Contents

Chapter 7: Evolutionary Games

After distinguishing between repeated games and evolutionary games, the chapter introduces Maynard Smith's famous Hawk-Dove game. Then it shows that there are only two other generic types of single population 2x2 games: Dominant strategy (DS), and Coordination (CO). The chapter mentions briefly higher dimensional single population games and multipopulation games. It focuses on deterministic replicator dynamics, but mentions other sorts.

1 Repeated games vs evolutionary games.

These are two quite different ways of modeling strategic interaction extending over time, and they serve quite different purposes. Three slogans summarize the difference.

1. *Survival of the fittest.* In evolutionary games, strategies that currently have higher payoff (are "fitter") become more prevalent over time, and lower payoff strategies become less prevalent. In repeated games, current stage game payoffs can be traded off against future payoffs, and the Folk Theorem shows that there is no automatic advantage to higher payoff strategies.

2. *Evolution, not revolution.* Change is continuous in evolutionary games, with no jumps in continuous time. As we will see, the population shares may ultimately converge to some particular Nash equilibrium. The focus in repeated games is on strategies that are *already in equilibrium*, and large jumps can occur from one period to the next, e.g., when punishments are triggered.

3. *"Price-taking."* In repeated games, the key tension in choosing an action is between current payoff and the impact on future opportunities, as other players react. That tension is absent in evolutionary game theory; players simply adapt to current payoffs. In perfect competition, players do not try to influence price (or their opportunity set), but take it as given. Likewise in evolutionary games, individual

players do not try to influence the future population shares that will govern future payoffs; they take these as given.

We will focus on continuous time in this chapter. In the previous chapter, we focused on discrete time, but that is not an essential difference. One can readily model evolutionary games in discrete time, and (although it is less common) can model repeated games in continuous time.

2 Evolutionary Hawk-Dove

To get started, let's work informally with the simplest non-trivial case: a single population ($k = 1$) symmetric game with two alternative actions, as described by a symmetric 2x2 bimatrix. Consider this parametric example, known as Hawk-Dove.

	H	D
H	$\frac{v-c}{2}, \frac{v-c}{2}$	$\underline{v},\underline{0}$
D	$\underline{0},\underline{v}$	$\frac{v}{2}, \frac{v}{2}$

The story is that there is a resource of value $v > 0$, which will be monopolized by players adopting an aggressive strategy (Hawk, or H) if they encounter players adopting a passive strategy (Dove, or D). Doves will split v evenly when they encounter each other. Hawks encountering other Hawks will fight over the resource, incurring cost $c > v$ due to injury or death. The winner of that conflict gains v and perhaps incurs some of the cost, most of which is borne by the loser. On average, then, Hawks evenly split both conflict cost and resource value, obtaining payoff $\frac{v-c}{2} < 0$.

Later we will consider the case in which players from one population are matched against players from a separate population. In this section we deal only with a single population, and so can rewrite the game in streamlined form.

	p	$1-p$
H	$\frac{v-c}{2}$	v
D	0	$\frac{v}{2}$

- Here p is the fraction of Hawks in the population, hence the probability that the player will encounter H, and $(1-p)$ is the population share of Doves.

- For example, you might take $v = 8$ and $c = 12$, and get the payoff matrix

$$A = \begin{pmatrix} -2 & 8 \\ 0 & 4 \end{pmatrix}$$

- Let $D(p) \equiv E\pi(H,p) - E\pi(D,p)$ be the expected payoff advantage to H. In the numerical example,

$$\begin{aligned} E\pi(H,p) &= -2p + 8(1-p) \\ E\pi(D,p) &= 0p + 4(1-p) \end{aligned}$$

- Thus,

$$\begin{aligned} D(p) &= -2p + 4(1-p) \\ D(p) &= 4 - 6p \end{aligned}$$

- The slope of $D(p) = -6 < 0$, and $D(p) = 0$ results in the mixed NE: $p^* = \frac{2}{3}$. The $p = 0$ intercept is 4 and the $p = 1$ intercept is -2, as illustrated in Figure 1.

- There are 2 pure NE $p = 0$ and $p = 1$, but neither is evolutionarily stable.

- The unique evolutionary equilibrium (EE) is $p^* = \frac{2}{3}$.

- Here is the key argument. Because it is a downcrossing, we see that $D(p) > 0$ for $p < p^*$. That is, H has a positive payoff advantage over D. By the first principle of evolution, this means that the H share, i.e., p, should increase, as indicated by the arrows in Figure 1.

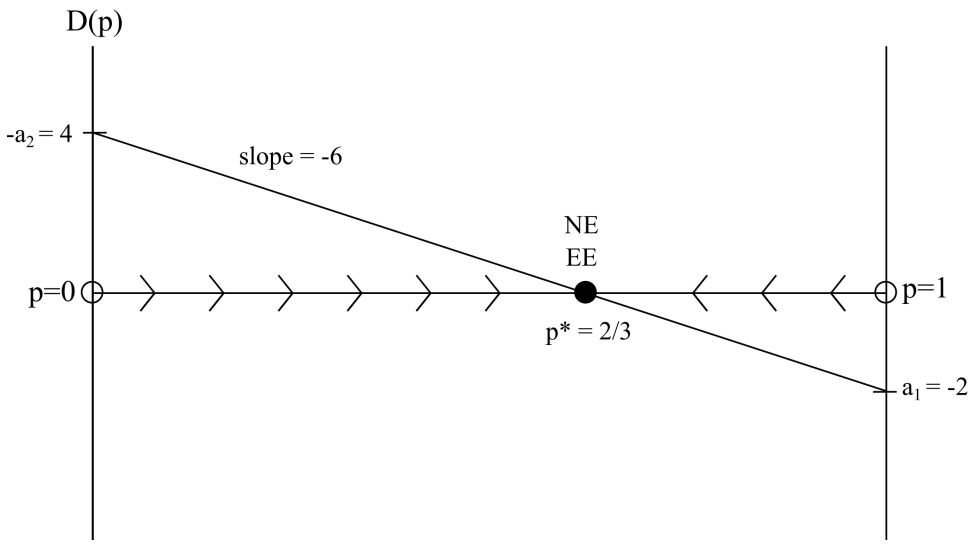

Figure 1: The payoff advantage of H over D has a downcrossing at the mixed NE $p^* = 2/3$, which is therefore the unique EE.

- By the same token, $D(p) < 0$ for $p > p^*$, so here the H share p should decrease, again as indicated by the arrows.
- Thus the current population share (or state) $p(t)$ will converge to $p^* = \frac{2}{3}$ from any initial condition.

To check your understanding, pick different values of $c > v > 0$ (or, even better, solve for general values), compute $D(p)$, verify that you get a downcrossing at $p^* = \frac{v}{c}$, and argue that this is a NE mixture of Hawks and Doves, and it is the only mixture that is evolutionarily stable.

3 Elements of Evolutionary Games

With this simple example in mind, we can now consider the general specification of Evolutionary Games. They consist of:

1. Player populations $k = 1, ..., K$, each with a continuum of players.

- In most applications, $K = 1$ or 2 populations. If everyone has the same strategic situation (as in a symmetric game), then there is just one population. Use 2 populations to model situations with two distinct roles, say buyers and sellers. Or home and Foreign firms, North and South (trade) models, etc.
- The continuum makes the formulas and the math easier. Substantively, it enforces the third slogan ("price taking"), since individual players have negligible influence in a continuum.

2. Strategy or state spaces, to describe the population shares at each moment of time.

- In the Hawk-Dove game, the state was denoted p and the state space was the line segment $[0, 1]$.
- In single population games with 3 pure strategies, the state $s = (p, q, 1-p-q)$ is a point in the familiar triangle simplex.
  - For example, the state $s = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) \in \Delta_3$ tells us that currently $\frac{1}{2}$ of the population is taking the first action, $\frac{1}{3}$ is taking the second action, and $\frac{1}{6}$ is taking the third action.
- The square $[0, 1]^2$ is the state space for two populations when each has two actions.
  - For example, the state $s = (\frac{1}{3}, \frac{1}{4}) \in [0, 1]^2$ tells us that currently $\frac{1}{3}$ of the first population is taking the first action (so $\frac{2}{3}$ takes the second action), while $\frac{1}{4}$ of the second population is taking its first action (and $\frac{3}{4}$ of that population takes its other action).
- The state s does "double duty."
  - It is the state variable, and it also has a mixed strategy interpretation.
  - As a player in the game, s represents the distribution of pure strategies that you currently will meet.

- Your expected payoff is the same as playing against a single other player (in a 2-player game) using the mixed strategy given by s.

3. A payoff function, usually in normal form.

   - Notation: $\pi_k(r,s)$ is the payoff for a player from population k playing strategy r (possibly mixed) when the population state is $s = (s_1, ..., s_K)$.
     - Ex: Let A be a 3×3 payoff matrix for single population $K = 1$. Then

$$\pi(r,s) = \begin{array}{c} (r_1, r_2, r_3) \\ 1 \times 3 \end{array} \quad \begin{array}{c} A \\ 3 \times 3 \end{array} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}_{3\times 1} = \text{scalar}$$

       if the player chooses mixed strategy r when the current state is s.
     - For two populations, the bimatrix is broken into two matrices A and B respectively, where typically $B^T \neq A$.

4. Dynamic adjustment process.

   - The state adjusts according to the payoffs. The basic idea is "survival of the fittest."
   - For example, in a single population 3 strategy game, whenever the payoff is highest for the second pure strategy ($r = (0,1,0)$) and lowest for the third ($r = (0,0,1)$), then $\frac{ds_2}{dt} = \dot{s}_2 > 0$, while $\dot{s}_3 < 0$.
   - What happens to the shares with middling payoffs — in the previous example, what is the sign of $\dot{s}_1$? That depends on the specific dynamics, and on details of shares and fitnesses.
   - The most common choice of dynamics is called *Replicator*. We will see below that for matrix games it expresses the $\dot{s}_i$'s as cubic polynomials.

- We are especially interested in rest points of the dynamics, where all $\dot{s}_i = 0$. We will see that these rest points generally include all NE, and that the *stable* rest points pick out the most empirically useful NE.

4 Classifying 2x2 games

Let us return to the simplest possible case, 1 population with two alternative actions. Here $s = (p, 1-p)$ is the state, where $p \in [0,1]$ is the proportion of the population choosing the first pure strategy. Let $r = (x, 1-x)$ be a particular player's strategy where x is the mixing probability on the first pure strategy.

The payoff function is described by matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, so

$$\pi(r,s) = rAs^T = \quad (x, 1-x) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \text{scalar}$$

You can easily see how the Hawk-Dove game in the first section can be re-expressed in this notation. As a second example, consider the bimatrix

2, 2	3, 0
0, 3	4, 4

- This is a coordination game with two pure NE and one mixed NE.
- The NE with payoffs (2, 2) is risk-dominant and the NE with payoffs (4, 4) is payoff dominant.
- The game is symmetric, thus the payoff matrix for both players can be represented by:

$$A \quad = \quad \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

To analyze any 2x2 game A, note that the first pure strategy $r = (1, 0)$ (the top row, also denoted $s1$ below) has payoff

$$\pi(s1, p) = \quad (1, 0) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

while the second pure strategy $r = (0, 1)$ (or $s2$, the bottom row) has payoff

$$\pi(s2, p) = \quad (0, 1) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix}.$$

Thus we can write the payoff difference $D(p) = \pi(s1, p) - \pi(s2, p)$ as:

$$D(p) = \quad (1, -1) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix}.$$

Carrying out the matrix multiplication, we obtain

$$\begin{aligned} D(p) &= (a_{11} - a_{21})p + (a_{12} - a_{22})(1-p) \\ &\equiv a_1 p - a_2(1-p) \end{aligned} \tag{1}$$

Equation 1 tells us that:

- $D(p)$ graphs as a straight line segment connecting intercept $-a_2 \equiv a_{12} - a_{22}$ at $p = 0$ to $a_1 \equiv a_{11} - a_{21}$ at $p = 1$, as in Figure 2.
- The slope of the line segment is $\frac{\partial D(p)}{\partial p} = a_1 + a_2$, and
- the solution to $D(p) = 0$ is $p^* = \frac{a_2}{a_1 + a_2}$.

Let's apply these insights to the coordination game, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$.

- Here the $p = 0$ intercept is $-a_2 = -1$ and the $p = 1$ intercept is $a_1 = 2$.
- The mixed strategy NE is the solution to $D(p) = 0$, so it is $p^* = \frac{a_2}{a_1 + a_2} = \frac{1}{3}$.

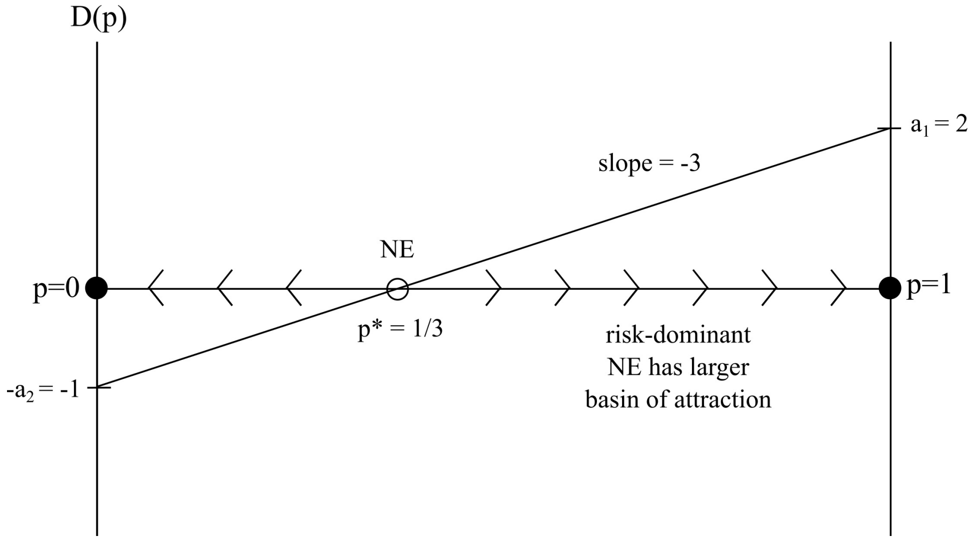

Figure 2: Coordination Game example. The payoff advantage $D(p)$ of the first strategy has an upcrossing at the mixed NE $p^* = 1/3$, which is therefore separates the basins of attraction of the pure strategy EE. Typo: slope is +3, not -3.

- Is this dynamically stable? The arrows say no — the payoff advantage is negative, so p decreases, when it is smaller than p^*, and it increases when it is larger that p^*.

- The underlying reason for the instability is that $D(p)$ has a positive slope: $\frac{\partial D(p)}{\partial p} = 3 > 0$.

- The two pure NE are EE and the mixed NE is a source, separating the basins of attraction of the two EEs.

- Now the initial condition matters; the state will ultimately converge to the risk-dominant NE/EE if initially it is above $p^* = \frac{1}{3}$, or to the payoff-dominant NE/EE if initially it is below p^*.

- Generally, as in the current example, the risk-dominant EE of a coordination game has the larger basin of attraction, lending further support to this refinement.

So now we have seen two possible types of 2x2 games. It turns out that, from an evolutionary perspective, there is only one other type.

We can classify any 2x2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in terms of its reduced parameters $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$ as follows.

1. **HD type**. If $a_1, a_2 < 0$, then there is a mixed NE $p^* = \frac{a_2}{a_1+a_2} \in (0,1)$, and it is a downcrossing since $\frac{\partial D(p)}{\partial p} = a_1 + a_2 < 0$. Therefore p^* is stable and it is the unique EE — from any initial conditions, the state converges to p^*.

2. **CO type**. If $a_1, a_2 > 0$, then there is a mixed NE $p^* = \frac{a_2}{a_1+a_2} \in (0,1)$, and it is an upcrossing since $\frac{\partial D(p)}{\partial p} = a_1 + a_2 > 0$. Therefore p^* is unstable — from any lower initial condition, the state converges to $p = 0$, and from any higher initial condition it converges to the other pure NE $p = 1$. The pure NE are both EE, and the mixed NE separates their basins of attraction.

3. **DS type.** If a_1 and a_2, have opposite signs, then $\frac{a_2}{a_1+a_2} \notin (0,1)$ so there is no mixed NE. $D(p)$ is always positive so the first pure strategy is dominant if $a_1 < 0 < a_2$, and likewise D is negative and the second pure strategy is dominant if $a_1 > 0 > a_2$. Evolutionary dynamics always push the state towards the dominant strategy from any initial condition.[Dan should we also mention the $\frac{\partial D(p)}{\partial p} = 0$ possibility as well?]

Although this evolutionary analysis gives real insight into stability, hence the sort of NE we might expect to observe, it doesn't tell us everything of interest. For example, it doesn't tell us whether an NE is efficient or not: the evolutionary dynamics are the same for Prisoner's dilemma as for a team game where the DS equilibrium is efficient. (see exercise xxx).

5 3x3 Games, and replicator dynamics

Consider an arbitrary linear payoff function for a symmetric game with three strategies a, b, c played by a single population. The state is denoted $s = (s_a, s_b, s_c) \in \Delta_3$, the triangle simplex. Using biologists' preferred notation w for fitness, we have a general 3x3 payoff matrix:

$$\mathbf{w} = \begin{pmatrix} w_{aa} & w_{ab} & w_{ac} \\ w_{ba} & w_{bb} & w_{bc} \\ w_{ca} & w_{cb} & w_{cc} \end{pmatrix}. \tag{2}$$

Given the current state $s = (s_a, s_b, s_c)$, the current payoff w_i of strategy i is its weighted average payoff, using the current shares $s_j,\ j \in \{a, b, c\}$ as weights. Thus

$$w_i(s_a, s_b, s_c) = s_a w_{ia} + s_b w_{ib} + s_c w_{ic} \quad \text{for } i = a, b, c. \tag{3}$$

.

The current weighted average (or mean) payoff turns out to be useful. It is $\bar{w}(s_a, s_b, s_c) = s_a w_a + s_b w_b + s_c w_c$. In matrix notation it is

$$\bar{w}(\mathbf{s}) = \mathbf{sw} \cdot \mathbf{s} = \sum_{i,j=a,b,c} s_i s_j w_{ij}. \tag{4}$$

Replicator dynamics for any evolutionary game (with any number of populations and pure strategies) equate the growth rate of each strategy share to its relative payoff. In biology, taking payoff as fitness, this is a tautology: fitness can be *defined to be* the growth rate. In economics, replicator dynamics are a reasonable approximation for many purposes, but they are not canonical.

The growth rate of strategy share i is $\frac{dln s_i}{dt} = \frac{1}{s_i}\frac{ds_i}{dt} = \dot{s}_i/s_i$. Relative payoff here means $w_i - \bar{w}$. Hence replicator dynamics are given by a system of cubic ordinary differential equations. For the current general 3x3 single population game, the system is

$$\dot{s}_i = s_i(w_i(\mathbf{s}) - \bar{w}(\mathbf{s})),\ \ i = a, b, c. \tag{5}$$

To analyze behavior for any 3x3 payoff matrix, it will help to rewrite equation (5) as

$$\begin{aligned}\dot{s}_i/s_i &= w_i - \bar{w} = w_i \sum_j s_j - \sum_j w_j s_j = \sum_j (w_i - w_j) s_j \\ &= \sum_j s_j D_{i-j}, \end{aligned} \tag{6}$$

where $D_{i-j}(s) = w_i(s) - w_j(s)$ is a generalization of the $D(p)$ used for the 2x2 games. Here we must distinguish the fitness difference, not just between a and b, but also between a and c, and between b and c.

Equation (6) provides a useful insight: the growth rate of any strategy's share, $\dot{s}_i/s_i$, comes at the expense of every other strategy's share. Strategy i grows at the expense of strategy j to the extent that j has a large share s_j and a disadvantage $-D_{i-j}$ that is large and positive. From this you can see that, under replicator dynamics, the highest payoff strategy definitely increases share ($\dot{s}_i/s_i > 0$) since both relevant D_{i-j}s are positive, and the lowest payoff strategy definitely loses share ($\dot{s}_i/s_i < 0$) since both relevant D_{i-j}s are negative. The share middling fitness strategy could go either way since it has terms of opposite sign.

Zeeman's 3x3 game. A numerical example may help fix ideas. Zeeman (1980) used the following 3x3 matrix game to make a point about the nature of evolutionary equilibria (EE, i.e., dynamically stable NE).

$$\mathbf{w} = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}. \tag{7}$$

The payoff advantage of the first strategy s_1 over the second s_2 is

$$D_{1-2}(\mathbf{s}) = (1, -1, 0)\mathbf{w} \cdot \mathbf{s} = (3, 6, -9) \cdot \mathbf{s} = 3(s_1 + 2s_2 - 3s_3). \tag{8}$$

As shown in Panel A of Figure (3), the line $D_{1-2}(\mathbf{s}) = 0$ intersects the $s_1 = 0$ (or bottom) edge of the simplex at $(0, 3, 2)/5$ and the $s_2 = 0$ (or right) edge at $(3, 0, 1)/4$. Hence

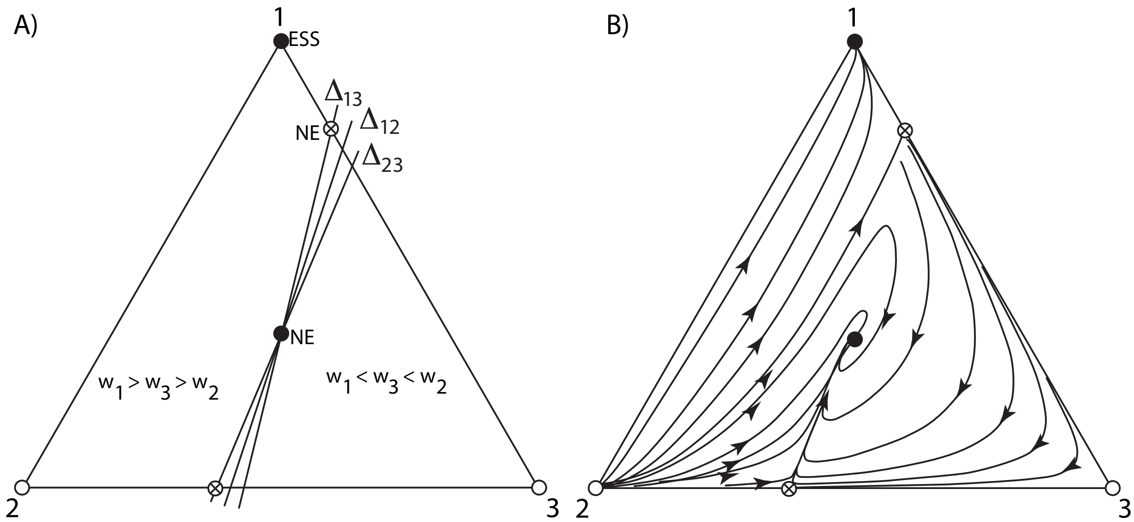

Figure 3: Zeeman game (7). Panel A: Sectoring, using notation Δ instead of D. Panel B: Replicator dynamics.

$w_1 > w_2$ at points in the simplex to the left of the straight line connecting these points. Similarly,

$$D_{1-3}(\mathbf{s}) = (1, 0, -1)\mathbf{w} \cdot \mathbf{s} = (1, 3, -4)\cdot\mathbf{s} = s_1 + 3s_2 - 4s_3, \tag{9}$$

and $D_{1-3}(\mathbf{s}) = 0$ intersects the bottom edge at $(0, 4, 3)/7$ and the right edge at $(4, 0, 1)/5$, so $w_1 > w_3$ at points in the simplex to the left of the straight line connecting these points. Also,

$$D_{2-3}(\mathbf{s}) = (0, 1, -1)\mathbf{w} \cdot \mathbf{s} = (-2, -3, 5)\cdot\mathbf{s} = -2s_1 - 3s_2 + 5s_3, \tag{10}$$

and $D_{2-3}(\mathbf{s}) = 0$ intersects the bottom edge at $(0, 5, 3)/8$ and the right edge at $(5, 0, 2)/7$, so $w_2 > w_3$ at points in the simplex to the right (not left, since (10) has coefficients with signs on s_i opposite to those of the other two D expressions) of the straight line connecting these points.

In view of equation (5), the only candidates for steady states are intersections of the edges and the $\Delta_{ij} = 0$ lines. Corners are where two edges intersect, and they are automatically steady states for replicator dynamics. However, only the first corner $i = 1$

(i.e., $\mathbf{e}^1 = (1,0,0)$) satisfies the (strict) Nash inequalities $w_i > w_j \quad \forall j \neq i$, by virtue of the fact that the first entry in the first column of $\mathbf{w}$ exceeds the other entries in that column. Indeed, we will later see that that fact ensures that this corner is also an ESS.

We have already found the six intersections of $\Delta_{ij} = 0$ lines with edges. Of these, only $(4,0,1)/5$ is a NE because it satisfies $w_1 = w_3$ (since it is on the $\Delta_{13}(s) = 0$ line) and also satisfies $w_1 > w_2$ (since it is to the left of the $\Delta_{12}(s) = 0$ line which hits that edge at $(3,0,1)/4$). The other 5 intersections either fail to equate the two payoffs pertaining to their edge, or else the remaining payoff is higher. For example, $(0,5,3)/8$ does equate the pertinent payoffs w_2 and w_3, but w_1 is higher. Hence it is not a NE, but it is a saddle point for replicator dynamics, with unstable saddle path leading into the interior of the simplex, as can be seen in Panel B of Figure (3).

The final candidate for equilibrium is the intersection of any pair (hence all three) $\Delta_{ij} = 0$ lines. This intersection occurs at $(1,1,1)/3$, the center of the simplex, as is easy to verify from equations (8 - 10). Since all three payoffs are equal at this point, it is by definition a mixed NE. See the technical appendix to this chapter for further analysis, which establishes that it is dynamically stable but not an ESS.

6 Simple two population games

Any payoff bimatrix specifies a game played between two distinct populations that interact with each other but do not directly within their own population.

[[insert a 2x2 bimatrix and run with it.]]

- If you make this a two population Hawk-Dove game, as specified at the beginning of section 2, then an interesting result emerges.

- One becomes completely Hawk or Dove.

- This is due to the matching assumption (one player is randomly matched with a player from the other population).
  - Ex: Fisher and Kakkar *JIE* (2004) international trade model.
- does not have to occur if there are own-population effects (some McGinty articles).

7 More general models

This section should mention, but not really elaborate, on

- own- vs cross-population effects in 2 pop games
- nonlinear games
- higher dimensional games
- different dynamics, e.g., noisy BR, stochastic approx, ...
- possibly mention assortative matching.

8 Evol Game Applications

This section should say something about when EvG models are appropriate in applications, e.g., when there are multiple NE, picking out which of them, if any, are relevant.

It might mention lab evidence, e.g., Oprea et al (2014) on one vs 2 pop HD, and Cason et al (RPS, and convergent vs nonconvergent cycles).

Most important, it should give a few sentences on the following applied theory papers.

- Dan and KC, Bouwe and Frans, many others

9 History and Notes

From the preface of Friedman and Sinervo (2016).

Classic evolutionary theory goes back to biologists Charles Darwin (1859) and Alfred Wallace (1858), and to precursors such as economist Thomas Malthus (1798). In the 20th century that theory became the unifying principle of biology, and is sometimes referred to as the Neo-Darwinian Synthesis or Modern Synthesis. **Classic game theory**, founded in 1944 by mathematician John von Neumann and economist Oskar Morgenstern, studies strategic interaction among agents with differing objectives. As developed in the second half of the 20th century by numerous researchers (including twelve Nobel laureates), game theory focuses on equilibrium. It provides a common language across many of the social sciences.

Evolutionary game theory (EGT) is a cross-fertilization of the two classic theories. EGT focuses on how fitness emerges from strategic interaction, and on the resulting evolutionary dynamics. Launched by biologist John Maynard Smith and mathematician George Price in the 1970s, EGT has been developed in the last few decades by theorists in biology, social science, mathematics and, most recently, in computer science and engineering. In the last two decades, EGT has been trumpeted in numerous surveys and special issues of academic journals and in several textbooks.

More history:

1. Nash dissertation (1950) mass action, many individuals acting rationally and everyone knows this. (See forward in Weibull's evol game book for exact quote.)

2. Alchian (1950) *JPE*

    - Assumptions of rationality and perfect foresight are not needed in Economic modeling.

    - Rather, we just need individuals to respond to payoff differences and those actions that yield a payoff advantage increase in prevalence.

3. Shapley 1950's Fictitious play model with a counter-example.
    - A 3×3 matrix failed to come to a mixed strategy NE.
    - Fictitious play means to play against a historical probability distribution. (form of an evolutionary game)
4. Maynard Smith (1974) evolutionary biologist.
    - Mutations as deviations, selection advantage, survival of the fittest.
5. 1960's to 1985 they fell off the map in Economics.
6. 1985 Kreps and Selton revived interest in fictitious play (version of evolutionary games)
7. 1991 D. Friedman *Econometrica* "Evolutionary Games in Economics."
8. Recently application are beginning to come out.

10 Further Reading

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Technical Appendix

Zeeman's example, continued

To check stability under replicator dynamics, we compute the Jacobian matrix $((\frac{\partial \dot{s}_i}{\partial s_j}))$, evaluate it at $(1,1,1)/3$, and multiply it by the projection matrix $\mathbf{P} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, as explained in Appendix A below. The result is $\mathbf{J} = \frac{1}{9}\begin{pmatrix} 4 & 9 & -13 \\ -5 & -9 & 14 \\ 1 & 0 & -1 \end{pmatrix}$. The characteristic equation reduces (after more tedious algebra) to $3\lambda^3 + 2\lambda^2 + \lambda = 0$, with roots (i.e., eigenvalues) 0 (corresponding to the eigenvector (1,1,1) normal to the simplex, hence irrelevant) and $\frac{-1\pm\sqrt{-2}}{3}$. Since the real parts of the two relevant eigenvalues are negative (at -1/3), we conclude that this NE is indeed a stable steady state for replicator dynamics.

However, $(1,1,1)/3$ is not an ESS. To see this, consider an "invasion" of the first strategy, i.e., set $\mathbf{x} = \mathbf{e}^1 = (1, 0, 0)$ in the last version of the ESS definition. Since we are at an interior state, we must have $f(\mathbf{s}, \mathbf{s}) = f(\mathbf{x}, \mathbf{s})$ so part (i) is false. By part (ii), then, ESS requires that $f(\mathbf{s}, \mathbf{x}) > f(\mathbf{x}, \mathbf{x})$, but in fact $f(\mathbf{s}, \mathbf{x}) = \frac{1}{3}(1, 1, 1)\mathbf{w} \cdot \mathbf{e_1} = \frac{1}{3}(-4, 9, 1) \cdot (1, 0, 0) = \frac{-4}{3} < 0 = \mathbf{e_1}\mathbf{w} \cdot \mathbf{e_1} = f(\mathbf{x}, \mathbf{x})$.

The general intuition is that ESS is essentially a negative definiteness condition, which requires that all trajectories in its neighborhood to come closer and closer together, monotonically. Panel B of the Figure suggests that trajectories actually move farther apart as they approach their NNE (or SSW) turning points, although they ultimately do converge.