3. Cost and Technology

Varian, Chapters 1-5

I. Describing the Firm

A. The neoclassical description of the firm is really just a description of the firm's production possibilities.

1. Which outputs can be obtained from given inputs
2. How much has to be expended to get those inputs
3. How these two factors generate cost curves for the firm.
4. These cost curves themselves completely describe *everything we need to know about the firm*, if we are neoclassical.

B. Input/Output

1. The firm produces a vector $\mathbf{y}$ of product quantities.
   a. We'll usually focus on a firm with a single product with quantity y.
2. The firm has a set of inputs it can use to create these products.
   a. We describe these inputs as a vector (or a bundle) $\mathbf{x} = (x_1, x_2, ..., x_n)$.
      - The vector components are the quantities of each input being utilized in production.

C. Technology

1. What output y can the firm produce with a bundle $\mathbf{x}$?
2. This is described by the firm's technology.
3. The **input requirement set** V(y) consists of all of the bundles $\mathbf{x}$ that can produce output quantity y.

   Ex: Activity analysis and production plans.
   Basically, recipes. For spaghetti sauce, for 16Gb memory chips, for 100 rides to SFO, ...

4. The **production function** $y = f(\mathbf{x})$ describes the maximum output that can be produced with any input bundle .

   Ex: Cobb-Douglas technology, $y = a_o x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

   Ex: Leontief technology , $y = \min\{a_1x_1, a_2x_2, ..., a_nx_n\}$.

5. The **isoquant** for given output level y^* is the set of input bundles that can produce y^* .

   Analogous to an indifference curve, but the label here (y^*) is meaningful.

D. Common assumptions about technology

1. Monotone
   a. More input enables at least as much output.
   b. Say this using V's: if you can produce y' with $\mathbf{x}$ you can still produce y' with a bigger bundle $\mathbf{x}' > \mathbf{x}$.
   c. This is innocuous if extra inputs can be thrown away, "free disposal."
2. Convex
   a. If plans $\mathbf{x}$ and $\mathbf{x}'$ are in $V(y)$ (i.e. can produce y), then so is the mixture $\alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$, for any mixing proportion $0 < \alpha < 1$.
   b. If a production plan can be replicated, then it is reasonable to say that the technology is convex.

      Ex: Replicating two production plans to create any convex sets.
3. Non-empty
   a. With enough of and the right kinds of inputs, you can create any level of output y.
4. Closed: a boring technical condition.

E. Trade Offs in Production Plans

1. Assume we have a "smooth" production technology.

2. At what rate can we substitute one of our inputs for another in producing a particular output, y?

   a. Called the **technical rate of substitution**.

   b. With a smooth technology with two inputs, it is just the slope of the isoquant.

   c. TRS is a direct analogue of MRS.

   Ex: Using the production function and the **implicit function theorem** to find the technical rate of substitution.

$$TRS_{ij} = \frac{dx_j}{dx_i}|_{[f(\cdot)=y^*]} = -\frac{mp_i}{mp_j} = -\frac{\partial f}{\partial x_j}/\frac{\partial f}{\partial x_j} \tag{1}$$

   Ex: A Cobb-Douglas example.

3. Elasticity of substitution σ is elasticity of $[x_j/x_i]$ wrt $|TRS|$. It is a measure of isoquant curvature. See Varian for ugly details (optional).

F. Returns to Scale

1. The returns to scale tells us what happens when we try to scale up a production plan.

2. If we multiply $\mathbf{x}$ by $t > 0$, what happens to y?

3. Three cases:

   a. Constant returns to scale.

      - If $y = f(x_1, x_2)$, then for $f(tx_1, tx_2) = ty$
      - Output is proportional to the inputs.

   b. Increasing returns to scale.

      - If $y = f(x_1, x_2)$, then $f(tx_1, tx_2) > ty$ for $t > 1$.
      - We get more bang for our buck (at fixed prices of course) at higher scales of production.

- May be inherent in a technology or possibly related to learning from doing.

c. Decreasing returns to scale.

- If $y = f(x_1, x_2)$, then $f(tx_1, tx_2) < ty$ for $t > 1$.
- We get diminishing returns from scaling our plans up.
- A major reason for DRS: there is some fixed input (not in the list), such as CEO attention, or planetary resources, or ...

Ex: Cobb-Douglas and returns to scale.

4. Homogeneity degree 1 as CRS. Homogeneous functions of degree $d = 0, 1, ...$

G. Long run and short run

1. Short run means that at least one component of the input bundle $\mathbf{x}$ is fixed.

   a. Note that (for standard production functions) this implies DRS at large scale in the short run.

2. Long run means that all inputs are choice variables for the firm.

II. Cost Minimization

A. Behavior of the firm

1. We assume that firms economize in production.
2. Assume they choose technologies which minimize the cost of producing their output.
3. When is this reasonable to assume?

B. The firm's problem.

1. To derive cost function, take as given the desired output quantity y, and the input price vector $\mathbf{w} = (w_1, w_2, ..., w_n)$. Sometimes also called factor prices.
2. Firms choose an input bundle $\mathbf{x}$.

a. For convenience we will often write $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_1, w_2)$, but the reasoning extends to any finite vector of inputs.

3. The firm's main constraint (aside from factor prices) is technological.

   a. Can be summarized with the production function: $y = f(x_1, y_2)$.

4. So the firm's problem is simply:

$$c(\mathbf{w}, y) = \min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2 \text{ s.t. } f(x_1, x_2) = y \tag{2}$$

5. The four conditions noted earlier, including a strict version of convexity, allow us to say that if the problem above has an interior solution x^*, then it is unique and is characterized by the first order conditions.

6. The same algebra and calculus used previously show that the first order conditions for this problem state that the technical rate of substitution is equal to the ratio of the factor prices: $|TRS_{ij}(x^*)| = \frac{w_i}{w_j}$.

7. The intuition is appealing: at the optimum input vector x^*, the isocost curve has the same slope (i.e., market tradeoff rate given by the price ratio) as the isoquant (the production tradeoff rate, TRS).

C. Conditional factor demand

1. The firm's cost minimizing problem yields the firm's demand for each input as a function of prices and the scale of output.

2. Conditional factor demand for input i is $x_i^*(w_1, w_2, y)$.

D. The cost function

1. Cost functions represent the lowest cost of production available to a firm at a given set of factor prices. So we can rewrite equation (2) as

$$c(\mathbf{w}, y) = \mathbf{w} \cdot \mathbf{x}^*(\mathbf{w}, y) \tag{3}$$

2. With only two factors: $c(\mathbf{w}, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$

Ex: Constant Elasticity technology.

Special cases: Cobb-Douglas technology, Leontief technology, Linear technology

E. Relationship between cost and conditional factor demand

1. If the cost function is differentiable, then you can use it to recover the input (or factor) demand functions.

2. This is known as **Shephard's lemma**: $x_i^*(\mathbf{w}, y) = \frac{\partial c(\mathbf{w},y))}{\partial w_i}$.

3. To verify, differentiate equation (3) wrt w_i. The main effect is as in Shepard's lemma, but there is also an indirect effect $\mathbf{w} \cdot \frac{\partial x_i^*(\mathbf{w},y))}{\partial w_i}$. By the FOC, this indirect effect is proportional to $(mp_1, ...mp_n) \cdot \frac{\partial x_i^*(\mathbf{w},y))}{\partial w_i} = 0$, as can be seen by differentiating the isoquant identity $f(\mathbf{x}^*(\mathbf{w}, y)) = y$.

   Geometrically, the idea is that the production function gradient (i.e., marginal product vector) is normal to (aka orthogonal or perpendicular to) the isoquant surface, and therefore also normal to the isocost surface, so the indirect effect is zero. This is an example of the envelope theorem. See Varian p. 74 for further discussion.

F. Duality and properties of cost functions

1. Suppose that you have a function $c(\mathbf{w}, y)$ with the following properties

   a. monotone increasing in each argument,

   b. homogeneous degree 1 in $\mathbf{w}$,

   c. concave in $\mathbf{w}$, and

   d. continuous and (at least piecewise) differentiable

   Then there is some nice [monotone, ..., closed] production function for which c is the cost function!

   Conversely, if c is the cost function for some nice production function, then it satisfies the four properties just mentioned.

2. Implications for applied work:

   (a) usually you can skip estimating a production function, especially if not all data on input quantities are available, and just estimate the cost function directly, using data on input prices and output levels, which usually is easier to collect.

   (b) when estimating a cost function, consider imposing homogeneity and monotonicity as coefficient restrictions, and testing for concavity.

Ex: CES cost function

Ex: Translog cost function

III. Cost Curves

A. Cost curves

1. It turns out we can summarize all of the important (non-behavioral) facts about the firm using the cost functions we derived in the last section.
2. The main ideas come through most clearly when we assume just two inputs to production which we will rename f (for fixed in SR) and v (for freely variable).
   a. $\mathbf{x} = (x_f, x_v)$
   b. $\mathbf{w} = (w_f, w_v)$

B. The short run cost curve

1. In the short run, some of the costs are fixed. That is, the firm is unable to mitigate costs by varying all of its inputs.
   a. In the short run we can't buy new machinery, build new factories, hire new management, or change union contracts.
   b. Those costs are fixed – we represent them as F.
2. Total costs incurred by the firm consist of both variable costs and fixed costs (which are in turn simple to express in the two inputs model).

a. Variable Cost: c_v

- $c_v(y) = w_v x_v(w_v, y, x_f)$

Ex: Variable Cost Curve

b. Fixed Cost: F

- $F = w_f x_f$

Ex: Fixed cost curve

c. Total Cost: $c_v(y) + F$

- $c(y) = w_v x_v(w, y, x_f) + w_f x_f$

Ex: Total cost curve

3. Average Costs

a. There are three main types of average costs that are used in studying firm behavior.

- Average Total Cost: $AC(y) = \frac{c(y)}{y} = \frac{c_v(y)+F}{y}$
  - U-shaped
- Average Variable Cost: $AVC(y) = \frac{c_v(y)}{y}$
  - Eventually rising
- Average Fixed Cost
  - Always falling

b. The three average cost functions are related by a simple equation:

c. $AC(y) = AFC(y) + AVC(y)$

Ex: Average cost curves.

4. Marginal Costs

a. $MC(y) = \frac{\partial c}{\partial y} = \frac{\partial c_v}{\partial y}$

b. Relationship between MC and AC.

- When AC is decreasing, MC must be smaller than AC.
- When AC is increasing, MC must be larger than AC.
- MC intersects AC at the minimum point of the AC.
  - Minimum efficient scale
- The same must be true with AVC!
- Integral of (area under) MC gives VC.

Ex: Some cost curves.

C. Long run cost curve

1. As we saw in the last section, decreasing returns set in when a factor is fixed.
2. Eventually, the cost of short run factors force the firm to change long run factors.
3. Relationship between long and short term curves

a. Suppose costs are given as a function of output and the corresponding demand for the fixed factor, $x_f(y)$: $c(y, x_f(y))$.

b. Fix y at some level $\bar{y}$ and refer to $\bar{x}_f$ as the optimal amount of the fixed factor for producing $\bar{y}$.

c. Now lets differentiate $c(\bar{y}, x_f(\bar{y}))$ with respect to y at $\bar{y}$.

- This gives us the following:

$$\frac{dc(\bar{y},x_f(\bar{y}))}{dy} = \frac{\partial c(\bar{y},\bar{x}_f)}{\partial y} + \frac{\partial c(\bar{y},\bar{x}_f)}{\partial x_f}\frac{\partial x_f(\bar{y})}{\partial y}$$

d. **Envelope theorem**

- The conditional factor demand, x_f is the cost minimizing factor choice for producing $\bar{y}$.
- Thus $\frac{\partial c(\bar{y},\bar{x}_f)}{\partial x_f} = 0$

- Plugging this into the math above we get a great expression for the relationship between long run cost and short run cost.

$\frac{dc(\bar{y}, x_f(\bar{y}))}{dy} = \frac{\partial c(\bar{y}, \bar{x}_f)}{\partial y}$

- That is the slope of long run cost equals the slope of short run cost! (at the point of tangency)
- The long term costs are the lower envelope of all of the short run costs at various levels of production.

Ex: Multiple plants