Solutions to Recitation 14

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This is my own solutions to the problems from recitation 14 of 6.041SC.

1. You are visiting the rainforest, but unfortunately your insect repellent has run out. As a result, at each second, a mosquito lands on your neck with probability 0.5. If one lands, with probability 0.2 it bites you, and with probability 0.8 it never bothers you, independently of other mosquitoes.

   (a) What is the expected time between successive mosquito bites? What is the variance of the time between successive mosquito bites?

   At each second, the probability that a mosquito lands on your neck and bites you is $0.5 \times 0.2 = 0.1$. As the length of first string of losing days discussed in the lecture, the time between successive mosquito bites have the same model. It is a geometric variable with parameter 0.1. So the expected time is $1/0.1 = 10$ and the variance is $(1-0.1)/0.1^2 = 90$.

   (b) In addition, a tick lands on your neck with probability 0.1. If one lands, with probability 0.7 it bites you, and with probability 0.3, it never bothers you, independently of other ticks and mosquitoes. Now, what is expected time between successive bug bites? What is the variance of the time between successive bug bites?

   This is a merging of two Bernoulli processes into another Bernoulli process. At each second, the probability that a bug bites you is $1-(1-0.5\times 0.2)(1-0.1\times 0.7) = 0.163$. The time between successive bug bites is a geometric random variable with parameter 0.163. Thus the expecation is $1/0.163 \approx 6.135$ and the variance $(1-0.163)/0.163^2 \approx 31.503$.

2. Al performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of three fair coins.

   (a) Given that Al has just had a trial with 3 *tails*, what is the probability that both of the next two trials will also have this result?

   Since these trials are independent, what happened in the past has no influence on what will happen in the future. So the probability is simply $(1/2)^3(1/2)^3 = 1/64$.

   (b) Whenever all three coins land on the same side in any given trial, Al calls the trial a success.

      i. Find the PMF for K, the number of trials up to, but *not* including, the second success.

      At each trial, the probability of success is $2\times(1/2)^3 = 1/4$. Thus the PMF is

      $$p_K(k) = \left(\frac{3}{4}\right)^k \frac{1}{4} = \frac{3^k}{4^{k+1}}.$$

      ii. Find the expectation and variance of M, the number of tails that occur *before* the first success.

      Let G be the geometric random variable with parameter $1/4$ and X_i is the number of tails in trial i. Trials are independent from each other and so are X_i. Conditioned on G, we have

$$
\begin{aligned}
\mathrm{E}[M \mid G = g] &= \mathrm{E}[\sum_{i=1}^{g-1} X_i \mid G = g] \\
&= \sum_{i=1}^{g-1} \mathrm{E}[X_i \mid G = g] \\
&= \sum_{i=1}^{g-1} \sum_{j=1}^{2} j \, p_{X_i|G}(j \mid g) \\
&= \sum_{i=1}^{g-1} \left(\frac{\binom{3}{1}(1/2)^3}{1 - 2(1/2)^3} + 2\frac{\binom{3}{2}(1/2)^3}{1 - 2(1/2)^3} \right) \\
&= \sum_{i=1}^{g-1} \frac{3}{2} = \frac{3}{2}(g-1),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{var}(M \mid G = g) &= \mathrm{var}(\sum_{i=1}^{g-1} X_i \mid G = g) \\
&= \sum_{i=1}^{g-1} \mathrm{var}(X_i \mid G = g) \\
&= \sum_{i=1}^{g-1} \left(\left(1 - \frac{3}{2}\right)^2 \frac{1}{2} + \left(2 - \frac{3}{2}\right)^2 \frac{1}{2} \right) \\
&= \sum_{i=1}^{g-1} \frac{1}{4} = \frac{1}{4}(g-1).
\end{aligned}
$$

By applying law of iterated expectation, we have

$$
\begin{aligned}
\mathrm{E}[M] &= \mathrm{E}[\mathrm{E}[M \mid G]] \\
&= \mathrm{E}[\frac{3}{2}(G-1)] \\
&= \frac{3}{2}(\mathrm{E}[G] - 1) = \frac{9}{2}.
\end{aligned}
$$

By applying law of total variance, we have

$$
\begin{aligned}
\mathrm{var}(M) &= \mathrm{E}[\mathrm{var}(M \mid G)] + \mathrm{var}(\mathrm{E}[M \mid G]) \\
&= \mathrm{E}[\frac{1}{4}(G-1)] + \mathrm{var}(\frac{3}{2}(G-1)) \\
&= \frac{1}{4}(\mathrm{E}[G] - 1) + \left(\frac{3}{2}\right)^2 \mathrm{var}(G) \\
&= \frac{3}{4} + 27 = \frac{111}{4} = 27.75.
\end{aligned}
$$

(c) Bob conducts an experiment like Al's, except that he uses 4 coins for the first trial, and then he obeys the following rule: Whenever all of the coins land on the same side in a trial, Bob permanently removes one coin from the experiment and continues with the trials. He follows this rule until the *third* time he removes a coin, at which point the experiment ceases. Find $\mathrm{E}[N]$, where N is the number of trials in Bob's experiment.

It can be easily shown that N is a sum of three geometric random variables G_1, G_2, G_3 with parameter 1/8, 1/4, and 1/2 respectively. Thus $\mathrm{E}[N] = \mathrm{E}[G_1 + G_2 + G_3] = \mathrm{E}[G_1] + \mathrm{E}[G_2] + \mathrm{E}[G_3] = 8 + 4 + 2 = 14$.

3. Suppose there are n papers in a drawer. You draw a paper and sign it, and then, instead of filing it away, you place the paper back into the drawer. If any paper is equally likely to be drawn each time, independent of all other draws, what is the expected number of papers that you will draw before signing all n papers? You may leave your answer in the form of a summation.

   Let N be such the number of papers. This also can be seen as a sum of n geomeric random variables $N_1, N_2, \dots, N_n$ with parameter n/n, $(n-1)/n$, …, $1/n$. Notice that the first random variable is degenerate. The expected number is

$$\begin{aligned} \mathrm{E}[N] &= \mathrm{E}[N_1 + N_2 + \cdots + N_n] \\ &= \mathrm{E}[N_1] + \mathrm{E}[N_2] + \cdots + \mathrm{E}[N_n] \\ &= \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} \\ &= n \sum_{k=1}^{n} \frac{1}{k} = n H_n. \end{aligned}$$