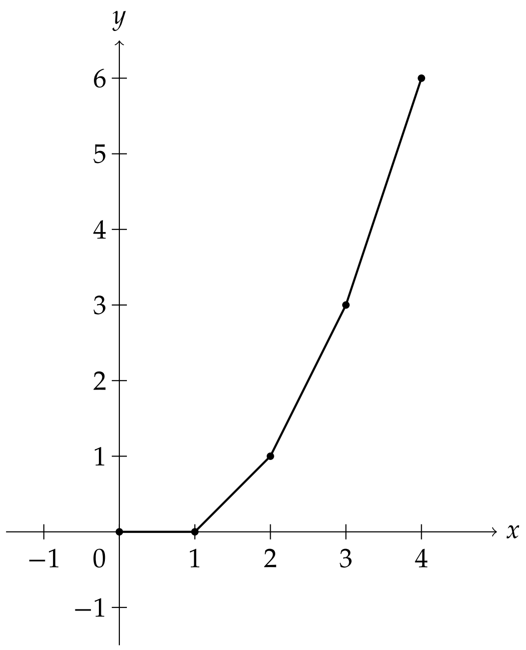
1. (a) If n is a positive integer, prove that $\int_0^n [t]\,dt = n(n-1)/2$.

   Proof. Let $P = \{0, 1, \dots, n\}$. Then P is partition for $[t]$ on $[0, n]$ and $s_k = k - 1$ on each open subinterval. Hence

$$
\begin{aligned}
\int_0^n [t]\,dt &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) && \text{by definition,}\\
&= \sum_{k=1}^{n} (k-1)\{k - (k-1)\} && \text{by our choice of partion,}\\
&= 0 + \sum_{k=1}^{n-1} k && \text{by simple algebra and reindexing,}\\
&= \frac{n(n-1)}{2}.
\end{aligned}
$$

□

   (b) If $f(x) = \int_0^x [t]\,dt$ for $x \geqslant 0$, draw the graph of f over the interval $[0, 4]$.



2. Given a positive integer p. A step function s is defined on the interval $[0, p]$ as follows: $s(x) = (-1)^n n$ if x lies in the interval $n \leqslant x < n + 1$, where $n = 0, 1, 2, \dots, p - 1$; $s(p) = 0$. Let $f(p) = \int_0^p s(x)\,dx$.

   (a) Calculate $f(3)$, $f(4)$, and $f(f(3))$.
   It's easy to see that

$$
f(p) = \int_0^p s(x)\,dx = \sum_{k=1}^{n} (-1)^{k-1}(k-1).
$$

Then we have

$$\begin{aligned} f(3) &= 0 - 1 + 2 = 1, \\ f(4) &= f(3) - 3 = -2, \end{aligned}$$

and

$$f(f(3)) = f(1) = 0.$$

(b) For what value (or values) of p is $|f(p)| = 7$?
Since p is assumed to be a positive integer, we have

$$f(p) = \sum_{k=1}^{n} (-1)^{k-1}(k-1) = (-1)^{p+1}[\tfrac{p}{2}].$$

Thus $|f(p)| = 7$ if and only if $f(p) = \pm 7$. It is only when $p = 14$ or $p = 15$ that the original equation is satisfied.
As an exercise, we can derive a formula without the greatest-integer function by summation by parts. The result is simply

$$\sum_{k=1}^{n} (-1)^{k-1}(k-1) = \tfrac{1}{2}\{(p - \tfrac{1}{2})(1 - (-1)^p) - p\}.$$

3. Prove Theorem 1.5 (the comparison theorem).

Proof. Notice that it is assumed that $a < b$. Since s and t are step functions, so are $-s$ and $t - s$. Furthermore, $t(x) - s(x) > 0$ for every x in $[a, b]$. Let $u = t - s$. Then there exists a partition $\{x_0, x_1, \dots, x_n\}$ such that u is constant on each open subinterval. Let u_k denote the constant on the kth subinterval. Hence

$$\begin{aligned} \int_a^b (t - s) &= \sum_{k=1}^{n} u_k(x_k - x_{k-1}) && \text{by definition,} \\ &> 0 && \text{by Axiom 7 and Theorem I.25.} \end{aligned}$$

By linearity of integral, we have

$$\int_a^b t - \int_a^b s = \int_a^b (t - s) > 0. \qquad \square$$