Contents

1 Math Notation

Notation	*Meaning*
$\Longrightarrow$	implies.
$\Leftrightarrow$	implies and is implied by, also denoted iff (if and only if)
$\rightarrow$	denotes a mapping from the set on the left hand side (LHS) to the set on the right hand side (RHS).
$\Re$	the set of all real numbers.
$\exists$	there exists.
$\forall$	for all.
$\succ$	is strictly preferred to.
I_e	the indicator function for event e, so $I_e = 1$ in event e and otherwise $I_e = 0$.
$s \in S$	state s is an element of the set of possible states S.
	Example: $S = \{\text{rain, sun}\}$, $s = \text{rain}$.
π	probability distribution, $\pi : S \rightarrow [0, 1]$.
	Example: $\{\pi(\text{rain}) = 0.3, \pi(\text{sun}) = 0.7\}$
E_π	expected value over the probability distribution given by π
$a \in A$	action a is an element of the set of possible actions A.
	Example: $A = \{\text{carry umbrella, don't carry umbrella}\}, a = \text{carry umbrella}$.
$x \in X$	outcome x is an element of the set of possible outcomes X.
	oucome depends on state and action $(a : S \rightarrow X)$.
	Example: $x = $ carrying umbrella on a sunny day.
U	utility function, $U : X \rightarrow \Re$.
	For two possible outcomes $y, z \in X$, $y \succ z \Leftrightarrow U(y) > U(z)$.
	Preferences map outcomes to the set of real numbers via a utility function.
U_s	state dependent utility function, $U_s : \{S, X\} \rightarrow \Re$

2 Static Decision Theory

Decision theory answers the basic question: "Which action $a \in A$ to choose?"

Begin with the simplest setting:

- a single once-and-for-all decision,
- among a finite set of alternative actions $A = \{a_1, ..., a_M\}$,
- given a finite number of alternative states $S = \{s_1, ..., s_N\}$
- with known probabilities $(p_1, ..., p_N)$.

The actions $a \in A$ represent alternative choices with typically uncertain outcomes, such as which pension plan to buy, if any, or where to live while looking for a job. The states $S \subset \Re^m$ represent uncertain events that the decider cares about, for example political outcomes; other examples include health events, or tomorrow's weather, or possible job offers. A leading example is that the states $S \subset \Re$ are possible wealth increments or, a bit more generally, the monetary values that the decider assigns to possible outcomes. In that case, the choice is over monetary gambles, or "lotteries."

In terms of notation introduced earlier, we have $p_i = \pi(s_i)$. Of course, the probabilities must be non-negative and sum to 1.0. The set of all possible probability vectors is called the *N-simplex* and denoted $\mathbf{\Sigma}^N = \{(q_1, ..., q_N) : q_i \geq 0, q_1 + ... + q_N = 1\}$.

The goal for the next several subsections is to formulate the choice problem in the wealth increments case (the results generalize, but this case is the most useful), and to show how expected utility theory answers the "basic question."

2.1 Preferences over lotteries

Define a *lottery* $L = (S, P)$ as a finite list of monetary outcomes $S = \{s_1, s_2, ..., s_N\} \subset \Re$ together with corresponding probabilities $P \in \mathbf{\Sigma}^N$. The *space of all lotteries* is denoted

$\mathcal{L}$.

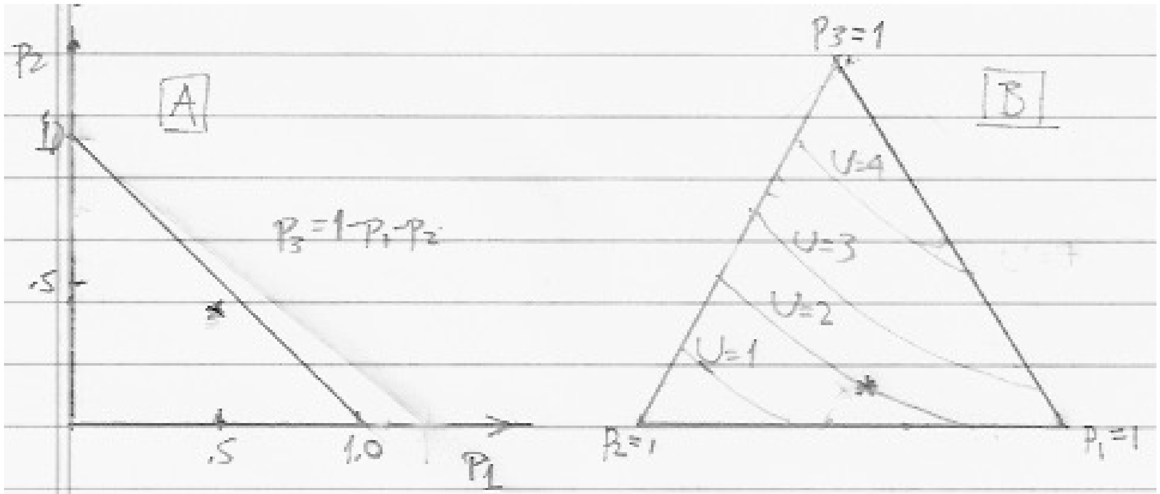

Figure 1: The N-Simplex for $N = 3$. These represent all lotteries over given states $s_2 < s_1 < s_3 \in \Re$. Panel A shows the first two probabilities using the usual coordinate system, while Panel B uses barycentric coordinates (in which lines of constant p_i are parallel to the edge where $p_i = 0$). The point $p^* = (.5, .4, .1)$ is marked by an asterisk (*) in both panels. Indifference curves for some (non-Bernoulli) utility function U are shown in Panel B.

The *expected value* of lottery $L = (S, P)$ is $E_P S = \sum_{i=1}^{N} p_i s_i$. For example, the lottery $(S, P) = ([0, -1, 10], [.5, .4, .1])$ has expected value $.5(0) + .4(-1) + .1(10) = 0.6$.

A utility function over monetary outcomes (henceforth called a *Bernoulli function*) is a strictly increasing function $u : \Re \rightarrow \Re$. Given Bernoulli function u, the *expected utility* of lottery $L = (S, P)$ is $E_P u = \sum_{i=1}^{k} p_i u(s_i)$. For example, if $u(x) = 1 - \exp(-2x)$, then the expected utility of the lottery just mentioned is $1 - [.5e^0 + .4e^2 + .1e^{-20}] \approx -2.46$.

Preferences $\succeq$ over any set refer to a complete and transitive binary relation. A utility function U *represents* preferences $\succeq$ if $x \succeq y \iff U(x) \geq U(y) \quad \forall\, x, y$. In particular, a utility function $U : \mathcal{L} \rightarrow \Re$ represents preferences $\succeq$ over $\mathcal{L}$ if

$$L \succeq L' \iff U(L) \geq U(L') \quad \forall\, L, L' \in \mathcal{L}. \tag{1}$$

Preferences $\succeq$ over $\mathcal{L}$ have the *expected utility property* if they can be represented by a utility function U that is the expected value of some Bernoulli function u. That is, there is some Bernoulli function u, such that for all $L, L' \in \mathcal{L}$, we have

$L \equiv (M, P) \succeq (M', P') \equiv L' \iff$

$$U(L) \equiv E_P u \equiv \sum_{i=1}^{N} p_i u(s_i) \geq \sum_{i=1}^{N} p_i' u(s_i') \equiv E_{P'} u \equiv U(L'). \tag{2}$$

2.2 Expected utility theorem

It might seem that preferences with the expected utility property are quite special, and indeed they are. For example, their indifference surfaces are parallel and flat. Thus preferences over for lotteries with $k = 3$ given monetary outcomes but varying probabilities (as in Figure 1) must have indifference curves on the probability simplex that are all straight lines with the same slope. Thus the preferences indicated by the curved indifference curves in Panel B of the Figure do *not* have the expected utility property.

The expected utility theorem (EUT) is therefore surprising. It states that preferences over lotteries that satisfy a seemingly mild set of conditions will automatically satisfy the expected utility property, and thus be representable via a Bernoulli function.

Over the decades since the original results of Von Neumann and Morgenstern, many different sets of conditions have been shown to be sufficient. Mas-Colell et al. [2010] work with the following four conditions that an individual's preferences $\succeq$ on $\mathcal{L}$ should satisfy:

1. Rationality: Preferences $\succeq$ are complete and transitive on $\mathcal{L}$.

2. Continuity: The precise mathematical expressions are rather indirect (they state that certain subsets of real numbers are closed sets), but they capture the intuitive idea that U doesn't take jumps on the space of lotteries. This axiom rules out lexicographic preferences.

3. Reduction of Compound Lotteries: Compound lotteries have outcomes that are themselves lotteries in $\mathcal{L}$. One obtains the *reduced* lottery, a simple lottery in $\mathcal{L}$

over all possible outcomes in the final lotteries, using the probabilities obtained in the obvious way (multiplication). The axiom states that the person is indifferent between any compound lottery and the corresponding reduced lottery.

4. Independence: Let $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0, 1)$. Suppose that $L \succeq L'$. Then $\alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$. "In other words, if we mix two lotteries with a third one, then the preference ordering of the resulting two mixtures does not depend upon (is independent of) the particular third lottery used." (Mas-Colell et al. p. 171).

Theorem 1 (EUT) *Let preferences $\succeq$ on $\mathcal{L}$ satisfy axioms 1-4 above. Then $\succeq$ has the expected utility property, i.e., there is a Bernoulli function u such that (2) holds.*

As Mas-Colell et al. point out, all four axioms seem innocuous. Someone who cares only about the ultimate monetary payoffs and whose calculations are not affected by indirect ways of stating the probabilities will satisfy the third axiom. For example, such a person would be indifferent between the compound lottery "get 0 with probability 0.5 and with probability 0.5 play the lottery that pays 10 with independent probability 0.5 and 0 otherwise," and the reduced lottery "get 0 with probability 0.75 and 10 with probability 0.25." The fourth axiom enforces a degree of consistency by requiring that preference rankings over lotteries are not changed by nesting each of those lotteries within a generic compound lottery. The first two axioms are even less controversial or problematic.[1]

For a proof of the EUT, see Mas-Colell et al. and the references cited therein. Here is a sketch of how the function u can be constructed for given preferences. Denote by s_+ and s_- the maximum and minimum monetary outcomes in the lottery. Set $u(s_+) = 1$ and $u(s_-) =$

[1]On the other hand, the EUT's conclusion is quite strong, and not consistent with some actual choice data. One response is to accommodate some of the anomalous data by weakening the axioms, usually the third or fourth. For an extensive and skeptical review of these matters, see *Risky Curves* (Routledge, 2014) by D. Friedman, M. Isaac, D. James and S. Sunder.

0. Consider any other monetary outcome s, and the set of lotteries $\{([s_+, s_-], [p, (1-p)]) : p \in [0, 1]\}$. For $p = 1$ the lottery is preferred to s and for $p = 0$ the outcome s is preferred to the lottery. Using the continuity axiom, one can show that for some intermediate p^* the person is indifferent between s and that lottery. Set $u(s) = p^*$. Then use the other axioms to verify that the Bernoulli function so constructed indeed represents the given preferences.

2.3 Expected utility hypothesis (EUH)

The expected utility *theorem* (EUT) is a mathematical result with a rigorous proof. There can be no doubt that it is true. But what are the empirical implications?

The expected utility *hypothesis* (EUH) states that actual people choose in lottery-like situations as if maximizing some personal Bernoulli function. More specifically,

Definition 1 (EUH) *Each person has some Bernoulli function* $u : \Re \to \Re$ *and always chooses an action* $a \in A$ *that maximizes her expected utility, i.e., that solves*

$$\max_{a \in A} \{E_\pi u\,[x(a(s))]\} = \sum_{s \in S} \pi(s) u\,[x(a(s))]\,. \tag{3}$$

In particular, if the action is to chose a lottery from $A = \mathcal{L}$, then the EUH would follow from the EUT if the four axioms hold, which seems reasonable. But how can we know the Bernoulli function for a particular person? The EUT proof sketch offers a suggestion: for each outcome s_i, vary the probabilities on best and worst outcomes to try to find a person's point of indifference. There are many variations on this theme; one of the currently more popular is the Multiple Price List scheme introduced by Holt and Laury (*American Economic Review*, 2002).

Of course, in many (if not most) choice problems in our uncertain world, the state the probabilities π are not given to us. How can you then calculate (3) and apply the EUH?

The fallback position is called subjective expected utility (SEU). Leonard Savage (1954) first developed a list of sufficient conditions for the expected utility property, and others since have tweaked them. For example, Kreps' (1990) list includes asymmetry, negative transitivity, substitution and Archimedian — so far, all very plausible — plus a fifth condition noted below. That is, if choices over A satisfy Kreps' five conditions, then there is some some Bernoulli function $u : \Re \to \Re$ and some (subjective) probability distribution π such that (3) holds. This result is called the Strong SEU Theorem.

- "strong" refers to state-independent utility, a strong assumption that is not always valid.
- Example: utility for ice cream on a hot, sunny day is different than utility for ice cream in the winter.

Weak SEU allows for state dependent utility, i.e., the Bernoulli function u_s can vary with the state of the world s. Weak SEU then gives the expected utility property as

$$a \succ a' \Leftrightarrow E_\pi u_s \left[x(a(s)), s\right] > E_\pi u_s \left[x(a(s)), s\right] \tag{4}$$

The SEU Theorem says that Kreps' first four (very plausible) conditions are sufficient to ensure (4); here we drop the fifth condition, which basically just says that preferences are state-independent.

Before turning to other matters, we now mention something that will be helpful later in the course. The so-called *Harsanyi doctrine* connects the beliefs π across two or more decision makers:

- The π used in SEU comes via Bayes' Rule from a common prior belief and person-specific information.
- This means that differences in beliefs arise solely from differences in information to which different deciders have been exposed.
- As we will see later in this chapter, Bayes' rule is the only logically consistent way to combine new information with prior beliefs.

- Bayes' Rule (and, implicitly, the Harsanyi doctrine) is the foundation for "rational expectations" in macroeconomics.

2.4 Do people really behave this way?

Three questions:

Q1. Is EUH (or SEU) a good normative theory?

A1: These theories offer prescriptions for how best to achieve your goals. A rational person should indeed act as if maximizing the expected value of a Bernoulli function when the decision is stand-alone. Here's an important caveat: the prescription doesn't apply to *partial* decisions. In particular,

(a) portfolio effects. For example, if you look at the choice of which financial asset to choose in isolation, it would be irrational to pick gold because its returns are first-order stochastically dominated. (We'll see soon what that means.) But gold probably has a negative covariance with the rest of your portfolio (returns are not independent), so if you look at the overall decision of which entire portfolio to choose, you might rationally include some gold, depending on your Bernoulli function.

(b) path dependence. Your current action may affect the outcomes of future decisions. For example, the return to holding cash is first order stochastically dominated by returns to holding bonds, but its liquidity better enables you to cope with opportunities and risks that might turn up later. We'll deal with such matters later in this chapter.

Q2. Do actual people really behave according to EUH (or SEU)?

A2: Well, a few come pretty close, especially if they get expert advice, or have good MBA training. But most people tend to depart in some systematic ways, at least in unfamiliar situations. These are called choice anomalies, and we will discuss some later, when we return to the questionnaire distributed earlier.

Q3. As an economist, how should I react to choice anomalies?

A3: This is an active and controversial area of research.

(a) Orthodox behavioral economists seek generalized models of EUH (or SEU); a leading example is cumulative prospect theory.

(b) Hardline neoclassical economists believe that anomalies are ignorable because the markets reflect "smart money."

(c) The authors of these notes believe that under favorable conditions, deciders learn to behave more consistently with EUH (or Bayesian SEU), and so it is a mistake to work with static generalized (weaker) models, but it is also a mistake to ignore anomalies. They think that the way forward is to supplement neoclassical models with dynamic models of adaptation and learning.