GEMOETRIC POINT OF VIEW OF P-INTEGRAL

The following result is given in the lecture,

$$\int_{x=1}^{+\infty} x^{-p}\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } 0 < p \le 1. \end{cases}$$

Once we have this, we can argue from a geometric point of view that $\int_0^1 x^{-p}\,dx$ is symmetric to the above result.

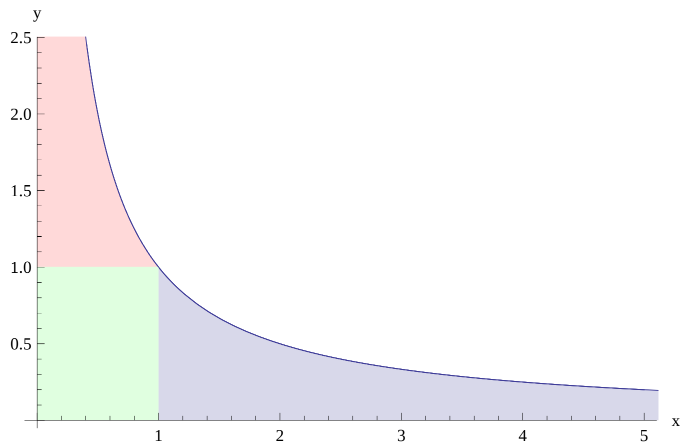

FIGURE 1.

The blue area of figure 1 can depict the definite integral $\int_{x=1}^{+\infty} x^{-p}\,dx$, whereas $\int_0^1 x^{-p}\,dx$ can be interpreted as the sum of green and red area. The green area is finite and it equals one. This way, we can see that $\int_0^1 x^{-p}\,dx$ converges if and only if the red area is finite.

If we can exchange the x-axis and the y-axis and the red area becomes the blue area, then we can reduce the problem to one that is already solved. Simply view the screen as a paper and flip (or mirror, reflect) the paper around the line $y = x$. Another way to visualize the process is this. View the screen as an odd page in a book (odd pages are always right pages, see recto and verso), turn the page and rotate 90° clockwise. You will get something like figure 2 on next page.

We shall now view x as a function of y and the relationship is

$$\begin{aligned} y &= x^{-p} \\ 1/y &= x^{p} \\ x &= y^{-1/p}. \end{aligned}$$

So the $1/p$ becomes the new p. The red area converges if and only if $1/p > 1$, this is the same as saying it converges if and only if $p < 1$. Moreover, when $p < 1$, the sum of green area and red area is

$$1 + \frac{1}{1/p - 1} = 1 + \frac{p}{1-p} = \frac{1}{1-p}.$$

Finally, we have

$$\int_0^1 x^{-p}\, dx = \begin{cases} \dfrac{1}{1-p}, & \text{if } 0 < p < 1 \\ \text{diverges}, & \text{if } p \geq 1. \end{cases}$$

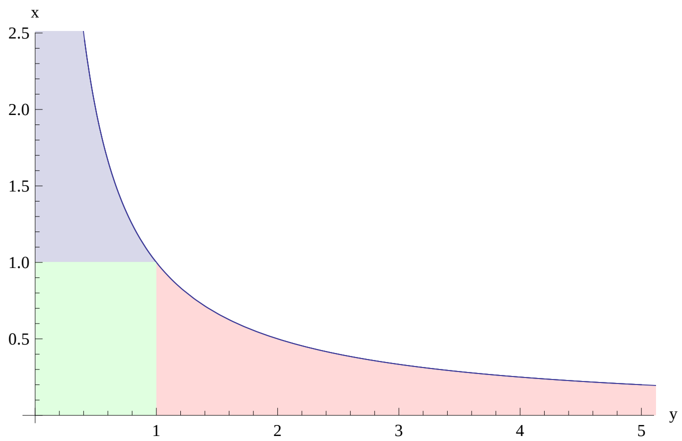

FIGURE 2.