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Hilbert Spaces with Applications

Prepared by :

Ohood Talat Zakour	3201772
Shahad Mohammed Madani	3657342
Haneen Hamed Alofi	3542051
Leen Abdullah Alharbi	3657191

Department of Mathematics, Science College, Taibah University

Supervisor:

Dr. Abdessalem Gasmi

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Table of Contents

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Abstract

The aim of this project is to introduce the Hilbert spaces and study the linear operators on this spaces and gives same applications to integral and differential equations .

Introduction

The concept of Hilbert space was put forward by David Hilbert in his work on quadratic forms in infinitely many variables. It's a generalization of Euclidean space to infinite dimensions.
The Hilbert space is a wide field in pure and applied mathematics and has been widely used in other fields, for example quantum mechanics, economists,
The aim of this project is to introduce the Hilbert spaces and study the linear operators on this spaces and gives same applications to integral and differential equations.
This work is organized as follows:
In chapter 1, we present basic definitions of the Hilbert spaces .
The Chapter 2 is devoted to Linear Operators on Hilbert Spaces
In chapter 3, we gives same applications to integral and differential equations.

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Chapter 1

Hilbert Spaces and Orthonormal Systems

Chapter 1: Hilbert Spaces and Orthonormal Systems

In this chapter, we present the basic definitions of the Hilbert spaces and we study the linear mappings in normed spaces. (see [1]).

1.1 Vector Spaces

Definition 1.1.1 ***(Vector space)***

A vector space is a nonempty set E mean with two operations:

$$E \times E \to E$$
$$(x, y) \to x + y \quad \textit{(addition)}$$
$$\mathrm{F} \times E \to E$$
$$(\lambda, y) \to \lambda x \quad \textit{(multiplication by scalars).}$$

Where F *is* $\mathbb{R}$ *or* $\mathbb{C}$.

For al $x, y, z \in E$ *and* $\alpha, \beta \in \mathbb{F}$, we have

(a) $x + y = y + x$

(b) $(x + y) + z = x + (y + z)$

(c) *For every* $x, y \in E$ *there exists a* $z \in E$ *such that* $x + y = z$

(d) $\alpha(\beta x) = (\alpha\beta)x$

(e) $(\alpha + \beta)x = \alpha x + \beta x$

(f) $\alpha(x + y) = \alpha x + \alpha y$

.(g) $1x = x$

Example 1.1.2.

.■ $\mathbb{R}^{\mathrm{N}} = \{(\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{N}}) : \mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{N}} \in \mathbb{R}\}$

.■ $\mathbb{C}^{\mathrm{N}} = \{(\mathrm{z}_1, \ldots, \mathrm{z}_{\mathrm{N}}) : \mathrm{z}_1, \ldots . \mathrm{z}_{\mathrm{N}} \in \mathbb{C}\}$

Example 1.1.3. (Function spaces)

Let Ω be an open subset of $\mathbb{R}^N$.

$\mathcal{C}(\Omega) = \{f: \Omega \to \mathbb{C};\ f\ is\ a$ continuous function $\}$.

$\mathcal{C}^k(\Omega) = \{f: \Omega \to$
$\mathbb{C};\ f$ hase a continuous partial derivatives of order k $\}$.

$\mathcal{C}^\infty(\Omega) =$ the space of infinitely differentiable functions.

$\mathcal{P}(\Omega) =$ the space of all polynomials of N variables .

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Example 1.1.5. (Sequence spaces)

The space of all sequences (x_n) of complex numbers is a vector space.

Example 1.1.6. (($\boldsymbol{l^p}$)- spaces)

For $p \geq 1$, $\boldsymbol{l^p} = \{(z_n)$ of complex numbers such that $\sum_{n=1}^{\infty} |z_n|^p < \infty\}$

Theorem 1.1.7. (Hölder's inequality)

Let $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $(x_n) \epsilon l^p and\ (y_n)\ \epsilon\ l^q$, then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{1/q}$$

Theorem 1.1.8. (Minkowski's inequality)

Let p ≥ 1. if $(x_n), (y_n) \in l^p$,then

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}.$$

Let $x_1, \dots\dots, x_k$ be elements of a vector space E. A vector $x \epsilon E$ is called a linear combination of vectors $x_1, \dots., x_k$ if there exist scalars $\alpha_1, \dots\dots, \alpha_k$ such that $x = \alpha_1 x_1 + \dots\dots\dots + \alpha_k x_k$.

For example, any element of $\mathbb{R}^N$ is linear combination of vectors

$e_1 = (1,0,0, \dots,0)$, $e_2 = (0,1,0, \dots,0)$, $\dots\dots\dots$, $e_N = (0,0, \dots.,0,1)$.

Similarly, any polynomial of degree K is a linear combination of monomials $1, x, x^2, \dots\dots., x^k$.

A finite collection of vectors $\{x_1, \dots.., x_k\}$ is called linearly independent if $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$ implies $\alpha_1 = \alpha_2 = \dots\dots.. = \alpha_k = 0$. An infinite collection of vectors $\mathcal{A}$ is called linearly independent if every finite sub collection of $\mathcal{A}$ is linearly independent.

Let $\mathcal{A}$ be a subset of a vector space E. By span $\mathcal{A}$ we denote the set of all finite linear combinations of vectors from $\mathcal{A}$,

A set of vectors $\mathcal{B} \subset$ E is called a basis of E if $\mathcal{B}$ is linearly independent and span $\mathcal{B} = E$.

The following are examples of set of vectors which are bases in $\mathbb{R}^3$

$$\mathcal{A} = \{(1,0,0), (0,1,0), (0,0,1)\} \qquad (1.2)$$

$$\mathcal{B} = \{(0,1,1), (1,0,1), (1,1,0)\} \qquad (1.3)$$

$$\mathcal{C} = \{(1,2,3), (1,3,5), (3,2,3)\} \qquad (1.4)$$

We have dim $\mathbb{R}^3 = 3$, and in general dim $\mathbb{R}^N = N$ Spaces $\mathcal{C}(\Omega), \mathcal{C}^k(\mathbb{R}^N), \mathcal{C}^\infty(\mathbb{R}^N)$ are infinite dimensional. Note that the dimension of the real vector space $\mathbb{C}^N$is $2N$, while the dimension of the complex vector space $\mathbb{C}^N$ is N.

1. 2 Normed Spaces

Definition 1.2.1. (Norm)

A function $x \mapsto \|x\|$ from a vector space E into $\mathbb{R}$ is called a norm if it satisfies the following conditions:

(a) $\|x\| = 0$ *implies* $x = 0$

(b) $\|\lambda x\| = |\lambda|\|x\|$ *for every* $x \in \mathrm{E}$ *and* $\lambda \in \mathbb{F}$

(c) $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in \mathrm{E}$

Example 1.2.2.

The following are norms on $\mathbb{C}^N$:

■ $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_N|^2}$

■ $\|z\| = |z_1| + \ldots\ldots + |z_N|$

■ $\|z\| = \max\{|z_1|, \ldots., |z_N|\}$.

Example 1.2.3.

Let Ω be a closed bounded subset of $\mathbb{R}^N$. The function $\|f\| = max_{x\in\Omega}|f(x)|$ defines a norm on $\mathcal{C}(\Omega)$.

Example 1.2.4.

Let $z = (z_n) \in l^p$. The function defined by $\|z\| = \left(\sum_{n=1}^{\infty}|z_n|^p\right)^{1/p}$ is a norm on l^p for any $p \geq 1$.

Definition 1.2.5. (Normed space)

A vector space with a norm is called a normed space.

Definition 1.2.6. (Convergence in a normed space)

Let $(E, \|.\|)$ be a normed space. A sequence (x_n) of E converges to some $x \in \mathrm{E}$, if $\lim_{n\to\infty}\|x_n - x\| = 0$, we write $\lim_{n\to\infty} x_n = x$ or simply $x_n \longrightarrow x$.

Theorem 1.2.7. (Equivalence of norms)

Let $\|.\|_1$ *and* $\|.\|_2$ are equivalent if and only if there exist positive numbers α and β such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \quad for\ all\ x \in \mathrm{E}.$$

Theorem 1.2.8

If E is finite dimensional, then any two norms on E are equivalent.

Definition 1.2.9. (open ball, closed ball, sphere)

Let $x \in E$, and let r be a positive number. We use the following notation:

$$B(x,r) = \{y \epsilon \text{ E}: \|y - x\| < r\} \qquad (open\ ball)$$

$$\bar{B}(x,r) = \{y \epsilon \text{ E}: \|y - x\| \leq r\} \qquad (closed\ ball)$$

$$S(x,r) = \{y \epsilon \text{ E}: \|y - x\| = r \qquad (sphere)$$

Definition 1.2.10. (open and closed sets)

A subset S of a normed space E is called open if for every $x \epsilon$ S there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq S$. A subset S is called closed if E$/S$ is open.

Example 1.2.11.

Open balls are open sets. Closed ball and spheres are closed sets.

Theorem 1.2.12.

(a) The union of any collection of open sets is open.

(b) The intersection of a finite number of open sets is open.

(c) The union of a finite number of closed sets is closed.

(d) The intersection of any collection of closed sets is closed.

(e) The empty set and the whole space are both open and closed.

Theorem 1.2.13.

A subset S of a normed space E is closed if and only if every sequence of elements of S convergent in E has its limit in S that is,

$x_1, x_2, \ldots \epsilon\ S$ *and* $x_n \to x$ *implies* $x \in S$.

Definition 1.2.14. (Closure)

Let S be a subset of a normed space E. The closure of S, denoted by cl S, is the intersection of all closed sets containing S.

Theorem 1.2.15

Let S be a subset of a normed space E. The closure of S is the set of limits of all convergent sequences of elements of S, that is,

$$\text{cl}\, S = \{\, x \epsilon \text{E}:\ there\ exist\ x_1, x_2, \ldots . \epsilon\, S\ such\ that \quad x_n \to x\,\}.$$

Theorem 1.2.16.

The Weierstrass theorem says that every continuous function on an interval $[a, b]$, can be uniformly approximated by polynomials. This can also be expressed as follows: The closure of the set of all polynomials on $[a, b]$ is the whole space $C\ ([a, b])$.

Definition 1.2.17. (Dense subset)

A subset S of a normed space E is called dense in E if cl $S = E$.

Example 1.2.18.

The set all polynomials on $[a, b]$ is dense in $\mathcal{C}\ ([a, b])$. The set of all sequences of coplex number, which have only a finite number of nonzero terms is dense in l^p for all $p \geq 1$.

Theorem 1.2.19.

Let S be a subset of a normed space E. The following conditions are equivalent:

(a) S is dense in E,

(b) For every $x \,\epsilon\, E$ there exist $x_1, x_2, \ldots\ \epsilon\, S\ such\ that\ x_n \to x,$

(c) Every nonempty open subset of E contains an element of S.

Definition 1.2.20. (Compact set)

A subset S of a normed space E is called compact if every sequence (x_n) in S contains a convergent subsequence whose limit belongs to S.

Example 1.2.21.

In $\mathbb{R}^N$ or $\mathbb{C}^N$, a set is compact if and only if it is bounded and closed.

Definition 1.2.22. (Bounded subset)

A subset S of a normed space E is called bounded if $S \subseteq B(0, r)$ for some $r > 0$.

It is easy to show that S is bounded if and only if $\|\lambda_n x_n\| \to 0$ for any $x_n \,\epsilon\, S$ and any scalars $\lambda_n \to 0$.

Theorem 1.2.23.

Compact sets are closed and bounded.

The next example shows that closed and bounded sets in infinite dimensional spaces need not be compact.

Example 1.2.24.

Consider the space $\mathcal{C}\ ([0,1])$. The closed unit ball $\bar{B}(0,1)$ is a closed and bounded set, but it is not compact. To see this, consider the sequence of functions defined by $x_n(t) = t^n$. Then $x_n \epsilon\ \bar{B}(0,1) for\ all\ n\ \epsilon\ \mathbb{N}$. Clearly, no subsequence of (x_n) converges in $\mathcal{C}\ ([0,1])$.

It turns out that finite dimensional spaces can be characterized by the property that the unit ball is compact. The proof of this fact is based on a theorem that is usually referred to as Riesz's lemma.

Theorem 1.2.25. (Riesz's lemma)

Let X be a closed proper subspace of a normed space E. For every $\varepsilon\ \epsilon\ (0,1)$ there exists an $x_\varepsilon \epsilon$ E such that $\|x_\varepsilon\| = 1$ and $\|x_\varepsilon - x\| \geq \varepsilon\ for\ all\ x\ \epsilon\ X$.

Theorem 1.2.26.

A normed space E is finite dimensional if and only if the closed unit ball in E is compact.

1.3 Banach Spaces

Definition 1.3.1. (Cauchy sequence)

A sequence of vectors (x_n) in a normed space is called a Cauchy sequence if for every $\varepsilon > 0$ there exists a number M such that

$\|x_m - x_n\| < \varepsilon \; for \; all \; m, n > M.$

Theorem 1.3.2.

The following conditions are equivalent:

(a) (x_n) is a Cauchy sequence

(b) $\|x_{p_n} - x_{q_n}\| \to 0 \; as \; n \to \infty$, for every pair of increasing sequences of positive integers (p_n) and (q_n)

(c) $\|x_{p_{n+1}} - x_{p_n}\| \to 0 \; as \; n \to \infty, for \; every \; increasing \; sequence \; of \; positive \; integers \; (p_n)$

Observe that every convergent sequence is a Cauchy sequence. In fact, if $\|x_n - x\| \to 0$, then

$$\|x_{p_n} - x_{q_n}\| \le \|x_{p_n} - x\| + \|x_{q_n} - x\| \to 0$$

For every pair of increasing sequences of indices p_n and q_n. The converse is not true, in general.

Example 1.3.3.

Let $\mathcal{P}$ $([0,1])$ be the space of polynomials on $[0,1]$ with the norm of uniform convergence $\|p\| = max_{[0,1]}|p(x)|$. Define

$$p_{n(x)} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

For $n = 1, 2, \ldots$. Then (p_n) is a Cauchy sequence, but it does not converge in $\mathcal{P}$ $([0,1])$ because its limit is not a polynomial.

Lemma 1.3.4.

If (x_n) is a Cauchy sequence in a normed space, then the sequence of norms $(\|x_n\|)$ converges.

If (x_n) is a Cauchy sequence then is a number M such that $\|x_n\| \leq M$ for all n.

Definition 1.3.5. (Banach space)

A normed space E is called complete if every Cauchy sequence in E converges to an element of E. A complete normed space is called Banach space.

Example 1.3.6.

- The space l^2 and $C\ ([a, b])$ are Banach spaces.

Definition 1.3.7. (Convergent and absolutely convergent series)

A series $\sum_{n=1}^{\infty} x_n$ in a normed space E is called convergent if there exists $x \epsilon \text{ E } such\ that\ \|x_1 + x_2 + \ldots + x_n - x\| \to 0\ as\ n \to \infty$.

In that case we write $\sum_{n=1}^{\infty} x_n = x, if\ \ \sum_{n=1}^{\infty} \|x_n\| < \infty$, then the series is called absolutely convergent.

In general, an absolutely convergent series need not converge.

Theorem 1.3.9.

A normed space is complete if and only if every absolutely convergent series converges.

Theorem 1.3.10.

A closed vector subspace of a Banach space is a Banach space itself.

Let $(E, \|.\|)$ be a normed space. A normed space $(\widetilde{E}, \|\ .\ \|_1)$ is called a completion of $(E, \|.\|)$ if

(a) There exists a one-to-one mapping $\Phi: E \longrightarrow \widetilde{\text{E}}$ such that

$$\Phi(\alpha x + \beta y) = \alpha\Phi(x) + \beta\Phi(y) \quad for\ all\ x, y\ \epsilon\ E\ and\ \alpha, \beta\ \epsilon\ F,$$

(b) $\|x\| = \|\Phi(x)\|_1$ for every $x\ \epsilon$ E,
(c) $\Phi(E)$ is dense in $\widetilde{\text{E}}$,
(d) $\widetilde{\text{E}}$ is complete.

1.4 Linear Mappings

First, we introduce some notations :

Let $E_1 and\ E_2$ be a vector spaces, and let L be a mapping from $E\ _1\ into\ E_2$.

If A is a subset of $E_1\ and\ \ B$ is a subset of E_2, we denoted by

- $L(A) = \{L(x) : x \epsilon A\}$,
- $L^{-1}(B) = \{x \epsilon E_1 : L(x) \epsilon B\}$.
- $D(L)$ the domain of L .
- $\mathcal{R}(L) = \{y \epsilon E_2 : L(x) = y$ for some $x \epsilon \mathcal{D}(L)\}$ the range of L .
- $N(L) = \{x \epsilon \mathcal{D}(L) / L(x) = 0\}$ the null space of L.
- $\mathcal{G}(L) = \{(x, y) : x \epsilon \mathcal{D}(L) and\ y = L(x)\} the\ graph\ of\ L.$

Definition 1.4.1. (linear mapping)

A mapping $L: \mathrm{E}_1 \to \mathrm{E}_2$ is called a linear mapping if

$L\ (\alpha x + \beta y) = \alpha L(x) + \beta L(y)\ for\ all\ x, y \epsilon E_1\ and\ all\ scalars\ \alpha, \beta.$

Definition 1.4.2. (Continuous mapping)

Let E_1 and E_2 be normed spaces. A mapping F from E_1 into $E_2$is called continuous at $x_0 \epsilon E_1$if, for any sequence (x_n) of elements of E_1convergent to x_0, the sequence $\left(F(x_n)\right)$ converges to $F(x_0)$.

F is continuous on E_1 if F is continuous at every $x \epsilon E_1$.

Example 1.4.3.

The norm on a normed space E is a continuous mapping from E into $\mathbb{R}$. Indeed, if $\|x_n - x\| \to 0$, then$|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0$.

Theorem 1.4.4.

Let $F: E_1 \to E_2$. The following conditions are equivalent:

(a) F is continuous
(b) The inverse image $F^{-1}(U)$ of any open subset U of E_2 is open in E_1
(c) The inverse image $F^{-1}(S)$ of any closed subset S of E_2 is closed in E_1.

Theorem 1.4.5.

If a linear mapping $L: E_1 \to E_2$ is continuous at some $x_0 \epsilon\, E_1$, then it is continuous.

Definition 1.4.6. (Bounded linear mapping)

A linear mapping $L: E_1 \to E_2$ is called bounded if there exists a number $\alpha > 0$ such that $\|Lx\| \leq \alpha\|x\|$ for all $x \epsilon E_1$.

Note that the condition in this definition is equivalent to saying that L is bounded by α on the unit sphere in E_1. More precisely, $\|Lx\| \leq \alpha$ for all $x \epsilon E_1$ such that $\|x\| = 1$.

Theorem 1.4.7.

A linear mapping is continuous if and only if it is bounded

Theorem 1.4.8.

If E_1 and E_2 are normed space, then $B(E_1, E_2)$ is a normed space with the norm defined by

$$\|L\| = sup_{\|x\|=1}\|Lx\|. \qquad (1.14)$$

Theorem 1.4.9.

If E_1 is a normed space and E_2 is a Banach space, then $B(E_1, E_2)$ is a Banach space.

Theorem 1.4.10.

If L is a continuous linear mapping from a subspace of a normed space E_1 into a Banach E_2, then L has a unique extension to a continuous linear mapping defined on the closure of the domain $D(L)$. In particular, if $D(L)$ is dense in E_1, then L has a unique extension to a continuous linear mapping defined on the whole space E_1.

Theorem 1.4.11.

If $L: E_1 \to E_2$ is a continuous linear mapping, then the null space $N(L)$ is a closed subspace of $E_1 \times E_2$.

Theorem 1.4.12. (Banach-Steinhaus theorem)

Let T be a family of bounded linear mappings from a Banach space X into a normed space Y. If for every $x \epsilon X$ there exists a constant M_x such that $\|TX\| \leq M_x$ for all $T \epsilon \mathcal{T}$, then there exists a constant $M > 0$ such that $\|T\| \leq M$ for all $T \epsilon \mathcal{T}$.

1.5 Contraction Mapping and the Banach Fixed point Theorem

The name fixed point theorem is usually given to a result which says that, if a mapping f satisfies certain conditions, then there is a point z such that $f(z) = z$. Such a point z is a called a fixed point of f.

Example 1.5.1.

Let $E = \mathcal{C}([0,1])$ be the space of complex-valued continuous functions defined on the closed interval $[0,1]$. let T be defined by

$$(Tf)(t) = f(0) + \int_0^t f(\tau)d\tau.$$

Clearly, for any $a \epsilon \mathbb{C}$, the function $f(t) = ae^t$ is a fixed point of T.

Definition 1.5.2. (Contraction mapping)

A mapping f from a subset A of a normed space E into E is called a contraction if there exists a positive number $c < 1$ such that

$$\|f(x) - f(y)\| \leq c\|x - y\| \quad \text{for all } x, y \epsilon A. \qquad (1.16)$$

Note that contraction mappings are continuous.

Theorem 1.5.3. (Banach fixed point theorem)

Let F be a closed subset of a Banach space E and let f be a contraction mapping from F into F. Then there exists a unique $x \epsilon F$ such that f(x) = x.

1.6 Inner Product Spaces

Definition 1.6.1. (Inner product space)

Let V be a real vector space. A mapping <u, v> is called an inner product in V if for any u, v, and w in V and all scalars k, if the following axioms are satisfied:

1) < u, v > = < v, u >

2) < u+ v, w > = < u, w > + < v, w >

3) < k u, v > = k < u, v >

4) < v, v > ≥ 0***, and < v ,v > = 0 if and only if v=0***

A real vector space with an inner product is called a real inner product space.

Example 1.6.1:

The space C ([a,b]) of all continuous valued functions on the interval [a ,b] , with the inner product

< f, g > = $\int_a^b \mathbf{f(x)g(x)dx}$.

Is an inner product space.

Example 1.6.2:

The vector space $\mathbf{M_{nxn}}$ **of all real matrices can be made into an inner product space under the inner product** $\langle \mathrm{A}, \mathrm{B} \rangle = \mathrm{tr}\ (\mathrm{B}^{\mathrm{T}} \mathrm{A})$ **, where A,B in** $\mathbf{M_{nxn}}$.

Theorem 1.6.1. (Schwarz's inequality)

For any two elements x and y of an inner product space, we have

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\|\ \|\boldsymbol{y}\|$$

The equality $|\langle x, y \rangle| = \|x\| \|y\|$ holds if and only if x and y are linearly dependent.

Corollary: (Triangle inequality)

For any tow elements x and y of an inner product space we have

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$$

Definition 1.6.2. (Norm in an inner product space)

By the norm in an inner product space E we mean the functional defined by

.$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$

1.7 Hilbert Spaces

Definition 1.7.1. (Hilbert space)

A complement inner product space is called a Hilbert space .

Example 1.7.1.

$\mathbb{R}^N$ is complement , it is a Hilbert space.

Example 1.7.2.

Any finite dimensional inner product space is a Hilbert space .

Definition 1.7.2. (Strong convergence)

A sequence (x_n) of vectors in an inner product space E is called strongly convergent to a vector x in E if $\|x_n - x\| \to 0$

Definition 1.7.3. (Weak convergence)

A sequence (x_n) of vectors in an inner product space E is called weakly convergent to a vector x in E $\|x_n - x\| \to 0 \text{ as } n \to \infty$

1.8 Orthogonal and Orthonormal Systems

Definition 1.8.1. (Orthogonal and orthonormal systems)

A family S of nonzero vectors in an inner product space E is called an orthogonal system if x , y are perpendicular to each other . In addition, || x || = 1 for all xϵS , then S is called an orthonormal system .

Theorem 1.8.1.

Orthogonal systems are linearly independent .

Definition 1.8.2. *(Orthonormal sequence)*

A sequence of vectors which constitutes an orthonormal system is called an orthonormal sequence.

Example 1.8.1.

For $\mathrm{e}_n = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \dots)$ **with 1 in the nth position , the set S =** $\{\mathrm{e}_1, \mathrm{e}_2, \dots\}$ **is an orthonormal system in** $\boldsymbol{\ell}^2$.

Chapter 2

Linear Operators on Hilbert Spaces

CHAPTER 2 Linear Operators on Hilbert Spaces

In this chapter we study the linear operator on Hilbert space .

2.1 Linear Operators

Definition 2.1.1.

A linear operator $\boldsymbol{A}$ is a linear mappings between subspaces of a Hilbert space and the operator norm is defined by

$$\|\boldsymbol{A}\| = \sup_{\|\boldsymbol{x}\|=\mathbf{1}} \|\boldsymbol{Ax}\|$$

and is a bounded norm.

Examples of operators

- The identity operator is defined by

$$Ix = x\,, for\ all\ x \in E.$$

- The null operator is defined by

$$Ox = 0\,,\ for\ all\ x \in E.$$

- The differential operator is a operators defined on a space of differentiable functions by

$$(Df)(x) = \frac{df}{dx}(x) = f'(\mathrm{x})$$

- The integral operator T is defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt\,.$$

  Where a and b are finite or infinite ,$a < b$ and k is the kernel of T.
- **Multiplication operator**
  A function $z \in C([a,b])$ is called the multiplier if we have an operator A on $L^2([a,b])$ defined by $(Ax)(t) = z(t)x(t)$ that is

$$\|Ax\|^2 = \int_a^b |x(t)|^2|z(t)|^2dt \leq \max_{[a,b]}|z(t)|^2 \int_a^b |x(t)|^2dt$$

We have

$$\|Ax\| \le \max_{[a,b]}|z(t)|\,\|x\|$$

Thus, A is bounded

- In this example we will show that every operator on $C^N$and every operator on any finite dimensional Hilbert space is bounded

Let A be an operator on $C^N$and let $\{e_1, \dots e_N\}$ be the standard orthonormal base in C^N

Define for $i, j \in \{1,2, \dots, N\}$

$$\alpha_{ij} = \langle Ae_j, e_i \rangle$$

Then for $x = \sum_{j=1}^{N} \lambda_j e_j \in C^N$ we have $Ax = \sum_{j=1}^{n} \lambda_j Ae_j$and hence

$$\langle Ax, e_i \rangle = \sum_{j=1}^{N} \lambda_j \langle Ae_j, e_i \rangle = \sum_{j=1}^{N} \alpha_{ij} \lambda_i$$

Thus, every operator on the space C^N can be defined by an $N \times N$ matrix. Conversely, for every $N \times N$ matrix (α_{ij}), this formula defines an operator on C^N. If operator A is defined by the matrix (α_{ij}),then

$$\|A\| \le \sqrt{\sum_{i=1}^{N} \sum_{J=1}^{N} |\alpha_{ij}|^2}$$

Which is the definition of bounded operator

Some properties of operators

Let A and B are two operators

- A and B are said to be equal if

  $D(A) = D(B)\ and\ Ax = Bx\ , \forall x\ in\ that\ common\ domain$

- The sum of two operators defined by

$$(A + B)x = Ax + Bx$$

- $D(A + B) = D(A) \cap D(B)$ as we mentioned in the introduction, the domain of an operator is a vector subspace, so $D(A) \cap D(B)$ is never empty
- The product of a scalar λ and an operator A is defined by

$$(\lambda A)x = \lambda(Ax)$$

- The product AB is defined by

$$(AB)x = A(Bx)$$

- The domain of the product AB

$$D(AB) = \{x \in D(B): Bx \in D(A)\}$$

- Non commuting operators

Consider the operators

$$Af(x) = xf(x) and\ D = \frac{d}{dx}$$

Its easy to check that $AD \neq DA$

Theorem 2.1.2

The product AB of bounded operators A and B is bounded and

$$\|AB\| \leq \|A\|\|B\|$$

Theorem 2.1.2

A bounded operator on a separable infinite dimensional Hilbert space can be represented by an infinite matrix

2.2 Bilinear functionals and quadratic forms

2.2.1 Bilinear functional.

Definition 2.2.1.

A bilinear functional is a mapping $\varphi: E \times E \to C$ satisfying the following

- $\varphi(\alpha x_1 + \beta x_2,\ y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2. y)$
- $\varphi(x,\ \ \alpha y_1 + \beta y_2) = \bar{\alpha}\varphi(x, y_1) + \bar{\beta}\varphi(x, y_2)$

For any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in E$.

Examples of bilinear functional

- Inner product is a bilinear functional
- Let A and B be operators on an inner product space E . then

$\varphi_1(x, y) = \langle Ax, y\rangle, \varphi_2(x, y) = \langle x, By\rangle, and\ \varphi_3(x, y) = \langle Ax, By\rangle$ are bilinear functional

- Let *f* and *g* be linear functionals on a vector space E. then

$\varphi(x,y) = f(x)g(y)$ is bilinear functional on E .

Symmetric , Positive, Strictly positive, and Bounded bilinear functionals

Let φ be a bilinear functional on E

- φ is called symmetric if $\varphi(x,y) = \overline{\varphi(\mathrm{x},\mathrm{y})}$ for all $x, y \in E$
- φ is called positive if $\varphi(x,x) \geq 0$ for every $x \in E$
- φ is called strictly positive and $\varphi(x,x) > 0$ for all $x \neq 0$
- If E is a normed space , then φ is called bounded if $|\varphi(x,x)| \leq K\|x\|\|y\|$ for some $K > 0$ and all $x, y \in E$. The norm of a bounded bilinear functional is defined by

$$\|\varphi\| = \sup_{\|x\|=\|y\|=1} |\varphi(x,y)|$$

Quadratic Form

A function $\Phi: E \to C$ defined by $\Phi(x) = \varphi(x,x)$ is called the quadratic form associated with φ , where φ is a bilinear functional on a vector space E

A quadratic form on a normed space is called bounded if there exist a constant $K > 0$ such that $|\Phi(x)| \leq K\|x\|^2$for all $x \in E$. The norm of a bounded quadratic form is defined by

$$\|\Phi\| = \sup_{\|x\|=1} |\Phi(x)|$$

Let φ be a bilinear functional on E , and let Φ be the quadratic form associated with φ

Theorem 2.2.1

$$4\varphi(x,y) = \Phi(x+y) - \Phi(x-y) = i\Phi(x+iy) - i\Phi(x-iy)$$

For all $x, y \in E$

Theorem 2.2.2

φ is symmetric if and only if the associated quadratic form Φ is real

Theorem 2.2.3

φ is bounded if and only if Φ is bounded. Moreover,

We have

$$\|\Phi\| \leq \|\varphi\| \leq 2\|\Phi\|$$

Theorem 2.2.4

If φ is symmetric and bounded, then $\|\varphi\| = \|\Phi\|$

Theorem 2.2.5

Let A be a bounded operator on a Hilbert space H. then the bilinear functional defined by $\varphi(x, y) = \langle Ax, y\rangle$ is bounded and $\|A\| = \|\varphi\|$

Theorem 2.2.6

Let φ be a bounded bilinear functional on a Hilbert space H. there exists a unique bounded operator A on H such that

$$\varphi(x, y) = \langle x, Ay\rangle \text{ for all } x, y \in H$$

2.3 Adjoint and self-adjoint operators

From theorem 2.5.5 we know that $\varphi(x, y) = \langle Ax, y\rangle$is bounded.

There exists a unique bounded operator A^*such that

$\langle Ax, y\rangle = \varphi(x, y) = \langle x, A^*y\rangle$ for all $x, y \in H$

Adjoint operator (Hilbert adjoint)

An adjoint operator of A defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for all } x, y \in H$$

Where A is a bounded operator on a Hilbert space H. and $A^*: H \to H$

Theorem 2.3.1

The adjoint operator A^* of a bounded operator A is bounded. Moreover, we have $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$

Self-adjoint operator

If $= A^*$, then A is self-adjoint.

Example of self-adjoint

Let A be the operator on $L^2([a, b])$ defined by $(Ax)(t) = tx(t)$ since

$$\langle Ax, y\rangle = \int_a^b tx(t)\overline{y(t)}dt = \int_a^b x(t)\overline{ty(t)}dt = \langle x, Ay\rangle$$

Theorem 2.3.2

Let φ be a bounded bilinear functional on H and let A be an operator on H such that $\varphi(x, y) = \langle x, Ay\rangle$ for all$x, y \in H$. Then A is self-adjoint if and only if φ is symmetric.

Theorem 2.3.3

The product of two self-adjoint operators is self-adjoint if and only if the operators commute.

Theorem 2.3.4

Let T be a self-adjoint operator on a Hilbert space H. then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

2.4 Positive Operators

A positive operator on a Hilbert space is a linear operator A for which the corresponding quadratic form $\langle Ax, x\rangle$ is non-negative.

Definition 2.4.1

If A and B are self-adjoint operators on a Hilbert space H, then we write $A \geq B$ if $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for all $x \in H$.

Example 2.4.1. Let φ and ψ be non-negative continuous function on $[a,b]$ and let A and B be multiplication operators on $L^2([a, b])$ defined by $Ax = \varphi x$ and $Bx = \psi x$. If $\varphi(t) \geq \psi(t)$ for every $t \in [a, b]$, then $A \geq B$. In fact, for any $x \in L^2([a, b])$, we have

$$\langle Ax, x\rangle = \int_a^b \zeta(t)x(t)\overline{x(t)}dt = \int_a^b \zeta(t) \mid x(t) \mid^2 dt \geq \int_a^b \psi(t) \mid x(t) \mid^2 dt = \int_a^b \psi(t)x(t)\overline{x(t)}dt = \langle Bx, x\rangle$$

Theorem 2.4.2. *If A is a self-adjoint operator on H and* $\|A\| \leq 1$, *then* $\leq I$.

Definition 2.4.3. (*Positive operator*)

An operator A is called *positive* if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

Example 2.4.4. Let K be a positive continuous function defined on $[a, b] \times [a, b]$. The integral operator T on $L^2([a, b])$ defined by

$$\langle Tx, x\rangle = \int_a^b \int_a^b K(s,t)x(t)\overline{x(t)}dtds = \int_a^b \int_a^b K(s,t) \mid x(t) \mid^2 dtds \geq 0$$

for all $x \in L^2([a,b])$.

Theorem 2.4.5. *For any bounded operator A on H, the operators* A^*A *and* $AA*$ are positive.

Theorem 2.4.6. *Let* $A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots$ *be self-adjoint operators on* H such that $A_n A_m = A_m A_n$ for all m, n $\in \mathbb{N}$. If B is a self-adjoint operator on H such that $A_n B = BA_n$ and $A_n \leq$ B for all n $\in \mathbb{N}$, then there exists a self-adjoint operator A such that

$$for \lim_{x \to 0} A_n x = Axevery\ x \in H$$

and

$$A_n \leq A \leq B \ for\ every\ n \in \mathbb{N}.$$

Definition 2.4.7. (*Square root*)
By a square root of a positive operator A we mean a self-adjoint operator B satisfying B^2 = A.

Theorem 2.4.8. *Every positive operator A has a unique positive square root B.Moreover, B commutes with every operator commuting with A.*

Definition 2.5.12. (*Strictly positive operator*)
A self-adjoint operator is called strictly positive or positive definite if $\langle Ax,x \rangle > 0$ for all $x \in H$, $x \neq 0$.

2.5 Projection Operators

Definition 2.5.1. (*Orthogonal projection operator*)
Let S be a closed subspace of a Hilbert space H. The operator P on H defined by

$$Px = y \ if\ x = y + z, y \in S, and\ z \in S^{\perp},$$

is called the orthogonal projection operator onto S, or simply projection onto S. The vector y is called the projection of x onto S. Projection onto a subspace S will be usually denoted by P_S.

Example 2.5.2. Let S be a closed subspace of a Hilbert space H, and let $\{e_1, e_2, \ldots\}$ be a complete orthonormal system in S. Then the projection operator P_S can be defined by

$$p_S x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

In particular, if S is of dimension 1 and $v \in S$, $\|v\| = 1$, then
$P_S x = \langle x, v \rangle v$.

Example 2.5.3. Let $H = L^2([-\pi,\pi])$ and let P be an operator on H defined by

$$(Px)(t) = \begin{cases} 0 & if \quad t \leq 0 \\ x(t) & if \quad t > 0 \end{cases}$$

Then P is the projection operator onto the space of all functions that vanish for $t \leq 0$.

Definition 2.5.4. (Idempotent operator)
An operator T is called idempotent if $T^2 = T$.

Example 2.5.5. Consider the operator T on $\mathbb{C}^2$ defined by
$T(x, y) = (x-y, 0)$.
Obviously, T is idempotent. On the other hand, since

$$\langle T(x, y), (x, y) - T(x, y) \rangle = x\overline{y} - |y|^2,$$

$T(x, y)$ need not be orthogonal to $(x, y) - T(x, y)$, and thus T is not a projection.

Theorem 2.5.6. *A bounded operator is a projection if and only if it is idempotent and self-adjoint.*

Two projection operators P and Q are called orthogonal if $PQ = 0$.

Theorem 2.6.8. Two projection operators P_R and P_S are orthogonal if and only if R $\perp$ S.

Example 2.5.9. The operators defined by matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

are projections in $\mathbb{C}^2$. It is easy to check that AB is not a projection.

Theorem 2.5.10. *Let R and S be two closed subspaces of a Hilbert space H, and let P_R and P_S be the respective projections. The following conditions are equivalent:*

(a) $R \subset S$;

(b) $P_S P_R = P_R$;

(c) $P_S P_R = P_R$;

(d) $\|P_R x\| \leq \|P_S x\|$ *for all* $x \in H$.

2.6 Compact Operators

Definition 2.6.1. (*Compact operator*)
An operator A on a Hilbert space H is called a compact operator (or completely continuous operator) if, for every bounded sequence (x_n) in H, the sequence (Ax_n) contains a convergent subsequence.

Example 2.6.2. Let y and z be fixed elements of a Hilbert space H. Define

$$Tx = \langle x, y\rangle z\ .$$

Let (x_n) be a bounded sequence, that is $\|x_n\| \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$. Since

$$|\langle x_n, y\rangle| \leq \|x_n\|\|y\| \leq M\|y\|,$$

the sequence $(\langle x_n, y\rangle)$ contains a convergent subsequence $(\langle x_{p_n}, y\rangle)$. Denote the limit of that subsequence by α. Then

$$Tx_{p_n} = \langle x_{p_n}, y\rangle z \to \alpha z \text{ as } n \to \infty\ .$$

Therefore, T *is* a compact operator.

Example 2.6.3. Let S be a finite dimensional subspace of a Hilbert space H. The projection operator P_S is a compact operator.

Theorem 2.6.4. *Compact operators are bounded.*

Theorem 2.9.5. *The collection of all compact operators on a Hilbert space H is a* vector space.

Theorem 2.6.6. *Let A be a compact operator on a Hilbert space H, and let B be a bounded operator on H. Then AB and BA are compact.*

Definition 2.6.7. (*Finite-dimensional operator*)
An operator is called finite-dimensional (or a finite rank operator) if its range is of finite dimension.

Theorem 2.6.8. *Finite-dimensional bounded operators are compact.*

Theorem 2.6.9. *The limit of a uniformly convergent sequence of compact operators is compact. More precisely, if* $T_1, T_2, \ldots$ *are compact operators on a Hilbert space H and* $\|T_n - T\| \to 0$ *as* $n \to \infty$ *for some operator T on H, then T is compact.*

Theorem 2.6.10. *The adjoint of a compact operator is compact.*

Theorem 2.6.11. *An operator T on a Hilbert space H is compact if and only if it maps weakly convergent sequences into strongly convergent sequences. More precisely, T is compact if and only if* $x_n \xrightarrow{w} x$ *implies* $Tx_n \to Tx$ *for any* $x_n, x \in H$.

CHAPTER 3

Applications to Integral and Differential Equations

The integral equations appears in most applied areas and are as important as differential equations. In this chapter we discuss some applications of the theory of Hilbert spaces to integral and differential equations **.(see [2], [3]).**

3.1 The Integral Equations

Definition3.1.1

A general integral equation for an unknown function y *can be written as*

$$f(x) = a(x)y(x) + \int_a^b k(x.t)y(t)dt.$$

where f, a *and* k *are given functions. The function* k *is called the kernel .*

- *If* $a = 0$, *the equation is said to be of the first kind and otherwise of the second kind .*
- *If* $f = 0$. *the equation are said to be homogeneous otherwise inhomogeneous.*
- *If the integration limits* a *and* b *are constants, the equation is said to be a* ***Fredholm equation****.*
- *If the integration limits* a *and* b *are functions of* x, *the equation is said to be a* ***Volterra equation****.*
-

  Example 3.1.1
- $\int_0^x (x-t)y(t)dt = x$, *inhomogeneous equation of the first kind .*
- $y + \int_0^1 (1-3xt)y(t)dt = 0$, *homogeneous equation of the second kind .*
- $\int_a^b k(x.t)y(t)dt + a(x)y(x) = f(x)$, ***Fredholm equation.***
- $\int_a^x k(x.t)y(t)dt + a(x)y(x) = f(x)$, **volterra equation.**

3.2 Basic Existence Theorems

Theorem 3.2.1. (Contraction mapping theorem)

Let S be a closed subset of a Banach space, and let $T: S \to S$ be a contraction mapping. Then

$(a) the\ equation\ Tx = x\ has\ one\ and\ only\ one\ solution\ in\ S, and$

$(b) the\ unique\ solution\ x\ can\ be\ obtained\ as\ the\ limit\ of$

$the\ sequence\ (x_n) of\ elements\ of\ S\ defined\ by\ x_n = Tx_{n-1}, n = 1,2, \ldots \ldots,$

$where\ x_0\ is\ an\ arbitrary\ element\ of\ S$:

$$x = \lim_{n\to\infty} T^n x_0.$$

Theorem 3.2.2.

Let E be a Banach space, and let $T: E \to E$. If T^m is a contraction for some $m \in \mathbb{N}$, then T has a unique fixed point

$$x_0 \in E\ and\ x_0 = \lim_{n\to\infty} T^n x\ for\ any\ x \epsilon E.$$

Theorem 3.2.3.

If A is a bounded linear operator on a Banach space $E, and\ \varphi$ is an arbitrary element of E, then the operator defined by

$$Tf = \alpha Af + \varphi$$

If k is a positive constant such that $\|Af\| \leq k\|f\|$. $\quad for\ all\ f \epsilon E,$

Then $Tf = f$ has a unique solution whenever $|\alpha|k < 1$.

Corollary 3.2.4.

Let A be a bounded linear operator in a Banach space. Then the equation

$$x = x_0 + \alpha Ax$$

Has a unique solution given by

$$x = \sum_{n=0}^{\infty} \alpha^n A^n x_0$$

Whenever $|\alpha|\|A\| < 1$.

Theorem 3.2.5. ***(Picard's existence and uniqueness theorem).***

Consider the initial value problem for the ordinary differential equation

$$\frac{dy}{dx} = f(x,y) \qquad (E1)$$

With the initial condition

$$y(x_0) = y_0, \quad (E2)$$

Where f is a continuous function in some closed domain

$$R = \{(x,y): a \le x \le b, c \le y \le d\}$$

Containing the point (x_0, y_0) in its interior, if f satisfies the Lipschitz condition

$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|$$

For some $K \in R$ and all $(x,y_1),(x,y_2) \in R$, then there exists a unique solution $y = \varphi(x)$ of the problem (E1) to (E2) defined in some neighborhood of x_0.

Proof: Observe that every solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)dt \qquad (E3)$$

Satisfies $(E1)$ and $(E2)$, and conversely. Consider the operator T defined on $\mathcal{C}([a,b])$ by

$$(T\varphi)(x) = y_0 + \int_{x_0}^{x} f\big(t, \varphi(t)\big)dt. \qquad (E4)$$

Let

$$M = \sup\{|f(x,y)|: (x,y) \in R\},$$

and select $\varepsilon > 0$ such that $K\varepsilon < 1$ *and* $[x_0 - \varepsilon, x_0 + \varepsilon] \subset [a,b]$. If

$$S = \{\varphi(x) \in \mathcal{C}([x_0 - \varepsilon, x_0 + \varepsilon]): |\varphi(x) - y_0| \le M\varepsilon \ \ for\ all\ x \in [x_0 - \varepsilon, x_0 + \varepsilon]\},$$

Then S is a closed subset of the Banach space $\mathcal{C}([x_0 - \varepsilon, x_0 + \varepsilon])$ with the sup-norm

$$\|\varphi\| = sup_{[x_0-\varepsilon, x_0+\varepsilon]}|\varphi(x)|.$$

Furthermore, if $\varphi \epsilon S$ *and* $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$, then

$$|(T\varphi)(x) - y_0| = \left| \int_{x_0}^{x} f\big(t, \varphi(t)\big) dt \right| \leq M\varepsilon,$$

And thus T maps S onto itself. Finally, for any $\varphi_1, \varphi_2 \in S,$ we have

$$\|T\varphi_1 - T\varphi_2\| = sup_{[x_0-\varepsilon, x_0+\varepsilon]} \left| \int_{x_0}^{x} (f(t, \varphi_1(t)) - f\big(t, \varphi_2(t)\big) dt \right|$$
$$\leq K\varepsilon\|\varphi_1 - \varphi_2\|.$$

Theorem 3.2.6. (Fred Holm alternative for self-adjoint compact operators)

Let A be a self-adjoint compact operator on a Hilbert space H. Then the nonhomogeneous operator equation

$$f = Af + \varphi \qquad (E5)$$

has a unique solution for every $\varphi \in H$ if and only if the homogeneous equation

$$g = Ag \qquad (E6)$$

has only trivial solution $g = 0$.

3.3 Fredholm Integral Equations

In 1900 the famous Swedish mathematician Ivar Fredholm (1866-1927) first introduced the Fredholm integral equations.

Theorem 3.3.1 (*Existence and uniqueness of solution of the Fredholm non-homogeneous linear integral equation of the second kind)*

The equation $f(x) = \alpha \int_a^b k(x,y) f(y) dy + \varphi(x)$

has a unique solution $f \in L^2([a,b])$*provided the kernal* K *is continuous in* $[a,b] \times [a,b], \varphi \epsilon L^2([a,b])$, *and* $|\alpha|k < 1$, *where*

$$k = \sqrt{\int_a^b \int_a^b |K(x,y)|^2 dx dy}$$

Example 3.3.2

Consider the integral equation

$$f(x) = \alpha \int_a^b e^{(x-y)/2} f(y)dy + \varphi(x), \quad (1)$$

Where φ *is a given function. Since*

$$\int_a^b \int_a^b (e^{(x-y)/2})^2 \, dxdy = \frac{(e^b - e^a)^2}{e^{a+b}}$$

Equation (1) has a unique solution whenever

$$|\alpha| < \frac{e^{(a+b)/2}}{e^b - e^a}$$

Method of Successive Approximations

Example 3.3.3 We obtain the Neumann series solution of the integral equation

$$f(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x) f(t) dt.$$

First we set $f_0(x) = x. Then$

$$f_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x) t dt = x + \frac{1}{3}.$$

Substituting f_1 *back into the original equation, we find*

$$f_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\left(t + \frac{1}{3}\right) dt = x + \frac{1}{3} - \frac{1}{3}x.$$

Continuing this process, we obtain

$$f_3(\mathrm{x}) = \mathrm{x} + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2},$$

$$f_4(x) = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2} + \frac{x}{3^2},$$

.

.

.

$$f_{2n} = x + \sum_{m=1}^{n} (-1)^{m-1} 3^{-m} - x \sum_{m=1}^{n} (-1)^{m-1} 3^{-m}.$$

By letting $n \to \infty, we\ get$

$$f(x) = \frac{3}{4}x + \frac{1}{4}.$$

It is easy to verify this solution by direct substitution into the original equation .

3.4 Linear Volterra Integral Equations

In 1896 the Italian mathematician Vito VOLTERRA (1860-1940) first introduced the volterra integral equations.

Theorem 3.4.1

Consider the following linear Volterra integral equation

$$(L.V.E)\ \ y(t) = f(t) + \int_0^t K(t,s)y(s)\,ds,\ \ t \in I = [0,T].$$

Assume that the kernel K of this equation is continuous on $D = \{(t,s) : 0 \le s \le t \le T\}$. Then for any function f that is continuous on I, the equation $(L.V.E)$possesses a unique solution y $\in$ C(I) . This solution can be written in the form $y(t) = f(t) + \int_0^t R(t,s)f(s)\,ds,\ t \in I.$

for some $R \in C(D)$. The function R is called the resolvent kernel of the given kernel .

Proof

First choose $\mathrm{y}_0(\mathrm{t}) = \mathrm{f}(\mathrm{t})$ and define the sequence $(y_n)_n$ by

$y_n(t) = f(t) + \int_0^t K(t,s)y_{n-1}(s)\,ds\ , t \in I$

By Picard's theorem we have

$y_1(t) = f(t) + \int_0^t K(t,s) f(s)\, ds$

and for any $n \geq 1$,

$$y_n(t) = f(t) + \int_0^t \sum_{j=1}^{n} k_j\,(t,s) f(s) ds, t \in I,$$

where $K_1(t,s) = K(t,s)$ and

$$K_j(t,s) = \int_s^t K(t,u) K_{j-1}(u,s)\, du \;(j \geq 2)$$

The function K is continuous and there exist a constant M such that $|K(t,s)| \leq M$ for all $(t,s) \in D)$. then the series

$$\sum_{j=1}^{\infty} K_j(t,s)$$

Converges uniformly to $R(t,s) = \lim_{k\to\infty} \sum_{j=1}^{k} Kj(t,s)\big((t,s) \in D\big)$

Therefore:

$$\lim_{n\to\infty} y_n(t) = z(t), t \in I$$

Is a solution of the equation

$$(L.V.E)\; y(t) = f(t) + \int_0^t K(t,s) y(s)\, ds, \;\; t \in I = [0,T].$$

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