

TWO MICRODYNAMIC MODELS OF EXCHANGE

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Two models are presented of real-time exchange and price adjustment for a system of interrelated markets mediated by trade specialists. The first (MBM) features partially decentralized barter exchange in which specialists continuously maintain inventories and periodically adjust prices. A 'correspondence principle' yields existence and efficiency of (steady-state) equilibrium. Local stability follows from a direct argument. The second model (MMM) features fully decentralized money-mediated exchange as well as specialists. Existence, approximate-efficiency, local-stability and asymptotic-neutrality results are derived.

1. Introduction

Walrasian auctioneers are valuable creatures, allowing economic theorists to study the effects of tastes and technology without worrying about market organization. In actual economies, however, Walrasian auctioneers are extremely rare, and one observes a variety of devices employed to arrange transactions and to adjust prices. Perhaps the most common are *trade specialists* (middlemen or dealers) who maintain inventories and choose the prices at which they will buy from producers and sell to consumers. An obvious and crucial question, seldom addressed in the literature, then is the extent to which outcomes in such trade specialist markets resemble Walrasian outcomes. Clower and Friedman (1986) propose two models of specialist-mediated exchange and state some reassuring results on existence, stability and money neutrality. The purpose of the present paper is formally to specify these models and to derive precise results.

As a first attempt to formalize organized exchange in a set of interrelated markets, the present paper does not seek the greatest possible generality; indeed, many assumptions are merely for convenience and undoubtedly could be greatly relaxed without substantially affecting the ultimate results. For similar reasons, I usually employ only well-known techniques in the proofs, although this makes the argument a bit longer at times. Some

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motivation is provided for definitions and assumptions, but the interested reader should consult Clower and Friedman for a fuller intuitive picture as well as references to the literature.

The main lesson of this exercise appears to be that it is possible to implement Walrasian outcomes in a simple logistically plausible market structure. The MMM in particular always allows transactions to proceed even when not mutually consistent (although grossly inconsistent plans require rationing), and each Walrasian equilibrium has its counterpart as a locally stable steady state of a corresponding Monetary Microdynamic Model (MMM). Moreover, money is neutral in the long-run in this model of fully decentralized exchange: the real effects of a small change in the money stock, irrespective of its initial distribution, all vanish asymptotically, leaving merely proportional changes in prices and all money holdings.

There is the hint of a different lesson, however. Major shocks (i.e., initial conditions far from Walrasian equilibrium values) may lead to effective demand failure equilibria, which involve rationing and efficiency losses. One is reminded of Leijonhufvud's (1973) 'corridor effects': agents' buffer stocks of cash or inventories, once wiped out, may be difficult (or in extreme cases, impossible) to rebuild and normal trading patterns are disrupted in the meantime.

The next section establishes notation and quickly reviews the discrete-time standard Walrasian model, the point of departure, and specifies the first model of a dynamic exchange process organized by 'market specialists'. This barter process is only defined locally (because it cannot deal with stock outages), but some simple existence and stability results are established. Section 3 presents a more complicated but globally defined process of money-mediated exchange and obtains fairly strong existence, stability and neutrality results.

2. Notation and Walrasian model

Time is discrete and indexed by $t=0, 1, 2, \dots$

The economy consists of $n < \infty$ households, indexed by superscripts $j=1, \dots, n$. They exchange $l < \infty$ goods, indexed by subscripts $i=1, \dots, l$. Summation over an index is indicated by a T (for 'total'), e.g., $x^T = \sum x^j$. We employ the standard vector ordering notation: for $x, y \in R^l$, $x \geq y$ means $x_i \geq y_i$, all i ; $x > y$ means $x \neq y$ and $x \geq y$; $x \gg y$ means $x_i > y_i$, all i . The (closed) positive orthant is denoted $R_+^l = \{x \in R^l \mid x \geq 0\}$; its boundary is $\partial R_+^l = \{x \in R_+^l \mid x_i = 0, \text{ some } i\}$; and its interior is $R_{++}^l = R_+^l \setminus \partial R_+^l = \{x \in R^l \mid x \gg 0\}$. We also use the standard inner product: for $x, y \in R^l$, $x \cdot y = x' y = \sum_{i=1}^l x_i y_i \in R$.

Each household is characterized by periodic *endowments* in the l goods, $s^j(t) = (s_1^j(t), \dots, s_l^j(t))$, and by *preferences* represented by a utility function U^j defined on consumption bundles $d^j \in R_+^l$. We shall maintain the following assumptions:

(H1) For all t and j ,

- (a) $s^j(t) = s^j(0) \equiv s^j \in R_+^l \setminus \{0\}$, i.e., endowments do not vary over time, are non-negative and never zero.
- (b) $s^T = \sum s^j \gg 0$, i.e., at least one household is positively endowed in each good.

(H2) For each j , U^j is also time independent and satisfies the strong neoclassical conditions

- (a) $U^j \in C^2(R_+^l, R)$, i.e., preferences are *smooth* (twice continuously differentiable real valued functions on the positive orthant).
- (b) $x > y \in R_+^l$ implies $U^j(x) > U^j(y)$, i.e., strict *monotonicity*.
- (c) $(U^j)^{-1}(c, \infty)$ is strictly convex for all $c \in R$, i.e., strict *convexity* of preferences, or strict quasi-concavity of the utility functions.
- (d) $(U^j)^{-1}[c, \infty) \cap \partial R_+^l = \emptyset$ for all $c \in U^j(R_+^l)$, i.e., indifference curves don't intersect the boundary of the positive orthant.

Prices $p(t) = (p_1(t), \dots, p_l(t)) \in R_{++}^l$ are given exogenously. If prices are restricted to lie in a subspace of R^l , e.g., $p(t) \in S_+^l = \{q \in R_+^l \mid q \cdot q = 1\}$, we refer to them as *normalized prices*. For the moment, prices are *not* normalized.

Household j 's *budget set* is $B^j(p, s^j) = \{z \in R_+^l \mid p \cdot z \leq p \cdot s^j\}$. Given our strong assumptions (H1–2), U^j achieves a maximum value on $B^j(p, s^j)$ at a unique point d^{*j} , which is a continuously differentiable function of (p, s^j) . This *demand point* $d^{*j} \in R_{++}^l$, and gives rise to the household's *desired net trade* $x^{*j} = d^{*j} - s^j$, which satisfies 'Say's Principle' $p \cdot x^{*j} = 0$.

We may now define a *Walrasian Economy* \mathcal{E} as a specification $\{(s^j, U^j)\}_{j=1}^n$ of household characteristics consistent with (H1–2). The *Walrasian Equilibria* $WE(\mathcal{E})$ comprise the set of pairs (\bar{p}, \bar{x}) , where $\bar{p} \in R_{++}^l$ and $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in (R^l)^n$, such that

- (a) $\bar{x}^j = x^{*j}(\bar{p}, s^j)$ for all j , i.e., \bar{x}^j is j 's desired net trade at \bar{p} , and
- (b) $\bar{x}^T = \sum_{j=1}^n \bar{x}^j = 0$, i.e., the desired trades are mutually consistent.

The *existence* of *WE* is well-known under conditions much weaker than H1–2, see Debreu (1958), for instance. It is obvious from the definition of $B(p, s^j)$ that if $c > 0$ and $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$, then also $(c\bar{p}, \bar{x}) \in WE(\mathcal{E})$. Hence, Walrasian equilibria are never unique. Even if we normalize p , we need stronger conditions than those imposed so far (e.g., 'Gross Substitutes') to ensure uniqueness. On the other hand, (H1–2) do ensure (static) *efficiency* in the sense that $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$ implies that $\bar{p}_i / \bar{p}_k = MRS_{ik}^j(s^j + \bar{x}^j)$ for every i, j, k , where $MRS_{ik}^j(y)$ is j 's marginal rate of substitution of good i for good k at allocation y .

Let us now turn to our first microdynamic model. We supplement the static Walrasian model \mathcal{E} by specifying a transactions structure and

dynamics. We think of trade specialists (for simplicity, a representative specialist for each good) as posting prices in the 'morning' of each day t . The households receive their endowments, compute desired net trade vectors $x^j(t)$, and by afternoon arrive at the market place with the goods they wish to sell and shopping lists of goods they wish to buy. Specialists deposit goods sold by households in inventory bins and withdraw households' purchases. Some central authority checks that no household violated its budget constraint. In the evening, households go home and consume $d^j(t)$, and specialists choose the prices for the next day, using a rule based on x =change in inventories ('flow demand') and X =gap between desired and actual inventory ('stock demand'). [Friedman (1984) provides an optimizing rationale for these rules.] The resulting dynamical process is defined as long as households' desired flow demand does not exhaust current inventories.

We still specify households by preferences and endowments satisfying (H1–2). Specialists are characterized by the axioms

(S1) For each i , there is a smooth (C^2) price adjustment rule $f_i: R^l \times R^l \rightarrow R$, $(x, X) \mapsto \Delta p_i$, such that $f_i(0, 0) = 0$ and $\text{grad } f_i(0, 0) \neq 0$. [This last is a mild non-degeneracy condition that not all partial derivatives of f_i vanish at $(x, X) = (0, 0)$.]

(S2) For each i , there is a desired inventory level $D_i > 0$.

We may now define a MBM economy as a pair $\mathcal{E}' = (\{(s^j, U^j)\}_{j=1}^n, \{(f_i, D_i)\}_{i=1}^l)$. A state of \mathcal{E}' is a pair (p, S) of prices $p \in R_+^l$ and inventory levels $S \in R_+^l$.

If a Walrasian economy \mathcal{E} and an MBM economy \mathcal{E}' have the same household sector $\{(s^j, U^j)\}_{j=1}^n$, we say that \mathcal{E} and \mathcal{E}' correspond.

A dynamical process for \mathcal{E}' is a function $F: V \subset R_+^l \times R_+^l \rightarrow R_+^l \times R_+^l$, $(p(t), S(t)) \mapsto (p(t+1), S(t+1))$, specifying the evolution of the economy over time, satisfying the following conditions for each $t \geq 0$ and $i = 1, \dots, l$:

(D1) $S_i(t+1) = S_i(t) - x_i^{*T}$, i.e., inventory stocks are depleted or replenished by total desired excess demand,

(D2) $p_i(t+1) = p_i(t) + f_i(x, X)$, where $x = x^{*T}(t)$ and $X = D - S(t+1)$, i.e., the price adjustment rule applies to total desired excess demand and discrepancies between actual and desired inventory stocks, and

(D3) $S_i(t+1) \geq 0$ and $p_i(t+1) > 0$, i.e., F yields admissible states of \mathcal{E}' .

The domain V of F consists of those states for which (D1) and (D2) are consistent with (D3). Clearly $\emptyset \neq V \subsetneq R_+^l \times R_+^l$, i.e., F is neither globally defined nor vacuous.

We are particularly interested in the rest states of this process, in which prices and inventories remain constant. Formally, the *barter steady-state equilibria* $BSE(\mathcal{E}')$ consist of the fixed points (\bar{p}, \bar{S}) of F . We may characterize them by the following 'correspondence principle'.

Proposition 1. For every Walrasian economy \mathcal{E} and any corresponding MBM economy \mathcal{E}' , there is a $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$ for each $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$. Conversely, for every MBM economy \mathcal{E}' and every $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$, there is an \bar{x} such that $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$, where \mathcal{E} is the Walrasian economy that corresponds to \mathcal{E}' .

Proof. Given $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$, pick any (f, D) satisfying (S1)–(S2), and set $\bar{S} = D$. Note that $D \gg 0$ by (S2), so $\bar{S} \in R^l_{++}$; that $\bar{p} \in R^l_{++}$ by hypothesis; and that $X = D - \bar{S} = 0$. Also $x = x^{*T} = \bar{x}^T = 0$ by hypothesis, so $\Delta p = f(x, X) = f(0, 0) = 0$, and $\Delta S = -\bar{x}^T = 0$. Hence $F(\bar{p}, \bar{S}) = (\bar{p}, \bar{S}) + (0, 0)$, so $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$. For the converse, assume $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$. Then $0 = \Delta S = -x^{*T} \equiv -\sum_{j=1}^n x^{*j}$, the first equality by hypothesis and second by (D1). Hence we may set $\bar{x}^j = x^{*j}$ and conclude $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$ if \mathcal{E} corresponds to \mathcal{E}' . Q.E.D.

Corollary. For any MBM economy \mathcal{E}' , equilibria exist (i.e., $BSE(\mathcal{E}') \neq \emptyset$) and are efficient.

Existence of *BSE* is not enough; to be economically relevant, an equilibrium must be stable in the sense that the economy returns to it following mild shocks.

In our stability analysis we employ the following definitions. A price adjustment rule $f_i: R^l \times R^l \rightarrow R$, $(x, X) \rightarrow \Delta p_i$ is *simple* if $x_i = y_i$, $X_i = Y_i$ implies $f_i(x, X) = f_i(y, Y)$. In this case we write $f_i: R \times R \rightarrow R$, $(x_i, X_i) \rightarrow \Delta p_i$. (That is, a specialist for good i looks only at stock and flow demand for good i , and ignores demand for other goods.) A price adjustment rule that is not simple is called *sophisticated*.

An equilibrium $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$ is *locally stable* if $S(t) \rightarrow \bar{S}$ and $p(t) \rightarrow c\bar{p}$ as $t \rightarrow \infty$ for some $c > 0$ whenever $(p(0), S(0))$ is sufficiently near (\bar{p}, \bar{S}) .

The *Hicksian matrix* at $p \in R^l_{++}$ can be defined for either a Walrasian or MBM economy as the $l \times l$ matrix A of gross substitution effects

$$A = \left(\frac{\partial x_i^{*T}(p)}{\partial p_k} \right).$$

For an arbitrary $l \times l$ matrix M , we say that M is *almost non-singular (ans)* if it has rank $l-1$, i.e., if it has a simple eigenvalue of 0. If the other eigenvalues of an *ans* matrix M have negative real part, M is called *almost Routhian (aR)*, and if it is also symmetric, it is called *symmetric almost negative definite (sand)*.

The following local stability result indicates that a wide class of pricing rules will stabilize an equilibrium if the substitution effects are not too wild. Much of the technical detail is due to the fact that Hicksian matrices do not have full rank because of price level indeterminacy.

Proposition 2. Let $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$, and suppose the Hicksian matrix A is *ans* at \bar{p} . Then there is an MBM economy \mathcal{E}' corresponding to \mathcal{E} and an $\bar{S} \gg 0$ such that $(\bar{p}, \bar{S}) \in BSE(\mathcal{E})$ and is locally stable. If A is *sand* at \bar{p} , then the pricing rule in \mathcal{E}' can always be chosen to be simple.

Proof. We begin with a given MBM \mathcal{E}' and $(\bar{p}, \bar{S}) \in BSE(\mathcal{E}')$, and examine the linearized dynamics of \mathcal{E}' at (\bar{p}, \bar{S}) . For (p, S) near (\bar{p}, \bar{S}) , let

$$x = x^{*T}(p) \simeq x^{*T}(\bar{p}) + A(p - \bar{p}) = 0 + A(p - \bar{p}), \quad \text{and}$$

$$X = D - S = \bar{S} - S,$$

where \simeq denotes 'equality up to first order' and the vectors are column vectors. Now

$$\begin{aligned} f(x, X) &\simeq f(0, 0) + \text{grad}_1 f(0, 0) \cdot x + \text{grad}_2 f(0, 0) \cdot X \\ &= 0 + \alpha x + \beta X, \end{aligned}$$

where α and β are the indicated $l \times l$ matrices of partial derivatives of f at $(0, 0) \in R^l \times R^l$. Hence the linearization of F at (\bar{p}, \bar{S}) is

$$S(t+1) = S(t) - x^{*T}(p(t)) \simeq S(t) - A(p - \bar{p}), \quad \text{and}$$

$$p(t+1) = p(t) + f(x^{*T}(p(t)), \bar{S} - S(t+1))$$

$$\simeq p(t) + \alpha A(p(t) - \bar{p}) + \beta(\bar{S} - S(t) + A(p(t) - \bar{p})).$$

We may express this more concisely in terms of the deviations $\hat{p}(t) = p(t) - \bar{p}$ and $\hat{S}(t) = S(t) - \bar{S}$ and the $l \times l$ identity matrix I_l as

$$\begin{pmatrix} \hat{S}(t+1) \\ \hat{p}(t+1) \end{pmatrix} = \begin{pmatrix} I_l & -A \\ -\beta & \underbrace{(\alpha + \beta)A + I_l}_M \end{pmatrix} \begin{pmatrix} \hat{S}(t) \\ \hat{p}(t) \end{pmatrix}. \quad (1)$$

Well-known theorems [see Samuelson (1947), for instance] assert that F is locally stable if (1) is stable, and that (1) is stable if M has all eigenvalues of modulus less than 1 and is unstable if M has an eigenvalue of modulus

greater than 1. We use the well-known formula for determinants of partitioned matrices

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |D - CA^{-1}B|$$

to obtain the eigenvalues of M from its characteristic equation

$$\begin{aligned} 0 = \det(M - \lambda I_{2l}) &= \begin{vmatrix} (1-\lambda)I_l & -A \\ -\beta & (\alpha+\beta)A + (1-\lambda)I_l \end{vmatrix} \\ &= |(1-\lambda)I_l| \left| (\alpha+\beta)A + (1-\lambda)I_l - \frac{(-\beta)(-A)}{(1-\lambda)} \right| \\ &= |(1-\lambda)(\alpha+\beta)A + (1-\lambda)^2 I_l - \beta A| \\ &= |\lambda^2 I - \lambda((\alpha+\beta)A + 2I_l) + (\alpha A + I_l)| \equiv |C_\lambda|. \end{aligned}$$

Suppose v is a real unit eigenvector of M for a given eigenvalue λ . Then

$$0 = C_\lambda v, \quad \text{and hence}$$

$$\begin{aligned} 0 &= v' C_\lambda v = \lambda^2 v' I_l v - \lambda(v'(\alpha+\beta)Av + 2v' I_l v) + (v' \alpha Av + v' I_l v) \\ &= \lambda^2 - \lambda(v'(\alpha+\beta)Av + 2) + (v' \alpha Av + 1) \\ &\equiv \lambda^2 + b\lambda + c. \end{aligned}$$

Now a quadratic expression as above has both roots less than 1 in modulus iff (i) $c - 1 < 0$, (ii) $c - b + 1 > 0$, and (iii) $c + b + 1 > 0$ [see Sargent (1979, p.183), for example]. In the present instance, these conditions are

- (i) $v'(\alpha A)v < 0$ (a negative definite condition on αA),
- (ii) $4 + v'(2\alpha + \beta)Av > 0$, $((2\alpha + \beta)A$ not too negative, a no-overshoot condition), and
- (iii) $v'(\beta A)v < 0$ (a negative definite condition on βA).

Before proceeding further, we need the following

Lemma. Suppose $(\bar{x}, \bar{p}) \in WE(\mathcal{E})$ and A is the Hicksian matrix at \bar{p} . Then $\lambda = 0$ is an eigenvalue of A , with eigenvector \bar{p} . If A is ans, then it has full rank on the subspace $H = \langle \bar{p} \rangle^\perp$. If A is symmetric, then $Av \in H$ for all $v \in R^l$.

The lemma warns that (i)–(iii) won't be satisfied for arbitrary v , but suggests that they may be satisfied for $v \in H$, which turns out to be sufficient to prove Proposition 2. The lemma may be proved as follows.

Recall Say's Principle, $0 = p \cdot x^{*j}(p)$, all j . Summing over j , we have $0 = p \cdot x^{*T}(p) = x^{*T}(p) \cdot p$, and differentiating and evaluating at \bar{p} yields $0 = A\bar{p} + x^{*T}(\bar{p}) = A\bar{p}$, which establishes the first part of the lemma. The space $H = \langle \bar{p} \rangle^\perp \equiv \{v \in R^l \mid v \cdot \bar{p} = 0\}$ has dimension $l-1$ and is orthogonal to the null space of A , so A must be of full rank on H if it is *ans.* For the last part of the lemma, let $Z: R^l \rightarrow H$ be orthogonal projection onto H , so $v = Zv + (\bar{p} \cdot v / \bar{p} \cdot \bar{p})\bar{p}$ for any $v \in R^l$. Applying this decomposition to Av , we obtain $Av = ZAv + (\bar{p} \cdot Av / \bar{p} \cdot \bar{p})\bar{p}$. If A is symmetric, $\bar{p} \cdot Av = v \cdot A\bar{p} = v \cdot 0 = 0$, in which case $Av = ZAv \in H$.

Returning now to Proposition 2, suppose that A is *sand*. The lemma now implies that A restricted to H is negative definite, so it is easy to pick f so as to satisfy (i)–(iii) for $v \in H$. For instance, we may construct a simple f by setting $\alpha = \beta = kI_l$, where $0 < k < 4/3\|A\|$. [Recall that $\|A\| \equiv \sup\{|Av|/|v| : 0 \neq v \in R^l\}$ is equal to the modulus of the largest eigenvalue of A ; it is easy then to see that $0 < k$ guarantees (i) and (iii), while $k < 4/3\|A\|$ guarantees (ii).]

From the last part of the lemma and the symmetry of A , it follows that $AZ\hat{p}(t) = ZA\hat{p}(t) = A\hat{p}(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$. Hence $x(t) \rightarrow 0$ and $\hat{s}(t) = -X(t) \rightarrow 0$ also, so $\Delta p(t) \rightarrow 0$. In fact, these converge to 0 exponentially as $t \rightarrow \infty$, so actually $p(t)$ converges, say $p(t) \rightarrow \bar{p}$ as $t \rightarrow \infty$. Hence we are done in the *sand* case as soon as we show $\bar{p} = c\bar{p}$. But $\bar{p} = Z\bar{p} + (\bar{p} \cdot \bar{p} / \bar{p} \cdot \bar{p})\bar{p} = c\bar{p}$, since $Z\bar{p} = \lim_{t \rightarrow \infty} Zp(t) = \lim_{t \rightarrow \infty} Z\hat{p}(t) = 0$.

Finally, assume merely that A is *ans.* Pick any f such that $\alpha = \beta = k(A'A)^{-1}A'$, where we use any generalized inverse of $A'A$ and $0 > k > -4/3$. Note that such f are typically sophisticated, and will satisfy (i) to (iii) on H . Hence by the argument in the previous paragraph, $A\hat{p}(t) \rightarrow 0$ exponentially, so $S(t) \rightarrow \bar{S}$ and $p(t) \rightarrow c\bar{p}$ for some $c > 0$ as $t \rightarrow \infty$. Q.E.D.

Remark 1. Near-symmetry of A evidently would be sufficient for a simple pricing rule to stabilize equilibrium if the $l-1$ non-zero eigenvalues of A have negative real parts. Indeed, it seems reasonable to conjecture that it would suffice for A to be *aR*.

Remark 2. The simple pricing rules described in the proof required k only as an upper bound for the speeds of price adjustment, to avoid 'overshooting'. Hence it is quite feasible to have different adjustment speeds for the different goods.

Recall that assumption (S2) in our definition of an MBM required that desired inventory levels D be constant over time. We end our examination of MBM economies by sketching the consequences of relaxing this assumption to

(S2') For each i , D_i is a smooth (C^2) function $g_i: R_+ \times R_+ \rightarrow R_{++}$ of desired purchases $x_i^{+T} = \sum_{j=1}^n \max\{0, x_i^{*j}\}$ and sales $x_i^{-T} = \sum_{j=1}^n \min\{0, x_i^{*j}\}$, i.e., $D_i(t+1) = g_i(x_i^{+T}(t), x_i^{-T}(t))$.

Clearly the argument in Proposition 1 is not affected by this change, so the existence and efficiency of *BSE* are retained in this modified MBM economy. The most interesting remaining question is: when can simple pricing rules ensure local stability? An answer is that the following conditions suffice:

- (a) Both $A_+ = (\partial x_i^{+T}(\bar{p})/\partial p_k)$ and $A_- = (\partial x_i^{-T}(\bar{p})/\partial p_k)$ are *sand*,
- (b) δ_{+i} and δ_{-i} , the first and second partial derivatives, respectively, of g_i evaluated at \bar{p} , are non-negative for each i , and
- (c) for some positive constant k depending on α , A_+ and A_- , $0 < \beta_{ii} \delta_{\pm i} < k$ for each i .

The proof involves a tedious repetition and expansion of the argument of Proposition 2 and is omitted. The idea is that (a) and (b) ensure that a condition like (i) holds, that (c) prevents 'overshooting' [condition (ii)], and that condition (iii) is not changed. We may conclude that the dynamics of the modified MBM are essentially the same as those of our basic MBM, so (S2) involves less loss of generality than one might suppose.

3. The Monetary Microdynamic Model (MMM)

The MBM establishes that trade specialists can implement Walrasian equilibria, but it has several shortcomings. The barter system it envisions requires a centralized authority to enforce the budget constraints, and is not globally defined. Also, specialists as yet have no source of revenue. As a remedy we now introduce money as a medium of exchange. Households then are constrained in their purchases only by cash balances, not by Say's Principle, eliminating the need for a central authority. Hence cash income (obtained by selling endowed goods), although not directly consumed, is valuable and appears in the utility function. (The ambitious reader can derive such a utility function in a stochastic intertemporal setting.) A second innovation in this monetary model (MMM) is to allow each specialist a spread between his buying and selling prices. Formally, we redefine *prices* as $p = (p^-, p^+) = (p_1^-, \dots, p_I^-, p_1^+, \dots, p_I^+) \in R_{++}^{2I}$.

Each household j is now characterized by (U^j, s^j, θ^j) where endowments s^j still satisfy (H1), but preferences are now defined over net income y as well as consumption d , and possibly affected by cash balances M (assumed non-negative) and prices p . Specifically, we assume that preferences are represented by a utility function $U^j: R_+^I \times R \times R_+ \times R_{++}^{2I} \rightarrow R$, $(d, y, M, p) \mapsto U^j(\cdot)$, such that (H2) holds, with (b)–(d) holding in d for every $(y, M, p) \in R \times R_+ \times$

R_{++}^{2l} . In addition, we assume

(H2)

- (e) U^j is *strictly monotonic* and *additively separable* in y , i.e., it can be written as $U^{j1}(d) + U^{j2}(y, M, p)$, and
 (f) U^j is *homogeneous of degree 0* jointly in its last three arguments, i.e., $\forall (d, y, M, p) \in \text{dom } U^j$ and $c \in R_+$, $U^j(d, cy, cM, cp) = U^j(d, y, M, p)$.

Once again, these conditions are far stronger than necessary, but will simplify the exposition. To close the model, we assume given 'ownership shares' θ^j satisfying

(H3) $\theta^j = (\theta_1^j, \dots, \theta_l^j) \in R_+^l$ and $\sum_{j=1}^n \theta_i^j = 1$, for each i .

The household's optimization problem yields desired net trades x^{*j} as follows. For simplicity we drop j and t indices for the time being, and use the notation $c^+ = \max\{0, c\}$, $c^- = \min\{0, c\}$, $c \in R$ and $v^+ = (v_1^+, \dots, v_n^+)$, $v \in R^n$. (This notation does *not* apply to p , of course.) Now let the *finance-constrained trade set* $T(s, p, M) = \{x \in R^l \mid p^+ \cdot x^+ \leq M \text{ and } x + s \geq 0\}$. For given p and M (and s), let x^* maximize $V(x) = U(s + x, y(x), M, p)$ on $T(s, p, M)$, where $y(x) = \pi^j - p^+ x^+ - p^- x^-$ is net income, and $\pi^j = \theta^j \cdot \pi$, for the vector $\pi \in R_+^l$ of 'payouts' by the specialists described below.¹ Assuming $p^+ \geq p^-$ (as we do in (S3) below) ensures that $y(x)$ is concave in x , and therefore (H2) (c) and (e) imply that $V(x)$ is strictly quasi-concave. (Evidently one could replace (c) and (e) by the direct assumption that U^j is strictly quasi-concave in (d, y) .) Since T is convex, we conclude that x^* exists and is unique. In fact, it can easily be verified that it is a continuous, piecewise differentiable function of all parameters and homogenous of degree 0 in (M, p) .

The characterization of specialists is also more complicated in this model. (S2) still applies, as does (S1) with the following convention regarding 'spread' or markup:

(S3) For some $\sigma_i \geq 0$, $p_i^+ = (1 + \sigma_i)p_i^-$; the rule f in (S1) applies to p_i^- ; and f_i is simple.

Additional structure is provided by

(S4) There is some payout rule $\pi_i: R_+ \times R \times R_+ \rightarrow R_+$, $(M_i, NCR_i, p_i^-) \mapsto (M_i + NCR_i - M^*(p_i^-))^+$, where $NCR_i = p_i^+ x_i^{+T} + p_i^- x_i^{-T}$ and $M_i^* = a_i p_i^-$ for some $a_i > 0$.

¹Reference to (S4–5) below reveals that π_i will in general depend on the x^{*j} 's, so it is not strictly correct to regard π^j as independent of x as we do here. However, the dependence will vanish in a 'large' economy, and even in a 'small' economy seems to have little substantial effect, although it does complicate the algebra. (In effect, households face slightly different prices p^+ and p^- depending on their shares θ^j .)

Finally, we have a rationing scheme

- (S5) For each i there is some $Q_i: R^n \times R_+ \times R_+^2 \times R_+ \rightarrow R^n$, $(x_i^*, M_i, (p_i^+, p_i^-), S_i) \mapsto x_i = (x_i^1, \dots, x_i^n)$ such that
- (a) for j such that $x_i^{*j} > 0$ (sellers): $0 \leq x_i^j \leq x_i^{*j}$, and if $M_i + p_i^+ x_i^{*+T} \geq p_i^- x_i^{*-T}$ (i.e., no cash shortfall), then $x_i^j = x_i^{*j}$,
 - (b) for j such that $x_i^{*j} < 0$ (buyers): $x_i^{*j} \leq x_i^j \leq 0$, and if $S_i + x_i^{*-T} \geq x_i^{*+T}$ (i.e., no inventory outage), then $x_i^j = x_i^{*j}$,
 - (c) if $x_i^{*j} > x_i^j$ for some j , then $M_i + p_i^+ x_i^{+T} = p_i^- x_i^{-T}$, and
 - (d) if $x_i^{*j} < x_i^j$ for some j , then $S_i + x_i^{-T} = x_i^{+T}$.

[In essence, (S5) calls for minimal short-side rationing when required to prevent M_i or S_i from becoming negative.]

The last element of our model is the *total money stock* $M_T^T = M^T + M_T > 0$ where $M^T = \sum M^j$ is the sum of household cash balances and $M_T = \sum M_i$ is the sum of specialist cash balances.

We may now define an *MMM economy* as a triple

$$\mathcal{E}'' = (\{(U^j, s^j, \theta^j)\}_{j=1}^n, \{(f_i, Q_i, \pi_i, \sigma_i, D_i)\}_{i=1}^l, M_T^T),$$

satisfying (H1–3) for each j and (S1–5) for each i . A *state* of \mathcal{E}'' is a vector $(p, S, M) \in R_+^{2l} \times R_+^l \times R_+^{l+n}$. Note that there is redundancy in this description of a state, since p^+ is determined by (S3) from p^- given σ . Hence we will often indulge in a slight abuse of notation and write $p \in R^l$ to denote p^- , with p^+ left implicit. Anticipating assumption (MD3) below, we take it that M_T^T is the same in any attainable state of \mathcal{E}'' . Hence we can take as our *state space* the $(3^l + n - 1)$ set

$$W = W(M_T^T, \sigma) = \left\{ (p, S, M) \in R_+^{2l} \times R_+^l \times R_+^{l+n} \mid \sum_{i=1}^l M_i + \sum_{j=1}^n M^j = M_T^T \text{ and } p_i^+ = (1 + \sigma_i) p_i^-, i = 1, \dots, l \right\}.$$

The *dynamical process* $F = F(\mathcal{E}'')$ for an MMM economy \mathcal{E}'' may now be defined as the map

$$F: W \rightarrow W, (p(t), S(t), M(t)) \mapsto (p(t+1), S(t+1), M(t+1)),$$

where, for all $i = 1, \dots, l$

$$(MDI) \quad S_i(t+1) = S_i(t) - x_i^T(t), \quad \text{where} \quad x_i^T(t) = \sum_{j=1}^n x_i^j(t) \quad \text{and} \quad x_i^j(t) = Q_i(x_i^*(t), M_i(t), p_i(t), S_i(t)),$$

(MD2) $p_i^-(t+1) = p_i^-(t) + f_i(x_i^T(t), X_i(t))$, where $X_i(t) = D_i - S_i(t+1)$, and $p_i^+(t+1) = (1 + \sigma_i)p_i^-(t+1)$; and

(MD3) $M_i(t+1) = M_i(t) + p_i^-(t)x_i^{-T}(t) + p_i^+(t)x_i^{+T}(t) - \pi_i(t)$, while for $j = 1, \dots, n$, $M^j(t+1) = M^j(t) + \theta^j \pi(t) - p^-(t)(x^j(t))^- - p^+(t)(x^j(t))^+$.

Intuitively, we can summarize the new structure as allowing householders to wander at will among decentralized trading specialists, buying or selling goods for cash. The specialist's markup σ_i is exogenous here; presumably it comes from competition, given inventory-carrying costs, etc. Specialists can ration supply or demand for their goods, when necessary, so the process is globally defined. Inventories adjust according to realized (possibly rationed) excess flow demand, and prices adjust according to simple rules. (Sophisticated rules are less interesting in this decentralized context.) Money balances M then evolve by the obvious accounting relations involving cash income, expenditures and specialists' payout. Note this last assumption (MD3) implies that total money stock is constant.

The *steady-state equilibria* of \mathcal{E}'' again are the fixed points of $F(\mathcal{E}'')$ on $W = W(M_T^T, \sigma)$; the set of all such equilibria is denoted $MSE(\mathcal{E}'')$.

Other useful definitions are: let $\mathcal{E} = \{(U^j, s^j)\}_{j=1}^n$, $\mathcal{E}' = (\{(U^{j'}, s^{j'}\}_{j=1}^n, \{(f'_i, D_i)\}_{i=1}^l)$ and $\mathcal{E}'' = (\{(U^{j''}, s^{j''}, \theta^{j''})\}_{j=1}^n, \{(f''_i, Q''_i, \pi''_i, \sigma''_i, D''_i)\}_{i=1}^l)$ be respectively Walrasian MBM and MMM economies. We say that \mathcal{E}'' corresponds to \mathcal{E} if, for each $j = 1, \dots, n$, $S^j = S^{j''}$ and $U^{j''}(d, y, M, p) = h^j(U^j(d), (y, M, p))$ for some smooth function h^j and all $(d, y, M, p) \in \text{dom}(U^{j''})$. We say that \mathcal{E}'' corresponds to \mathcal{E}' if they both correspond to the same Walrasian economy \mathcal{E} and if $(f'_i, D_i) = (f''_i, D_i)$ for $i = 1, \dots, l$.

We say that there is *no rationing* at a state $(p, S, M) \in W$ of an MMM economy \mathcal{E}'' if $x_i = Q_i(x_i^*, M, p, S_i) = x_i^*$ for $i = 1, \dots, l$.

Proposition 3. (Correspondence principle.) *Given a Walrasian economy \mathcal{E} , there is a MMM economy \mathcal{E}'' corresponding to \mathcal{E} such that for every $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$, there is an $\bar{S} \gg 0$ and $\bar{M} \gg 0$ such that $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E}'')$. Conversely, if a Walrasian \mathcal{E} corresponds to a given MMM \mathcal{E}'' with $\sigma = 0$, if $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E}'')$ and if there is no rationing at $(\bar{p}, \bar{S}, \bar{M})$, then there is an \bar{x} such that $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$.*

Proof. Fix $\mathcal{E} = \{(U^j, s^j)\}_{j=1}^n$, and let $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$. We know from intermediate price theory that, for all j , $\exists \lambda^j > 0$ such that the first-order conditions $MU_i^j(s^j + \bar{x}^j) = \lambda^j \bar{p}_i$, $i = 1, \dots, l$, characterize \bar{x} . Set $b^j = \lambda^j \bar{p} \cdot (\bar{x}^j)^+$, and let $V^j(d^j, y^j, M^j) = U^j(d^j) + (b^j/M^j)y^j$. Set $M_0^j = \bar{p} \cdot (\bar{x}^j)^+$, $\sigma = 0$, $S_i^0 = D_i$, $M_i^0 = M_i^*(\bar{p}_i)$ and complete the construction of \mathcal{E}'' essentially arbitrarily.

Lemma. $x^* \in R^l$ is a solution to $\max v(x) \equiv V^j(s^j + x, y(x), M^j)$ on $T(s^j, \bar{p}, M^j)$ for $M^j = \bar{p} \cdot (\bar{x}^j)^+$, iff $x^* = \bar{x}^j$.

Proof. Given our strong assumptions, the *unconstrained* solution to the max problem will be characterized by the first order conditions

$$0 = v_i(x) = MU_i^j(s^j + x) + (b^j/M^j)(-\bar{p}_i), \quad i = 1, \dots, l. \quad (2)$$

But the last term by definition is just $-\lambda^j \bar{p}_i$, so these are the *same* first order conditions as those that characterize \bar{x}^j . It is easy to check that the solution to (2) actually lies in $T(s^j, \bar{p}, M^j)$, so $x^* = \bar{x}^j$ and the lemma is proved.

Using the lemma, it is straightforward to verify that $(\bar{p}, S^0, M^0) \in MSE(\mathcal{E}'')$.

Suppose now that $(\bar{p}, \bar{x}) \in WE(\mathcal{E})$. We wish to show that the net trades \bar{x} also arise in \mathcal{E}'' for some price vector \hat{p} proportional to \bar{p} and some redistribution \hat{M} of M^0 , and that $(\hat{p}, S^0, \hat{M}) \in MSE(\mathcal{E}'')$. Let $M_i^* = a_i \bar{p}_i$ and for each $j = 1, \dots, l$, let $\hat{\lambda}_j = MU_i^j(s^j + \bar{x}^j)/\bar{p}_i$ for any i . Set $c = (\sum_{i=1}^l a_i \bar{p}_i + \sum_{j=1}^n b^j/\hat{\lambda}_j)/M_0^T$. Then take $\hat{p} = (1/c)\bar{p}$, $\hat{M}^j = b^j/c\hat{\lambda}_j$, and $\hat{M}_i = a_i \bar{p}_i/c$. By construction, $\hat{M}^T = M_0^T$, so $(\hat{p}, S^0, \hat{M}) \in W(M_0^T, 0)$. Again by construction, $(b^j/\hat{M}^j)\hat{p}^j = (b^j(\frac{1}{c}\bar{p})/b^j/(c\hat{\lambda}_j)) = \bar{\lambda}^j \bar{p}$, so another application of the lemma reveals that \bar{x} will indeed be the desired (and hence realized) net trade vector in \mathcal{E}'' at (\hat{p}, S^0, \hat{M}) . Since $\bar{X}^T = 0$, we have $\Delta S = 0$; since $S^0 = D$ also, $\Delta p = 0$; and since $\sigma = 0$, it follows that $\Delta M = 0$. Hence again $(\hat{p}, S^0, \hat{M}) \in MSE(\mathcal{E}'')$, and the first part of the proposition has been established.

The second part is straightforward. Given such a $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E}'')$, let $\bar{x} = x^*(\bar{p}, \bar{M}, S) = Q(x^*(\cdot), \bar{M}, \bar{p}, \bar{S})$. We have $\Delta S = 0$ so $\bar{x}^T = 0$, i.e., these net trades are mutually consistent in the \mathcal{E} corresponding to \mathcal{E}'' . A final application of the lemma shows that they are also the desired net trades in \mathcal{E} , so $(\bar{x}, \bar{p}) \in WE(\mathcal{E})$. Q.E.D.

Remark 1. The argument shows that if we normalize the prices in \mathcal{E} , and if $\sigma = 0$ and M_1^T is properly chosen in \mathcal{E}'' , then there is a 1:1 correspondence between $WE(\mathcal{E})$ and the subset of $MSE(\mathcal{E}'')$ for which no rationing occurs. In particular, steady-state equilibria exist and (at least some) are efficient in an MMM with $\sigma = 0$.

Remark 2. To understand the situation when $\sigma > 0$, imagine a family $\{\mathcal{E}_\sigma'' \mid \sigma \geq 0\}$ of MMM whose members differ only in σ . Evidently the sets $MSE(\mathcal{E}_\sigma'')$ depend continuously on σ , and our 'efficiency' relation for equilibria becomes

$$\bar{p}_i = \bar{p}_i^- \leq MRS_{iy}^j(s^j + x^{*j}(\bar{p}), y(x^{*j}(\bar{p})), \bar{M}^j, \bar{p}) \leq \bar{p}_i^+ = (1 + \sigma_i)\bar{p}_i,$$

where MRS_{iy}^j is j 's marginal rate of substitution of good i for income. For small σ we should therefore expect existence and near-efficiency of no-rationing equilibria; for very large σ we should expect autarky as an

equilibrium, at least if $s^j \in R_{++}^I$. A topological argument (which would be beyond the scope of this paper) seems required to establish these results rigorously, but at least the existence of equilibria for $\sigma > 0$ is clear.

Remark 3. The qualification of ‘no-rationing’ is not vacuous – there are generally states in $MSE(\mathcal{E}'')$ for which the efficiency relation fails because of rationing. Such states have no corresponding Walrasian equilibria; we refer to them as *effective demand failure equilibria*. An example is in order, and for this purpose we must construct a simple but reasonable rationing scheme Q . In *priority rationing* we assume $x_i^j = x_i^{*j}$ for all $j > J_i$, and $x_i^j = 0$ for $j < J_i$. (S5)(c–d) determine J_i as the smallest non-negative integer for which this may be done, and also yield $x_i^{J_i}$ as a residual.

Example. Let \mathcal{E} be an MMM economy with $\sigma = 0$, $Q = \text{priority rationing}$, $s^1 = (1, 0, \dots, 0)$, and $s^2 = (0, 1, 0, \dots, 0)$.

Let \mathcal{E}' be another MMM economy derived from \mathcal{E} by deleting households 1 and 2, and let $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E}')$. Then set $\bar{M} = \bar{M}$ except that $\bar{M}^j = \bar{M}_i = 0$ for $i, j = 1, 2$. It is straightforward to verify that $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E})$ and that $x_1^{*1} < x_1^1 = 0$ and $x_2^{*2} < x_2^2 = 0$. It appears that one could extend this example to the $\sigma > 0$ case at the cost of some complication if one relaxes (H2)(d).

Remark 4. The argument in Proposition 3 makes it clear that price levels are not so indeterminate in MMM economies. If $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E}'')$, then in general $(c\bar{p}, \bar{S}, \bar{M}) \notin MSE(\mathcal{E}'')$ for $c \neq 1$. On the other hand, $(c\bar{p}, \bar{S}, c\bar{M}) \in MSE(\mathcal{E}')$ for all $c > 0$, where \mathcal{E}' differs from \mathcal{E}'' only in its total money stock M_T^1 , because of the degree 0 homogeneity of demand. This observation leads to neutrality propositions, which we shall examine shortly as we study the stability properties of our economy.

We begin our stability analysis with a closer look at household expenditures.

Proposition 4. For given prices p , each household j has a ‘target cash balance’ $M^{*j}(p)$ such that desired net income $y(x^{*j}(p))$ is positive (resp. negative) whenever actual cash balances M^j are less than (exceed) $M^{*j}(p)$.

Proof. (Sketch.) We drop the superscript j to simplify notation. From the definition of x^* one can see that $M = 0$ implies that $(x^*(p))^+ = 0$ (no expenditures desired) and $-\bar{p}(x^*(p))^- > 0$ (positive desired sales revenue), so $y(x^{*j}(p)) > 0$. On the other hand, strict monotonicity of U in d implies that $d \gg s$ and in fact $y(x^*(p)) < 0$ for M sufficiently large. The quasi-concavity of $V(x)$ and the structure of $T(p, s, M)$ (defined in the beginning of this section) imply that $y(x^*(p, M))$ is increasing in M , and the continuity of $x^*(p, M)$ implies that $y(x^*(p, M))$ is continuous in M . Hence by the intermediate value theorem, there is some $M^*(p)$ with the desired properties. Q.E.D.

Proposition 4 shows that households have the 'homeostatic' property that they build up their cash balances when low, and run them down when high. To avoid overshooting when this process is near completion, we impose the additional, rather innocuous assumption

(H2)

(g) For each household j , $\partial y^j(x^{*j})/\partial M^j > -1$ if $y^j = 0$.

That is, if cash balances were to increase exogenously by \$1, then net expenditures would increase by less than \$1, *ceteris paribus*.

Specialists have a similar, built-in homeostat in their desired cash balances by (S4). These individual homeostats do not suffice for the economy as a whole, however. Certainly we need to know that the 'real sector' is stable. In analogy to the matrices of substitution effects of the previous section, we define² the $l \times l$ Hicksian matrix A_σ at $(p, S, M) \in W$ for an MMM economy with $\sigma \geq 0$ by

$$A_\sigma = \left(\begin{array}{c|c} \frac{\partial x_i^T}{\partial p_k} & \\ \hline & \end{array} \right)_{(p, S, M)}.$$

We say that a pricing rule f is *stabilizing* at (p, S, M) if conditions (i–iii) in the proof of Proposition 2 are satisfied on H . Recall that this is not a very stringent property.

A final condition that we will employ concerns the effects on the composition of demand of a cash redistribution among households. We say that an MMM economy \mathcal{E}'' has *distributional effects of order ε* at $(\bar{p}, \bar{S}, \bar{M})$ if

$$\left\| \left(\begin{array}{c|c} \frac{\partial x_i^T}{\partial M^j} & \\ \hline & \end{array} \right)_{(\bar{p}, \bar{S}, \bar{M}), M^T = \bar{M}^T} \right\| < \varepsilon.$$

That is, the norm of a certain rank $n-1$ matrix is small. The matrix is that which arises from taking directional derivatives of x^T in directions in M for which $M^T = \sum M^j$ remains constant. One can show, for instance, that homothetic identical preferences among households imply distribution effects of order 0.

We now define local stability and neutrality. An equilibrium $(\bar{p}, \bar{S}, \bar{M}) \in MSE(\mathcal{E})$ is *locally stable* if it has some open neighborhood N in W such that $p(t) \rightarrow \bar{p}$, $S(t) \rightarrow \bar{S}$ and $M(t) \rightarrow \bar{M}$ as $t \rightarrow \infty$ whenever

²Recall that the x^{*} 's are only piecewise differentiable, so there is a set of measure zero where A_σ is not defined. This leads to technical complications but does not prevent the argument from going through; we'll ignore this problem for the remainder of the paper.

$(p(0), S(0), M(0)) \in N$. The equilibrium $(\bar{p}, \bar{S}, \bar{M})$ has the *asymptotic neutrality property (ANP)* locally if it has some open neighborhood $N' \subset R^l \times R^l \times R^{l+n}$ such that $p(t) \rightarrow c\bar{p}$, $S(t) \rightarrow \bar{S}$ and $M(t) \rightarrow c\bar{M}$, where $c = M_T^T(0)/M_T^T$, as $t \rightarrow \infty$ whenever $(p(0), S(0), M(0)) \in N'$ and $S(0) = \bar{S}$.

Proposition 5. *A no-rationing equilibrium $(\bar{p}, \bar{S}, \bar{M}) \in \text{MSE}(\mathcal{E})$ of an MMM economy \mathcal{E} with $\sigma \gg 0$ is locally stable and has the ANP locally if the following conditions are satisfied:*

- (a) *the pricing rule f is stabilizing for the Hicksian matrix A_σ at $(\bar{p}, \bar{S}, \bar{M})$, and*
- (b) *distributional effects are of order ε for ε sufficiently small.*

Proof. Given $(p, S, M) \in W$, define $z = (\hat{p}, \hat{S}, \hat{M}) \in H_{\bar{p}} \times R^l \times H_{\bar{M}}$ as follows. Let $\hat{p} = Z_{\bar{p}}(p - \bar{p})$ be orthogonal projection of the price discrepancy $(p - \bar{p})$ onto $H = \langle \bar{p} \rangle^\perp$, i.e., the 'relative price' discrepancy. (See lemma for Proposition 2.) Let $\hat{S} = S - \bar{S}$, and $\hat{M} = Z_{\bar{M}}(M - \bar{M})$, where $M = (M^1, \dots, M^n)$, be orthogonal projections of cash balances onto $H_{\bar{M}} = \langle \bar{M} \rangle^\perp$; \hat{M} may be thought of as the cash gains and losses relative to \bar{M} associated with a redistribution of M among households. One can verify that $p = \hat{p} + c_1 \bar{p}$ and $M = \hat{M} + c_2 \bar{M}$ for some $c_1, c_2 \geq 0$.

We are interested in $z = (\hat{p}, \hat{S}, \hat{M})$ because the dynamics $F(\mathcal{E})$ on W (which is not a linear space) may be adequately represented in terms of z on the $(l-1) + l + (n-1)$ dimensional linear space $H_{\bar{p}} \times R^l \times H_{\bar{M}}$. The crucial step in our argument is to show that $z(t) \rightarrow 0$ exponentially if $z(0)$ is sufficiently close to 0. To this end, take a first-order Taylor expansion of $F(\mathcal{E})$ around $(\bar{p}, \bar{S}, \bar{M})$. We may write $z(t+1) \simeq Tz(t)$, where T is the partitioned matrix

$$T = \begin{bmatrix} B & \vdots & C \\ -D & \vdots & E \end{bmatrix} \begin{matrix} 2l-1 \\ n-1 \end{matrix}$$

It suffices to show that all eigenvalues of T have modulus < 1 . If D were the 0-matrix, a straightforward application of the partitioned matrix formula would show that we only need consider the eigenvalues of B and E . But D is a submatrix of

$$D' = \left(\frac{\partial p}{\partial M}, \frac{\partial S}{\partial M} \right) = \left(\frac{\partial p}{\partial x^T} \frac{\partial x^T}{\partial M} + \frac{\partial p}{\partial S} \frac{\partial S}{\partial M}, - \frac{\partial x^T}{\partial M} \right)$$

$$= (\alpha G - \beta G, -G), \quad \text{where}$$

$$G = \left(\frac{\partial x^T}{\partial M} \mid \right)_{(\bar{p}, \bar{M})}$$

is the matrix of 'distributional effects', and α and β are the positive diagonal matrices arising from the pricing rule. Hence D can be made arbitrarily small by an application of hypothesis (b) of the proposition.

Now (MD3) implies that $\hat{M}^j(t+1) = \hat{M}^j(t) + y^j(x^{*j}(M^j, \dots))$, so E is a submatrix of a diagonal matrix E' with diagonal entries (eigenvalues)

$$E'_{jj} \left(1 + \frac{\partial y^j}{\partial M^j} \bigg|_{y^j=0} \right),$$

which have modulus < 1 by the homeostatic properties of households. Hence the eigenvalues of E also have modulus < 1 .

Finally, hypothesis (a) implies that B is precisely the operator analyzed in Proposition 2, which was shown to have all eigenvalues of modulus < 1 when restricted to H . Hence the hypotheses of the proposition ensure that T has all eigenvalues < 1 in modulus, and therefore $z(t) \rightarrow 0$ exponentially.

As an immediate consequence, $S(t) \rightarrow \bar{S}$ and $x^T(t) \rightarrow 0$ exponentially, as $t \rightarrow \infty$. These two results imply that $\Delta p(t) \rightarrow 0$ exponentially, so $p(t)$ converges to some \bar{p} . Now for some c^* , we have $\bar{p} = c^* \bar{p}$ since $\hat{p}(t) \rightarrow 0$. On the other hand, $z(t) \rightarrow 0$ implies $M \cdot(t) - c(t) \bar{M} \rightarrow 0$ for some sequence $c(t)$ of positive numbers. Actually, we must have $c(t) \rightarrow c^*$: if not, say w.o.l.o.g. $c(t) > c^* + \varepsilon$ for infinitely many t , then for such t and some $\delta > 0$,

$$\begin{aligned} x^T(t) &= \sum_{j=1}^n x^{*j}(p(t), M^j(t)) \rightarrow \sum x^{*j}(c^* \bar{p}, c(t) \bar{M}^j) > \sum \{x^{*j}(c^* \bar{p}, c \bar{M}^j) \\ &\quad + \delta\} = n\delta > 0, \end{aligned}$$

a contradiction of $x^T(t) \rightarrow 0$ (in the last two steps, we used the continuity and homogeneity of x^{*j}). One can now verify that $c^* = M^T(\infty)/\bar{M}^T$, i.e., the ratio of cash held by all households at the end of the process to that held by households at the basic equilibrium. We now need only show that $M_i(t) \rightarrow c^* \bar{M}_i$, indeed, the exponential convergence of $p(t)$ (to $c^* \bar{p}$) and x^T to 0 imply that each $M_i(t)$ converges, and the payout rule ensures that $M_i(\infty) \leq a_i c^* \bar{p} = c^* \bar{M}_i$. At this point, we finally use the assumption $\sigma \gg 0$ to deduce that $NCR_i(\infty) > 0$, so $M_i(\infty) \geq c^* \bar{M}_i$. From this it follows that $(p(t), S(t), M(t)) \rightarrow (c^* \bar{p}, \bar{S}, c^* \bar{M})$ for all $(p(0), S(0), M(0))$ sufficiently close to $(\bar{p}, \bar{S}, \bar{M})$ and that $c^* = M^T(0)/\bar{M}^T$. The local ANP is the special case $S(0) = \bar{S}$ and local stability is the special case $M^T(0) = \bar{M}^T$. Q.E.D.

There are many directions in which these models might be extended. One could pose more sophisticated (intertemporal, stochastic) optimization problems for specialists [see Friedman (1984)] and households. One could

consider other organizational modes for markets (such as search, brokers, or auctions) either separately or in conjunction with a trading specialist sector. Ultimately, one would like to include financial institutions and government, so macroeconomic policy issues can be addressed. Evidently the work presented here is but a first step.

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