原问题

原两阶段鲁棒优化问题:

$$\min_{\boldsymbol{y}\in\boldsymbol{S_y}} \boldsymbol{c}^T\boldsymbol{y} + \max_{\boldsymbol{u}\in U} \min_{x\in F(\boldsymbol{y},\boldsymbol{u})} \boldsymbol{b}^T\boldsymbol{x} \tag{1}$$

s.t.

$$\boldsymbol{Ay} \geq \boldsymbol{d} \tag{2}$$
$$F(\boldsymbol{y},\boldsymbol{u}) = \{\boldsymbol{x}\in\boldsymbol{S_x}: \boldsymbol{Gx} \geq \boldsymbol{h} - \boldsymbol{Ey} - \boldsymbol{Mu}\} \tag{3}$$
$$\boldsymbol{S_x} \subseteq \mathbb{R}_+^n \tag{4}$$
$$\boldsymbol{S_y} \subseteq \mathbb{R}_+^m \tag{5}$$

Benders-dual

令π为对偶变量，则获得 Benders-dual 子问题:

$$\mathrm{SP}_1: Q(\boldsymbol{y}) = \max_{\boldsymbol{u},\boldsymbol{\pi}} \ (\boldsymbol{h} - \boldsymbol{Ey} - \boldsymbol{Mu})^T\boldsymbol{\pi} \tag{6}$$

s.t.

$$\boldsymbol{G}^T\boldsymbol{\pi} \leq \boldsymbol{b} \tag{7}$$
$$\boldsymbol{u}\in U, \boldsymbol{\pi} \geq \boldsymbol{0} \tag{8}$$

问题是一个双线性优化问题，求解策略见[8][19][21][15][13]。
对于给定的变量$\boldsymbol{y}_k^*$，最优解$(\boldsymbol{u}_k^*,\boldsymbol{\pi}_k^*)$得到 $Q(\boldsymbol{y}_k^*)$，其产生加到主问题的切平面为:

$$\eta \geq (\boldsymbol{h} - \boldsymbol{Ey} - \boldsymbol{Mu}_k^*)^T\boldsymbol{\pi}_k^* \tag{9}$$

主问题为:

$$\mathrm{MP}_1: \min_{\boldsymbol{y},\eta} \boldsymbol{c}^T\boldsymbol{y} + \eta \tag{10}$$

s.t.

$$\boldsymbol{Ay} \geq \boldsymbol{d} \tag{11}$$
$$\eta \geq (\boldsymbol{h} - \boldsymbol{Ey} - \boldsymbol{Mu}_k^*)^T\boldsymbol{\pi}_k^*, \forall l \leq k \tag{12}$$
$$\boldsymbol{y}\in\boldsymbol{S_y}, \eta\in\mathbb{R} \tag{13}$$

主问题MP_1的解为$(\boldsymbol{y}_{k+1}^*,\eta_{k+1}^*)$, $\boldsymbol{c}^T\boldsymbol{y}_{k+1}^* + \eta_{k+1}^*$产生一个下界, 而$\boldsymbol{c}^T\boldsymbol{y}_k^* + Q(\boldsymbol{y}_k^*)$产生一个上界, 通过不断迭代求解主子问题，最终收敛到最优解。

Proposition 1. $\boldsymbol{u}_k^*$，$\boldsymbol{\pi}_k^*$是其各自定义域的极点，设p是不确定集U的极点数量，q是$\{\boldsymbol{\pi}: \boldsymbol{G}^T\boldsymbol{\pi} \leq \boldsymbol{b}, \boldsymbol{\pi} \geq \boldsymbol{0}\}$的极点数，Benders-dual 法收敛次数为$O(pq)$。

C&CG

C&CG 方法会在约束中添加包含 rerourse decision variables 的约束，因此叫列（添加的

rerourse decision variable）与约束生成方法。令$U=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_l\}$，$\{\boldsymbol{x}^1,\boldsymbol{x}^2,\dots,\boldsymbol{x}^l\}$为 rerourse decision variables。主问题为：

$$\mathrm{MP}_2\colon \min_{\boldsymbol{y},\eta} \boldsymbol{c}^T\boldsymbol{y}+\eta \tag{14}$$

s.t.

$$\boldsymbol{A}\boldsymbol{y}\ge \boldsymbol{d} \tag{15}$$

$$\eta\ge \boldsymbol{b}^T\boldsymbol{x}^l, l=1,2,\dots,k \tag{16}$$

$$\boldsymbol{G}\boldsymbol{x}^l\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_l^*, l=1,2,\dots,k \tag{17}$$

$$\boldsymbol{y}\in \boldsymbol{S}_y,\eta\in\mathbb{R},\boldsymbol{x}^l\in \boldsymbol{S}_x, l=1,2,\dots,k \tag{18}$$

对于一个子问题识别的一个场景$\boldsymbol{u}_l^*$，会添加一个对应的变量$\boldsymbol{x}^l$。这是因为在更新新的变量$\boldsymbol{y}$时，对应的子问题识别的场景$\boldsymbol{u}_l^*$对应的决策变量和是$\boldsymbol{x}^{*l}$不同的，但是新的$\boldsymbol{x}^l$需要满足定义域要求，因此每个$\boldsymbol{u}_l^*$需要添加一个约束$\boldsymbol{G}\boldsymbol{x}^l\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_l^*$，在该约束里会添加变量$\boldsymbol{x}^l$。第$l$次迭代时主问题的变量逐渐增加为$\{\boldsymbol{y},\eta,\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_l\}$，变量列逐渐增多，这也是为啥叫做列与约束生成的原因。

子问题为：

$$\mathrm{SP}_2\colon Q(\boldsymbol{y})=\max_{\boldsymbol{u}\in U}\min_{x\in F(\boldsymbol{y},\boldsymbol{u})}\boldsymbol{b}^T\boldsymbol{x} \tag{19}$$

s.t.

$$\boldsymbol{G}\boldsymbol{x}\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u} \tag{20}$$

$$\boldsymbol{x}\in \boldsymbol{S}_x \tag{21}$$

求解步骤：

1. 设$LB=-\infty, UB=\infty, k=0$
2. 求主问题MP_2，获得最优解$(\boldsymbol{y}_{k+1}^*,\eta_{k+1}^*,\boldsymbol{x}^{*1},\boldsymbol{x}^{*2},\dots,\boldsymbol{x}^{*k})$，更新下界$LB=\boldsymbol{c}^T\boldsymbol{y}_{k+1}^*+\eta_{k+1}^*$
3. 求解子问题SP_2，更新上界$UB=min\{UB,\boldsymbol{c}^T\boldsymbol{y}_{k+1}^*+Q(\boldsymbol{y}_{k+1}^*)\}$
4. 如果满足收敛条件，返回$\boldsymbol{y}_{k+1}^*$。否则添加约束$\eta\ge \boldsymbol{b}^T\boldsymbol{x}^{k+1}$和$\boldsymbol{G}\boldsymbol{x}^{k+1}\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_{k+1}^*$约束，更新$k=k+1$，返回步骤 2。

事实上如果子问题SP_2可行，那么添加的割就是最优割$\eta\ge \boldsymbol{b}^T\boldsymbol{x}^{k+1}$和$\boldsymbol{G}\boldsymbol{x}^{k+1}\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_{k+1}^*$；如果子问题$\mathrm{SP}_2$不可行，那么添加可行割$\boldsymbol{G}\boldsymbol{x}^{k+1}\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_{k+1}^*$，但是由于更新$\boldsymbol{y}$时，$\eta\ge \boldsymbol{b}^T\boldsymbol{x}^l$这个约束也是需要满足的，因此不管子问题可行不可行，都添加$\eta\ge \boldsymbol{b}^T\boldsymbol{x}^{k+1}$和$\boldsymbol{G}\boldsymbol{x}^{k+1}\ge \boldsymbol{h}-\boldsymbol{E}\boldsymbol{y}-\boldsymbol{M}\boldsymbol{u}_{k+1}^*$这两条割。

Proposition 2. 设p是不确定集U的极点数量，C&CG 法收敛次数为$O(p)$。

与 Benders-dual 的对比：

1. 主问题的决策变量。在 C&CG 方法中是不断增大的，而在 Benders-dual 中不变。
2. 可行割。在 C&CG 方法统一处理，添加两条割。而 Benders-dual 视不同问题而定[2][14][18][15]。
3. 计算复杂度。C&CG 方法迭代复杂度为$O(p)$，而 Benders-dual 为$O(pq)$。
4. 算法适用性。Benders-dual 法需要第二阶段子问题为线性规划 LP 问题，而 C&CG 无需次需要，如[20]。
5. 割的有效性。对于同样的识别出的场景集，MP_1的最优值比MP_2的最优值小。即 C&CG 下界更紧。

Proposition 3. 对于同样的识别出的场景集，MP_1的最优值比MP_2的最优值小。

子问题SP_2的求解方法：
对于一个多面体不确定集，对原SP_2应用 KKT 条件，即：

$$max\ \boldsymbol{b}^T\boldsymbol{x} \tag{22}$$

s.t.

$$\boldsymbol{Gx} \ge \boldsymbol{h} - \boldsymbol{Ey} - \boldsymbol{Mu} \tag{23}$$

$$\boldsymbol{G}^T\boldsymbol{\pi} \le \boldsymbol{b} \tag{24}$$

$$(\boldsymbol{Gx} - \boldsymbol{h} + \boldsymbol{Ey} + \boldsymbol{Mu})_i\pi_i = 0, \forall i \tag{25}$$

$$(\boldsymbol{b} - \boldsymbol{G}^T\boldsymbol{\pi})_j\boldsymbol{x}_j = \boldsymbol{0}, \forall j \tag{26}$$

$$\boldsymbol{x} \in \boldsymbol{S}_x, \boldsymbol{u} \in U, \boldsymbol{\pi} \ge \boldsymbol{0} \tag{27}$$

式(25)(26)是完全松弛条件，采用大 M 法，可将其线性化，如：

$$x_j \le Mv_j, (\boldsymbol{b} - \boldsymbol{G}^T\boldsymbol{\pi})_j \le M\left(1 - v_j\right), v_j \in \{0,1\}$$

因此子问题SP_2被转换为 MILP 问题。

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