Decision making outside the laboratory

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Abstract

A model of a real-life decision problem encompasses assumptions that frame the problem as accurate as possible in three dimensions: demarcation of the probability space, definition of the target probability and construction of the information structure. A real-life decision problem can be modeled in different ways, due to assuming different interpretations of these dimensions. Each specific model imposes a specific rational decision. As a result, different models may impose different, even contradictory, rational decisions, creating choice 'anomalies' and 'paradoxes'. This aspect of decision making in real-life situations is different from decision making in a laboratory experiment. A laboratory is a designed setting according to an experimenter's model of the decision problem, while for a real-life situation it is not always obvious what the design is (solving the problem is tantamount to modeling the problem). This distinction between a real-life situation and a laboratory has also consequences for a laboratory experiment. A subject in an experiment may initially have a different model of the task than the experimenter and thus possibly make apparently irrational decisions from the experimenter's model perspective. As a consequence a choice anomaly can be eliminated by learning what the experiment's model is.

Introduction

A real-life decision problem can be modeled in various ways. Each model imposes a specific rational decision. The decision's rationality depends on the particular specification of the model. As a result, different models may impose different, even contradictory, rational decisions. There is no rationality beyond a model of the problem at hand.

A model of the real-life decision problem entails a formal description of the problem setting and a formal description of the problem itself. For probabilistic decision models this presupposes drawing on assumptions that frame the problem as accurate as possible in three dimensions: demarcation of the probability space, definition of the target probability and construction of the information structure.

Formal framing, however, is not an obvious task, and one can have different interpretations of what the problem entails, in particular in the case of a real-life problem. Moreover, there is no higher authority that decides which models is the most adequate formalization of a real-life problem. Experts may differ on interpretations, and so can come up with different solutions of how to act rationally.

This latter aspect of decision making in real-life situations is different from decision making in a laboratory, where most of today's experiments are run. A laboratory is a designed setting according to an experimenter's model of the decision problem. For the laboratory case, the experimenter is the expert who knows the accurate model of the problem (and when designing the experiment has verified whether this model is indeed accurate).

Another important difference between a real-life situation and a laboratory setting is that the latter is a 'small world' in Savage's (1954) terminology. In a small world the three above mentioned dimensions are known: the probability space, the target probability and the information structure. In the real world these are open for different interpretations as will be shown in this paper.

This distinction between a real-life situation and a laboratory setting has also consequences for a laboratory experiment. Although we can assume that the experimenter has an accurate model of the experiment, one cannot simply assume this is also the case for the subjects who participate in this experiment. They may have a different model in mind when making their decisions and so making irrational decisions according to the experimenter's model but not from their own perspective.

This paper will discuss these issues for three cases, of which two are published elsewhere and so only will be discussed briefly. The third case is the notorious Monty Hall (three doors) paradox. But before we will discuss these three cases, it should be emphasized that the possibility of having several models of a problem is not the same as the framing effect. Framing effects refer to the possibility that alternative ways of posing an *identical* problem may affect agent's choices (Camerer 1995). So, in the framing case, the problem is defined equally for different framings, but in the modeling case the definition of the problem may differ for each model.

1. Decision making in medicine

A classic example of a base rate fallacy is the Harvard Medical School Test. It appeared that, when a laboratory test result is given, physicians do not take account of the base rate, or pre-test probability, to reach a clinical decision. This Test, carried out by Casscells, Schoenberger and Graboys (1978), was a small survey to obtain some idea of how physicians interpret a laboratory result.

> We asked 20 house officers, 20 fourth-year medical students and 20 attending physicians, selected in 67 consecutive hallway encounters at four Harvard Medical School teaching hospitals, the following question: "If a test to detect a disease whose prevalence is 1/1000 has a false positive rate of 5 per cent, what is the chance that a person found to have a positive result actually has the disease, assuming that you know nothing about the person's symptoms or signs?" (Casscells, Schoenberger and Graboys 1978: 999)

Using Bayes' theorem, the 'correct' answer should be: 2%.[1] The result of this test was that only 11 of 60 participants gave this answer. The most common answer, given by 27, was 95%. The average of all answers was 55.9%, 'a 30-fold overestimation of disease likelihood' (p. 1000).

Four years later a similar result was published by David Eddy (1982). He discusses a more specific case of deciding whether to perform a biopsy on a woman who has a breast mass that might be malignant. Specifically, he studied how physicians process information about the results of a mammogram, an X-ray test used to diagnose breast cancer.

The prior probability, Pr(*ca*), 'the physician's subjective probability', that the breast mass is malignant is assumed to be 1%. To decide whether to perform a biopsy or not, the physician orders a mammogram and receives a report that in the radiologist's opinion the lesion is malignant. This is new information and the actions taken will depend on the physician's new estimate of the probability that the patient has cancer. This estimate also depends on what the physician will find about the accuracy of mammography. This accuracy is expressed by two figures: sensitivity, or true-positive rate $\Pr(+ \mid ca)$, and specificity, or true-negative rate $\Pr(- \mid benign)$. They are respectively 79.2% and 90.4%. Applying Bayes' theorem leads to the following estimate of the posterior probability: 7.7%.[2] In an informal sample taken by Eddy, most physicians (approximately 95 out of 100) estimated the posterior probability to be about 75%.

When Eddy asked the 'erring' physicians about this, they answered that they assumed that the probability of cancer given that the patient has a positive X-ray, $\Pr(ca \mid +)$, was approximately equal to the probability of a positive X-ray in a patient with cancer, $\Pr(+ \mid ca)$.

> The latter probability is the one measured in clinical research programs and is very familiar, but it is the former probability that is needed for clinical decision making. It seems that many if not most physicians confuse the two. (Eddy 1982: 254)

According to Eddy, it is not only the physicians who are erring, but a review of the medical literature on mammography reveals a 'strong tendency' to equate both probabilities, that is, to equate $\Pr(ca \mid +) = \Pr(+ \mid ca)$.

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[1] $\Pr(P \mid +) = \dfrac{\Pr(+ \mid P)\Pr(P)}{\Pr(+ \mid P)\Pr(P) + \Pr(+ \mid A)\Pr(A)} \approx 0.001/(0.001 + 0.05 \cdot 0.999) = 0.02$, where P: disease is present, A: disease is absent, and +: positive test result.

[2] $\Pr(ca \mid +) = \dfrac{\Pr(+ \mid ca)\Pr(ca)}{\Pr(+ \mid ca)\Pr(ca) + \Pr(+ \mid benign)\Pr(benign)}$
$= 0.792 \cdot 0.01/(0.792 \cdot 0.01 + 0.096 \cdot 0.99) = 7.7\%$

So, from the outside, it looks very much that physicians make irrational decisions, and if so this would be a real threat to the health care system. Seen from the inside, however, one arrives at a different view on physicians' decision making (Boumans 2008). What Casscells, Schoenberger and Graboys and Eddy overlooked was that a test is not like a drawing from an urn with coloured balls. Tests can be painful and/or risky, so a clinician only asks for a test after a well-considered evaluation of reliability, value and risk. The model for making this rational decision is based on Pauker and Kassirer (1980). This article describes a model that uses two thresholds to aid physicians in making clinical decisions:

1) a 'no treatment/test' threshold, T_t, which is the disease probability at which the expected utility of withholding treatment is the same as that of performing a test;
2) a 'test/treatment' threshold, T_{trx}, which is the disease probability at which the expected utility of performing is the same as that of administering treatment.

The decision not to treat, to test, or to treat is determined by pre-test disease probability and both thresholds, see figure 1. The best clinical decision for probabilities below the 'no treatment/test' threshold T_t is to refrain from treatment; for probabilities above the 'test/treatment' threshold T_{trx}, the best decision is to administer treatment. When the pre-test disease probability lies between the thresholds, the test result could change the probability of the disease enough to alter the decision, so the best decision would be to administer a test. So, for clinical decision making, estimates of base rates are crucial.

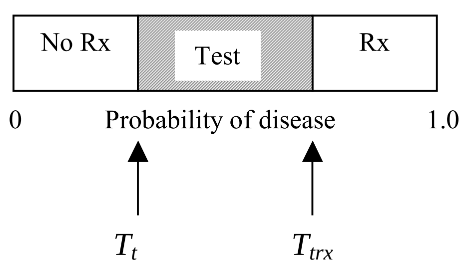

Figure 1 Test-treatment thresholds
(adapted from Pauker and Kassirer 1980: 1111)

From this threshold model test criteria can be inferred. First, according to Scherokman (1997), tests that do not change the probability of disease enough to cross the threshold probability T_{trx} are not useful and should not be ordered. This means that when the pre-test disease probability lies between the thresholds and we have a positive test result, the post-test disease probability should lie above the test/treatment probability: $\Pr(P \mid +) > T_{trx}$. A stronger test requirement is that it should be 'most informative'. A test is most informative when its predictive values, $\Pr(P \mid +)$ and $\Pr(A \mid -)$, are optimal. Generally, it is expected that a test is most informative when the pre-test probability of disease is between 40% and 60% (Scherokman 1997).

These demands on tests with respect to accuracy and applicability give new light on the interpretation by physicians of clinical laboratory results. First, let us take Eddy's figures:

$\Pr(+ \mid P) = 79.2\%$ and $\Pr(- \mid A) = 90.4\%$, and assume optimal conditions for using the test, $40\% < \Pr(P) < 60\%$, then $37.44\% < \Pr(+) < 51.36\%$,[3] and so

$$84.6\% < \Pr(P \mid +) < 92.5\%.$$

Most physicians estimated the post-test probability to be about 75%.

And secondly, the Harvard Medical School Test figure, $\Pr(+ \mid A) = 5\%$, leads even to higher post-test probabilities, when the prevalence is between 40% and 60% ($43\% < \Pr(+) < 62\%$):

$$93\% < \Pr(P \mid +) < 95\%.$$

Recall that most common answer, given by 27 of 60, was 95%.

Physicians are trained not to ask for diagnostic tests when prevalences are too small (or too large). Faced with test results they might have assumed automatically that the test was performed for the right conditions. So, they might have developed a heuristic to read the sensitivity and specificity as predictive values. Seen from this perspective, the physician's high estimates of the post-test probabilities in the case of the Harvard Medical School Test and in Eddy's test are not biased, but show rationality.

In a discussion of the Monty Hall problem (see section 4) in the *American Statistician*, one of the debaters used the Harvard Medical School test in his argumentation and arrived at a similar conclusion as here:

> This answer of 2% apparently assumes that everyone in the population, whether they have the disease or not, has an equally likely chance of receiving the test and that the false negative rate is zero […]. Neither of these assumptions is stated or clearly implied in the problem. Stating "you know nothing about the person's symptoms or signs" is not the same as stating that the test has an equal chance of being administered to people in the population, even if that was the intent of the phrase. The medical students and staff that were given this question would know full well that patients having a disease are almost more likely to have a test for their disease administered to them than the general public […]. The majority response of 95% is consistent with the assumption that persons having the test applied to them have a 50% chance of actually having the disease […]. Certainly these assumptions are more reasonable than those needed to support the 2% answer. Perhaps this illustration shows not that medically trained people don't understand probability but that some statisticians don't understand medicine. (Wendell 1992: 242)

In the case of the Harvard School Test (Casscells, Schoenberger and Grayboys 1978) and in the later test by Eddy (1982), it was simply assumed that both questioner and

[3] $\Pr(+) = \Pr(+ \mid P)\Pr(P) + \Pr(+ \mid A)\Pr(A)$

respondent had the same model in mind. However, both were trained differently and therefore had modelled the problem differently.

2. Existence of the hot hand

A second, nice but more briefly discussed here, example of how different models can lead to different decisions is the question of whether the hot hand in basketball exists. A basketball game is not a small world and features a complexity which cannot be easily brought into the laboratory (at least to my knowledge it has never been done). The problem here again is that there exist different models of the hot hand. Of those two will be discussed, one is claiming that the hot had is a cognitive illusion (Tversky and Gilovich 1989) and the other claims that it is okay to believe in it (Larkey, Smith and Kadane 1989).

According to Tversky and Gilovich (1989), the belief in the existence of the hot hand is caused by a misconception of randomness. To show this they define a hot hand or 'streak shooting' as a departure from coin tossing in two essential respects. First, the frequency of streaks (i.e., moderate or long runs of successive hits) must exceed what is expected by a chance process with a constant hit rate. Second, the probability of a hit should be greater following a hit that following a miss, yielding a positive serial correlation between the outcomes of successive shots:

1. # streaks hits > # streaks by chance process
2. P(H|H) > P(H|M), where H denotes a hit and M a miss

To investigate this, they first used the field-goal records of 9 major players of 48 home games of the Philadelphia 76ers during the 1980-1981 period. These data provided no evidence for the second claim. The second study of claims 1 and 2 were controlled experiments. Twenty-six members of the varsity teams at Cornell University were recruited to participate in shooting experiments, again with no evidence for claims 1 and 2.

Larkey, Smith and Kadane (1989) proposes a different conception of how observer's beliefs in streak shooting are based on National Basketball Association (NBA) player shooting performances, test this alternative conception on data from the 1987-1988 NBA season, and come to the conclusion that the hot hand exists. According to them, there is a problem with both Tversky and Gilovich's conceptualization and their data analyses for analyzing the origination, maintenance, and validity of beliefs about the streakiness of particular players. The shooting data that they analyze are in a very different form than the data usually available to observers of streak shooting. The data analyzed by Tversky and Gilovich consisted of isolated individual player shooting sequences by game. "The data available to observers including fans, players, and coaches for analysis are individual players' shooting efforts in the very complicated context of an actual game" (p. 24). The observers' focus on the unfolding sequence of shot opportunities rather than the activities of an individual player suggests, according to Larkey, Smith and Kadane, a very different model of player shooting activities than the one used by Tversky and Gilovich. Tversky and Gilovich's model ignores game context, how player's shooting activities interact with the activities of the other players. As a result Larkey, Smith and Kadane come up with a different hypothesis to be tested than the two claims for which Tversky and Gilovich found evidence:

> "The field goal shooting patterns of players with reputations for streakiness will differ from the patterns of reputationless players; a streak shooter will accomplish low-probability, highly noticeable and memorable events with greater frequency than reputationless players in the data set and with greater frequency than would be expected of him in the context of a game." (Larkey, Smith and Kadane 1989: 26)

To look not only at isolated player shooting sequences, they have to consider context which they hypothesized is what really enables observers of NBA basketball to differentiate streak shooters from the other players. Therefore context is defined as a sequence of 20 consecutive field goal attempts taken by all players in a game. In a game in which all players attempted N field goals, the total number of 20 shot contextual sequences is $N - 20 + 1$. Performance of a player is then expressed as a ratio of which the numerator is the number of times that a player accomplishes the sequence of a given length in context. The denominator is an expectation: the number of shot opportunities times the probability of a player taking T or more shots (where T is greater than or equal to the sequence length, L) in a 20 field goal context and of making r of L shot regardless of position in the T shots:

$$P_i^L \left\{ \sum_{j=L}^{m} \binom{m}{j} \gamma_i^j (1-\gamma_i)^{m-j} (j+1-L) \right\} S$$

where
P_i = probability of a hit given a shot by player i.
L = Length of run.
m = Number of possible shots (size of content) = 20
γ_i = Probability of player i taking a shot.
G = Number of games.

$$S = \sum_{g=1}^{G} (A_g - m + 1)$$

where
A_g = All field goal attempts in game g.

With the model to analyze the data, Larkey, Smith and Kadane found evidence for the existence of hot hands. In their conclusion they arrived at a comment on evidence outside the laboratory which has amore general validity:

> Attributing error in reasoning about chance processes requires at the outset that you know the correct model for observations about which subjects are reasoning. Before you can identify errors in reasoning and explain those errors as the product

> of a particular style of erroneous reasoning, you must first know the correct reasoning. It is much easier to know the correct model in an experimental setting than in a natural setting. In the experimental setting you can choose it. In a natural setting such as professional basketball you must first discover it. (Larkey, Smith and Kadane 1989: 30).

3. Experimental evidence for the Monty Hall choice anomaly

The third example is "one of the most persistent and best documented examples of irrational behavior" (Kluger and Friedman 2006: 1), and therefore chosen a subject for experimental research on economic decision making (Friedman 1998), the Monty Hall's Three Doors problem, or shortly the Monty Hall problem:

> Host Monty Hall of the once-popular TV game show "Let's Make A Deal" asked his final guest of the day to choose one of three doors (or curtains). One door led to the "grand prize" such as a new car and the other two doors led to "zonks" or worthless prizes such as goats. After the guest chose a door, Monty always opened one of the other two doors to reveal a zonk and always offered the guest the opportunity to switch her choice to the remaining unopened door. The stylized fact is that very few guests accepted the opportunity to switch. (Friedman 1998: 933)

According to Friedman, nonswitching is anomalous because in the game just described the probability of winning is 1/3 for nonswitchers and 2/3 for switchers. For this outcome he referred to Selvin (1975), Nalebuff (1987), vos Savant (1990) and Gillman (1992).

To test the three-door choice task, Friedman (1998) conducted the following laboratory experiment:

> One hundred four subjects were recruited […]. Each subject entered a quiet room and sat at a table opposite the conductor with no other subjects present. After reading the instructions, each subject completed a series of ten trials (or "periods"). In each trial, the subject initially picked one of three face-down cards. Then the conductor turned over a nonprize card that the subject did not choose and offered her the opportunity to switch to the other face-down card. Finally both face-down cards were turned over, one of which was the prize card. Each trial the subject earned 40 cents when her final choice turned out to be the prize-card and 10 cents otherwise. (Friedman 1998: 934)

Figure 2 shows the results. The switch rate started out extremely low (less than 10 percent in the first trial) and increased fairly steadily over the next several trials. But it stagnates at about 40 percent after the sixth trial and actually declines in the last trial to about 30 percent. The overall switch percentage is 28.7 percent.

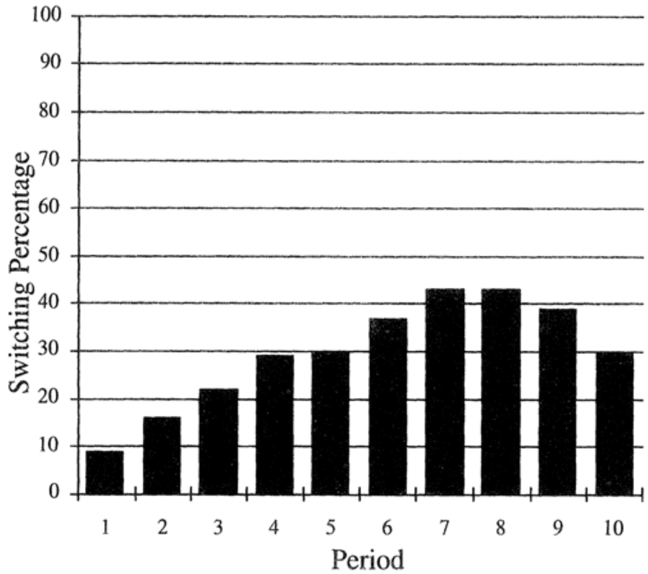

FIGURE 1. SWITCHING PERCENTAGE IN RUN1

Figure 2: Switch results of Monty Hall experiment
(Friedman 1998: 936)

This result was remarkable:

> The three-door task is now a true anomaly. The data are not hypothetical: more than 100 real people left lots of real money on the table in a controlled laboratory setting. The observed behavior is not "approximately" rational; most of the people most of the time made the irrational choice when the rational choice was just as convenient. And with ten trials, each laboratory subject had the opportunity to become familiar with the task and to find the rational decision. […]
> One conclusion seems clear. The three-door task now deserves a place among the leading choice anomalies. […] Indeed, I am not aware of any anomaly that has produced stronger departures from rationality in a controlled laboratory environment. (Friedman 1998: 935-936)

Friedman, however, could not explain this result, "the ultimate reasons for its strength remain unclear" (p. 936). So, the second part of his article was to deal with the following two questions: "Why are people so irrational? When (if ever) can people become more rational in tasks of this sort?"(p. 937). Therefore, he run the same baseline procedures, but now with additional alternative treatments (Run2), chosen to encourage more effective learning: higher incentives, track record, written advice and comparison of results, and now for 15 periods.

In particular, the written advice is of interest here. Before the first period of Run2, a subject in this treatment was handed a page with two paragraphs in random order. One paragraph recommended always switching and explained succinctly why switching improves the odds. The other paragraph, written in a similar style, recommended always remaining. The paragraphs were not presented separately to avoid confounding demand effects with learning through written advice. The text was as follows:

> *Advice treatment text*. - Please read the [following] two pieces of advice on how to earn money in this experiment. The pieces disagree on what you should do; you must make up your own mind.
>
> Advice S1: You have 1 chance in 3 of picking the Prize card initially. If you did pick it initially you will win the prize if you remain. You have 2 chances in 3 of not picking the Prize card initially. If you did not pick it initially you will win the Prize if you Switch. So, the overall chance of winning is 2 chances in 3 if you Switch and only 1 in 3 chances if you Remain. Therefore, you will do the best in the long run if you always Switch.
>
> Advice R1: The conductor may try to distract you by offers to Switch, but these offers are made for his/her own reasons and are not necessarily in your interest. When only 2 cards remain face down, you have at least 1 chance in 2 of picking the Prize card and Switching will not improve your chances. You should not let yourself be distracted. Therefore, you should always Remain with your initial choice and never Switch. (Friedman 1998: 942-942)

Even though the treatments increased the overall switch rate, the impact of these treatments was not big: it rose to a maximum of 55 percent in period 10 but trailed off somewhat in the last few periods. Nevertheless Friedman concluded that "every choice 'anomaly' can be greatly diminished or entirely eliminated in appropriately structured learning environments.

There is however, in my view, something unsatisfactory about this experimental result. In the history of the Monty Hall problem and the discussions that went with it (more about this below), two aspects are striking: there are only two camps and the dividing line is sharp: one camp, NS, believes very strongly that switching is irrelevant and the other camp, S, believes very strongly that switching is optimal. When people switch from camp NS to camp S, this is always permanent.

So, did Friedman's experiment show that people are "so irrational"? To investigate this question I studied the literature on this problem, how it is conceived and discussed, in mathematical and statistical journals. The reason for investigating this literature is that the participants in the discussions were experts in mathematical and probabilistic reasoning, and so cannot put aside as being 'naïve', 'intuitive' or 'subjective'.

4. The Monty Hall problem interpretation debate

Important for deciding which choice is rational is to figure out what model of the problem is. This is often not clear. One has to infer this from the verbal expression of the problem and the solution given as most optimal. In the literature one will find several different verbal expressions of the Monty Hall problem. Six different versions are given in the Appendix. The first version I could find is Selvin's (1975) phrasing of a problem in probability. It appears however that a structural similar problem, the so-called three-prisoners problem, was published in 1959 (see Appendix). The current standard version became only stabilized after vos Savant's discussion of it. Because most discussions of the Monty Hall problem refer to her wording of it we take hers as starting point:

> Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say #1, and the host, who knows what's behind the doors, opens another door, say #3, which has a goat. He says to you, "Do you want to pick door #2?" Is it to your advantage to switch your choice of doors? (vos Savant 1990: 13)

To better understand what kind of model she had in mind, her answer will also be given: "Yes; you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance" (vos Savant 1990, p. 13). She, subsequently, gave four explanations why this answer should be considered as the only correct one.

Before vos Savant's problem and solution will be modeled, let us first define a few symbols that will be used throughout this paper. Let C_i denote the event that the car is at door i, and H_j the event that the host opens door j (i, j = 1, 2, 3). Without any lost of generality we can say that the contestant chooses door 1. W_s denotes the event that you win the car if you switch. Let us define the following probabilities: $\pi_i = \Pr(C_i)$ and $p_{ij} = \Pr(H_j \mid C_i)$. Note that the probabilities p_{ij} represent the host's possible strategies.

Vos Savant's solution can now be modeled as follows. Assume that $\pi_i = 1/3$ and let us only assume the host will always open a door: $p_{i2} + p_{i3} = 1$.

The simplest proof of the vos Savant's answer is the following:

$$\begin{aligned}\Pr(W_s) &= \Pr(H_2 \textit{ and } C_3) + \Pr(H_3 \textit{ and } C_2) \\ &= \pi_3 \cdot p_{32} + \pi_2 \cdot p_{23} = \tfrac{1}{3}\cdot 1 + \tfrac{1}{3}\cdot 1 = \tfrac{2}{3}\end{aligned} \tag{1}$$

A more general result than vos Savant's can be obtained by permitting that the host might also open a door with a car behind it:

$$\begin{aligned}\Pr(W_s) &= \Pr(H_2 \textit{ and } C_2) + \Pr(H_2 \textit{ and } C_3) + \Pr(H_3 \textit{ and } C_2) + \Pr(H_3 \textit{ and } C_3) = \\ &= \tfrac{1}{3}\,(p_{22} + p_{32} + p_{23} + p_{33}) = \tfrac{2}{3}\end{aligned} \tag{2}$$

It should be noted that these results are independent of the strategy of the host. The same result would be obtained when the host opens only a door of which he knows it will not show a car: $p_{22} = p_{33} = 0$, and thus $p_{23} = p_{32} = 1$; or when he opens a door randomly $p_{22} = p_{23} = p_{32} = p_{33} = ½$.

Vos Savant's solution started a debate among mathematicians and statisticians about whether she was right or wrong. It is particularly this debate that shows that there is not simple one unique model of the Monty Hall, and thus one unique way to make a rational choice when faced with this problem. Only a few years after this debate, one specific model became *the* model of the Monty Hall problem dictating that switching is rational.

As a response to this debate caused by vos Savant's publication, Rao and Rao (1992) emphasized that it matters which target probability is chosen. They consider equation (1) as the solution to a so-called 'scenario 1': "The participant contacts a statistician as to the best course of action to be taken to maximize the probability of winning the automobile *before getting on to the stage*" (p. 90). 'Scenario 2' treats the situation in which "the participant is actually on the stage" (p. 91). For this scenario a *conditional* probability is worked out:

$$\Pr(C_1 \mid G_3) = \Pr(C_1 \mid C_2 \text{ or } C_1) = \Pr(C_1)/[\Pr(C_2) + \Pr(C_1)] = \tfrac{1}{3}/\tfrac{2}{3} = \tfrac{1}{2} \qquad (3)$$

where G_i denotes that a goat is shown behind door i. According to Rao and Rao, however, this conditional probability is not the correct interpretation of vos Savant's problem. They assumed that this interpretation was the one of those who criticized vos Savant's solution. If one is not explicit about the target probability, one may well choose advise S (in Friedman's experiment) when one assumes that $\Pr(Ws)$ is the target (equation 1 or 2) or advise NS, assuming that the conditional probability of equation 3 is the target probability.

Another important difference between scenario 1 and scenario 2 is whether the choice behavior of host is taken into account. Or in other words whether the host (H) is part of the probability space. This difference was also noted by Granberg and Brown (1995: 717): "Most people seem to ignore, or at least do not adequately take into account, the knowledge of the host as a cue". Their sample space does not include events H_j, but only C_i. In other words, they interpret the problem as having the sample space $\{C_1, C_2, C_3\}$. Most people seem to solve the following problem: What is the (conditional) probability of winning a car by switching doors when door 3 shows a goat? Any information about procedures to open doors or the specific role of the host is not taken into account.

Vos Savant, however, emphasized that the host's strategy is irrelevant for the target probability: "pure probability is the paradigm, and we published no significant reason to view the host as anything more than an agent of chance who always opens a losing door and offers the contestant the opportunity to switch" (vos Savant 1991: 347). So, in vos Savant's model, the host is in the sample space, but the host's strategy does not influence the target probability. See for this also my comment just below equation 2. Nevertheless, in the mathematical and statistical literature the idea took hold that the strategy of the host does actually matter. The publication that contributed particularly to this idea about the relevance of the host's strategy is (Morgan, Chaganty, Dahiya, and Doviak 1991a).

According to these authors, vos Savant was right about the answer to switch, but for the wrong reasons. They start with discussing six, in their view, false solutions. To see that these solutions are false, one should notice that the authors had changed the original vos Savant problem a tiny bit, but enough to make a difference:

> Suppose you're on a game show and given a choice of three doors. Behind one is a car; behind the others are goats. You pick door No. 1, and the host, who knows what's behind them, opens No. 3, which has a goat. He then asks if you want to pick No. 2. Should you switch? (Morgan et al. 1991a, p. 284)

Or in their own words:

> The player has chosen door 1, the host has then revealed a goat behind door 3, and the player is now offered the option to switch. (Morgan et al. 1991a, p. 284)

The crucial difference is that the host doesn't open any of the other doors with a goat behind it, which is indicated by using 'say', but a specific door, namely door No. 3. In other words, contrary to Rao and Rao (1992), Morgan et al. consider the conditional probability as the correct target probability of vos Savant's problem. This can be clearly seen from their discussion of the 'false solutions', in particular the first one:

> *Solution F1.* If, regardless of the host's action, the player's strategy is to never switch, she will obviously win the car ⅓ of the time. Hence the probability that she wins if she does switch is ⅔. (p. 284)

Their comments point clearly to the conditional probability as the, in their view, correct target probability: "F1 is a solution to the unconditional problem […] The distinction between the conditional and unconditional situations here seems to confound many, from whence much of the pedagogic and entertainment value is derived" (p. 285). Three other solutions (F2, F3 and F5) are also considered to be 'true' solutions but again to the unconditional problem, which is not, according to Morgan et al., the correct interpretation of the problem.

Solution F4 gives an answer to the conditional problem, but "the original sample space is incorrectly specified" with the result that the probability of winning by switching is ½.

> *Solution F4.* The original sample space and probabilities are as given in Solution F2. However, since door 3 has been shown to containing a goat, GGA is no longer possible. The remaining two outcomes form the conditional sample space, each having probability (⅓)/(1 - (⅓)) = ½. (Morgan et al. 1991a, p. 285)

If one defines the sample space as follows: $\{C_1, C_2\}$ the chance of winning a car is indeed ½.

Solution F6 gives, according to the authors, a correct specification but assumes falsely a certain strategy on the part of the host. They showed that a more general solution can be given. The general solution they give for the conditional problem is

$$\Pr(W_s \mid G_3) = \Pr(H_3 \textit{ and } C_2)/[\Pr(H_3 \textit{ and } C_1) + \Pr(H_3 \textit{ and } C_2)] = p_{23}/(p_{13} + p_{23}) \qquad (4)$$

The problem is then solved by the assignments of values for the p_{ij}'s, viewed as a quantification of the host's strategy. For example, in solution F6 the quantification: $p_{23} = 1$ and $p_{13} = ½$ was assumed.

Because the host never has the option of showing the car, which they call the "vos Savant scenario", the host's strategy can be specified as follows:

$$p_{12} = p,\; p_{13} = q = 1 - p,\; p_{22} = p_{33} = 0,\; p_{23} = p_{32} = 1 \tag{5}$$

So

$$\Pr(W_s \mid G_3) = 1/(1 + q) \tag{6}$$

and so they arrive at the solution that $\Pr(W_s \mid G_3) \geq ½$ for every q. In other words, when we have a vos Savant scenario, "we need not know, or make any assumption about, the host's strategy to state that the answer to the original question is yes. The player should switch, for she can do no worse and may well improve her chances" (p. 286).

In the last part of their article, the authors also give their solution to the unconditional problem, "for it evaluates the proportion of winners out of all games with the player following a switch strategy" (p. 286):

$$\Pr(W_s) = \Pr(W_s \mid G_2)\Pr(G_2) + \Pr(W_s \mid G_3)\Pr(G_3) =$$
$$\frac{p_{32}}{p_{12} + p_{32}}\frac{p_{12} + p_{32}}{3} + \frac{p_{23}}{p_{13} + p_{23}}\frac{p_{13} + p_{23}}{3} = \frac{p_{23} + p_{32}}{3}.$$

So, one cannot do better than ⅔ in the unconditional game, and the vos Savant scenario "maximizes the overall efficacy of the switch strategy" (p. 286).

If Morgan et al. are right then the unconditional probabilities depend on the host's strategy too, which contradicts the solutions of the unconditional probabilities discussed above. But they are not right. The first part of the above solution shows that the assumed sample space is $\{G_2, G_3\} = \{(H_2 \text{ and } C_1), (H_2 \text{ and } C_3), (H_3 \text{ and } C_1), (H_3 \text{ and } C_2)\}$. This sample space, implicitly, assumes the so-called 'vos Savant scenario', see (5), thus $p_{23} = p_{32} = 1$. Therefore, the unconditional probability is $\Pr(W_s) = ⅔$. So, even for these authors whom aimed at a "wider dissemination for instructional purposes of the pitfalls of conditional probability calculations and interpretations" (p. 284) could not help to add another pitfall to the literature. This pitfall arises whenever one doesn't make a distinction between the sample space and the host's strategy.

In a reply, vos Savant (1991) complaints about "strong attempts at misinterpretation" of the original question, and therefore gives the original once again. "Nearly all of my critics understood the intended scenario, and few raised questions of ambiguity" (vos Savant 1991, p. 347). In a rejoinder in the same rubric, however, Morgan et al. (1991b) maintain the claim that the problem should be considered as a conditional probability problem.

In 1992, the *American Mathematical Monthly* published a similar analysis of the three-doors problem by Gillman. Like Morgan et al., Gillman distinguishes between the conditional problem ('Game I'), $\Pr(W_s \mid G_3)$, and the unconditional problem ('Game II'), $\Pr(W_s)$. And the solution to the conditional problem "*depends on* [the host's] *selection strategy when he has this choice* – on the *probability q* that he will then open door #3" thereby explicitly stating that "Marilyn did not address this question" (Gillman 1992, p. 3): $\Pr(W_s \mid G_3) = 1/(1 + q)$, where $q = p_{13}$, cf. equation (6). Gillman, however, gives the correct solution for the unconditional problem: $\Pr(W_s) = 2/3$.

5. Information economics

The Monty Hall debate which started with vos Savant discussion shows that it is important to distinguish between the target probability and the probability space and that both should be made explicit. The importance of taking account of this distinction between these two modeling dimensions will be shown in a paper by Chun (1999), where the Monty Hall problem was discussed from an information economics perspective.

In the information economics approach, the decision maker is concerned with comparing several information structures and choosing the best one that gives highest expected utility under the optimal decision strategy. The typical decision situation in which the decision-maker, who cannot observe the actual states of the world, assign prior probabilities based on his or her own information and beliefs. After observing the signals, the decision-maker chooses the act that gives the highest expected payoff. Thus, there are four essential components: The information structure $P = [p_{ij}]$ is expressed as an $m \times n$ matrix of probabilities p_{ij}. The probability p_{ij} denotes the conditional probability that, for the ith event, the jth signal will be displayed. The decision matrix $D = [d_{jk}]$ is a $n \times r$ matrix, where r is the number of possible acts and n the number of signals. The element d_{jk} describes the decision-maker's decision rule; that is, the probability that the kth act will be chosen when the jth signal is received. The $r \times m$ payoff matrix is represented as $V = [v_{ki}]$, where v_{ki} is the payoff associated with the kth action and the ith state of nature. Finally, let π_i be the prior probability associated with the ith event and let Π be a square matrix containing the elements π_i in its main diagonal and zeros elsewhere. According to the theory of information economics, the decision-maker's expected payoff z resulting from the combination of an information structure P, a decision strategy D, a payoff matrix V, and prior probabilities Π is shown to be $z = \mathbf{tr}(PDV\Pi)$, where **tr** represents the trace operator.

For the Monty Hall problem, the probabilities of the information structure, p_{ij}, are defined in the same way as above, $\Pr(H_j \mid C_i)$, where $\sum_{j=1}^{3} p_{ij} = 1$. From now on, we refer to the host as "he" and the player as "she". The player's decision matrix in the 'generalized game show problem' is expressed as

$$D = \begin{array}{r} \\ He\ \ No.1 \\ opens\ No.2 \\ door\ \ No.3 \end{array} \begin{array}{c} \textit{She switches to door} \\ \begin{array}{ccc} No.1 & No.2 & No.3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1-x & 0 & x \\ 1-y & y & 0 \end{bmatrix} \end{array} \tag{7}$$

where the variables x and y are either zero or one. The set of binary variables x and y in the decision matrix represents the player's strategy.

The player's payoff matrix in the generalized problem is

$$V = \begin{array}{r} \\ She\ \ \ No.1 \\ switches\ No.2 \\ to\,door\ \ No.3 \end{array} \begin{array}{c} \textit{The car is behind door} \\ \begin{array}{ccc} No.1 & No.2 & No.3 \end{array} \\ \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \end{array}$$

The prior probabilities, car is behind door i, are $\pi_i = 1/3$, for $i = 1, 2, 3$. Chun makes a distinction between switching to door 2 and 3, indicated by x and y. However, this distinction is irrelevant for the literature we discuss here. So, we assume $x = y$. As a result, the expected utilities under a player's strategy (x) can be expressed as:

$x = 0$ (no switching) $$z_0 = \frac{1}{3}(v_{11} + v_{12} + v_{13})$$

$x = 1$ (switching) $$z_1 = \frac{1}{3}\sum_{i=1}^{3}(p_{i1}v_{1i} + p_{i2}v_{3i} + p_{i3}v_{2i})$$

To compare this result with the previous ones, let us assume that the player chooses an act which maximizes the probability of winning the car. Then, the diagonal elements v_{ii} are 1 and every other element, v_{ij}, $i \neq j$, is zero in the payoff matrix V. The unconditional probability z_x of winning the car under a player's strategy (x) can now be expressed as follows

$$z_0 = \frac{1}{3}$$

$$z_1 = \frac{1}{3}(p_{11} + p_{32} + p_{23})$$

In other words, if the player refuses to switch, the probability of winning the car is $z_0 = ⅓$ regardless of the host's strategy p_{ij}. If the player switches to a different door, the probability of winning is dependent on the host's strategy represented by p_{11}, p_{32} and p_{23}. This is a remarkable result, which contradicts earlier results.

This contradictory result is due to Chun's conceptualization of what decision matrix D (equation 7) represents. He gives the following reasoning for the specific choice of the decision matrix:

> When the player has selected door No. 1, the game ends immediately if (a) the host promptly opens door No. 1, or (b) the host opens one of the two remaining doors and shows the car behind it. If the host opens a different door that does not have the prize behind it, then the player is offered a chance to switch to the remaining, unselected door. (Chun 1999, p. 46)

The misconception here is that the decision matrix cannot represent the game's ending, because it doesn't involve the possible positions of the car, situation (b). The player will not have a chance to decide anymore when the game ends (the host opened door 1 or a door with a car). The representation of the events for which the game ends can only be done by defining the sample space carefully. The general sample space of the generalized game show problem is: $\mathbf{C} \times \mathbf{H}$, where $\mathbf{C} = \{C_1, C_2, C_3\}$, and $\mathbf{H} = \{H_1, H_2, H_3\}$. From this sample space, however, we have to separate a subspace for which events the game ends: $\mathbf{End} = \{H_1, C_2 \textit{ and } H_2, C_3 \textit{ and } H_3\}$. So, the sample space for this decision problem is $(\mathbf{C} \times \mathbf{H})\backslash\mathbf{End} = \{H_2 \cap C_3, H_3 \cap C_2, H_2 \cap C_1, H_3 \cap C_1\}$. This is the sample space in which the decisions are made, and thus the space in which can be analyzed which decision is optimal or not. This space, however, induces the following host's strategies: $p_{12} + p_{13} = 1$, $p_{23} = p_{32} = 1$.

The general expression of the probability of winning a car is $z_x = \frac{1}{3}\mathbf{tr}(PD) = \frac{1}{3}\sum_{i,j=1}^{3} p_{ij} d_{ji}$. Because of some events (**End**) the game will end, we have to reduce the range of this summation: $z_x = \frac{1}{3}\sum_{i=1}^{3}\sum_{\substack{j=2 \\ j\neq i}}^{3} p_{ij} d_{ji}$. Thus the probability of winning a car by sticking to door 1 is the sum of probabilities of events for which $x = 0$:

$$z_0 = \frac{1}{3}(p_{12} + p_{13}) = \frac{1}{3}.$$

The probability of winning a car by switching is the sum of probabilities of events for which $x = 1$:

$$z_1 = \frac{1}{3}(p_{32} + p_{23}) = \frac{2}{3}$$

The conditional probability z^c of winning the car under the optimal strategy is shown by Chun (1999) to be $z^c = \frac{\mathbf{tr}(PBV\Pi)}{\mathbf{tr}(PE\Pi)}$. In this case the host has opened door No. 3, so the situation for having to end the game does not occur. The matrices B and E as given by Chun are:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-x & x & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and the conditional probabilities become:

$$z^c{}_0 = p_{13}/(p_{13} + p_{23})$$
$$z^c{}_1 = p_{23}/(p_{13} + p_{23}) \qquad \text{(cf. equation 4)}$$

6. Modeling of the Monty Hall problem

To model the Monty Hall problem, three levels should be distinguished: 1) the target probability, 2) the host's strategy expressed as conditional probabilities $p_{ij} = \Pr(H_j \mid C_i)$, and 3) the sample space.

From the literature discussed above one can distinguish three different target probabilities: $\Pr(W_s)$, $\Pr(W_s \mid G_3)$, and $\Pr(W_s \mid G_2 \text{ or } G_3)$. We have already arrived at the expression for $\Pr(W_s \mid G_3)$, namely equation (4). The various interpretations of the Monty hall problem, however, do not rule out the following conditional probability as target probability:

$$\begin{aligned} \Pr(W_s \mid G_3 \text{ or } G_2) &= \frac{\Pr(H_3 \cap C_2) + \Pr(H_2 \cap C_3)}{\Pr(H_3 \cap C_1) + \Pr(H_3 \cap C_2) + \Pr(H_2 \cap C_1) + \Pr(H_2 \cap C_3)} \\ &= \frac{p_{23} + p_{32}}{p_{13} + p_{23} + p_{12} + p_{32}} \end{aligned} \qquad (8)$$

where the intersection of set H_j and C_i is denoted by $H_j \cap C_i$.

We have seen that the choice of the sample space is crucial for the solution of the target probability. The general sample space for the Monty Hall problem is $\mathbf{C} \times \mathbf{H}$, but most models are based on subspaces of this general space. Each assumed subspace put restrictions on the quantification of the conditional probabilities p_{ij}.

Both dimensions of subspaces and target probabilities define a table (Table 1), which shows the various possible correct solutions. The numerals in parentheses refer to the equations in the main text where they were first formulated. This table still leaves open the possible specific strategies of the host. The subsequent two tables show the solutions of the two strategies that were discussed most often in the literature: the so-called 'vos Savant strategy': $p_{12} = p_{13} = ½$, $p_{23} = p_{32} = 1$ (Table 2), and the host as random agent: $p_{i1} = p_{i2} = p_{i3}$ (Table 3).

Conclusions

Various authors considered the Monty Hall paradox as exemplary for the kind of difficulty when modeling real-world problems in contrast with textbook problems, e.g.: "Textbook problems are often set in artificial situations that neither require nor inspire real-world thinking on the student's part" (Morgan et al 1991a, p. 284). Modeling a real-world problem presupposes drawing up assumptions that formalize the problem as accurate as possible. Modeling of a real-world problem is not an obvious task, and one can have different interpretations of what the problem entails.

> Outside the never-never land of textbooks, however, conditioning events are not handed out on silver platters. They have to be inferred, determined, extracted. In other words, real-life problems (or textbook problems purporting to describe real life) need to be *modeled* before they can be solved formally. And for the selection of an appropriate model (i.e., probability space), the way in which information is obtained (i.e. the statistical experiment) is crucial. (Bar-Hillel and Falk 1982: 121)

The Monty Hall problem shows that beside the probability space and information structure, the target probability should also be specified.

Bohl, Liberatore and Nydick (1995) discuss the importance of assumptions in problem solutions.

> We believe that the issue of clearly stating, then questioning, and finally agreeing on, one or more sets of appropriate assumptions is critical to the success of any modeling effort. "Buying in" to the assumptions is tantamount to buying in to the solution. In the long run, you cannot have the second without the first. (Bohl, Liberatore and Nydick 1995: 4)

They give the following "reasonable" set of assumptions, which they consider as consistent with the solution that switching increases the probability of winning to 2/3.

1. The game show host will always open a door that was not selected by the contestant.
2. The opened door will always reveal a goat.
3. If the contestant selects a car, the game show host will select one of the other two doors with equal probability.
4. The car is more valuable than the goats.
5. The position of the car will not change once the game begins.
6. The contestant cannot offer to sell the door to the host or to anyone else.
7. The game show does not have the option to offer the contestant money to purchase his or her door.
8. The game show host will always offer the opportunity to switch.
9. The game is not repeatable, that is, the contestant only has the opportunity to play the game once.

10. The initial location of the car was randomly determined prior to the start of the game. (Bohl et al. 1995, pp. 4-5)

As we have seen above, modeling the problem by making explicit the problem-solver's interpretation, that is, drawing up the assumptions the problem solver considers to be appropriate, is not sufficient for a "reasonable", i.e. intuitive, solution. The kind of assumptions, given by Bohl et al. above, in fact show that the decisions on the three model dimensions have already been made. The model tells us which additional assumptions (like those of Bohl et al.) are needed. Another model will require other additional assumptions. As has been shown in this paper, modeling a real-world problem entails first deciding the target probability, the sample space, and the way information is provided. In all three directions, assumptions about their properties have to be made explicit before arriving at a definite solution.

So, confronted for the first time with a verbally expressed real-life decision problem, it is often not clear what the target probability, sample space or information structure is, unless stated explicitly. Different interpretations of these three dimensions lead to different rational outcomes. So, a debate about which decision is most rational should be a debate about which model is considered to be the most accurate representation of the problem and its settings.

This conclusion has also implications for running experiments on probabilistic decision making. Friedman (1998: 941) asserts that "Every choice 'anomaly' can be greatly diminished or entirely eliminated in appropriately structured learning environments". People are "not hardwired to behave irrationally" (p. 941), the task may not be easy to learn. According to him, "there is no such thing as an anomaly in the traditional sense of stable behavior that is inconsistent with rationality. There are only pseudo-anomalies describing transient behavior before the learning process has been completed" (p. 941). As a consequence, absence of adequate learning opportunities leads to "pseudo-anomalies". I can only but agree with this assertion, but have a somewhat different view on what this learning entails. When being subject in an experiment, it is often not clear to this subject what the experiment's model is. So, the subject will have his or her own interpretation of the task, which may deviate from the experimenter's design. As a result, the subject will make decisions that within the experimental set-up look irrational. But if the experimenter makes the model assumptions explicit by explaining the subject what these are or by playing the game (like the card game in the Monty Hall problem) enough times to give the subject the opportunity to grasp in a more intuitive way the models assumptions (learning by doing) then, like all of us, in the end will all learn what the most rational choice is for this specific task. Some people take more time for seeing the model than others, so the amount of periods should be sufficient needed for an average subject to learn. It even may be the case that in Friedman's experiment, the increase of the switch rate in Run2 was simply caused by the increase of the number of periods.

APPENDIX: Alternative Phrasings of The Monty Hall Problem

Monty Hall Problem 1

It is “Let’s Make a Deal” – a famous TV show starring Monte Hall.

Monte Hall: One of the three boxes labeled A, B, and C contains the keys to that new 1975 Lincoln Continental. The other two are empty. If you choose the box containing the keys, you win the car.

Contestant: Gasp!

Monte Hall: Select one of these boxes.

Contestant: I’ll take box B.

Monte Hall: Now box A and box C are on the table and here is box B (contestant grips box B tightly). It is possible the car keys are in that box! I’ll give you \$100 for the box.

Contestant: No, thank you.

Monte Hall: how about \$200?

Contestant: No!

Audience: No!

Monte Hall: remember that the probability of your box containing the keys to the car is 1/3 and the probability of your box being empty is 2/3. I’ll give you \$500.

Audience: No!

Contestant: No, I think I’ll keep this box.

Monte Hall: I’ll do you a favor and open one of the remaining boxes on the table (he opens box A). It’s empty! (Audience: applause). Now either box C or your box B contains the car keys. Since there are two boxes left, the probability of your box containing the keys is now ½. I’ll give you \$1000 cash for your box.

WAIT!!! Is Monte right? The contestant knows at least one of the boxes on the table is empty. He now knows it was box A. Does this knowledge change his probability of having the box containing the keys from 1/3 to ½? One of the boxes on the table has to be empty. Has Monte done the contestant a favor by showing him which of the two boxes was empty? Is the probability of winning the car ½ or 1/3?

Contestant: I’ll trade you my box B for the box C on the table.

Monte Hall: That’s weird!!

(Selvin 1975, p. 67)

Monty Hall Problem 2

Monty Hall, a thoroughly honest game-show host, has randomly placed a car behind one of three closed doors. There is a goat behind each of the other two doors. "First you point to a door," he explains. "Then I'll open one of the other doors to reveal a goat. After I've shown you the goat, you make your final choice, and you win whatever is behind that door." You want the car very badly. You point to Door Number 1. Mr. Hall opens another door to show you a goat. Now there are two closed doors remaining, and you have to make your choice: Do you stick with Door 1? Or do you switch to the other door? Or doesn't it matter? (Tierney 1991, p. 20)

Monty Hall Problem 3

On a TV show, a new car is hidden behind one of three screens; the MC has the contestant select one screen. He then opens one of the two remaining screens, revealing no car, and asks the contestant whether he wishes to change his initial selection. Can the contestant increase his odds by doing so? (Barbeau 1991, p. 308)

Monty Hall Problem 4

The host of a game show invites a member of the audience on to the stage. There are three closed doors on the stage. Behind one of the doors, there is a automobile, and behind the other two there are goats. The participant is asked to choose a door and selects one of them. The host knows exactly what is behind each door. The host always throws open one of the two remaining unchosen doors, revealing a goat. The host now offers the participant one of the following options.

Option 1. Stick to the initial selection of the chosen door.

Option 2. Switch doors.

The participant will win the automobile if his final choice of door reveals the automobile. (Rao and Rao 1992, p. 89)

Monty Hall Problem 5

A TV host shows you three numbered doors, one hiding a car (all three equally likely) and the other two hiding goats. You get to pick a door, winning whatever is behind it. You choose door #1, say. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and invites you to switch your choice if you so wish. Assume he opens door #3. Should you switch to #2?
(Gillman 1992, p. 3)

Monty Hall Problem 6

A contestant in a game show is given a choice of three doors. Behind one is a car; behind each of the other two, a goat. She selects Door A. However, before the door is opened, the host opens Door C and reveals a goat. He then asks the contestant: “Do you want to switch your choice to Door B?” Is it to the advantage of the contestant (who wants the car) to switch? (Barbeau 1993, pp. 149-150)

The Three-Prisoners Problem.

This latter problem is, like the Monty Hall problem, not so easy and unambiguously stated, so several versions exist (Bar-Hillel and Falk 1982: 118; Barbeau 1993: 150; Gardner 1959: 180, 182; Mosteller 1987: 4) of which the shortest is:

Of three prisoners, *A*, *B* and *C*, two have been chosen by lot for execution. *A* asks the guard, “Which of *B* and *C* is to be executed? One of them will be, and you give me no information about myself in telling me which it is”. The guard finds this reasonable and says, “*C* is to be executed”. And now *A* reasons, “I know that *C* is to be executed; the other one is either *B* or myself, and so my chance of being executed is now only ½ instead of 2/3, as it was before”. Apparently the guard *has* given him information. (Barbeau 1991, p. 308)

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Table 1 General

	$\Pr(W_s)$	$\Pr(W_s \mid G_3\ or\ G_2)$	$\Pr(W_s \mid G_3)$
	$\frac{1}{3}\sum_{j=1}^{3}(p_{2j}+p_{3j})$	$\frac{p_{23}+p_{32}}{p_{13}+p_{23}+p_{12}+p_{32}}$ (8)	$\frac{p_{23}}{p_{13}+p_{23}}$ (4)
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$. ($p_{12}+p_{13}=1$, $p_{23}=p_{32}=1$)	$\frac{2}{3}$ (1)	$\frac{2}{3}$	$\frac{1}{1+p_{13}}$ (6)
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_2 \cap C_2$, $H_3 \cap C_3$. ($p_{12}+p_{13}=1$, $p_{22}+p_{23}=1$, $p_{32}+p_{33}=1$)	$\frac{2}{3}$ (2)	$\frac{p_{23}+p_{32}}{1+p_{23}+p_{32}}$	$\frac{p_{23}}{p_{13}+p_{23}}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_1 \cap C_2$, $H_1 \cap C_3$. ($p_{12}+p_{13}=1$, $p_{21}+p_{23}=1$, $p_{31}+p_{32}=1$)	$\frac{2}{3}$	$\frac{p_{23}+p_{32}}{1+p_{23}+p_{32}}$	$\frac{p_{23}}{p_{13}+p_{23}}$
$H_3 \cap C_2$, $H_3 \cap C_1$. ($p_{23}=p_{13}=1$, $\pi_1=\pi_2=½$)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ ($F4$)
$H_3 \cap C_2$, $H_3 \cap C_1$, $H_3 \cap C_3$. ($p_{23}=p_{13}=p_{33}=1$)	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
C_1, C_2, C_3	$\pi_2+\pi_3=\frac{2}{3}$	$\frac{\pi_2+\pi_3}{2\pi_1+\pi_2+\pi_3}=\frac{1}{2}$	$\frac{\pi_2}{\pi_1+\pi_2}=\frac{1}{2}$ (3)

Table 2. vos Savant scenario

	$\Pr(W_s)$	$\Pr(W_s \mid G_3\ or\ G_2)$	$\Pr(W_s \mid G_3)$
H $\times$ **C**	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$.	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_2 \cap C_2$, $H_3 \cap C_3$.	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_1 \cap C_2$, $H_1 \cap C_3$.	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$H_3 \cap C_2$, $H_3 \cap C_1$.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$H_3 \cap C_2$, $H_3 \cap C_1$, $H_3 \cap C_3$.	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
C_1, C_2, C_3	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 3. Random Agent

	$\Pr(W_s)$	$\Pr(W_s \mid G_3 \; or \; G_2)$	$\Pr(W_s \mid G_3)$
H × C	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$.	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_2 \cap C_2$, $H_3 \cap C_3$.	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$H_2 \cap C_3$, $H_3 \cap C_2$, $H_2 \cap C_1$, $H_3 \cap C_1$, $H_1 \cap C_2$, $H_1 \cap C_3$.	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$H_3 \cap C_2$, $H_3 \cap C_1$.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$H_3 \cap C_2$, $H_3 \cap C_1$, $H_3 \cap C_3$.	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
C_1, C_2, C_3	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$