University of California Santa Cruz
Econ/Cmps 166a/272 - Evolutionary Game Theory
Profs. Friedman & Sinervo – Winter/2017

Answer Key Problem Set 2[1]

Question 1

Textbook (Friedman and Sienervo, 2016) problem 2.1, p. 60.

Solution

Following the book example (sections 2.1 and 2.2) we have the following assumptions:

- For the outcome of HH, each group contains a probability of 1/2 of losing, giving the payoff of $(v-c)/2$.
- The outcome DH gives a payoff of 0
- The outcome HD gives a payoff of v
- The outcome DD gives a payoff of $v/2$

It can be shown on the following table, as the book:

Table 1: H-D Parameter and Dynamics

	Encounter Rate:	
	s_H	$1-s_H$
Choice	H	D
H	$\frac{v-c}{2}$	v
D	0	$\frac{v}{2}$

Creating a dynamic replicator in excel spreadsheet to simulate the dynamics for general HD games will be as following, starting with the parameters' derivations:

[1] Thanks to Fernando Chertman

$W_H = s_H W_{HH} + (1 - s_H) W_{HD} = s_H \frac{v-c}{2} + (1 - s_H) v$
$W_D = s_H W_{DH} + (1 - s_H) W_{DD} = (1 - s_H) \frac{v-c}{2}$
$\overline{W} = s_H W_H + (1 - s_H) W_D$
Considering also the discrete time replicator equation as $s_i(t+1) = \frac{W_i}{\overline{W}} s_i(T)$

With the parameters v, c and s_H free to be changed over the excel sheet, we get some results as below (starting with c = 4, v = 6 and $s_H = 1/2$):

Table 2: Replicator Dynamics

t	v	c	s_H	s_D	W_H	W_D	$\bar{W}$	s'_H
0	4	6	0.5	0.5	1.5	1	1.25	0.60000
1	4	6	0.6000	0.4	1.0	0.8	0.92	0.65217
2	4	6	0.6521	0.3478	0.7391	0.6956	0.7240	0.6658
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5	4	6	0.666667	0.33333	0.666667	0.666667	0.666667	0.666667

With c = 6, v = 6 and $s_H = 0.5$

Table 3: Replicator Dynamics

t	v	c	s_H	s_D	W_H	W_D	$\bar{W}$	s'_H
0	6	6	0.5	0.5	3.0	1.5	2.25	0.6667
1	6	6	0.6667	0.3333	2.0	1.0	1.6667	0.8000
2	6	6	0.8000	0.2000	1.2000	0.6000	1.0800	0.8889
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	6	6	1.0000	0.0000	0.0000	0.0000	0.0000	1.0000

With c = 8, v = 6 and $s_H = 0.5$

Table 4: Replicator Dynamics

t	v	c	s_H	s_D	W_H	W_D	$\bar{W}$	s'_H
0	8	6	0.5	0.5	4.5	2.0	3.25	0.6923
1	8	6	0.6923	0.3076	3.1538	1.2307	2.5621	0.8521
2	8	6	0.85219	0.1478	2.0346	0.5912	1.8213	0.9520
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	8	6	1.0000	0.0000	1.0000	0.0000	1.0000	1.0000

Question 2

Textbook problem 2.2, p. 60.

Solution

Creating a dynamic replicator in excel spreadsheet to simulate the RPS and other 3 x 3 matrix games will be as following, starting with the parameters' derivations:

$W_R = s_R W_{R,R} + s_P W_{R,P} + s_S W_{R,S}$
$W_P = s_R W_{P,R} + s_P W_{P,P} + s_S W_{P,S}$
$W_S = s_R W_{S,R} + s_P W_{S,P} + s_S W_{S,S}$
Considering also the discrete time replicator equation as $s_i(t+1) = \frac{W_i}{\bar{W}} s_i(T)$

Checking the three payoff tables as disposed on pages 54 and 55, with their respective simulations:

Table 5: RPS Payoff Table

encounter rate:	s_r	s_p	s_s
Player:	R	P	S
R	1	1/2	3/2
P	3/2	1	1/2
S	1/2	3/2	1

This would give the following outward spiraling until it almost reaches the edges of the simplex (starting with $s_R = 0.4$, $s_P = 0.4$ and $s_S = 0.2$):

Table 6: Replicator Dynamics

t	s_R	s_P	s_S	W_R	W_P	W_S	$\bar{W}$	s'_R	s'_P	s'_S
0	0.40	0.40	0.20	0.90	1.10	1.0	1.00	0.3600	0.4400	0.2000
1	0.36	0.44	0.2000	0.8800	1.0800	1.0400	1.0000	0.3168	0.4752	0.2080
2	0.3168	0.4752	0.2080	0.8664	1.0544	1.0792	1.0000	0.2744	0.5010	0.22447
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
530	0.0000	1.0000	0.0000	0.5000	1.0000	1.5000	1.0000	0.0000	1.0000	0.0000

For the next payoff table (basically putting more weight in the winning payoff situation), we will see by the simulation that the cycle will die out and the state will spiral into the center of the simplex:

Table 7: RPS Payoff Table

encounter rate:	s_r	s_p	s_s
Player:	R	P	S
R	1	1/2	4
P	4	1	1/2
S	1/2	4	1

Starting with $s_R = 0.4$, $s_P = 0.4$ and $s_S = 0.2$)

Table 8: Replicator Dynamics

t	s_R	s_P	s_S	W_R	W_P	W_S	$\bar{W}$	s'_R	s'_P	s'_S
0	0.40	0.40	0.20	1.40	2.10	2.0	1.80	0.3111	0.4666	0.2222
1	0.3111	0.4666	0.2222	1.4333	1.8222	2.2444	1.7950	0.2484	0.4737	0.2778
2	0.2484	0.4737	0.2778	1.5967	1.6063	2.2969	1.7958	0.2287	0.4237	0.3553
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
270	0.3333	0.3333	0.3333	1.8333	1.8333	1.8333	1.8333	0.3333	0.3333	0.3333

For the next payoff table (basically putting more weight in the winning payoff situation as the first scenario, but less than the second), we will see that the cycling goes forever with the same amplitude:

Table 9: RPS Payoff Table

encounter rate:	s_r	s_p	s_s
Player:	R	P	S
R	1	1/2	2
P	2	1	1/2
S	1/2	2	1

Starting with $s_R = 0.4$, $s_P = 0.4$ and $s_S = 0.2$)

Table 10: Replicator Dynamics

t	s_R	s_P	s_S	W_R	W_P	W_S	$\bar{W}$	s'_R	s'_P	s'_S
0	0.40	0.40	0.20	1.00	1.30	1.20	1.16	0.3448	0.4482	0.2069
1	0.3448	0.4482	0.2068	0.9827	1.2413	1.2758	1.1593	0.2923	0.4800	0.2276
2	0.2923	0.4800	0.2276	0.9876	1.1784	1.3338	1.1580	0.2493	0.4884	0.2622
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
700	0.2552	0.4888	0.2559	1.0115	1.1272	1.3611	1.1576	0.2230	0.4759	0.3009

Question 3

Textbook problem 2.6, p. 61.

Solution

$$W = \begin{pmatrix} 1.00 & 1.18 & 0.88 \\ 0.85 & 1.00 & 1.16 \\ 1.13 & 0.86 & 1.00 \end{pmatrix}$$

A game is *true RPS* if the following strict inequalities hold:

- $w_{s,r} < w_{r,r} < w_{p,r}$
- $w_{r,p} < w_{p,p} < w_{s,p}$
- $w_{p,s} < w_{s,s} < w_{r,s}$

Looking for the three edge games (calling A - first row/column, B - second row/column and C - third row/column):
1. Strategies A and B

$$W1 = \begin{pmatrix} 1.00 & 1.18 \\ 0.85 & 1.00 \end{pmatrix}$$

In this sub-game A dominates B. 2. Strategies A and C

$$W2 = \begin{pmatrix} 1.00 & 0.88 \\ 1.13 & 1.00 \end{pmatrix}$$

In this sub-game C dominates A. 3. Strategies B and C

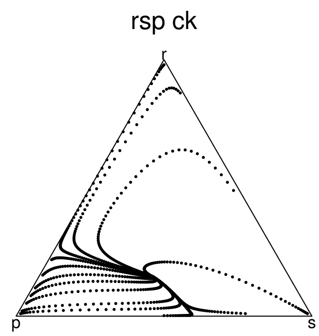
$$W3 = \begin{pmatrix} 1.00 & 1.16 \\ 0.86 & 1.00 \end{pmatrix}$$

In this sub-game B dominates C.

So we have this circular dominance among strategies A, B and C.

To visualize the dynamics of matrix W, we get the below picture generated in R (deSolve package):

Figure 1: Dynamics



Question 4

Extra credit: problems 2.3 - 2.5, p. 61 and 2.7, p. 62.

Solution

Question 2.3

$$P = \begin{pmatrix} 0 & 5 \\ -1 & 3 \end{pmatrix}$$

(a) The first strategy is dominant, because it gives a higher payoff (or fitness) against any strategy adopted by other players ($0 > -1 and 5 > 3$).

(b) $W_A = 0 * s_A + 5(1 - s_A) = 5(1 - s_A)$

$W_B = -1 * s_A + 3(1 - s_A) = 3 - 4s_A$
$\overline{W} = s_A W_A + (1 - s_A)W_B = 5(1 - s_A)s_A + (3 - 4s_A)(1 - s_A) = (1 - s_A)(s_A + 3)$
$\overline{W} = (3 - 2s_A - s_A^2)$

Setting $s_A = x$, we get the desired expression $\overline{W}(x) = 3 - 2x - x^2$.

(c) The function $\overline{W}(x) = 3 - 2x - x^2$ is decreasing because $0 > \frac{\partial \overline{W}(x)}{\partial x} = -2x - 2$ for all $x \in [0, 1]$.
As $\overline{W}$ is a decreasing function of the share x of the dominant strategy, the more individuals who adopt this strategy, the lower is everyone's payoff. This scenario implies that P has a subtype of Prisoner's Dilemma (PD).

(d) $Q = \begin{pmatrix} 3 & 5 \\ -1 & 0 \end{pmatrix}$
By the same logic of (a), the first strategy is still dominant (3 $>$ -1 and 5 $>$ 0).
The mean payoff will be calculated as:

$W_A = 3x + 5(1 - x) = 5 - 2x$
$W_B = -1x + 0x = -1x$
$\overline{W} = xW_A + (1 - x)W_B = (5 - 2x)x + (-1x)(1 - x)$
$\overline{W} = (4x - x^2)$

Now $\frac{\partial \overline{W}(x)}{\partial x} = 4 - 2x > 0$ for all $x \in [0, 1]$, so W is now increasing in x.
As $\overline{W}$ is a increasing function of the share x of the dominant strategy, the more individuals who adopt this strategy, the result is a higher total payoff. This scenario implies that Q has a subtype of Laissez-Faire (LF).

Question 2.4

$W = \begin{pmatrix} W_{AA} & W_{AB} \\ W_{BA} & W_{BB} \end{pmatrix}$
Let x $=$ s_A $W_A(x) = W_{AA} * x + W_{AB}(1 - x)$
$W_B(x) = W_{BA} * x + W_{BB}(1 - x)$
$\overline{W} = W_A * s_A + W_B * s_B = W_A * x + W_B * (1 - x) =$
$(W_{AA} * x + W_{AB}(1 - x)) * x + (W_{BA} * x + W_{BB}(1 - x)) * (1 - x)$
$(W_{AA} - W_{AB} - W_{BA} + W_{BB}) * x^2 + (W_{AB} + W_{BA} - 2W_{BB}) * x + W_{BB}$

Question 2.5

(a) Note that $\frac{\partial \overline{W}(x)}{\partial x} = 2x(W_{AA} - W_{AB} - W_{BA} + W_{BB}) + (W_{AB} + W_{BA} - 2W_{BB})$.

This expression linear in x so it is always positive for $x \in [0,1]$ iff it is positive at both endpoints, $x = 0, 1$. Checking $x = 0$, we get the condition

$$\begin{aligned} (W_{AB} + W_{BA} - 2W_{BB}) &> 0 & (1)\\ \iff \frac{1}{2}(W_{AB} + W_{BA}) &> W_{BB}. & (2) \end{aligned}$$

Checking $x = 1$, we get the condition

$$\begin{aligned} 2(W_{AA} - W_{AB} - W_{BA} + W_{BB}) &+ (W_{AB} + W_{BA} - 2W_{BB}) > 0 & (3)\\ \iff 2W_{AA} - W_{AB} - W_{BA} &> 0 & (4)\\ \iff W_{AA} &> \frac{1}{2}(W_{AB} + W_{BA}). & (5) \end{aligned}$$

So the mean payoff is always increasing (as in Laissez faire) iff inequalities (2) and (5) hold, and is always decreasing iff inequalities are reversed. Note that in the latter case we have essential condition (b) on p47 for Prisoner's Dilemma (after changing the notation for the strategies from A, B to C, D).

(b) Essential condition (a) $w_{DC} > w_{CC} > w_{DD} > w_{CD}$.
Breaking the condition in two parts: $w_{DC} > w_{CC}$ and $w_{DD} > w_{CD}$.
As s_C and $1 - s_C$ are > 0, their product won't affect the inequality. So, multiplying the first part by s_C and the second part by $(1 - s_C)$, we obtain
$w_{DC} * s_C > w_{CC} * s_C$ and $w_{DD} * (1 - s_C) > w_{CD} * (1 - s_C)$.
Adding the greater part of inequalities, we obtain $w_D(x)$, and the smaller parts give us $w_C(x)$. The result is $w_D(x) > w_C(x)$. We have already noted that mean payoff is decreasing iff (b) holds. So we are done: the two conditions are sufficient for a true social dilemma.

Question 2.7

- If w1 $=$ w2 $= 0$, it means that $\Delta W = 0$, so the line in Figure 2.2 lies on the horizontal axis. The fitness of strategy A is the same as fitness of strategy B no matter what. This is a degenerate case which the state of the system stays at its initial state, whatever it is, and the states don't mean anything since the strategies are the same.

- If w1 $>$ w2 $= 0$, we have a borderline case between HD and DS in Fig 2.2. The crossing is exactly at $s_A = 1$, so the first strategy is dominant, and the unique attractor is $s_A = 1$.

- If w1 < w2 = 0, we have a borderline case between CO and DS in Fig 2.2. The crossing is exactly at $s_A = 1$, but since $\Delta W < 0$ at all other points, the second strategy is dominant, and that the unique attractor is $s_A = 0$.