Contents

Chapter 4: Solving Normal Form Games

There are several different solution concepts for NFGs. Before presenting them, we first list nice properties that we might hope for in a solution, and run through several examples to build intuition. Next we present definitions on which the concepts are built, and define the first two solution concepts, DS and IDDS. The foremost solution concept, NE is presented in two parts, pure and mixed. Then we briefly mention other concepts, such as CE and QRE, that are sometimes useful.

[note for next edition: don't forget to use commented out HD material]

1 Desiderata and Examples

What should it mean to solve a game? What might we expect a solution to do?
Here is a list of some desirable properties.

A solution of a game should:

1. Be consistent with rationality (SEU, EUH).
2. Always exist.
3. Be unique.
4. Be efficient: either (a) maximize average or total payoff ("transferrable" utility), or perhaps just (b) be Pareto optimal.
5. Be "reasonable": sensible players could be persuaded to stick with it.
6. Have empirical support, and predict accurately what actual human players do in the lab or field.

Unfortunately, we will see that there is no solution concept that can satisfy all these desiderata. The leading candidate, Nash equilibrium, always satisfies the first two, but fairly simple games can be found where it fails to satisfy each of the others. Other solution concepts do better on some desiderata and worse on others.

1.1 Some examples

[[half class has Row role, other half has Column. Each person chooses what to do, votes are tallied, explanations elicited.]]

Game 1: Pigs

	L	D
l	1,4	-1,6
d	3,2	0,0

Game 2: IDDSb

	t_1	t_2	t_3
s_1	$4,3$	$2,\underline{7}$	$\underline{3},4$
s_2	$\underline{5},\underline{5}$	$\underline{5},-1$	$-4,2$

- Convention is that row player is called player 1 and column player is called player 2. Best-responses are underlined.
- Row has 2 possible strategies: s_1 and s_2.
- Column has 3 possible strategies: t_1, t_2, and t_3.
- One way to think about solving games is to see how each player responds best to each possible situation.
- Iteration (i)
  - Looking at Row player in Game 2, we can state the best-responses as:

$$\text{if } t_1 \text{ then } s_2 \text{ is best-response.}$$

$$\text{if } t_2 \text{ then } s_2 \text{ is best-response.}$$

$$\text{if } t_3 \text{ then } s_1 \text{ is best-response.}$$

- Thus, initially we can not eliminate any rows.

• Iteration (ii)

- Then look at Columns' best-responses:

$$\text{if } s_1 \text{ then } t_2 \text{ is best-response.}$$

$$\text{if } s_2 \text{ then } t_1 \text{ is best-response.}$$

- Thus, t_3 is never a best-response.
  * If s_1, then t_3 is dominated by t_2.
  * If s_2, then t_3 is dominated by t_1.
  * Since t_3 is not a best-response to either pure strategy, it is possible that t_3 is dominated by some mixture of t_1 and t_2.
  * Consider, for example, playing t_1 and t_2 with probability $p(t_1) = 0.6$ and $p(t_2) = 0.4$:
    · against $s_1 : \pi_2(0.6t_1 + 0.4t_2) = \frac{18+28}{10} = 4.6$ better than payoff 4 from t_3.
    · against $s_2 : \pi_2(0.6t_1 + 0.4t_2) = \frac{30+-4}{10} = 2.6$ better than payoff 2 from t_3.
  * you can check that any such mixed strategy with $0.5 < p(t_1) < 0.75, p(t_2) = 1 - p(t_1)$ dominates t_3.
- We can eliminate t_3 and obtain the reduced game:

	t_1	t_2
s_1	$4, 3$	$2,\underline{7}$
s_2	$\underline{5},\underline{5}$	$\underline{5}, -1$

- Iteration (iii)

    - Then look at row's best-responses:

if t_1 then s_2 is best-response.

if t_2 then s_2 is best-response.

-
    - Thus, now s_2 dominates s_1 and we can delete s_1 to obtain the reduced game:

	t_1	t_2
s_2	$5, 5$	$5, -1$

- Iteration (iv).

    - Now, column is facing only s_2, and clearly t_1 dominates t_2,
    - Thus we can eliminate t_2 to obtain the solution (s_2, t_1):

	t_1
s_2	$5, 5$

- Now follow the same procedure with Game 1.

    - The Row player (the "little pig") has a dominated strategy, l ("push the food **l**ever").
    - Assuming that Row always plays d ("wait by the food **d**ispenser"), the Column player ("big pig") best responds by playing L.
    - The solution (d,L) has payoff (3,2), illustrating that in game theory as in Scripture, things may work out well for the meek!

- We will see later that this solution concept is called iterated deletion of dominated strategies (IDDS) and that it is a special case of Nash equilibrium (NE).
- The IDDS procedure worked for the last 2 games, but there are equally simple games where it doesn't get us very far.

Game 3: Driving

	L $[p]$	R $[1-p]$
L	1,1	-10,-10
R	-10,-10	1,1

- No strategy is dominated in Game 3 — you want to make the same choice as the other players as to which side of the road to drive on.
- Population game interpretation: There are a large number of other drivers you may encounter, and fraction p of them choose L.
  - Your expected payoff is $p - 10(1-p) = 11p - 10$ if you play L, and $-10p + 1(1-p) = 1 - 11p$ if R.
  - You are better off choosing L if the expected payoff advantage to L $(11p - 10 - (1 - 11p) = 22p - 11$ is positive, i.e., if $p > 0.5$
  - Likewise, you are better off choosing R if more than half of other drivers choose R, i.e., if $1 - p > 0.5$.
  - If $p = 0.5$, as it seems to be in some cities you may have visited, then R and L are equally bad.
  - Indeed, if you could choose to stay home and get payoff 0, you'd prefer that if $p \in [\frac{1}{11}, \frac{10}{11}]$.
  - Thus we have multiple solutions. If everyone else plays R then you'd also prefer R, but if everyone else plays L then you'd prefer that. We will see that these

solutions are examples of pure NE. And the $p = 0.5$ bad solution is an example of a mixed NE.

This sort of situation is called a **coordination game.**

- Basic idea is that there is a payoff advantage to being on the "same page" as the other players.
- There are various sorts of coordination games, as we will see. Consider one famous example, known to economists as "Battle of the Sexes":
  - Two players, each with two possible actions.
  - traditionally, a heterosexual couple choosing Boxing Match or Opera, but you can make it more interesting if you like. The idea is that they would rather be together, but each has a preferred event.

Game 4: Battle of the Sexes

	t_1	t_2
s_1	$0, 0$	$\underline{20},\underline{40}$
s_2	$\underline{40},\underline{20}$	$0, 0$

- Looking at the best-responses, starting with Row:

  if t_1 then s_2 is best-response.

  if t_2 then s_1 is best-response.

- Thus, row does not have any dominated strategies.
- Column's best-responses:

  if s_1 then t_2 is best-response.

  if s_2 then t_1 is best-response

- Thus, column does not have any dominated strategies either and the IDDS procedure has no traction.

- Another solution idea is that nobody wants to change unilaterally, as in Nash equilibrium (as we will see). We have 2 NE in pure strategies $NE = \{(s_2, t_1), (s_1, t_2)\}$ (and a third lurking in mixed strategies, to be discussed below).

- Note that player 1 prefers (s_2, t_1) and player 2 prefers (s_1, t_2). We have two predictions, at least. In games in the lab with distinct populations for row players and column players, typically one of these two will emerge eventually as the convention that almost everyone follows.

- There are also other sorts of coordination games we will see later where there is a different tension between predicted NE (risk-dominant vs. payoff dominant).

1.2 Mixed strategies

[[maybe move this to the previous chapter?]]

A **mixed strategy** is a probability distribution over the set of pure strategies. There are two interpretations of mixed strategies. Both have their uses.

1. each pure strategy is followed by a given fraction of a large population, possibly 0. The fractions are non-negative and sum to 1.0.

2. There is only one player in each role (e.g., Row or Column), and those players make choices as if flipping a coin, spinning a roulette wheel, or rolling dice so as to pick each pure strategy with a given probability.

   In some games, there is an advantage to being unpredictable,

   - (American) football example: running 30% of the time on first down.
   - (soccer) football example: penalty kicks (L,M,R)

Example:

Game 5: Conflict

	t_1	t_2
s_1	$\underline{3}, 1$	$1,\underline{3}$
s_2	$0,\underline{5}$	$\underline{4}, 2$

- Look at the best-responses, first for Row.

  s_1 is best-response to t_1.

  s_2 is best-response to t_2.

- Similarly, looking at columns' best-responses:

  t_2 is best-response to s_1.

  t_1 is best-response to s_2 .

- No dominated strategies, no NE in pure strategies. IDDS is no help.

So we need to find a mixed NE.

This seems like a good juncture to discuss mixed strategies in more detail.

- The set of pure strategies is $S = \{s_1, s_2\}$ for the case of two pure strategies.

- $\Delta(S)$ denotes the set of all possible mixed strategies. The set of mixed strategies $\Delta(S)$ is a simplex:

$$\Delta(S) = \{ps_1 + (1-p)s_2\} \text{ where } p \in [0,1]$$

  - If $p = 0$ then pure strategy s_2 is obtained. This means that the set of pure strategies is a proper subset of the set of mixed strategies: $S \subset \Delta(S)$.

- For three pure strategies then $\Delta(S)$ is a three dimensional triangle.

$$\Delta(S) = \{ps_1 + qs_2 + (1-p-q)s_3\} \text{ where } p, q \in [0,1] \text{ and } p+q \in [0,1]$$

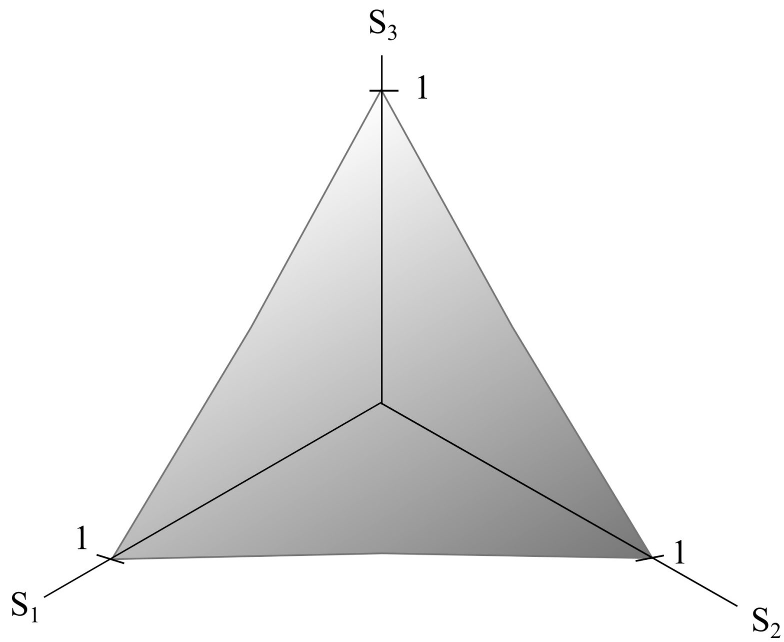

Figure 1: When a player has three pure strategies, the set of mixed strategies is the equilateral triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

- For n possible pure strategies the simplex is a hypertriangle.
- So, a mixed strategy is a probability distribution across the set of pure strategies.
  - Note that the realization of a mixed strategy is one of the pure strategies.
- Picking the mixing probabilities p and q is the trick.
- We will see that even games (as above) with no pure strategy NE still have a NE in mixed strategies.

2 DS and IDDS

With these examples in mind, we can introduce the ideas more formally.

2.1 Basic Definitions

A pure strategy $s_i \in S_i$ is **weakly dominant** if:

$$f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i}) \qquad \forall \left\{ \begin{array}{c} s_i' \in S_i \\ s_{-i} \in S_{-i} \end{array} \right\} \tag{1}$$

Thus, a weakly dominant strategy is a best-response to *all* possible strategy profiles of the other players.

- Fix what the other players are doing and then choose your best strategy.
- Then vary what everyone else is doing, and if that strategy always is still a best response, then it is weakly dominant.

A pure strategy is said to be **strictly dominant** if the inequality in equation (1) is strict for all other pure strategies $s_i' \neq s_i \in S_i$.

Of course, most games that you will encounter will not have dominant strategies — only in special situations does a player have a single strategy that will always work well. The opposite case is more common, when some players have some strategies that they would never want to use.

A pure strategy $s_i \in S_i$ is **weakly dominated** if:

$$\exists \quad s_i' \in S_i \quad \text{such that} \quad f_i(s_i, s_{-i}) \leq f_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i} \tag{2}$$

- Thus, a strategy is weakly dominated if there exists a different pure strategy that always yields at least as high a payoff.
- A strategy is **strictly dominated** if the inequality in (2) is strict.
- A pure strategy $s_i \in S_i$ can be dominated by a mixed strategy x, if

$$\exists \quad x \in \Delta\left(S_i\right) \quad \text{such that} \quad f_i(s_i, s_{-i}) \leq f_i(x, s_{-i}) \quad \forall s_{-i} \in S_{-i} \tag{3}$$

  - Like our Game 2 example: $0.6t_1 + 0.4t_2$ dominates t_3.

Now we are ready for the key definition.

A pure strategy $s_i' \in S_i$ is a **best-response** to s_{-i} if:

$$f_i(s_i', s_{-i}) \geq f_i(s_i, s_{-i}) \quad \forall s_i \in S_i \tag{4}$$

If equation (4) holds, we write $s_i' \in B_i(s_{-i})$. So $B_i(s_{-i})$ is the set of all pure strategy best responses that player i has to a particular strategy profile by the other players, s_{-i}. If there is just one such strategy, we sometimes (a bit carelessly) write $s_i' = B_i(s_{-i})$.

2.2 DS

If all players have a strictly dominant strategy, then the game is said to have a **dominant strategy (DS)** solution.

DS definitely satisfy desiderata 1) [consistent with rationality] and 3) [unique]. DS also typically satisfy 5) [reasonable], although this might be arguable in a few cases. Of course, desideratum 2) [existence] fails badly for DS since dominant strategies are a bit special for any single player, much less for all players.

Prisoner's dilemma is the most famous game with a DS solution.

- Original story: s_1 and t_1 are remain silent, s_2 and t_2 are confess. (Other stories concern arms races, cartel behavior, etc. etc.)

- The original payoffs are negative (years in jail) but since even cardinal utility is defined up to a positive affine transformation (adding a constant in this case), we can make the payoffs positive and still represent the same set of preferences. We'll return later to the question of cardinal vs ordinal payoffs.

Game 6: Prisoner's Dilemma

	t_1	t_2
s_1	$8, 8$	$0,\underline{10}$
s_2	$\underline{10},0$	$\underline{2},\underline{2}$

- The DS solution is: (s_2, t_2).

- How does this solution fare according to our desiderata?

  - It exists for this game, and so satisfies desideratum 2) as well as 1) and 3).

  - Does it seem reasonable to you? (desideratum 5).

  - The solution's payoff sum is $2+2 = 4$, only 25% of the maximum sum $8+8 = 16$, so the solution is inefficient in the transferrable utility sense, and it is also Pareto dominated, so either way, desideratum 4) fails.

  - It predicts well in some settings, as we will see later in the course. So the last desideratum is partially satisfied.

2.3 IDDS

Recall the process illustrated earlier of eliminating strictly dominated strategies, then iterating in the reduced game. If, at the end of this process, only a single profile remains, then it is called the **iterated deletion of dominated strategies IDDS** solution of that game, and the game is said to be dominance solvable.

- IDDS satisfies desideratum 1) — it is always consistent with rationality, and the belief that others are rational. More on this point later.

- we already know from earlier examples that desideratum 2) fails – the IDDS procedure does not always produce a single profile that we can deem to be the solution of the game.

- Apropos 3) [uniqueness], there is a proposition due to Zermelo $\sim$ 1890. Almost every extensive form game (EFG) of perfect information has a unique IDDS solution.
  - The proof idea: Use BI to find the optimal action for each player at each decision node and take expectations across nature moves.
  - Each decision step of BI in the EFG involves eliminating a subset of dominated strategies in the NFG, namely, those strategies involving inferior actions at the contingency (player node or info set) considered at that step.
  - The "almost" in the proposition comes from the fact that there may exist "ties" in the relevant player's payoffs, in which case the BI solution may not be unique.
- Technical points
  - What if there are several different strategies that are dominated at some iteration — does it matter which you eliminate first? It turns out that the answer is no, it doesn't matter. See [cite].
  - What if the IDDS procedure halts with some reduced game with at least two players having at least two remaining pure strategies — did we waste our time? The answer is again no; the real action is in the reduced game.
    * When multiple such profiles remain, they can be refined slightly and considered to be a set of equilibria.
    * In the reduced game, are any remaining strategies never-best-responses? If so, throw them out to reduce the profile set. Iterate that procedure until no more never-best-replies can be found.
    * The remaining profiles are called rationalizable equilibria.
    * They satisfy Des 1) in the strong sense that rationality is "common knowledge."

    * That last piece of jargon means that each player is rational, and believes that all other players are rational, and believes that all players believe that all players are rational, etc, etc.

  See [cite] for a fuller explanation.

- Caveats:

  – IDDS may not be feasible. For example, chess is a finite game of perfect information, complete information (win≻draw≻loss), and perfect recall. Try BI? Chess has an astronomical number of branches. Deep Blue: compute x moves ahead and estimate expected payoffs. This is a "brute force" approach that remains far from IDDS.

  – IDDS may not reasonable or empirically valid in certain cases. For example, consider the Centipede game, invented by Robert Rosenthal. It is a finite EFG of complete and perfect information, as in Figure 2.

  – In this game there are 2 piles. One pile has two coins, the other zero, in period 1. Two actions: take a pile, (leaving the other pile for the other player and ending the game), or push the two piles across the table to the other player in which case one coin is added to each pile. The game continues for 100 periods or until one player takes a pile.

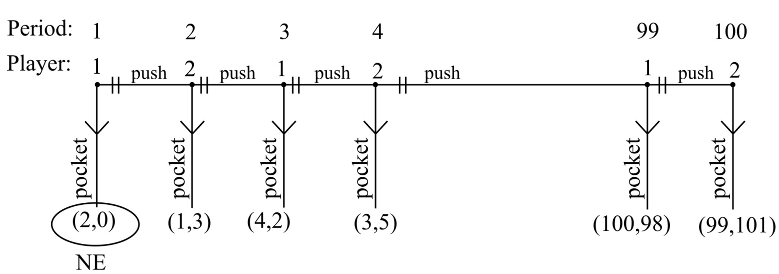

Figure 2: caption goes here.

– Empirical results: Starting with McKelvey and Palfrey (1992, *Econometrica*), NE play is rarely observed. Explanation is still somewhat controversial, as discussed in the last section of this chapter.

3 Nash equilibrium: Pure Strategies

Nash equilibrium (NE) is the leading solution concept for NFGs. In brief, it is a profile where every player is making a best response to the other players.

3.1 Equal payoff property

- Recall that pure strategy $s_i \in B_i(s_{-i})$, i.e., s_i is a best-response to opponents' profile s_{-i}, if:

$$f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i}) \quad \forall s_i' \in S_i. \tag{5}$$

- If this holds for several different pure strategies $s_i \in B_i(s_{-i})$, then they have the same (maximal) payoff, call it m, against s_{-i}. Any weighted average of numbers equal to m is itself equal to m. Therefore, if x_i is a mixture involving only pure strategies in $B_i(s_{-i})$, then $x_i \in B_i(s_{-i})$.

3.2 Definition and existence

A profile $s^* = (s_1^*, ..., s_n^*)$ of pure strategies in an $n-$player game is a NE if

$$s_i^* \in B(s_{-i}^*), i = 1, ..., n. \tag{6}$$

Likewise, a profile $x^* = (x_1^*, ..., x_n^*)$ of (possibly) mixed strategies is a NE if

$$x_i^* \in B(x_{-i}^*), i = 1, ..., n. \tag{7}$$

Theorem. John Nash (1951). Let the strategy set for player i, $S_i \subset \Re^m$ be convex and compact (which implies the set is closed and bounded), the payoff function f_i be continuous in $S = S_1 \times \ldots \times S_n$ and, for the restricted payoff function $f_i(\cdot, s_{-i})$ be quasi-concave $i = 1, \ldots, n$. Then the game has a Nash equilibrium (not necessarily unique).

Proof sketch:

$$B(s) = (B_1(s_{-1}), ..., B_n(s_{-n})) \tag{8}$$

defines the best-response correspondence. Can verify that B satisfies the assumptions of Kakutani's fixed point theorem, and therefore has a fixed point. But a fixed point $s^* \in B(s^*)$ is, by definition, a NE.

- Even if $f_i(\cdot, s_{-i})$ is not continuous in $S = S_1 \times \ldots \times S_I$ or quasi-concave in $i = 1, ..., n$, there still may be a NE; see for example pg. 253 MCWG.

- But are the strategy sets convex and compact? The sets of pure strategies are not. In the next section, we'll see that the sets of mixed strategies are, and so we have a

- *Corollary* to this theorem: Every finite normal form game (NFG) has a NE, possibly in mixed strategies.

- Unlike the previous solution concepts, NE does very well on desideratum 2) [existence], in light of Nash's theorem and its corollary.

- It also does well in terms of 5) [reasonable] and 1) [rational] because if a profile is not a NE, then at least one player is not choosing a best-response (not acting rationally).

Here is the first part of a recipe for finding DS, IDDS and pure NE solutions of NFGs. (The second part is for mixed NE.)

1.

0. If given an EFG, write out all players' pure strategy sets (remember: complete contingency plans!) and payoff functions to get the NFG. Collapse identical rows (or columns) to get the R[educed]NFG.

2. Eliminate strictly dominated strategies, including those dominated by mixtures, and write down the reduced NFG. If only one profile remains, it is the DS solution.

3. Repeat step 3) as many times as possible. If only one profile remains, it is IDDS solution.

4. Find each player's pure strategy best responses to each profile of other players' pure strategies. [In bimatrix games there is a handy way to do this by underlining the corresponding payoffs]. Inspect for mutual best responses, i.e., the pure strategy NE.

Let's try out the recipe on

Game 7: CoordinationB

	L	R
U	$2,\underline{2}$	$\underline{2},\underline{2}$
D	$\underline{3},\underline{1}$	$0,0$

1. There are no strictly dominated strategies, although R is weakly dominated by L for Column player.

2. Moot.

3. Underlining the BR's we find two pure strategy NE: $\{(U, R), (D, L)\}$.

It is hard to argue that Column would play the weakly dominated strategy R so it is tempting to eliminate (U, R). However, Column earns a payoff of 2 at this NE and only 1 at (D, L). You can't eliminate a weakly dominated strategy without additional criteria or "refinements" (to be discussed later).

4 Nash Equilibrium: Mixed

The corollary establishing the existence of NE in finite games follows from checking the "mixed extension." It has strategy sets $\Delta(S_i)$ for player i, and uses the expected payoff as the payoff function.

That is, the strategy sets are simplexes as we have seen, and so convex and compact. We now will see that the payoff functions are linear, hence continuous and quasi-concave.

4.1 Payoffs for mixed strategies.

The payoff function f_i for each player (sometimes denoted u_i or π_i) extends linearly to cover mixed strategies.

- Suppose player i's pure strategy set is $S_i = \{t_1, ..., t_n\}$, and she chooses mixture

$$\sigma_i = p_1 t_1 + ... + p_n t_n.$$

- Player i's (expected) payoff when the other players use pure strategy profile s_{-i} is

$$f_i(\sigma_i, s_{-i}) = \sum_{k=1}^{n} p_k f_i(t_k, s_{-i}). \tag{9}$$

- What if some or all of the other players use mixed strategies? Say the profile is

$$\sigma_{-i} = q_1 s_{-i}^1 + ... + q_m s_{-i}^m.$$

- Then i's (expected) payoff when playing own pure strategy t_k is

$$f_i(t_k, \sigma_{-i}) = \sum_{j=1}^{m} q_j f_i(t_k, s_{-i}^j). \tag{10}$$

- If all players are using mixed strategies, i's (expected) payoff is obtained by substituting equation (10) into (9).

- Later we will see [in evgame chapter] that there are nice compact ways to express such payoffs using matrix algebra.

[[Remember to put in something here (or maybe in the previous chapter) about the need for cardinal utility when computing mixed NE. I think ordinal is fine for pure NE.]]

The last step in recipe, once you have finished the IDDS process and checked for pure NE, is

4. In the remaining strategy profiles, consider each possible combination of subsets of two or more strategies for each player. Set up the system of equations that, for each player, equates her payoffs with the subset of her strategies. If there is solution for which the probabilities are non-negative and sum to 1.0 for each player, then it is a mixed NE.

For example, in Game 7, both pure strategies remain for both players. Thus there is only one relevant subset combination: both strategies for both players.

- Call Row's mixed strategy x where U is chosen with probability p. Thus, $x = pU + (1-p)D$.

- Call Column's mixed strategy y where L is chosen with probability q. Thus, $y = qL + (1-q)R$.

- The mixed NE is the simultaneous solution to the set of equations:

$$\begin{aligned} f_1(U, y) &= f_1(D, y) \\ f_2(x, L) &= f_2(x, R) \end{aligned} \tag{11}$$

  if the solution satisfies $0 \leq x, y \leq 1$, then it is a mixed NE (and completely mixed if $0 < x, y < 1$).

- If you think about it, there is something peculiar about these two equations.

- Row's mixing probability p is determined by Column's indifference condition, i.e., Column's payoff function not his own payoffs. Likewise, Column's mixing probability seems independent of her own payoffs, but directly dependent on Row's payoffs.
- This seems counter-intuitive at first. As we will argue later in more detail, this peculiarity comes from the nature of the best-response correspondence. If $q < q^*$ then pure strategy U is the unique best-response. If $q > q^*$ then pure strategy D is the unique best-response. Only if $q = q^*$ row indifferent between U and D, and therefore willing to mix U and D.

- Row player solves

$$\begin{aligned}
f_1(U, y) &= f_1(D, y). \text{ Since} \\
f_1(U, y) &= 2q + 2(1-q) = 2 \text{ , and} \\
f_1(D, y) &= 3q + 0(1-q) = 3q, \text{ we see that} \\
f_1(U, y) &= f_1(D, y) \quad \Rightarrow q^* = \frac{2}{3}.
\end{aligned}$$

  - Thus, if Column chooses L with probability $\frac{2}{3}$ and R with probability $\frac{1}{3}$ then Row is indifferent between U and D, and therefore willing to mix U and D. Note the expected payoff of D if $q = q^* = \frac{2}{3}$ is: $\frac{2}{3}(3) + \frac{1}{2}(0) = 2$, which is the payoff of U regardless of q. If $q < q^*$ then pure strategy U is a best-response.

- Column player then solves

$$\begin{aligned}
f_2(x, L) &= f_2(x, R) \\
f_2(x, L) &= 2p + 1(1-p) = 1 + p \\
f_2(x, R) &= 2p + 0(1-p) = 2p \\
f_2(x, L) &= f_2(x, R) \Rightarrow p^* = 1
\end{aligned}$$

– Thus, Column is only indifferent between pure strategies if $p^* = 1$ since R is weakly dominated.

- Therefore we have a mixed NE: (x, y), where $x = U$ and $y = \frac{2}{3}L + \frac{1}{3}R$.

Let's apply the recipe to another coordination game

Game 8: CoordinationC

		(q) L	$(1-q)$ R
(p)	U	$\underline{2},\underline{5}$	1,0
$(1-p)$	D	0,1	$\underline{5},\underline{2}$

Again, we can go right to step 3, and find the best responses.

- If $q = 1$ then U (i.e., $p = 1$) is BR
- If $q = 0$ then D (i.e., $p = 0$) is BR
- If $p = 1$ then L (i.e., $q = 1$) is BR
- If $p = 0$ then R (i.e., $q = 0$) is BR

There are two pure NE $= \{(U, L), (D, R)\}$. Clearly Row prefers (D, R) while Column prefers (U, L).

- Mixed NE. Player 1 solves:

$$\begin{aligned} f_1(U, y) &= f_1(D, y) \\ 2q + 1(1-q) &= 0q + 5(1-q) \\ 2q &= 4(1-q) \\ 6q &= 4 \\ q^* &= \frac{2}{3} \end{aligned} \tag{12}$$

- Note: if $q < \frac{2}{3}$, D (i.e., $p = 0$) is BR; if $q > \frac{2}{3}$, U (i.e., $p = 1$) is BR.

- Player 2 solves:

$$\begin{aligned} f_2(x, L) &= f_2(x, R) \\ 5p + 1(1-p) &= 0p + 2(1-p) \\ 5p &= 1 - p \\ 6p &= 1 \\ p^* &= \frac{1}{6} \end{aligned} \tag{13}$$

  - Note: if $p < \frac{1}{6}$, R (i.e., $q = 0$) is BR; if $p > \frac{1}{6}$, L (i.e., $q = 1$) is BR.

- The mixed NE is where player 1 chooses U with probability $p = \frac{1}{6}$ and D with probability $\frac{5}{6}$ while player 2 chooses L with probability $\frac{2}{3}$ and R with probability $\frac{1}{3}$.

4.2 Best-response diagram

- We can draw the best-response correspondences and find the mixed NE as their intersection. (Not best-response *functions*, since the graphs include vertical lines.)

- We have, $f_2(x, L) - f_2(x, R) = 1 + 4p - (2 - 2p) > 0$ if $6p > 1$ or $p > p^* = \frac{1}{6}$.

  - Thus, for any $p > p^* = \frac{1}{6}$ pure strategy L ($q = 1$) is a best-response and for any $p < p^* = \frac{1}{6}$ pure strategy R ($q = 0$) is a best-response.

  - Any $q \in [0, 1]$ is a best-response if $p = p^* = \frac{1}{6}$.

$$p^*(q) = BR_1 = \left\{ \begin{array}{c} 0 \text{ if } q < \frac{2}{3} \\ 1 \text{ if } q > \frac{2}{3} \\ [0,1] \text{ if } q = \frac{2}{3} \end{array} \right\}$$

- Similarly,

$$q^*(p) = BR_2 = \left\{ \begin{array}{c} 0 \text{ if } p < \frac{1}{6} \\ 1 \text{ if } p > \frac{1}{6} \\ [0,1] \text{ if } p = \frac{1}{6} \end{array} \right\}$$

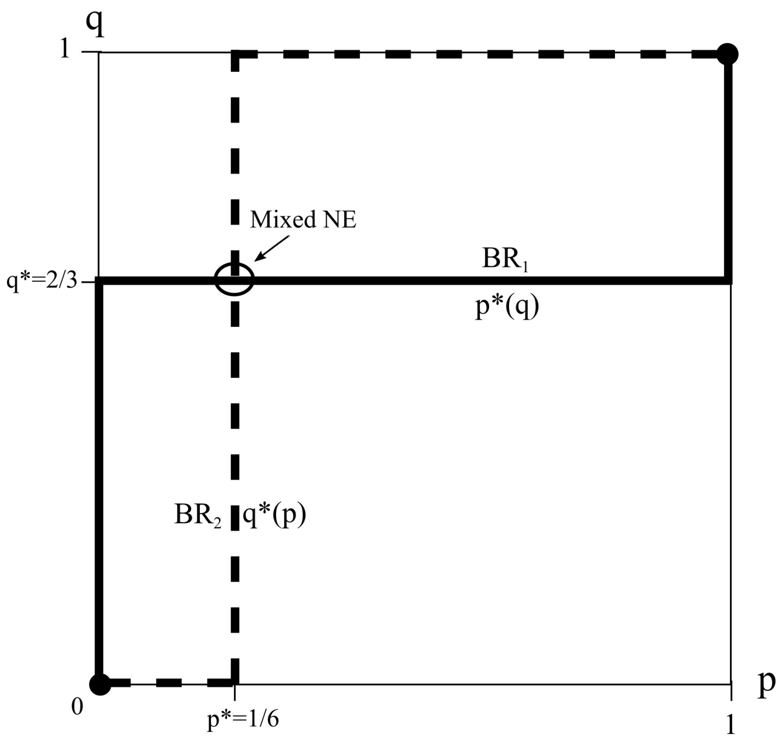

Figure 3: Best response correspondences for Game 8. The pure NE are the corner points where the two correspondences intersect, and the mixed NE is the interior intersection.

- The NFG looks like it could be symmetric, but the mixing probabilities are different.
  - Players 1 and 2 have similar payoffs at the two pure NE (5 or 2 for a total of 7) and at the off diagonal elements (0 and 1).
  - Later we will define a symmetric game and show that this game is not symmetric, thus the mixing probabilities are different.
- Each player's payoffs at the mixed NE are equal across the pure strategies that she mixes: $(x = \frac{1}{6}U + \frac{5}{6}D, y = \frac{2}{3}L + \frac{1}{3}R)$.

– Player 1:

$$\begin{aligned} E\left[\pi_1\right] &= \frac{1}{6}\left[\frac{2}{3}(2)+\frac{1}{3}(1)\right]+\frac{5}{6}\left[\frac{2}{3}(0)+\frac{1}{3}(5)\right] \\ E\left[\pi_1\right] &= \frac{1}{6}\left[\frac{5}{3}\right]+\frac{5}{6}\left[\frac{5}{3}\right]=\frac{5}{3} \end{aligned}$$

– Player 2:

$$\begin{aligned} E\left[\pi_2\right] &= \frac{2}{3}\left[\frac{1}{6}(5)+\frac{5}{6}(1)\right]+\frac{1}{3}\left[\frac{1}{6}(0)+\frac{5}{6}(2)\right] \\ E\left[\pi_2\right] &= \frac{2}{3}\left[\frac{10}{6}\right]+\frac{1}{3}\left[\frac{10}{6}\right]=\frac{5}{3} \end{aligned}$$

* Note: $\frac{5}{3}<2<5$, so mixed NE payoff is lower than either pure NE payoff.
* Note: $\frac{5}{3}$ is an *expectation* and not a *realization* outcome.
    · If the NFG came from an EFG with Nature moves, then the NFG payoffs are already expectations, and the mixed NE payoffs are expected values of expected values.

4.3 Mixed NE as maximum expected payoff

Continuing with Game 8,

- Maximizing expected payoff under uncertainty about the other players mixed strategy offers an alternative method of achieving the same mixed NE.
  - Again, let player 1 (row player) choose U with probability p and player 2 (column player) choose L with probability q.
  - Player 1 chooses p to maximize expected payoff as a function of q.

$$\begin{aligned}
\max_p E(\pi_1) &= q\,[2p + 0(1-p)] + (1-q)\,[1p + 5(1-p)] \\
&= 2pq + p - pq + 5(1-q)(1-p) \\
&= pq + p + 5 - 5q - 5p + 5pq \\
&= 6pq - 4p - 5q + 5 \qquad (14)
\end{aligned}$$

    * The first order condition is

$$\begin{aligned}
\frac{\partial E(\pi_1)}{\partial p} &= 6q - 4 = 0 \\
q^* &= \frac{2}{3} \qquad (15)
\end{aligned}$$

    * Note:

$$\begin{aligned}
\frac{\partial E(\pi_1)}{\partial p} &< 0 \text{ if } q < \frac{2}{3} \Longrightarrow p^* = 0 \\
\frac{\partial E(\pi_1)}{\partial p} &> 0 \text{ if } q > \frac{2}{3} \Longrightarrow p^* = 1.
\end{aligned}$$

    * This is the exact same result we obtained in equations (13) and (12). Note the SOC are zero and don't confirm (or contradict) a maximum.

- Player 2 maximizes expected payoff by choosing q.

$$
\begin{aligned}
\max_q E(\pi_2) &= p\,[5q + 0(1-q)] + (1-p)\,[1q + 2(1-q)] \\
&= 5pq + q - pq + 2(1-p)(1-q) \\
&= 4pq + q + 2 - 2p - 2q + 2pq \\
&= 6pq - q - 2p + 2 \qquad (16)
\end{aligned}
$$

  * The first order condition is

$$
\begin{aligned}
\frac{\partial E(\pi_2)}{\partial q} &= 6p - 1 = 0 \\
p^* &= \frac{1}{6} \qquad (17)
\end{aligned}
$$

  * Note:

$$
\begin{aligned}
\frac{\partial E(\pi_2)}{\partial q} &< 0 \text{ if } p < \frac{1}{6} \Longrightarrow q^* = 0 \\
\frac{\partial E(\pi_2)}{\partial q} &> 0 \text{ if } p > \frac{1}{6} \Longrightarrow q^* = 1.
\end{aligned}
$$

4.4 Symmetric NFG

I think we will move this material to the EvG chapter.

- This part draws on Weibull's 1995 text.

- Any NFG game can be decomposed to a payoff matrix for each player.

  - Denote the payoff matrix for Row (player 1) as A, and the payoff matrix for Column (player 2) as B.

  - The bi-matrix:

Column

		L	R
Row	U	$\underline{2},\underline{5}$	1,0
	D	0,1	$\underline{5},\underline{2}$

- - becomes:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$$

- A two-player normal form game, $G = (I, S, \pi)$ is symmetric if:
  - $I = \{1, 2\}$,
  - $S_1 = S_2$ and
  - $\pi_2(s_1, s_2) = \pi_1(s_2, s_1) \forall (s_1, s_2) \in S$.
    * Read this as player 2's payoff when player 1 uses pure strategy 1 and player 2 uses pure strategy 2 is equal to player 1's payoff from using pure strategy 2 and player 2 uses pure strategy 1 (for this matrix this requires 0 equals 0, which is OK, but not 1 against 1 or 2 against 2).
    * This last symmetry requirement on the payoff functions is equivalent to: $B^T = A$.

- For our example:

$$B^T = \begin{pmatrix} 5 & 1 \\ 0 & 2 \end{pmatrix} \neq A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

  - Thus, the game is not symmetric and the mixing probabilities are not identical.

- An example of a symmetric game with $B^T = A$ and identical mixing probabilities is the Hawk-dove game with bi-matrix:

Column (2)

		a	D
Row (1)	a	$\frac{v-c}{2}, \frac{v-c}{2}$	$\underline{v},\underline{0}$
	D	$\underline{0},\underline{v}$	$\frac{v}{2}, \frac{v}{2}$

- - The mixed NE occurs when each player chooses a with probability $\frac{v}{c} \in (0, 1)$.

- The expected payoff for each player at the mixed NE is:

$$
\begin{aligned}
E(\pi_1) &= E(\pi_2) = \left(\frac{v}{c}\right)^2 \left(\frac{v-c}{2}\right) + \left(\frac{v}{c}\right)\left(1-\frac{v}{c}\right)v + 0 + \left(1-\frac{v}{c}\right)^2 \frac{v}{2} \\
&= \frac{1}{c^2}\left[\frac{v^3 - v^2 c}{2} + \frac{2v^2(c-v)}{2} + \frac{(c-v)^2 v}{2}\right] \\
&= \frac{1}{2c^2}\left[v^3 - v^2 c + 2v^2 c - 2v^3 + vc^2 - 2v^2 c + v^3\right] \\
&= \frac{1}{2c^2}\left[-v^2 c + vc^2\right] \\
&= \frac{v}{2c}[c-v] \\
&> 0 \text{ since } c > v.
\end{aligned}
$$

- Note:

$$
\begin{aligned}
E(\pi_1) &= E(\pi_2) = \frac{v}{2c}[c-v] < \frac{v}{2} \\
&\Leftrightarrow \frac{c-v}{c} < 1 \\
&\Leftrightarrow c - v < c \\
&\Leftrightarrow 0 < v
\end{aligned}
$$

  always holds.

- If a game is symmetric then both players have equal mixing probabilities, but the converse is not always true.
  - say $2A = B$
  - so $B^T \neq A, a_1 \neq b_1,\ a_2 \neq b_2$
  - but $\frac{b_1}{b_1+b_2} = \frac{a_1}{a_1+a_2}$.
- A NFG is doubly symmetric if: $A^T = A$.
  - Since symmetric requires $B^T = A$, doubly symmetric requires $A = B$.
  - An example of doubly symmetric is the coordination game:

		Column (2)	
		t_1	t_2
Row (1)	s_1	$\underline{2},\underline{2}$	$0,0$
	s_2	$0,0$	$\underline{1},\underline{1}$

- - This game has two pure NE and a mixed NE where both players choose pure strategy 1 with probability $\frac{1}{3}$.
    - * The expected payoff for each player at the mixed NE is: $E(\pi_1) = E(\pi_2) = \left(\frac{1}{9}\right) 2 + \left(\frac{4}{9}\right) 1 = \frac{2}{3}$, less than either of the pure NE payoffs.

4.5 Beliefs

I think we will move this material to the next chapter.[Sounds good to me, Matt]

- Suppose the following bimatrix and the set of rational (permissible) beliefs by player 2.

			Column (2)	
			(q)	$(1-q)$
			t_1	t_2
Row (1)	s_1	(p)	$2,-$	$3,-$
	s_2	$(1-p)$	$\underline{5},-$	$\underline{6},-$

- - Strategy s_2 strictly dominates s_1 so the only rational belief that player 2 could have is $p = 0$.

- However, with the following bimatrix any belief $p \in [0, 1]$ by player 2 is permissible.

		Column (2)	
Row (1)		t_1 (q)	t_2 $(1-q)$
p	$s_1 = U$	$2,-$	$\underline{4},-$
$(1-p)$	$s_2 = D$	$\underline{5},-$	$3,-$

- - Since:
    - $E\pi_1(U) = 2q + 4(1-q) = 4 - 2q$
    - $E\pi_1(D) = 5q + 3(1-q) = 2q + 3$
    - $E\pi_1(U) - E\pi_1(D) = 1 - 4q$
  - Thus, if $q < \frac{1}{4}$ then U is a best-response, and if $q > \frac{1}{4}$ then D is a best-response.
  - Since there is no dominant strategy for player 1, player 2 could have any beliefs regarding p.

4.6 Dominance by mixtures

I suggest dropping this material.[Agreed, Matt]

- When searching for strictly dominated strategies to eliminate, you should focus on never-best-responses. They may be dominated by pure or mixed strategies [[Matt, maybe put it like this: when searching for strictly dominated strategies to eliminate, you should focus on never-best-responses. They may be dominated by pure or mixed strategies.]][Changed, Matt]
  - For example,

			Column	
			L (q)	R $(1-q)$
Row	(p)	U	$\underline{4}$,-	0,-
	$(1-p)$	M	0,-	$\underline{4}$,-
		D	1,-	1,-

- - Strategy D is never a best-response, however it not dominated by either U or M.
  - If $q < \frac{1}{4}$ then D is better than U, and $q > \frac{3}{4}$ then D is better than M.

- However, D is strictly dominated by mixed strategy $x = pU + (1-p)M$, for a range of p.
    * For example, if $p = \frac{1}{2}$ the expected payoff of x is: $E\pi_1(x) = 2q + 2(1-q) = 2 > E\pi_1(D) = 1$.
- 0The set of mixed strategies that strictly dominate D in this example are all $p \in (\frac{1}{4}, \frac{3}{4})$, which yield expected payoff greater than 1 for all $q \in [0, 1]$.

5 Refinements, QRE and CE

5.1 Overview of NE refinements

- The set of NE is often non-singleton.
    - Numerous attempts have been made to reduce this set by adding additional criteria to the NE concept.

1. Payoff dominance vs. risk-dominance.

- - The higher payoff NE may be riskier.

2. Trembling-hand perfection (THPNE)

- - How robust are the NE to small perturbations?

3. Subgame perfection (SGPNE)

- - Trying to eliminate non-credible threats.
    - Looks for sequentially rational behavior.

4. Bayesian NE (BNE).

- - Mixed strategy beliefs must be rational.
  - Add incomplete information.
  - Payoffs are unknown since type is unknown.

5. Perfect Bayesian NE (PBE)

- - Combines BNE with subgame perfection.

- Note: 3,4 and 5 try to eliminate "irrational" NE.

5.2 Payoff vs. risk-dominance

- The definition of payoff dominance is simply the NE that has the highest total payoff.

- Risk dominance means that a NE is more robust to uncertainty.

- Consider the following symmetric game:

		Column (2)	
		L	R
Row (1)	T	$\underline{5},\underline{5}$	$-1, 4$
	B	$4, -1$	$\underline{3},\underline{3}$

- - There are two pure NE: (T, L) and (B, R)
  - Since the NE (T, L) has a higher payoff than the NE (B, R) for both players (T, L) is called payoff dominant.
  - The NE (B, R) is risk dominant.
    * Players are not hurt as much by deviations from this equilibrium.
  - We can ask how robust each of the pure NE are to deviations.

* 1) The pure strategy NE (B, R).
    - Player 2 is intending to choose R, but deviates with probability ϵ and chooses L.
    - The payoff advantage for player 1 from choosing B is now:

$$\begin{aligned} E\pi_1(B) - E\pi_1(T) &= [4\epsilon + 3(1-\epsilon)] - [5\epsilon - 1\,(1-\epsilon)] \\ &= [\epsilon + 3] - [6\epsilon - 1] \\ &= 4 - 5\epsilon > 0 \forall \varepsilon < 0.2 \end{aligned} \tag{18}$$

    - Note: Mixed NE $\Longrightarrow p = 0.8$ compliment to $\varepsilon = 0.2$.
    - So, for any $\epsilon < 0.8$ the expected payoff from playing B is greater than T, thus (B, R) is robust of to an 80% deviation rate by player 2.
    - Compare this with the other pure NE.
* 2) Player 2 deviates from the NE (T, L) and chooses R.
    - The payoff advantage for player 1 from choosing T is now:

$$\begin{aligned} E\pi_1(T) - E\pi_1(B) &= [5(1-\epsilon) - 1\epsilon] - [4(1-\epsilon) + 3\epsilon] \\ &= [5 - 6\epsilon] - [4 - \epsilon] \\ &= 1 - 5\epsilon \end{aligned} \tag{19}$$

    - So, for any $\epsilon < .2$ the expected payoff from playing T is greater than B, thus (T, L) is robust of to an 20% tremble rate by player 2.
* Thus, (B, R) is risk-dominant since it remains a best-response for a larger degree of uncertainty (or larger deviation rate) by the other player.

- One definition of risk dominance states that NE players are still making their best responses as long as opponents probability of a tremble is less than 0.5 (for 2 pure strategies).

- When there is more than one pure NE there are usually mixed NE.
  - Using the expected payoff method in equations (16) through (15) we get:

$$\begin{aligned}
\max_p E(\pi_1) &= p\,[5q + -1(1-q)] + (1-p)\,[4q + 3(1-q)] \\
&= p\,[6q - 1] + (1-p)\,[q + 3] \\
&= 5pq - p + q - 3p + 3 \\
\frac{\partial E(\pi_1)}{\partial p} &= 5q - 4 = 0 \\
q^* &= \frac{4}{5} \qquad (20)
\end{aligned}$$

  - Since this game is symmetric $p^* = \frac{4}{5}$ as well. So, we have 3 NE for this game.
  - The mixed NE $(x, y)(x = \frac{4}{5}T + \frac{1}{5}B, y = \frac{4}{5}L + \frac{1}{5}R)$.
  - Note that the $\epsilon = 0.2$ is the complement to the mixed strategy NE.
- In a sense, a mixed strategy is an intentional deviation.

5.3 Normalization of symmetric NFG

move this to EvG chapter...and don't give Weibull credit for this approach...I invented it![Agreed and Sorry! Matt]

- Another way to think about risk-dominance comes from Weibull (1995)
- Given a symmetric ($B^T = A$) two-player NFG with payoff matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

  - We can define the payoff advantage to pure strategy 1 played against itself as: $a_1 \equiv a_{11} - a_{21}$ and the payoff advantage to pure strategy 2 played against itself as: $a_2 \equiv a_{22} - a_{12}$.

– Then the normalized payoff matrix A' is:

$$A' = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix}$$
$$A' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

* Note that this is just row operations, subtracting a_{21} from the first column and subtracting a_{12} from the second.

- Now, we have only three types of games that are possible:

1. 1) If $a_1 > 0$ and $a_2 > 0$ then we have a coordination game.
2. 2) If $a_1 < 0$ and $a_2 < 0$ then we have a hawk-dove game.
3. a) If $a_1 > 0$ and $a_2 < 0$ then we have a prisoner's dilemma, with pure strategy 1 dominant.

b) If $a_1 < 0$ and $a_2 > 0$ then we have a prisoner's dilemma, with pure strategy 2 dominant.

- Using our symmetric two player NFG example:

		Column (2)	
		L	R
Row (1)	T	$\underline{5},\underline{5}$	$-1, 4$
	B	$4, -1$	$\underline{3},\underline{3}$

– Pure NE (T, L) is payoff dominant and (B, R) is risk dominant. The payoff matrix is:

$$A = \begin{pmatrix} 5 & -1 \\ 4 & 3 \end{pmatrix} = B^T$$

- The normalized payoff matrix is:

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

- Thus, we have a coordination game with mixed NE where pure strategy 1 is played with probability $p = \frac{a_2}{a_1+a_2} = \frac{4}{5}$.

- Harsayni and Selton (1988) book "A General Theory of Equilibrium Selection in Games" say the equilibrium (B, R) risk dominates (T, L).

- (From Weibull) For any symmetric 2×2 game with normalized payoffs $a_1, a_2 > 0$, and where Θ^{NE} is the set of pure NE we can say: (B, R) risk dominates (T, L) if $a_2 > a_1$.

  - That is, if the pure NE risk dominates the other if, after payoffs are normalized, it strictly Pareto dominates the other.
  - Many would argue that rational players should play the risk-dominant NE (B, R)

- What actually happens: Friedman (1996) Economic Journal. Found more evidence of payoff dominant NE behavior.

5.4 Correlated equilibrium

- A mixed strategy NE assumes that the mixing probabilities p and q are statistically independent.

  - Given two mixed strategies, the "probability matrix" showing the weight of the utility outcomes in the expected payoffs is:

		Column (2)	
Row (1)		t_1 (q)	t_2 $(1-q)$
p	s_1	$pq(\pi_1, \pi_2)$	$p(1-q)(\pi_1, \pi_2)$
$(1-p)$	s_2	$(1-p)\,(q)(\pi_1, \pi_2)$	$(1-p)\,(1-q)(\pi_1, \pi_2)$

- In a correlated equilibrium, the probabilities would differ from above.
- In a correlated equilibrium the probability of s_1 is not p, but rather depends on the realization of q, that is: $prob(p|q) \neq p$ and $prob(q|p) \neq q$.

material from Parkes and Sueken (forth) – be sure not to plagiarize. The correlated equilibrium is another kind of equilibrium that is widely used in game theory. A Nash equilibrium (pure or mixed) is also a correlated equilibrium, but the set of correlated equilibria can include additional solutions. Rather than insisting that each player independently randomizes in selecting an action, a correlated equilibrium allows for a joint distribution on actions that is not a product distribution. To illustrate the idea, we return to the game of Chicken. Example 2.7. We introduce a coordinating signal into the game of Chicken. The signal is a random variable that takes on value 0 or 1 and is visible to each player. Each agent selects an action based on the signal's value: player 1 player 2 ...

6 Behavioral Considerations.

Centipede stresses common knowledge of rationality – you have to [believe that your opponent believes]^{100} that both are rational, exactly as in the finitely repeated PD. The BR is to defect just one period before your opponent, but it is a epsilon-dominant strategy to just wait until near the end.