Basics and Arithmetic of the Real Numbers

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THEOREM.

(1) *Let* n *be a whole positive number that grows without limit. If the unique function* $\varphi(n)$ *grows (decreases) continuously with* n *from a value* $n = n_1$*, and thus has a limit at* $\lim = +\infty$ *according to No. 6, which is supposed to be infinite; then it follows from the existence of the limit*

$$\lim_{n=+\infty} \frac{f(n+1) - f(n)}{\varphi(n+1) - \varphi(n)} = K,$$

where $f(n)$ *also means a unique function of* n*, that there is also a limit for the fraction* $f(n) : \varphi(n)$ *at* $\lim n = +\infty$*, namely that it is equal to* K*.*

(2) *The same theorem applies if the unique functions* $f(n), \varphi(n)$ *each have the limit* 0 *for* $\lim n = +\infty$ *and* $\varphi(n)$ *grows (decreases) continuously with* n *from* $n = n_1$*.*

PROOF. For the proof, it is sufficient to assume that $\varphi(n)$ increases continuously with n and that K does not have the sign $-$. Then two cases can be distinguished for the first theorem, to which we restrict the following development.

a) $K \geqq 0$. According to the assumption, for every number $\varepsilon > 0$ there belongs a number G such that

$$\left| \frac{f(n+1) - f(n)}{\varphi(n+1) - \varphi(n)} - K \right| < \varepsilon \qquad n > G.$$

From this we conclude that, since $\varphi(m+1) > \varphi(n)$,

$$(K - \varepsilon)[\varphi(n+1) - \varphi(n)] < f(n+1) - f(n) < (K + \varepsilon)[\varphi(n+1) - \varphi(n)].$$

Now let m be a certain integer $> G$, so that in this relation we can set $n = m, m+1, \dots, m+r-1$ one after the other. If we add the r inequalities obtained in this way, we get

$$(K - \varepsilon)[\varphi(m+r) - \varphi(m)] < f(m+r) - f(m) < (K + \varepsilon)[\varphi(m+r) - \varphi(m)]$$

and

$$-\varepsilon[\varphi(m+r) - \varphi(m)] < f(m+r) - f(m) - K[\varphi(m+r) - \varphi(m)] < \varepsilon[\varphi(m+r) - \varphi(m)].$$

Since $\varphi(m)$ and $\varphi(m+r)$ can be assumed to be positive, we also find:

$$-\varepsilon\varphi(m+r) < f(m+r) - f(m) - K[\varphi(m+r) - \varphi(m)] < \varepsilon\varphi(m+r),$$

$$-\varepsilon < \frac{f(m+r)}{\varphi(m+r)} - K - \frac{f(m) - K\varphi(m)}{\varphi(m+r)} < \varepsilon.$$

However large the number m may be; then, because $\lim \varphi(n) = +\infty$ we can find a number $\varrho > 0$ such that

(1)
$$\frac{|f(m) - K\varphi(m)|}{\varphi(m+r)} < \varepsilon \quad \text{for} \quad r > \varrho.$$

We therefore have

(2)
$$\left| \frac{f(m+r)}{\varphi(m+r)} - K \right| < 2\varepsilon$$

or $|f(n) : \varphi(n) - K| < 2\varepsilon$ for all $n > m + \varrho$. Now then, the number 2ε, which can be any positive number, indeed corresponds to the number $m + \varrho$, since m results from ε initially and after m the number ϱ from ε. Thus we have

$$\lim f(n) : \varphi(n) = K$$

for $\lim n = +\infty$.

b) $K = +\infty$. Let $0 < H < H'$. According to the assumption, for $H' > 0$ there should be a number G' such that for $n > G'$

$$f(n+1) - f(n) > H'\{\varphi(n+1) - \varphi(n)\},$$

from which we find as above—$m > G'$—

(3)
$$f(m+r) - f(m) > H'\{\varphi(m+r) - \varphi(m)\},$$

$$\frac{f(m+r)}{\varphi(m+r)} > H' + \frac{f(m) - H'\varphi(m)}{\varphi(m+r)}.$$

Now a number $\varrho > 0$ can be specified such that

$$r > \varrho \qquad |f(m) - H'\varphi(m)| : \varphi(m+r) < H' - H;$$

therefore, for all $n > m + \varrho$

$$\frac{f(n)}{\varphi(n)} > H' - (H' - H) = H \quad \text{i.e.} \quad \lim \frac{f(n)}{\varphi(n)} = +\infty.$$

At the same time, we can see from (3) that $\lim f(n) = +\infty$, so that we could also conclude as follows: According to theorem a),

$$\lim\{\varphi(n) : f(n)\} = 0,$$

therefore, since $\varphi(n) : f(n) > 0$,

$$\lim\{f(n) : \varphi(n)\} = +\infty. \qquad \square$$

Remark. The above theorems can be extended to the case that the independent variable grows continuously over all limits.