Answer Key for Econ 204b PS 1, Winter 2014

Problem 1

(a) Our expected payoff is given by

$$E[u(a)] = E[(1-(a-s)^2)|a] = E[(1-a^2+2as-s^2))|a]$$
$$= 1-a^2+2aE[s]-E[s^2]$$

So we maximize this value with respect to a,

$$\Rightarrow -2a + sE[s] = 0$$
$$\Rightarrow a = E[s]$$

Since $s \in [0,1]$, then $E[s] = \frac{1}{2}$ and thus $\mathbf{a}^* = \frac{\mathbf{1}}{\mathbf{2}}$. The maximum expected payoff is then:

$$E[u(a^*)] = 1 - {a^*}^2 + 2a^*E[s] - E[s^2] = \frac{\mathbf{11}}{\mathbf{12}}$$

(b) Now assume that s has distribution $F(s)$. Then $a^* = E[s] = \int sdF(s)$ which gives:

$$a^* = \begin{cases} 20 & E[s] > 20 \\ \int sdF(s) & \text{otherwise} \\ -10 & E[s] < -10 \end{cases}$$

With a maximum expected payoff of:

$$E[u] = 1 - {a^*}^2 + 2a^* \int sdF(s) - \int s^2 dF(s) = 1 - {a^*}^2 + 2a^*E[s] - Var[s]$$

.

Problem 2

(a)

State	$P(S)$	$P(t1+\|S)$	$P(t1-\|S)$	$P(t2+\|S)$	$P(t2-\|S)$
A	0.6	0.7	0.3	0.2	0.8
B	0.4	0.4	0.6	0.5	0.5

Joint probabilities $\quad P(t1+, t2+, S) = p(t1+|S) * p(t2+|S) * P(S)$
Note that this holds because given S, t1 and t2 are conditional independent.

State	$P(t1+, t2+, S)$	$P(t1+, t2-, S)$	$P(t1-, t2+, S)$	$P(t1-, t2-, S)$	Sum
A	0.084	0.336	0.036	0.144	$\mathbf{P(A) = 0.6}$
B	0.08	0.08	0.12	0.12	$\mathbf{P(B) = 0.4}$
Sum=$\mathbf{P(t1, t2)}$	**0.164**	**0.416**	**0.156**	**0.264**	1

(b) $\quad P(t1+, S) = p(t1+|S)P(S)$

State	$P(t1+, S)$	$P(t1-, S)$	$P(t2+, S)$	$P(t2-, S)$
A	0.42	0.18	0.12	0.48
B	0.16	0.24	0.2	0.2
Sum	$\mathbf{P(t1+) = 0.58}$	$\mathbf{P(t1-) = 0.42}$	$\mathbf{P(t2+) = 0.32}$	$\mathbf{P(t2-) = 0.68}$

(c) Posterior probabilities

$$P(A|T1+) = \frac{P(A) * P(T1+|A)}{[P(A) * P(T1+|A) + P(B) * P(T1+|B)}$$

$$\mathbf{P(A|T1+) = 0.724} \qquad \mathbf{P(A|T1-) = 0.429} \qquad \mathbf{P(A|T2+) = 0.375} \qquad \mathbf{P(A|T2-) = 0.706}$$

(d) Posterior probabilities

$$P(A|T1+, T2+) = \frac{P(A) * P(T1+, T2+|A)}{(P(A) * P(T1+, T2+|A) + P(B) * (PT1, T2+|B)}$$

$$\mathbf{P(A|T1+, T2+) = 0.512} \qquad \mathbf{P(A|T1+, T2-) = 0.808} \qquad \mathbf{P(A|T1-, T2+) = 0.231} \qquad \mathbf{P(A|T1-, T2-) = 0.545}$$

Problem 3

(a) The expected payoff without message is $0.1*20+0.5*5+0.4*(-10) = \mathbf{0.5}$, as hiring the new person is better than not hiring, which brings payoff 0.

The expected payoff under perfect information is $0.1*20+0.5*5+0.4*0 = \mathbf{4.5}$, because once we know the person's type, we will hire him if he is great (payoff 20) or good (payoff 5), but if he is a poor salesman, we will not hire and get payoff 0. So the value of perfect information is $4.5-0.5 = \mathbf{4}$ (Please check the decision trees at attachment).

(b) Daily sales follow the Possion distribution $\lambda = (1/2, 1/4, 1/8)$ for type =(great, good, poor). The likelihood probabilities are $p(n|type) = \frac{e^{(-\lambda * t)}(\lambda * t)^n}{n!}$

$$p(m=n|type=great) = p(n|\lambda * t) = \frac{2^n * e^{-2}}{n!}$$

$$p(m=n|type=good) = p(n|\lambda * t) = \frac{1^n * e^{-1}}{n!}$$

$$p(m=n|type=poor) = p(n|\lambda * t) = \frac{2^{-n} * e^{-1/2}}{n!}$$

Assume the salesman sell n cars for one week, then the posterior probabilities of salesman's type are

$$p(type=great|m=n) = \frac{\frac{1}{10}\frac{2^n}{n!}e^{-2}}{\frac{1}{10}\frac{2^n}{n!}e^{-2}+\frac{5}{10}\frac{1^n}{n!}e^{-1}+\frac{4}{10}\frac{2^{-n}}{n!}e^{-0.5}} = \frac{2^n e^{-2}}{2^n e^{-2}+5e^{-1}+2^{2-n}e^{-0.5}}$$

$$p(type=great|m=n) = \frac{\frac{5}{10}\frac{1^n}{n!}e^{-1}}{\frac{1}{10}\frac{2^n}{n!}e^{-2}+\frac{5}{10}\frac{1^n}{n!}e^{-1}+\frac{4}{10}\frac{2^{-n}}{n!}e^{-0.5}} = \frac{5e^{-1}}{2^n e^{-2}+5e^{-1}+2^{2-n}e^{-0.5}}$$

$$p(type=great|m=n) = \frac{\frac{4}{10}\frac{2^{-n}}{n!}e^{-1}}{\frac{1}{10}\frac{2^n}{n!}e^{-2}+\frac{5}{10}\frac{1^n}{n!}e^{-1}+\frac{4}{10}\frac{2^{-n}}{n!}e^{-0.5}} = \frac{2^{2-n}e^{-0.5}}{2^n e^{-2}+5e^{-1}+2^{2-n}e^{-0.5}}$$

The posterior probabilities values

p(type|m=n)	n=0	n=1	n=2	n=3	n=4	n≥ 13
great	0.031	0.081	0.181	0.336	0.521	0.998
good	0.418	0.554	0.616	0.570	0.443	0.001
poor	0.551	0.365	0.203	0.094	0.036	0.000

Once the car selling n is observed, the dealer can decide whether to hire by calculating expected payoff, using posterior probabilities.

$$E_n(w) = \max\{0, p(great|n)\cdot 20 + p(good|n)\cdot 5 + p(poor|n)\cdot(-10)\}$$

The diagram gives expected payoff. Note that when the dealer observes $n=0$, he will not hire (and just lose hiring cost), otherwise, he will hire. At here, $E_w(bd)$ means total expected payoff before decision made and $E_w(ad)$ means total expected payoff after decision made. The total E_w is the final expected payoff (multiplied prior probability $P(m=n)$).

	$n=0$	$n=1$	$n=2$	$n=3$	$n\geq 4$
$E_w(bd)$	0(-2.808)	0.746	4.673	8.624	13.325
Decision	not hire	hire	hire	hire	hire
$E_w(ad)$	-0.04	0.706	4.633	8.584	13.285
Total E_w	-0.176	2.480	13.959	27.818	50.988

The value of information is realized only if information makes the dealer **change decision**. Here, only when $n=0$, the dealer change his decision. In this case, if hired, the expected loss is $0.031\cdot 20+0.418\cdot 5+0.551\cdot(-10) = -2.808$

This equation shows the loss avoid by information. Thus, the value of information is:

$$V'_{n|n=0} = 2.808 * 0.4401 = 1.236$$

and the net value of information is

$$V_{n|n=0} = 1.236 - 0.04 = \mathbf{1.196}$$

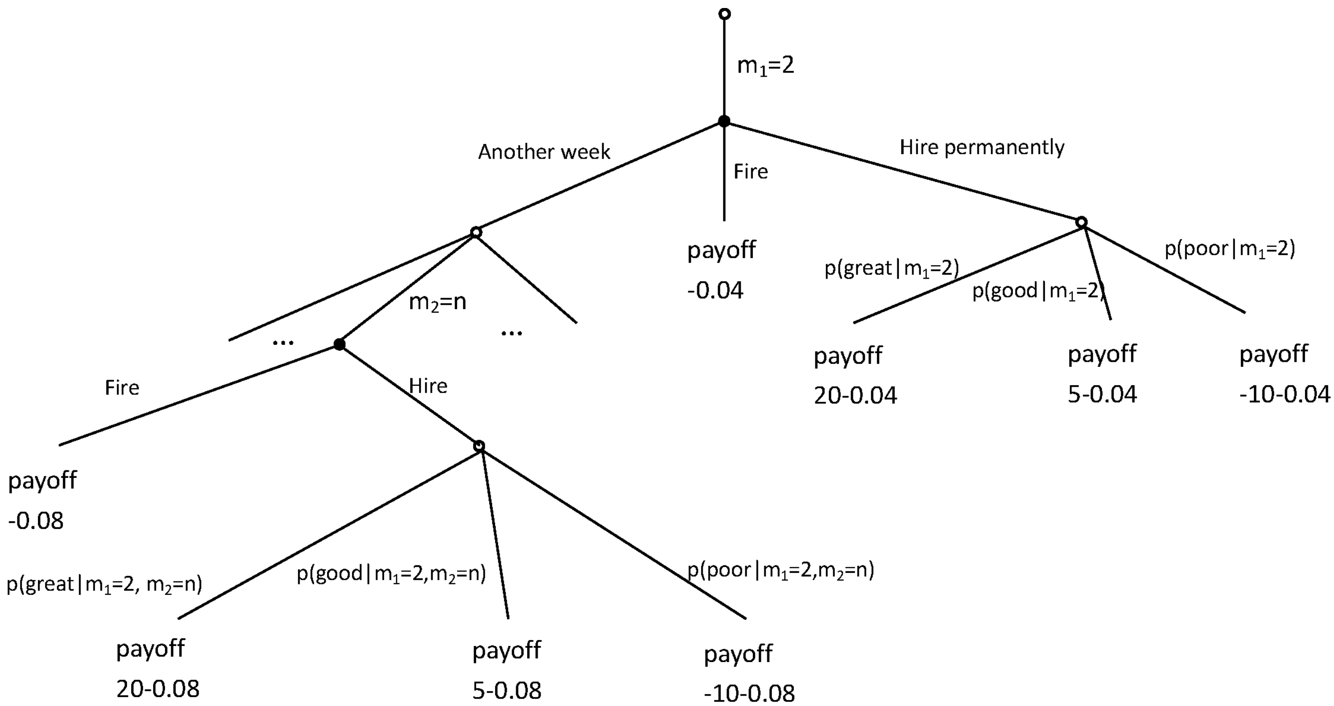

Figure 1: The decision tree in part (c).

(c) Denote n as the sales in the second week. The posterior probability in part b) becomes the new prior probability, thus the new posterior probability of n is

$$p(great|m_2 = n) = \frac{p(m_2 = n|great)p(great|m_1 = 2)}{p(m_2|great)p(great|m_1 = 2) + p(m_2|good)p(good|m_1 = 2) + p(m_2|poor)p(poor|m_1 = 2)}$$

The values are listed below.

P(type—m=n)	n=0	n=1	n=2	n=3	n=4
great	0.065	0.145	0.276	0.447	0.626
good	0.606	0.672	0.638	0.517	0.362
poor	0.329	0.183	0.087	0.035	0.012

Skip some similar calculation, the following table gives the expected payoff after observing second week's sales.

	n=0	n=1	n=2	n=3	n=4
$E_w(bd)$	1.047	4.441	7.836	11.184	14.203
Decision	hire	hire	hire	hire	hire
Total E_w	0.392	1.497	2.785	4.898	8.895

As we can see, no matter how many cars were sold, the dealer will always hire this salesman, i.e., the decision after the second week are the same as after the first week. This indicates that the value of information of the second week's test is 0. Considering the hiring cost, the dealer's best choice is hire permanently at the end of the first week.

Problem 4

Denote s_1 = rival with new technology, s_2 = rival without new technology and a = the reported productivity. The prior probabilities $P(s_1) = 0.1$ and $P(s_2) = 0.9$, and reported productivity is normally distributed $P(a|s_1) \sim N(12, 1)$ and $P(a|s_2) \sim N(10, 1)$.

Then the posterior probability

$$p(s_1|a) = \frac{p(a|s_1) \cdot p(s_1)}{p(a|s_1) \cdot p(s_1) + p(a|s_2) \cdot p(s_2)} \geq 0.5$$

which implies

$$\frac{0.1 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-12)^2}{2}}}{0.1 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-12)^2}{2}} + 0.9 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-10)^2}{2}}} \geq 0.5$$

$$0.1 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-12)^2}{2}} \geq 0.9 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-10)^2}{2}}$$

$$e^{-\frac{(a-12)^2}{2} + \frac{(a-10)^2}{2}} \geq 9$$

$$e^{2a-22} \geq 9$$

Taking log on both sides,

$$2a - 22 \geq 2ln3$$

$$\mathbf{a \geq 11 + \ln 3 \approx 12.099}$$

Problem 5

(a) The agent judges whether the firm has adequate or substandard quality control. Before judging the quality, the agent decides the number of tests to conduct before receiving the first test result.

The information of N times of tests is summarized by the mean, $\bar{X}$, of the outcomes $\{X_1, \cdots, X_n\}$. Since each test result follows normal distribution with variance 9 and mean $+1$ if adequate(A) (-1 if substandard(S)), The distribution of $\bar{X}$ is,

$$\bar{X} \sim \begin{cases} N(1, \frac{9}{N}) & \text{(if adequate(A))} \\ N(-1, \frac{9}{N}) & \text{(if substandard(S))} \end{cases}$$

Given the information, the agent judge whether the firm has adequate(a) of substandard quality(s). The agent lose C_{Sa} if the choice is adequate(a) and the true quality is substandard(S), and lose C_{As} if the choice is substandard(s) and the true quality is adequate(A). Also, each test cost 1. Then, the expected payoff of each action is,

$$E\left[U|a\right] = -nP(A|\bar{X}) + (-C_{Sa} - n)P(S|\bar{X})$$

$$E\left[U|s\right] = -nP(S|\bar{X}) + (-C_{As} - n)P(A|\bar{X})$$

Using $P(A|\bar{X}) + P(S|\bar{X}) = 1$, adequate(a) is optimal if

$$\frac{P(S|\bar{X})}{C_{As}} \leq \frac{P(A|\bar{X})}{C_{Sa}}$$

By Bayes' theorem,

$$P(S|\bar{X}) = \frac{P(\bar{X}|S)P(S)}{P(\bar{X}|S)P(S) + P(\bar{X}|A)P(A)}$$

$$P(A|\bar{X}) = \frac{P(\bar{X}|A)P(A)}{P(\bar{X}|A)P(A) + P(\bar{X}|S)P(S)}$$

Thus, adequate(a) is optimal if

$$\frac{P(\bar{X}|S)P(S)}{C_{As}} \leq \frac{P(\bar{X}|A)P(A)}{C_{Sa}}$$

Now, $P(A) = P(S) = \frac{1}{2}$, $C_{Sa} = C_{As} = 1000$, so the expression simplifies to

$$P(\bar{X}|S) \leq P(\bar{X}|A) \quad \Leftrightarrow \quad \frac{P(\bar{X}|A)}{P(\bar{X}|S)} \geq 1 \quad \Leftrightarrow \quad \ln P(\bar{X}|A) - \ln P(\bar{X}|S) \geq 0$$

Using the pdf of normal distribution, adequate(a) is optimal if

$$\ln\left(\frac{1}{\sqrt{2\pi(9/N)}} e^{-\frac{1}{2}(\frac{\bar{X}-1}{3/\sqrt{N}})^2}\right) - \ln\left(\frac{1}{\sqrt{2\pi(9/N)}} e^{-\frac{1}{2}(\frac{\bar{X}+1}{3/\sqrt{N}})^2}\right) \geq 0 \quad \Leftrightarrow \quad \bar{X} \geq 0$$

So, the agent's optimal decision is to choose adequate(a) if $\bar{X} \geq 0$ and choose substandard(s) if $\bar{X} < 0$.

The expected payoff from this optimal cutoff strategy given N is

$$\begin{aligned} E[U|N] = &\left\{-NP(A|\bar{X} \geq 0) + (-1000 - N)P(S|\bar{X} \geq 0)\right\} P(\bar{X} \geq 0) \\ &+ \left\{-NP(S|\bar{X} \leq 0) + (-1000 - N)P(A|\bar{X} \geq 0)\right\} P(\bar{X} \leq 0) \end{aligned}$$

By Bayes' theorem,

$$P(A|\bar{X} \geq 0) = \frac{P(\bar{X} \geq 0|A)P(A)}{P(\bar{X} \geq 0|A)P(A) + P(\bar{X} \geq 0|S)P(S)}$$
$$P(A|\bar{X} < 0) = \frac{P(\bar{X} < 0|A)P(A)}{P(\bar{X} < 0|A)P(A) + P(\bar{X} < 0|S)P(S)}$$

For the ease of derivation, express the probability using cdf of standardized normal distribution.

$$\begin{aligned} P(\bar{X} \geq 0|A) &= P\left(z \equiv \frac{\bar{X} - 1}{3/\sqrt{N}} \geq \frac{-1}{3/\sqrt{N}}\right) \\ &= 1 - \Phi\left(\frac{-1}{3/\sqrt{N}}\right) \\ P(\bar{X} < 0|A) &= 1 - P(\bar{X} \geq 0|A) \\ &= \Phi\left(\frac{-1}{3/\sqrt{N}}\right) \\ P(\bar{X} \geq 0|S) &= P\left(z \equiv \frac{\bar{X} + 1}{3/\sqrt{N}} \geq \frac{1}{3/\sqrt{N}}\right) \\ &= 1 - \Phi\left(\frac{1}{3/\sqrt{N}}\right) \\ P(\bar{X} < 0|S) &= 1 - P(\bar{X} \geq 0|S) \\ &= \Phi\left(\frac{1}{3/\sqrt{N}}\right) \end{aligned}$$

Since standard normal distribution is symmetric around 0,

$$\begin{aligned} P(\bar{X} \geq 0|A) &= P(\bar{X} < 0|S) \\ P(\bar{X} < 0|A) &= P(\bar{X} \geq 0|S) \end{aligned}$$

Therefore, the maximal expected utility given N simplifies to

$$\begin{aligned} E[U|N] &= -N - 1000\Phi\left(\frac{-1}{3/\sqrt{N}}\right) \\ &= -N - 1000 \int_{\infty}^{\frac{-1}{3/\sqrt{N}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \end{aligned}$$

The agent choose N to maximize this expected utility. Using Leibnitz's rule, F.O.C. is

$$\begin{aligned} \frac{dE[U|N]}{dN} &= 0 \\ \Leftrightarrow 0 &= -1 - 1000\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{-1}{3/\sqrt{N}})^2} \left(-\frac{1}{6}N^{-\frac{1}{2}}\right) \\ \Leftrightarrow 0 &= \frac{N}{18} + \frac{1}{2}\ln N - \ln 500 + \frac{1}{2}\ln 18\pi \end{aligned}$$

So, the optimal number of tests is

$$\mathbf{N = 42}$$

(b) First, given $\bar{X}$, the agent chooses adequate(a) or substandard(s). As derived in part a), the optimal decision is adequate(a) if

$$\frac{P(\bar{X}|S)P(S)}{C_{As}} \leq \frac{P(\bar{X}|A)P(A)}{C_{Sa}}$$

Now, $P(A) = 0.8$, $P(S) = 0.2$, $C_{Sa} = 1000$, $C_{As} = 400$, so the expression simplifies to

$$\begin{aligned} \frac{P(\bar{X}|A)}{P(\bar{X}|S)} &\geq \frac{5}{8} \\ \bar{X} \geq -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{N} &\equiv c(N) \end{aligned}$$

The expected utility under this optimal cutoff strategy given N is

$$\begin{aligned} E\left[U|N\right] = &\left\{-NP(A|\bar{X} \geq c(N)) + (-1000 - N)P(S|\bar{X} \geq c(N))\right\} P(\bar{X} \geq c(N)) \\ &+ \left\{-NP(S|\bar{X} \leq c(N)) + (-400 - N)P(A|\bar{X} \leq c(N))\right\} P(\bar{X} \leq c(N)) \end{aligned} \tag{1}$$

By Bayes' theorem,

$$\begin{aligned} P(A|\bar{X} \geq c(N)) &= \frac{P(\bar{X} \geq c(N)|A)P(A)}{P(\bar{X} \geq c(N)|A)P(A) + P(\bar{X} \geq c(N)|S)P(S)} \\ P(A|\bar{X} < c(N)) &= \frac{P(\bar{X} < c(N)|A)P(A)}{P(\bar{X} < c(N)|A)P(A) + P(\bar{X} < c(N)|S)P(S)} \end{aligned}$$

Thus, the expression of the expected utility simplifies to

$$\begin{aligned} E\left[U|N\right] = &-NP(\bar{X} \geq c(N)|A)P(A) + (-1000 - N)P(\bar{X} \geq c(N)|S)P(S) \\ &-NP(\bar{X} \leq c(N)|S)P(S) + (-400 - N)P(\bar{X} \leq c(N)|A)P(A) \end{aligned}$$

For the ease of derivation, express the probability using cdf of standardized normal distribution.

$$\begin{aligned} P(\bar{X} \geq c(N)|A) &= P\left(z = \frac{\bar{X} - 1}{3/\sqrt{N}} \geq \frac{c(N) - 1}{3/\sqrt{N}}\right) \\ &\equiv 1 - \Phi\left(\frac{c(N) - 1}{3/\sqrt{N}}\right) \\ P(\bar{X} < c(N)|A) &= 1 - P(\bar{X} \geq c(N)|A) \\ &= \Phi\left(\frac{c(N) - 1}{3/\sqrt{N}}\right) \\ P(\bar{X} \geq c(N)|S) &= P\left(z = \frac{\bar{X} + 1}{3/\sqrt{N}} \geq \frac{c(N) + 1}{3/\sqrt{N}}\right) \\ &\equiv 1 - \Phi\left(\frac{c(N) + 1}{3/\sqrt{N}}\right) \\ P(\bar{X} < c(N)|S) &= 1 - P(\bar{X} \geq c(N)|S) \\ &= \Phi\left(\frac{c(N) + 1}{3/\sqrt{N}}\right) \end{aligned}$$

Also, using the fact that

$$\begin{aligned} &P(\bar{X} \geq c(N)|A)P(A) + P(\bar{X} \geq c(N)|S)P(S) \\ &\qquad + P(\bar{X} \leq c(N)|S)P(S) + P(\bar{X} \geq c(N)|A)P(A) = 1, \end{aligned}$$

the final expression of expected utility under the optimal cutoff strategy given N is

$$E\left[U|N\right] = -N - 1000\left(1 - \Phi\left(\frac{c(N) + 1}{3/\sqrt{N}}\right)\right)\frac{1}{5} - 400\Phi\left(\frac{c(N) - 1}{3/\sqrt{N}}\right)\frac{4}{5}$$

where, as derived before, the optimal cutoff value is

$$c(N) = -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{N}$$

In the end, the optimal N is the value which maximizes this expected utility. The optimal N is

$$\mathbf{N = 32}$$

(c) Part (c) is beyond course materials.

Problem 6

The optimal rule is: Sort methods in ascending order of cost/benefit ratio (c_i/p_i). Try the methods in order, and stop whenever either (a) success is achieved or (b) all remaining methods have $c_i/p_i > 1$.

To see that this rule is optimal, suppose that (b) holds. It is easy to see that trying any remaining method i changes expected payoff by $1p_i - c_i < 0$ and hence reduces expected payoff. If (a) holds, of course, trying another method incurs its cost with no gain. Hence the stopping rule is optimal.

To see that ascending order of benefit/cost is optimal, suppose to the contrary that at some stage the plan calls for a higher cost/benefit method k to be tried just before a lower cost benefit method $k+1$. We will show that switching the order increases expected payoff. This will establish the rule, since such a k exists iff the rule is violated. Remark: this idea is behind the bubble-sort algorithm used in spreadsheets.

To complete the argument, for notational convenience (and no loss of generality) write $k = 1$ so we have $\frac{c_1}{p_1} > \frac{c_2}{p_2}$, i.e.

$$c_1 p_2 - c_2 p_1 > 0. \tag{2}$$

Solving the tree in Figure 2 via backward induction, we see that the left side has expected payoff

$$L = p_1 - p_1 c_1 + (1 - p_1)[p_2(1 - c_1 - c_2) + (1 - p_2)(-c_1 - c_2)] = p_1 + p_2 - p_1 p_2 - c_1 - c_2 + p_1 c_2,$$

while the right side has payoff

$$R = p_2 - p_2 c_2 + (1 - p_2)[p_1(1 - c_1 - c_2) + (1 - p_1)(-c_1 - c_2)] = p_1 + p_2 - p_1 p_2 - c_1 - c_2 + p_2 c_1.$$

The gain in expected payoff from switching plans is $R - L = c_1 p_2 - c_2 p_1$, which is positive by equation (2). Thus switching to ascending order indeed does increase expected payoff, so the rule is indeed optimal. QED.

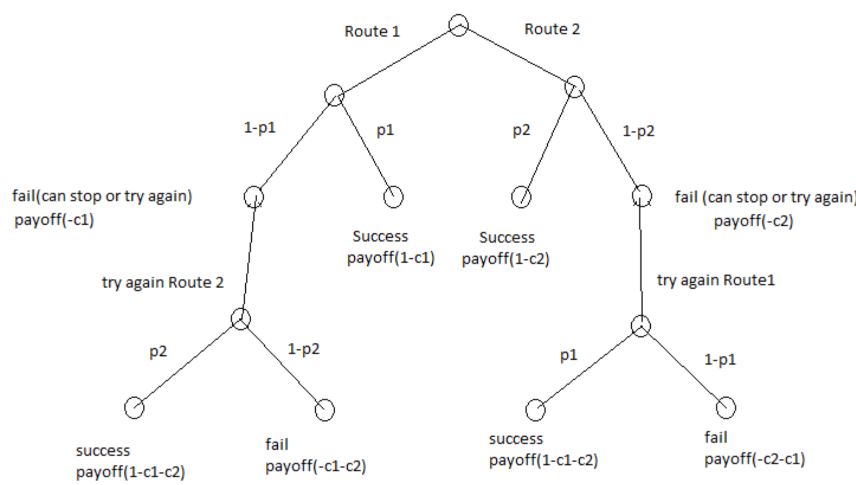

Figure 2: Given equation (2), the left side (try Route or method 1 first) is inconsistent with the proposed rule.

Textbook Problems

Problem 6.B.4

1. Given:
   Since $A \succ D$, normalise payoff such that $u(A) = 1$ and $u(D) = 0$.

$u(B) = p(1) + (1-p)0 = p$
Similarly $u(C) = q$
I assume u(C) $\gg$ u(B), then $A \succ C \succ B \succ D$

2. Let $s_1 = flood$ and $s_2 = no\ flood$. Let action $a_1 = evacuate$ and $a_2 = not\ evacuate$.
$EV[C_1] = 0.891(1) + q(0.009) + p(0.099) + 0 = .0891 + 0.009q + 0.099p$
$EV[C_2] = 0.9405 + 0.095q + 0.0495p$
Assuming u(C) $\gg$ u(B), then **criterion 2 is preferred**.

Criterion	s	$P(s_i)$	$P(a_1\|s_i)$	$P(a_2\|s_i)$	$P(a_1, s_i)$	$P(a_2, s_i)$
C_1	s_1	0.01	0.9	0.1	0.009	0.001
C_1	s_2	0.99	0.1	0.9	0.099	0.891
C_2	s_1	0.01	0.95	0.05	0.095	0.0005
C_2	s_2	0.99	0.05	0.95	0.0495	0.9405

Problem 6.C.1

max $(1-\pi)u(w-\alpha q) + \pi u(w - \alpha q - 1 + \alpha)$
F.O.C.

$$-q(1-\pi)u'(w-\alpha q) + \pi(1-q)u'(w - D + \alpha * (1-q)) \le 0$$

When $\alpha^* > 0$, this holds with equality, so

$$\frac{u'(w - D + \alpha * (1-q))}{u'(w - \alpha * q)} = \frac{q(1-\pi)}{\pi(1-q)}$$

When $q > \pi$, $u'(w - D + \alpha^*(1-q))$ is greater than $u'(w - \alpha^* q)$, the agent does not insure completely.

Problem 6.C.2

1. An individual with quadratic form Bernoulli utility function $u(x) = \beta x^2 + \gamma x$ will have an expected utility

$$E[u] = \int_{-\infty}^{-\gamma/2\beta} (\beta x^2 + \gamma x) dF(x)$$

Here, β should be negative and the upper limit of integral cannot exceed $-\gamma/2\beta$ in order to get the concavity of u. Note that:

$$E[u] = \int_{-\infty}^{-\gamma/2\beta} (\beta x^2 + \gamma x) dF(x) = \beta \int_{-\infty}^{-\gamma/2\beta} x^2 dF(x) + \gamma \int_{-\infty}^{-\gamma/2\beta} x dF(x)$$
$$= \beta E[x^2] + \gamma E[x] = \beta Var[x] + \beta (E[x])^2 + \gamma E[x]$$

This shows the expected utility is only determined by the mean and variance of distribution.

2. Let u($\cdot$) be the Bernoulli utility function and G($\cdot$) be the distribution function that puts probability one on x,

$$u(x) = U(G) = E[G] - \gamma Var[G] = x - \gamma \times 0 = x$$

Then, we have $u(x) = x$, which implies risk neutral. Therefore, $u(x) = x$ is not a Bernoulli utility function which should be concave (risk-averse).

Problem 6.C.18

1. Arrow-Pratt Absolute Risk Aversion: $-\frac{u''(x)}{u'(x)}$
Given $u(x)=\sqrt{x}$, then $u'(x)=\frac{1}{2}x^{-\frac{1}{2}}$ and $u''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$
Then, $-\frac{u''(x)}{u'(x)}=-\frac{\frac{-1}{4}x^{-\frac{3}{2}}}{\frac{1}{2}x^{-\frac{1}{2}}}=\frac{2}{4}x^{-\frac{3}{2}+\frac{1}{2}}=\frac{1}{2x}$
When $x = 5$, then $-\frac{u''(x)}{u'(x)} = \mathbf{0.1}$

Arrow-Pratt Relative Risk Aversion:
$-\frac{xu''(x)}{u'(x)}=x\frac{1}{2x} = \mathbf{0.5} \Rightarrow$ constant over all level of wealth

2. The expected utility of gamble $(16, 4; \frac{1}{2}, \frac{1}{2})$ is $E(u) = \frac{1}{2} \cdot u(16) + \frac{1}{2} \cdot u(4) = 3$, thus the certainty equivalent is $\mathbf{c = 9}$, and the probability premium satisfies

$$u(\frac{16+4}{2}) = (\frac{1}{2} + p) \cdot u(16) + (\frac{1}{2} - p) \cdot u(4) \to \mathbf{p} = \frac{\mathbf{\sqrt{10} - 3}}{\mathbf{2}}$$

3. The expected utility of gamble $(36, 16; \frac{1}{2}, \frac{1}{2})$ is $E(u) = \frac{1}{2} \cdot u(36) + \frac{1}{2} \cdot u(16) = 5$, thus the certainty equivalent is $\mathbf{c = 25}$, and the probability premium satisfies

$$u(\frac{36+16}{2}) = (\frac{1}{2} + p) \cdot u(36) + (\frac{1}{2} - p) \cdot u(16) \to \mathbf{p} = \frac{\mathbf{\sqrt{26} - 5}}{\mathbf{2}}$$

4. Interpretation: according to the absolute risk aversion, $-\frac{u''(x)}{u'(x)} = \frac{1}{2x}$, the measurement of risk aversion decreases in x, Therefore, the higher the expected wealth level, the lower risk aversion. As we can see in (b), the percentage of insurance $\frac{10-9}{10} = \frac{1}{10}$, and in (c), $\frac{26-25}{26} = \frac{1}{26}$. Thus, the individual pays more insurance in terms of wealth. In addition, p=0.08 in (b) $>$ p=0.05 in (c). Then, more risk-averse individual requires higher probability premium for compensation.

Problem 6.C.20

Expected value of the lottery $= 1/2(x + \epsilon) + 1/2(x - \epsilon) = x$

Let x_c be the certainty equivalent of this gamble, $u(x_c) = \frac{1}{2}u(x + \epsilon) + \frac{1}{2}u(x - \epsilon)$

Take derivative w.r.t. ϵ,

$$u'(x_c) \cdot \frac{dx_c}{d\epsilon} = \frac{1}{2}u'(x + \epsilon) - \frac{1}{2}u'(x - \epsilon) \tag{1}$$

Take derivative w.r.t. ϵ again,

$$u'(x_c) \cdot \frac{d^2 x_c}{d\epsilon^2} + u''(x_c) \cdot \frac{dx_c}{d\epsilon} \cdot \frac{dx_c}{d\epsilon} = \frac{1}{2}u''(x + \epsilon) + \frac{1}{2}u''(x - \epsilon) \tag{2}$$

Plugging (1) into (2),

$$u'(x_c) \cdot \frac{d^2 x_c}{d\epsilon^2} + u''(x_c) \cdot \frac{\frac{1}{2}u'(x+\epsilon) - \frac{1}{2}u'(x-\epsilon)^2}{u'(x_c)^2} = \frac{1}{2}u''(x + \epsilon) + \frac{1}{2}u''(x - \epsilon)$$

Take limit on both sides as $\epsilon \to 0$, note that $x_c \to x$,

$$\lim_{\epsilon \to 0}[u'(x_c) \cdot \frac{d^2 x_c}{d\epsilon^2} + u''(x_c) \cdot \frac{\frac{1}{2}u'(x+\epsilon) - \frac{1}{2}u'(x-\epsilon)^2}{u'(x_c)^2}] = \lim_{\epsilon \to 0}[\frac{1}{2}u''(x + \epsilon) + \frac{1}{2}u''(x - \epsilon)]$$

Thus,

$$\lim_{\epsilon \to 0} \frac{d^2 x_c}{d\epsilon^2} = \lim_{\epsilon \to 0} \frac{\frac{1}{2}u''(x+\epsilon) + \frac{1}{2}u''(x-\epsilon)}{u'(x_c)} = \frac{u''(x)}{u'(x)} = -r_A(x)$$