A simple testable model of double auction markets

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We propose a model of price formation in Double Auction markets which employs the strong simplifying assumption that agents neglect strategic feedback effects and regard themselves as playing a Game against Nature. Agents otherwise are strict expected utility maximizers employing Bayesian updating procedures. We prove the optimality of simple ('aggressive reservation price') strategies in our general model and propose a parametric form that yields very detailed and computable predictions of market behavior.

1. Introduction

The Double Auction market institution, in which participants may continually make and accept public offers to buy (i.e., bids) and to sell (asks), has long been favored in major financial markets,[1] and also in laboratory experiments, for its evident efficiency and logistical simplicity. Yet it has so far attracted little attention from theorists, despite several important and intriguing questions[2] regarding its performance posed by Smith (1982) and Plott (1982). Difficulties in formalizing tractable models of the Double Auction doubtless account for this lack of attention. We begin with a very brief review of the three approaches that have been employed so far, and then introduce a new approach.

Easley and Ledyard (1986) postulate plausible but ad hoc behavioral rules

[1] Indeed, the major U.S. 'money markets' for Federal funds, NCDs, etc., commodities markets and foreign exchange markets all employ this DA institution, as do some of the newer stock markets such as NASDAQ. The main holdout is the New York Stock Exchange, which still employs a restriction of the DA in which 'specialists' are given sole access to other participants' bids and offers. Here we shall only consider the basic DA.

[2] In particular, why does the market outcome generally converge so rapidly to Competitive Equilibrium, even when the number of traders is small? Standard theory (based on a Tatonnement mechanism, rather than the Double Auction) suggests that a large number of traders is required for convergence to Competitive Equilibrium.

regarding agents' actions – i.e., bid, ask and acceptance choices. The rules assume reservation prices that are not tied to agents' induced value parameters. Over repetitions of the market the reservation prices adjust, and the authors show that the transaction prices remain in a fairly narrow interval around the competitive equilibrium price after sufficiently many repetitions. Some weak implications of the model are tested successfully against laboratory data.

Wilson (1987) postulates a sequential equilibrium in parameter-contingent strategies: Buyers and sellers draw their valuation parameters from a known (common-knowledge) joint distribution, and at each moment use likelihood functions derived from knowledge of the equilibrium strategy profile to update beliefs regarding others' parameters and subsequent actions; relative to this system of beliefs, strategies are in Nash equilibrium and are consistent with the beliefs. Such a sequential equilibrium is shown to exist and to have some features consistent with observed laboratory behavior (e.g., near-efficient outcomes). Wilson focuses on a single trading round, omits computational details, and does not attempt to demonstrate uniqueness of his equilibrium.

Various theorists, including Wilson himself, have noted that such an approach presumes agents possess an incredible amount of prior knowledge and computing skill. Friedman (1984) evades this difficulty by posing a condition[3] that many strategies and learning rules will satisfy, and demonstrates that this condition (together with some mild technical conditions) ensures efficient outcomes. Unfortunately that analysis provides few testable predictions regarding the actual process of price formation.

The present paper seeks detailed testable predictions of bid, ask and acceptance behavior. It employs the following strong simplifying assumption: Agents behave as if playing a 'Game against Nature'. That is, we assume each agent neglects the possibility that his current bid òr ask may affect other agents' contingent behavior and thus affect his own future trading opportunities. Otherwise agents are good Bayesian decision makers and chose feasible actions so as to maximize expected final wealth given all available information.

A model of this sort invites a priori criticism from two sides. Its neglect of strategic interaction is clearly a departure from full rationality as understood by game theorists.[4] On the other side, enthusiasts of behavioral models may

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[3]The condition is called no-congestion equilibrium (NCE). In essence, it requires that agents do not necessarily regret their final actions, ruling out cases such as an agent who *tries* to accept the outstanding bid just at the close of trade, but is unable to do so because of a flurry of last-second activity by other traders. See Friedman (1984, p. 65) for the formal definition.

[4]It is perhaps worth noting here that some game theorists are beginning to question the appropriateness of standard rationality assumptions. Binmore (1987) and Anderlini (1989) argue that full rationality is logically impossible: even Turing machines of unbounded complexity can't guarantee Nash equilibrium behavior. Crawford (1989) emphasizes 'strategic uncertainty' that prevents actual game players from achieving Nash equilibrium immediately, at least in some games with multiple equilibria.

complain that even the residual Bayesian inference and decision problems are unrealistically complex; although solvable by a trained economist (as we will show in the next several sections), the problems may be beyond the grasp of real-world traders.

We have two lines of defense against these criticisms. The first is that the combination of naïveté (on the strategic side) and sophistication (on the inference/decision side) does have some a priori plausibility in the context of Double Auction markets. The neglected strategic effects are difficult for an agent to predict (except in a specific strategic equilibrium) and are probably not regarded as systematic by most Double Auction participants. For instance, a buyer may believe that sellers will raise their price aspiration levels and be less inclined to sell later if he now raises the current market bid. On the other hand, such a raise might not damage (and could even enhance) his future trading opportunities if it 'scared off' other buyers so they bid less aggressively. Such ambiguities suggest that, at least in early repetitions of a Double Auction market, an agent trader may not miss any profit opportunities if he ignores strategic feedback effects. On the other hand, the usual 'as if' argument would seem to apply even in early trading periods with respect to inference and decision: A trader who ignores important information or takes significantly suboptimal actions will suffer for it, and the Double Auction mechanism (especially the oral form of it) is likely to make him aware of his loss. Consequently Bayesian rationality should be a reasonable approximation of actual behavior.

The second, and ultimately more important, line of defense is empirical. The assumptions allow construction of a relatively tractable and parsimonious model. The degree of success achieved by this model in predicting agent behavior and market outcomes then can be used as a benchmark against which more sophisticated models of the Double Auction can be judged. We suspect that it will not be easy to formulate equally specific and parsimonious models which predict more accurately. In any case, the value of a benchmark can hardly be disputed.

Our goal then is to exposit a computable model of agent behavior in Double Auction markets, suitable for testing on existing laboratory data. Along the way, however, we encounter several interesting technical issues, e.g., how best to model continuous-time strategies, and interactions between Bayesian learning and the optimal control of a stochastic process. In the interests of simplicity and brevity, we will pursue such issues here only far enough to properly specify our model. Thus, for example, we will (except in the Appendix) specify the absence of clairvoyance by a direct restriction on a function rather than by the more general abstract measurability conditions.

In section 2, we present the Double Auction institution and the basic assumptions of rationality, Bayesian inference, and the Game against Nature view of strategic interaction. Section 3 derives some analytical results on

general existence of optimal strategies, the simple ('aggressive reservation price') strategies agents will follow under present assumptions, and the efficiency of market outcomes. Section 4 suggests a convenient parametrization and discusses some computational issues. Section 5 offers some suggestions on testing the model.

2. Basic specifications

In experimental Double Auction markets for perishables, agents are assigned specialized roles as *buyers* or *sellers*. Sellers are endowed with cost schedules and can sell but not (re)purchase units of the good, while buyers are endowed with cash and redemption-value schedules, and can purchase but not (re)sell. For simplicity in the next several sections we assume (as do our predecessors) that agents have only a *single* unit to transact; that is, each seller has a finite *cost* $c_i > 0$ for his first unit of the good and infinite cost for subsequent units, and each buyer has a positive finite *redemption value* d_j for his first unit and zero value for subsequent units. We always assume that there are $n \geq 3$ buyers, each endowed with at least d_j in cash, and $m \geq 3$ sellers.

Trade is allowed during some finite interval of time $[0, T]$. For $t \in [0, T]$, each buyer j can post and freely adjust his *bid price* $b_j(t)$, representing the amount of cash he is prepared to pay for a unit of the good. Likewise, each seller can post and adjust her *ask price* $a_i(t)$, the amount of cash she is prepared to accept in exchange for a unit of the good. The *market* (referred to by some as the 'best' or 'standing') *bid* and *ask* then are the highest bid price $b(t)$ and the lowest ask price $a(t)$, respectively. We say an agent *holds* the market bid (ask) at time t if he or she was the first to announce that price. Ties have zero probability here, but for completeness, say the agent with lower index number holds the market bid (ask) if more than one announce the same best price at the same time. A seller (resp. buyer) can *accept* the current market bid (resp. ask) in which case she immediately consummates the indicated transaction with the holder of the market bid (resp. ask) at the latter's posted price $p(t) = b(t)$ (resp. $p(t) = a(t)$). The two transactors then become *inactive* and no longer participate for the remainder of the trading period. Agents who have not yet transacted are referred to as *active*.[5]

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[5]There are several variants of the Double Auction, which differ according to how much information agents have (e.g., do they know *which* agent holds the market bid? Do they know the other bid prices?), how one acquires the market bid or ask (e.g., does the second-highest bid automatically become the market bid after a transaction? Does the NYSE convention, described in section 4, apply?), and other details (e.g., does one have to confirm a transaction before it becomes final?). The analysis below applies to all these variants, so they are not emphasized here.

See Friedman (1984) for a presentation of an *asset* version of the Double Auction. An asset market differs from the perishables market analyzed here mainly in allowing resale or repurchase, and in assigning a trader-specific *constant* unit redemption value for multiple units (i.e., each agent has a constant MRS of the goods for cash).

Typically experiments consist of several *stationary repetitions* of the trading period – at the beginning of each period agents are re-endowed and the c_i and d_j parameters are usually kept constant throughout the experiment. Such repetition surely affects traders' beliefs held *prior* to the start of a trading period; in the following analysis of behavior during a given trading period we ignore any other possible effects of stationary repetition.

More formally, let $a(t) = \min\{a_i(t):i=1,\ldots,n\}$, $b(t) = \max\{b_j(t):j=1,\ldots,m\}$ and $p(t) = $ most recent accepted $a(u)$ or $b(u)$, $u \leq t$, if any transactions have occurred [say $p(t) = 0$ for $t <$ first transaction time] be the histories of market ask, bid and transaction prices for a trading period, i.e., for t in $[0, T]$. Let $P = (0, \bar{C}]$, where $\bar{C} \geq \max d_j$ is an a priori upper bound on possible trans-actions prices; e.g., $\bar{C}$ may be taken as the sum of cash endowments if that is common knowledge. Let $Q \subset 2^P$ be a given collection of subsets of P which is compact in the Hausdorff pseudometric topology. In cases where an indivisible unit of cash is employed (e.g., $\varepsilon = \$0.01$ or $\varepsilon = 1$ franc), we assume Q contains all subsets of $\{0, \varepsilon, 2\varepsilon, \ldots, [\bar{C}/\varepsilon]\varepsilon\}$. Let $A_i(t)$ and $B_j(t) \in Q$ denote the sets of market prices acceptable to seller i and buyer j at time t. Let $\mathbf{1}_i$ denote activity: $\mathbf{1}_i(t) = 1$ means agent i is active and $\mathbf{1}_i(t) = 0$ means he or she is inactive.

The set of conceivable *price histories* is $H = \{h:[0,T] \to P^3 \mid h$ is right-continuous and piecewise constant$\}$. We impose the following restrictions on strategies $S_i = (a_i, A_i)$ for sellers, $i=1,\ldots,n$. Exactly similar restrictions are imposed on buyers' strategies, $S_j = (b_j, B_j)$, $j=1,\ldots,m$.

Assumption 1. $a_i:H \times [0, T] \to P$ and $A_i:H \times [0, T] \to Q$ are Borel measurable functions. There is some $\delta > 0$ such that for every $h \in H$ the set $D_i \subset [0, T]$ of discontinuity points of $A_i(h, \cdot)$ and $a_i(h, \cdot)$ satisfies $\inf\{s - t:s > t \in D_i\} < \delta$. The function a_i is piecewise constant and right-continuous in t for every h.

Assumption 2. For all $t \in [0, T]$, if $h(u) = g(u)$ for $0 \leq u \leq t$, then $S_i(h, t) = S_i(g, t)$.

Assumption 3. There is some (small) $\delta > 0$ such that $b(u) \in A_i(u)$ for $u \in (t - \delta, t)$ implies $\mathbf{1}_i(s) = 0$ for $s \geq t$, i.e., i transacts and then becomes inactive by time t.

If a strategy satisfies Assumptions 1–3 we shall call it *admissible*. The first condition for admissibility is a mild regularity condtion, the most restrictive clause being the existence of a minimum adjustment time. One can think of this δ as representing the cycle time of the computer if trading is automated (say 1 millisecond), or the time it takes to be recognized in an oral auction

(say 0.1 second). The second conditon prevents current actions from depending on future events, i.e., no clairvoyance allowed. The third condition is a link between outcomes and strategies. It requires that each agent's reaction time for acceptances is at worst some specified $\delta > 0$. The set of admissible strategies is endowed with the product topology inherited from $P \times Q$ and is denoted for sellers by Σ^s and for buyers by Σ^b. We often will abuse notation by writing Σ indifferently for Σ^b and Σ^s, and by writing $a_i(t)$ for $a_i(h, t)$, etc.

Later we will focus on admissible strategies with the property that $A_i(t)$ [resp. $B_j(t)$] is always of the form $[v_{it}, \bar{C}]$ [resp., $[0, v_{jt}]]$, i.e., 'accept any market bid of v_{it} (resp., ask of v_{jt}) or better'. Such a v_{it} (v_{jt}) is called a *reservation price* at time t, and such strategies are called *reservation price strategies*. A reservation price strategy is called *aggressive* if the seller (buyer) always seizes the market ask (bid) – by shaving (raising) the price slightly – whenever (i) he does not already hold it, (ii) the current bid (ask) is not acceptable and (iii) he can do so at a price not below (above) his reservation value. Aggressive reservation price strategies are diagrammed in fig. 1.

Our basic behavioral assumption is that agents try to maximize the expected utility of trading profits. Specifically, assume each agent i has a von Neumann–Morgenstern utility function $U_i(\pi)$, normalized so $U_i(0) = 0$, with $U_i' > 0$. Here π represents *trading profit*, defined for a seller i (resp. buyer j) who transacts at $p(t) = p$ at some $t \in [0, T]$ by $\pi = p - c_i$ (resp. $\pi = d_j - p$). Agents who fail to transact by time T have $\pi = 0$.

The distinctive aspect of the present model is how expectations are formed. We shall refer below to *non-dogmatic* beliefs, by which we mean that the relevant probability distribution has a density and its support is P. Our three Bayesian Game Against Nature assumptions are:

BGAN1. Each agent regards market asks and bids that he does not hold as realizations of random processes ('Nature'). The perceived processes are unaffected by his own actions. An acceptance of the market ask (resp. bid) by another agent is regarded as a realization of the bid (ask) process that exceeds (falls below) the current market ask (bid).

BGAN2. Each agent begins with non-dogmatic priors regarding Nature's bid and ask processes and uses Bayes' Rule to update after each process realization. Expectations are taken with respect to the corresponding posterior distributions.

BGAN3. Nature's random processes generate an exchangeable sequence of new bid (ask) prices at random times. The distributions of prices have unknown parameters, but agents know enough about the distribution of times to compute $\bar{N}(t) = EN(t)$, the expected number of new prices available
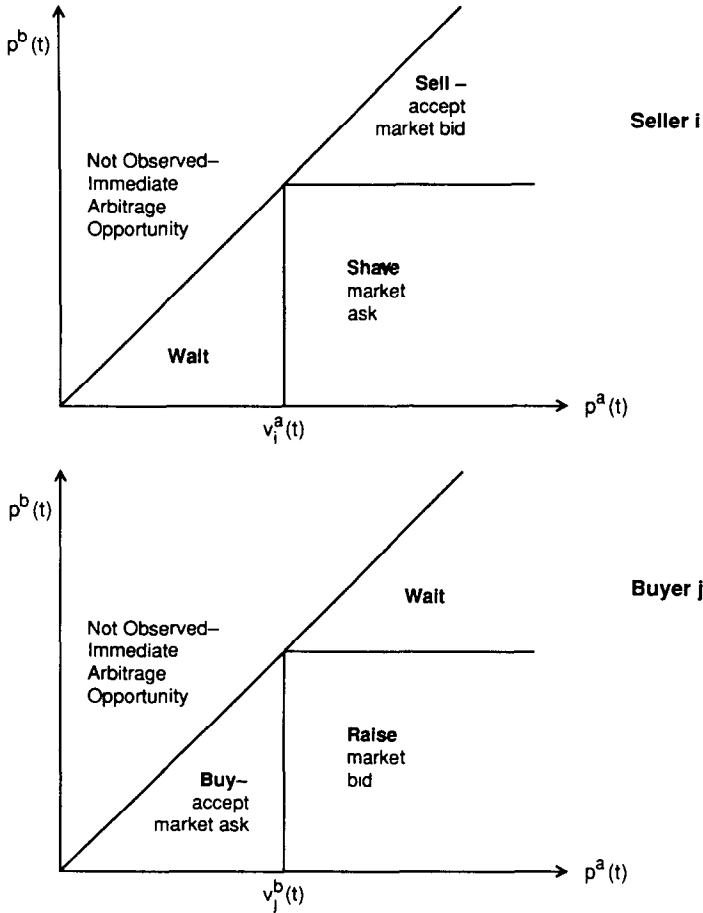
Aggressive Reservation Price Strategies

Fig. 1

to an agent before the end of the trading period. $\bar{N}$ is differentiable on $[0, T]$, with $\bar{N}' < 0$ and $\bar{N}(T) = 0$.

The first assumption was motivated in the introduction. We shall see that agents are interested primarily in the parameters of the process driving the *other side* of the market – the bid process for sellers and the ask process for buyers. Each seller (buyer) begins with a prior cumulative distribution function for new prices, denoted F_0^b (F_0^a), with positive densities on P. Such non-dogmatic priors allow agents to avoid the embarrassment of updating

after events of zero probability density and permit computation of the posterior CDFs F_t^b and F_t^a. As for BGAN3, the important point is that agents are aware that time is running out, i.e., the number of new prices $N(t) \rightarrow 0$ a.s. as $t \rightarrow T$. There are computational conveniences from having $\bar{N}(t)$ known and differentiable since in this case one can verify directly from the definition that $(N(0) - N(t))$ is a Poisson process with known (time dependent but bounded) intensity parameter. We shall exploit this last property in section 4.

We note in passing some less restrictive alternatives to BGAN1 and BGAN2, short of allowing true strategic interdependence. A seller, for instance, might notice that it is not his *current* perception of Nature's bid process that matters, but his perception *when he transacts*; the very act of transacting may alter his perception in a foreseeable manner. Thus, one might distinguish between the 'true' but unknown process envisaged by a seller, and perceived process which will vary with observations by Bayes' Rule. We will refer to the assumption that BGAN1 holds, except that it is 'true' rather than perceived process that remain unaffected, as BGAN1'. In a similar vein, BGAN2 does not explicitly recognize that 'Nature' will be better understood later in the trading round and expectations could take this into account, a possibility we refer to as BGAN2'. We will later return briefly and informally to these alternatives.

3. Basic results

Although detailed predictions of the model require further assumptions, we already have sufficient structure to obtain some general convergence and optimality results. We say a strategy $S \in \Sigma$ is *optimal* if no other admissible strategy yields higher expected utility at $t = 0$.

Proposition 1. Suppose an agent's prior expected trading profit is a continuous function on the set of admissible strategies. Then he has an optimal strategy.

Proof. Let $\phi(S) = E_{i0} = E_i(\pi | S)$ be the prior expected utility of profit under strategy S for agent i. Note that $U_i(\bar{C})$ is an upper bound for ϕ on the set of admissible strategies, so ϕ has a least upper bound $\bar{u}$ and therefore *near-optimal* strategies always exist – that is, for each $\varepsilon > 0$ there is some admissible S such that $\phi(S) > \bar{u} - \varepsilon$ – even if expectations are not continuous. Under present hypothesis, however, ϕ is continuous (since U_i and the expectation operator are continuous) so it suffices to show that the set of admissible strategies is compact. Note that the range of each strategy is $P \times Q$, a compact set, and the domain is contained in $H \times [0, T]$ so by

Tychonoff's theorem, the set of functions $(P \times Q)^{H \times [0, T]}$ is compact in the product topology. By definition, such a function $S: H \times [0, T] \to P \times Q$ is a strategy if it satisfies Assumptions 1 and 2, but set of functions satisfying these constraints is closed. We conclude that the set of admissible strategies is a closed subset of a compact set, and therefore compact. □

The continuity hypothesis of Proposition 1 is rather mild and will be satisfied by any reasonably well-behaved prior expectations. Certainly BGAN2 suffices (as we argue at the beginning of the next proof), but the hypothesis would also apply to most cases in which agents' expectations incorporate beliefs regarding other agents' strategies. Existence of optimal strategies (or best responses in the case of strategic interdependence) then is quite general, but so far one has little insight into their structure. It turns out this structure is quite simple under BGAN1–2 as we now demonstrate.

Proposition 2. Suppose BGAN1–2 hold. Then each agent has an aggressive reservation price strategy which is optimal.

Proof. Consider the case of a seller (index i suppressed for convenience); the case of a buyer is entirely analogous. First note that a small change in any A_t or a_t produces only a small change in $E_0 U(\pi)$ given non-dogmatic priors, so BGAN2 and Proposition 1 imply that an optimal strategy S^* exists. By Bellman's principle[6] one can characterize optimality by the inequality $E_t U(\pi | S^*) \geqq E_t U(\pi | S)$ for each $t \in [0, T]$ and each admissible strategy S that agrees with S^* up to time t. For arbitrary fixed $t \in [0, T]$, it therefore suffices to find some $V \in P$ such that an aggressive reservation price strategy employing this V satisfies the inequality.

If the seller has already sold his unit before time t, then $V = \bar{C}$ trivially yields an optimal reservation price strategy since no further trading profits are possible (recall that only one unit can be produced at finite cost). If he has not yet sold his unit, then let $\hat{S}_t$ represent the strategy of refusing to sell

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[6]Some technical justification for applying Bellman's principle in a continuous time stochastic setting may be in order. Rishell (1970) establishes a version of the principle [his eq. (28)] that applies even to 'adaptive' settings (e.g., our BGAN1'–2'). His key condition, called 'relative completeness', holds in our model as a simple corollary of Proposition 1. His version of Bellman's principle explicitly mentions an arbitrarily short time interval h; our assumption (Assumption 1) of right-continuity and piecewise constancy provides such a parameter since it implies that each ask is held for some minimum time before being altered. Rishel works with a Lagrange-type minimization problem, but conversion of such problems into the Mayer-type maximization problem we consider here appears standard [cf. Fleming and Rishell (1975)].

A simpler justification of our result may be found in Chow, Robbins and Siegmund (1971). It can be shown (pp. 113–118) that under BGAN3 our model reduces to a discrete-time problem, which (given BGAN1–2) satisfies the assumptions of their Theorem 4.5, establishing the existence of optimal reservation price strategies for acceptance of 'Nature's' bids (or asks).

at time t (e.g., setting $A_t = \varnothing$ and $a_t = \bar{C}$) and following S^* thereafter.[7] Let $g(p) = U(p-c) - E_t U(\pi | \hat{S}_t)$. The second term (representing the opportunity cost of selling at time t) is independent of p by BGAN1, but the first term (representing the gain from selling at price p) is strictly increasing in p. Since $U(\pi | \hat{S}_t)$ is a non-negative random variable, the second term is negative (or possibly zero for $t = T$), so $g(p) < 0$ for $p < c$. Now $g(p) > 0$ for p sufficiently large, i.e., for $p > c + \pi^*$, where π^* is the certainty-equivalent of $U(\pi | \hat{S}_t)$. Hence by monotonicity and the Intermediate Value Theorem, there is a unique V such that $g(V) = 0$, and $g(p) \geq 0$ if $p \geq V$. That is, the seller gains at time t if he sells at a price exceeding V, and loses if he sells at a lower price. In a Double Auction, there are only two ways for him to transact: to accept the market bid – so $A_t = [V, \bar{C}]$ is an optimal acceptance set – or to hold the market ask when some buyer accepts. Therefore, it is optimal for the seller to seize the market ask (when he does not already hold it and the market bid is unacceptable) if he can do so at a price of V or higher, given the assumption (BGAN1) that this will not damage his subsequent trading opportunities. Hence an aggressive reservation price strategy with $a_t = V$ is optimal.   □

The heart of this argument is that a buyer or seller optimally employs an intuitively plausible stopping rule of the form: 'stop (i.e., transact) if the current price realization is better than some threshold value V', where V depends on the amount of the remaining, the redemption value or cost of the unit, and risk preferences. We will compute such V's for the risk-neutral case in the next section.

However, it has been known for some time that such a stopping rule may not be optimal if the opportunity cost of transacting explicitly recognizes the foregone opportunity to acquire more information on the parameters,[8] as in BGAN1' and 2'. This is referred to in the optimal control literature as the 'dual control problem' [see e.g. Kumar (1985, page 342)].

Under BGAN1', a seller (for instance) would set his reservation price so that *if it were accepted*, his posterior distribution (after observing the realized bid exceeding his reservation price) would indicate that he had been wise to set his reservation price at that level.[9] That is, the seller anticipates how his posterior would respond to observing a bid realization of p, and uses this

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[7]Given our definition of admissibility (see previous footnote), this is equivalent to saying that, for some small time increment $h > 0$, $\hat{S}_t(\cdot, u) = S^*(\cdot, u)$ for $u \geq t + h$, but $\hat{S}_t(\cdot, u) = (\bar{C}, \varnothing)$ for $t \leq u < t + h$.

[8]Although not consistent with our assumptions, the following example makes the point clearly: a seller knows that Nature's bid distribution is *either* Uniform $[0, 1]$ *or* Uniform $[2, 3]$. Then he will accept a realization near 1 but reject a 'better' realization near 2.

[9]This is reminiscent of the 'winner's curse' problem in first-price sealed bid common-value auctions: the bidder there optimally must set his bid as if his information set included the knowledge his bid was going to be accepted (i.e., was the highest).
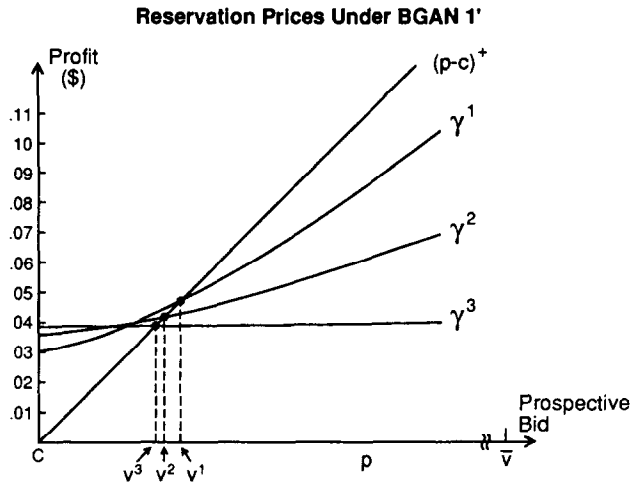
Reservation Prices Under BGAN 1'

Fig. 2

posterior distribution to compute his certainty-equivalent opportunity cost $\gamma(p) = U^{-1}(E_t U(\pi | \hat{S}_t, b(t+h) = p))$.

A formal analysis of BGAN1'-2' would take us very far afield, but a brief discussion might clarify a few issues. Three curves, calculated from this BGAN1' modification of the parametric model of section 4, are sketched in fig. 2: γ^1 arises from a diffuse prior for the location parameter of Nature's bid distribution, γ^2 from a less diffuse prior (after observing 5 bids) and γ^3 from a rather tight prior (10 bids observed). The reservation price V^3 is almost indistinguishable from that of Proposition 2, for which $\gamma(p)$ would be horizontal at $\pi^* = U^{-1}(E_t U(\pi | \hat{S}_t))$. Note that $V^2 > V^3$ and $V^1 > V^2$, reflecting the value of expected learning. One obtains simple adjustments to the reservation prices in this manner as long as $0 \leq \gamma' < 1$, but other cases are possible.[10]

We close this section with a brief investigation of the outcome of trade, indicating how our approach fits with previous work on the subject. Easley and Ledyard's behavioral rules are essentially aggressive reservation price strategies, so Proposition 2 provides choice-theoretic underpinnings to their

[10]For example, one might have an uncertain scale parameter as well as location parameter, in which case γ' might increase for large p and γ could cross $(p-c)^+$ a second time (from below) at, say $\bar{V}$. Then the acceptance set would be a finite interval $[V, \bar{V}]$ and no reservation price might exist. For very diffuse priors in the scale parameter, one might even have γ everywhere above $(p-c)^+$, in which case the acceptance set would be empty, since any bid, no matter how high, suggests the possibility that even better bids might be forthcoming. Finally, a bimodal likelihood function (as in footnote 7) could lead to a non-monotonic γ and an acceptance set with several disconnected components. However, the only cases we have actually encountered in our simulations appear to satisfy $0 \leq \gamma' < 1$.

approach. Their main proposition perhaps could be adapted to show in our setting that the market outcome converges to a competitive equilibrium under stationary repetition. There are several subtleties connected with stationary repetition in a Bayesian framework [see Harrison, Smith and Williams (1983)], so in the interests of simplicity we will not pursue such convergence issues here.

On the other hand, some of the single-trading round efficiency results of Friedman (1984) apply here. Note that the outcome of a perishables market is Pareto optimal if each buyer active at $t = T$ has a lower redemption value than the cost of any active seller, i.e., $\max\{d_j : j \text{ s.t. } 1_j(T) = 1\} \leq \min\{c_i : i \text{ s.t. } 1_i(T) = 1\}$, because otherwise gains from trade remain. Following Friedman (1984), call the outcome *almost Pareto optimal* if at most one transaction is required to make it Pareto optimal. Note that NCE is the key condition for an almost Pareto Optimal outcome [see footnote 3 above and Proposition 2 of Friedman (1984)]. It is straightforward to verify that NCE holds if sellers' (buyers') reservation prices are pursued aggressively and decay to $c_i(d_i)$ reasonably rapidly – linearly will certainly do. But under present assumptions the opportunity cost of waiting clearly decays at least a linear rate; for instance for seller i, with subjective CDFs F_t and $\hat{F}_t$ for the *next* and *best* market bids in the time remaining, one computes

$$\mathrm{E}_t U_i(\pi \,|\, \hat{S}_t) = \int_{c_t}^{\bar{c}} U(p - c_i)\, \mathrm{d}\hat{F}_t = \bar{N}(t) \int_{c_1}^{\bar{c}} U(p - c_i)\, \mathrm{d}F_t + o(T - t)$$

$$\leq k(T - t) \text{ for some constant } k > 0,$$

because by BGAN3, $\bar{N}(t) \to 0$ as $t \to T$ and $\bar{N}'(T)$ is finite. Consequently, $(V_{it} - c_i) < k_1(T - t)$, where $k_1 = \sup\{U'(\pi)^{-1} : \pi \in [0, \bar{C}]\}k$, and the following Proposition has been established.

Proposition 3. Suppose all agents pursue aggressive reservation price strategies and BGAN3 holds. Then the market outcome is almost Pareto optimal.

4. Computable models

Our main purpose in constructing the current model is to provide detailed predictions of events in a Double Auction market, so our general assumptions must be augmented by some parametric assumptions (henceforth PAs) that allow computation of the reservation prices. On grounds of tractability and reasonableness, we a priori select the following:

PA1. Agents are all risk neutral, so $U_i(\pi) = U_j(\pi) = \pi$ for $i = 1, \ldots, m$, and $j = 1, \ldots, n$.

PA2. $\bar{N}$ declines linearly in t, so $\bar{N}(t) = \bar{N}(0)(T-t)/T$, where $\bar{N}(0)$ is the average number of transactions per trading period in previous trading periods, and is a free parameter in the first trading period. That is, the intensity parameter in the Poisson process $[N(0) - N(t)]$ is the constant $\lambda = \bar{N}(0)/T$.

PA3. Buyers (sellers) have a common prior F_0^a (F_0^b) for the distribution of Nature's new asks (bids), say Normal with unknown mean m^a (m^b) and known standard deviation equal to the sample standard deviation in the previous trading period (a free parameter in the first trading period). The conjugate prior on m^a (m^b) is also Normal, initially diffuse (i.e., improper).

PA4. Unaccepted market bids (asks) that are shaved (raised) within one second or that are at least three standard deviations from the estimated mean are ignored. Otherwise the response lag δ is negligible; in particular, all other bids (asks) are immediately used to update the posterior distributions F_t^b (F_t^a).

A quick estimate of a seller's reservation price V may be obtained from the assumption that he (overoptimistically) believes he can recognize and accept the best bid in the time remaining, and regards the expected number $\bar{N}$ of remaining new bids as non-stochastic. Then, for F^b the current estimate of Nature's bid distribution, the best (highest) bid has distribution $\hat{F} = (F^b)^{\bar{N}}$, so the (risk-neutral) expected profit is approximately[11]

$$\pi^* = \int_{-\infty}^{\infty} (p-c)^+ \, \mathrm{d}\hat{F}^b(p) = \bar{N} \int_c^{\infty} (p-c)(F^b(p))^{\bar{N}-1} \, \mathrm{d}F^b(p), \tag{1}$$

and so $V = \pi^* + c$, where c is the seller's cost. Similarly, a quick estimate of a buyer's reservation price is $V = d - \pi^{**}$, where d is the buyer's redemption value and

$$\pi^{**} = \int_{-\infty}^{\infty} (d-p)^+ \, \mathrm{d}\hat{F}^a(p) = \bar{N} \int_0^d (d-p)(1 - F^a(p))^{\bar{N}-1} \, \mathrm{d}F^a(p), \tag{2}$$

since the best (lowest) ask has distribution $\hat{F}^a = 1 - (1 - F^a)^{\bar{N}}$. These approximations clearly are first-order in $\bar{N}$, a fact used in the proof of Proposition 3.

Precise numerical evaluation of the reservation price for all $t \in [0, T]$ can be obtained as follows. Since Proposition 2 proves a reservation price strategy is optimal, the price $V(t)$ can be characterized by a functional equation which equates the expected gross profit from following that strategy to the function $V(t)$ itself. This equation can be reduced to a differential

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[11]This π^* is DeGroot's (1970, chapter 11.8) transform of the extreme value distribution evaluated at c, i.e. $\pi^* = T_{\hat{F}}(c)$. This transform recurs later in our formula (3).

equation with boundary condition $V(T) = c$ for a seller (or $V(T) = d$ for a buyer). The differential equation can be solved and evaluated numerically. A seller's optimal reservation price thus can be written as $V(t) = \alpha^{-1}(\bar{N}(t))$, where α is a monotone increasing function defined in terms of the function

$$\phi(z) = \int_z^\infty (p - z)\, dF^b(p) > 0$$

by

$$\alpha(v) = \int_c^v dz/\phi(z). \tag{3}$$

Similarly, a buyer's optimal reservation price is $V(t) = \beta^{-1}(\bar{N}(t))$ where β is defined in terms of

$$\psi(z) = \int_0^z (z - p)\, dF^a(p)$$

by

$$\beta(V) = \int_V^d dz/\psi(z). \tag{4}$$

Appendix 1 contains the derivation of these equations for acceptance strategies.

It is not difficult to verify from (1) and (2), or from (3) and (4), that active sellers (buyers) with higher costs (redemption values) generally have higher reservation values; e.g., $c_1 \geqq c_2$ implies $V_1(t) \geqq V_2(t)$ for all $t \in [0, T]$. It would require differences in risk-preferences or in estimates of Nature's distributions, contra PA1 and PA3,[12] to reverse this conclusion. It is also clear that $V_i(t) \to c_i$ and $V_j(t) \to d_j$ as $t \to T$.

Note that formulas (1)–(4) require only the assumption of risk neutrality, and any other convenient assumptions that specify $\bar{N}(t)$ and F_t^a or F_t^b can be substituted for PA2–3. Risk aversion could be incorporated in a first-order approximation to V by deriving a coefficient of risk aversion for U and an approximate variance for π_t, but exact formulas seem harder to come by. In the context of experimental markets, our assumption (PA1) of risk-neutrality seems to us quite reasonable, since the stakes in each trading round are normally quite small relative to a subject's wealth, and many rounds are played.

One can check the robustness of V to our BGAN assumptions by computing V under BGAN1'. We use the following iterative procedure for

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[12] Actually BGAN1 permits agent-specific distributions (and therefore possible reorderings of reservation prices) inasmuch as an agent ignores his own bids and asks when he updates. This aspect is ignored for notational simplicity in this exposition.
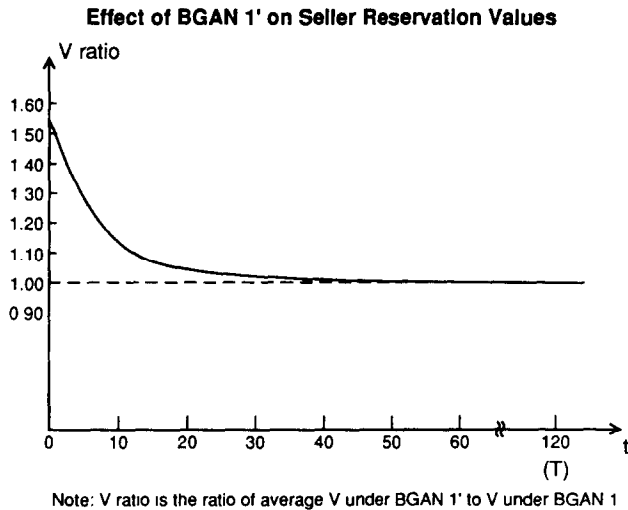
Effect of BGAN 1' on Seller Reservation Values

Note: V ratio is the ratio of average V under BGAN 1' to V under BGAN 1

Fig. 3

sellers and an analogous one for buyers. Let $F = F_t^b$ be the current perceived bid distribution, and let V be computed from eq. (3). Set $V^0 = V$. For $k = 1, 2, 3, \ldots$, use eq. (5) below to update F after observing a bid at V^{k-1}, and substitute this updated F into eq. (3) to obtain V^k. Proposition 2 and the remarks following it show that *if* an optimal reservation price exists under BGAN1' then it must satisfy $v = \gamma(v) + c$. If it exists, $V^* = \lim_{k \to \infty} V^k$ has this property by construction in the risk-neutral case. As can be seen from fig. 3, which is typical of the data we have examined, relaxing BGAN1 to BGAN1' under this procedure seems to have a negligible effect on $V(t)$ except for t near 0. In the latter case, one obtains more cautious strategies (higher V's for sellers, lower for buyers) than under BGAN1.

The other important computational aspect is the updating of perceived distributions, F^a or F^b. Under assumptions like PA3–PA4, the matter is quite straightforward when the ask or bid price is sharply observed. Winkler (1972, p. 169) gives the formulas for the PA3 case, for example, and also contains formulas for other parametric cases for which conjugate priors exist.

However, BGAN1 introduces a complication that may be important: sellers who see an acceptance of the current market ask $a(t)$ will infer a realization $p \geq a(t)$ from F_t^b that they do *not* sharply observe. Such observations are referred to in the literature as 'censored'. Bayes' theorem then requires that the conjugate prior density $g(m^b)$ be updated after an acceptance of $a(t) = a_0$ by the formula

$$g(m^b \mid p \geq a_0) = k(1 - F^b(a_0 \mid m^b)) g(m^b). \tag{5}$$

The constant k normalizes the expression to have integral 1, and in the likelihood function F^b is (under PA3) the normal CDF with mean m^b evaluated at a_0. Similarly, an accepted market bid of $b(t) = b$ leads buyers to update the conjugate prior density $h(m^a)$ for Nature's bids by the formula

$$h(m^a | p \leqq b_0) = kF^a(b_0 | m^a)h(m^a). \tag{6}$$

Another complication in updating arises from the so-called NYSE convention employed in many experimental markets. This convention allows only new offers that 'improve' the current market offer, in which case bids below $b(t)$ and asks above $a(t)$ are not observed ('missing' observations). Appendix 2 contains the relevant formulas.

5. Discussion

Following any bid, ask or acceptance, one can compute the posterior distributions F_t^a and F_t^b and the reservation prices $V_i(t)$ and $V_j(t)$ for active agents numerically from eqs. (3)–(6), augmented by the equations from Appendix 2 if the NYSE convention applies. Then one can check (see fig. 1) whether these reservation price strategies lead to immediate acceptance of the market bid or ask by any active agent, and whether aggressive strategies will lead any buyer to raise the market bid or seller to shave the market ask. If several buyers (or sellers) are predicted to raise (shave), then the model predicts that the one with the highest (lowest) reservation price will obtain the market bid (ask) at the level of the second highest $V_i(t)$ [second lowest $V_j(t)$]. If no immediate action is predicted for any agent, then one can fix the current distributions F_t^a and increment t in the $\bar{N}(t)$ formula (PA2) to deduce from eqs. (3) and (4) which agent is predicted to make the next bid, ask or acceptance and at what time. Thus the model generates specific predictions as to the time and nature of the next event in the Double Auction. Given a detailed log of a Double Auction experiment, these predictions can be tested, event for event, against the data.

Coarser tests of the model are also possible and may facilitate comparisons with existing models of the Double Auction. One can break the trading round into 'individual unit subauctions', aggregating all market bids and asks between one acceptance (or the start of the round) and the next acceptance (or the end of the round), and check whether an empirical model correctly predicts which agents seize the market bid and ask, what prices are posted, and which agents finally transact. A third and coarsest level of testing would summarize the whole trading round by the number of transactions and the distributions of bid, ask and transaction prices and compare these theoretical predictions to the data.

The present model can be expected to fit data from later trading periods of

standard Double Auction experiments rather closely. A more challenging task is to fit the more erratic data from earlier periods. We suspect that the present model will fare well here relative to more strategic models for reasons suggested in the introduction. For example, in otherwise standard experiments with all gains from trade going to sellers [the 'swastika' experiments of Smith and Williams (1988)], the model predicts approximately exponential convergence from below to the competitive equilibrium price of ask and transaction prices; and this seems to accord reasonably well with the data. On the other hand, experiments with random cost and redemption values in each period provide the most favorable environment for the Wilson (1987) and related models. Nevertheless, the present model could do relatively well even in this setting.

Many of the arbitrary parametric assumptions can be altered in response to systematic prediction errors. For instance, a log-normal, gamma or beta distribution for F^a and F^b is computationally feasible.[13] One could postulate that realizations of Nature's random processes are serially correlated, and introduce another free parameter. For this reason and others, one might also wish to examine specifications for $\bar{N}(t)$ other than PA2. Similarly, PA1 could be altered to incorporate risk aversion.

Most experiments allow agents to transact several units of the good. The present model could be tested directly against such data under the convention that a seller with costs (say) c_1 and c_2 for his first and second units is treated as two agents with costs c_1 and c_2, etc. This convention ignores the fact that a seller who sells his first unit now can contemplate selling his second unit at the highest (rather than second-highest) subsequent market bid. Such considerations will lower the opportunity costs of transacting early and therefore reduce the difference between reservation values and endowed parameters (c_i and d_j). However, this effect is probably small except when $\bar{N}(t)$ is small (i.e., except late in the trading round) and may be ignored in initial tests of the model. Similarly, the effects of risk preferences (including wealth effects in the multiple unit case) are probably small enough to be ignored in initial tests.

Some experiments ('asset markets') allow agents to participate on both sides of the market [see Forsythe, Palfrey and Plott (1982), Plott and Sunder (1982) and Friedman, Harrison and Salmon (1983, 1984) for illustrations]. Several significant modifications to the present model, such as agent-specific distributions and no-self-arbitrage constraints, seem important in that context, as discussed in Friedman and Harrison (1984) and Friedman, Harrison and Salmon (1984, Appendix A). Although one sees the Double Auction

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[13]Such distributions assign zero probability to negative prices and therefore may be preferred on a priori grounds. The conventions suggested in section 4 on missing and censored observations reduce the importance of this property, as well as the importance of serially correlated prices noted below.

institution employed more often in ongoing markets for assets than for perishables, there are more experimental data and more theoretical models for the perishables case so it probably should be examined first. Once an empirically-based consensus arises as to the best way to model price formation in Double Auction laboratory markets for perishables, it can be further tested against behavior in laboratory asset markets[14] and then perhaps in some of the major ongoing asset markets.

Appendix 1: Derivation of reservation prices

Let $\underline{N}(t)$ here denote the number of new bid prices observed by sellers up to time $t \in [0, T]$. By BGAN3, $\underline{N}(t)$ is a Poisson process, i.e.,

$$\Pr(\underline{N}(t+h) = \underline{N}(t) + 1] = \lambda h + o(h) \quad \text{as} \quad h \to 0, \tag{A1.1}$$

and $\Pr[\underline{N}(T) < \infty] = 1$. By a suitable change of variables (a simple change of scale, $t \to t' = t \, E_0 \underline{N}(T)/T$, under PA2) we may assume wolog that $\underline{N}(t)$ has constant intensity $\lambda = 1$. Let $\tau_0, \tau_1, \ldots, \tau_{N(T)}$ denote the arrival times for new prices, i.e., $\tau_0 = 0$ and

$$\tau_{n+1} = \{\inf t \geqq \tau_n : \underline{N}(t) > \underline{N}(\tau_n) = n\}.$$

Since the distribution $F = F^b$ of new bid prices b_n has finite mean μ and support in $[0, \infty]$, we have

$$E_0 \sup \{b_n : n = 1, \ldots, N(T)\} \leqq E_0 \sum b_n = T\mu < \infty.$$

Let $\underline{A}$ be the class of 'stopping rules', i.e., random variables $\underline{s}$ with values in $\{1, 2, \ldots, \underline{N}(T) + 1\}$ for which the event $[\underline{s} = n]$ belongs to the Borel σ-algebra generated by $\underline{N}(u)$, $0 \leqq u \leqq \tau_n$ and $b_1, \ldots, b_n$. (Intuitively, this last is a no-clairvoyance condition: the decision to stop at n, i.e., accept b_n, depends only on past observations. $[\underline{s} = \underline{N}(T) + 1]$ denotes the event that the seller does not transact.) Let $\underline{A}(t) = \{\underline{s} \in \underline{A} : \Pr[\underline{s} > \underline{N}(t) = 1\}$ denote the class of stopping rules that have not yet stopped by time $t \in [0, T]$. For

[14]There are several other nonstandard laboratory settings in which it would be interesting to test extensions of the present model if it does well in standard settings. When only one multiunit seller (or buyer) is present the results of section 3 are no longer valid. Nevertheless one could assume (now with negligible theoretical support) that traders employ aggressive reservation price strategies and use the parametric model to predict outcomes. The fit might not be too bad, given the results summarized in Smith (1982). A more speculative extension would be to the multiperiod asset markets of Smith, Suchanek and Williams (1988). One could try to explain their bubbles by applying Bayesian updating techniques to price changes (rather than price levels). A more theoretically justified extension which does not predict bubbles is sketched in Appendix A of Friedman, Harrison and Salmon (1984).

$n=1,\ldots,\underline{N}(T)$ let $\pi_n = b_n - c$ denote the profit from accepting the nth bid realization, and let $\pi_0 = 0 = \pi_{\underline{N}(T)+1}$. For $\underline{s} \in \underline{A}$, define the random variable $\pi_{\underline{s}}$ by $\pi_{\underline{s}} = \pi_n$ whenever $[\underline{s} = n]$.

We may now define

$$V(t) = \underset{\underline{s} \in \underline{A}(t)}{\mathrm{ess\ sup}}\, E\pi_{\underline{s}} + c.$$

Note that $V(T) = c$, and that $V(t)$ is non-increasing and differentiable as a result of (A1.1). Also, $V(0) \leq E_0 \sup b_n + c < \infty$. Let $\underline{\sigma} \in \underline{A}$ be the reservation price stopping rule using $V(t)$, i.e., $\underline{\sigma} = $ first n such that

$$b_n \geq \underset{\underline{s} \in \underline{A}(\tau_n)}{\sup}\, E\pi_{\underline{s}} + c = V(\tau_n).$$

Theorem 4.5 or 5.2 of Chow et al. (1971) proves that $\underline{\sigma}$ is optimal. Following their argument on pages 113–118, we note that V must satisfy the functional equation

$$V(t) = E\pi_{\underline{\sigma}(t)} \tag{A1.2}$$

where $\underline{\sigma}(t) \in \underline{A}(t)$ is the rule $\underline{\sigma}(t) = $ first $n > N(t)$ s.t. $\pi_n \geq V(\tau_n)$.

We expand (A1.2) by writing $E\pi_{\underline{\sigma}(t)}$ as the integral over $u \in [t, T]$ of the *expected profit conditioned on trading at time u* times the *probability density of trading at time u* under $\underline{\sigma}(t)$. The first factor in the integrand is:

$$\int_{V(u)}^{\infty} (b - c)\, dF(b)/(1 - F(V(u))).$$

The density we want is $-y_t'(u)$, where $y_t(u) = \Pr[\tau_{\sigma(t)} > u]$. It may be calculated by observing from (A1.1) and from the definition of $\underline{\sigma}(t)$ that

$$y_t(u+h)/y_t(u) = \Pr[\tau_{\sigma(t)} > u + h \mid \tau_{\sigma(t)} > u] = 1 - h(1 - F(V(u))) + o(h),$$

so $y_t'(u)/y_t(u) = F(V(u)) - 1$. Since $y_t(t) = 1$, we obtain on integration

$$y_t(u) = \exp\left\{\int_t^u (F(V(z)) - 1)\, dz\right\}.$$

Consequently, (A1.2) may be written as the integral equation

$$V(t) = \int_t^T g(u, t) \, \mathrm{d}u + c,$$ (A1.3)

where

$$g(u, t) = \left\{ \int_{V(u)}^{\infty} (b - c) \, \mathrm{d}F(b)/(1 - F(V(u))) \right\} \left\{ -(F(V(u)) - 1) \right.$$

$$\times \exp \left\{ \int_t^u (F(V(z)) - 1) \, \mathrm{d}z \right\} \right\}$$

or, simplifying.

$$g(u, t) = \int_{V(u)}^{\infty} (b - c) \, \mathrm{d}F(b) \exp \int_t^u (F(V(z)) - 1) \, \mathrm{d}z.$$ (A1.4)

Differentiating (A1.3) one obtains

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -g(t, t) + \int_t^T g_2(u, t) \, \mathrm{d}u$$

$$= -\int_{V(t)}^{\infty} (b - c) \, \mathrm{d}F(b) \exp(0) + \int_t^T g(u, t) \, \mathrm{d}u (1 - F(V(t)))$$

$$= -\int_v^{\infty} (b - c) \, \mathrm{d}F(b) + (V - c) \int_v^{\infty} \mathrm{d}F(b) = -\int_v^{\infty} (b - V) \, \mathrm{d}F(b).$$

Therefore, defining $\phi(z) = \int_z^{\infty} (b - z) \, \mathrm{d}F(b)$, we obtain

$$\mathrm{d}V/\mathrm{d}t = -\phi(V).$$ (A1.5)

Since, as previously noted, $V(T) = c$ and V is monotone decreasing, we can integrate its inverse function $u(V)$ from T (back) to t and obtain

$$T - t = -\int_{V(T)}^{V(t)} \frac{\mathrm{d}u}{\mathrm{d}z} \, \mathrm{d}z = +\int_c^V \frac{1}{\phi(z)} \, \mathrm{d}z.$$ (A1.6)

Finally, defining $\alpha(V) = \int_c^V (1/\phi(z)) \, \mathrm{d}z$ and inverting (A1.6), we obtain

$$V(t) = \alpha^{-1}(T - t).$$ (A1.7)

If the arrival intensity of new prices is not unity, we can replace $T - t$ by the expected number of such new prices in the time remaining, denoted $\bar{N}(t)$ in the text, and obtain

$$V(t) = \alpha^{-1}(\bar{N}(t)). \tag{A1.7'}$$

An analogous argument for Nature's ask process as observed by a buyer yields the differential equation and boundary condition

$$\mathrm{d}V/\mathrm{d}t = \psi(V), \qquad V(T) = d, \tag{A1.8}$$

where ψ is the transform of the perceived ask distribution $F(a) = F^a(a)$ defined by $\psi(z) = \int_0^z (z - a)\,\mathrm{d}F(a) \geq 0$, and d is the buyer's redemption value. Since V is therefore increasing, we may write in the case of unit arrival intensity

$$T - t = \int_{V(t)}^{V(T)} \frac{\mathrm{d}u}{\mathrm{d}z}\,\mathrm{d}z = \int_V^d \frac{1}{\psi(z)}\,\mathrm{d}z = \beta(V) \tag{A1.9}$$

and obtain

$$V(t) = \beta^{-1}(T - t). \tag{A1.10}$$

In more general case of non-unit arrival intensity we have

$$V(t) = \beta^{-1}(\bar{N}(t)). \tag{A1.10'}$$

Eqs. (A1.7') and (A1.10') can be evaluated numerically.

Appendix 2: Bayesian updating procedures

In this appendix we provide explicit procedures for updating conjugate priors and predictive PDF's under the 'NYSE convention' that bids (asks) which do not better the market bid (ask) are *not* transmitted to market participants.

Let $a_0 = a(t)$ be the current market ask and $b_0 = b(t)$ be the current market bid at some fixed $t \in [0, T]$. Then there are six possible 'Natural' events that might occur: a bid $p \geq a_0$, which registers as an accepted market ask; a bid $p \in (b_0, a_0)$, which registers sharply as a new bid; a bid $p \leq b_0$, which is not observed; an ask $p \leq b_0$, which registers as an accepted market bid; an ask $p \in (b_0, a_0)$, which registers sharply as a new market ask; and finally an ask $p \geq a_0$, which is not observed under the NYSE convention. See fig. 4 for a depiction of these six cases.

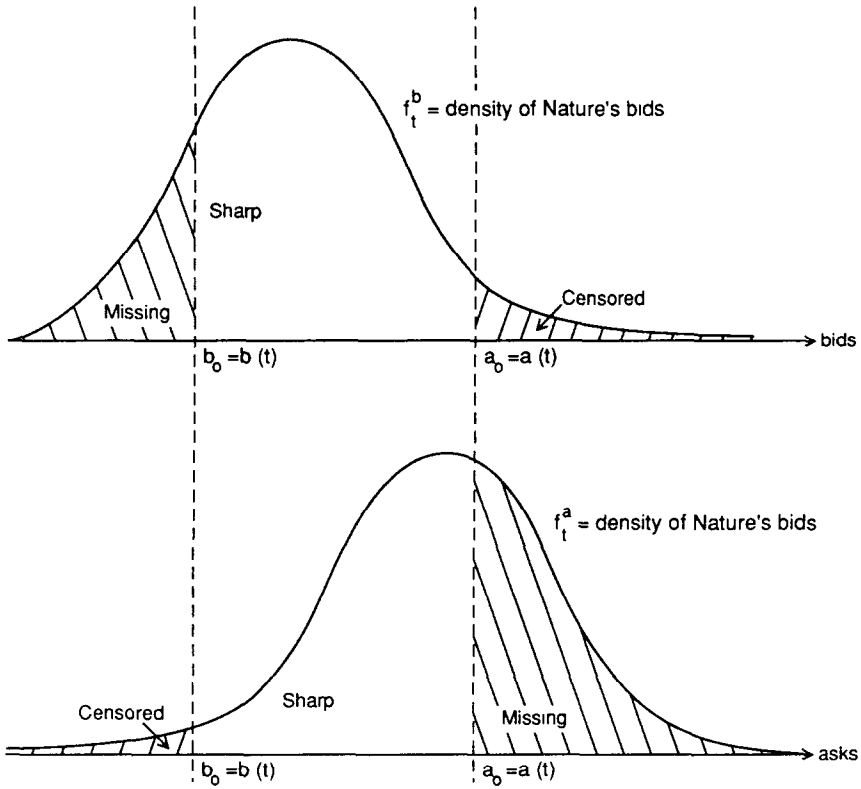The likelihood functions for the four observable events must take into

Fig. 4

account the 'missing' or unobservable events that might have occurred. Essentially, one must truncate the relevant probability distributions and renormalize. Let m^b be the unknown parameter (or parameter vector) for 'Nature's true' bid distribution F^b (whose density f^b exists by BGAN2), and let m^a be the unknown parameter for the ask distribution F^a (density f^a). Then the likelihood functions for observed bids are:

$$H(m^b|a_0, b_0) = (1 - F^b(a_0|m^b))/(1 - F^b(b_0|m^b)) \qquad (A2.1)$$

in case of an accepted market ask, or

$$H(m^b|p, b_0) = f^b(p|m^b)/(1 - F^b(b_0|m^b)) \qquad (A2.2)$$

in the case of an observed new bid $p \in (a_0, b_0)$. Similarly, the likelihood functions for observed asks are:

$$H(m^a | a_0, b_0) = F^a(b_0 | m^a) / F^a(a_0 | m^a) \qquad (A2.3)$$

if the ask is inferred from an accepted market bid, or

$$H(m^a | p, a_0) = f^a(p | m^a) / F^a(a_0 | m^a) \qquad (A2.4)$$

in the case of an observed new market ask $p \in (a_0, b_0)$.

One then can use the likelihood functions in the usual manner to obtain updated conjugate priors on m^b (or m^a) and predictive distributions F^a and F^b. Specifically, if $g(m^b)$ is the conjugate prior density on the unknown bid parameter m^b then its updated version after observing an event giving the likelihood function $H(m^b)$ is $k H(m^b) g(m^b)$, where $k^{-1} = \int_{-\infty}^{\infty} H(m^b) g(m^b) \, dm^b$. The formula for updating the conjugate prior for m^a is exactly similar. The updated predictive density function for bids is then obtained from the updated conjugate prior, still denoted by $g(m^b)$, and the (conditional) bid density $f^b(\cdot | m^b)$ by integration:

$$f^b(b) = \int_{-\infty}^{\infty} f^b(b | m^b) g(m^b) \, dm^b. \qquad (A2.5)$$

Similarly,

$$f^a(a) = \int_{-\infty}^{\infty} f^a(a | m^a) g(m^a) \, dm^a \qquad (A2.6)$$

is the predictive density for 'Nature's next ask'. In view of assumption BGAN2, the distributions referred to in the paper as F^a and F^b are the CDFs corresponding to the densities obtained in (A2.5) and (A2.6).

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