Midterm Exam

Econ 200

1. A firm has production function ln y = (1/3)ln x_1 + (1/2)ln x_2. Input 2 is unchangeable for the moment at the level $x_2 = 8$. Prices are \$8 and \$5 respectively for inputs 1 and 2.

(a) (6 points) What is the firm's variable cost? fixed cost (if any)? total cost? marginal cost?

Solution: In the short run, total costs are the sum of variable costs and fixed costs:

$$c(y) = c_v(y) + F$$

Therefore, we can find variable costs and fixed costs from total costs $c(y)$ which is simply the cost function $c(w_1, w_2, y)$ where $w_1 = 8$ and $w_2 = 5$.

The conditional factor demand $x_1^*(w_1, w_2, y)$ can be obtained simply by solving $\ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln 8$ for x_1 to get:

$$x_1^*(8, 5, y) = 8^{-\frac{3}{2}}y^3$$

Normally, we would have to set up the minimization problem to find the conditional factor demand but here with only 1 variable (x_1) and 1 constraint, the value of x_1 is pinned down. Variable costs are $w_1 x_1^*$ and fixed costs are $w_2\bar{x}_2$.

Suppose we did not notice this "trick" to find x_1^*. How else could we obtain the cost function? Let's try to solve

$$\min_{x_1 \geq 0} 8x_1 + 5(8) \quad \text{subject to} \quad \ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln 8$$

The easiest way to solve this is to rewrite $\ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln 8$ as $x_1 = 8^{-\frac{3}{2}}y^3$ and substitute this into $8x_1 + 5(8)$. The problem then becomes

$$\min_{x_1 \geq 0} 8 \cdot 8^{-\frac{3}{2}}y^3 + 5(8) = \min_{x_1 \geq 0} 8^{-\frac{1}{2}}y^3 + 40$$

There's nothing to minimize! Again, with only 1 input, the firm does not get the choose how much of x_1 to use to produce y units of output. The solution of this "minimization" problem is just $8^{-\frac{1}{2}}y^3 + 40$. This gives us our cost function:

$$c(8, 5, y) = 8^{-\frac{1}{2}}y^3 + 40$$

implying that variable costs are $c_v(y) = 8^{-\frac{1}{2}}y^3$ and fixed costs are $F = 40$. Marginal costs are

$$\text{MC}(y) = \frac{dc}{dy} = 3 \cdot 8^{-\frac{1}{2}}y^2$$

(b) (2 points) What is the firm's supply function?

Solution: The firm's supply curve is the upward-sloping portion of the marginal cost curve that lies above the average variable cost curve. In this problem, the marginal cost curve is always upward-sloping (and thus marginal cost is always higher than average variable cost). Therefore, the firm's supply function is given by its marginal cost:

$$p = 3 \cdot 8^{-\frac{1}{2}}y^2 \quad \Rightarrow \quad y^* = 2^{\frac{3}{4}}3^{-\frac{1}{2}}p^{\frac{1}{2}}$$

(You should verify that this is the same supply function that you get when you apply Hotelling's Lemma to the profit function.)

(c) (6 points) Now assume that both inputs can be adjusted freely. What are the firm's conditional input demands? What is its average cost? Marginal cost? Supply function?

Solution: To solve

$$\min_{x_1, x_2 \geq 0} w_1x_1 + w_2x_2 \quad \text{subject to} \quad \ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln x_2$$

we use the Lagrangian

$$\mathcal{L} = w_1x_1 + w_2x_2 - \lambda(\frac{1}{3}\ln x_1 + \frac{1}{2}\ln x_2 - \ln y)$$

which gives the first-order conditions

$$w_1 - \lambda \frac{1}{3x_1} = 0$$
$$w_2 - \lambda \frac{1}{2x_2} = 0$$

When we combine these two conditions to get rid of λ, we get $3w_1x_1 = 2w_2x_2$. We substitute this back into the constraint to get the conditional factor demands:

$$\begin{aligned} \ln y &= \frac{1}{3}\ln x_1 + \frac{1}{2}\ln\left(\frac{3w_1}{2w_2}x_1\right) \\ \Rightarrow \quad y &= x_1^{\frac{1}{3}}\left(\frac{3w_1}{2w_2}x_1\right)^{\frac{1}{2}} = \left(\frac{3w_1}{2w_2}\right)^{\frac{1}{2}} x_1^{\frac{5}{6}} \\ \Rightarrow \quad x_1^*(w_1, w_2, y) &= \left(\frac{2w_2}{3w_1}\right)^{\frac{3}{5}} y^{\frac{6}{5}}, \\ x_2^*(w_1, w_2, y) &= \left(\frac{3w_1}{2w_2}\right)^{\frac{2}{5}} y^{\frac{6}{5}} \end{aligned}$$

With these conditional factor demands, the cost function is

$$\begin{aligned} c(w_1, w_2, y) &= w_1x_1^* + w_2x_2^* \\ &= \left[\left(\frac{2}{3}\right)^{\frac{3}{5}} + \left(\frac{3}{2}\right)^{\frac{2}{5}}\right] w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}}\left(1 + \frac{3}{2}\right) w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \end{aligned}$$

Long-run average cost is

$$\mathrm{LAC}(y) = \frac{c}{y} = \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}}$$

and long-run marginal cost is

$$\begin{aligned} \mathrm{LMC}(y) &= \frac{dc}{dy} = \frac{6}{5}\left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \\ &= 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \end{aligned}$$

The firm's supply function is once again given by its marginal cost:

$$\begin{aligned} p &= 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \\ \Rightarrow \quad y^*(w_1, w_2, p) &= \frac{p^5}{72 w_1^2 w_2^3} \end{aligned}$$

(Alternatively, we could have found the supply function by maximizing profit. You should verify that the same supply function is found by applying Hotelling's Lemma to the profit function!)

2. Suppose that 150 firms in an industry each have marginal cost function $c'(y_i) = 3y_i$ and they face industry demand curve $y_T = 180 - 10p$.

(a) (6 points) What is the competitive equilibrium price p^*? Firm output $y_i(p^*)$? Industry output $y_T(p^*)$?

Solution: Each firm produces along its marginal cost curve which means that the firm's supply function is $y_i = \frac{p}{3}$. Industry supply is the sum of each firm's supply, $y_T = 150 y_i = 50p$. Industry supply and demand intersect at $p^* = 3$ which gives $y_i(p^*) = 1$ and $y_T(p^*) = 150$.

(b) (4 points) Compute the producer surplus (PS) and consumer (CS) and total surplus (TS) at competitive equilibrium.

Solution:

$$\begin{aligned} \text{PS} &= \frac{1}{2} \cdot 3 \cdot 150 = 225 \\ \text{CS} &= \frac{1}{2}(18 - 3)150 = 1125 \\ \text{TS} &= 225 + 1125 = 1350 \end{aligned}$$

(c) (4 points) Suppose a tax $t = 0.50$ per unit of output is imposed. Compute the impact on the price paid by consumers, the net price to firms, and the change in PS, CS, and TS.

Solution: With the tax,

$$\begin{aligned} & p^d = p^s + 0.5 \\ \Rightarrow \quad & 18 - \frac{1}{10}y_T = \frac{1}{50}y_T + 0.5 \\ \Rightarrow \quad & y_T^* = \frac{875}{6} \end{aligned}$$

The price paid by consumers increases to $p^d = 3.42$ and the price that firms receive falls to $p^s = 2.92$.

Producer surplus falls by $\frac{875}{6}(3 - 2.92) + 0.5(150 - \frac{875}{6})(3 - 2.92) = 11.83$. Consumer surplus decreases by $\frac{875}{6}(3.42-3)+0.5(150-\frac{875}{6})(3.42-3) = 62.13$. Total surplus falls by the amount of the deadweight loss, $0.5(150 - \frac{875}{6})0.5 = 1.04$.

(d) (2 points) Returning to the situation with no tax, suppose that each firm has to join an industry association to stay in business. What is the maximum fee that the association could extract from each firm?

Solution: Each firm is willing to pay a fee less than or equal to its surplus, $\frac{225}{150} = 1.5$.

3. A consumer has preferences represented by the utility function $u(x_0, x_1) = x_0 + 4\sqrt{x_1}$. Normalize so that the price of good 0 is always 1, and let $p > 1$ be the price of good 1.

(a) (2 points) Compute her marginal rate of substitution of good 1 for good 0.

Solution:

$$\text{MRS}_{x_1,x_0} = \frac{\text{MU}_{x_1}}{\text{MU}_{x_0}} = \frac{2x_1^{-\frac{1}{2}}}{1} = \frac{2}{\sqrt{x_1}}$$

(b) (4 points) Compute her Marshallian demand for good 1, assuming that income $m > 10$.

Solution: The first-order conditions imply

$$\text{MRS}_{x_1,x_0} = \frac{p}{1} \quad \Rightarrow \quad x_1^* = \frac{4}{p^2} > 0$$

However, the first-order conditions are only good for *interior* solutions. To check, we must also make sure x_0^* is positive.

$$x_0^* = m - px_1^* = m - \frac{4}{p}$$

Since $p > 1$ and m is assumed to be greater than 10, it must be true that $m > \frac{4}{p}$. Thus, $x_0^* > 0$ and we are indeed at an interior solution.

(Without the assumption that $m > 10$, then for $m \leq \frac{4}{p}$ we have a corner solution where $x_0^* = 0$ and $x_1^* = \frac{m}{p}$.)

(c) (6 points) Compute her expenditure function, her indirect utility function and her Hicksian demand for good 1.

Solution: The indirect utility function is

$$v(p,m) = u(x_0^*, x_1^*) = m - \frac{4}{p} + 4\sqrt{\frac{4}{p^2}} = m + \frac{4}{p}$$

(This is not homogenous of degree 0 since the price of x_0 is fixed at 1.)

The indirect utility function is the maximum utility the consumer can obtain given p and m. This is the dual of the expenditure function, which is the minimum expenditure given p and utility u. We can exploit this relationship to find the expenditure function: knowing the maximum utility v that can be obtained by spending m also tells us the minimum spending m needed to obtain utility v. By solving $u = v(p,m) = m + \frac{4}{p}$ for m, we obtain the expenditure function

$$e(p,u) = m = u - \frac{4}{p}$$

with Hicksian demand for x_1

$$h_1^*(p,u) = \frac{\partial e}{\partial p} = \frac{4}{p^2}$$

(You should verify that these are the same Hicksian demand and expenditure functions that are obtained from the expenditure minimization problem.)

4. Suppose that the firm in problem 1 realizes that their production function and cost function are not really very close to what was specified there. The firm hires you to estimate their cost function so that they can make contingency plans to deal with possible changes in input prices. What data would you request, and (*very* briefly) how would you begin to analyze that data? (8 pts)

Solution: We could estimate the production function in log or translog form but accurate input data is often difficult to obtain. Good data on input *prices* is often easier to obtain, so use the dual approach and estimate the cost function in log or translog specifications:
Step 1: Ask for weekly data on output quantity y_t, cost of production c_t, and input price data w_{it}.
Step 2: In log specification, regress

$$\ln c_t = \alpha_0 \ln y_t + \alpha_1 \ln w_{1t} + \ldots + \alpha_n \ln w_{nt} + \varepsilon_t$$

and impose $\sum_{i=1}^{n} \alpha_i = 1$ (since cost functions are homogenous of degree 1 in w_1, w_2, ..., w_n). If we instead wanted a translog specification, we would add interaction terms to the regression along with additional restrictions on the coefficients.