UCSC Evolutionary Game Theory Practice Midterm Solutions

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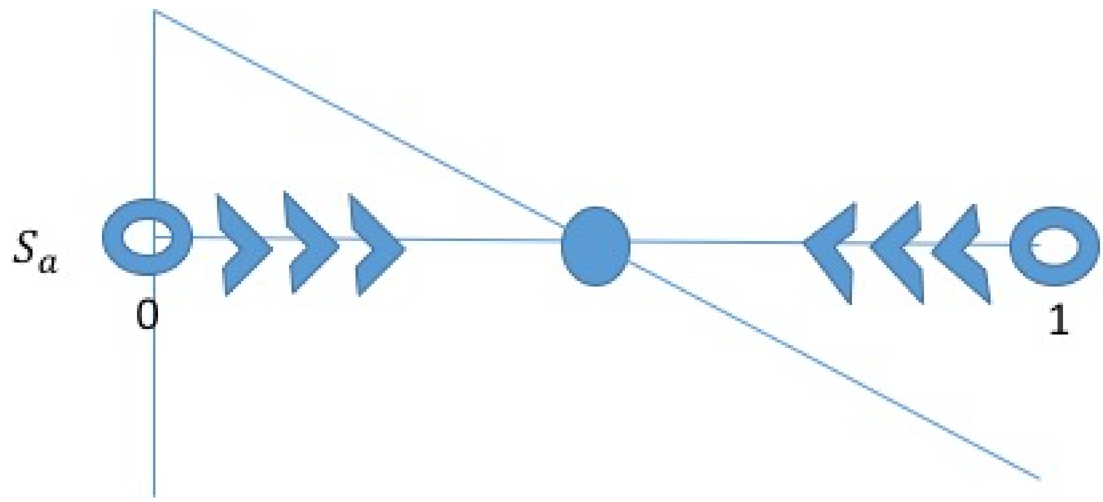
1. Using single digit integers, e.g., −1, 0, 1, 2

   (a) Write down a HD-type 2x2 fitness matrix with 0 entries on the main diagonal.[8 points]
   Hawk-Dove game follows the following matrix:

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\begin{aligned} w_a &= 0 + 2 - 2S_a \\ w_b &= 2S_a \\ w_a - w_b &= 2 - 4S_a \end{aligned}$$

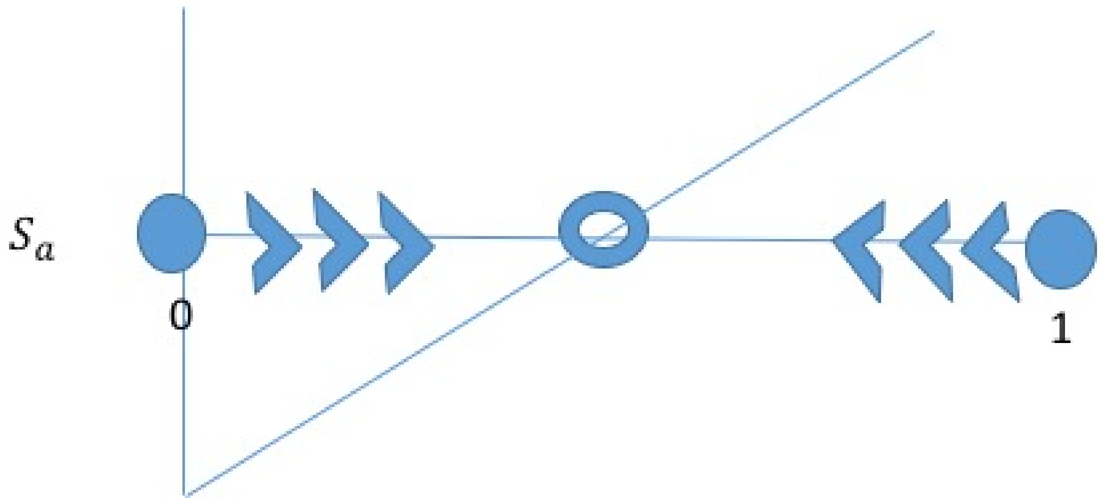


Interior steady state at $S_a^* = 1/2$. We can see that this is a down-crossing so it has an interior equilibrium.

   (b) Write down a Co-type 2x2 fitness matrix, again with 0 entries on the main diagonal.[8 points]
   Coordination game follows the following matrix:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{aligned} w_a &= -1 + S_a \\ w_b &= -S_a \\ w_a - w_b &= 2 - 4S_a \end{aligned}$$



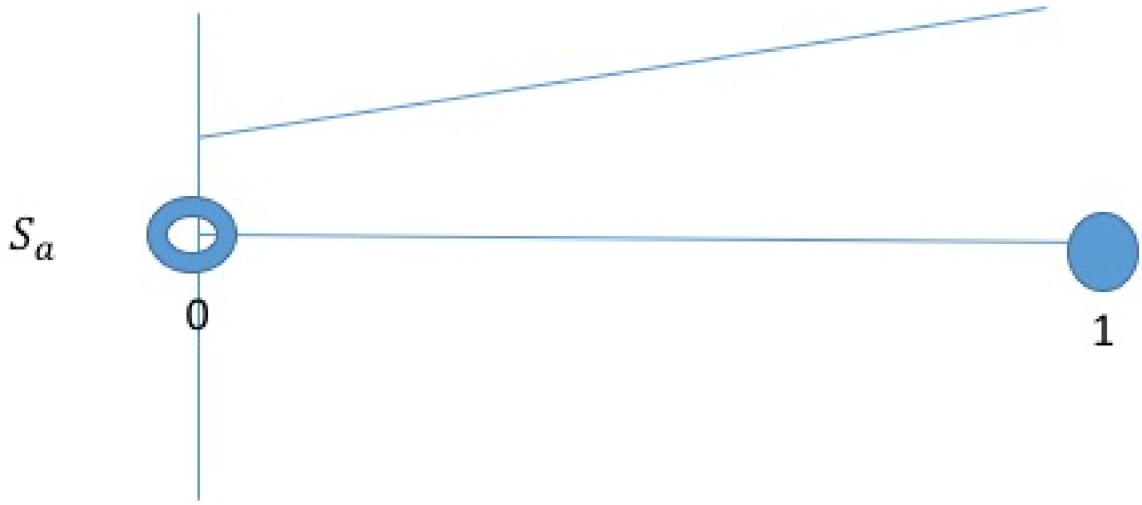
So the interior steady state at $S_a^* = 1/2$ is an up-crossing and this is a coordination game.

(c) Write down a DS-type 2x2 fitness matrix, with (0 entries on the main diagonal.[8 points]

A **dominant strategy** game occurs when w_a and w_b have opposite signs, implying that one of the strategies can invade the other without being invaded. The invading strategy will always have the higher fitness. There are two possibilities: $w_a > 0 > w_b$ and $w_b > 0 > w_a$. Consider:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$w_a = 1 - S_a$$
$$w_b = -S_a$$
$$w_a(s) - w_b(s) = 1$$



So a dominates b, and this games is indeed of type DS.

2. Using your answers to the preceding question, write down a 3x3 fitness matrix whose three (2x2) edge games are HD, Co and DS.[10 Points]

$$\begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

(a) Write down the three Delta functions in terms of the state variable s = (S_1, S_2, S_3) = (x, y, 1-x-y). [10 points].
First we calculate w_1, w_2, w_3 and $\overline{w}$.

$$\begin{aligned} w_1 &= 2S_2 - S_3 \\ w_2 &= 2S_1 + S_3 \\ w_3 &= -S_1 - S_2 \\ \overline{w} &= 4S_1S_2 - 2S_1S_3 \end{aligned}$$

$$\dot{S}_i = S_i(w_i - \overline{w})$$

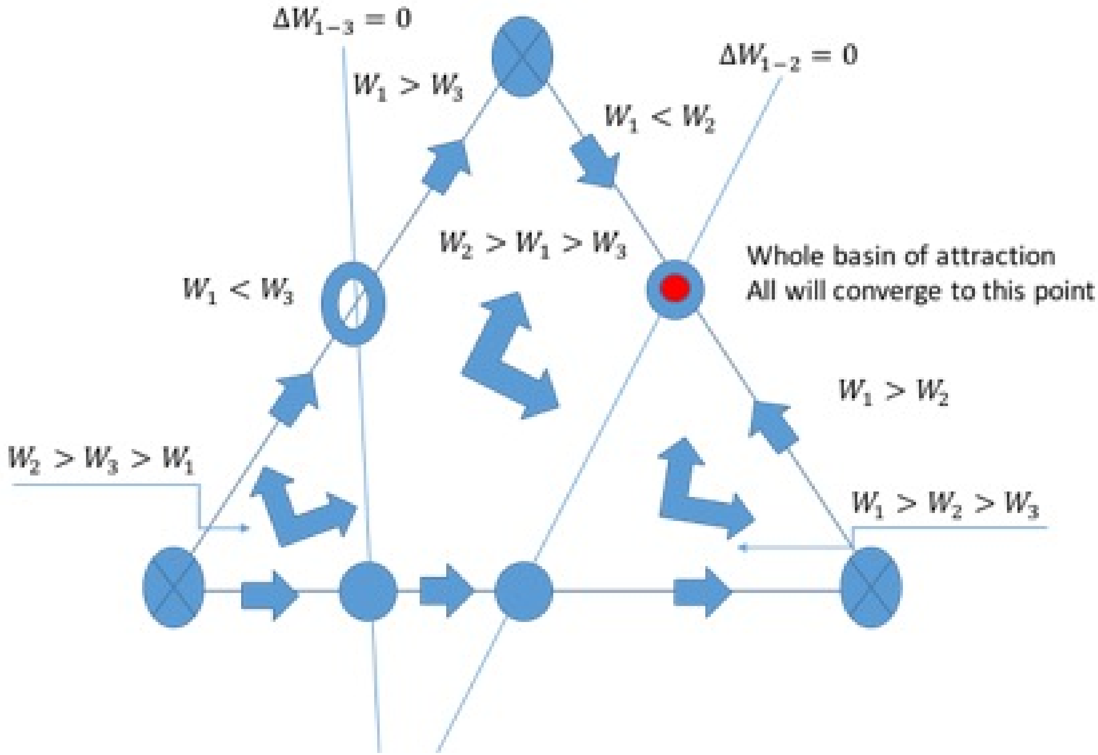
$$\begin{aligned} \dot{S}_1 &= S_1(2S_2 - S_3 - 4S_2S_1 + 2S_3S_1) \\ \dot{S}_2 &= S_2(2S_1 + S_3 - 4S_2S_1 + 2S_3S_1) \\ \dot{S}_3 &= S_3(-S_1 - S_2 - 4S_2S_1 + 2S_3S_1) \end{aligned}$$

$$\begin{aligned} \Delta w_{1-2} &= 2S_2 - 2S_3 - 2S_1 \\ \Delta w_{2-3} &= 3S_1 + S_3 + S_2 \\ \Delta w_{1-3} &= S_1 + 3S_2 - S_3 \end{aligned}$$

(b) Is there an interior equilibrium? Find it or show that none exists [10 points]

Setting $\Delta w_{1-2} = \Delta w_{2-3} = \Delta w_{3-1}$ yields no interior equilibrium because the equation $\Delta w_{2-3} = 3S_1 + S_3 + S_2 > 0$ has no zeros in the simplex. i.e. row 2 is dominated by row 3.
For completeness, we find the other Delta lines, and where they intersect the edges corresponding to $s_1 = 0$, $s_2 = 0$ and $s_3 = 0$.

$$\begin{aligned} &\Delta w_{2-3} = 3S_1 + S_3 + S_2 > 0 \implies \text{ no edge intersection.} \\ &\Delta w_{1-2} = 2S_2 - 2S_3 - 2S_1 = 0 \implies \text{edge intersections at:} \\ &\quad S_3 = 0, S_1 = \frac{1}{2}, S_2 = \frac{1}{2} \\ &\quad S_1 = 0, S_2 = \frac{1}{2}, S_3 = \frac{1}{2} \\ &\Delta w_{1-3} = S_1 + 3S_2 - S_3 = 0 \implies \text{edge intersections at:} \\ &\quad S_1 = \frac{1}{2}, S_2 = 0, S_3 = \frac{1}{2} \\ &\quad S_1 = 0, S_2 = \frac{1}{4}, S_3 = \frac{3}{4} \end{aligned}$$



In the figure, the top corner is $s_1 = 1$, bottom right is $s_2 = 1$ and bottom left is $s_3 = 1$. Directional arrows are very rough.

(c) Explain how you would determine the stability of the interior equilibrium if it exists and if you had plenty of time. [5 points]

There is no interior equilibrium. However when there is an interior equilibrium you would solve it as presented in section d below. Linearity of delta functions ensures that setting them all equal to zero will give us only one unique EQ or none at all, as long as they come from a nondegenerate payoff matrix.

This changes however, if we include non-linear fitness functions. In that case, there could be several intersections of the $\Delta w_{i-j} = 0$ lines.

This is true for 3×3 as well as for any $n \times n$ matrix.

(d) Explain how you would find all stable steady states under replicator dynamics for the present example if you had plenty of time. [5 points]

We can determine an interior equilibrium's stability for replicator or other dynamics using the eigenvalue technique. If the dynamics are expressed as ODEs $\dot{s}_i = V_i(s)$, then

1.) At steady state $s^* \in S$ of interest for dynamics V, compute the $n \times n$ matrix DV$(s^*) = ((\frac{\partial V_i(s)}{\partial s_j}\Big|_{s=s^*}))$ and the projection matrix $P_0 = I - \frac{1}{n} 1_{nxn}$ and set $J = P_0 DV(s^*)$.

2.) Compute the eigenvalues Λ_1,...,Λ_2 of J_0. Drop a 0 eigenvalue (whose existence is guaranteed by the projection matrix).

3.) Sort the remaining $n-1$ eigenvalues by their real parts from largest to smallest. By a simple extension of the Hartman-Grobman theorem, we know that s^* is

- locally stable if the largest remaining real part is negative,
- a source (i.e., completely unstable) if the smallest remaining real part is positive, and

- a saddle if some real parts are positive and some are negative.

(e) For extra credit, find all stable steady states and their basins of attraction, if time does permit [5 points max]

3. Beginning with the DS-type 2x2 matrix from Problem 1c above

(a) Add a single digit integer (e.g., -3) to both entries in one of the columns so that the new matrix defines a PD-subtype game.[8 points]

DS $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is neither PD nor LF.

But add 1 to entries in the second column to get $\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$

to see that this game is PD, just compute $\bar{w}(s)$ and note that it is decreasing in the share s_1 of the dominant strategy.

(b) Find a different single digit integer (and possibly a different column) so that adding it to both entries in one of the columns instead defines a LF-subtype game. [8 points]

Add instead 1 to both entries in the first column to get $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

The mean fitness is increasing as the shares of the dominant strategy increases

(c) Argue that monotone dynamics for all 3 games (the original one from 1c above plus the other two from a and b in the present problem) are identical, but the implications for population fitness are completely different. Hint: compare mean fitness functions. [5 points] Evolution doesn't respond to mean fitness, it only responds to fitness differences. (This point is easily checked in the preceding example.) However, we care about mean fitness.

4. (a) Describe a situation (biological, social or virtual) where the HD game might approximate the key strategic interaction. [5 points]

The Hawk-Dove game describes two strategies or types of players: "Hawk" refers to the player that chooses an aggressive strategy that may escalate into a fight, where it can either win the entire resource against a Dove, or some of it with the cost of being injured (or killed). "Dove" refers to the player that chooses a more peaceful strategy, where it can leave with nothing from a confrontation with a Hawk or share the resources evenly with another dove. A real-life biological example of the Hawk-Dove game are the Australian Gouldian finches. Red-headed finches are much more aggressive than their black-headed counterparts, and therefore are more likely to win a fight over a nesting place. Because of their tendency to instigate fights with other birds, red-headed finches spend little time raising their young than they do fighting over the best nesting places, while the black-headed finches avoid confrontations altogether and spend the majority of their time caring for their young. The Hawk-Dove model predicts that the ratio of red finches to black is 30:70, which is exactly what is observed in nature.

(b) Then explain exactly how the shares S_H and $S_D = 1 - S_H$ should enter the payoffs or fitnesses. In particular, do pairwise encounters actually occur at random? Or do individuals "play the field"? Or are there only local interactions? Can assortativity (or an adjustment matrix) play a role? Your answer should demonstrate your familiarity with these concepts. [10 points]

In the classical Hawk-Dove game, the equilibrium Hawk share of the population is equal to the resource divided by the cost of confrontation (v/c), and the Dove share is 1-(v/c). In this case, the reward is equal to 3 and the cost is 10, giving two red-headed finches a payoff of -3.5 each [(3-10)/2]. This gives a red-headed finch a payoff of 3 against a black-headed finch, who backs down and receives 0. Two black-headed finches receive a payoff of 1.5 each.

The main point here is that this assumes random matching; the probability of interaction is proportional to the share s_i of each strategy. But with negative assortativity, reds would interact less frequently with other reds and more often with blacks. This would enable reds to gain a larger equilibrium share of the population by avoiding each other and incurring a negative payoff. This makes logical sense; a red-headed finch will see and probably avoid another red-headed finch, knowing the costs could be great (in other words, the fight is worth avoiding because they both know they are "hawks").

Seeing a black-headed finch is a different story, and would motivate the red-headed finch to take their nesting place knowing the black-headed finch would back down. The black headed finch loses nothing by playing dove against a hawk, and walking away. He gains by cooperating with other "doves" and sharing resources or nesting places, a behavior which would point to positive assortativity and give the black-headed finches a higher equilibrium share.