Solutions to Recitation 4

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This is my own solution to the Recitation 4 of 18.014, which includes Exercise 3 from Munkres's Course Note A and Exercise 3 and 1achi from Apostol's *Calculus* (9; 1: 28, 43).

1. Show that if a set A of integers is bounded above, then A has a largest element.

   Proof. By the least-upper-bound axiom, there exists a real number u such that $u = \sup A$. If $u \in A$, we are done since u is the largest element. If $u \notin A$, then by definition there must exist an integer $n \in A$ such that $u - 1 < n < u$. The last inequality is the same as $u < n + 1 < u + 1$. If $n + 1 \in A$, then u is not a upper bound for A. It follows that $k \notin A$ for all integers $k \geqslant n + 1$. Then we find the largest element in A, which is simply n. (Furthermore, $u = n$.) $\square$

2. If $x > 0$, prove that there is a positive integer n such that $1/n < x$.

   Proof. Simply apply Theorem I.30 with $y = 1$ and we have $nx > 1$. Divide both side of the inequality with n and we obtain $1/n < x$. $\square$

3. Prove each of the following properties of absolute values.

   (a) $|x| = 0$ if and only if $x = 0$.

   Proof. If $|x| = 0$, then either $x = -|x| = -0$ or $x = |x| = 0$. In both cases, we have $x = 0$. If $x = 0$, then clearly we have $|x| = x = -x = 0$. $\square$

   (b) $|x - y| = |y - x|$.

   Proof. We have $|x - y| = |-(y - x)| = |y - x|$. $\square$

   (c) $|x - y| \leqslant |x| + |y|$.

   Proof. We have

$$\begin{aligned} |x - y| &= |x + (-y)| \\ &\leqslant |x| + |-y| \\ &= |x| + |y|. \end{aligned}$$

$\square$

   (d) $|x| - |y| \leqslant |x - y|$.

   Proof. We have

$$\begin{aligned} |x| &= |x - y + y| \\ &\leqslant |x - y| + |y|. \end{aligned}$$

   Substract $|y|$ from both sides of the above inequality and we obtain the result. $\square$