Problem Set 2, Question 4

Econ 200

1. A firm has production function ln y = (1/3)ln x_1 + (1/2)ln x_2. Input 2 is unchangeable for the moment at the level $x_2 = 8$. Prices are \$8 and \$5 respectively for inputs 1 and 2.

(a) (6 points) What is the firm's variable cost? fixed cost (if any)? total cost? marginal cost?

> **Solution:** In the short run, total costs are the sum of variable costs and fixed costs:
>
> $$c(y) = c_v(y) + F$$
>
> Therefore, we can find variable costs and fixed costs from total costs $c(y)$ which is simply the cost function $c(w_1, w_2, y)$ where $w_1 = 8$ and $w_2 = 5$.
>
> In order to derive cost function, the first step is to derive conditional factor demand function. The cost minimization problem is,
>
> $$\min_{x_1 \geq 0} 8x_1 + 5(8) \quad \text{subject to} \quad \frac{1}{3}\ln x_1 + \frac{1}{2}\ln 8 \geq \ln y$$
>
> Since the firm has no incentive to use more x_1 than necessary to achieve the fixed output level, the solution of this optimization problem is implicitly defined as
>
> $$\frac{1}{3}\ln x_1^* + \frac{1}{2}\ln 8 = \ln y$$
>
> By solving this equation explicitly for x_1 yields $x_1^*(8, 5, y) = 8^{-\frac{1}{2}}y^3$. The second step is to substitute this factor demand function into cost. This gives us our cost function:
>
> $$c(8, 5, y) = 8^{-\frac{1}{2}}y^3 + 40$$
>
> implying that variable costs are $c_v(y) = 8^{-\frac{1}{2}}y^3$ and fixed costs are $F = 40$. Marginal costs are
>
> $$\text{MC}(y) = \frac{dc}{dy} = 3 \cdot 8^{-\frac{1}{2}}y^2$$

(b) (2 points) What is the firm's supply function?

Solution: The firm's supply curve is the upward-sloping portion of the marginal cost curve that lies above the average variable cost curve. In this problem, the marginal cost curve is always upward-sloping (and thus marginal cost is always higher than average variable cost). Therefore, the firm's supply function is given by its marginal cost:

$$p = 3 \cdot 8^{-\frac{1}{2}} y^2 \quad \Rightarrow \quad y^* = 2^{\frac{3}{4}} 3^{-\frac{1}{2}} p^{\frac{1}{2}}$$

(You should verify that this is the same supply function that you get when you apply Hotelling's Lemma to the profit function.)

(c) (6 points) Now assume that both inputs can be adjusted freely. What are the firm's conditional input demands? What is its average cost? Marginal cost? Supply function?

Solution: To solve

$$\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad \ln y = \frac{1}{3} \ln x_1 + \frac{1}{2} \ln x_2$$

we use the Lagrangian

$$\mathcal{L} = w_1 x_1 + w_2 x_2 - \lambda\left(\frac{1}{3} \ln x_1 + \frac{1}{2} \ln x_2 - \ln y\right)$$

which gives the first-order conditions

$$\begin{aligned} w_1 - \lambda \frac{1}{3x_1} &= 0 \\ w_2 - \lambda \frac{1}{2x_2} &= 0 \end{aligned}$$

When we combine these two conditions to get rid of λ, we get $3w_1 x_1 = 2w_2 x_2$. We substitute this back into the constraint to get the conditional factor demands:

$$\ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln\left(\frac{3w_1}{2w_2}x_1\right)$$

$$\Rightarrow \quad y = x_1^{\frac{1}{3}}\left(\frac{3w_1}{2w_2}x_1\right)^{\frac{1}{2}} = \left(\frac{3w_1}{2w_2}\right)^{\frac{1}{2}} x_1^{\frac{5}{6}}$$

$$\Rightarrow \quad x_1^*(w_1, w_2, y) = \left(\frac{2w_2}{3w_1}\right)^{\frac{3}{5}} y^{\frac{6}{5}},$$

$$x_2^*(w_1, w_2, y) = \left(\frac{3w_1}{2w_2}\right)^{\frac{2}{5}} y^{\frac{6}{5}}$$

With these conditional factor demands, the cost function is

$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1^* + w_2 x_2^* \\ &= \left[\left(\frac{2}{3}\right)^{\frac{3}{5}} + \left(\frac{3}{2}\right)^{\frac{2}{5}}\right] w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}}\left(1 + \frac{3}{2}\right) w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \end{aligned}$$

Long-run average cost is

$$\mathrm{LAC}(y) = \frac{c}{y} = \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}}$$

and long-run marginal cost is

$$\begin{aligned} \mathrm{LMC}(y) &= \frac{dc}{dy} = \frac{6}{5}\left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \\ &= 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \end{aligned}$$

The firm's supply function is once again given by its marginal cost:

$$p = 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}}$$

$$\Rightarrow \quad y^*(w_1, w_2, p) = \frac{p^5}{72 w_1^2 w_2^3}$$

(Alternatively, we could have found the supply function by maximizing profit. You should verify that the same supply function is found by applying Hotelling's Lemma to the profit function!)