ECON 200 - Fall 2016 - Problem Set 3
Answer Key

November 6, 2016

Part I
Short Case Study Problems

1 Lottery Ticket

Your Bernoulli function is $u(m) = \sqrt{m}$, and currently your wealth is $m = 4$. You have a lottery ticket that will pay 12 with probability 0.1 and otherwise will pay 0.

a. What is the mean and variance of the lottery ticket value?
The mean is the possible values times their probabilities:

$$\mathbb{E}[\text{ticket}] = p_1\text{outcome}_1 + p_2\text{outcome}_2$$
$$= 0.1 \cdot 12 + 0.9 \cdot 0 = 1.2$$

The variance formula is as follows:

$$\text{Var}[\text{ticket}] = \sum p_i \cdot (\text{outcome}_i - \mathbb{E}[\text{ticket}])^2$$
$$= 0.1 \cdot (12 - 1.2)^2 + 0.9 \cdot (0 - 1.2)^2$$
$$= 12.96$$

b. What is your expected utility with the ticket? Without the ticket?
Expected utility with the ticket is the probability-weighted sum of possible outcomes:

$$\mathbb{E}[U(\text{ticket})] = p_1 \cdot u(m + \text{outcome}_1) + p_2 \cdot u(m + \text{outcome}_2)$$
$$= 0.1 \cdot \sqrt{4 + 12} + 0.9 \cdot \sqrt{4 + 0}$$
$$= 0.1 \cdot \sqrt{16} + 0.9 \cdot \sqrt{4}$$
$$= 0.1 \cdot 4 + 0.9 \cdot 2 = 2.2$$

Without the ticket, expected utility is your utility of the only possible outcome:

$$\mathbb{E}[U(m)] = u(m) = \sqrt{4} = 2$$

c. What is the maximum price that you rationally would turn down if someone offered to buy your ticket?
You would rationally turn down any price that makes you worse off. To find out what range of prices that is, suppose you are given a price that makes you equally well off. That is:

$$u(m + p') = \mathbb{E}[U(\text{ticket})]$$
$$\sqrt{4 + p'} = 2.2$$
$$4 + p' = 4.84$$
$$p' = 0.84$$

For any price greater than $p' = 0.84$, that is $p > 0.84$, you would be happier with the money than the ticket. For any price lower than 0.84, you would be happier with the ticket, so you would turn down that offer. The maximum price is 0.84.

d. What is your risk premium for the lottery ticket?

The risk premium is the amount you are willing to pay for certainty. Here, the lottery ticket has a higher expected payoff than its value to us (or the value of that payoff in utility terms). The difference between the valuation we discovered in (c) and the expected value of the lottery is our risk premium:

$$\text{Risk Premium} = \mathbb{E}\left[\text{ticket}\right] - p'$$
$$= 1.2 - 0.84 = 0.36$$

2 Mean-Variance and Bernoulli EU

2. Finance textbooks often assume that utility takes the form $U(L) = \mathbb{E}[L] - c \cdot \text{Var}[L]$, for some $c > 0$ parametrizing the degree of risk aversion. Consider two investors, A and B. Investor A maximizes $U[L]$ for $c = 0.01$, while B maximizes the expected value of the Bernoulli function $u(m) = m^{0.9}$. Consider two lotteries: S pays \$1 for sure, while R pays \$1 with probability 0.999 and pays \$1000 with probability 0.001.

a. Compute the mean and variance of each lottery.

As before.

$$\mathbb{E}[S] = 1$$
$$\mathbb{E}[R] = 0.999 \cdot 1 + 0.001 \cdot 1000$$
$$= 0.999 + 1 = 1.999$$

$$\text{Var}[S] = 0$$
$$\text{Var}[R] = 0.999 \cdot (1 - 1.999)^2 + 0.001 \cdot (1000 - 1.999)^2$$
$$= 0.997 + 996.006 = 997.003$$

b. Compute U (for investor A) for each lottery.

We are given

$$U_A(L) = \mathbb{E}[L] - 0.01 \cdot \text{Var}[L]$$

$$U_A(S) = \mathbb{E}[S] - 0.01 \cdot \text{Var}[S]$$
$$= 1 - 0.01 \cdot 0 = 1$$

$$U_A(R) = \mathbb{E}[R] - 0.01 \cdot \text{Var}[R]$$
$$= 1.999 - 0.01 \cdot 997.003 = -7.971$$

c. Compute Eu (for investor B) for each lottery.

We are given

$$u(m) = m^{0.9}$$

Expected utility is then:

$$\mathbb{E}[U_B(L)] = \sum_i p_i u(m(\lambda_i))$$

Which is just saying that the expected utility is a probability-weighted sum of the utility of each possible outcome.

$$\mathbb{E}[U_B(S)] = \sum_i p_i u(m(s_i))$$
$$= 1 \cdot 1^{0.9} = 1$$

$$\mathbb{E}\left[U_B\left(R\right)\right] = \sum_i p_i u\left(m\left(r_i\right)\right)$$
$$= 0.999 \cdot (1)^{0.9} + 0.001 \cdot (1000)^{0.9}$$
$$= 0.999 + 0.501 = 1.5$$

d. If faced with the choice between R and S, which would each investor choose?

Investor A would choose S, while investor B would choose R.

e. Now do Short Essay 1 below.

To be shown below, the point is that R stochastically dominates S, or in simpler terms, R is weakly better than S for any probabilities and preferences. The mean-variance approximation falters under certain circumstances, such as this one.

3   Bayesian Updating

A patient has either disease A or disease B. Diagnostic test 1 is positive with probability 0.7 (and negative with probability 0.3) with disease A, and is positive with probability 0.4 with disease B. Similarly, diagnostic test 2 is positive with probability 0.2 when the disease is A, and with probability 0.5 when the disease is B. Overall, the relative incidence of the two diseases is 60% A and 40% B.

a. Compute the joint probabilities $p(d, t_1, t_2)$ for each disease$d = $ A, B and each test result $t_1 = $ pos, neg and $t_2 = $ pos, neg.

The joint probabilities are

$$p\left(A, t_1^+, t_2^+\right) = p\left(t_1^+|A\right) \times p\left(t_2^+|A\right) \times p\left(A\right) = 0.7 \cdot 0.2 \cdot 0.6 = 0.084$$
$$p\left(A, t_1^+, t_2^-\right) = p\left(t_1^+|A\right) \times p\left(t_2^-|A\right) \times p\left(A\right) = 0.7 \cdot 0.8 \cdot 0.6 = 0.336$$
$$p\left(A, t_1^-, t_2^+\right) = p\left(t_1^-|A\right) \times p\left(t_2^+|A\right) \times p\left(A\right) = 0.3 \cdot 0.2 \cdot 0.6 = 0.036$$
$$p\left(A, t_1^-, t_2^-\right) = p\left(t_1^-|A\right) \times p\left(t_2^-|A\right) \times p\left(A\right) = 0.3 \cdot 0.8 \cdot 0.6 = 0.144$$
$$p\left(B, t_1^+, t_2^+\right) = p\left(t_1^+|B\right) \times p\left(t_2^+|B\right) \times p\left(B\right) = 0.4 \cdot 0.5 \cdot 0.4 = 0.08$$
$$p\left(B, t_1^+, t_2^-\right) = p\left(t_1^+|B\right) \times p\left(t_2^-|B\right) \times p\left(B\right) = 0.4 \cdot 0.5 \cdot 0.4 = 0.08$$
$$p\left(B, t_1^-, t_2^+\right) = p\left(t_1^-|B\right) \times p\left(t_2^+|B\right) \times p\left(B\right) = 0.6 \cdot 0.5 \cdot 0.4 = 0.12$$
$$p\left(B, t_1^-, t_2^-\right) = p\left(t_1^-|B\right) \times p\left(t_2^-|B\right) \times p\left(B\right) = 0.6 \cdot 0.5 \cdot 0.4 = 0.12$$

b. Compute the prior probabilities of each test result and each disease.

First of all, this should say the message probabilities of each test result and the prior probabilities of each disease. The priors are given, .6 for A and .4 for B. The message probabilities are just the marginals

$$p\left(m = t_1^+\right) = p\left(t_1^+|\text{has A}\right) \cdot p\left(\text{has A}\right) + p\left(t_1^+|\text{has B}\right) \cdot p\left(\text{has B}\right)$$
$$= 0.7 \cdot 0.6 + 0.4 \cdot 0.4 = 0.42 + 0.16 = 0.58$$
$$p\left(m = t_1^-\right) = p\left(t_1^-|\text{has A}\right) \cdot p\left(\text{has A}\right) + p\left(t_1^-|\text{has B}\right) \cdot p\left(\text{has B}\right)$$
$$= 0.3 \cdot 0.6 + 0.6 \cdot 0.4 = 0.18 + 0.24 = 0.42$$
$$p\left(m = t_2^+\right) = p\left(t_2^+|\text{has A}\right) \cdot p\left(\text{has A}\right) + p\left(t_2^+|\text{has B}\right) \cdot p\left(\text{has B}\right)$$
$$= 0.2 \cdot 0.6 + 0.5 \cdot 0.4 = 0.12 + 0.2 = 0.32$$
$$p\left(m = t_2^-\right) = p\left(t_2^-|\text{has A}\right) \cdot p\left(\text{has A}\right) + p\left(t_2^-|\text{has B}\right) \cdot p\left(\text{has B}\right)$$
$$= 0.8 \cdot 0.6 + 0.5 \cdot 0.4 = 0.48 + 0.2 = 0.68$$

Note that the probability of receiving a message here is unconditional of the message for the other test. Thus, the probability of receiving a positive result plus the probability of receiving a negative result should equal 1.

Table 1: Posterior Probabilities

	t_2^+	t_2^-
t_1^+	0.512	0.808
t_1^-	0.231	0.545

c. Calculate the posterior probability of disease A when t_2 is not performed and t_1 is positive (and also t_1 negative). Similarly, when t_1 is not performed and t_2 is positive (and also t_2 negative).

The formula for calculating the posterior is as follows:

$$p\left(A|t_1^+\right) = \frac{p\left(t_1^+|A\right)p\left(A\right)}{p\left(t_1^+\right)} = \frac{0.7 \cdot 0.6}{0.58} = 0.724$$

$$p\left(A|t_1^-\right) = \frac{p\left(t_1^-|A\right)p\left(A\right)}{p\left(t_1^-\right)} = \frac{0.3 \cdot 0.6}{0.42} = 0.429$$

$$p\left(A|t_2^+\right) = \frac{p\left(t_2^+|A\right)p\left(A\right)}{p\left(t_2^+\right)} = \frac{0.2 \cdot 0.6}{0.32} = 0.375$$

$$p\left(A|t_2^-\right) = \frac{p\left(t_2^-|A\right)p\left(A\right)}{p\left(t_2^-\right)} = \frac{0.8 \cdot 0.6}{0.68} = 0.706$$

d. Calculate and put into a table the posterior probability of disease A for each possible combination of the test results (t_1, t_2) when both tests are performed.

These can be calculated as follows:

$$
\begin{aligned}
p\left(A|t_1^+, t_2^+\right) &= \frac{p\left(t_1^+, t_2^+|A\right)p\left(A\right)}{p\left(t_1^+, t_2^+\right)} \\
&= \frac{p\left(t_1^+, t_2^+|A\right)p\left(A\right)}{p\left(t_1^+, t_2^+|A\right)p\left(A\right) + p\left(t_1^+, t_2^+|B\right)p\left(B\right)} \\
&= \frac{0.14 \cdot 0.6}{0.164} = 0.512
\end{aligned}
$$

See Table 1 for all for posteriors.

Part II
Short Essay

4   Commentary on problem 2

In problem 2 above, which investor's choice (if either) makes sense to you? How can you generalize beyond the specific values (e.g., c = 0.01 and the probs and payoffs of R and S) in that problem to make a general point? Write about 100 words intelligible to your TA.

4.1   Answer Guidance

The purpose here is to identify the flaws of the mean-variance formula as an approximation for expected utility theory. While the mean-variance formulation works for gambles with small variance and utility functions with constant relative or absolute risk aversion, it breaks down with other cases. The lottery R stochastically dominates R for all probabilities and utility functions, since the expected gain can only be greater (or equal for $p_2 = 0$). The choice of investor A makes no sense in this context.