Solutions to Recitation 6

L. F. Jaw

March 16, 2021

This is my own solution to the Recitation 6 from 18.014, which includes Exercise 1bcde, 2, 3, and 13 from Apostol's *Calculus* (1: 70–71).

1. Compute the value of each of the following integrals. You may use theorems of Section 1.13 whenever it is convenient to do so. The notation $[x]$ denotes the greatest integer $\leqslant x$.

   (a) $\int_{-1}^{3}[x+\frac{1}{2}]\,dx$.

   $$\int_{-1}^{3}[x+\tfrac{1}{2}]\,dx = \int_{-0.5}^{3.5}[x]\,dx = -1\times 0.5 + 0 + 1 + 2 + 3\times 0.5 = 4.$$

   (b) $\int_{-1}^{3}([x]+[x+\frac{1}{2}])\,dx$.

   $$\int_{-1}^{3}([x]+[x+\tfrac{1}{2}])\,dx = \int_{-1}^{3}[x]\,dx + \int_{-1}^{3}[x+\tfrac{1}{2}]\,dx = 2+4 = 6.$$

   (c) $\int_{-1}^{3}2[x]\,dx$.

   $$\int_{-1}^{3}2[x]\,dx = 2\int_{-1}^{3}[x]\,dx = 2\times 2 = 4.$$

   (d) $\int_{-1}^{3}[2x]\,dx$.

   $$\int_{-1}^{3}[2x]\,dx = \frac{1}{2}\int_{-2}^{6}[x]\,dx = \frac{1}{2}\left(\int_{-2}^{-1}[x]\,dx + \int_{-1}^{3}[x]\,dx + \int_{3}^{6}[x]\,dx\right) = \frac{1}{2}(-2+2+3+4+5) = 6.$$

2. Give an example of a step function s, defined on the closed interval $[0,5]$, which has the following properties:

   $$\int_{0}^{2}s(x)\,dx = 5 \quad\text{and}\quad \int_{0}^{5}s(x)\,dx = 2.$$

   Simply define

   $$s(x) = \begin{cases} \frac{5}{2}, & 0\leqslant x<2,\\ -1, & 2\leqslant x\leqslant 5.\end{cases}$$

3. Show that $\int_a^b[x]\,dx + \int_a^b[-x]\,dx = a-b$.

   Proof. By linearity, we have

   $$\int_{a}^{b}[x]\,dx + \int_{a}^{b}[-x]\,dx = \int_{a}^{b}([x]+[-x])\,dx.$$

   Let us define $f(x)=[x]+[-x]$. Notice that

   $$f(x) = \begin{cases} 0, & x\text{ is an integer},\\ -1, & \text{otherwise}.\end{cases}$$

The case for $a = b$ is trivial. When $a < b$, we can find a partitation $P = \{x_0, x_1, \dots, x_n\}$, where $x_0 = a$, $x_n = b$, and $x_k \in \mathbf{Z}$ for all $a < k < b$. Then we have

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \sum_{k=1}^{n} (-1)(x_k - x_{k-1}) && \text{by definition of integral of step functions,}\\
&= (-1)\sum_{k=1}^{n} (x_k - x_{k-1}) && \text{by homogeneity of summation,}\\
&= (-1)(x_n - x_0) && \text{by telescoping property of summation,}\\
&= a - b && \text{by our choice of partition and simple algebra.}
\end{aligned}
$$

When $a > b$, we have

$$
\begin{aligned}
\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx && \text{by our convention,}\\
&= -(b-a) && \text{by the fact that } b < a,\\
&= a - b && \text{by simple algebra.} \qquad \square
\end{aligned}
$$

4. Prove Theorem 1.2 (the additive property).

Proof. When $a = b$, the proof is trivial since all are zeros. Then, we prove the property holds when $a < b$. We can find the common refinement $P = \{x_0, x_1, \dots, x_n\}$ of the partitions P_1 and P_2 corresponding to step functions s and t respectively. Simply let $P = P_1 \cup P_2$. Hence

$$
\begin{aligned}
\int_a^b s + \int_a^b t &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) + \sum_{k=1}^{n} t_k(x_k - x_{k-1}) && \text{by definition,}\\
&= \sum_{k=1}^{n} \{s_k(x_k - x_{k-1}) + t_k(x_k - x_{k-1})\} && \text{by additivity of summation,}\\
&= \sum_{k=1}^{n} (s_k + t_k)(x_k - x_{k-1}) && \text{by distributivity,}\\
&= \int_a^b (s+t) && \text{by definition.}
\end{aligned}
$$

When $a > b$, we have

$$
\begin{aligned}
\int_a^b (s+t) &= -\int_b^a (s+t) && \text{by our convention,}\\
&= -\left(\int_b^a s + \int_b^a t\right) && \text{by the fact that } b < a,\\
&= \left(-\int_b^a s\right) + \left(-\int_b^a t\right) && \text{by distributivity,}\\
&= \int_a^b s + \int_a^b t && \text{by our convention again.} \qquad \square
\end{aligned}
$$