(a) Basis: $1 = 1(1+1)/2$

Induction: $1+\cdots+k = k(k+1)/2$

$$1+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \left(\frac{k}{2}+1\right)(k+1)$$
$$= \frac{(k+2)(k+1)}{2}$$
$$= \frac{(k+1)[(k+1)+1]}{2} \quad \square$$

(b) Basis: $1 = 1^2$.

Induction: $1+3+\cdots+(2k-1) = k^2$

$$1+3+\cdots+(2k-1)+(2k+1)$$
$$= k^2+2k+1$$
$$= (k+1)^2. \quad \square$$

3. (c) Basis: $1^3 = 1^2$

Induction: $1^3+\cdots+k^3 = (1+\cdots+k)^2$

$$1^3+\cdots+k^3+(k+1)^3 = (1+\cdots+k)^2+(k+1)^3$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\cancel{=(1+\cdots+k)^2+k^3+3k^2+3k+1}$$
$$\cancel{=(1+\cdots+k)^2+k^3+2k^2+k+k^2+2k+1}$$
$$\cancel{=(1+\cdots+k)^2+k(k^2+2k+1)+(k+1)^2}$$
$$=(1+\cdots+k)^2+k(k+1)^2+(k+1)^2$$
$$=(1+\cdots+k)^2+2(1+\cdots+k)(k+1)+(k+1)^2$$
$$\cancel{=(1+\cdots+k)+(k+1)}$$
$$=[1+\cdots+k+(k+1)]^2 \quad \square$$

Alternatively

$$=\frac{[k(k+1)]^2}{4}+(k+1)^3$$
$$=\frac{(k+1)^2}{4}(k^2+4k+4)$$
$$=\frac{(k+1)^2}{4}(k+2)^2$$
$$=\frac{(k+1)^2(k+2)^2}{4}$$
$$=\left[\frac{(k+1)(k+1+1)}{2}\right]^2$$
$$=[1+\cdots+k+(k+1)]^2. \quad \square$$

(d) We can use results from previous parts.

$$[1+\cdots+(n-1)]^2 = \frac{n^2(n-1)^2}{4} = \frac{n^4}{4}-\frac{n^3}{2}+\frac{n^2}{4}$$
$$(1+\cdots+n)^2 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

$$1<2n \quad \forall n\in\mathbb{N}$$
$$\Leftrightarrow \frac{1}{4}<\frac{n}{2} \Leftrightarrow \frac{n^2}{4}<\frac{n^3}{2}$$
$$\Leftrightarrow -\frac{n^3}{2}+\frac{n^2}{4}<0$$
$$\Leftrightarrow \frac{n^4}{4}-\frac{n^3}{2}+\frac{n^2}{4}<\frac{n^4}{4}<\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$
$$\Leftrightarrow 1^3+\cdots+(n-1)^3<\frac{n^4}{4}<1^3+\cdots+n^3 \quad \square$$

Alternatively, we induct.

Basis. $0^3<\frac{1}{4}<1^3$

Induction

$$1^3+\cdots+(n-1)^3<\frac{n^4}{4}<1^3+\cdots+n^3$$
$$1^3+\cdots+(n-1)^3+n^3<\frac{n^4}{4}+n^3<$$
$$1^3+\cdots+n^3+n^3<1+\cdots+(n+1)^3$$

$$\frac{n^4}{4}+n^3=\frac{n^3(n+4)}{4}=\frac{n^4+4n^3}{4}$$
$$\frac{(n+1)^4}{4} \qquad <\frac{n^4+4n^3+6n^2+4n+1}{4}$$
$$=\frac{(n+1)^4}{4}$$

$$\frac{n^4}{4}+(n+1)^3=\frac{n^4+4n^3+12n^2+12n+4}{4}$$
$$>\frac{n^4+4n^3+6n^2+4n+1}{4}$$
$$=\frac{(n+1)^4}{4}$$

$$1^3+\cdots+n^3<\frac{(n+1)^4}{4}<1^3+\cdots+(n+1)^3 \quad \square$$

2. $1=1,$

$1-4=-(1+2),$

$1-4+9=1+2+3,$

$1-4+9-16=-(1+2+3+4)$

$$\sum_{k=1}^{n}(-1)^{k+1}k^2=(-1)^{n+1}\sum_{k=1}^{n}k$$

Basis: $1=1$

Induction: $\sum_{k=1}^{n}(-1)^{k+1}k^2=(-1)^{n+1}\sum_{k=1}^{n}k$

$$\left(\sum_{k=1}^{n}(-1)^{k+1}k^2\right)+(-1)^{n+2}(n+1)^2$$

$$=(-1)^{n+1}\left(\sum_{k=1}^{n}k\right)+(-1)^{n+2}(n+1)^2$$

$$=(-1)^{n+1}\left[-(n+1)^2+\sum_{k=1}^{n}k\right]$$

$$=(-1)^{n+1}\left[-n^2-2n-1+\sum_{k=1}^{n}k\right]$$

$$=(-1)^{n+2}\left[(n+1)^2-\frac{n(n+1)}{2}\right]$$

$$=(-1)^{n+2}\left[(n+1)\left(n+1-\frac{n}{2}\right)\right]$$

$$=(-1)^{n+2}\left[\frac{(n+1)(n+2)}{2}\right]$$

$$=(-1)^{n+2}\sum_{k=1}^{n+1}k. \qquad \square$$

3. $1+\frac{1}{2}=2-\frac{1}{2}$,

$1+\frac{1}{2}+\frac{1}{4}=2-\frac{1}{4}$,

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=2-\frac{1}{8}$,

$\vdots$

$$\sum_{k=0}^{n}2^{-k}=2-2^{-n}.$$

Basis: $1+\frac{1}{2}=2-\frac{1}{2}$.

Induction:

$$\sum_{k=0}^{n}2^{-k}+2^{-(n+1)}=2-2^{-n}+2^{-(n+1)}$$

$$=2-2^{-n}(1-2^{-1})$$

$$=2-2^{-n}\cdot 2^{-1}$$

$$=2-2^{-(n+1)}. \qquad \square$$

4. $1-\frac{1}{2}=\frac{1}{2}$,

$(1-\frac{1}{2})(1-\frac{1}{3})=\frac{1}{3}$,

$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})=\frac{1}{4}$,

$\vdots$

$$\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)=\frac{1}{n+1}.$$

Basis: $1-\frac{1}{2}=\frac{1}{2}$

Induction:

$$\left[\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)\right]\left(1-\frac{1}{n+2}\right)$$

$$=\frac{1}{n+1}\left(1-\frac{1}{n+2}\right)$$

$$\cancel{=\frac{(n+2)-(n+1)}{(n+1)(n+2)}}$$

$$\cancel{=\frac{1}{(n+1)(n+2)}}$$

$$=\frac{(n+2)-1}{(n+1)(n+2)}$$

$$=\frac{\cancel{n+1}}{\cancel{(n+1)}(n+2)}$$

$$=\frac{1}{n+2}. \qquad \square$$

Alternatively, we can provide a direct proof as follows.

$$\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)=\prod_{k=1}^{n}\frac{k}{k+1}$$

$$=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}.$$

$$=\frac{1}{n+1} \qquad \square$$

5. $$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{1}{2}\left(1+\frac{1}{n}\right).$$

Basis: $1-\frac{1}{4}=\frac{1}{2}\left(1+\frac{1}{2}\right)$.

Induction:

$$\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)=\left[\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(n+1)^2}\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)$$

$$=\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{(n+1)^2-1}{(n+1)^2}$$

$$=\frac{1}{2}\,\frac{n+1}{n}\,\frac{n(n+2)}{(n+1)^2}$$

$$=\frac{1}{2}\,\frac{n+2}{n+1}=\frac{1}{2}\left(1+\frac{1}{n+1}\right). \qquad \square$$

Alternatively, we have

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\prod_{k=2}^{n}\frac{(k-1)(k+1)}{k^2}=\frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdots\frac{(n-1)(n+1)}{n^2}$$

$= \frac{1}{2}\frac{n+1}{n} = \frac{1}{2}(1+\frac{1}{n})$. □

6.
(a) $\frac{1}{8}(2n+1)^2 + (n+1)$
$= \frac{1}{8}(4n^2+4n+1+8n+8)$
$= \frac{1}{8}(4n^2+12n+9)$
$= \frac{1}{8}(2n+3)^2 = \frac{1}{8}[2(n+1)+1]^2$

(b) Of course, it's well-known that ~~that~~ $1+2+\cdots+n = (n+1)n/2 \neq (2n+1)^2/8$ for all $n \in N$.
The statement is ~~wro~~ false and the argument is invalid simply because the basis does not hold

$$1 \neq \frac{1}{8}(2+1)^2 = \frac{9}{8}.$$

(c) Basis: $1 < \frac{9}{8} = \frac{1}{8}(2+1)^2$

Induction:
We claim that
$1+\cdots+n+(n+1) < \frac{1}{8}(2n+1)^2+(n+1)$
$= \frac{1}{8}[2(n+1)+1]^2$. □

7. $n_1 = 3$.

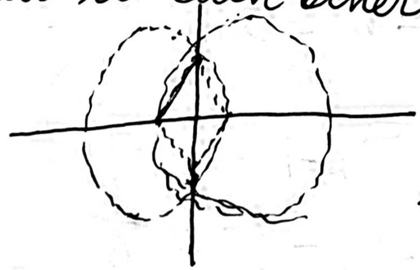
Basis: $(1+x)^3 = 1+3x+3x^2+x^2$
$> 1+3x+3x^2$
for all $x > 0$.

Induction: ~~$(1+x)^n$~~ $(1+x)^{n+1} = (1+x)^n(1+x)$
$= (1+x)^n + x(1+x)^n$
$> 1+nx+nx^2 + x(1+nx+nx^2)$
$= 1+(n+1)x+2nx^2+nx^3$
$> 1+(n+1)x+(n+1)x^2+nx^3$
$> 1+(n+1)x+(n+1)x^2$ □
$\forall n \geq 3$.

8.
Basis: $a_1 \leq a_1 = a_1 c^{1-1}$

Induction
$a_{n+1} \leq c a_n$ $\forall n \geq 1$
$\leq c(a_1 c^{n-1})$
$= a_1 c^n$. □

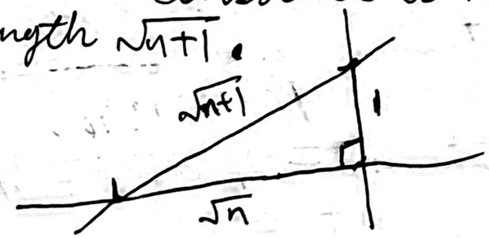
9. By the result of elementary geometry, we can construct two lines that are perpendicular to each other as follows.



two circles and two equilateral triangles

Basis: $1 = \sqrt{1}$
and the ~~uni~~ unit ~~lengh~~ length is given.

Induction:
Hypothesis is that we already have the length $\sqrt{n}$ and we are going to construct $\sqrt{n+1}$ from ~~this~~. First, we put $\sqrt{n}$ on the first line. Then we construct a second line ~~that~~ is perpendicular to the first line and ~~inser~~ intersects with the first line at one vertex of the length $\sqrt{n}$. At the intersection, we construct an unit ~~leg~~ length on the second line. We joins the other two vertices from the ~~rest~~ two length and by Pythagorean theorem we have construct a new length $\sqrt{n+1}$. □



10. Basis: When $n=0$, we can set $q=r=0$ and it's trivial to verify that
~~n~~ $0 = 0 \cdot b + 0$, $0 \leq 0 < b$.

Induction:
$n = q_n b + r_n$, $0 \leq r_n < b$
Since $r_n < b$, we have
$r_n + 1 \leq b$.

$r_{n+1} = \begin{cases} r_n+1 & r_n+1 \leq b \\ 0 & r_n+1 = b \end{cases}$ $q_{n+1} = \begin{cases} q_n+1 & r_n+1 = b \\ q_n & r_n+1 < b \end{cases}$

It's easy to verify that

$$n+1 = q_n b + r_n + 1$$
$$= q_{n+1} b + r_{n+1}$$

and $0 \le r_{n+1} < b$. □

11. Basis: It's easy to verify that 2 is a prime and its only positive divisors are 1 and 2.

Induction: Assume n is either a prime or a product of primes. We are going to show ~~that~~ $n+1$ is also ~~either~~ either a prime or a product of prime. Suppose $n+1$ is ~~neither~~ not a prime ~~nor a product of primes~~. Then

~~$\exists c, d \in N$ s.t. $c \neq 1, d \neq n+1$, $n+1 = cd$.~~

$\exists\, 1 < c, d < n+1$ s.t.

$$n+1 = cd.$$

Since $1 < c, d < n+1$, both c and d are either a prime or a product of primes, ~~so are $n+1$. This contradicts our supposition.~~ $n+1$ is a product of primes. Thus, $n+1$ is either a prime or a product of primes. □

Strong Mathematical Induction

12. The fallacy lies in "repeating the process with G_1, G_2, and $G_4, \ldots$". There's no way you can do that since our assuption only concerns the specified 3 girls. Once fixed, you simply could not repeat the process.

I4.7

1.(a) 10

(c) $$\sum_{r=0}^{3} 2^{2r+1} = 2\sum_{r=0}^{3} 4^r = 2\cdot\frac{1-4^4}{1-4} = 2\times\frac{255}{3} = 2\times 85 = 170.$$

(b) $$\sum_{n=2}^{5} 2^{n-2} = \sum_{n=0}^{3} 2^n = \frac{1-2^4}{1-2} = 15.$$

(d) $$\sum_{n=1}^{4} n^n = 1+4+27+256 = 288.$$

(e) $$\sum_{i=0}^{5} (2i+1) = 6^2 = 36.$$

(f) $$\sum_{k=1}^{5} \frac{1}{k(k+1)} = \sum_{k=1}^{5}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{6} = \frac{5}{6}.$$

2.(a) Additive

$$\sum_{k=1}^{n} (a_k + b_k) = a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n$$

Bubbling due to commutativity and associativity

$$= a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

(b) Homogeneous

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n$$

Due to distributivity

$$= c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k.$$

(c) Telescoping

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_1 - a_0 + a_2 - a_1 + \cdots + a_n - a_{n-1}$$

Commutativity and associativity

$$= a_{n-1} - a_0.$$

3. $$\sum_{k=1}^{n} 1 = \sum_{k=1}^{n} [k - (k-1)] = n - (1-1) = n - 0 = n.$$

4. $$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} (k+k-1)[k-(k-1)] = \sum_{k=1}^{n} [k^2 - (k-1)^2] = n^2 - 0^2 = n^2.$$

5. $$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}.$$

$$\sum_{k=1}^{n} (2k-1) = 2\left(\sum_{k=1}^{n} k\right) - n = n^2$$

$$\Leftrightarrow \sum_{k=1}^{n} k = \frac{n^2+n}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

6. $$\sum_{k=1}^{n} (3k^2 - 3k + 1) = \sum_{k=1}^{n} [k^3 - (k-1)^3] = n^3 - 0^3 = n^3$$

$\iff 3\left(\sum_{k=1}^{n} k^2\right) - 3\left(\sum_{k=1}^{n} k\right) + n = n^3$

$\iff \sum_{k=1}^{n} k^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{3}{2}n - n\right)$

$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$

7. $\sum_{k=1}^{n} (4k^3 - 6k^2 + 4k - 1)$

$= \sum_{k=1}^{n} [k^4 - (k-1)^4]$

$= n^4 - 0^4 = n^4$

$\iff 4\left(\sum_{k=1}^{n} k^3\right) - 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) - n = n^4$

$\iff \sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n\right)$

$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$

$= \left(\frac{n^2}{2} + \frac{n}{2}\right)^2.$

8. (a) $(1-x)\sum_{k=0}^{m} x^k = \sum_{k=0}^{n} [(1-x)x^k]$

$= \sum_{k=0}^{n} (x^k - x^{k+1})$

$= -\sum_{k=0}^{n} (x^{k+1} - x^k)$

$= -\sum_{k=1}^{n+1} (x^k - x^{k-1})$

$= -(x^{n+1} - 1)$

$= 1 - x^{n+1}$

$\iff \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad \forall x \neq 1.$

(b) $\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} 1^k = \sum_{k=0}^{n} 1 = n$

when $x = 1$. (Degenerate case)

9. Basis: When $n=1$, we have

$\sum_{k=1}^{2} (-1)^k(2k+1) = -3 + 5 = 2 \times 1.$

We can guess that the constant of proportionality is 2.

Induction: Suppose

$\sum_{k=1}^{2n} (-1)^k (2k+1) = 2n,$

then $\sum_{k=1}^{2(n+1)} (-1)^k (2k+1) = \left[\sum_{k=1}^{2n} (-1)^k (2k+1)\right] + (-1)^{2n+1}(4n+3) + (-1)^{2n+2}(4n+5)$

$= 2n + (-1)(4n+3) + (4n+5)$

$= 2n + 2 = 2(n+1).$ $\square$

10. (a)

i. Define

$\sum_{k=m}^{m} a_k = \sum_{k=m}^{m+0} a_k = a_m.$

ii. Assume $\sum_{k=m}^{m+n} a_k$ is defined for a fixed $n \geq 0$. Then define

$\sum_{k=m}^{m+(n+1)} a_k = \left(\sum_{k=m}^{m} a_k\right) + a_{n+1}.$

(b) Basis: When $n=1$, we have

$\sum_{k=1+1}^{2} \frac{1}{k} = \frac{1}{2} = 1 - \frac{1}{2}$

$= \sum_{m=1}^{2} \frac{(-1)^{m+1}}{m}.$

Induction:

$\sum_{k=(n+1)+1}^{2(n+1)} \frac{1}{k} = \left(\sum_{k=n+1}^{2n} \frac{1}{k}\right) + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$= \left(\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m}\right) + \frac{(-1)^{2n+2}}{2n+1} - \frac{(-1)^{2n+3}}{2n+2}$

$= \sum_{m=1}^{2(n+1)} \frac{(-1)^{m+1}}{m}.$ $\square$

11. (a) True. $\sum_{n=0}^{100} n^4 = 0^4 + \sum_{n=1}^{100} n^4$

$= \sum_{n=1}^{100} n^4.$

(b) ~~True~~ False. $\sum_{j=0}^{100} 2 = 2\sum_{j=0}^{100} 1 = 2\sum_{j=1}^{101} 1$

$= 2 \times 101 = 202 \neq 20$

(c) False. $\sum_{k=0}^{100} (2+k) = \left(\sum_{k=0}^{100} 2\right) + \left(\sum_{k=0}^{100} k\right)$

$= 202 + \sum_{k=0}^{100} k$

$\neq 2 + \sum_{k=0}^{100} k.$

(d) False.

$\sum_{i=1}^{100} (i+1)^2 = \sum_{i=2}^{101} i^2 = 100^2 + 101^2 - 0^2 - 1^2 + \sum_{i=0}^{99} i^2 \neq \sum_{i=0}^{99} i^2.$

(e) False.

$$\sum_{k=1}^{100} k < \sum_{k=1}^{100} k^2$$

$$\sum_{k=1}^{100} k^3 = \left(\sum_{k=1}^{100} k\right)^2 < \left(\sum_{k=1}^{100} k\right)\left(\sum_{k=1}^{100} k^2\right).$$

(f) $\sum_{k=0}^{100} k^3 = \left(\sum_{k=0}^{100} k\right)^2 < \left(\sum_{k=0}^{100} k\right)^3.$

Thus, the original equality is false.

12. $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{(k+1)-k}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}. \quad \square$$

13. It's easy to verify that

(1) $2 < \sqrt{1+\frac{1}{n}} + 1$ and (2) $1 + \sqrt{1-\frac{1}{n}} < 2$

for all $n \geq 1$. Notice that

$$2 = 2\cdot 1 = 2[(n+1) - n] = 2(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})$$

and

$$2 = 2\cdot 1 = 2[n - (n-1)] = 2(\sqrt{n} - \sqrt{n-1})(\sqrt{n} + \sqrt{n-1}).$$

Divide (1) with $\sqrt{n+1} + \sqrt{n}$ and (2) with $\sqrt{n} + \sqrt{n-1}$, and we obtain

$$2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}). \quad \square$$

From the above inequality, we get

$$\sum_{n=1}^{m}[2(\sqrt{n+1} - \sqrt{n})] < \sum_{n=1}^{m} \frac{1}{\sqrt{n}} < \sum_{n=1}^{m}[2(\sqrt{n} - \sqrt{n-1})]$$

$$\overset{\text{homogeneity}}{\iff} 2\sum_{n=1}^{m}(\sqrt{n+1} - \sqrt{n}) < \sum_{n=1}^{m}\frac{1}{\sqrt{n}} < 2\sum_{n=1}^{m}(\sqrt{n} - \sqrt{n-1})$$

$$\overset{\text{telescoping}}{\iff} 2(\sqrt{m+1} - 1) < \sum_{n=1}^{m}\frac{1}{\sqrt{n}} < 2\sqrt{m}.$$

Since $m \geq 2$, we also have

$$\sum_{n=1}^{m}\frac{1}{\sqrt{n}} = 1 + \sum_{n=2}^{m}\frac{1}{\sqrt{n}} < 1 + 2\sum_{n=2}^{m}(\sqrt{n} - \sqrt{n-1}) = 1 + 2(\sqrt{m} - 1) = 2\sqrt{m} - 1.$$

Thus,

$$2\sqrt{m+1} - 2 < \sum_{n=1}^{m}\frac{1}{\sqrt{n}} < 2\sqrt{m} - 1$$

for all $m \geq 2$. $\square$

When $m = 10^6$, we have

$$1998 = 2\sqrt{10^6} - 2 < 2\sqrt{10^6+1} - 2 < \sum_{n=1}^{10^6}\frac{1}{\sqrt{n}} < 2\sqrt{10^6} - 1 = 1999.$$

I 4.9

1. (a) If $|x| = 0$, then either $x = -|x| = -0$ or $x = |x| = 0$. In both cases, we have $x = 0$.

If $x = 0$, then clearly we have $|x| = x = -x = 0$. $\square$

(b) If $x = 0$, the case is trivial.

If $x > 0$, $|x| = x$ and $-x < 0$, $|-x| = -(-x) = x$. Thus, $|-x| = |x|$. $\square$

(c) Simply apply the result from the previous exercise.

$|x-y| = |-(y-x)| = |y-x|$. $\square$

(d) Either $|x| = x$ or $|x| = -x$, in both case, we have $|x|^2 = x^2$. $\square$

(e) If $x = 0$, it's trivial.

If $x > 0$, $|x| = x = \sqrt{x^2}$.

If $x < 0$, $-x > 0 \Rightarrow \sqrt{x^2} = \sqrt{(-x)^2} = -x = |x|$. $\square$

(f) Use the result from the previous exercise, we have

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|. \quad \square$$

(g) Since $1/y$ and y have the same sign for $y \neq 0$, thus $|1/y| = 1/|y|$.

Use the result from the previous exercise, we have

$$\left|\frac{x}{y}\right| = \left|x \cdot \frac{1}{y}\right| = |x|\left|\frac{1}{y}\right| = |x| \cdot \frac{1}{|y|} = \frac{|x|}{|y|}. \quad \square$$

(h)
$$|x-y| = |x+(-y)| \le |x| + |-y| = |x| + |y|. \qquad \square$$

(i)
$$|x| = |x-y+y| \le |x-y| + |y|$$
$$\Longleftrightarrow \quad |x| - |y| \le |x-y|. \qquad \square$$

(j) ~~Two~~ Two ways to prove this.

The logical approach:

If $|x|-|y| \ge 0$, then
$$\big||x|-|y|\big| = |x|-|y| \le |x-y|.$$
If $|x|-|y| < 0$, then
$$\big||x|-|y|\big| = |y|-|x| \le |y-x| = |x-y|.$$
In all cases, $\big||x|-|y|\big| \le |x-y|$. $\square$

The algebraic approach:
$$xy \le |xy| = |x||y|$$
$$\Longleftrightarrow \quad -2|x||y| \le -2xy$$
$$\Longleftrightarrow \quad |x|^2 - 2|x||y| + |y|^2 \le x^2 - 2xy + y^2$$
$$\Longleftrightarrow \quad (|x|-|y|)^2 \le (x-y)^2$$
$$\Longleftrightarrow \quad \sqrt{(|x|-|y|)^2} \le \sqrt{(x-y)^2}$$
$$\Longleftrightarrow \quad \big||x|-|y|\big| \le |x-y|. \qquad \square$$

Combine results from (h) through (i) ~~and~~ we write
$$|x|-|y| \le \big||x|-|y|\big| \le |x-y| \le |x|+|y|.$$
Notice that there is no persistent ordering between $|x+y|$ and $|x-y|$ and you can construct examples to make the former greater than, equal to, or less than the latter. But it's only when at least one of x or y is zero that the equality holds.

2. $(a_1) \equiv (b_2)$, $(a_2) \equiv (b_5)$, $(a_3) \equiv (b_7)$, $(a_4) \equiv (b_{10})$, $(a_5) \equiv (b_3)$, $(a_6) \equiv (b_8)$, $(a_7) \equiv (b_9)$, $(a_8) \equiv (b_4)$, $(a_9) \equiv (b_6)$, $(a_{10}) \equiv (b_1)$.

There is always a geometric interpretation as well as an algebraic interpretation for those inequalities. We can always apply theorem I.38 to the above inequalites and strip off the absolute value symbol, then apply algebraic tricks and tidy up results with logic. Pure logical approach becomes cumbersome very quickly when the situation becomes complicated. However cumbersome it is, it is still doable. You cannot avoid logic completely. The geometric interpretation is as follows. Suppose $b \ge 0$, the number $|x-a|$ represents the distance of $x-a$ from zero, which is the same as the distance of x from a. The inequality $|x-a| \le b$ can be interpreted as finding all x of which the distance from a is less than or equal to b; similarly the ~~number~~ inequality $|x-a| \ge b$ can be interpreted as finding all x of which the distance from a is larger than or equal to b. For (a_1) through (a_7), the geometry can be used with mild twists to obtain the answers. For (a_8), the algebraic approach is very neat.
$$|x-5| < |x+1|$$
$$\Longleftrightarrow \quad \sqrt{(x-5)^2} < \sqrt{(x+1)^2}$$

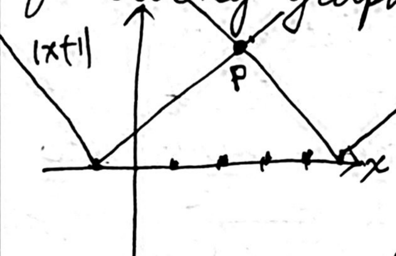
$\Leftrightarrow \quad (x-5)^2 < (x+1)^2$

$\Leftrightarrow \quad x^2-10x+25 < x^2+2x+1$

$\Leftrightarrow \quad 24 < 12x$

$\Leftrightarrow \quad 2 < x.$

There are two ways to look at ~~this~~ this from a geometric perspective. The first is linear programing. Let $f(x)=|x-5|$ and $g(x)=|x+1|$, then we plot these two functions on the Cartesian plane and obtain the ~~follo~~ following graph.

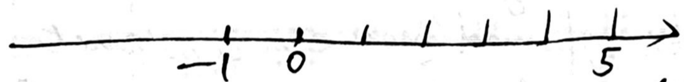


~~\$~~ The original inequality is the same as $f(x) < g(x)$, and this is satisfied ~~for all the points where~~ when g is above f. The graph indicates that all x to the right of the ~~\$~~ intersection of f and g satisfies the condition. The intersection ~~is~~ can be given by

$$x+1=5-x \Rightarrow x=2.$$

Thus the original inequality is equivalent to $x>2$.

Finally, the alternative geometric approach is to look at the real axis.



The number $|x+1|$ ~~can~~ is the distance of x from -1 and ~~$|x+5|$~~ $|x-5|$ ~~a~~ is the distance of x from 5. Geometrically, if x is to the left of -1, ~~the dis~~ $|x+1|$ is always strictly less than $|x-5|$; similarly if x is to the right of 5, $|x+1|$ is always larger than $|x-5|$. Thus the critical point must lies between -1 and 5. In fact, it's the ~~cet~~ midpoint that is away from -1 and 5 at equal distance. And it can be found by

$$\frac{-1+5}{2} = \frac{4}{2} = 2.$$

Thus, ~~all~~ only $x > 2$ ~~a~~ satisfies the original inequality. ~~and only~~

For (a_9), after some simplification we have $1 \le x^2 \le 3$. Due ~~the~~ to the symmetry of squre, we have also $1 \le (-x)^2 \le 3$ and this exhausts ~~the~~ all the possible values that x could take, i.e., $x \ge 0$ and $x < 0$. ~~Simple~~ Similarly, if we were to solve $||x|-2| \le 1$, we have ~~the~~ to consider the symmetry of $|x|$.

For (a_{10}), we ~~rew~~ rewrite the inequality into

$(x-4)(x+3) > 0$ and

$(x-6)(x+2) < 0$.

We then ~~use~~ use logic to tidy up the result.

3. (a). False. When $x=-10$, the antecedent is true but the consequent is false.

(b) True. $|x-5|<2 \Leftrightarrow$ $-2 < x-5 < 2 \Leftrightarrow 3 < x < 7$.

(c) ~~False~~ True. $|1+3x| \le 1$ $\Leftrightarrow -\frac{2}{3} \le x \le 0 \Rightarrow -\frac{2}{3} \le x$.

(d) ~~F~~ False. Consider when $x = \frac{3}{2}$.

(e) False. Consider when $x=3$.

4. Use the ~~na~~ notation and the setting from the textbook. We have that the equality sign holds iff $AC = B^2$ iff $Ax^2 + 2Bx + C = A(x + \frac{B}{A})^2$. Let $x = -\frac{B}{A}$, then $Ax^2 + 2Bx + C = 0$, which means

$$\sum_{k=1}^{n} (a_k x + b_k)^2 = 0.$$

But a sum of squares ~~a~~ is zero only when every item is zero. This is the same as saying that x satisfies that $a_k x + b_k = 0$ for every $k = 1, 2, \cdots, n$.

Conversely, if there exists an x s.t. ~~$a_k x + b$~~ $a_k x + b_k = 0$ for all k, then ~~$\sum_{k=1}^{n}(a_k x + b_k)^2 = 0$, which means~~

~~$A(x+\frac{B}{A})^2 + \frac{AC-B^2}{A} = 0$~~

~~$x = -\frac{b_k}{a_k}$ and~~ $b_k = -a_k x$ for all k, which implies

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left[\sum_{k=1}^{n} (-x a_k^2)\right]^2$$

$$= x^2 \left(\sum_{k=1}^{n} a_k^2\right)^2$$

$$= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} a_k^2 x^2\right)$$

$$= \left(\sum_{k=1}^{n} a_k^2\right)\left[\sum_{k=1}^{n} (-a_k x)^2\right]$$

$$= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \square$$

I 4.10

1. (a) $\binom{5}{3} = \frac{5\times4\times3}{1\times2\times3} = 10$, (b) $\binom{7}{0} = 1$, (c) $\binom{7}{1} = 7$, (d) $\binom{7}{2} = 21$,

(e) $\binom{17}{14} = \binom{17}{3} = \frac{17\times16\times15}{2\times3} = 17\times8\times5 = 680.$

(f) $\binom{0}{0} = 1$.

2. (a) $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k}$.

~~(c)~~ (c)

~~$\Leftrightarrow$~~ $\binom{14}{k} = \binom{14}{k-4} \Leftrightarrow \frac{14!}{k!(14-k)!} = \frac{14!}{(k-4)!(18-k)!}$

$\Leftrightarrow \frac{k!}{(k-4)!} = \frac{(18-k)!}{(14-k)!}$

$\Leftrightarrow k(k-1)(k-2)(k-3) = (18-k)(17-k)(16-k)(15-k)$.

Solve the equation and we obtain ~~of~~ the solution. However, this is computationally intensive and cumbersome.

Another approach is to utilize the symmetry of binomial coefficient. We have

$$\binom{14}{k} = \binom{14}{14-k} = \binom{14}{k-4}$$

$\Rightarrow 14 - k = k - 4 \Rightarrow k = 9.$

This gives a quick way to find such k.

(b) By symmetry, we have

$$\binom{n}{10} = \binom{n}{n-10} = \binom{n}{7}.$$

Let $n - 10 = 7 \Rightarrow n = 17$.

(d) Suppose that there is such a k, then we have

$$\binom{12}{k} = \binom{12}{12-k} = \binom{12}{k-3} \Rightarrow$$

$12 - k = k - 3 \Rightarrow k = 7.5.$

But k must be an integer, which contradicts our result. Thus, no.

3. ~~$\binom{n+1}{k}$~~

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \frac{n!(n+1-k+k)}{k!(n-k+1)!}$$

$$= \frac{n!\,k}{k![n-(k-1)]!} + \frac{n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!}{(k-1)![n-(k-1)]!} + \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k-1} + \binom{n}{k}. \square$$

4. Basis:

$$(a+b)^0 = 1 = \binom{0}{0} a^0 b^0.$$

Induction:

$$(a+b)^{n+1} = (a+b)^n (a+b)$$

$$= \left(\sum_{k=0}^{n} a^k b^{n-k}\right)(a+b)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \binom{n}{0}a^0 b^{n+1} + \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n-k+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k+1}$$

$$= \binom{n+1}{0}a^0 b^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]a^k b^{n-k+1} + \binom{n}{n}a^{n+1}b^0$$

$$= \binom{n+1}{0}a^0 b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n+1-k} + \binom{n+1}{n+1}a^{n+1}b^0$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}. \quad \square$$

$$\sum_{k=0}^{n}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k} = (1+1)^n = 2^n.$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = (-1+1)^n = 0^n = \begin{cases} 0, & \text{for } n>0, \\ 1, & \text{for } n=0. \end{cases}$$

5. We can begin with one, but I choose to start from zero.

(a) $\prod_{k=1}^{0} a_k = 1,$

(b) $\prod_{k=1}^{n+1} a_k = \left(\prod_{k=1}^{n} a_k\right) a_{n+1}$

6. Basis: $\prod_{k=1}^{0}(a_k b_k) = 1 = 1\cdot 1 = \left(\prod_{k=1}^{0} a_k\right)\left(\prod_{k=1}^{0} b_k\right).$

Induction:

$$\prod_{k=1}^{n+1}(a_k b_k) = \left[\prod_{k=1}^{n}(a_k b_k)\right] a_{n+1} b_{n+1} = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right) a_{n+1} b_{n+1}$$

$$= \left[\left(\prod_{k=1}^{n} a_k\right) a_{n+1}\right]\left[\left(\prod_{k=1}^{n} b_k\right) b_{n+1}\right] = \left(\prod_{k=1}^{n+1} a_k\right)\left(\prod_{k=1}^{n+1} b_k\right). \quad \square$$

$$\prod_{k=1}^{n}(c a_k) = c^n \prod_{k=1}^{n} a_k.$$

7. Basis: $\prod_{k=1}^{0}\frac{a_k}{a_{k-1}} = 1 = \frac{a_0}{a_0}.$

Induction:

$$\prod_{k=1}^{n+1}\frac{a_k}{a_{k-1}} = \left(\prod_{k=1}^{n}\frac{a_k}{a_{k-1}}\right)\frac{a_{n+1}}{a_n} = \frac{a_n}{a_0}\cdot\frac{a_{n+1}}{a_n} = \frac{a_n}{a_0}. \quad \square$$

8. Basis: $\prod_{k=1}^{0}(1+x^{2^{k-1}}) = 1 = \frac{1-x}{1-x} = \frac{1-x^{2^0}}{1-x}.$

Induction:

$$\prod_{k=1}^{n+1}(1+x^{2^{k-1}}) = \left[\prod_{k=1}^{n}(1+x^{2^{k-1}})\right](1+x^{2^n}) = \frac{1-x^{2^n}}{1-x}\cdot(1+x^{2^n}) = \frac{1^2-(x^{2^n})^2}{1-x} = \frac{1-x^{2^{n+1}}}{1-x}. \quad \square$$

When $x=1$, we have

$$\prod_{k=1}^{n}(1+x^{2^{k-1}}) = \prod_{k=1}^{n} 2 = 2^n.$$

9. The inequality doesn't hold for the general case and we can easily construct counterexamples as follows.

$a_1 = -4,\ b_1 = 1 \quad a_2 = -2,\ b_2 = 2$

Then $a_1 < b_1$ and $a_2 < b_2$,

but $a_1 a_2 = 8 > 2 = b_1 b_2$.

However, if we restrict the condition to $0 < a_k < b_k$ for each $k = 1, 2, \dots, n$, then the inequality does hold and can be easily proved by induction.

If we restrict the condition to $a_k < b_k < 0$, then the inequality holds for odd n but the direction is reversed for even n.

10. First, we consider when $x>1$.

Basis: Since $0<1<x$, we have

$$0<x-1 \text{ and } 0<x,$$

which implies

$$0<x(x-1)=x^2-x$$

$$\Leftrightarrow \quad x^2>x.$$

Induction:

$$\because \quad x^n>x$$

$$\therefore \quad x^{n+1}>x^2>x. \qquad \square$$

Now we consider when $0<x<1$.

Basis: Since $0<x<1$, we have

$$1-x>0 \text{ and } x>0,$$

which implies

$$(1-x)x>0 \Leftrightarrow x^2<x.$$

Indution:

$$x^n<x \Rightarrow x^{n+1}<x^2<x. \qquad \square$$

11. We claim $2^n<n!$ for all $n\geq 4$.

Since $n!$ is only defined for nonnegative integers, we can list the first few terms for both expressions.

	0	1	2	3	4	5	...
2^n	1	2	4	8	16	32	...
$n!$	1	1	2	6	24	120	...

Basis: $2^4=16<24=4!$.

Induction:

Since $2^n<n!$ and

$$2<4\leq n<n+1,$$

we have

$$2^{n+1}<n!(n+1)=(n+1)!.$$

We have verified that it's not the case that $2^n<n!$ for all $n<4$. This means all $n\geq 4$ are the only integers that makes $2^n<n!$ true. $\square$

12. (a)

$$\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}$$

$$=1+\sum_{k=1}^{n}\frac{n!}{k!(n-k)!\,n^k}$$

$$=1+\sum_{k=1}^{n}\frac{1}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{\underbrace{n\cdot n\cdots n}_{k\text{ times}}}$$

$$=1+\sum_{k=1}^{n}\frac{1}{k!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$=1+\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}. \qquad \square$$

(b) We deduce the inequalities in three steps. First, we want to establish

$$2<\left(1+\frac{1}{n}\right)^n \text{ for } n>1.$$

When $n>1$, we have

$$\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}$$

$$=1+\sum_{k=2}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}$$

Since $\frac{1}{k!}>0$ and $1-\frac{r}{n}>0$ for all possible values taken here, we have

$$\sum_{k=2}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}>0,$$

which implies

$$\left(1+\frac{1}{n}\right)^n=1+\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}$$

$$=2+\sum_{k=2}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}$$

$$>2.$$

Second, we want to establish

$$\left(1+\frac{1}{n}\right)^n<1+\sum_{k=1}^{n}\frac{1}{k!}.$$

Conditioned on $2\leq k\leq n$, we have

$$\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)<1,$$

which means

$$\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)<\frac{1}{k!}$$

$$\Rightarrow \sum_{k=2}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}<\sum_{k=2}^{n}\frac{1}{k!}$$

Then, we have

$$\left(1+\frac{1}{n}\right)^n = 2+\sum_{k=2}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}$$
$$< 2+\sum_{k=2}^{n}\frac{1}{k!}$$
$$= 1+\sum_{k=1}^{n}\frac{1}{k!}$$
for all $n>1$.

Finally, we want to show that
$$1+\sum_{k=1}^{n}\frac{1}{k!}<3.$$
This is equivalent to
$$\sum_{k=1}^{n}\frac{1}{k!}<2.$$
Since $\frac{1}{k!}\leq\frac{1}{2^{k-1}}$ for $k\geq1$, thus
$$\sum_{k=1}^{n}\frac{1}{k!}\leq\sum_{k=1}^{n}\frac{1}{2^{k-1}}$$
$$=\sum_{k=0}^{n-1}\frac{1}{2^k}$$
$$=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$
$$=2-\frac{1}{2^{n-1}}$$
$$<2.$$
□

13. (a) Although this is so obvious can be shown by directly multiplying out the RHS, we decide to use the telescoping property as per the hint.
$$b^p-a^p=\sum_{k=1}^{p}\left(b^k a^{p-k}-b^{k-1}a^{p-(k-1)}\right)$$
$$=\sum_{k=1}^{p}(b-a)b^{k-1}a^{p-k}$$
$$=(b-a)\sum_{k=1}^{p}b^{k-1}a^{p-k}$$
$$=(b-a)(b^{p-1}+b^{p-2}a+b^{p-3}a^2+\cdots+ba^{p-2}+a^{p-1}).$$

(b) Notice that
$$\frac{(n+1)^{p+1}-n^{p+1}}{p+1}=\frac{(n+1)-n}{p+1}\sum_{k=0}^{p}(n+1)^{p-k}n^{k}$$
$$=\frac{1}{p+1}\sum_{k=0}^{p}(n+1)^{p-k}n^{k}$$
notice that
$$n<n+1$$
$$\Rightarrow n^k<(n+1)^k \text{ for } k>0$$
$$\Rightarrow (n+1)^{p-k}n^k<(n+1)^p \text{ for } k>0$$
$$\Rightarrow \sum_{k=1}^{p}(n+1)^{p-k}n^k<\sum_{k=1}^{p}(n+1)^p$$
and
$$(n+1)^{p-0}n^0=(n+1)^p$$
thus,
$$\sum_{k=0}^{p}(n+1)^{p-k}n^k<\sum_{k=0}^{p}(n+1)^p$$
$$=(n+1)^p(p+1)$$
$$\Rightarrow \frac{1}{p+1}\sum_{k=0}^{p}(n+1)^{p-k}n^k<(n+1)^p.$$
Similarly, since
$$n<n+1$$
$$\Rightarrow n^{p-k}<(n+1)^{p-k} \text{ for } k<p$$
$$\Rightarrow \sum_{k=0}^{p-1}n^p<\sum_{k=0}^{p-1}(n+1)^{p-k}n^k$$
And
$$n^p n^{p-p}=(n+1)^{p-p}n^p,$$
therefore
$$\sum_{k=0}^{p}n^p<\sum_{k=0}^{p}(n+1)^{p-k}n^k.$$
Multiply both side with $1/(p+1)$, we obtain
$$\frac{1}{p+1}\sum_{k=0}^{p}(n+1)^{p-k}n^k>\frac{1}{p+1}n^p(p+1)$$
$$=n^p.$$
Thus, we have
$$n^p<\frac{(n+1)^{p+1}-n^{p+1}}{p+1}<(n+1)^p.$$

(c) Basis:
$$0<\frac{1}{p+1}<1.$$
Induction:

IH is
$$\sum_{k=1}^{n-1}k^p<\frac{n^{p+1}}{p+1}<\sum_{k=1}^{n}k^p.$$
Add the result from (b), we get
$$\left(\sum_{k=1}^{n-1}k^p\right)+n^p<\frac{n^{p+1}}{p+1}+\frac{(n+1)^{p+1}-n^{p+1}}{p+1}$$
$$<\left(\sum_{k=1}^{n}k^p\right)+(n+1)^p$$
$$\Leftrightarrow \sum_{k=1}^{n}k^p<\frac{(n+1)^{p+1}}{p+1}<\sum_{k=1}^{n+1}k^p.$$
□

14. Basis: $1+a_1\geq1+a_1$

Induction:

$$\prod_{k=1}^{n+1}(1+a_k) = \left[\prod_{k=1}^{n}(1+a_k)\right](1+a_{n+1})$$

$$\geq \left(1+\sum_{k=1}^{n} a_k\right)(1+a_{n+1})$$

$$= 1+\sum_{k=1}^{n} a_k + \left(1+\sum_{k=1}^{n} a_k\right)a_{n+1}$$

We want to show that

$$\left(1+\sum_{k=1}^{n} a_k\right)a_{n+1} \geq a_{n+1}.$$

Since a_k all have the same sign, let us suppose they are positive. Then

$$\sum_{k=1}^{n} a_k > 0$$

$$\Leftrightarrow \quad 1+\sum_{k=1}^{n} a_k > 1$$

$$\Rightarrow \quad \left(1+\sum_{k=1}^{n} a_k\right)a_{n+1} > a_{n+1}.$$

Suppose they are all negative.

$$\sum_{k=1}^{n} a_k < 0$$

$$\Leftrightarrow \quad 1+\sum_{k=1}^{n} a_k < 1$$

$$\Rightarrow \quad \left(1+\sum_{k=1}^{n} a_k\right)a_{n+1} > a_{n+1}.$$

The situation is trivial when they are all zero.

$$\left(1+\sum_{k=1}^{n} a_k\right)a_{n+1} = 0 = a_{n+1}.$$

Thus, we have shown that

$$\prod_{k=1}^{n+1}(1+a_k) \geq 1+\left(\sum_{k=1}^{n} a_k\right) + a_{n+1}$$

$$= 1+\sum_{k=1}^{n+1} a_k. \quad \square$$

The meaning of "all having the same sign" could have different interpretations. In the above treatment, I treated positive, negative, and zero as three signs. We could merge zero into positive or negative, or both. It does not matter for the final results. The result is the same anyway.

It's trivial to show that

$$(1+0)^n = 1 = 1+n0$$

for $n>1$. But to show that $x=0$ is the only number ~~the~~ for ~~wich~~ which the equality sign holds, we can look at the induction step. We have shown that

$$\left(1+\sum_{k=1}^{n} a_k\right)a_{n+1} > a_{n+1}$$

for nonzero a_k. This means that when $n>1$ the equality sign holds only for $x=0$.

15.

Basis:

$$\frac{2!}{2^2} = \frac{1}{2} \leq \left(\frac{1}{2}\right)^1.$$

notice that $k=[n/2]=[2/2]=1$.

Induction:

$$\frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!(n+1)}{(n+1)^n(n+1)}$$

$$= \frac{n!}{(n+1)^n}.$$

We are going to show

$$(n+1)^n > n^n \cdot 2.$$

This is necessarily so since we have already established in exercise 12 that

$$\left(1+\frac{1}{n}\right)^n > 2 \quad \text{for } n \geq 2$$

$$\Leftrightarrow \quad \left(\frac{n+1}{n}\right)^n > 2$$

$$\Leftrightarrow \quad (1+n)^n > n^n \cdot 2$$

$$\Rightarrow \quad \frac{n!}{(n+1)^n} < \frac{n!}{n^n \cdot 2}$$

$$\leq \left(\frac{1}{2}\right)^{k+1}$$

$$< \left(\frac{1}{2}\right)^k.$$

~~When~~ Notice that

$$\left[\frac{n+1}{2}\right] = \begin{cases} k, & n \text{ is even}, \\ k+1, & n \text{ is odd}. \end{cases}$$

In any case, we have

$$\frac{(n+1)!}{(n+1)^{n+1}} \leq \left(\frac{1}{2}\right)^{[(n+1)/2]}. \qquad \square$$

The question starts from $n \geq 2$, but it's easy to show that we can start from $n \geq 0$.

16. Basis:

$$1 < \frac{1+\sqrt{5}}{2}$$

and

$$2 < \left(\frac{1+\sqrt{5}}{2}\right)^2.$$

~~Since~~ This is due to

$$1 < \sqrt{5}$$
$$\iff 2 < 1+\sqrt{5}$$
$$\iff 1 < \frac{1+\sqrt{5}}{2}.$$

and

$$1 < \sqrt{5}$$
$$\iff 2 < 2\sqrt{5}$$
$$\iff 8 < 6+2\sqrt{5}$$
$$\iff 2 < \frac{5+2\sqrt{5}+1}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^2.$$

Induction:

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} \\ &< \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}. \qquad \square \end{aligned}$$

17. The pth-power mean is defined as

$$M_p := \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}.$$

M_1 is arithmetic mean, M_2 is root mean square, and M_{-1} is the harmonic mean.

As per the hint, we let $a_k = x_k^p$ and $b_k = 1$, then by Cauchy-Schavarz

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \leq \left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n b_k^2\right)$$
$$\iff \left(\sum_{k=1}^n x_k^p\right)^2 \leq n\sum_{k=1}^n x_k^{2p}$$
$$\iff \frac{1}{n}\left(\sum_{k=1}^n x_k^p\right)^2 \leq \sum_{k=1}^n x_k^{2p}$$

The equality sign holds if and only if $\exists x$ s.t. $a_k x + b_k = 0$ for all k. Then

$$x_k^p = -\frac{1}{x}.$$

Since x_k are all positive, ~~there is a unique~~ thus

$$x_k = \left(-\frac{1}{x}\right)^p.$$

This means x_k are all equal which contradicts our condition. The equality sign doesn't hold. We have

$$\frac{1}{n}\left(\sum_{k=1}^n x_k^p\right)^2 < \sum_{k=1}^n x_k^{2p}$$
$$\iff \left(\frac{1}{\sqrt{n}}\sum_{k=1}^n x_k^p\right)^2 < \sum_{k=1}^n x_k^{2p}.$$

Both sides of this inequality are positive and we can take (2p)th root of both side and obtain

$$\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n x_k^p\right)^2\right]^{1/2p} < \left(\sum_{k=1}^n x_k^{2p}\right)^{1/2p}$$
$$\iff \left(\frac{1}{\sqrt{n}}\sum_{k=1}^n x_k^p\right)^{1/p} < \left(\sum_{k=1}^n x_k^{2p}\right)^{1/2p}$$
$$\iff \left(\frac{\sqrt{n}}{n}\sum_{k=1}^n x_k^p\right)^{1/p} < \left(\sum_{k=1}^n x_k^{2p}\right)^{1/2p}$$
$$\iff n^{1/2p}\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{1/p} < \left(\sum_{k=1}^n x_k^{2p}\right)^{1/2p}$$

$$\Leftrightarrow \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{p}\right)^{1/p} < \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{2p}\right)^{1/2p}$$

$$\Leftrightarrow \quad M_p < M_{2p}. \quad \square$$

18. $$\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \le \left(\frac{a^4+b^4+c^4}{3}\right)^{1/4}$$

$$\Leftrightarrow \quad \left(\frac{8}{3}\right)^{1/2} \le \left(\frac{a^4+b^4+c^4}{3}\right)^{1/4}$$

$$\Rightarrow \quad \left(\frac{8}{3}\right)^{4/2} \le \frac{a^4+b^4+c^4}{3}$$

$$\Leftrightarrow \quad \frac{64}{3} \le a^4+b^4+c^4. \quad \square$$

The equality sign holds if and only if $a=b=c$.

19. ~~Instead of proving~~

$$\cancel{\prod_{k=1}^{n} a_k = 1 \Rightarrow \sum_{k=1}^{n} a_k \geq n,}$$

~~we are going to prove~~

$$\cancel{\prod_{k=1}^{n} a_k \geq 1 \Rightarrow \sum_{k=1}^{n} a_k \geq n,}$$

~~which entails the former.~~

~~Basis:~~ $\cancel{a_1 \geq 1 \Rightarrow a_1 \geq 1,}$

$\cancel{a_1 a_2 \geq 1 \Leftrightarrow 2a_1 a_2 \geq 2,}$

~~and since~~ $\cancel{(a_1-a_2)^2 \geq 0 \Rightarrow a_1^2+a_2^2 \geq 2a_1a_2 \geq 2} \Rightarrow a_1^2+2a_1a_2+a_2^2 \geq 4a_1a_2 \geq 4$ $\cancel{\Rightarrow (a_1+a_2)^2 \geq 4} \Rightarrow$

$\Leftrightarrow a_1+a_2 \geq 2.$

Induction: ~~We have when $n \geq 2$~~ $\cancel{\prod_{k=1}^{n+1} a_k \geq 1}$; ~~if every a_k is less than one, then clearly $\prod_{k=1}^{n+1} a_k < 1$, which contradicts our assumption. There is at least one $a_k \geq 1$, let's denote its index as m. Then~~

$$\cancel{\left(\prod_{\substack{1\le k\le n+1\\ k\neq m}} a_k\right) + a_m \geq 2,}$$

~~this is simply due to~~

$$\cancel{\prod_{k=1}^{n+1} a_k = \left(\prod_{\substack{1\le k\le n+1\\ k\neq m}} a_k\right) a_m \geq 1.}$$

~~Since $a_m \geq 1 \Leftrightarrow -a_m \leq -1$~~

Using the notation from the hint, we have the following.

Basis: $a_1 = 1 \Rightarrow a_1 \geq 1.$

Induction:

If all $a_k = 1$, then $\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n+1} 1 = n+1 \geq n+1$. If ~~not all a_k~~ not all $a_k = 1$, there is at least one $a_i \neq 1$. Either $a_i > 1$ or $a_i < 1$. Suppose $a_i > 1$, then the rest of factors cannot be greater ~~than or equal~~ to one, if that is the case, the entire product would exceed one. This means there must be a factor $a_j < 1$. We can reshuffle the factors such that $i = 1$ and $j = n+1$. Then

$$a_1 - 1 > 0 \quad \text{and} \quad a_{n+1} - 1 < 0,$$

which implies

$$(a_1-1)(a_{n+1}-1) < 0$$

$$\Leftrightarrow \quad a_1 a_{n+1} - a_1 - a_{n+1} + 1 < 0$$

$$\Leftrightarrow \quad a_1 a_{n+1} + 1 < a_1 + a_{n+1}$$

Let $b_1 = a_1 a_{n+1}$, then we have

$$b_1 \prod_{k=2}^{n} a_k = 1 \Rightarrow b_1 + \sum_{k=2}^{n} a_k \geq n$$

and thus

$$n+1 \leq b_1 + \left(\sum_{k=2}^{n} a_k\right) + 1 < a_1 + \left(\sum_{k=2}^{n} a_k\right) + a_{n+1} = \sum_{k=1}^{n+1} a_{n+1}.$$

If $a_i < 1$, we will find another factor $a_j > 1$ from the rest factors. ~~Rest~~ Reshuffle such that $i = n+1$ and $j = 1$ and the above reasoning still holds.

All in all, we have

$$\sum_{k=1}^{n+1} a_k \geq n+1 \quad \text{for } n \geq 1.$$

If not all $a_k = 1$, then $\sum_{k=1}^{n+1} a_k > n+1$. This means the equality sign holds if and only if all $a_k = 1$. $\square$

20. (a)

Since
$$1 = \frac{\prod_{k=1}^{n} x_k}{\prod_{j=1}^{n} x_j} = \prod_{k=1}^{n} \frac{x_k}{\left(\prod_{j=1}^{n} x_j\right)^{1/n}},$$
We have
$$\sum_{k=1}^{n} \frac{x_k}{\left(\prod_{j=1}^{n} x_j\right)^{1/n}} \ge n$$
$$\Leftrightarrow \quad \frac{1}{n}\sum_{k=1}^{n} x_k \ge \left(\prod_{j=1}^{n} x_j\right)^{1/n}$$
$$\Leftrightarrow \quad G \le M_1.$$
The equality sign holds only when $\frac{x_k}{G} = 1$ for all k, which is the same as $x_1 = x_2 = \cdots = x_n = G$. □

(b)

Since
$$1 = \frac{\prod_{k=1}^{n} x_k^p}{\prod_{j=1}^{n} x_j^p} = \frac{\prod_{k=1}^{n} x_k^p}{\left(\prod_{j=1}^{n} x_j\right)^p} = \prod_{k=1}^{n} \frac{x_k^p}{\left(\prod_{j=1}^{n} x_j\right)^{p/n}}$$
and $x_1, x_2, \ldots, x_n$ are not all equal, which implies $x_k^p / \left(\prod_{j=1}^{n} x_j\right)^{p/n}$ are not all equal to one, we have
$$n < \sum_{k=1}^{n} \frac{x_k^p}{\left(\prod_{j=1}^{n} x_j\right)^{p/n}}$$
$$\Leftrightarrow \quad \left(\prod_{j=1}^{n} x_j\right)^{p/n} < \frac{1}{n}\sum_{k=1}^{n} x_k^p$$
Since $p > 0$, we take on both sides the pth power and the direction of inequality is preserved. Thus,
$$\left(\prod_{j=1}^{n} x_j\right)^{1/n} < \left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{1/p}$$
$$\Leftrightarrow \quad G < M_p.$$
We can do the same to the case of $q < 0$, except that in the last step the direction of inequality is reversed since q is negative. Overall, we have
$$M_q < G < M_p. \quad \square$$

21.
$$\frac{a+b+c}{3} \ge \sqrt[3]{abc} = 2$$
$$\Leftrightarrow \quad a+b+c \ge 6.$$
$$\frac{3}{1/a + 1/b + 1/c} \le \sqrt[3]{abc} = 2$$
$$\Leftrightarrow \quad \frac{3}{2} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$
$$\Leftrightarrow \quad 12 \le ab + ac + bc.$$
The last step is obtained by multiplying both sides with $abc = 8$.

22.
$$\cancel{M_1 \ge M_{-1}}$$
$$\cancel{\Leftrightarrow \quad \frac{1}{n}\sum_{k=1}^{n} x_k \ge n \sum_{k=1}^{n} y_k}$$
Multiply the LHS out and we get
$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) = \sum_{1 \le i,j \le n} x_i y_j.$$
Since
$$\prod_{1 \le i,j \le n} x_i y_j = 1,$$
thus,
$$\sum_{1 \le i,j \le n} x_i y_j \ge n^2.$$
The RHS is n^2 since there are n^2 terms such that $1 \le i,j \le n$ by multiplication principle. Thus,
$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \ge n^2. \quad \square$$

23.
$$\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \le \frac{a+b+c}{3} = \frac{1}{3}$$
$$\Leftrightarrow \quad 9 \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$
$$\Leftrightarrow \quad 9abc \le bc + ac + ab$$
$$\Leftrightarrow \quad 1 - a - b - c + ab + ac + bc - abc \ge 8abc$$
$$\Leftrightarrow \quad (1-a)(1-b)(1-c) \ge 8abc. \quad \square$$