Problem 1

Question: $\displaystyle\int \frac{x^3+10x^2+33x+36}{x^2+4x+3}\,dx.$

Answer: $\displaystyle\frac{1}{2}x^2+6x+6\ln|x+1|+C.$

$$\begin{aligned}\int \frac{x^3+10x^2+33x+36}{x^2+4x+3}\,dx &= \int\left(x+6+\frac{6}{x+1}\right)dx\\ &= \frac{x^2}{2}+6x+6\ln|x+1|+C.\end{aligned}$$

Problem 2

Question: $\displaystyle\int \frac{x+2}{(x-1)^2}\,dx.$

Answer: $\displaystyle\ln|x-1|-\frac{3}{x-1}+C.$

$$\int \frac{x+2}{(x-1)^2}\,dx = \int\left(\frac{A}{x-1}+\frac{B}{(x-1)^2}\right)dx$$
$$\Downarrow$$
$$A(x-1)+B=x+2$$
$$\Downarrow$$
$$A=1 \quad B=3$$
$$\Downarrow$$
$$\begin{aligned}\int \frac{x+2}{(x-1)^2}\,dx &= \int\left(\frac{1}{x-1}+\frac{3}{(x-1)^2}\right)dx\\ &= \ln|x-1|-\frac{3}{x-1}+C.\end{aligned}$$

Problem 3

Question: $\displaystyle\int \frac{dx}{x^4-6x^3+12x^2}$

Answer:

$$\begin{aligned}\int \frac{dx}{x^4-6x^3+12x^2} &= \int \frac{dx}{x^2(x^2-6x+12)}\\ &= \int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-6x+12}\right)dx\end{aligned}$$
$$\Downarrow$$
$$Ax(x^2-6x+12)+B(x^2-6x+12)+(Cx+D)x^2=1$$
$$\Downarrow$$
$$\left\{\begin{array}{rrrrr|l} A & & +C & & =0 & x^3\\ -6A & +B & & +D & =0 & x^2\\ 12A & -6B & & & =0 & x^1\\ & 12B & & & =1 & x^0\end{array}\right.$$
$$\Downarrow$$
$$A=\frac{1}{24} \quad B=\frac{1}{12} \quad C=-\frac{1}{24} \quad D=\frac{1}{6}$$
$$\Downarrow$$

$$\int \frac{dx}{x^4 - 6x^3 + 12x^2} = \frac{1}{24}\ln|x| - \frac{1}{12x} - \frac{1}{24}\int \frac{x-4}{(x-3)^2+3}\,dx$$

$$\Downarrow$$

$$\int \frac{x-4}{(x-3)^2+3}\,dx = \int \frac{(\sqrt{3}\tan\theta - 1)\cancel{\sqrt{3}\sec^2\theta}}{\cancel{3\sec^2\theta}^{\sqrt{3}}}\,d\theta \qquad \langle \text{Let } x = \sqrt{3}\tan\theta + 3 \rangle$$

$$= \int \left(\tan\theta - \frac{1}{\sqrt{3}}\right) d\theta$$

$$= \int \tan\theta\,d\theta - \frac{\theta}{\sqrt{3}}$$

$$\Downarrow$$

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,dx$$

$$= -\int \frac{d\cos\theta}{\cos\theta}$$

$$= -\ln|\cos\theta| + C$$

$$\Downarrow$$

$$\int \frac{x-4}{(x-3)^2+4}\,dx = -\ln|\cos\theta| - \frac{\theta}{\sqrt{3}} + C$$

$$= -\ln\left|\cos\left(\arctan\frac{x-3}{\sqrt{3}}\right)\right| - \frac{1}{\sqrt{3}}\arctan\frac{x-3}{\sqrt{3}} + C \qquad \langle\, \theta = \arctan\frac{x-3}{\sqrt{3}} \,\rangle$$

$$= \frac{1}{2}\ln\left(\left(\frac{x-3}{\sqrt{3}}\right)^2 + 1\right) - \frac{1}{\sqrt{3}}\arctan\frac{x-3}{\sqrt{3}} + C \qquad \langle\, \cos(\arctan z) = 1/\sqrt{x^2+1} \,\rangle$$

$$= \frac{1}{2}\ln(x^2 - 6x + 12) - \frac{1}{\sqrt{3}}\arctan\frac{x-3}{\sqrt{3}} + C \qquad \langle\, -\ln 3 \text{ is absorbed by } C \,\rangle$$

$$\Downarrow$$

$$\int \frac{dx}{x^4 - 6x^3 + 12x^2} = \frac{1}{24}\left[\ln|x| - \frac{2}{x} - \frac{1}{2}\ln(x^2 - 6x + 12) + \frac{1}{\sqrt{3}}\arctan\frac{x-3}{\sqrt{3}}\right] + C.$$