Game Theory Midterm Answer Key

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1

Sectoring the simplex for payoff matrix

$$W = \begin{pmatrix} 0 & 3 & \text{-1} \\ 3 & 0 & \text{-1} \\ 1 & 1 & 0 \end{pmatrix}$$

A

The Delta functions are
$\Delta w_{1-2} = (1, -1, 0)\, W s^T$
$\Delta w_{1-2} = (3, -3, 0)\, s^T$
$\Delta w_{1-2} = 3s_2 - 3s_1$

$\Delta w_{1-3} = (1, 0, -1)\, W s^T$
$\Delta w_{1-3} = (-1, 2, -1)\, s^T$
$\Delta w_{1-3} = 2s_2 - s_1 - s_3$

$\Delta w_{2-3} = (0, 1, -1)\, W s^T$
$\Delta w_{2-3} = (2, -1, -1)\, s^T$
$\Delta w_{2-3} = 2s_1 - s_2 - s_3$

D

1-2 game: $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$
HD type

1-3 game: $\begin{pmatrix} 0 & \text{-1} \\ 1 & 0 \end{pmatrix}$

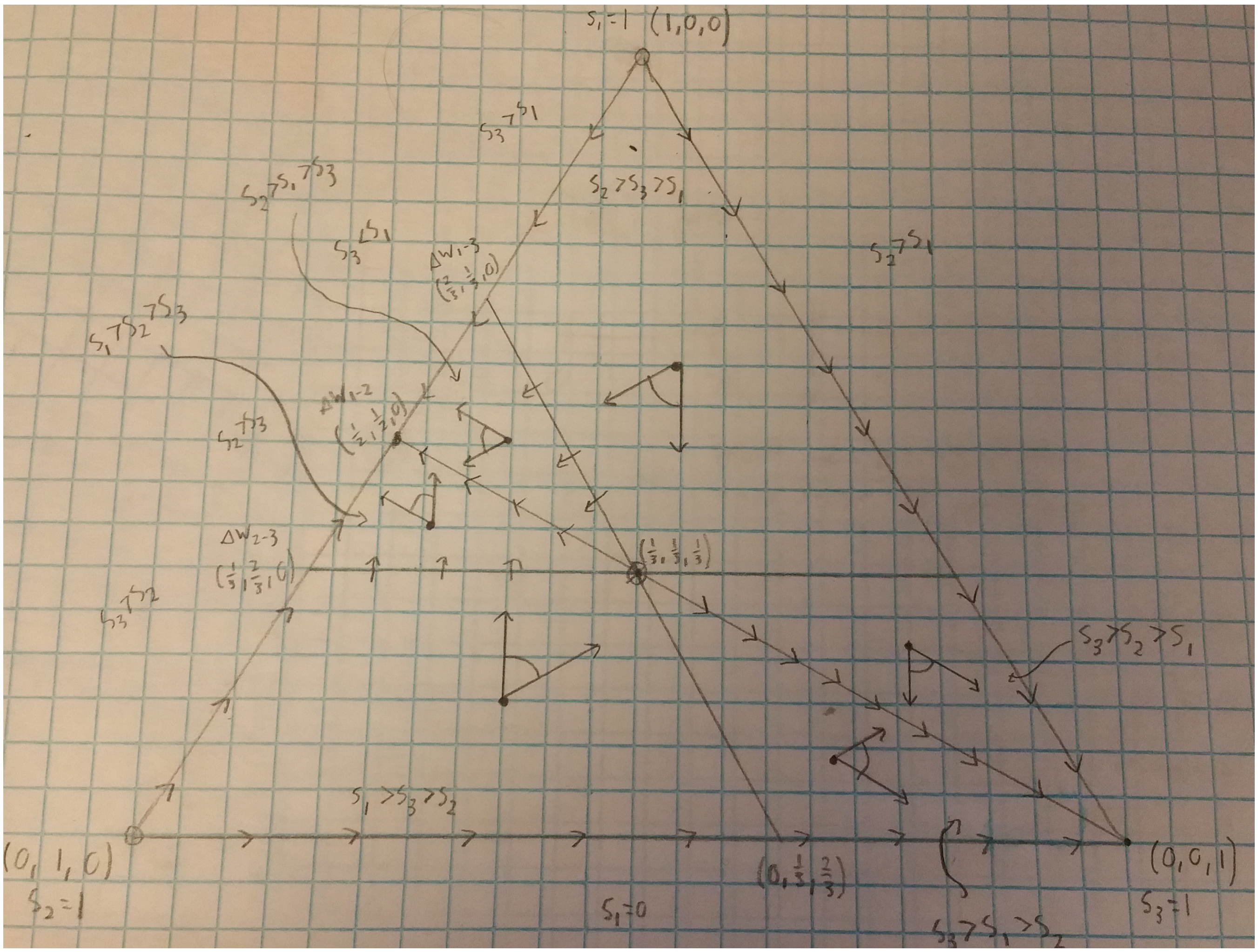

Figure 1: Sectoring: parts B, C, ...

DS type

2-3 game: $\begin{pmatrix} 0 & \text{-1} \\ 1 & 0 \end{pmatrix}$

DS type

E

As can be seen from Figure 1, $(1,0,0)$: Source
$(0,1,0)$: Source
$(0,0,1)$: Sink
$\left(\frac{1}{2},\frac{1}{2},0\right)$: Sink
$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$: Saddle

F

The Nash equilibria automatically include the sinks and the interior saddle:
$\left(\frac{1}{2},\frac{1}{2},0\right), (0,0,1)$, and $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.

Of these, the sinks are the only candidates for ESS, and we verify them as follows
$\pi(3,3) = 0 > \pi(1,3) = \pi(2,3) = -1$, so $(0,0,1)$ is also an ESS.

$\pi\left((\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},0)\right) = \frac{3}{2} > \pi\left((0,0,1),(\frac{1}{2},\frac{1}{2},0)\right) = 1$
$\pi\left((\frac{1+d}{2},\frac{1-d}{2},0),(\frac{1}{2},\frac{1}{2},0)\right) = \frac{3(1-d^2)}{2} < \pi\left((\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},0)\right) = \frac{3}{2}$ so $\left(\frac{1}{2},\frac{1}{2},0\right)$ is also an ESS.

G

The replicator equations are:
$\dot{s_1} = s_1\left(W_1 s^T - \bar{W} s^T\right)$
$\dot{s_1} = s_1\left((1,0,0)W s^T - sW s^T\right)$
$\dot{s_1} = s_1\left((1-s_1,-s_2,-s_3)W s^T\right)$
$\dot{s_1} = s_1\left((-3s_2-s_3, 3-3s_1-s_3 1-s_1, s_1-1+s_2)s^T\right)$
$\dot{s_1} = s_1\left(-3s_1s_2 - s_1s_3 + 3s_2 - 3s_1s_2 - s_2s_3 + s_1s_3 - s_3 + s_2s_3\right)$
$\dot{s_1} = -6s_1^2 s_2 + 3s_1s_2 - s_1s_3$

$\dot{s_2} = s_2\left((-s_1, 1-s_2, -s_3)W s^T\right)$
$\dot{s_2} = s_2\left((3-s_2-s_3, 3-3s_1-s_3, s_1-1+s_2)s^T\right) s^T$
$\dot{s_2} = s_2\left(3s_1 - 3s_1s_2 - s_1s_3 - 3s_1s_2 - s_2s_3 + s_1s_3 + s_2s_3 - s_3\right)$
$\dot{s_2} = 3s_1s_2 - 6s_1s_2^2 - s_2s_3$

$\dot{s_3} = s_3 \left((-s_1, -s_2, 1 - s_3)Ws^T\right)$
$\dot{s_3} = s_3 \left((1 - 3s_2 - s_3, 1 - 3s_1 - s_3, s_1 + s_2 - 1)s^T\right)$
$\dot{s_3} = s_3 \left(-3s_1s_2 + s_1 - s_1s_3 - 3s_1s_2 - s_2s_3 + s_1s_3 + s_2s_3 - s_3\right)$
$\dot{s_3} = -6s_1s_2s_3 + s_1s_3 + s_2s_3$

H

Time permitting I would use the eigenvalue method to test whether the steady states were stable:

1. Create the V matrix for the dynamic and use it to find the DV matrix $= ((\frac{dV_i}{ds_j}))$, which is then evaluated at the point in question s^* and multiplied by the projection matrix $P_0 = I - \frac{1}{n}1_{n\times n}$ to get the jacobian matrix.
2. Solve the jacobian for eigenvalues and drop a zero which will be in the list
3. If the largest real part of the eigenvalues is negative, it is a sink; if the smallest real part of the eigenvalues is positive, it is a source; and if there are positive and negative real parts of eigenvalues, it is a saddle.

2

a.

For a game with 1 population and 2 pure strategies a and b, the payoff matrix takes the form:

$$W = \begin{bmatrix} Waa & Wab \\ Wba & Wbb \end{bmatrix}$$

b.

For a game with 1 population and 3 strategies:

$$U = \begin{bmatrix} Uaa & Uab & Uac \\ Uba & Ubb & Ubc \\ Uca & Ucb & Ucc \end{bmatrix}$$

c.

For a game with two populations, 1 with 2 strategies a and b, and 1 population with 3 strategies c, d, and e, to specify the first population's payoff we use the matrix:

$$H = \begin{bmatrix} Hca & Hcb \\ Hda & Hdb \\ Hea & Heb \end{bmatrix}$$

and to specify the second population's payoff we use a similar 2 row x 3 column matrix, call it $Y = ((y_{ij}))$.

d.

Using the notation from part c, the payoff to players in the first population using possibly mixed strategy $x = (x_a, x_b)$ is $xW \cdot (s_a, s_b) + xH \cdot (s_c, s_d, s_e)$. For the second population using strategy $z = (z_a, z_b, z_c)$, the payoff is $zU \cdot (s_c, s_d, s_e) + zY \cdot (s_a, s_b)$.

e.

With a third population, the state space is a 4-dimensional object, the cartesian product of a square with a triangle. With no own-population effects, the payoffs can be written as stacked trimatrices (3 entries per cell), e.g., a stack of 3 2x2 multi-matrices. It's a bit of a mess, but is a little nicer using tensor notation (beyond the scope of this course).

3

This problem is about assortative matching.

a.

What is the implicit assumption in the previous problems about how strongly players using a given strategy interact with players who use different strategies? [4pts]

In many games we use the assumption that there are uniformly random encounters. E.g., Player i using strategy H will face sH (share of players using H) and 1 - sH (share of players using a different strategy).

b.

The text mentions three different ways to relax this assumption. Briefly discuss the advantages of each. Which (if any) seems most relevant to your group's project? [6 pts]

Sometimes we relax this assumption. Four ways to depart from random encounters are: Assortativity, adjustment matrix (using Hadamard product with the fitness matrix), Price Equation and Cellular Automata.

Assortativity: it specifies how often one sort of strategist encounters another. For example, in the Hawk-Dove game, $x = s_H$ shows the current fraction of H play in the population and $p(x) \in [0, 1]$ indicates the probability that a given H player will be matched with another H player. Likewise, $q(x) \in [0, 1]$ shows the probability that a given D player will be matched with a H player. A large p means that H players tend to hang out together, if $p(x) < x$, they tend to avoid each other. When $p(x) = 0$, H players will find only D players, when $p(x) = 1$, H players will only interact with other H players. The index of assortativity $a(x) = p(x) - q(x)$ captures this distortion in matching probability. This works for symmetric 2 x 2 games, but it is hard to generalize $a(x)$ to higher dimensions.

An n x n adjustment matrix A is used to capture twists in n-strategy symmetric games. You take the Hadamard product of A with the payoff matrix. If each entry of A is 1 there is no departure from simple random matching. Entries a_{ij} less (or greater) than 1 indicate that strategy i matches less (or more) often with strategy j than random.

Price Equation: it models group-mediated social interactions. Each individual belongs to some group, and interactions within the group are different than interactions outside the group.

Cellular Automaton: CA is a discrete spatial simulation. Each location (or "cell") has a current state and a set of neighbors. Each cell transitions to a new state each period according to well-defined rules involving the nearest neighbors. Therefore a player, instead of interacting directly with an entire population, interacts directly only with a set of nearest neighbors. Those interactions have a domino effect as all the neighbors interacts with their own neighbors as they interact with theirs and so on. If the grid is finite, its endpoints need special treatment because they have only one neighbor. The torus is widely used in CA simulations to avoid the boundary effects that arise in finite grids in two dimensions.