Solutions to Recitation 3

L. F. Jaw

March 1, 2021

This is my own solution to the Recitation 3 of 18.014, which includes Theorem I.24 and Exercise 1a, 11, and 6 from Apostol's *Calculus* (1: 20, 35–36, 40).

1. If $ab > 0$, then both a and b are positive or both negative.

   Proof. If either a or b is zero, then $ab = 0$ by Theorem I.6. But by Axiom 9, $0 \not> 0$. Thus, neither a nor b is zero. If one is negative and the other positive, say, $a < 0$ and $b > 0$, then $-a > 0$ by Theorem I.23 and $-ab = (-a)b > 0$ by Theorem I.12 and Axiom 7. However, this contradicts with Axiom 8 since both $ab > 0$ and $-ab > 0$. It is easy to verify that no contradiction will be derived in the remaining scenarios. □

2. Prove $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ by induction.

   Proof. The basis is clearly true for $n = 1$ since $1 = 1 \cdot (1 + 1)/2$. Now suppose the above holds for some $k \geqslant 1$. Then

$$
\begin{aligned}
1 + 2 + 3 + \cdots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\
&= \left(\frac{k}{2} + 1\right)(k + 1) \\
&= \frac{(k + 2)(k + 1)}{2} \\
&= \frac{(k + 1)[(k + 1) + 1]}{2}.
\end{aligned}
$$

□

3. Let n and d denote integers. We say that d is a *divisor* of n if $n = cd$ for some integer c. An integer n is called a *prime* if $n > 1$ and if the only positive divisors of n are 1 and n. Prove, by induction, that every integer $n > 1$ is either a prime or a product of primes.

   Proof. It's easy to verify that 2 is a prime and its only positive divisors are 1 and 2. Now suppose the above statement holds for some $n > 1$. We are going to establish that $n + 1$ is either a prime or a product of primes. Suppose $n + 1$ is not a prime. Then by definition and simple reasoning there exist integers c and d such that $1 < c, d < n + 1$. Then by the induction hypothesis, c and d are either a prime or a product of primes, which implies $n + 1$ is a product of primes. Thus, $n + 1$ is either a prime or a product of primes (constructive dilemma). □

4. Derive the formula

$$\sum_{k=1}^{n} k^2 = \frac{n^2}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

   We have

$$
\begin{aligned}
3\left(\sum_{k=1}^{n} k^2\right) - 3\left(\sum_{k=1}^{n} k\right) + n &= \sum_{k=1}^{n} (3k^2 - 3k + 1) && \text{by linearity,} \\
&= \sum_{k=1}^{n} [k^3 - (k - 1)^3] \\
&= n^3 - 0^3 && \text{by telescoping property,} \\
&= n^3.
\end{aligned}
$$

We already know that $\sum_{k=1}^{n} k = n^2/2 + n/2$. Substitute this into the above equation, do some simple algebra, and we obtain

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{3}{2}n - n\right) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$