# Solutions to Chapter 2 Quiz: Functions

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## Problem 1

**Question:** If $f(x) = x^{2x}$, compute $\dfrac{df}{dx}$.

**Answer:** $2x^{2x}(1 + \ln x)$.

$$
\begin{aligned}
\ln(y &= x^{2x}) \\
d(\ln y &= 2x \ln x) \\
\frac{dy}{y} &= \left(\frac{2x}{x} + \ln x\right) dx \\
\frac{dy}{dx} &= y(1 + \ln x) \\
\frac{dy}{dx} &= x^{2x}(1 + \ln x).
\end{aligned}
$$

This problem tests your understanding of the topics in lecture 16 Differentiation Operator. This is a function of super-exponential type.

The function of the type $u(x)^{v(x)}$ in fact can be transformed into $e^{v(x)\ln u(x)}$, differentiate this and you get

$$\frac{d}{dx}u(x)^{v(x)} = u(x)^{v(x)}\left(\frac{v(x)}{u(x)}u'(x) + v'(x)\ln u(x)\right).$$

## Problem 2

**Question:** Consider the function $f(x) = \sqrt{3}x^2e^{1-x}$. Use the formula for curvature,

$$\kappa = \frac{|f''|}{(1 + |f'|^2)^{3/2}}$$

to compute the curvature of the graph of $f$ at the point $(1, \sqrt{3})$.

**Answer:** $\dfrac{\sqrt{3}}{8}$.

$$
\begin{aligned}
f'(x) &= 2\sqrt{3}xe^{1-x} - \sqrt{3}x^2e^{1-x} \\
&= \sqrt{3}xe^{1-x}(2 - x) \\
f''(x) &= \sqrt{3}e^{1-x}(1 - x)(2 - x) - \sqrt{3}xe^{1-x} \\
&= \sqrt{3}e^{1-x}[(1 - x)(2 - x) - x] \\
f'(1) &= \sqrt{3} \\
f''(1) &= -\sqrt{3} \\
\kappa\Big|_{(1,\sqrt{3})} &= \frac{\sqrt{3}}{(1 + 3)^{3/2}} = \frac{\sqrt{3}}{8}.
\end{aligned}
$$

## Problem 3

**Question:** Assume that $x$ and $y$ are related by the equation $y \ln x = e^{1-x} + y^3$. Compute $\frac{dy}{dx}$ evaluated at $x = 1$.

**Answer:** 0.

$$
\begin{aligned}
d(y \ln x &= e^{1-x} + y^3) \\
\frac{y\,dx}{x} + \ln x\,dy &= -e^{1-x}\,dx + 3y^2\,dy \\
\frac{dy}{dx} &= \frac{\frac{y}{x} + e^{1-x}}{3y^2 - \ln x}
\end{aligned}
$$

solve the original equation about $x = 1$ and get

$$
\begin{aligned}
x &= 1 \\
y &= -1
\end{aligned}
$$

substitute $x$ and $y$ into the above formula about $\frac{dy}{dx}$ and get

$$
\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-1+1}{3} = 0.
$$

## Problem 4

**Question:** Use the linear approximation of the function $f(x) = \arctan(e^{3x})$ at $x = 0$ to estimate the value of $f(0.01)$.
**Hint:** remember that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

**Answer:** $\frac{\pi}{4} + \frac{3}{200}$.

$$
\begin{aligned}
f(x+h) &\approx f(x) + f'(x)h \\
f(0.01) &\approx f(0) + f'(0) \times 0.01 \\
f(0) &= \frac{\pi}{4} \\
f'(x) &= \frac{3e^{3x}}{1+e^{6x}} \\
f'(0) &= \frac{3}{2} \\
f(0.01) &\approx \frac{\pi}{4} + \frac{3}{200}.
\end{aligned}
$$

## Problem 5

**Question:** A rectangular picture frame with total area $50000\,\text{cm}^2$ includes a border which is $1\,\text{cm}$ thick at the top and the bottom and $5\,\text{cm}$ thick at the left and right side. What is the largest possible area of a picture that can be displayed in this frame?

**Answer:** $98\,\text{cm} \times 490\,\text{cm}$.

$S$ is the total area of the picture being displayed by this frame and $x$ is the width of the top and bottom side of the rectangle that actually displays the picture (i.e. frame width minus the left and right border size). Similarly, the height of the left and right side of the rectangle displaying the picture is $50000/(x+10)-2$. So

$$\begin{aligned} S &= \left(\frac{50000}{x+10}-2\right)x \\ &= \frac{50000(x+10)-500000}{x+10}-2x \\ &= 50000-\frac{500000}{x+10}-2x \\ S' &= \frac{500000}{(x+10)^2}-2. \end{aligned}$$

$x$ only makes sense when it's positive. $S'$ is monotonically decreasing on $(0,+\infty)$ and has a zero at $x=490$. Thus, $S'$ is positive on $(0,490)$ and negative on $(490,+\infty)$, and $S$ is monotonically increasing on $(0,490)$ and monotonically decreasing on $(490,+\infty)$. Therefore, $S$ has a maximum at $x=490$. At this time, the corresponding height is $50000/(490+10)-2=98$. Hence, $98\,\text{cm}\times 490\,\text{cm}$.

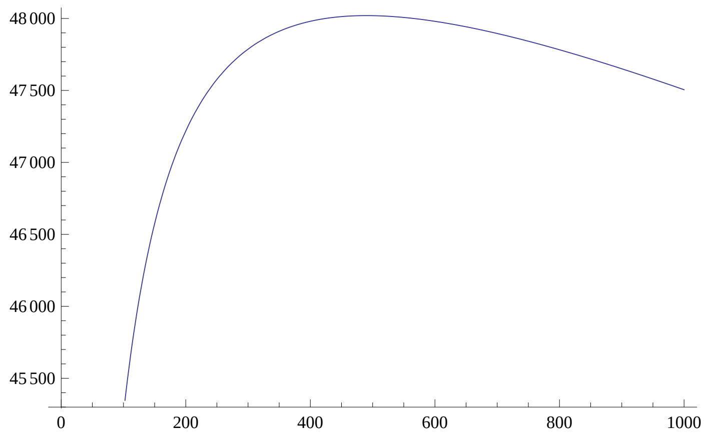


Figure 1: the area $S$ with respect to $x$

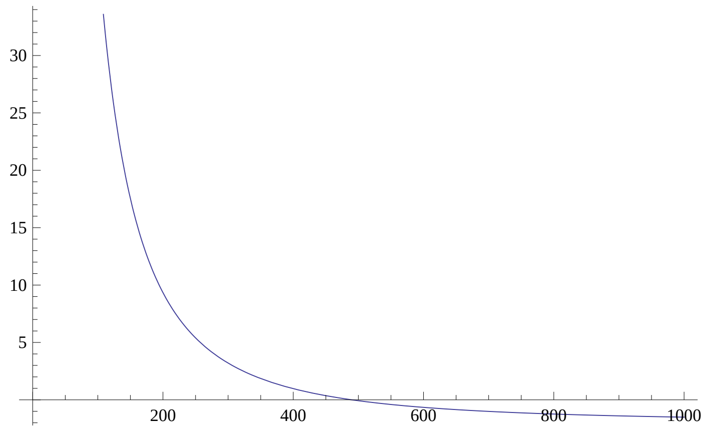


Figure 2: the rate of change of area, $S'$, with respect to $x$

This is an optimization problem. Please refer to lecture 14 Optimization.

## Problem 6

**Question:** Which of the following statements are true for the function $f(x)=\frac{4}{x}+x^4$? In order to receive full credit for this problem, you must select all the true statements (there may be many) and none of the false statements.

**Answer:**

| | | |
|---|---|---|
| | a. | The global maximum of $f$ for $-3/2 \le x \le 2$ is at $x = -1$. |
| ✓ | b. | The global maximum of $f$ for $-2 \le x \le -1$ is at $x = -2$. |
| | c. | The critical points of $f$ are at $x = -1$ and $x = 1$. |
| ✓ | d. | The global maximum of $f$ for $-1 \le x \le -1/2$ is at $x = -1$. |
| | e. | The global minimum of $f$ for $-2 \le x \le 2$ is at $x = 1$. |
| ✓ | f. | $f$ is not differentiable at $x = 0$. |
| ✓ | g. | The global minimum of $f$ for $1/2 \le x \le 2$ is at $x = 1$. |
| | h. | The global minimum of $f$ for $-1 \le x \le 2$ is at $x = 1$. |

$f$ is not defined at $x = 0$, and this hints us it has a blow-up or oscillation around $x = 0$. In fact, $f$ is continuous everywhere except at $x = 0$, and has a blow-up to positive infinity as $x \to 0^+$ and a blow-up to negative infinity as $x \to 0^-$. So $f$ is not differentiable at $x = 0$ and any intervals containing zero as its interior point have neither maximum nor minimum since $f$ can be arbitrary large toward both positive and negative infinity around $x = 0$. Therefore, a., e., and h. are all incorrect and f. is correct.

Take the derivative and we get $f'(x) = 4x^3 - 4/x^2$. The first term $4x^3$ has the same sign as $x$ and is monotonically increasing on $\mathbb{R}$, the second term $-4/x^2$ is always negative and monotonically increasing on $(0, +\infty)$; therefore, $f$ is always negative on $(-\infty, 0)$ and monotonically increasing on $(0, +\infty)$. $f'$ has a zero at $x = 1$; and thus $f'$ is negative on $(0, 1)$ and positive on $(1, +\infty)$. Therefore, $f$ is monotonically decreasing on $(-\infty, 0) \cup (0, 1)$ and monotonically increasing on $(1, +\infty)$. Hence, b., d., and g. are all correct.

$f'(x) = 0$ when $x = 1$ and $f'(x)$ doesn't exist when $x = 0$. So the critical points of $f$ are at $x = 0$ and $x = 1$. Therefore, c. is incorrect.

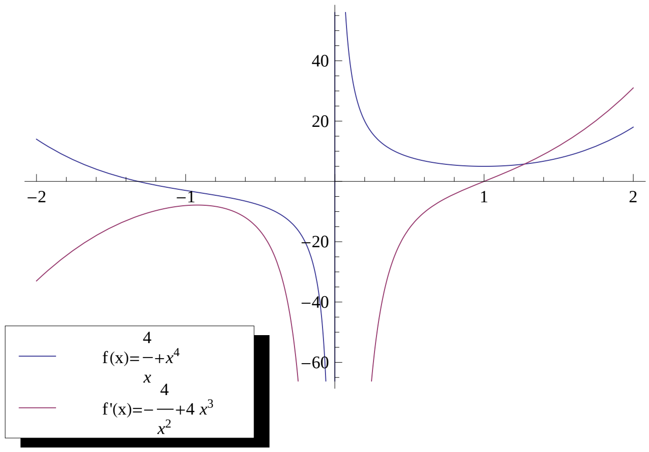


Figure 3: The plot of $f$ and $f'$

# Problem 7

**Question:** To approximate $\sqrt[3]{15}$ (the cube root of 15) using Newton's method, what is the appropriate update rule for the sequence $x_n$?

**Answer:** $x_{n+1} = \dfrac{2x_n}{3} + \dfrac{5}{x_n^2}$.

$\sqrt[3]{15}$ is the solution to the equation $x^3 - 15 = 0$. So let $f(x) = x^3 - 15$ and

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x)}{f'(x)} \\ f'(x) &= 3x^2 \\ x_{n+1} &= x_n - \frac{x_n^3 - 15}{3x_n^2} \\ &= \frac{3x_n^3 - x_n^3 - 15}{3x_n^2} \\ &= \frac{2x_n}{3} - \frac{5}{x_n^2}. \end{aligned}$$

Please refer to lecture 12 Linearization.

# Problem 8

**Question:** Fill in the blank:

$$\ln^2(x+h) = \ln^2 x + \underline{\quad\quad} \cdot h + O(h^2)$$

Here, $\ln^2 x$ means $(\ln x)^2$.

**Answer:** $2\dfrac{\ln x}{x}$.

The stronger version of derivative is defined by the second order variation. The above blank happen to be in the position of the second order variation, so it's just the derivative of $\ln^2 x$, which is

$$(\ln^2 x)' = \frac{2 \ln x}{x}.$$

Please refer to lecture 10 Derivatives.

# Problem 9

**Question:** Recall that the kinetic energy of a body is

$$K = \frac{1}{2}mv^2$$

where $m$ is mass and $v$ is velocity. Compute the relative rate of change of kinetic energy, $\dfrac{dK}{K}$, given that the relative rate of change of mass is $-7$ and the relative rate of change of velocity is $+5$.

**Answer:** $\dfrac{dK}{K} = 3$.

The relative rate of change of mass is $\dfrac{dm}{m}$ and the relative rate of change of velocity is $\dfrac{dv}{v}$. Therefore,

$$\begin{aligned} d(K &= \frac{1}{2}mv^2) \\ dK &= mv\,dv + \frac{1}{2}v^2\,dm \\ \frac{dK}{K} &= \frac{mv\,dv + \frac{1}{2}v^2\,dm}{\frac{1}{2}mv^2} \\ &= 2\frac{dv}{v} + \frac{dm}{m} \end{aligned}$$

substitute $\frac{dv}{v} = 5$ and $\frac{dm}{m} = -7$ into the above formula and get

$$\frac{dK}{K} = 2 \times 5 - 7 = 3.$$

Please refer to lecture 15 Differentials.

## Problem 10

**Question:** Compute the ninth derivative of $(x-3)^{10}$ with respect to x.

**Answer:** $10!(x-3)$.

$$\begin{aligned} \frac{d}{dx}(x-3)^{10} &= 10(x-3)^9 \\ \frac{d^2}{dx^2}(x-3)^{10} &= 10 \cdot 9(x-3)^8 \\ \frac{d^3}{dx^3}(x-3)^{10} &= 10 \cdot 9 \cdot 8(x-3)^7 \\ &\vdots \\ \frac{d^9}{dx^9} &= 10!(x-1) \end{aligned}$$