2. Preferences and Demand

Microeconomic Analysis, Chapters 7-9

I. Preference Orderings

A. Bundles

1. Our main goal in this section is to model how people choose among consumption opportunities.

2. Each opportunity is called a **bundle**.

3. A bundle is represented as a vector $\mathbf{x}$ with a dimension equal to the number of goods.

4. If consumers only make choices involving two goods (1 and 2), we could represent this as a vector $\mathbf{x} = (x_1, x_2)$

   a. $\mathbf{x} = (7, 4)$ represents a bundle consisting of 7 units of good 1 and 4 units of good 2.

   b. For simplicity we'll often assume there are only two goods in this section

   c. These results, however, extend to an arbitrarily large number of goods, e.g., everything you can find on Amazon.

Ex: Bundles in a 2-dimensional consumption space $\mathbf{R}^2_+$.

B. Preferences Over Bundles

1. We assume that people can compare pairs of bundles.

2. We call this comparison a **preference**.

3. In other words, a preference is really a *relation(ship)* between between pairs of bundles.

4. Taking any two bundles $\mathbf{x}$ and $\mathbf{y}$ we write

   - $\mathbf{x} \sim \mathbf{y}$ if the consumer is indifferent between $\mathbf{x}$ and $\mathbf{y}$, i.e., she is just as happy with $\mathbf{x}$ as she is with $\mathbf{y}$.

     OR

   - $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{x} \prec \mathbf{y}$ if the consumer either prefers $\mathbf{x}$ to $\mathbf{y}$ or prefers $\mathbf{y}$ to $\mathbf{x}$, i.e., she is happier with $\mathbf{x}$ than $\mathbf{y}$ or is happier with $\mathbf{y}$ than $\mathbf{x}$.

     This is called **strict preference.** Given the choice, the consumer would always choose one to the other.

   - $\mathbf{x} \succeq \mathbf{y}$ (or $\mathbf{x} \preceq \mathbf{y}$) if the consumer is either indifferent between $\mathbf{x}$ and $\mathbf{y}$ or strictly prefers $\mathbf{x}$ to $\mathbf{y}$ (alternatively the consumer is either indifferent between y and $\mathbf{x}$ or strictly prefers $\mathbf{y}$ to $\mathbf{x}$). This is called **weak preference**.

Ex: Indifference curves and better sets in a 2-dimensional consumption space.

C. Preferences are *always* assumed to satisfy three properties:

1. Complete: Either $\mathbf{x} \succeq \mathbf{y}$, $\mathbf{y} \succeq \mathbf{x}$ or both (in which case $\mathbf{x} \sim \mathbf{y}$).
2. Reflexive: $\mathbf{x} \succeq \mathbf{x}$
3. Transitive: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$ then $\mathbf{x} \succeq \mathbf{z}$.

Ex: Preference no-nos [Indifference curves that cross, ...]

D. Preferences are often assumed to have two extra properties (when someone talks about **well behaved preferences** this is what they mean).

1. (positive) Monotone: More is better: $x_i \geq y_i \;\; \forall i \implies \mathbf{x} \succeq \mathbf{y}$.
2. Convex: If $\mathbf{x} \sim \mathbf{y}$ then for any $0 \leq \alpha \leq 1$, $(\alpha x_1 + (1-\alpha y_1), \alpha x_2 + (1-\alpha y_2)) \succeq (x_1, x_2)$ (Mixtures are better)

Ex: Not well behaved preferences

II. Utility Functions

A. It seems pretty hard to model decision making using the preference theory we just talked about because we'd have to specify the preference order between every pair of bundles.

B. Representation — Luckily, if we assume that preferences (which are automatically complete, reflexive and transitive) are also continuous (a property awkward to define formally, but intuitively clear) and monotone, then it is known that we can represent those preferences by a continuous utility function. [Varian, p.97 sketches a proof.]

C. A **utility function** u assigns a real number $u(\mathbf{x})$ to each bundle $\mathbf{x}$.

D. The utility function u **represents** preferences $\succeq$ if

1. $\mathbf{x} \succ \mathbf{y}$ if and only if $u(\mathbf{x}) > u(\mathbf{y})$, and
2. $\mathbf{x} \sim \mathbf{y}$ if and only if $u(\mathbf{x}) = u(\mathbf{y})$.

E. The same preferences can be represented by several different utility functions.

1. Any monotone increasing transformation v of a utility function u represents the same preferences as u, i.e. you can use either one, whichever is more convenient.
2. No harm in choosing a smooth (continuously differentiable) utility function.

F. The partial derivative of a utility function is its **marginal utility**: $mu_i \equiv \frac{\partial u}{\partial x_i}$. It depends on the choice of a utility function, but, more importantly...

G. The **marginal rate of substitution** between two goods i and j is $MRS_{ij} \equiv -mu_j/mu_i \equiv -\frac{\partial u}{\partial x_j}/\frac{\partial u}{\partial x_i}$.

1. MRS_{ij} is the consumer's trade-off rate between those two goods. Think of it as the number of micro-units of i that the consumer is just willing accept to give up a micro-unit of j.

2. MRS_{ij} is invariant to the choice of utility function to represent given preferences!

3. MRS_{21} is just the slope of the indifference curve in (x_1, x_2) space.

4. In higher dimensions, there is an indifference hypersurface, and there MRS_{ij} is the slope of that surface in the $j - i$ plane.

Ex: Perfect substitutes: $u(x_1, x_2) = x_1 + cx_2$. Then $MRS_{12}(x_1, x_2) = c > 0$; the tradeoff rate is constant.

Ex: Cobb-Douglas utility: $u(x_1, x_2) = \ln x_1 + c \ln x_2$. Then $MRS_{12}(x_1, x_2) = \frac{cx_1}{x_2} > 0$. Convex, Inada. What can we say about $v(x_1, x_2) = \exp(u(x_1, x_2))$?

Ex: CES utility: $u(x_1, x_2) = \frac{1}{\rho} \ln(x_1^\rho + cx_2^\rho)$, where $\rho \in (-\infty, 1]$. Can show that this nests the previous two cases, which correspond respectively to $\rho = 1, 0$. Generalizes directly to more than 2 goods. Very useful in applied work.

Ex: Quasilinear utility: $U(x_0, x_1) = x_0 + g(x_1)$. Think of good 0 as money, or purchasing power. $MRS_{ij}(x_0, x_1) = g'(x_1)$. Very very useful in applied work.

III. The Direct Consumer Problem and (Marshallian) Demand

A. Putting this machinery to use, we can model choice and figure out where demand comes from.

B. Let me follow textbooks for the next hour. Later I will show you a streamlined approach I developed with Jozsef Sakovics (with roots in Marshall).

C. Suppose that the consumer is constrained by available income and by the prices of the goods.

1. Let m be the available money to spend, while p_1 the price of good 1, p_2 the price of good 2 etc.

2. Then the **budget constraint** is

$$m \geq \mathbf{p} \cdot \mathbf{x} = p_1 x_1 + p_2 x_2 + \ldots \tag{1}$$

(i.e. you can't spend more than you have)

3. If she has strictly monotone preferences, a consumer will spend all of her money — putting money in a savings account could count as one of the x_i's, and m as such brings no utility. So we can safely assume that equation (1) holds as an equality:

$m = \mathbf{p} \cdot \mathbf{x} = p_1 x_1 + p_2 x_2 + \ldots$

Ex: Budget constraints and budget sets.

D. Then we have the following constrained optimization problem.

$\max u(x_1, x_2, \ldots)$ s.t. $m = p_1 x_1 + p_2 x_2 + \ldots$

1. We form the Lagrangian

$\mathcal{L} = u(x_1, x_2, \ldots) + \lambda(m - \mathbf{p} \cdot \mathbf{x})$

2. Differentiating, we get first order conditions:

a. $\frac{\partial u}{\partial x_1} - \lambda p_1 = 0$

b. $\frac{\partial u}{\partial x_2} - \lambda p_2 = 0$

c. ...

3. Simultaneously solving this FOC we can get the exact consumption decisions, assuming that the SOC holds (as well as strong monotonicity, smooth indifference curves and an interior optimum).

4. But even in this abstract form we get a property of optimal choice that may look familiar:

$$p_i/p_j = \frac{\partial u}{\partial x_i} / \frac{\partial u}{\partial x_j} = MRS_{ji} \tag{2}$$

5. The indifference curve will be tangent to the budget line – their slopes ($-$ price ratio and the MRS) will be equal.

Ex: An example with Cobb-Douglas preferences.

Ex: Direct consumer problem with indifference curves.

Ex: Corner solutions. Non-convex prefs. non-monotone prefs.

E. Solving this problem (which we will call the *direct consumer problem*) for x_i gives us the quantity demanded for good i as a function of prices and income. (The solution will be unique if preferences are *strictly* convex.)

F. This is the individual's **Marshallian demand** curve, $x_i^*(\mathbf{p}, m)$.

1. If we hold the price of other goods and income constant, we get the old demand curve we studied in the last section: $x_i(p_i)$

Ex: Marshallian demand from Cobb-Douglas preferences. Obtain

$$\ln x_i = A + e_{ii} \ln p_i + e_{ij} \ln p_j + \eta_i \ln m \tag{3}$$

with $e_{ii} = -1, e_{ij} = 0, \eta_i = 1$, and $A = \ln$ expenditure share.

G. See Varian on **duality**, which is quite useful in empirical work on both supply and demand side, especially when expenditure (or cost) is better measured than physical quantities.

H. **Indirect utility** is $v(\mathbf{p}, m) = [\max u(\mathbf{x}) \text{ s.t. } m = \mathbf{p} \cdot \mathbf{x}] = u(\mathbf{x}^*(\mathbf{p}, m))$.

I. Dual problem to utility max is expenditure min:

$$\min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \text{ s.t. } u(\mathbf{x}) \geq u_o. \tag{4}$$

J. Solution to (3) is called expenditure function, denoted $e(\mathbf{p}, u_o)$, and argmax's are called Hicksian demands, denoted $h_i(\mathbf{p}, u_o)$.

Ex: indirect utility, expenditure and Hicksian demand from Cobb-Douglas preferences.

IV. see Varian for a bunch of general properties and identities, some of which can help in applied work.

E.g., $h_i(\mathbf{p}, u_o) = x_i^*(\mathbf{p}, e(\mathbf{p}, u_o))$, interpreted as compensated demand, as we will see shortly.

V. Income Changes: $\frac{\partial x_i}{\partial m}$

A. We just saw that m affects the quantity demanded. What happens when income (actually, expenditure) changes? Of course, it depends on the structure of preferences – we'll go through a few of the possible cases, drawing the Income Expansion Paths (IEPs or, better, EEPs):

B. **Normal goods** are goods that consumers consume more of as income increases

$\frac{\partial x_i}{\partial m} > 0$

C. **Inferior goods** are goods that consumers consume less of as income increases

$\frac{\partial x_i}{\partial m} < 0$

D. If preferences are **quasilinear**, then income doesn't affect your consumption of one of the goods. For some function g:

$u(x_0, x_1) = x_0 + g(x_1)$

1. All extra money gets spent on good 0 (i.e., kept as cash) and none on good 1, once enough x_1 has been purchased so that $g'(x_1) \leq p_1$.

E. If preferences are **homothetic**, then IEPs are straight lines, so the proportion of total consumption represented by each good remains the same regardless of income.

1. If m doubles, then consumption of each good doubles.
2. If you multiply m by t, then consumption of each good gets multiplied by t

Ex: Cobb-Douglas preferences are homothetic.

VI. Price Changes: $\frac{\partial x_i}{\partial p_i}$

A. We also know that the price of a good affects the quantity demanded. What happens when the price changes? Two effects work (sometimes in opposite directions) to change the quantity demanded.

1. **Substitution effects**: The rate at which consumers can trade off one good for another changes.
2. **Income effects**: The consumer's spending power (her real income) changes because goods are cheaper (if the price went down) or more expensive (if the price went up).

B. Income Effect

1. These effects are actually exactly like what we talked about in the last section.
2. As in the last section, the income effects of price changes can be either positive or negative depending on whether the price increase or decreases and whether the good is normal or inferior.

C. Substitution Effect

1. You can isolate the substitution effect by imagining what a consumer would choose if the price change happened and then we either gave the consumer some money (if the price increased) or took away some money (if the price decreased).
2. In other words we imagine what the consumer would choose if we compensated for the income effect, leaving only the substitution effect. There are several methods of substitution that "decompose" the substitution effect from the income effect; we'll just do one.
3. **Hicks decomposition**
    a. What if we give the consumer enough so that his optimal consumption after the price change, though different, was just as pleasing to him?
    b. The change in consumption due to the change in price *after* factoring this

compensation is a measure of the substitution effect.

Ex: Graphical example: roll and shift.

c. Hicksian Demand

- There is actually a special alternate demand curve (called **Hicksian demand**) that consists only of this Hicks decomposed demand, $h_i(\mathbf{p}, u)$.
- It is formally defined as the quantity of the good a consumer demands when choosing the cheapest (expenditure minimizing) bundle she could buy while maintaining a given level u of utility.
- Hicksian demand charts only changes in quantity demanded that are due to substitution effects.
- Hicksian demand is always inversely related to price.

D. **The Slutsky Equation**

1. This is the identity that formally shows how a price change ($\frac{\partial x_i}{\partial p_i}$) consists of both an income and substitution effect:
2. start with the identity $h_i(\mathbf{p}, u) = x_i^*(\mathbf{p}, e(\mathbf{p}, u))$.

   Differentiate both sides with respect to p_i and rearrange terms to get the Slutsky equation

$$\frac{\partial x_i(\mathbf{p}, m)}{\partial p_i} = \frac{\partial h_i(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_i} - \frac{\partial x_i^*}{\partial m} x_i^*(\mathbf{p}, m). \tag{5}$$

3. There are two terms on the right hand side of the Slutsky equation:

   a. $\frac{\partial h_i(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_i}$ is the substitution effect. It is actually the derivative of Hicksian demand evaluated at $u =$ the maximum level of utility achievable given the prices and income (and so we can substitute using the indirect utility function $u = v(\mathbf{p}, m)$).

b. $\frac{\partial x_i}{\partial m} x_i(\mathbf{p}, m)$ is the income effect. It is just the effect of income on demand described in the last section, weighted by the quantity currently demanded.

E. **The Law of Demand**

1. If a good is normal, then an increase in its price will result in a decrease in the quantity demanded (e.g. demand slopes downward, $\frac{\partial x_i(\mathbf{p},m)}{\partial p_i} < 0$)

2. Why?

   a. The substitution effect is always negative (see the stuff on Hicksian demand above).

   b. If the good is normal $\frac{\partial x_i}{\partial m} x_i(\mathbf{p}, m) > 0$ meaning $-\frac{\partial x_i}{\partial m} x_i(\mathbf{p}, m) < 0$. Since $x_i(\mathbf{p}, m)$ can't be negative, the right hand side is overall negative.

F. Giffen Goods

1. Usually substitution effects are greater than income effects and so, even if goods aren't normal, demand still slopes downward.

2. However, it is theoretically possible for a good to be so inferior that they overcome the substitution effect and, in places demand slopes up.

3. These theoretical constructs are called **Giffen goods**.

Ex: Graphical example of a Giffen good.

VII. Additional Tools

A. **Elasticity form of the Slutsky equation**: Multiply both sides of equation (4) by $\frac{p_i}{x_i}$, also multiply the last term by $\frac{m}{m}$, and simplify, using the usual expressions for elasticities. Writing $s_i = \frac{p_i x_i}{m}$ as the expenditure share of good i we get

$$\epsilon_i = \epsilon_i^h - s_i \epsilon_m. \tag{6}$$

where ϵ_i is the usual own-price elasticity, ϵ_i^h is the Hicksian (or income-compensated or pure substitution) elasticity, and ϵ_m is income elasticity.

This equation is useful because you may have estimates of some of these elasticities and want to know others, which you can get from equation (5).

B. **Roy's Identity** shows how you can recover the ordinary (Marshallian) demand function from the indirect utility function $v(\mathbf{p}, m)$. It says

$$x_i^*(\mathbf{p}, m) = \frac{-\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}} \tag{7}$$

See Varian p. 106-8 for three (!) proofs and some general remarks.

C. **Shepherd's Lemma** tells us how to recover the Hicksian (income compensated) demand functions from the expenditure function. We'll later see that an analogous expression (by the same name) on the supply side is very very useful. Anyway, the demand side version is

$$h_i^*(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}. \tag{8}$$

D. **Elasticities identity.** There is a useful formula that connects the values of the various demand elasticities. It is based on a formula discovered by the mathematician Leonhard Euler (1707-1783). The formula applies to homogeneous functions, i.e., functions that for all $\mathbf{y}$ and all $a > 0$ satisfy the identity

$$f(a\mathbf{y}) = a^k f(\mathbf{y}) \tag{9}$$

for some nonnegative integer k. For example, Cobb-Douglas functions are homogeneous of degree k = sum of exponents. Euler showed that any such function can be written

$$y_1 \frac{\partial f}{\partial y_1} + ... + y_n \frac{\partial f}{\partial y_n} = kf(y_1, ..., y_n). \tag{10}$$

It is easy to verify (9) if you wish, simply by totally differentiating both sides of equation (8) with respect to a and evaluating at $a = 1$.

Note that Marshallian demand functions $x_i^*(p_1, ..., p_n, m)$ are homogeneous of degree 0: doubling all prices and income, for example, will have no effect on the optimal quantities of goods purchased, as you can see by inspecting the budget constraint. Applying (9) with $k = 0$ to $x_i^*(p_1, ..., p_n, m)$, we get

$$p_1 \frac{\partial x_i^*}{\partial p_1} + ... + p_n \frac{\partial x_i^*}{\partial p_n} + m \frac{\partial x_i^*}{\partial m} = 0. \tag{11}$$

Denote price elasticities by $e_{ij} = \frac{\partial x_i^*}{\partial p_j} \frac{p_j}{q_i}$ — if $i = j$ this is called the own-price, otherwise the cross-price elasticity of demand — and income elasticity by $\eta_i = \frac{\partial x_i^*}{\partial m} \frac{m}{q_i}$. Now divide all terms in (10) by the quantity demanded $q_i = x_i^*$ to obtain the desired identity

$$e_{i1} + ... + e_{in} + \eta_i = 0, \quad i = 1, ..., n. \tag{12}$$

That is, for any good i, the sum of its income elasticity and own price and all cross price elasticities are zero! You can impose this constraint directly on estimated log-linear demand functions to obtain more accurate results. Another way to think of it is that own price elasticity is interesting to the extent that it differs from -1.0, cross price from from 0 and income elasticity from 1.0.

VIII. Corners and notches. These notes emphasized interior solutions to the standard (or dual) consumer choice problem. Of course, a choke price p_i^c represents a point above which $\mathbf{x}^*$ is at the $x_i = 0$ corner (or face).

Sometimes additional constraints on consumer choice can be analyzed using only slight modifications. For example, the opportunity set is a natural modification of the budget set when there is a ceiling or floor on the amount purchased, or a subsidy or tax or matching grant.

Ex: Opportunity set with rationing (ceiling).