Evolutionary Dynamics for Playing the Field *

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Abstract

Any piecewise-smooth, symmetric two-player game can be extended to define a population game in which each player interacts with a large representative subset of the entire population, i.e., is "playing the field." Assuming that players respond to the payoff gradient over a continuous action space, we obtain nonlinear integro-partial differential equations that are often numerically tractable and sometimes analytically tractable. Economic applications include oligopoly, growth theory, and financial bubbles and crashes.

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1 Introduction

> "There are many situations...in which an individual is, in effect, competing not against an individual opponent but against the population as a whole, or some section of it. Such cases can be loosely described as 'playing the field.' ... In fact, such contests against the field probably are more widespread and important than pairwise contests."
> —John Maynard Smith (1982, p. 23).

Classic game theory is formulated in terms of two or more distinct individuals who interact strategically. Population games, by contrast, consider anonymous interactions of large numbers of strategically identical players. Classic games and population games are nevertheless complementary and mutually illuminating, as noted in John Nash's 1950 dissertation, by Maynard Smith (1982), and by numerous more recent authors.

The current paper examines an important connection between the two theoretical traditions: single population games that arise from a symmetric two player game with a continuous action space. To generate a player's payoff in the population game, one takes the weighted average payoff that player would receive in the two player game, with weights given by the current population distribution of actions. That is, in Maynard Smith's terminology, everyone is playing the field.

Our contribution is to derive and analyze natural dynamics for such games. The idea is that each player adjusts his choice continuously within an interval of possible actions, seeking higher ground according to the payoff gradient. Of course, such adjustments change the distribution of play and thus the payoff gradient. The resulting dynamics are described by an integro-partial differential equation, where the integral is over the action space. Solutions exhibit a considerable range of behavior, and there is a wide variety of applications in the social sciences as well as in evolutionary biology.

Section 2 begins the exposition by showing how to obtain a population game with payoff ϕ from an arbitrary symmetric two-player game g over a continuous action space A. It then derives the key equation, the gradient dynamics PDE, and notes its connection to the Fokker-Planck-Kolmogorov equation. The section also presents a simple family of examples that illustrates the diversity of possible behaviors arising from the PDE.

Section 3 explores games that yield linear gradient dynamics PDEs. The first theorem establishes that solutions will be "classical" — that is, smooth — for a broad class of underlying games g, including those that lead to linear PDEs, when the action space A has no boundary points. It then introduces the method of characteristics from partial differential equation theory, and shows how to apply this method to understand the solution of linear gradient dynamics PDEs, including how population mass can pile up at a boundary point of the action space A. The section then works out dynamics for several notable examples in social science: Cournot oligopoly, Keynes' beauty contest, and growth theory. It concludes by fully characterizing population behavior when players' choices consist of strategy mixtures for an underlying 2×2 game.

Section 4 explores games which yield non-linear gradient dynamics PDEs. The solution in these cases can develop kinks or discontinuities in finite time, sometimes known as "shocks." The PDE, of course, requires re-interpretation in such non-differentiable cases, but we show how to extend the method of characteristics to do so. The section includes applications to Bertrand competition and to financial bubbles and crashes. Section 5 offers a brief concluding discussion, and Appendix A collects the more technical proofs.

The current paper builds on Friedman and Ostrov (2010). After discussing relevant existing literature in economics, biology and mathematics, that paper justifies gradient dynamics as the optimal response to quadratic adjustment cost, and analyzes in detail a class of "local" population game payoff functions that cannot typically be obtained from two-player games. It also obtains some fairly general results on existence and asymptotic convergence of solutions to the PDEs.

2 Extending two-player games

In this paper we study a class of games with a single population of strategically identical players. Each player has the same continuous action space A, which is a closed interval of real numbers or, at most, a finite union of such intervals. In economic applications, $x \in A$ might represent price, or output quantity, or location, or product quality. In biological applications, it might represent a continuous trait such as beak size or migration date. Our analysis will focus on two main cases: the entire set of real numbers $A = \mathbb{R} = (-\infty, \infty)$, and the unit interval $A = [0, 1]$. A prominent subcase, analyzed in section 3.7 below, is that $x \in A = [0, 1]$ represents a mixture $xs_1 + (1 - x)s_2$ of two pure strategies.

Time is also continuous, denoted by $t \in [0, \infty)$. At any particular time, the distribution of action choices within the population is represented by a cumulative distribution function $F(t,\cdot)$, where $F(t,x)$ denotes the fraction of the population choosing actions $y \le x \in A$. The distribution $F(t,\cdot)$ encapsulates the present state of the system.

2.1 Payoff functions and landscapes

The population games we study in this paper arise from symmetric two-player games g in normal form, where $g(x,y) \in \mathbb{R}$ is the payoff to a player choosing action $x \in A$ when the other player chooses action $y \in A$. Classical game theory routinely extends the payoff function to mixtures in y chosen by the other player; one simply averages g over the mixture distribution. Similarly, it is natural to write the payoff ϕ of the population game obtained from g as the average, or "expected value", of g given the current population distribution $F(t,\cdot)$,

$$\phi(x, F(t,\cdot)) = E_F g = \int_A g(x,y)dF(t,y). \tag{1}$$

Here the Stieltjes integral, $\int_A g(x,y)dF(t,y)$, can, of course, be rewritten as $\int_A g(x,y)f(t,y)dy$ if the density $f(t,x) = F_x(t,x)$ exists.

Population dynamics hinge on V, the adjustment velocity function, which is the gradient of the payoff function ϕ,

$$V(x, F(t,\cdot)) = \phi_x = \int_A g_x(x,y)dF(t,y), \tag{2}$$

at any interior point x in A. For example, suppose that, at the current distribution of competitors' prices, a firm finds that profit ϕ is locally an increasing function of its own price x, i.e., that $V > 0$. Then the firm will increase price, and do so more rapidly the steeper the profit gradient.

To ensure that V is well-defined, we impose the restriction that, for any fixed y, the function g is absolutely continuous and piecewise smooth in x. This guarantees that g_x is defined almost everywhere, and that its righthand limit $\lim_{h\searrow 0} g_x(x+h,y)$ exists at all interior points x in A and any left endpoints of intervals in A. To conform to the right continuity convention for cumulative distribution functions, we define $g_x(x,y)$ to be equal to this righthand limit on the set of measure zero where it is otherwise undefined. At right endpoints of intervals in A, the right hand limit does not exist, so we define g_x to be its lefthand limit.

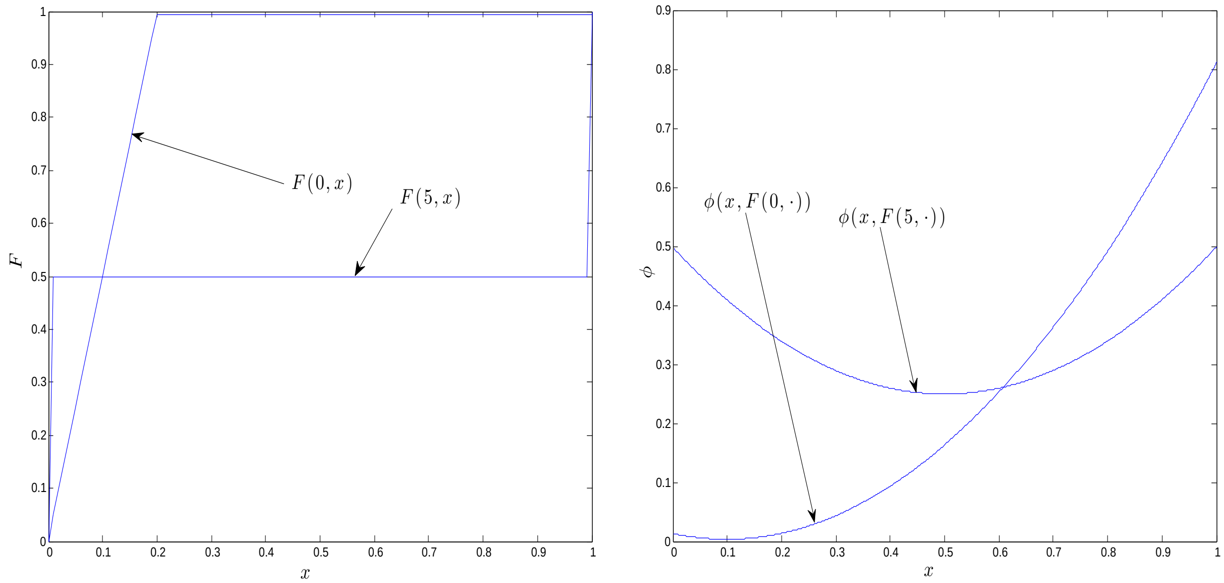

Figure 1: Panel A. The cumulative distributions $F(0, x)$ and $F(5, x)$ described in the text. Panel B. The corresponding landscapes, $\phi(x, F(0, \cdot))$ and $\phi(x, F(5, \cdot))$.

We also use (2) to define V at left and right endpoints of intervals in A, except when this definition would cause population mass to escape from A, in which case we redefine V to equal 0. Specifically, at each left endpoint of any interval in A, if $\int_A g_x(x, y)dF(t, y) < 0$ then we redefine V to equal 0 there, and, at each right endpoint, if $\int_A g_x(x, y)dF(t, y) > 0$ then we redefine V to equal 0 there.

Geometric intuition comes from thinking about the "landscape" at time t, which is the graph of ϕ at time t as a function of its first argument, x. Figure 1 shows simple examples arising from $A = [0, 1]$ and $g(x, y) = (x - y)^2$. Suppose that the distribution at $t = 0$ is uniform over the interval $[0,\ 0.2]$. Then the landscape is $\phi(x, F(0, \cdot)) = \int_0^{0.2}(x - y)^2 5dy = x^2 - 0.2x + 0.01333$. Any action choice above $x = 0.1$ will be improved by increasing it, and the gradient $V = \phi_x$ steepens at larger choices of x. Hence players in this region have the incentive to increase x.

Now suppose that, at some much later time $t = 5$, we have half the players bunched at the right endpoint $x = 1$ and the other half at the left endpoint $x = 0$. The landscape now is $\phi(x, F(5, \cdot)) = \int_0^1(x - y)^2 dF(5, y) = 0.5x^2 + 0.5(x - 1)^2$. Incentives are now symmetric to move away from the current mean $\mu_F = 0.5$, but at the current distribution, $F(5, \cdot)$, there is no room to do so. Hence that distribution should persist as a (stable) steady state.

Section 2.3 below will elaborate on this example, but for now we point out that the geometric intuition is quite general. When the current distribution $F(t,\cdot)$ is concentrated at a single point y_0 then the landscape is simply the relevant slice $\phi(x, F(t,\cdot)) = g(x, y_0)$ of the two-player payoff function. In general, the landscape is the weighted average of such slices, using $F(t,\cdot)$ as the weighting function over y. Varying $F(t,\cdot)$, even for a fixed g, clearly can generate a huge variety of landscapes.

The landscape's slope, i.e., the payoff gradient V, is key, because players have the incentive to adjust their actions in order to move uphill. Of course, such adjustments change the distribution F, which alters the landscape. These alterations in turn will provoke further adjustments by the players and further alterations of the landscape. This dynamic process may or may not converge to a steady state.

2.2 The gradient dynamics PDE

To formalize this geometric intuition, we now derive equations characterizing how the current state $F(t,\cdot)$ changes over time. Since F depends on x as well as t, we will end up with partial differential equations, or in light of equation (2), integro-PDE's.

The key behavioral assumption is that all players continuously adjust their actions so as to increase their own payoffs. More specifically, defining the payoff gradient V in (2) and then imposing the boundary constraints, we assume that each individual player systematically adjusts her choice $x \in A$ according to V evaluated at x. Theorem 1 of Friedman and Ostrov (2010) shows that such behavior is rational in the presence of quadratic adjustment costs.

Suppose for the moment that players' adjustments might also include a small random component, so that the adjustment of x is described by the stochastic differential equation

$$dx = V dt + \sigma dB.$$

Here dB is the change in the Brownian motion (which is also commonly written as dW or dw for Weiner process), scaled by the small constant volatility parameter σ.

The Fokker-Planck-Kolmogorov equation tells us that, for this evolution equation, the density $f(t,x)$ must be governed by the following fundamental PDE:

$$f_t = -(Vf)_x + \frac{1}{2}\sigma^2 f_{xx}. \tag{3}$$

This equation represents conservation of population mass and is an exact parallel of the conservation of momentum equation in fluid dynamics, where $\frac{1}{2}\sigma^2$ is the viscosity. See

Section 6.1 of the Appendix for a formal derivation and further discussion of this famous equation. Our focus here is the limit of equation (3) as the volatility σ goes to zero:

$$f_t = -(Vf)_x. \tag{4}$$

To resolve non-differentiability issues in section 4, we will need to recall that equation (4) represents the limit of equation (3) as the volatility vanishes.

Integrating (4) from $-\infty$ to x we obtain an alternative form of the PDE:

$$F_t = -VF_x. \tag{5}$$

To obtain intuition about this form, note that the left hand side of (5) evaluated at a point $x \in A$ is the rate of change in the fraction of the population that chooses actions $y \leq x$. The right hand side represents the flux, which is the adjustment speed times density of players moving their actions downward (hence the minus sign) from above x to below. Thus the equation says that the distribution changes via continuous adjustment by individual players—no population mass is gained or lost overall, and nobody jumps.

We will refer to either (4) and (5) as the *gradient dynamics PDE*.[1] Note the implicit assumption that the population is large. That all players face the same distribution $F(t, \cdot)$ presumes that each individual player has a negligible impact on overall distribution. Both the absence of a random component in (1), and taking the limit $\sigma \to 0$ in (3), also presume that the player is matched against a large representative sample of the population.

2.3 A simple family of games with diverse behavior

Consider the parametrized family of pairwise payoff functions $g(x, y) = b|x - y|^a$ over the action space $A = [0, 1]$, where the parameter $a > 0$. When the parameter $b = 1$ (or any other positive number), the function g and the corresponding population payoff function ϕ embody congestion in the sense that each player has an incentive to choose x as distant as possible from other players' choices.

$\mathbf{g(x, y) = |x - y|^2 = (x - y)^2}$: The consequences for $a > 1$ can be seen in the analytically tractable case $a = 2$ mentioned in Subsection 2.1 above. Let $\mu_F(t) = \int_0^1 y dF(t, y)$

[1] In physics and applied mathematics, these conservation of mass equations are referred to as *continuity equations*. When V has only local dependence on t and x, equations of the form (4) are called *conservation laws*, while equations of the form (5) are called the *Hamilton-Jacobi* equations.

and $\sigma_F^2(t) = \int_0^1 (x - \mu_F(t))^2 dF(t,y)$ denote the mean and variance of the distribution $F(t,\cdot)$. Then

$$\phi(x,F) = \int_0^1 (x-y)^2 dF(t,y) = x^2 - 2x\mu_F(t) + [(\sigma_F(t))^2 + (\mu_F(t))^2] = (x-\mu_F(t))^2 + (\sigma_F(t))^2. \tag{6}$$

Clearly payoff (6) is maximized when the population mass is as far away as possible from the mean, and when the variance is as large as possible. This occurs when half the population mass is concentrated on each endpoint, as in the second distribution graphed in panel A of Figure 1. Defining the Heaviside step function

$$\Theta_z(x) = \begin{cases} 0 & \text{if } x < z \\ 1 & \text{if } x \geq z, \end{cases}$$

the payoff maximizing state can be written $F^* = \frac{1}{2}\Theta_0 + \frac{1}{2}\Theta_1$.

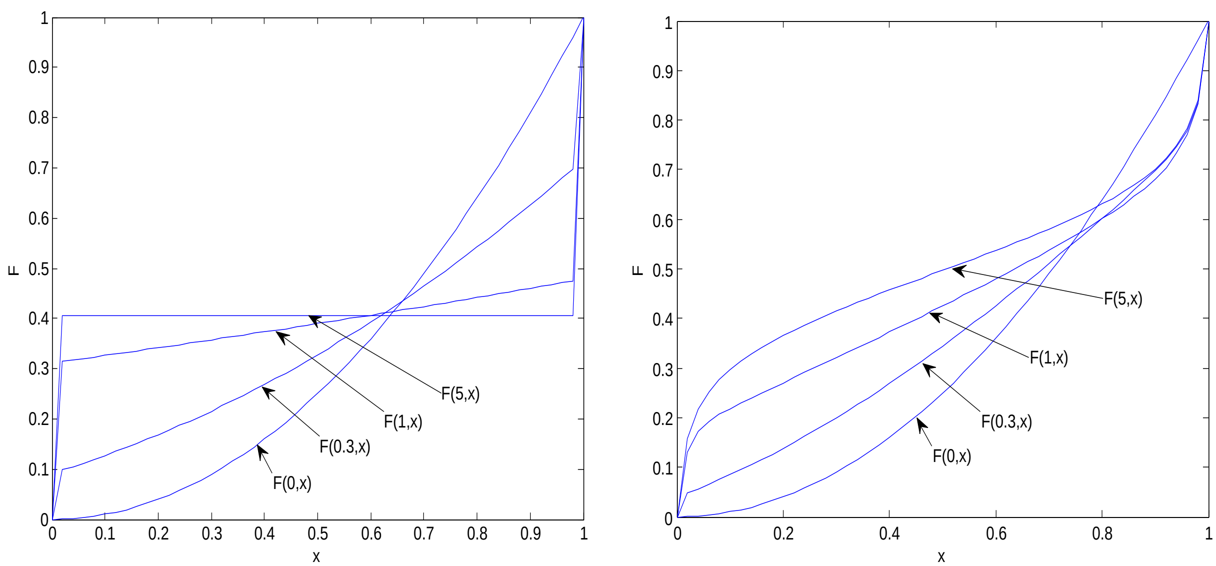

Figure 2: Evolving population distributions for gradient dynamics on $A = [0,1]$ with initial distribution $F(0,x) = x^2$, solved numerically with $\Delta x = 0.02$. Panel A uses $g(x,y) = (x-y)^2$. By time $t = 5$, the state closely approximates a degenerate steady state distribution of the form $F = p\Theta_0 + (1-p)\Theta_1$. Panel B uses $g(x,y) = |x-y|^{0.5}$. By time $t = 5$, the distribution closely approximates a non-degenerate steady state with higher density near the endpoints.

However, panel A of Figure 2 shows that other long run outcomes are possible. To analyze, first compute the gradient $V = \phi_x = 2(x - \mu_F(t))$ of (6), so the gradient dynamics

PDE (5) reads

$$F_t(t,x) = 2(\mu_F(t) - x)F_x(t,x). \tag{7}$$

For $x \in (0, \mu_F(t))$ we have $F_t(t,x) > 0$ at points where the density, $f = F_x$, is positive, i.e., the accumulated population in $[0,x]$ increases, meaning mass moves to smaller x values. Equation (7) similarly implies for each $x \in (\mu_F(t), 1)$ where the mass density is positive, the accumulated mass in $[x,1]$ increases, meaning mass moves to larger x values. Thus, as illustrated in panel A of Figure 2, all mass moves away from the current mean and towards the endpoints.

It follows that the distribution $F = p\Theta_0 + (1-p)\Theta_1$ is a steady state for any $0 < p < 1$. These states are locally stable, since any deviation that doesn't push mass past the mean $\mu_F(t) = p$ will shrink over time according to (7). Also, for any $z \in A$, the degenerate distribution Θ_z is a steady state, since $F_x(t,x) = 0$ for $x \neq z$ and $(\mu_F(t) - x) = 0$ for $x = z$, so in (7) we have $F_t(t,x) = 0$ everywhere. But these steady states are unstable, because small deviations to a non-degenerate distribution will, according to (7), increase over time.

$\mathbf{g(x,y) = |x-y|}$: The borderline case $a = 1$ is also analytically tractable. Here $\phi(x,F) = \int_0^x (x-y)dF(t,y) - \int_x^1 (x-y)dF(t,y) = \int_0^1 (x-y)dF(t,y) + 2\int_x^1 (y-x)dF(t,y) = x - \mu_F(t) + 2[(y-x)F(t,y)|_x^1 - \int_x^1 dF(t,y)] = 2 - x - \mu_F(t) - 2\int_x^1 dF(t,y)$, with gradient $2F(x) - 1$. The median of F is, by definition, the solution to $2F(x) - 1 = 0$, so the gradient dynamics PDE simply says that mass flows away from the median. Thus, starting from any initial distribution with finite density at the median, the distribution converges to the maximal payoff state $F^* = \frac{1}{2}\Theta_0 + \frac{1}{2}\Theta_1$.

$\mathbf{g(x,y) = |x-y|^{0.5}}$: Numerical methods that will be explained in Subsection 4.5 are necessary to approximate the solution for non-integer values of a. As illustrated in panel B of Figure 2, the steady state distributions are non-degenerate when $a < 1$. The intuition in this case is that the marginal benefit of increasing distance from other players' chosen actions is greater when those actions are nearby, in contrast to the case $a > 1$ where the more distant actions are more important.

$\mathbf{g(x,y) = -|x-y|}$: Now consider the case where $b = -1$, so $g(x,y) = -|x-y|^a$. Here, as with any negative b value, players want to be nearer to other players, not farther away, so we should expect that population mass will converge to an interior point in A. Yet behavior turns out to have some surprising aspects that depend on the value of the exponent a.

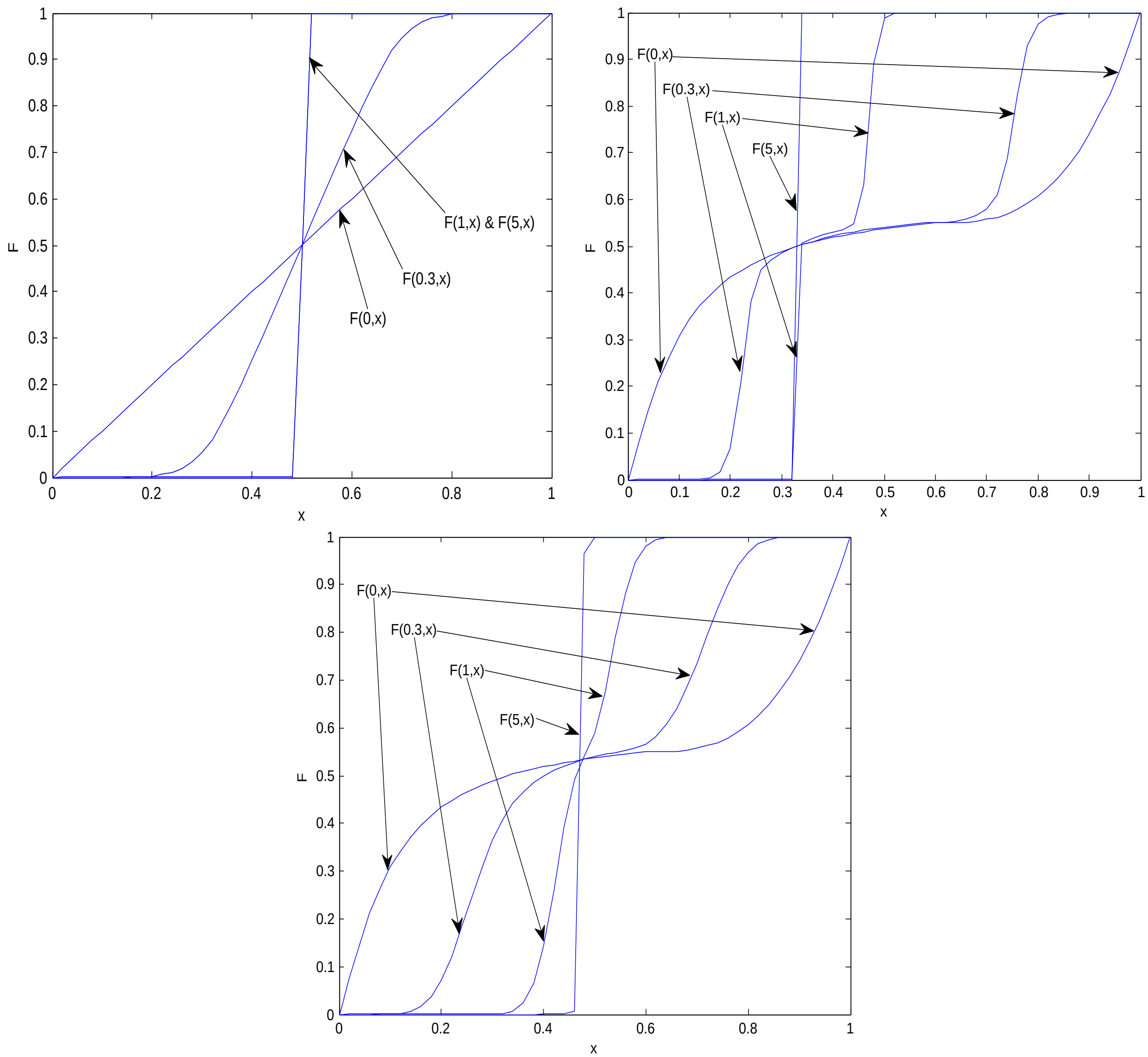

Figure 3: Evolving population distributions for gradient dynamics on $A = [0, 1]$ with mesh size of $\Delta x = .02$. Panel A uses $g(x, y) = -|x - y|$ and a uniform initial condition. The distribution reaches an exact steady state at $t = 0.5$, with all mass at $x = 0.5$. The discontinuity of F at $x = 0.5$ is called a *shock*. (The finite mesh size causes the appearance of a finite slope in F at the shock and curved segments in $F(0.3, x)$, neither of which actually exist.) Panel B uses the same g but a different initial condition, $F(0, x)$. By $t = 0.3$, two shocks have appeared, and they travel until they merge at the initial median, near $x = 0.32$, before $t = 5$. Panel C uses $g(x, y) = -(x - y)^2$. Mass converges to a single point, but only asymptotically as $t \to \infty$, so a shock is not formed. (The finite mesh size in simulations blurs the crucial distinction between shocks and asymptotic convergence to a mass point.)

For the case $a = 1$, we have $g_x(x, y) = 1$ if $x < y$ and otherwise $g_x(x, y) = -1$, so here (2) gives us

$$V = \int_{-\infty}^{x} (-1)dF(t, y) + \int_{x}^{\infty} (1)dF(t, y) = -F(t, x) + (1 - F(t, x)) = 1 - 2F(t, x),$$

and the gradient dynamics PDE (5) is

$$F_t + (1 - 2F)F_x = 0. \tag{8}$$

If the initial distribution is uniform on A, so $F(0, x) = x$ for all $x \in [0, 1]$, then (as can be verified by direct calculation of the partial derivatives) the solution to the given initial value problem for (8) when $t < \frac{1}{2}$ is

$$F(t, x) = \begin{cases} 0 & \text{if} \quad x < t \\ \frac{x-t}{1-2t} & \text{if } t \leq x \leq 1 - t \\ 1 & \text{if} \quad x > 1 - t. \end{cases} \tag{9}$$

In other words, the distribution remains uniform, but the width of its support shrinks linearly with time until, in the limit $t \nearrow \frac{1}{2}$, the distribution converges to the Heaviside step function $\Theta_{\frac{1}{2}}(x)$. That is, the entire population mass converges to action $x = \frac{1}{2}$ at $t = \frac{1}{2}$.

What then? The nature of the game suggests that all players continue to choose $x = \frac{1}{2}$ when $t \geq \frac{1}{2}$, and this is borne out by the numerical simulation in panel A of Figure 3, but there is a serious conceptual problem. When $t < \frac{1}{2}$, the solution is *classical*, i.e., the partial derivatives are defined and satisfy the PDE. However, when $t \geq \frac{1}{2}$, the distribution $F(t, x) = \Theta_{\frac{1}{2}}(x)$ is not continuous, much less differentiable, at $x = 0$. Thus there is no classical solution after $t = \frac{1}{2}$, and it is not yet clear in what sense equation (8) is satisfied.

This evolution of discontinuities in the solution is not due to a special initial distribution. In fact, as illustrated in panel B of Figure 3, a more typical smooth initial distribution leads to multiple discontinuities, and, further, these discontinuities travel over time. Nor is the problem due to a peculiar payoff function. Section 4 will show that discontinuities, called *shocks*, arise for a wide class of payoff functions. That section will show how to interpret the PDE and to find solutions for a wide class of functions where shocks arise and move over time.

$\mathbf{g(x, y) = -|x - y|^2 = -(x - y)^2}$: One might guess that setting $b = -1$ is responsible for the occurrence of shocks, but that is not the case. When $g(x, y) = -|x - y|^2$, we will always have smooth solutions at any given time. These smooth solutions only converge to a degenerate steady state as $t \to \infty$ as shown in panel C of Figure 3.

So the following questions are unavoidable when we explore gradient dynamics: How do we know when shocks can or cannot arise? How do we define the solution when they do arise? How do we find the solution analytically or simulate it numerically? We answer these questions in a number of contexts in the remainder of this paper.

3 Classical solutions and Linear PDEs

For what classes of two-player games, $g(x, y)$, can we guarantee that shocks, that is, discontinuities in the solution, do not occur? In this section we show that it suffices for g to be smooth. We then explore the properties of some important polynomial g functions within this class.

3.1 Smooth g implies no shocks

In general, shocks can easily occur when the function V in (4) depends on $f(t, x)$, even when the dependence is quite smooth. But the global dependence of V on the distribution and a two-player game g via (2) provides special structure that often prevents shock formation. The following theorem shows that when the action space $A = \mathbb{R}$, shocks are impossible if g is sufficiently smooth:

Theorem 1 *Let $A = \mathbb{R}$ and let $g : A \times A \to \mathbb{R}$ have bounded first, second and third partial derivatives in its first argument. Also, let the initial density $f_0(x)$ have a bounded first derivative. Then neither the solution, $f(t, x)$, nor the partial derivative of the solution, $f_x(t, x)$, to the integro-partial differential equation defined by (2) and (4) can become unbounded in finite time.*

The proof appears in Section 6.2 of Appendix A. It uses the bounds on g and its derivatives to obtain bounds on f and f_x. This implies shocks cannot exist. That is, f cannot become unbounded nor discontinuous, or, equivalently from the cumulative distribution perspective, neither F nor F_x can become discontinuous.

In particular, note that the two cases $g(x, y) = \pm(x - y)^2$ discussed above section 2.3 fit the requirements of this theorem. The only remaining obstacle to applying the theorem here is the fact that $A = \mathbb{R}$ in the theorem, but $A = [0, 1]$ in section 2.3. We will show

how to overcome this obstacle at the end of the next subsection by using the method of characteristics.

3.2 The method of characteristics

A standard technique for solving first order partial differential equations involves the equation's *characteristic curves* (also called *characteristics*). From each spatial location $x_0 \in A$ at $t = 0$, there is a characteristic curve $\xi(t)$ that evolves forward in time into the (t, x) plane. The solution can be determined along each of these characteristic curves, which can then be combined to form the full solution.

For our gradient dynamics PDE, let's consider the case where the velocity field does not depend explicitly on the distribution. Assume that $V = v(t, x)$ is a known function of t and x that is twice differentiable in x. The gradient dynamics PDE (4) then becomes $f_t = -(v(t, x)f)_x$, which, upon applying the product rule to the righthand side and rearranging slightly, becomes

$$f_t + v f_x = -v_x f. \tag{10}$$

The evolution of the characteristic path and the solution along the characteristic path are given by the following system of ODEs along with their corresponding initial conditions:

$$\begin{aligned} \frac{d\xi(t)}{dt} &= v\,(t, \xi(t)) & \xi(0) &= x_0 & (11) \\ \frac{df\,(t, \xi(t))}{dt} &= -v_x\,(t, \xi(t))\, f\,(t, \xi(t)) & f\,(0, \xi(0)) &= f_0(x_0). & (12) \end{aligned}$$

The reason for this ODE system is made clear by (10): by choosing $\xi(t)$ so that it obeys $\frac{d\xi(t)}{dt} = v$ as given in (11), we see that the chain rule and (10) dictate the equation for $\frac{df(t,\xi(t))}{dt}$ given in (12). Note that the solution, f, at any specific point $(\hat{t}, \hat{x})$ depends upon the initial condition at only one point: the point at $t = 0$ to which $(\hat{t}, \hat{x})$ is connected by a characteristic path. Put another way, information about the evolution of the solution of the gradient dynamics PDE travels out from the initial condition solely along the characteristics.

The term $v(t, x)f$ has a linear dependence on the unknown function f, which leads to the gradient dynamics PDE (10) being called a *linear* PDE. The ODEs, (11) and (12), are decoupled, so we first solve the ODE for $\xi(t)$ in (11) and use it to solve the ODE for $f(t, \xi(t))$ in (12), thereby obtaining the population density f along each characteristic:

$$f\,(t, \xi(t)) = f_0(x_0) e^{-\int_0^t v_x(s,\xi(s))ds}. \tag{13}$$

In the case $A = \mathbb{R}$, equation (13) fully specifies the solution to the gradient dynamics PDE with given initial condition f_0.

To illustrate, consider a payoff of the form $\phi = 1 - \frac{1}{2}(x - t + 1)^2$, a game against nature in which a player's payoff is higher the closer her action is to $x = t - 1$. Thus at time $t = 0$, the landscape is a parabolic hill with summit at $x = -1$, and over time the summit action shifts upward, e.g. to $x = 0$ at $t = 1$ and to $x = 1$ at $t = 2$. The gradient is $\phi_x = V = v(t, x) = t - x - 1$.

From (11) we have that $\frac{d\xi(t)}{dt} = t - \xi - 1$ and $\xi(0) = x_0$, so if $A = \mathbb{R}$, then

$$\xi(t) = t - 2 + (x_0 + 2)e^{-t}. \tag{14}$$

Since $v_x = -1$, equation (13) gives the solution along a characteristic as $f(t, \xi(t)) = f_0(x_0)e^t$, or, solving (14) for x_0 in terms of $x = \xi(t)$, the explicit solution to the gradient dynamics PDE is $f(t, x) = e^t f_0[(x - t + 2)e^t - 2]$.

Complications arise when the action space is $A = [0, 1]$. In the current example, the slope $v(t, 0)$ of the characteristics at $x = 0$, is negative when $t < 1$ but positive when $t > 1$. Therefore, before $t = 1$, the characteristics intersect the $x = 0$ boundary and population mass accumulates at $x = 0$. After $t = 1$, the characteristics switch direction and now push that mass back into the $x \in (0, 1)$ region. The problem is that the density f by then is infinite at $x = 0$, so the gradient dynamics PDE for f in (4) breaks down.

To deal with the problem, we use the alternative form of the gradient dynamics PDE given in (5),

$$F_t = -vF_x, \tag{15}$$

while imposing the natural boundary conditions[2] $F(t, 0) = 0$ and $F(t, 1) = 1$. This yields the following ODEs for the characteristics:

$$\begin{aligned} \frac{d\xi(t)}{dt} &= v\left(t, \xi(t)\right) & \xi(0) &= x_0 & (16)\\ \frac{dF\left(t, \xi(t)\right)}{dt} &= 0 & F\left(0, \xi(0)\right) &= F_0(x_0), & (17) \end{aligned}$$

[2] Although we generally follow the standard convention that $F(t, x)$ is continuous from the right in x and has limits from the left in x, in the interests of simplicity when $A = [0, 1]$, we slightly abuse this by writing that $F(t, 0) = 0$. Of course, when a finite mass is located at $x = 0$, this is not technically true. However, using $F(t, 0) = 0$ will generate the correct solution to the PDEs in our paper everywhere except at $x = 0$, and then the solution can be made correct at $x = 0$ by just replacing it with its limit from the right in x.

where, of course, $F_0(x) = \int_{-\infty}^{x} f_0(y)dy$. Note from (17) that F stays constant along each characteristic, and therefore the full set of characteristics, as shown in Figure 4, yields the solution for any given initial condition, $F(0,x)$. The interpretation from (17), but not clear from (11), is that the characteristics follow individual players' actions and players don't change their relative positions. For example, by following the characteristic $\xi(t)$ emanating from the x_0 where $F_0(x_0) = 0.75$, we follow the actions of the player at the 75th percentile. When V is a form that is not $v(t,x)$, as will be the case in Section 4, we do not always have that F is constant along characteristics, and so the interpretation of characteristics as action paths of individual players breaks down in the general case.

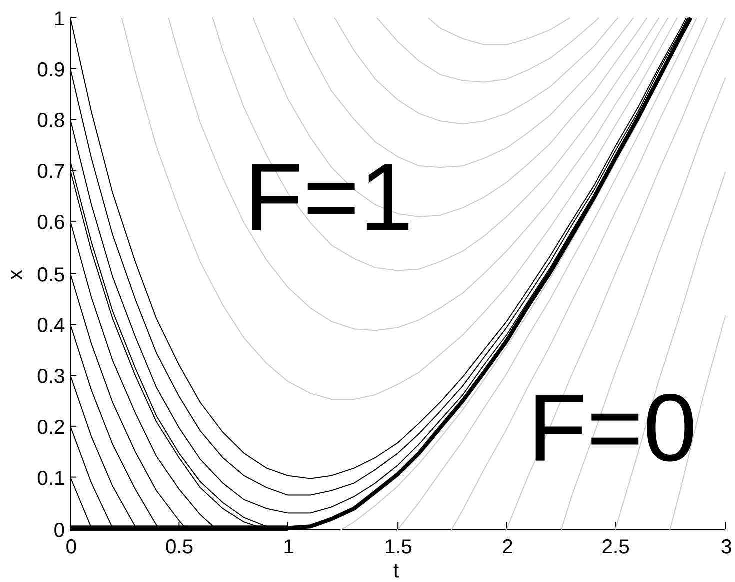

Figure 4: Characteristics for gradient dynamics on $A = [0,1]$ for the payoff function $\phi(t,x,f(\cdot)) = 1 - \frac{1}{2}(x - t + 1)^2$. On each gray characteristic curve at the bottom, we have $F = 0$. On each gray characteristic curve at the top, we have $F = 1$. On the black characteristic curves, the value of F is constant (and dictated by the initial condition). The bold curve accumulates mass until $t = 1$ as the black characteristics collide into it. After $t = 1$, the bold curve is a discontinuity in F, below which $F = 0$.

Now we return to our two cases from Section 2.3. When $g(x,y) = (x-y)^2$, we have that $V = 2(x-\mu_F(t))$, and, of course, when $g(x,y) = -(x-y)^2$, we have that $V = -2(x-\mu_F(t))$. Since $\mu_F(t) \in A = [0,1]$, when $g(x,y) = (x-y)^2$ we cannot have $V > 0$ at $x = 0$, nor $V < 0$ at $x = 1$. That is, information from the boundaries, $x = 0$ and $x = 1$, cannot have any effect on the solution on $(0,1)$, the interior of A, at any time. The choice of $A = [0,1]$, as opposed to $A = \mathbb{R}$, is irrelevant to the solution on $(0,1)$, the interior of A, since it depends solely on the initial condition specified in $(0,1)$. Therefore, we can apply Theorem 1 and conclude that the solution cannot have shocks in the interior of A. Note that this conclusion only applies to the interior; mass can, and does, build on the boundaries of $A=[0,1]$, as we have already seen in Section 2.3.

The reasoning for $g(x,y)=-(x-y)^2$ is slightly different. Here we cannot have $V<0$ at $x=0$, nor $V>0$ at $x=1$, so information from the boundaries (that $F(t,0)=0$ and $F(t,1)=1$) definitely could affect the solution on the interior of A. But if we extend our domain to $A=\mathbb{R}$ and extend the initial condition so that $F(0,x)=0$ when $x<0$ and $F(0,x)=1$ when $x>1$, then the solution for $x\in(0,1)$ is unaffected. This is because $V\geq 0$ when $x\leq 0$ and F doesn't change along characteristics, so $F(t,0)=0$ is guaranteed. Similar logic guarantees that $F(t,1)=1$. Given this equivalent formulation of the problem, Theorem 1 guarantees that our solution has no shocks, and so we knew in Section 2.3 that the convergence of mass to a point could not occur in finite time, as that would imply the presence of a shock.

3.3 Quadratic g and their corresponding V functions

The method of characteristics allows us to solve the gradient dynamics PDE explicitly for quadratic two-player games, i.e., when $g(x,y)=ax+\frac{b}{2}x^2+cxy+h(y)$. In this case $V(x,F(t,\cdot))=\int_{-\infty}^{\infty}g_x(x,y)dF(t,y)=a+bx+c\mu_F(t)$, and we have the following key result:

Theorem 2 *Let*

$$g(x,y)=ax+\frac{b}{2}x^2+cxy+h(y) \tag{18}$$

or let the velocity field take the form

$$V(x,F(t,\cdot))=a+bx+c\mu_F(t), \tag{19}$$

where $x\in A=\mathbb{R}$, $t\geq 0$, and $a,b,$ and c are real constants. Then, given an initial probability density, $f_0(x)$, the gradient dynamics PDE (4) has the following unique solution:

$$f(t,x)=e^{-bt}f_0\left(\mu_F(0)+\left(x+\frac{a}{b+c}\right)e^{-bt}-\left(\mu_F(0)+\frac{a}{b+c}\right)e^{ct}\right) \tag{20}$$

and the mean is

$$\mu_F(t)=\left(\mu_F(0)+\frac{a}{b+c}\right)e^{(b+c)t}-\frac{a}{b+c}. \tag{21}$$

This solution is a classical solution if f_0 is differentiable.

The proof is spelled out in Section 6.3 of Appendix A.[3] One integrates the gradient dynamics PDE to obtain an ODE for $\mu_F(t)$. Inserting its solution into the expression for V yields an explicit function, $v(t, x)$. The result then follows from equations (11) and (13).

3.4 Keynes' Beauty Contest

As a quick illustration of the economic implications of Theorem 2, recall that the winner in the famous beauty contest of Keynes (1936, Ch 12) is the player who guesses most closely the average guess of all players. We consider a popular generalization in which the objective of every player is to guess a multiple $c > 0$ of the average guess $\mu_F(t)$. We assume a quadratic penalty for guessing incorrectly, and obtain the payoff function $\phi(x, F) = k - \frac{1}{2}(c\mu_F(t) - x)^2$, where k is a constant that has no effect on our analysis.

The game is tricky because most people think of a fixed distribution of play and choose c times the mean. For example, if $c = 2/3$ and players contemplate uniform choices over $A = [0, 1]$, then the mean would be .50 and their choices would cluster around .33. But in this case, a more sophisticated or experienced player would choose .22. If a player believes that others are sophisticated enough to choose .22, then she should choose about .14, etc.

We now apply gradient dynamics to the given payoff function. The gradient is

$$V = c\mu_F(t) - x,$$

a special case of (19) with $a = 0$ and $b = -1$. Thus, if $A = \mathbb{R}$, (20) yields

$$f(t, x) = e^t f_0 \left(\mu_F(0) + xe^t - \mu_F(0)e^{ct}\right). \tag{22}$$

Inspection of (22) reveals that the support of f shrinks as t increases, so we have clumping as $t \to \infty$ and all mass converges exponentially to $\mu_F(0)e^{(c-1)t}$. If $c = 1$, then convergence is to $\mu_F(0)$. If $c < 1$, the guesses converge to $x = 0$. If $c > 1$, the guesses diverge to ∞ if $\mu_F(0) > 0$ (or to $-\infty$ if $\mu_F(0) < 0$). Such behavior is broadly consistent with laboratory results since Nagel (1995), who ran experiments with 15–18 paid human subjects, and a payoff function similar to ϕ with $c = 2/3$. See Figure 5 for an example of the evolution of this game's solution according to gradient dynamics.

[3] Section 6.4 sketches how to generalize Theorem 2 to any $g(x, y)$ that is the sum of terms of the form $c_{nm}x^n y^m$, where c_{nm} are constants, $n = 1$ or 2, and m is any nonnegative integer.

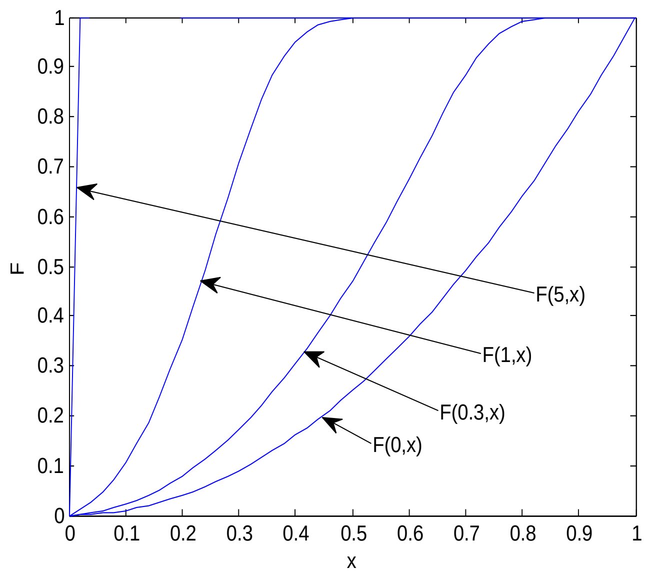

Figure 5: Keynes beauty contest with $c = 2/3$. By time $t = 5$, the distribution closely approximates the unique steady state θ_0, i.e., all mass at $x = 0$.

3.5 Cournot Duopoly

As another application, consider the first mathematical model in economics, the Cournot (1838) duopoly. There are two firms that simultaneously choose output quantities x and y and face a linear demand function with slope scaled to -1 and intercept scaled to 1. The two firms have zero fixed costs, and identical constant marginal cost $m \in [0, 1)$. Then price is $1 - x - y$, and the restriction $x, y \in A = [0, 1]$ is natural. Unit profit is price minus marginal cost and the payoff function is quantity times unit profit, so $g(x, y) = x(1 - x - y - m)$. Equation (18) holds with $a = 1 - m$, $b = -2$ and $c = -1$, and, were $A = \mathbb{R}$, equation (20) from Theorem 2 would give

$$f(t, x) = e^{2t} f_0 \left(0.5 + \left(x - \frac{1-m}{3}\right) e^{2t} - \left(0.5 - \frac{1-m}{3}\right) e^{-t}\right). \tag{23}$$

But now consider a uniform initial distribution on $A = [0, 1]$ with $0 < m < \frac{1}{2}$. Since $V = (1 - m) - 2x - \mu_F(t)$ and, from (21), we have that $\mu_F(t)$ monotonically decreases from $\frac{1}{2}$ to $\frac{1-m}{3}$, we know that at all times $V > 0$ at $x = 0$ and $V < 0$ at $x = 1$. Therefore, by the argument in the last paragraph of Subsection 3.2, we can still apply (23), even though $A = [0, 1]$. In particular, as time t increases, the distribution remains uniform but on a shrinking subinterval containing $x = \frac{1-m}{3}$. Indeed, for any reasonable initial condition, the distribution converges asymptotically to $\theta_{\frac{1-m}{3}}$, the degenerate distribution with all mass concentrated at the Nash-Cournot equilibrium output quantity $x = \frac{1-m}{3}$. In some cases part of the population mass will build up at $x = 0$ before then moving towards $x = \frac{1-m}{3}$.

We see three possible interpretations (none of them entirely convincing) for the population game dynamics described by (23). Although g describes the rivalry between our focus firm and a single actual rival, the focus firm might face a large number of potential rivals whose output choices have distribution $F(t, \cdot)$. As firms marginally adjust output they see on average the local profit gradient ϕ_x, and therefore (23) describes the adjustment dynamics with the population of potential duopolists. A second interpretation is that the firm produces a wide variety of similar but distinct products and faces a different single rival for each product. It adjusts the output quantity for each product in response to experience gained on all products. Then $F(t, \cdot)$ describes the current cross-sectional distribution of rivals' choices, which adjusts via (23). A third interpretation is that $\phi(x, F) = E_y[g(x, y)|F(t, \cdot)]$ represents the subjective expected profit of a firm contemplating (but not yet committed to) output x, where F summarizes management's anticipations of their rival's future actions. Gradient dynamics represent an internal process of modifying beliefs as the firm contemplates the potential profit consequences to itself and its rivals, and only when the process converges to an invariant distribution does the firm actually commit to produce output.

3.6 An example from growth theory

At least since Romer (1986), economists have modeled economic growth via complementarities and increasing returns technologies. Perhaps the simplest version is captured in the two person game $g(x, y) = xy - c(x)$, where the personal benefit xy is increasing in the other player's choice $y \in A$ as well as one's own choice $x \in A$, while the personal cost $c(\cdot)$ is assumed to be increasing and convex on $A = [0, M]$, where $M > 0$ represents the maximum feasible contribution to the public good.

It is natural to think of economic growth as a population game. Inserting g as above into equations (1) and (2), we obtain the payoff function $\phi(x, F) = x\mu_F(t) - c(x)$ and its gradient

$$V = \phi_x = \mu_F(t) - c'(x). \tag{24}$$

Suppose first that marginal cost is constant, so $c'(x) = a_1 > 0$. If the initial mean $\mu_F(0) < a_1$, then thereafter $V \leq \mu_F(0) - a_1 < 0$ everywhere in A except at the lower endpoint $x = 0$, so the distribution converges in finite time to θ_0, where nobody contributes anything to the public good. On the other hand, if $\mu_F(0) > a_1$, then $V > 0$ everywhere on A except at the upper endpoint M, so in finite time the distribution converges to θ_M, where everyone contributes maximally. Thus constant marginal cost leads to extreme hysteresis, with the outcome depending entirely on whether the initial mean contribution exceeds or falls short of the threshold a_1.

Behavior is quite different in the case of very convex cost, with $c'(0) = 0$ and $c'(x) \to \infty$ as $x \to M$. Here the Intermediate Value Theorem and the convexity of c ensures that equation (24) has a unique interior root $z(t) \in A$. Clearly $V > 0$ in (24) iff $x < z(t)$, and $V < 0$ iff $x > z(t)$, so population mass moves towards $z(t)$. We should expect to see asymptotic convergence to a degenerate steady state distribution $\theta_{\hat{z}}$, where $\hat{z}$ is the unique root of $x - c'(x)$.

Such convergence can be shown explicitly in the case of quadratic cost, $c(x) = a_0 + a_1 x + \frac{1}{2}a_2 x^2$, when $A = \mathbb{R}$. Then (24) satisfies (19) with $a = -a_1, b = -a_2$ and $c = 1$, so (20) yields

$$f(t,x) = e^{a_2 t} f_0 \left(\mu_F(0) + \left(x - \frac{a_1}{1-a_2} \right) e^{a_2 t} - \left(\mu_F(0) - \frac{a_1}{1-a_2} \right) e^t \right), \tag{25}$$

and (21) yields

$$\mu_F(t) = \left(\mu_F(0) - \frac{a_1}{1-a_2} \right) e^{(1-a_2)t} + \frac{a_1}{1-a_2}. \tag{26}$$

Because $z(t) = \frac{\mu_F(t) - a_1}{a_2}$, equation (26) implies

$$z(t) = \frac{1}{a_2} \left(\mu_F(0) - \frac{a_1}{1-a_2} \right) e^{(1-a_2)t} + \frac{a_1}{1-a_2}.$$

Thus if $a_2 > 1$, we get enough convexity for the mass to converge exponentially to the mean, which exponentially converges to the finite value $\hat{z} = \frac{a_1}{1-a_2}$. On the other hand, if $a_2 < 1$ then (25-26) say that population mass diverges exponentially either to $+\infty$ or to $-\infty$, depending on the sign of $\left(\mu_F(0) - \frac{a_1}{1-a_2} \right)$. For $A = [0, M]$, the usual reasoning tells us that convergence is to the upper endpoint $x = M$ or the lower endpoint $x = 0$ when $a_2 < 1$, and also when $a_2 > 1$ but $\hat{z} \notin A$.

3.7 The bilinear case

Consider now the special case that $x \in A = [0, 1]$ represents the mixture $xs_1 + (1-x)s_2$ of two pure strategies, s_1 and s_2. Let m_{ij} be the payoff to a player using s_i when matched with another player using s_j for $i, j = 1, 2$. We will say that g is the *mixed extension* of the symmetric two-player game $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ if it is the payoff given the player's mix x and the opponent's independent mix y, i.e., if

$$g(x,y) = \mathbf{x}^T M \mathbf{y} = xym_{11} + (1-x)ym_{21} + x(1-y)m_{12} + (1-x)(1-y)m_{22}, \tag{27}$$

where $\mathbf{x}^T = [x, 1-x]$ and $\mathbf{y}^T = [y, 1-y]$. Defining the composite parameters $m_1 = m_{11} - m_{21}$ and $m_2 = m_{22} - m_{12}$, we have

$$g_x(x,y) = (1,-1)M\mathbf{y} = ym_{11} - ym_{21} + (1-y)m_{12} - (1-y)m_{22} = (m_1+m_2)y - m_2, \tag{28}$$

and so the corresponding population game has gradient

$$V(x, F(t,\cdot)) = \phi_x = \int_0^1 g_x(x,y)dF(t,y) = (m_1+m_2)\mu_F(t) - m_2. \tag{29}$$

Thus, in the interior of A, the slope V of each characteristic curve is independent of x and depends only on the mean, $\mu_F(t)$. That is, these bilinear payoff functions g produce landscapes with constant slope, and the state evolves via *censored translation* — that is, the distribution slides rigidly left or right with velocity V until it hits an endpoint, where mass piles up. By (29), that velocity is zero for any distribution $F(t,\cdot)$ such that $\mu_F(t) = \mu^*$, where $\mu^* = \frac{m_2}{m_1+m_2}$, so such distributions are steady states. Moreover, inspection of (29) reveals that $\mu(t)$ will move towards μ^* if $(m_1+m_2) < 0$, but will move away from $\mu(t)$ if $(m_1+m_2) > 0$. Of course, by its definition, $\mu^* \in (0,1)$ iff m_1 and m_2 have the same sign.

Once again (29) is a case of (19), with $a = -m_2, b = 0$ and $c = m_1+m_2$. Therefore, ignoring censoring, equations (20-21) yield the exact solution

$$f(t,x) = f_0\left(x + (\mu_F(0) - \mu^*)\left(1 - e^{(m_1+m_2)t}\right)\right), \tag{30}$$

with mean

$$\mu_F(t) = (\mu_F(0) - \mu^*)\, e^{(m_1+m_2)t} + \mu^*. \tag{31}$$

This conforms to our intuitions. If $(m_1+m_2) < 0$, the solution just translates monotonically so that $\mu_F(t)$ converges exponentially to μ^*. If $(m_1+m_2) > 0$, the solution translates off to ∞ if $\mu_F(0) > \mu^*$, or off to $-\infty$ if $\mu_F(0) < \mu^*$.

Taking censoring into account, i.e., mass building up at an endpoint, does not drastically alter the conclusions. Since $\mu_F(t)$ never crosses μ^*, equation (29) tells us that either $V \leq 0$ for all t and x or $V \geq 0$ for all t and x. It follows that mass can only build up at one of the two endpoints and any mass that builds up at an endpoint will always stay at that endpoint. See Figure 6, for example.

Now we can characterize behavior quite precisely. When $\mu^* \in (0,1)$ and $(m_1+m_2) < 0$, the distribution converges exponentially via censored translation to a distribution with mean μ^*. When $(m_1+m_2) > 0$, all mass reaches the right endpoint $x = 1$ in finite time if $\mu_F(0) > \mu^*$ (and a fortiori if μ^* is negative), and all mass reaches the left endpoint $x = 0$ in finite time if $\mu_F(0) < \mu^*$ (and a fortiori if $\mu^* > 1$).

Thus we have proved the following theorem:

Theorem 3 *Let $\phi(x, F(t,\cdot)) = \int_0^1 g(x,y)dF(t,y)$ where g is the mixed extension of $M = ((m_{ij}))_{i,j=1,2}$, as defined in (27), and let F_0 be an arbitrary initial state. Then the solution $F(t,\cdot)$ to the gradient dynamics PDE (5) converges monotonically to an asymptotic distribution $F^* = \lim_{t\to\infty} F(t,\cdot)$ that depends on the composite parameters as follows:*

a. If $m_1, m_2 < 0$ then F^* is the censored translation of F_0 with mean $\mu^* := \frac{m_2}{m_1+m_2}$, and the convergence of $F(t,\cdot)$ to F_0 is exponential.

b. If $m_1, m_2 > 0$ then $F^* = \Theta_1$ (or $F^* = \Theta_0$) whenever $\mu_{F_0} >$ (or $\mu_{F_0} <$) $\mu^* = \frac{m_2}{m_1+m_2}$, and in either case convergence is complete in finite time.

c. If $m_1 > 0 > m_2$ (or $m_1 < 0 < m_2$) then $F(t,\cdot) \to F^* = \Theta_1$ (or $F(t,\cdot) \to F^* = \Theta_0$) in finite time.

Case (a) of the Theorem occurs in 2×2 games like Hawk-Dove or Chicken. As illustrated in Figure 6, the initial distribution shifts up (if $V > 0$ because the initial mean is below μ^*) or shifts down (if $V < 0$ because the initial mean is above μ^*) until $\mu_F(t)$ hits μ^*. The shift slows as the current mean approaches μ^* and V approaches zero.[4]

[4]This suggests a resolution to a conundrum in evolutionary game theory. The standard prediction is that mean play converges to μ^*, but one school of thought (called monomorphic) holds that all players use the same mixed strategy μ^*, while another school (called polymorphic) holds that a fraction μ^* of players use the first pure strategy and the other players all use the second pure strategy, and nobody mixes. Our approach predicts that the initial dispersion of mixes will persist, but the mean will shift over time in the appropriate direction. The laboratory data seem roughly consistent with our prediction.

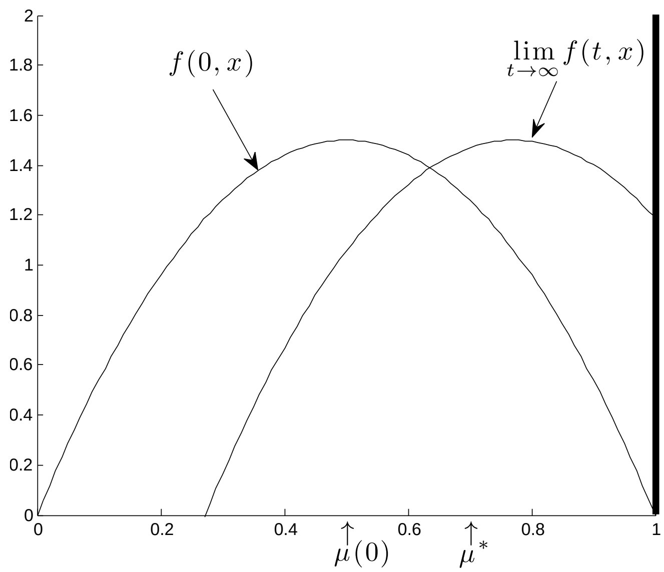

Figure 6: Initial density and asymptotic distribution for bilinear example with $m_1 = -3, m_2 = -7$. Note that some of the mass accumulates at $x = 1$ in the asymptotic distribution.

Case (b) occurs in 2×2 games like Coordination or Stag Hunt. Here any discrepancy between μ^* and the initial mean increases over time, and V increases as well. Hence we get convergence in finite time to the pure strategy equilibrium that initially had a higher expected payoff.

Case (c) occurs in 2×2 games with dominant strategies, e.g., Prisoner's dilemma. Here the expression for μ^* lies outside $A = [0, 1]$, and all population mass piles up at the endpoint corresponding to the dominant strategy.

4 Solutions with Shocks and Nonlinear PDEs

Behavior is especially interesting with two-player payoffs $g(x, y)$ that yield nonlinear gradient dynamics PDEs, as their solutions are generally not smooth. (Note from Theorem 1 that since the solution is not smooth, the payoff function $g(x, y)$ cannot be smooth.) To describe the resulting population behavior from these games, we first derive equations for the characteristic curves — now generalized for a broad class of nonlinear PDE — and illustrate their implications using the Bertrand pricing model. Then, within a more specific context, we show what happens when characteristics collide and form shocks. The rest of the section applies this knowledge to a model of financial market dynamics.

4.1 Characteristics for Nonlinear PDE

We consider two-player games $g(x, y)$ that yield nonlinear gradient dynamics PDEs of the form

$$F_t(t, x) + G(t, x, F(t, x), F_x(t, x)) = 0. \tag{32}$$

Note from (5) that $G = VF_x$ where V, in addition to depending on t and x as before, can now also depend on $F(t, x)$ and $F_x(t, x)$. Specification is completed by an initial distribution $F(0, x) = F_0(x)$ for all $x \in A$, together with the value of $F(t, x)$ at any boundary points x of A for all $t \in [0, \infty)$. To focus on the main new issues, we will assume for the moment that there are no boundary points, so $A = \mathbb{R}$.

To solve such an initial value problem, we must extend the method of characteristics from the linear case, given in (16) and (17), to our new nonlinear context. As before, we let $x = \xi(t)$ denote a characteristic path in the (t, x) plane emanating from some initial point $(0, x_0)$. Now let $u(t) = F(t, \xi(t))$ and $p(t) = F_x(t, \xi(t))$ denote the values of F and F_x along this characteristic. Using this notation, the PDE (32) tells us that $F_t(t, \xi(t)) + G(t, \xi(t), u(t), p(t)) = 0$ on this characteristic.

The system of ODEs defining any characteristic curve for (32) and the behavior along it is

$$\begin{aligned}
\frac{d\xi}{dt} &= G_{(F_x)}(t, \xi, u, p) && \xi(0) = x_0 && (33)\\
\frac{du}{dt} &= pG_{(F_x)}(t, \xi, u, p) - G(t, \xi, u, p) && u(0) = F_0(x_0) && (34)\\
\frac{dp}{dt} &= -G_x(t, \xi, u, p) - pG_F(t, \xi, u, p) && p(0) = [F_0]_x(x_0). && (35)
\end{aligned}$$

To see this, differentiate the definition of $u(t)$ using the chain rule and then substitute using (32) and (33) to obtain (34). To obtain (35), first differentiate the definition of $p(t)$ using the chain rule, to get $\frac{dp}{dt} = F_{xt} + F_{xx}\frac{d\xi}{dt} = [F_t]_x + F_{xx}\frac{d\xi}{dt}$. Next, substitute $F_t = -G$ from (32), and apply the chain rule to the x differentiation of G to get $\frac{dp}{dt} = -G_x - G_F F_x - G_{(F_x)}F_{xx} + F_{xx}\frac{d\xi}{dt}$. Now (33) implies (35). This last step explains why (33) is the correct ODE for the characteristic path $\xi(t)$ — it allows us to cancel the F_{xx} terms, thus making (33)–(35) a closed system of ODEs in t, ξ, u, and p.

Finally, we verify that (33)–(35) generalizes the characteristic equations (16) and (17) from the linear PDE case. We have $G = v(t,x)F_x$ in the linear case, so equation (33) becomes $\frac{d\xi}{dt} = v(t,x)$, which is equation (16), while equation (34) becomes $\frac{du}{dt} = pv - vp = 0$, which is equation (17). Equation (35) for $\frac{dp}{dt}$ is unnecessary in the linear case, as (16) and (17) form a closed system in t, ξ, and u, which is solvable without knowledge of p.

The method of characteristics in the nonlinear case (32) can be used to generate the solution forward in time as long as F is twice continuously differentiable. In the linear case when $A = \mathbb{R}$, we can guarantee that degree of smoothness by making $v(t,x)$ and the initial condition sufficiently smooth. By contrast, in the nonlinear case, regardless of how smooth G in (32) and the initial condition F_0 are, F or F_x can blow up in finite time or characteristic curves can colide, causing F or F_x to become discontinuous. We give an example in the next subsection.

4.2 Preemption and Bertrand pricing

Consider the following foraging scenario, based on Rogers (1995). A resource (say, a berry patch) has gross value 1 when fully ripe and a lesser value x at earlier times, scaled so that $x \in [0,1]$ represents time as well as resource value. By incurring a specified cost $c \in [0,1)$, a player (or forager) can visit and attempt to harvest the resource at any value (or time) x of her choosing. In the two-player version of the game, a second player has exactly the same opportunity, but only the first to visit the resource gains the gross value x; the latecomer receives nothing.

Thus we have the pairwise payoff function $g(x,y) = x - c$ if $x < y$ and $= 0$ if $x \geq y$; for completeness say that $g(x,y) = (x-c)/2$ if $x = y$. The corresponding population game has payoff $\phi(x, F(t,\cdot)) = \int_0^1 g(x,y)dF(t,y) = (x-c)\int_x^1 dF(t,y)$, with gradient

$$V = \phi_x = 1 - F(t,x) - (x-c)F_x(t,x), \tag{36}$$

and so the gradient dynamics PDE (5) becomes

$$F_t + F_x - FF_x - (x-c)(F_x)^2 = 0. \tag{37}$$

These dynamics can be thought of as taking place in evolutionary time, in which the forager species adapts its behavior in response to accrued fitness. Thus, if x is thought of as time rather than ripeness, it runs on a time scale incomparably faster than adaptation time t.[5]

[5] Another interpretation is Bertrand duopoly with inelastic demand for a single unit. Here x and y are

Equation (37) is of the form (32) with $G(t, \xi, u, p) = p - up - (\xi - c)p^2$, a nice, smooth polynomial. Characteristic equation (35) now is:

$$\frac{dp}{dt} = 2(p(t))^2 \qquad p(0) = [F_0]_x(x_0). \tag{38}$$

Assume that the initial distribution is smooth, take any starting point $x_0 \in A$, and let $p_0 = p(0) = [F_0]_x(x_0)$ be the density at this point. The solution to (38) is $p(t) = \frac{p_0}{1-2tp_0}$, which blows up at $t = \frac{1}{2p_0}$. That is, despite the smooth initial condition and G function, there is a singularity in the solution at $t = \frac{1}{2p_0}$, unless the chosen characteristic has already collided with another characteristic. If it has collided with another characteristic, the point in the (t, x) plane where the collision occurs will be part of a *shock curve.* A shock curve, often simply called a *shock*, is a curve of discontinuity in F or F_x.[6] The next subsection further explores shocks within an important and useful class of PDEs called balance equations.

4.3 Balance Equations

To understand examples of what happens at shocks, consider the following class of nonlinear PDE, which is somewhat less general than the form in (32):

$$F_t + [H(t, x, F)]_x = Q(t, x, F). \tag{39}$$

PDEs of this form are called *balance equations* in the PDE literature, where H is called the *flux function* and Q is called the *source term.* The characteristic equations (33) and (34) for the balance equation (39) are

$$\frac{d\xi}{dt} = H_F(t, \xi, u) \qquad \xi(0) = x_0 \tag{40}$$

$$\frac{du}{dt} = -H_x(t, \xi, u) + Q(t, \xi, u) \qquad u(0) = F_0(x_0). \tag{41}$$

Note that, as in the linear case, the right hand sides of these two equations do not involve p. It again follows that we do not need the equation (35) for $\frac{dp}{dt}$ to specify the characteristic ξ, or the solution u along the characteristic.

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prices, c is avoidable cost, and g is profit in the two-player game. The corresponding population game has the same interpretations as it does for the Cournot model.

[6]Another interesting behavior can arise in this model when A has one or more boundary points $\hat{x}$. The slope of the characteristics emanating from the initial condition as $x_0 \to \hat{x}$ may not agree with the slope of the characteristics emanating from the boundary $(t, \hat{x})$ as $t \to 0$. This can give rise to a "rarefaction fan" as explained in, e.g., Friedman and Ostrov (2010), Evans (1998), and Smoller (1994).

As with the general nonlinear PDE form in (32), characteristics for the balance equation (39) can easily collide, which form shocks. As colliding characteristics for balance equations carry different values of F, shocks represent discontinuities in the distribution, F.

To illustrate, recall from Subsection 2.3 the two-player game $g(x, y) = -|x - y|$, which yielded the gradient dynamics PDE $F_t + (1 - 2F)F_x = 0$. Rewritten in balance equation form, this is

$$F_t + [H(F)]_x = 0, \text{ where } H(F) = F - F^2. \tag{42}$$

The corresponding characteristic equations (40) and (41) are

$$\frac{d\xi}{dt} = 1 - 2u(t) \qquad \xi(0) = x_0 \tag{43}$$

$$\frac{du}{dt} = 0 \qquad u(0) = F_0(x_0). \tag{44}$$

Equation (44) tells us that $u(t) = F_0(x_0)$; that is, the value of F remains constant along each characteristic, and so (43) says that the characteristics are straight lines whose slopes are $1 - 2F_0(x_0)$.

Suppose that $A = [0, 1]$. Since $F = 0$ at $x = 0$, we have that all the characteristics that emanate from the boundary $x = 0$ (for all $t \in [0, \infty)$) have a constant slope of 1 and that $F = 0$ on them. Similarly, all the characteristics emanating from the boundary $x = 1$ have a constant slope of -1 and $F = 1$ on them. So, given the uniform initial distribution $F_0(x) = x$ assumed in Subsection 2.3, the characteristic equations (43) and (44) clearly generate the solution given in (9), which is valid until all the characteristics emanating from the initial condition collide at $t = 1/2$.

But what happens after $t = 1/2$? Here, the meaning of the PDE is not immediately clear, nor is the uniqueness of the solution once we allow for discontinuities. As noted in section 2.2, the way forward is to go back to the second order PDE (3) and take the limit of solutions as the volatility parameter σ goes to zero. The literature shows (see Friedman and Ostrov (2010), Dafermos (2005), or Smoller (1994), for example) that the limiting solution to this or any other balance equation is uniquely defined by two restrictions on the shocks:

1. The *entropy condition* must be satisfied at all shocks. That is, characteristic curves can terminate on shocks but cannot emanate from shocks. The idea is that information about the solution, which evolves via the characteristics, can be destroyed, but not created, at shocks.

2. The *Rankine-Hugoniot jump condition* must be satisfied at all shocks. That is, at any shock curve $x = s(t)$, the slope $\frac{ds}{dt}$ must satisfy

$$\frac{ds}{dt} = \lim_{\varepsilon \to 0^+} \frac{H\left(t, s(t), F(t, s(t)+\varepsilon)\right) - H\left(t, s(t), F(t, s(t)-\varepsilon)\right)}{F(t, s(t)+\varepsilon) - F(t, s(t)-\varepsilon)}. \tag{45}$$

The Rankine-Hugoniot condition follows from the fact that F is a *weak* (or *generalized* or *distributional*) solution in the following sense: As explained in the literature cited above, we can multiply the balance equation by a smooth test function and then integrate over a neighborhood of the (t, x) plane that includes any shock. Then we can use integration by parts to transfer the partial derivatives to the test function, so that no derivatives of F remain. The solution is weak in that we require the resulting integral equation to be satisfied instead of the PDE itself. Requiring that this equation be satisfied directly yields the Rankine-Hugoniot condition at the shock.

Returning to the example, the unique solution requires that for $t \geq \frac{1}{2}$ there is a single shock where $F = 0$ below the shock and $F = 1$ above the shock so, by the Rankine-Hugoniot condition, $\frac{ds}{dt} = \frac{(1-1^2)-(0-0^2)}{1-0} = 0$. Thus the shock curve must be $s(t) = \frac{1}{2}$ for $t \geq \frac{1}{2}$. That is, in the unique solution, when $t \geq \frac{1}{2}$, we have that $F(x, t) = 0$ if $x < \frac{1}{2}$ and $F(x, t) = 1$ if $x \geq \frac{1}{2}$. Note that it is the characteristics which emanate from the two boundaries that terminate on this shock when $t > \frac{1}{2}$.

4.4 Financial Market Shocks

To illustrate the application of these techniques, we examine a model of a financial market populated by fund managers. Each manager chooses leverage $x \in A$, where $x < 0$ indicates short selling of the risky asset (or market portfolio) and $x > 1$ indicates borrowing the safe asset in order to buy more of the risky asset. The probability of bankruptcy and the associated costs both are approximately linear in $|x|$, so investors incur a quadratic cost approximated by $0.5cx^2$ for some fixed parameter $c > 0$.

Let $r(t)$ be the excess return on the risky asset.[7] Historically, $r(t)$ has averaged about 5% per annum, but has sometimes been negative. On average, a manager choosing leverage x receives gross return $xr(t)$.

[7] A crude empirical proxy is the yield over the last 12 months on the S&P 500 index less the 10 year Treasury Bill yield. An exponential average of the daily yield differential is a more refined proxy.

Fund managers are largely concerned with *relative* performance.[8] To capture such concerns, we say that, relative to a rival manager with leverage y, a manager choosing x obtains "pride" component $g^P(x,y) = \max\{0, (x-y)r(t)\}$, and "envy" component $g^E(x,y) = \min\{0, (x-y)r(t)\}$. The overall two-player relative effect is $g = ag^E + bg^P$, with weights $a > b \geq 0$.[9]

Assume for the moment that $A = \mathbb{R}$; that is, there are no constraints on leverage. When $r(t) > 0$, the population game payoff arising from g is

$$\begin{aligned}
\int_A g(x,y)dF(t,y) &= r(t)\left[b\int_{-\infty}^{x}(x-y)dF(t,y) + a\int_{x}^{\infty}(x-y)dF(t,y)\right] \\
&= r(t)\left[a\int_{-\infty}^{\infty}(x-y)dF(t,y) + (b-a)\int_{-\infty}^{x}(x-y)dF(t,y)\right] \\
&= r(t)\left[a(x-\mu_F(t)) + (b-a)\int_{-\infty}^{x}F(t,y)dy\right].
\end{aligned}$$

The last expression uses integration by parts, assuming that the support of the current distribution has a finite lower bound. The expression is the same when $r(t) < 0$ except that the coefficient on the last term is $(a-b)$ instead of $(b-a)$. Thus, including the risk cost $0.5cx^2$, the payoff function in the population game is

$$\phi(x,F) = a(x-\mu_F(t))r(t) - (a-b)|r(t)|\int_{-\infty}^{x}F(t,y)dy - 0.5cx^2, \tag{46}$$

with gradient

$$V(x,F) = \phi_x = ar(t) - (a-b)|r(t)|F(t,x) - cx. \tag{47}$$

Gradient dynamics are quite natural for this population game.[10] The resulting gradient dynamics PDE (5) can be written in balance equation form (39)

$$F_t + [H(t,x,F)]_x = Q(F), \tag{48}$$

[8] The Wall Street Journal publishes funds rankings four times each year, and Lipper Analytics and Morningstar do so more frequently. Higher rank brings bonuses and competing job offers, and also indirectly increases managers compensation by attracting more customers; see Chevalier and Ellison (1997), Sirri and Tufano (1998), and Karceski (2002).

[9] That is, relatively low performance is more damaging than relatively high performance is beneficial. Del Guercio and Tkac (2002) find empirical support for this assumption of the model.

[10] Besides being intuitive, they can be justified formally. Per-share trading cost increases with the net amount traded in a given short time interval. If the increase is linear, then the adjustment cost (net trade times per share trading cost) is quadratic. Theorem 1 of Friedman and Ostrov (2010) therefore applies, and states that in an appropriate sense it is optimal for managers to adjust their leverage proportionately to the payoff gradient.

where

$$H(t,x,F) = \left[ar(t)F - \frac{1}{2}(a-b)|r(t)|F^2 - cxF\right] \text{ and } Q(F) = -cF, \tag{49}$$

so the characteristic equations (40) and (41) are

$$\frac{d\xi}{dt} = ar(t) - (a-b)\,|r(t)|\,u(t) - c\xi(t) \qquad \xi(0) = x_0 \tag{50}$$

$$\frac{du}{dt} = 0 \qquad u(0) = F_0(x_0). \tag{51}$$

Equation (51) tells us that $u = F$ is constant along characteristics, so each characteristic follows the leverage path of a particular manager.

A simple analysis of equation (50) shows that shocks are almost inevitable. Consider any two characteristics, $\xi_1(t)$ and $\xi_2(t)$, that start apart: $\xi_1(0) < \xi_2(0)$. Because F is montonically increasing, we have that $u_1 \leq u_2$. Assume we have a case where $u_1 < u_2$. Since $a - b > 0$, (50) shows that as long as $\xi_1(t) \leq \xi_2(t)$, then $\frac{d(\xi_2-\xi_1)}{dt} < -k < 0$ for some constant k — as long as $r(t)$ doesn't hover near 0 — and this guarantees a shock by time $t = \frac{\xi_2(0)-\xi_1(0)}{k}$. In other words, the more leveraged managers increase their leverage more slowly (or decrease it more rapidly) than their less leveraged peers, which pushes their characteristics to collide.

We now consider behavior when the market excess return $r(t) = \hat{r} > 0$ is constant and positive for a long time. In this case (50) can be solved using integrating factors to yield the characteristic equation

$$\xi(t) = \xi(0)e^{-ct} + \frac{(a-(a-b)u)\hat{r}}{c}(1-e^{-ct}). \tag{52}$$

From this equation (or the logic of the previous paragraph), we see that all the characteristics move towards each other as time evolves. Assuming that the mass of the distribution is initially contained within a finite interval $[X_l, X_u]$ on the x-axis, we conclude that all the characteristics emanating from $[X_l, X_u]$ at $t = 0$ must eventually hit the inexhaustible set of characteristics that emanate from $x < X_l$ for which $F = 0$ or that emanate from $x > X_u$ for which $F = 1$. In other words, for awhile we may have a complicated set of shocks forming and moving, but there must be some time, τ, after which we have a single shock where $F = 0$ below the shock and $F = 1$ above the shock. By the Rankine-Hugoniot condition (45) and our expression for the flux H in (49), we know that this shock, $x = s(t)$, moves by

$$\frac{ds}{dt} = \frac{1}{2}(a+b)\hat{r} - cs(t).$$

Solving this ODE, we have that

$$s(t) = \frac{1}{c}\left(\frac{1}{2}(a+b)\hat{r}\left(1 - e^{-c(t-\tau)}\right) + cs(\tau)e^{-c(t-\tau)}\right),$$

so we see that the shock exponentially converges to $x = \frac{(a+b)\hat{r}}{2c}$. In other words, after a finite amount of time, all the investors have herded themselves to the common leverage value $x = \frac{(a+b)\hat{r}}{2c}$. Panel A of Figure 7 illustrates the evolution of the leverage distribution.

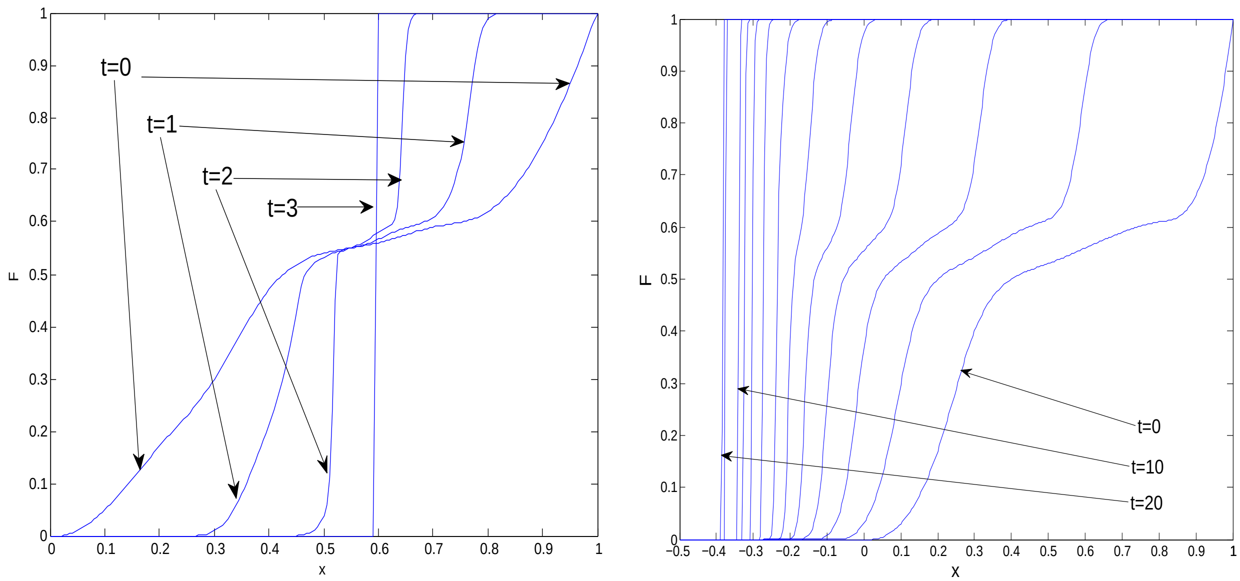

Figure 7: Market behavior with $a = 8, b = 1, c = 0.3$. The numerical simulation is an upwind scheme that uses $\Delta x = 0.005$. Panel A: We set $r = 0.04$. By $t = 2$, we see the formation of two shocks, which then combine by $t = 3$ to form one stable shock at $x = \frac{(a+b)r}{2c} = 0.6$. Panel B: We set $r = -0.01$. The panel shows $t = 0, 1, 2, ..., 8, 9, 10, 15,$ and 20. An analysis of the characteristics guarantees a shock as soon as $F > 0$ at $x = -0.266$, which first happens near $t = 6$. By $t = 20$, we have converged to a single shock that will remain at $x = \frac{(3a-b)r}{2c} = -0.383$.

Behavior is even more interesting when we have constant negative returns, $r(t) = \hat{r} < 0$ for a long time. In this case the characteristics are

$$\xi(t) = \xi(0)e^{-ct} + \frac{(a + (a-b)u)\hat{r}}{c}(1 - e^{-ct}), \tag{53}$$

so starting with positive leverage x leads to rapid deleveraging, where as noted earlier, the more highly leveraged managers deleverage faster. Thus we have a "race to the bottom" that again typically involves shock waves. As illustrated in Panel A of Figure 7, a slight variant on the Rankine-Hugoniot argument above tells us that, when $A = \mathbb{R}$ all managers will clump together at the short position $\hat{x} = \frac{(3a-b)\hat{r}}{2c} < 0$. When short-selling is prohibited (or prohibitively expensive), then managers will pile up at $x = 0$; that is, they all will exit the financial market until returns once again become positive.

More realistically, excess returns $r(t)$ will fluctuate. In this case, investors still tend to converge towards a common leverage $\hat{x}(t)$ that fluctuates in sympathy with $r(t)$. Idiosyncratic returns and behavior across investors, captured by $\sigma > 0$ in (3), would counterbalance this push towards conformity. We will not pursue such extensions here.[11]

4.5 Numerical simulations

We provide some comments about the numerical scheme used to generate the solutions to (48) shown in Figure 7 and to generate the PDE solutions shown in some other figures in this paper. Since the actual solution for (48) evolves strictly along characteristics, the numerical solution for (48) must also evolve along this characteristic flow. This was accomplished with what is known as a "first-order upwind scheme," where the time step is adapted to conform to the Courant-Friedrichs-Lewy (CFL) condition. The term "upwind" means that the algorithm determines the solution at a point $(t_0 + \Delta t, x_0)$ using data at time t_0 where either $x \le x_0$ or $x \ge x_0$, depending on whether the characteristics point upwards or downwards on the (t, x) plane near the point. The CFL condition prevents using a time step, Δt, that is so large that $(t_0 + \Delta t, x_0)$ lies outside of the reach of the characteristics emanating from the data employed at time t_0. Further, this numerical scheme has been proven in Crandall and Majda (1980) to converge to the correct unique solution, where all shocks obey the entropy condition and Rankine-Hugoniot jump condition. Simple modifications of this scheme were used to produce all the figures in Subsection 2.3, including the case $g(x, y) = |x - y|^{0.5}$ that can only be solved numerically, and in Subsection 3.4 for Keynes' Beauty Contest.

[11] Here we have assumed that excess returns are exogenous. However, when the entire population of fund managers is selling off their risky assets, then asset prices (and, for a while, asset returns) should be driven down, creating an even more vicious cycle of deleveraging. Building a model complementary to ours, Friedman and Abraham (2009) consider a finite population of heterogeneous managers and endogenize $r(t)$ (and the risk cost parameter c), but de-emphasize relative performance considerations. Their specification seems too complicated for analytic treatment, but their simulations exhibit impressive bubbles and crashes.

5 Discussion

Much of this paper can be summed up in the following recipe for model construction:

- Take any symmetric two-player game $g(x, y)$ with a continuous action space $A \subset \mathbb{R}$.
- Extend to a population game $\phi(x)$ by averaging $g(x, y)$ over the distribution $F(t, y)$ of all opponents' actions.
- Impose gradient dynamics: each player adjusts his or her own action $x \in A$ to move up the payoff gradient at a rate proportional to the slope.
- Use techniques borrowed from fluid dynamics (and suitably extended) to characterize solutions to the resulting partial differential equations (PDEs).

We showed that these PDEs have classical solutions when g is a smooth function of x, and that the solution can be written explicitly when g is quadratic in x. Examples include several famous games in economics as well as strategy mixtures in any symmetric 2×2 game. On the other hand, we showed that for games g with discontinuities or kinks, one can get nonlinear PDEs with solutions that involve shock waves. After reviewing some techniques for addressing such complications, we analyzed a population of fund managers who interact in a financial market. There we saw how concern for relative performance can lead to a form of herding behavior, including shocks and, when returns become negative, a race to deleverage.

The assumption of gradient dynamics can be relaxed somewhat. When the velocity field V is proportional to, rather than exactly equal to, the payoff gradient ϕ_x, we can retain most analytic results by rescaling time t. Less obviously, the literature on differential inclusions or cone fields (e.g., Aubin and Celina (1984), and Smale (1976)) shows that most qualitative results for gradient dynamics continue to hold as long as the velocity field V is commensurate with ϕ_x, in the sense that their ratio, $\frac{V}{\phi_x}$, will always fall between two positive constants.

The recipe and techniques presented here can be extended to wider classes of models. The underlying strategy space A could be a compact subset of $\mathbb{R}^n$. In a Hotelling model, for example, if the players choose price as well as location along a circle or line segment, then $n = 2$. Also, the underlying game g could be asymmetric, or involve more than two players, resulting in a population game with two or more strategically distinct player populations. For example, a population of buyers might interact with a population of sellers, each with a continuous action space. These models can lead to PDEs with more than one spatial dimension or to systems of PDE that must be solved simultaneously. Fortunately the mathematics literature contains many results about such PDEs, opening a vast set of possible applications in biology and the social sciences.

Conversely, gradient dynamics PDE models present interesting new mathematical questions. Balance equations have shocks that correspond to discontinuities in their solutions, and there are other classes of PDEs whose solutions have discontinuous spatial derivatives at shocks, but not discontinuities in the solution itself; see Crandall and Lions (1983) and Crandall, Evans, and Lions (1984). Both of these types of PDEs have specific structures that correspond to specific, but distinct, types of problems in the physical sciences. However, the economic model for preemption and Bertrand pricing in Subsection 4.2 leads to a gradient dynamics PDE, (37), that has aspects of both of the structures previously only seen separately. This makes this equation unique and unstudied. When characteristics cross, it is unclear if this corresponds to discontinuities in F or in F_x (although simulations suggest that the answer is F_x). Nor is uniqueness or even existence established for the solution of this equation after shocks form. Thus the study of gradient dynamics can provide a wide variety of interesting economic, biological, and social science models, as well as unique mathematical frontiers to explore.

6 Appendix A: Proofs

6.1 Derivation of the Fokker-Planck-Kolmogorov equation

Let $f(t,x)$ be the probability density function for the actions $x \in \mathbb{R}$ at time t. Assume for an individual player, the action $x = X(t)$ evolves by the stochastic dynamics

$$dX = V(t, X(t), f(t,\cdot))dt + \sigma dB \tag{54}$$

where $B(t)$ is a Brownian motion. We will show that this implies the Fokker-Planck-Kolmogorov equation

$$f_t = -(Vf)_x + \frac{1}{2}\sigma^2 f_{xx}.$$

Before we proceed to the derivation of this equation, we state three mathematical properties that we will need:

1. The law of iterated expectation: Let $E^{x,t}[\cdot]$ be the expected value operator conditioned on an player having action x at time t. For any $\Delta t > 0$, any $T \geq t + \Delta t$ and any function $G(x)$,

$$E^{X(t),t}\left[E^{X(t+\Delta t),t+\Delta t}\left[G\left(X(T)\right)\right]\right] = E^{X(t),t}\left[G\left(X(T)\right)\right].$$

   In other words, the current expectation already incorporates what is currently known about future expectations.

2. Ito calculus fact 1: $(dB)^2 = dt$ with a probability equal to one when integrated over any time interval.

3. Ito calculus fact 2: $E^{x,t}\left[\int_t^{t+\Delta t} h(t, X(t))dB\right] = 0$ for any function h. Because Brownian motion has a normal distribution with mean zero, we can think of $E[dB] = 0$. Since, in Ito calculus, h is determined at the beginning of any time period, h and dB are independent which implies the integral is like a sum of each $E[h]E[dB]$ from t to $t + \Delta t$, each of which is clearly zero since $E[dB] = 0$.

Proof: Define $u(x,t) = E^{x,t}\left[g\left(X(T)\right)\right].$ From Taylor's theorem, we have that

$$du = u_t dt + u_x dx + \frac{1}{2}u_{xx}(dx)^2.$$

We must keep track of the $(dx)^2$ term as it contributes to the expression, specifically through Ito calculus fact 1 and the dynamics (54), we see that $(dx)^2 = \sigma^2 dt$ (since all terms with dt^2 and $dtdB$ contribute nothing), so

$$du = \left(u_t + \frac{1}{2}\sigma^2 u_{xx}\right)dt + u_x dx.$$

Substituting (54) gives

$$du = \left(u_t + Vu_x + \frac{1}{2}\sigma^2 u_{xx}\right)dt + \sigma u_x dB. \tag{55}$$

If we integrate (55) between t_0 and $t_0 + \Delta t$ and then take E^{x,t_0} of both sides we get

$$E^{x,t_0}\left[\int_{t_0}^{t_0+\Delta t} du\right] = E^{x,t_0}\left[\int_{t_0}^{t_0+\Delta t}\left(u_t + Vu_x + \frac{1}{2}\sigma^2 u_{xx}\right)dt\right] + E^{x,t_0}\left[\int_{t_0}^{t_0+\Delta t}\sigma u_x dB\right]. \tag{56}$$

The expected value of the second integral on the right-hand side of (56) is zero by Ito calculus fact 2. The left-hand side is zero because

$$E^{x,t_0}\left[\int_{t_0}^{t_0+\Delta t} du\right] = E^{x,t_0}\left[u\left(X(t_0+\Delta t), t_0+\Delta t\right) - u\left(X(t_0), t_0\right)\right],$$

which by the definition of u is equal to $E^{x,t_0}\left[E^{X(t_0+\Delta t),t_0+\Delta t}\left[g\left(X(T)\right)\right]-E^{X(t_0),t_0}\left[g\left(X(T)\right)\right]\right]$ and this is zero by the law of iterated expectation. Therefore, we have the first term on the right-hand side of (56) is zero. If we divide by Δt and take the limit as $\Delta t \to 0$, we have that the integrand evaluated at x and t_0 is zero. Since t_0 can represent any time, we can replace it with t to get for $u(x,t)$ that

$$u_t + Vu_x + \frac{1}{2}\sigma^2 u_{xx} = 0. \tag{57}$$

This is called the backwards Kolmogorov equation.

To get the Fokker-Planck-Kolmogorov equation we first note that

$$\int_{-\infty}^{\infty} u(x,t)f(x,t)dx = E\left[G\left(X(T)\right)\right].$$

Since the right-hand side is not a function of time, the left-hand side cannot depend on time either. Therefore, the derivative of the left-hand side with respect to time is zero. Moving the time derivative inside the integral and applying the product rule yields

$$\int_{-\infty}^{\infty} uf_t + u_t f dx = 0.$$

Substituting the backwards Kolmogorov equation (57) gives

$$\int_{-\infty}^{\infty} uf_t - \left(Vu_x + \frac{1}{2}\sigma^2 u_{xx}\right) f dx = 0.$$

Now we apply integration by parts to all x derivatives. Assuming the function G is bounded and that $\lim_{x\to\pm\infty} f(x,t) = \lim_{x\to\pm\infty} f_x(x,t) = 0$, this leads to

$$\int_{-\infty}^{\infty} u\left(f_t + (Vf)_x - \frac{1}{2}\sigma^2 f_{xx}\right) dx = 0. \tag{58}$$

Since u is essentially arbitrary given that G is any bounded function, satisfying (58) necessitates the parenthetical term in its integrand to be zero, and therefore, we have the Fokker-Planck-Kolmogorov equation. $\diamondsuit$

6.2 Proof of Theorem 1: Classical Solutions for smooth g

Theorem 1: Let $A = \mathbb{R}$ and let $g : A \times A \to \mathbb{R}$ have bounded first, second and third partial derivatives in its first argument. Also, let the initial density $f_0(x)$ have a bounded first derivative. Then neither the solution, $f(t,x)$, nor the partial derivative of the solution, $f_x(t,x)$, to the integro-partial differential equation defined by (2) and (4) can become unbounded in finite time.

Proof. Since we have assumed that g_{xxx} exists and that it, and all its lower derivatives in x, are bounded in absolute value, we can define B to denote the largest of these bounds. Also, since $[f_0]_x$ is bounded and $\int_A f_0(x)dx = 1$, we have that f_0 is bounded. We first establish that f is bounded in finite time, and then we use that to establish that f_x is bounded in finite time.

Combining (13) and (2), we have that along any characteristic

$$f\left(t,\xi(t)\right) = f_0(x_0)e^{-\int_0^t \int_A g_{xx}(\xi(s),y)f(s,y)dyds},$$

so, since $|g_{xx}| \leq B$ and $\int_A f(s,y)dy = 1$, we have our bound for f :

$$\left|f\left(t,\xi(t)\right)\right| \leq \left|f_0(x_0)\right| e^{Bt}.$$

To obtain the bound on f_x we need to use the PDE for f_x instead of f, which we obtain by differentiating (4) with respect to x and then applying the product rule to obtain

$$(f_x)_t + V(f_x)_x = -V_{xx}f - 2V_x f_x. \tag{59}$$

The characteristic curves for f_x are the same as the curves for f, of course. Therefore, we still have that $\frac{d\xi}{dt} = V$ and so, by the chain rule, $\frac{df_x(t,\xi(t))}{dt}$ equals the left hand side of (59), which, after substituting, yields

$$\begin{aligned}\frac{df_x(t,\xi(t))}{dt} &= -V_{xx}(t,\xi(t))f(t,\xi(t)) - 2V_x(t,\xi(t))f_x(t,\xi(t)) \\ &= -\left[\int_A g_{xxx}(\xi(t),y)f(t,y)dy\right] f(t,\xi(t)) - 2\left[\int_A g_{xx}(\xi(t),y)f(t,y)dy\right] f_x(t,\xi(t)).\end{aligned}$$

Now we apply to this equation our bounds for the derivatives of g along with our obtained bound for f :

$$\left|\frac{df_x(t,\xi(t))}{dt}\right| \leq B\left|f_0(x_0)\right| e^{Bt} + 2B\left|f_x(t,\xi(t))\right|. \tag{60}$$

Consider the case where the three quantities in (60) contained within the absolute value signs are positive. In this case, we subtract the $2Bf_x$ term from both sides, multiply by the integrating factor e^{-2Bt}, and then apply the product rule to obtain

$$\frac{d\left[e^{-2Bt}f_x(t,\xi(t))\right]}{dt} \leq Bf_0(x_0)e^{-Bt}.$$

Then, integrating, we have that

$$f_x(t,\xi(t)) \le e^{2Bt}\left([f_0]_x(x_0) + f_0(x_0)\left(1-e^{-Bt}\right)\right) \le e^{2Bt}\left([f_0]_x(x_0) + f_0(x_0)\right). \tag{61}$$

Applying a similar process to the cases where various quantities within the absolute value signs are negative, we obtain the following modification of (61):

$$|f_x(t,\xi(t))| \le e^{2Bt}\left(|[f_0]_x(x_0)| + |f_0(x_0)|\right). \tag{62}$$

Since f_0 and $[f_0]_x$ are bounded, we see from (62) that f_x is also bounded in finite time.

Therefore, shocks (that is, discontinuities) cannot form in either $F(t,x)$ or $f(t,x)$ for these integro-partial differential equations. $\diamondsuit$

6.3 Proof of Theorem 2: Solution for Quadratic g

Theorem 2: Let

$$g(x,y) = ax + \frac{b}{2}x^2 + cxy + h(y) \tag{63}$$

or let the velocity field take the form

$$V(x, F(t,\cdot)) = a + bx + c\mu_F(t), \tag{64}$$

where $x \in A = \mathbb{R}$, $t \ge 0$, and a, b, and c are real constants. Then, given an initial probability density, $f_0(x)$, the gradient dynamics PDE (4) has the following unique solution:

$$f(t,x) = e^{-bt} f_0\left(\mu_F(0) + \left(x + \frac{a}{b+c}\right)e^{-bt} - \left(\mu_F(0) + \frac{a}{b+c}\right)e^{ct}\right) \tag{65}$$

and the mean is

$$\mu_F(t) = \left(\mu_F(0) + \frac{a}{b+c}\right)e^{(b+c)t} - \frac{a}{b+c}. \tag{66}$$

This solution is a classical solution if f_0 is differentiable.

Proof. We need to show that if $A = \mathbb{R}$, that equation (4) with

$$V(t, x, f(t,\cdot)) = a + bx + c\mu_F(t)$$

yields (65). Equation (4) in this case is

$$f_t = -\left[(a + bx + c\mu_F(t))f\right]_x.$$

Integrating x from $-\infty$ to y and then integrating y from $-\infty$ to ∞ gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{y} f(t,x)dxdy = -a-(b+c)\mu_F(t).$$

Applying integration by parts to the left hand side of this equation gives $\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{y} f(t,x)dxdy = -\frac{d}{dt}\int_{-\infty}^{\infty} yf(t,y)dy = -\frac{d\mu_F(t)}{dt}$ and then solving $\frac{d\mu_F(t)}{dt} = a+(b+c)\mu_F(t)$ by separation of variables gives

$$\mu_F(t) = \left(\mu_F(0)+\frac{a}{b+c}\right)e^{(b+c)t} - \frac{a}{b+c}.$$

With $\mu_F(t)$ now known explicitly, we have that $V = v(t,x)$, an explicit function of just t and x. This means we can apply (11), the characteristic equation for $\xi(t)$ from Subsection 3.2:

$$\frac{d\xi}{dt} = v(t,\xi) = a+b\xi+c\left[\left(\mu_F(0)+\frac{a}{b+c}\right)e^{(b+c)t} - \frac{a}{b+c}\right].$$

Since this is a linear ODE, we subtract $b\xi$ from both sides, multiply by the integrating factor e^{-bt}, apply the product rule to the left hand side and integrate, yielding

$$\xi(0) = \mu_F(0)+\left(\xi(t)+\frac{a}{b+c}\right)e^{-bt} - \left(\mu_F(0)+\frac{a}{b+c}\right)e^{ct}.$$

Finally, we apply (13), the evolution of f along a characteristic, and the fact that $x=\xi(t)$ to obtain our desired result:

$$f(t,x) = e^{-bt}f_0\left(\mu_F(0)+\left(x+\frac{a}{b+c}\right)e^{-bt} - \left(\mu_F(0)+\frac{a}{b+c}\right)e^{ct}\right).\diamondsuit$$

6.4 Generalization of Theorem 2 Results

We obtain a solution to the continuity equation when $A=\mathbb{R}$, f has compact support, and $g(x,y) = ax+\frac{b}{2}x^2+cxy+dxy^2+h(y)$. Then $g_x(x,y) = a+bx+cy+dy^2$ and $V(t,x) = \int_A g_x(x,y)f(t,y)dy$, so the continuity equation is of the form

$$[f(t,x)]_t = -\left[\int_{-\infty}^{\infty}(a+bx+cy+dy^2)f(t,y)dyf(t,x)\right]_x. \tag{67}$$

If we define $\mu_1(t)$ and $\mu_2(t)$ to be the first and second moments of x:

$$\begin{aligned}\mu_1(t) &= \mu_F(t) = \int_{-\infty}^{\infty} xf(t,x)dx \\ \mu_2(t) &= \int_{-\infty}^{\infty} x^2 f(t,x)dx,\end{aligned}$$

the continuity equation (67) can be rewritten in the form

$$[f(t,x)]_t = -\left[(a + bx + c\mu_1(t) + d\mu_2(t))f(t,x)\right]_x . \tag{68}$$

Recalling that f has compact support, if we integrate (68) from $-\infty$ to y in x and then integrate from $-\infty$ to ∞ in y, we have that

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{y} f(t,x)dxdy &= -\int_{-\infty}^{\infty}(a + by + c\mu_1(t) + d\mu_2(t))f(t,y)dy \\ &= -a - (b+c)\mu_1(t) - d\mu_2(t). \end{aligned} \tag{69}$$

Applying integration by parts to the exterior integral on the left-hand side of (69) and using the compact support of f yields

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{y} f(t,x)dxdy = -\frac{d}{dt}\int_{-\infty}^{\infty} yf(t,y)dy = -\mu_1'(t) \tag{70}$$

and combining (69) and (70) gives the ODE

$$\mu_1'(t) = a + (b+c)\mu_1(t) + d\mu_2(t). \tag{71}$$

On the other hand, if we integrate (68) from $-\infty$ to y in x, multiply by y, and then integrate from $-\infty$ to ∞ in y we have that

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} y\int_{-\infty}^{y} f(t,x)dxdy &= -\int_{-\infty}^{\infty} y(a + by + c\mu_1(t) + d\mu_2(t))f(t,y)dy \\ &= -a\mu_1(t) - b\mu_2(t) - c\left(\mu_1(t)\right)^2 - d\mu_1(t)\mu_2(t). \end{aligned} \tag{72}$$

Applying integration by parts to the exterior integral on the left-hand side of (72) yields

$$\frac{d}{dt}\int_{-\infty}^{\infty} y\int_{-\infty}^{y} f(t,x)dxdy = -\frac{d}{dt}\int_{-\infty}^{\infty}\frac{y^2}{2}f(t,y)dy = -\frac{1}{2}\mu_2'(t) \tag{73}$$

so combining (72) and (73) gives the ODE

$$\mu_2'(t) = 2a\mu_1(t) + 2b\mu_2(t) + 2c\left(\mu_1(t)\right)^2 + 2d\mu_1(t)\mu_2(t). \tag{74}$$

We can solve this system of two autonomous ODEs, (71) and (74), numerically and then insert the now known functions for $\mu_1(t)$ and $\mu_2(t)$ back into the continuity equation (68). With $\mu_1(t)$ and $\mu_2(t)$ inserted, (68) is now reduced to a time-varying exogenous landscape and can be easily solved numerically through the methods provided in Subsection 4.5.

Remark: This method can be extended to any $g(x,y)$ that is the sum of terms of the form cx^ny^m where c is constant, n is 0, 1 or 2 and m is a nonnegative integer. To do this, one determines and then solves the ODE system for the first M moments where M is the largest value of m appearing in the terms. So, for example, $g(x, y) = 7x^2 - 15y^4 + 23xy^6$ can be handled by solving the six ODEs for the first six moments, but $g(x, y) = 7x^3$ cannot be accommodated since the power of x is greater than 2.

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