Problem Set 3: Solutions

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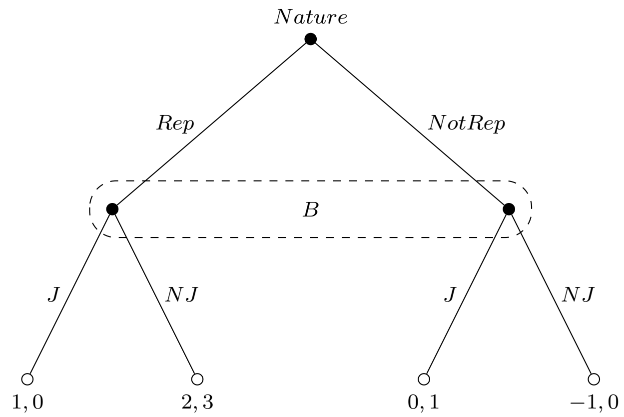
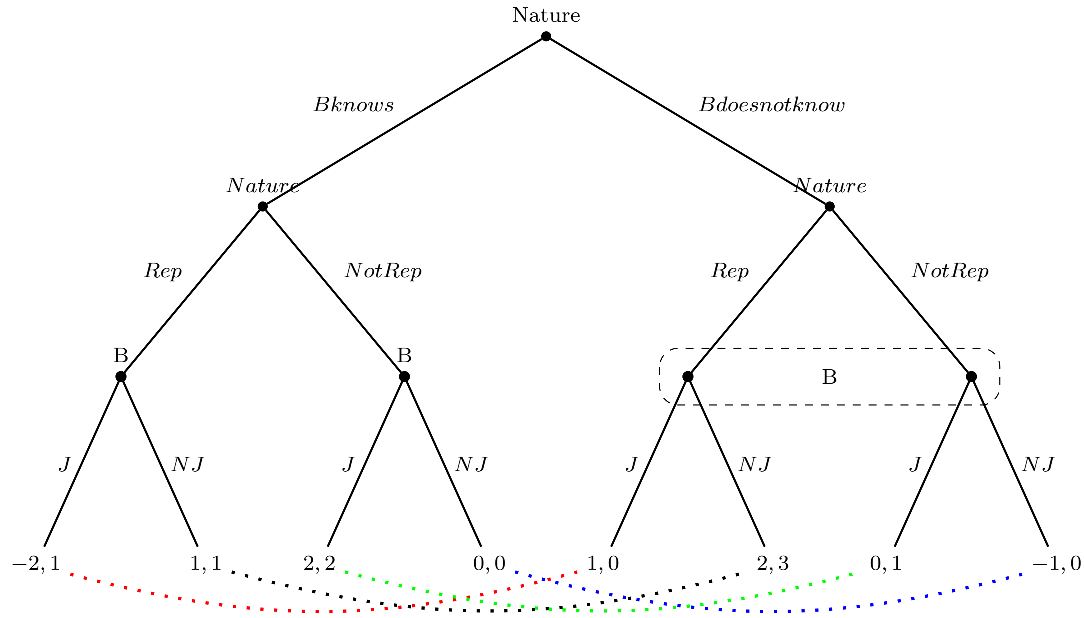
Problem 1

(a)



(b)



Legend:
Rep: A is a Republican
NotRep: A is not a Republican
J: B makes the joke
NJ: B does not make the joke

Note: I developed a what seems like a plausible and coherent reasoning to come up with the payoffs. You can do the same!

Problem 2

(a) Pure strategy Nash equilibrium - (H,T)
Efficiency = 1/8

(b) Generally d is determined by the market interest rate, the length of the game and the player's patience (or its lack thereof). In this case, d may be determined by how much the duke values present consumption relative to future consumption of corn. Other factors include the probability of continuation of the game. For instance, an impending war or increased probability of the duke being killed by invading barbarians will lower d. The serf's discount factor can be affected by the probability he assigns to continuation of the game. For instance, if the serf is likely to succumb to an epidemic of the plague, he will value today's consumption much more greatly than future consumption (and hence would have a low discount factor).

(c) The efficient stage game outcome is (P, S). This is the largest payoff that the serf can assure himself. Thus, the serf has no incentive to deviate from this strategy. The duke can earn a higher payoff by defecting to play T. Let U_d denote the duke's present value of the duke's lifetime earnings and d his discount factor:

Case 1: Duke plays plays efficient stage game outcome forever:
$U_d = 4 + d4 + d^2 4 + ... = \frac{4}{1-d}$

Cae 2: Duke defects: $U_d = 8 + d0 + ... = 8$

In order to sustain the efficient stage game outcome
$\frac{4}{1-d} > 8 \implies d > 1/2$

(d) The outcome is efficient so long as the duke is patient and has a high discount factor for future consumption of corn (d > 1/2). If the duke's discount factor is less than 1/2 (for instance with a barbarian attak), the serf will probably switch to H.

Problem 3

	u	d
U	0,0	4,1
D	1,4	2,2

Table 1: Stage game

	p	1-p
U	0	4
D	1	2

Table 2: One Population

(a) Nash equilibria of the stage game : (1/2, 1/2; D,u), (1/2, 1/2; U, d) , (2/3, 1/3 ; U,D)
Payoff to playing U against p = U(U, p) = 0p + 4(1-p) = 4-4p
Payoff to playing D against p = U(D, p) = (1)p + 2(1-p) = 2 - p
Payoff difference Δ(p) = U(U, p) - U(D, p) = 2-3p

When p= 2/3 $\implies \Delta(p) = 0$
When p > 2/3 $\implies \Delta$ (p) is negative $\implies \dot{p}$ is decreasing.
When p < 2/3 $\implies \Delta$ (p) is positive $\implies \dot{p}$ is increasing.

A Nash equilibrium is an evolutionary equilibrium if it is stable under the dynamics. Using sign preserving dynamics we see that p=2/3 is the evolutionary equilibria of this game. Thus, over time, 2/3 of the population will play U and 1/3 of the population will play D/

Population 2	p	1-p
Population 1		
q	0, 0	4,1
1-q	1, 4	2, 2

(b) Using sign-preserving dynamics: $\dot{q} \geq 0 \iff \Delta(q) \geq 0$

For population 1:
$\Delta(q)$ = 2-3p
At steady state: $\Delta(q) = 0 \implies p^* = 2/3$

When p<2/3 (say p=1/3)
$\Delta(q) = 1 \implies \dot{q}$ is increasing

When p>2/3 (say p=5/6)
$\Delta(p) = -1/2 \implies \dot{q}$ is decreasing

For population 2:
$\Delta(p) =$ 2-3q
At steady state: $\Delta(p) = 0 \implies q^* = 2/3$

By similar analysis, when q<2/3 $\dot{p}$ is increasing. When q>2/3 $\dot{p}$ is decreasing

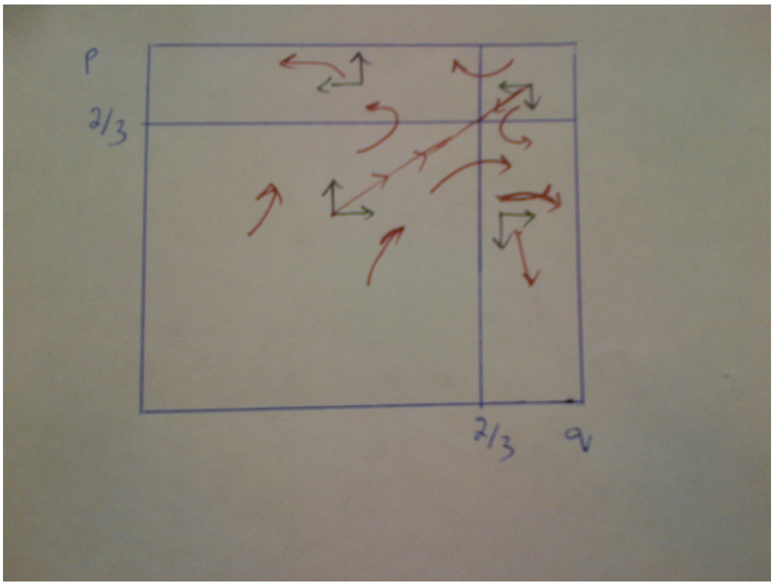


Figure 1: Dynamics for 2 population (Eilin likes to use her colour pens)

There is one saddle point (2/3, 2/3) – not an EE since nearby points will move away – and two stable equilibria points (0,1) and (1, 0). Their basins of attraction are separated by the saddle path (the diagonal).

(c) For one population, p=q. Thus, the dynamics is confined to the saddle path, along which the population reaches the saddle point equilibrium of 2/3.

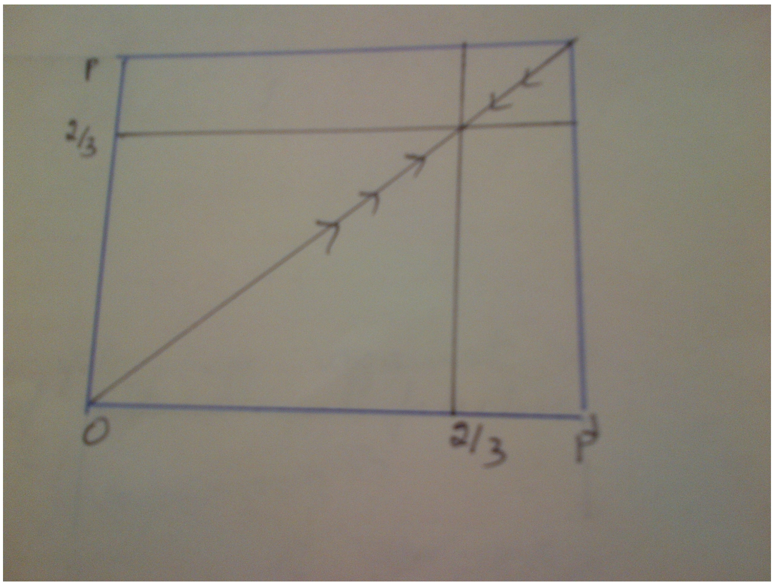


Figure 2: Dynamics for 1 population

Problem 4

(a) Relative to status quo, the sum of possible gains for countries A and B is 10 which can be split any way. Given this information, we calculate the Nash Bargaining Solution by solving the maximization problem:

$$\max \prod_{i=1}^{2}(u_i - \bar{u}_i)$$

$$\text{subject to } \sum_{j=1}^{2} u_j \leq 10$$

using the Lagrangian and assuming an interior solution:

$$\mathcal{L} = (u_A - \bar{u}_A)(u_B - \bar{u}_B) + \lambda(10 - u_A - u_B)$$

where u_A, u_B are the utilities/payoffs to A and B respectively, and $\bar{u}_A, \bar{u}_B$ are the "threat point" values or the "status quo" values. To solve, first take the first order conditions:

$$\frac{\partial \mathcal{L}}{\partial u_A} = u_B - \bar{u}_B - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial u_B} = u_A - \bar{u}_A - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 10 - u_A - u_B = 0$$

Solving gives: $10 - \bar{u}_A + \bar{u}_B = 2u_B + 2\bar{u}_B$ if we make the assumption that $\bar{u}_A = \bar{u}_B$, then it is easy to see that $u_B = 5$. Since $10 - u_A - u_B = 0$, this implies that $u_A = 5$ as well.

(b) Now a third country, C, enters the trade negotions. We have the following (the second value is normalized to a [0,1] scale):

$$\nu(\{B,C\}) = 5 \Rightarrow \frac{1}{3}$$
$$\nu(\{A,C\}) = 15 \Rightarrow 1$$
$$\nu(\{A,B\}) = 10 \Rightarrow \frac{2}{3}$$
$$\nu(\{A,B,C\}) = 15 \Rightarrow 1$$

I will make the assumption that each country doesn't get much by itself (no trading$\Rightarrow$ no benefit). So:

$$\nu(\{A\}) = \nu(\{B\}) = \nu(\{C\}) = 0$$

This characterizes our characteristic function.

(c) Is this game convex? In MCWG, convex is defined as: given coalitions $S, T \subset N$, then $\nu(S)+\nu(T) \leq \nu(S \cap T)+\nu(S \cup T)$. So in order to show that it is convex, we must show that this is true for every combination OR find a counterexample (note that, in general, we look at $S \neq T$ as otherwise trivial):

$$\nu(\{A,B\}) + \nu(\{A,C\}) = 25 \geq 15 = \nu(\{A,B,C\}) + \nu(\{A\})$$

Since the result is contrary to the definition of convex, we needn't go any further. Counterexample $\Rightarrow$ **this game is not convex**

(d) I will calculate the core using that the following must be true:

$u_A + u_B + u_C \geq 1$
$u_A + u_B \geq \frac{2}{3}$
$u_A + u_C \geq 1$
$u_B + u_C \geq \frac{1}{3}$

Since values which do not meet these conditions will be "blocked," we may draw the simplex below in order to find our resulting core (if one exists). **See the end of this problem.**

(e) The Shapley value of this three player game may be calculated using this table:

Where ϕ is calculated by dividing the sum by $\frac{1}{n!}$ ($n = 3$ in this case). Thus we see that **the Shapley Value is: $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$**

	MC_A	MC_B	MC_C
ABC	0	$\frac{2}{3}$	$\frac{1}{3}$
ACB	0	0	1
BAC	$\frac{2}{3}$	0	$\frac{1}{3}$
BCA	$\frac{2}{3}$	0	$\frac{1}{3}$
CAB	1	0	0
CBA	$\frac{2}{3}$	$\frac{1}{3}$	0
SUM	3	1	2
ϕ	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$

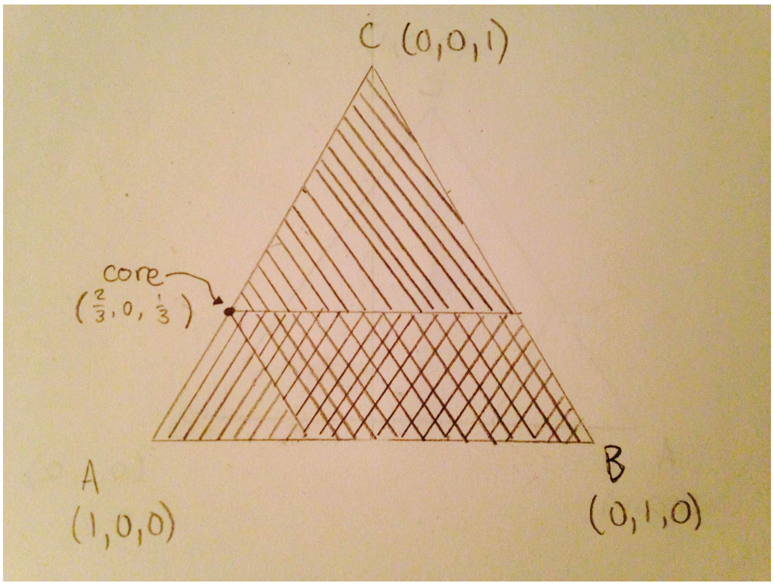


Figure 3: Core in the Simplex (Rachel likes to use her black pens)

Problem 5

(a) The payoff function for buyer

$$u_b(p,q) = \int_0^q (210 - p - q)dq = 210q - pq - \frac{q^2}{2}$$

The payoff for seller

$$u_s(p,q) = (p-10)q$$

(b) The Pareto frontier in (p,q) space is $p = 210 - q$, $100 \leq q \leq 200$.
Plugging into the payoff functions and we have

$$(u_b, u_s) = (210q - pq - \frac{q^2}{2}, (p-10)q) = (\frac{q^2}{2}, 200q - q^2), \qquad 100 \leq q \leq 200$$

This implies the Pareto frontier in (u_b, u_s) space is

$$u_s = 200\sqrt{2u_b} - 2u_b, \qquad 5000 \leq u_b \leq 20000$$

This analysis assumes non-transferable utility (NTU).

With transferrable utility (TU):
The point $(p,q) = (10, 200)$ (which corresponds to $(u_b, u_s) = (20000, 0)$) maximizes the sum of profits.

(c) Suppose the seller chooses price, then for the buyer, given p, he maximizes his payoff so that marginal benefit equals to marginal cost, which implies $q = 210 - p$.

For the seller, knowing that the buyer will respond as $q = 210 - p$, he chooses p to maximize u_s, i.e.,

$$\max_p (p-10)q = (p-10)(210-p)$$

F.O.C implies

$$(210 - p) - (p - 10) = 0$$

which solves $p^* = 110$ and $q^* = 100$. The resulting profits are $u_b(110, 100) = 5000$ and $u_s(110, 100) = 10000$.

(d) The buyer can maximise his utilty by choosing the lowest price at which the seller is willing to sell. The seller will sell steel as long as $p \geq 10$. Thus $p^* = 10$ and $q^* = 200$. The resulting profits are $u_b(10, 200) = 20000$ and $u_s(10, 200) = 0$.

(e) With NTU:
Denote the threat point as $(\underline{u}_b, \underline{u}_s) = (5000, 0)$[1]. The Nash-bargaining solution maximizes

$$\max_{\{u_b, u_s\}} (u_b - \underline{u}_b)(u_s - \underline{u}_s)$$

$$s.t. \quad u_s = 200\sqrt{2u_b} - 2u_b, \quad 5000 \leq u_b \leq 20000$$

Rewrite the maximization problem as

$$\max_{5000 \leq u_b \leq 20000} (u_b - 5000)(200\sqrt{2u_b} - 2u_b)$$

F.O.C implies

$$200\sqrt{2u_b} - 2u_b + (u_b - 5000)(100\sqrt{\frac{2}{u_b}} - 2) = 0, 5000 \leq u_b \leq 20000$$

which solves $u_b \approx 9355$ and the corresponding $u_s \approx 8648$.

With TU:
The Nash-bargaining solution maximizes

$$\max_{\{u_b, u_s\}} (u_b - \underline{u}_b)(u_s - \underline{u}_s)$$
$$s.t. u_b + u_s = 20000$$

Rewrite the maximization problem as

$$\max (u_b - 5000)(2000 - u_b)$$

with solution $u_b = 125,000$

Textbook Problems

Problem 12.B.1

(a) For a monopolist:

$$q^m p'(q^m) + p^m = c'(q^m) \tag{1}$$

$\frac{-dp}{dq} = \frac{p^m - c'(q^m)}{q^m}$

Multiplying both sides of the quation by $\frac{p}{q^m}$ $\frac{-dq}{dp}\frac{p}{q^m} = \frac{p^m}{p^m - c'(q^m)}$

The price elasticity of demand at price $p^m is \frac{-dq}{dp}\frac{p}{q^m}$. Thus the monopolist's price-cost margin is the inverse of the price elasticity of demand.
The intuition here is this: With inelastic or unit elastic demand, reducing output will increase revenue or keep it unchanged, while reducing cost, thus increasing profit. Therefore a profit-maximizing monopolist will reduce output until demand is sufficiently elastic.

(b) We want to show that $\eta = \frac{p^m}{p^m - c'(q^m)} > 1$.
From first it is evident that th denominator is positive. It follows that $p^m > p^m - c'(q^m)$. Thus η is greater than 1 and demand is elastic.

[1] Some people may argue that the threat point is (0,0) derived from no-transaction case, however, a rational buyer will never refuse buying from a rational seller, which leads to a minimum consumer surplus of 5000. Similarly, a rational seller will always sell and get non-negative profit.

Problem 12.B.3

Demand function is $x(p,\theta)$
Cost function is $c(q,\phi)$
Assume:
concave demand function $x_1 < 0, x_{11} < 0$; with θ representing an outward shift so $x_2 > 0$
convex cost function $c_1 > 0; c_{11} > 0$, with ϕ representing an upward shift so $c_2 > 0$.

Monopolists problem can be written as:
$max_p\{x(p,\theta) * p - c(x(p,\theta),\phi)\}$

FOC wrt p
$x_1(p,\theta) * p + x(p,\theta) - c_1(x(p,\theta),\phi) * x_1(p,\theta) = 0$

Totally differentiate FOC wrt θ

$\implies 0 = x_{12}(p,\theta) * p + x_1(p,\theta) * \frac{\partial p}{\partial \theta} + x_2(p,\theta) - [c_{11}(x(.),\phi) * x_1^2(p,\theta)$
$+c_1(x(.),\phi) * x_{12}(p,\phi) + c_{11}(x(.),\phi) * x_{11}(p,\theta) * \frac{\partial p}{\partial \theta} + c_1(x_1(p,\theta),\phi) * x_{11}(p,\theta) * \frac{\partial p}{\partial \theta}]$

$\implies \frac{\partial p}{\partial \theta} = \frac{-(p-c)*x_{12}+x_2*(x_1c_{11}-1)}{2x_1+(p-c_1)*x_{11}-x^2c_{11}}$

RHS $= \frac{term_1+term_2}{term_3+term_4-term_5}$
$term_2, term_3, term_4$ = negative by assumptions

$\implies \frac{\partial p}{\partial \theta} > 0$
if,
$term_1 = 0$ because $x_{12} = 0$ (parallel shift in demand)

Similarly,
Totally differentiate wrt ϕ

$\implies \frac{\partial p}{\partial \phi} = \frac{-c_{12}(p,\phi)*x_1(p,\phi)}{2x_1+(p-c_1)*x_{11}-x^2c_{11}}$

RHS $= \frac{term_1}{term_2+term_3+term_4}$

$term_{2,3,4}$ = negative by assumptions

$\frac{\partial p}{\partial \phi} > 0$
if $c_{12} > 0$
$\frac{\partial p}{\partial \phi} = 0$
if $c_{12} = 0$ (in the case of a parallel shift)

12.B.6

Call the government intervention (tax or subsidy) τ. Then the firm maximizes $\pi = p(q)q - c(q) - \tau q$. This is subject to their first order conditions:

$$p'(q)q + p(q) = c'(q) + \tau$$

In perfect competition, $p(q) = c'(q)$ (price equals to marginal cost). Thus in order to incentivize the monopoly to act competitively, the goverment would need to provide a subsidy equal to: $\tau = p'(q)q$.Since p' < 0 this is a negative tax (a subsidy).

12.D.4

Consider an infinitely repeated Bertrand oligopoly with discount factor $\delta \in [\frac{1}{2}, 1)$.

(a) This depends on whether or not this cost goes up or down. In the case where a Bertrand oligopoly is infinitely repeated, the most profitable price is the monopoly price, p^m. Generally, if cost goes up, the price goes up and if cost goes down, price goes down.

(b) Let π' be the first period profit and π_m be profit from the second period onwards.

When the players do not deviate:
U = $\frac{\pi'}{2} + [\frac{\delta\pi_m}{2} + \frac{\delta^2\pi_m}{2} + ...]$

$U = \frac{\delta}{1-\delta}\frac{\pi_m}{2}$

When the players deviate:
U = π' (Payoff in all other periods is Bertrand competitive profit which is 0).

The players collude if

$\frac{\delta}{1-\delta}\frac{\pi_m}{2} \geq \pi'$

There are two scenarios based on how much cost increases. If the increase in MC is not too large, then price continues to be at p^m even after the increase in cost. If the increase in MC is very large, then p^m cannot be sustained and the players collude to a price below p^m.

12.E.4

Since this is a perfect cartel, market price and quantity is at monopoly levels.

Cartel prices and quantities cannot be affected by the social planner. The planner can achieve socially optimum outcome by minimising n such that production is positive. Thus n=1.
In the second case, where the planner can't control entry and the number of firms is J:

$$J^* = \frac{\pi}{K}$$

Firms will enter until the aggregate entry cost equals the profits.