1) You, player A, are engaged in a team class project with your buddy player B. You each have to decide what level of work to put into the project, and your choices are 1-low, 2-medium, 3- high. Your work is designated w_a and that of your buddy w_b. The grade on the project is determined by the total work $W=w_a+w_b$ as

	2	3	4	5	6
Grade	50	70	83	95	100

(You will notice that your grade has diminishing returns as the amount of work goes up.) Each player i's payoff is Grade – $10w_i$.

a) Suppose you agree with your buddy to work independently. You will just staple your two parts together on the day it is due, without observing how hard your partner worked. Express the game's strategic form as a bimatrix.

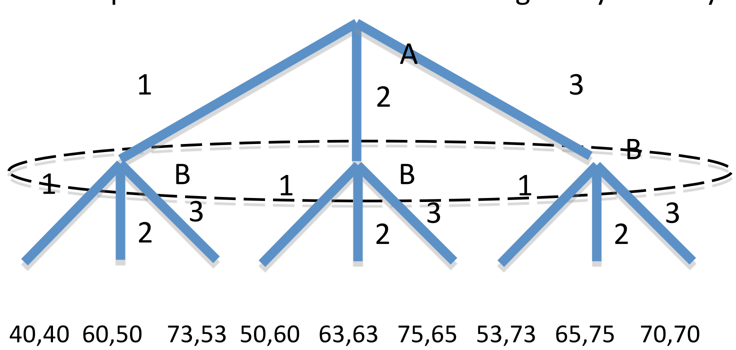
	1	2	3
1	40,40	60,50	73,53
2	50,60	63,63	75,65
3	53,73	65,75	70,70

b) What (pure) Nash equilibria can you find?

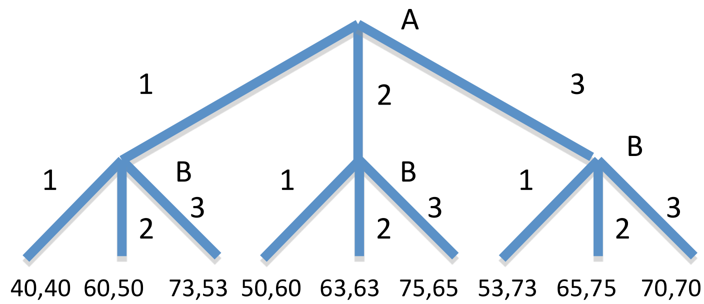
	1	2	3
1	40,40	60,50	73,53
2	50,60	63,63	75,65
3	53,73	65,75	70,70

The pure NE are (3,2) and (2,3).

c) What is the equivalent extensive form of the game you analyzed in parts a and b?

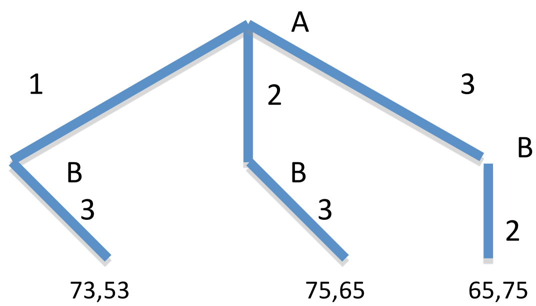


d) Now suppose you decide that you, player A, will finish your part and then pass it to B. B will see how hard you worked, and then decide how hard to work on their part. What is the extensive form of this game?

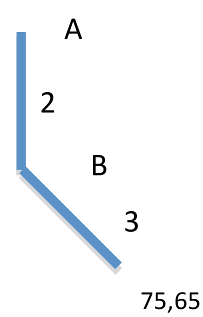


e) Can you find a backwards induction solution to the game of part d? Based on this analysis would you want to be the player who works on their part first or last?

Backwards induction allows us to eliminate branches to get:



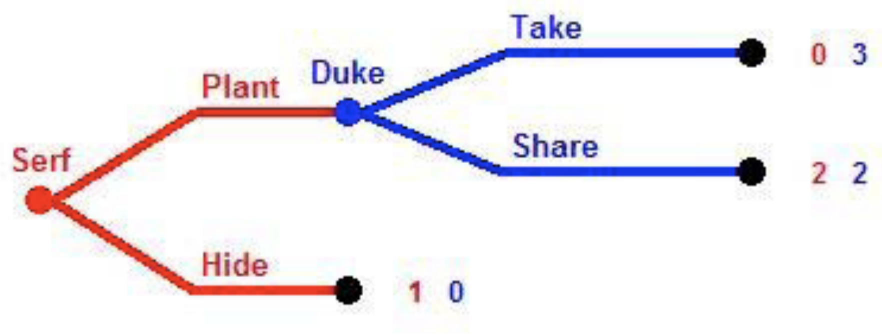
The next level of BI gets us to:



Thus the solution is (2,3). The first mover ends up with the higher payoff, so it is better to be him.

2) Player #1 (the Serf) first chooses either to plant crops (P) or to hide the seed (H). If he chooses P, then at harvest time Player #2 (the Duke) either takes the entire crop (T) resulting in payoffs (0, 3) for the two players, or else shares (S) resulting in payoffs (2, 2). If the Serf chooses H, then the game ends with payoffs (1, 0).

a) Draw the tree (i.e., the extensive form) for this game.



b) Find the backwards induction (BI) solution to this game, and the corresponding payoffs.

Begin at the last information set. The Duke must decide between "Take" or "Share". "Take" gives the Duke a greater payoff to him (3>2), so the Duke will choose "Take". Given this, the Serf must choose between Plant (which yields a payoff of 0) or Hide (which has a payoff of 1). The Serf will choose Hide. The backward induction solution for this game is (H,T) with payoff (1,0)

c) Write the game in normal form, and find all Nash equilibria.

	Take		Share	
Plant	0	3	2	2
Hide	1	0	1	0

The only pure strategy NE is (Hide,Take).

d) Find strategies that lead to an efficient outcome, i.e., one that maximizes the payoff sum.

The efficient outcome, the one that maximizes the payoff sum, is (Plant, Share)=(2,2). The payoff sum is 4.

e) Find a feasible "pre-play" move for the Duke that can improve his payoff; explain briefly.

e. If the Duke could commit to sharing the crop, or if the Duke signed an enforceable contract that fines him >1 for taking the entire crop, the Serf would know that the Duke would always share. Then the Serf would choose to Plant. This would result in a higher payoff for both. Some sort of commitment is required for this to occur.

3) Congratulations! You have been appointed to be the Car Czar of the United States. You are responsible for recommending a reorganization of the industry to reduce the losses the carmakers are suffering. Your demand forecaster tells you that you should think of all cars as being identical, and that the market price p for cars depends on the total quantity Q produced in the following way:

$$p = (30 - Q).$$

Suppose there are n car companies, and each company chooses a quantity q_i of cars to produce. It costs each car company 10 to make each car, and each car company has fixed "legacy" costs of 25 so that the payoff function for each company is

$$V_i = (p - 10)\, q_i - 25.$$

(Note on units: The prices are in units of thousands of dollars, quantities in units of millions of cars, and payoffs in units of billions of dollars. The formulas above are consistent with this, so you can work with the formulas as given without adding lots of zeros to the end of all the numbers!)

a) What is the best response function of company i? Express your answer in terms of q_{-i} , the sum of the quantities the companies other than i produce.

Player i's payoff has the form

$$V_i = (20 - q_i - q_{-i})q_i - 25.$$

This is quadratic in q_i, and hill shaped, so the best response is found by setting the partial derivative with respect to q_i to 0.

$$\frac{dV_i}{dq_i} = 20 - q_{-i} - 2q_i$$

$$\frac{dV_i}{dq_i}(q_i^*) = 0 \Rightarrow 20 - q_{-i} - 2q_i^* = 0$$

$$\Rightarrow q_i^* = 10 - \frac{1}{2}q_{-i}$$

The above value of q_i is the best response.*

b) Find a symmetric NE.

In a symmetric NE it must be that all players play best responses to each other, and that everybody produces the same quantity. This requires that

$$q_{-i} = (N-1)q_i^*$$

since there are N players, and thus every player faces N-1 opponents. Substituting this into the best response relation, we have

$$q_i^* = 10 - \frac{N-1}{2} q_i^*$$

$$\frac{N+1}{2} q_i^* = 10$$

$$q_i^* = \frac{20}{N+1}$$

So the symmetric NE is for all players to produce the quantity above.

c) What is the profit of each company in the equilibrium you find in part b?

This is found by substituting the above value into the payoff function:

$$V_i = (20 - Nq_i^*)q_i^* - 25$$

$$= \left(20 - \frac{20N}{N+1}\right)\frac{20}{N+1} - 25$$

$$= \frac{400}{(N+1)^2} - 25.$$

d) The president wants as much competition as possible while having an industry that doesn't lose money. What is the largest number companies in the car business possible and still have each company not lose money (have a non-negative payoff)?

From part C, we see that the profit of each company decreases in N in the symmetric NE, so the largest N that makes the payoff non negative should result in a zero payoff. (If this procedure gives us a noninteger N, we'd have to round down to find an integer that makes V_i nonnegative. However it turns out we get an integer N without rounding.) The algebra works as follows:

$$V_i = \frac{400}{(N+1)^2} - 25 = 0$$

$$\frac{400}{(N+1)^2} = 25$$

$$\frac{20}{N+1} = 5$$

$$N + 1 = 4$$

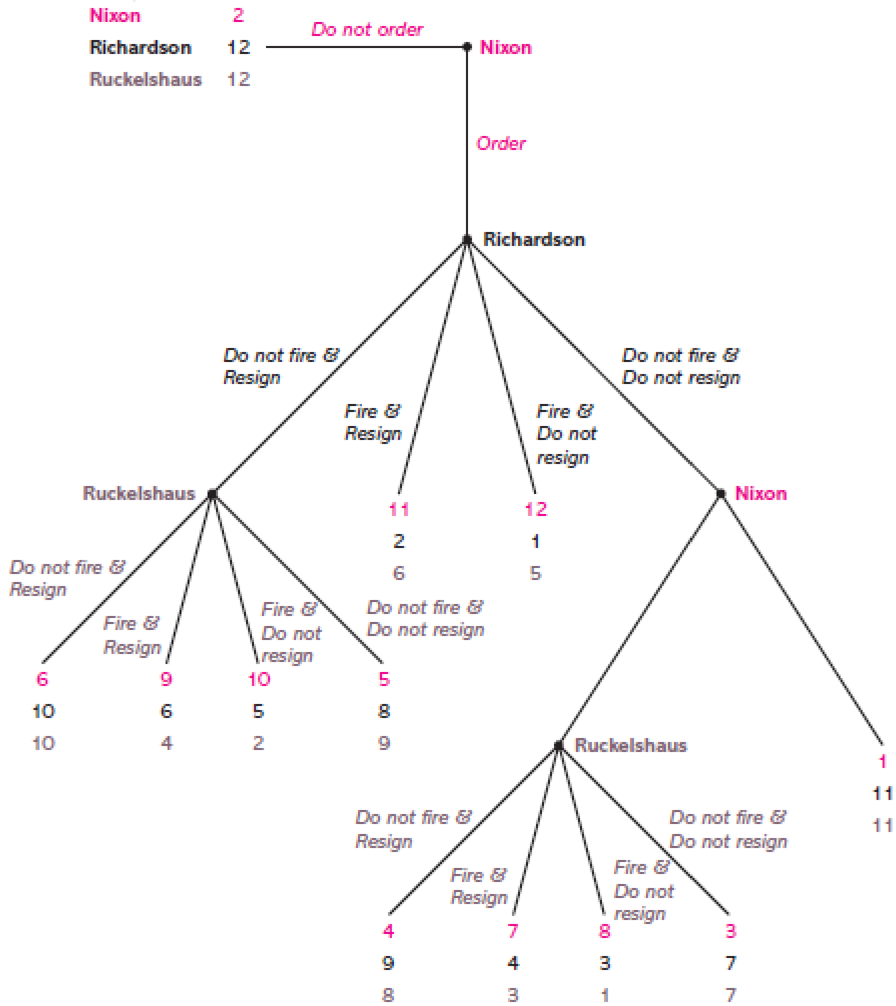
$$N = 3.$$

Harrington Chapter 8:

7. An infamous event that came to be known as the Saturday Night Massacre took place during the second term of the presidential administration of Richard Nixon. Though no one was fired *upon*, many were effectively fired *from* their high-level positions in the federal government. The Nixon White House was in the midst of covering up crimes committed by close aides to the president. As part of the investigation, Attorney General Elliot Richardson (who was not part of the cover-up) named Harvard Law professor Archibald Cox as a special prosecutor. During the investigation, President Nixon was acutely concerned with Cox's investigation and contemplated ordering Richardson to fire Cox (expressed as the initial decision node in FIGURE PR8.7). When Nixon's intent was expressed to Richardson, the latter conveyed that if he did fire Cox, he might feel compelled to resign, but also that he might be inclined not to fire Cox and, in that case, might also resign. Richardson's four possible combinations of firing Cox or not and resigning or not are depicted in the extensive form. If Richardson did choose to resign and not fire Cox, then Nixon would still be left with the matter of getting rid of Cox. And if Richardson chose not to fire Cox and did not resign, then Nixon would have to decide whether to fire Richardson. Upon Richardson's departure, Deputy Attorney General William Ruckelshaus would assume the position of acting attorney general and would face the same four options as Richardson. If Ruckelshaus also chose to resign and not fire Cox, then Solicitor General Robert Bork would become acting attorney general, and again, he would have the same four choices. To simplify matters, we'll not model Bork, even though what happened was that Richardson refused to fire Cox and resigned and Ruckelshaus did the same, at which point Bork came in and did fire Cox and did not resign. Find all subgame perfect Nash equilibria.

ANSWER: Let us begin with the choices of Ruckelshaus. If Richardson chooses not to fire Cox and resigns, then Ruckelshaus's optimal action is also to not fire Cox and resign. If Richardson chooses not to fire Cox but does not resign and is fired by Nixon, then again Ruckelshaus's optimal action is also to not fire Cox and resign. Moving to Nixon's decision node associated with the decision of whether or not to fire Richardson, Nixon realizes that if he fires Richardson, then Ruckelshaus will come in and still not fire Cox. Nixon's payoff is only 4, but that is higher from not firing Richardson, which is only 1, as it means Cox remains as special prosecutor. At Richardson's decision node, he can either not fire Cox and resign and get a payoff of 10 (as Ruckelshaus will act similarly), fire Cox and resign and get a payoff of 2, fire Cox and not resign and get a payoff of 1, or not fire Cox and not resign and get a payoff of 9 (as Nixon will fire Richardson and then Ruckelshaus will not fire Cox and resign). Thus, Richardson optimally does not fire Cox and resigns. Finally, at the initial decision node, Nixon can order Cox to be fired and receive a payoff of 6 (as both Richardson and then Ruckelshaus respond by not firing Cox and resigning) or not place the order and get a lowly payoff of 1. Thus, Nixon orders Richardson to fire Cox. The unique subgame perfect Nash equilibrium for Nixon is

FIGURE PR8.7 Saturday Night Massacre



order Cox to be fired/fire Richardson, for Richardson it is *do not fire and resign*, and for Ruckelshaus it is *do not fire and resign/do not fire and resign*.[1]

11. Consider the following passage from *Midnight in the Garden of Good and Evil*:[4]

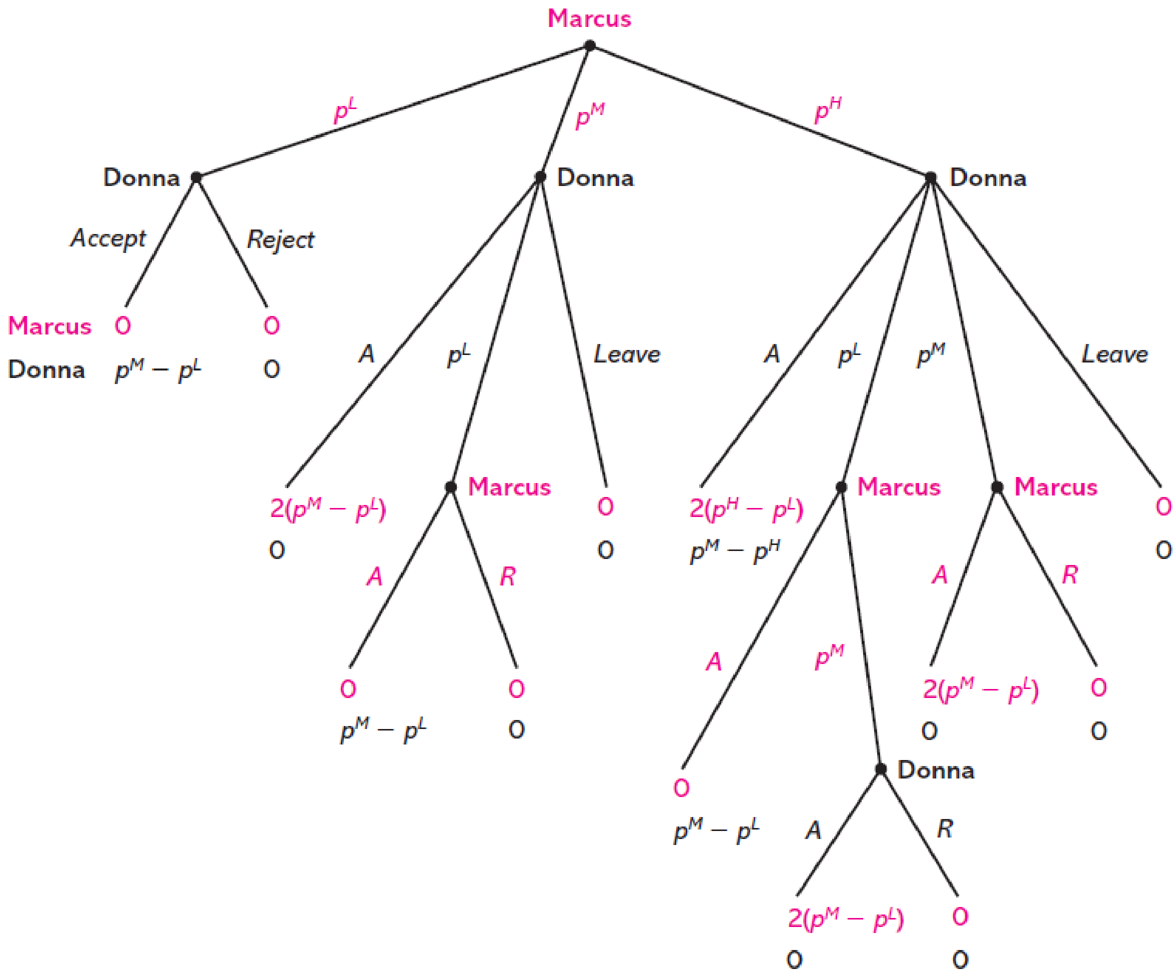
> *There's a woman here, a grande dame at the very apex of society and one of the richest people in the Southeast, let alone Savannah. She owns a copper mine. She built a big house in an exclusive part of town, a replica of a famous Louisiana plantation house with huge white columns and curved stairs. You can see it from the water. Everybody goes 'Oooo, look!" when they pass by it. I adore her. She's been like a mother to me. But she's the cheapest woman who ever lived! Some years ago she ordered a pair of iron gates for her house. They were designed and built especially for her. But when they were delivered she pitched a fit, said they were horrible, said they were filth. "Take them away," she said, "I never want to see them again!" Then she tore up the bill, which was for $1,400—a fair amount of money in those days. The foundry took the gates back, but they didn't know what to do with them. After all, there wasn't much demand for a pair of ornamental gates exactly that size. The only thing they could do was to sell the iron for its scrap value. So they cut the price from $1,400 to $190. Naturally, the following day the woman sent a man over to the foundry with $190, and today those gates are hanging on her gateposts where they were originally designed to go. That's pure Savannah. And that's what I mean by cheap. You mustn't be taken in by the moonlight and magnolias. There's more to Savannah than that. Things can get very murky.*

Using backward induction, can you explain where the foundry went wrong?

ANSWER: Once the foundry built the pair of custom iron gates, the value to the foundry if the grande dame did not buy them was nothing more than their value as scrap. Thus, the foundry would be willing to sell them for any price at least as great as their scrap value. The foundry failed to realize that once the gates were built, its bargaining position was seriously weakened. They should have written a contract that either required payment prior to the gates' construction or that specified that if the grande dame did not make the payment of $1,400 upon delivery, then the foundry would destroy them. With that latter contract, after the gates were built, the grande dame's choices would be to not have the gates or pay the $1,400 and have the gates. Presuming the latter is preferred, the foundry would receive payment. With either of these contracts, the negotiations are done upfront before the foundry builds the gates; at that point, the parties' bargaining powers are comparable. Without a contract and once the gates are built, it is the grande dame who has much of the bargaining power.

12. The haggling game from Chapter 2 is reproduced as **FIGURE PR8.12**. Solve for all subgame perfect Nash equilibria for which a player chooses *accept* whenever that is an optimal action. That is, if a player's payoff is maximized by either choosing *accept* or choosing some other action, he or she chooses *accept*.

FIGURE PR8.12 Haggling at the Auto Dealer



ANSWER: The strategy template for Marcus is:

At initial node, then ______
If Marcus proposed p^M and Donna rejected and proposed p^L, then ______
If Marcus proposed p^H and Donna rejected and proposed p^L, then ______
If Marcus proposed p^H and Donna rejected and proposed p^M, then ______

The strategy template for Donna is:

If Marcus proposed p^L, then ______
If Marcus proposed p^M, then ______
If Marcus proposed p^H, then ______
If Marcus proposed p^H and Donna rejected and proposed p^L and Marcus rejected and proposed p^M, then ______

When we state a players strategy as a 4-tuple of actions, the sequence of those actions will correspond with the sequence just given.

At the decision node in which Marcus proposed p^L, Donna's optimal action is to accept the offer.

At the decision node in which Marcus proposed p^M and Donna rejected his proposal and proposed p^L, Marcus gets a zero payoff from either accepting or rejecting Donna's proposal. Thus, we presume he accepts it, so the payoffs associated with Marcus proposing p^M are zero for Marcus and $p^M - p^L$ for Donna.

At the decision node in which Marcus proposed p^H, Donna rejected and proposed p^L, and Marcus rejected and proposed p^M, Donna is indifferent between accepting and rejecting p^M. Thus, we suppose she chooses to accept it. At the decision node in which Marcus proposed p^H and Donna rejected and proposed p^L, Marcus optimally chooses to reject p^L and propose p^M, as he gets a payoff of $2(p^M - p^L)$ from doing so (which presumes that Donna will accept Marcus's offer) and a payoff of 0 from accepting Donna's proposal of p^L.

At the decision node in which Marcus proposed p^H and Donna rejected and proposed p^M, Marcus optimally accepts p^M.

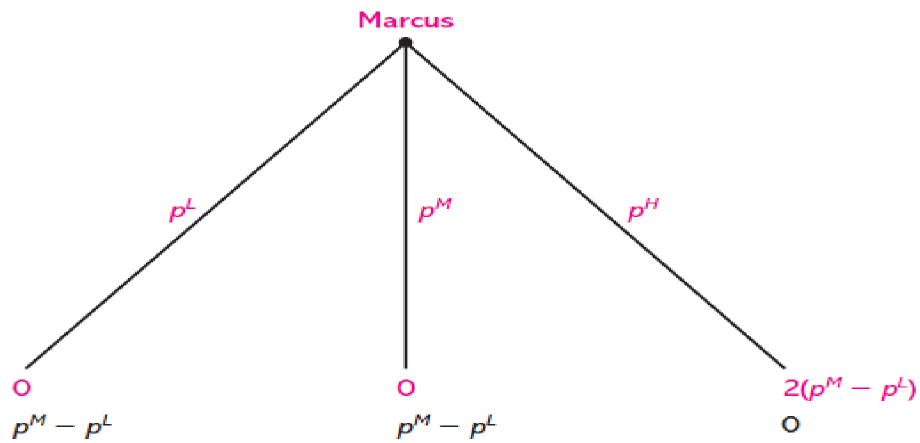
With that derived behavior, we can now examine the decision node in which Marcus proposed p^H. Donna has four choices: (1) if she accepts, her payoff is $p^M - p^H < 0$; (2) if she rejects and proposes p^L, then Marcus rejects and proposes p^M and Donna accepts, so her payoff is zero; (3) if she rejects and proposes p^M, then Marcus accepts and her payoff is zero; and (4) if she chooses to leave, her payoff is zero. The last three choices all produce a zero payoff, which is superior to accepting the proposal. Thus, there are (at least) three subgame perfect Nash equilibria; they differ in terms of Donna's strategy at the decision node in which Marcus proposed p^H.

Suppose that if Marcus proposed p^H, Donna's strategy has her reject it and propose p^L. The game faced by Marcus at the initial node is then as shown in FIGURE SOL8.12.1. Marcus optimally proposes p^H. The subgame perfect Nash equilibrium is

Marcus: Propose p^H/Accept/Reject and propose p^M/Accept.

Donna: Accept/Reject and propose p^L/Reject and propose p^L/Accept.

FIGURE SOL8.12.1



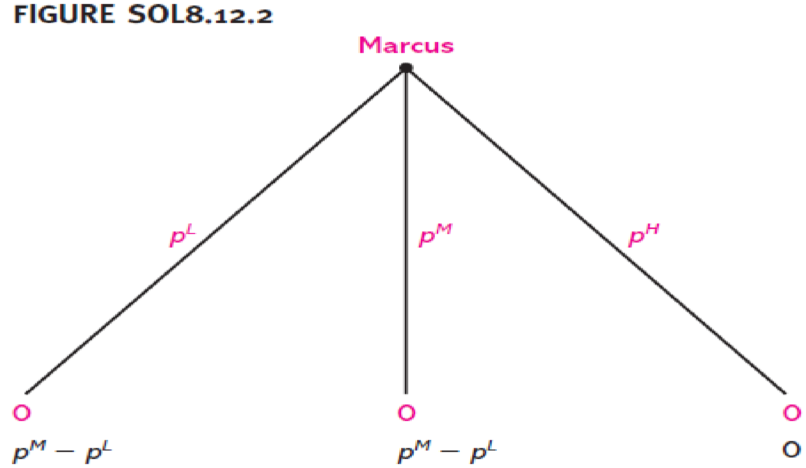
Suppose that if Marcus proposed p^H, Donna's strategy has her reject it and propose p^M. The game faced by Marcus at the initial node is still that shown in FIGURE SOL8.12.1. Marcus optimally proposes p^H. The subgame perfect Nash equilibrium is

Marcus: Propose p^H/Accept/Reject and propose p^M/Accept.

Donna: Accept/Reject and propose p^L/Reject and propose p^M/Accept.

Suppose that if Marcus proposed p^H, Donna's strategy has her leave. The game faced by Marcus at the initial node is then as shown in FIGURE SOL8.12.2. Any

FIGURE SOL8.12.2



of the three prices are optimal for Marcus. There are then three subgame perfect Nash equilibria:

(1) Marcus: Propose p^L/Accept/Reject and propose p^M/Accept.

Donna: Accept/Reject and propose p^M/Leave/Accept.

(2) Marcus: Propose p^M/Accept/Reject and propose p^M/Accept.

Donna: Accept/Reject and propose p^M/Leave/Accept.

(3) Marcus: Propose p^H/Accept/Reject and propose p^M/Accept.

Donna: Accept/Reject and propose p^M/Leave/Accept.