Part I. Problems

Problem 1

(a)

How many different action profiles can be observed in the first stage?
Suppose Player 1 has strategies {A, B}, Player 2 has strategies {C, D, E}. Then each action profile is a choice for each player, so 6 different action profiles can be observed in the first stage: {(A,C),(A,D),(A,E),(B,C),(B,D),(B,E)}.

(b)

How many different pure strategies does each player have at stage 2?
Since Player 1 has two different actions for each history, with 6 different action profiles from part (a), Player 1 has 2^6 pure strategies. Likewise, Player 2 has 3^6 pure strategies at stage 2.

(c)

How many different action histories can there be after 3 stages of play?
Since in each stages, 6 different action profiles can be observed, there can be $6^3 = 216$ different action histories after 3 stages of play.

(d)

How many different pure strategies does each player have at stage 4?
Based on the different action histories from part (c), Player 1 has 2^{216} pure strategies and Player 2 has 3^{216} pure strategies at stage 4.

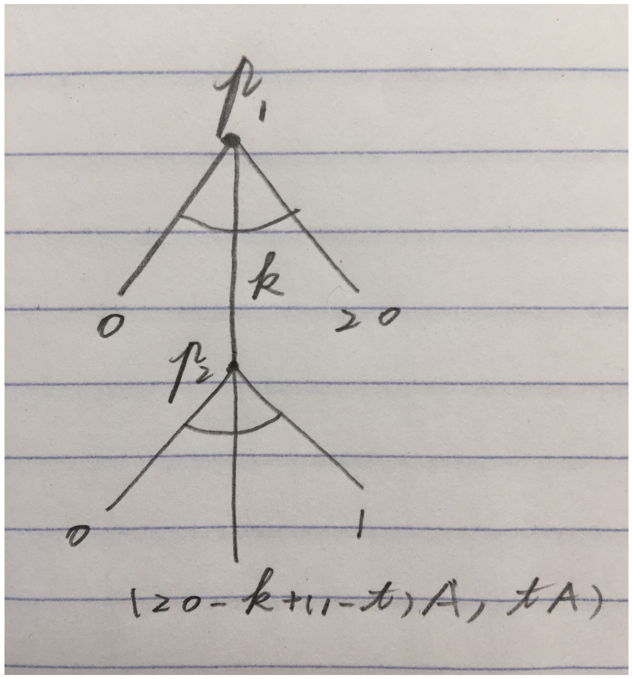
Problem 2

(a)

If $20 - k + (1-t)A \geq 20$, then a BR for player 1 is to invest, i.e. $k > 0$. So
$20 - k + (1-t)(1+r)k \geq 20 \iff$
$k[(1-t)(1+r) - 1] \geq 0 \iff$
$=> (1-t)(1+r) - 1 \geq 0 \iff$
$1 + r - t - tr - 1 \geq 0 \iff$
$t(1+r) \leq r$
$=> t \in [0, \frac{r}{1+r}]$

(b)

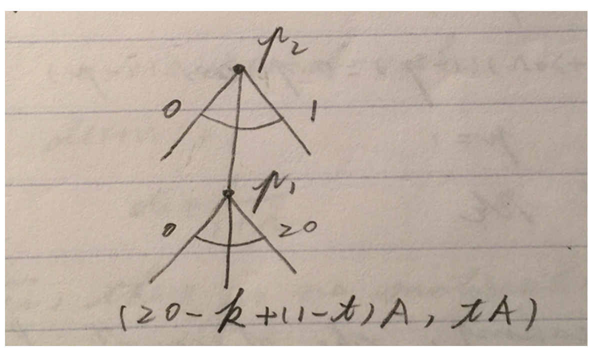
Government's payoff is $T = tA = t(1+r)k$, given that $k > 0$, government's best response is $t = 1$, since $\frac{\partial T}{\partial t} = (1+r)k > 0$.



(c)

According to the tree at the beginning and what we did in part b>, given $k > 0$, government's best response is $t = 1$, then we have $20 - k + (1-t)A = 20 - k < 20$, then investor's best response is $k = 0$ therefore, the unique SPNE is $(k, t) = (0, 1)$.

(d)



According to what we did in part a, there is a range of SPNE: $(k, t) = (k, \frac{r}{1+r})$ for any $k \in [0, 20]$.

(e)

Payoff sum is

$$
\begin{aligned}
u_1 + u_2 &= 20 - k + (1-t)A + tA \\
&= 20 - k + A \\
&= 20 - k + (1+r)k \\
&= 20 + rk
\end{aligned}
$$

where $r > 0, k \in [0, 20]$.
The sum is maximized when $k^* = 20$, which will only happen if $t \leq t^* = \frac{r}{1+r}$.

(g)

Generally d is determined by the market interest rate, the probability that the game continues, and the players' patience. In this case, d is might be affected by the rate of return r (relative to his time preference and the market interest rates), and by the probabilities that the government remains in power and that the investor's productivity stays at r (or higher).

(h)

The efficient stage game outcome is ($k = 20, t = \frac{r}{1+r}$). This is the largest payoff that the investor can make. Thus, the investor has no incentive to deviate from this strategy. The government can earn a higher payoff by defecting to play $t = 1$. Let Π_g denote the government's present value of the government's lifetime earnings and d as her discount factor:
Case 1: government plays efficient stage game outcome forever:

$$\Pi_{g1} = 20r + d20r + d^2 20r + \cdots = \sum_{t=0}^{\infty} d^t 20r = \frac{20r}{1-d}$$

Case 2: government defects:

$$\Pi_{g2} = 20(1+r) + d0 + d^2 0 + \cdots = 20(1+r)$$

In order to sustain the efficient stage game outcome, we need:

$$\tfrac{20r}{1-d} \geq 20(1+r) \Rightarrow d \geq \tfrac{1}{1+r}$$

(i)

The efficient outcome $k = 20, t \leq \frac{r}{1+r}$ can be supported as a NE of the repeated game if the government is sufficiently patient, i.e., if it values a stream of payments generated by moderate taxes ($t \leq \frac{r}{1+r}$ more than the one-time gain it could get from grabbing the asset ($t = 1$). Such patience is more likely when the government expects to stay in power for a long time. If the government shows signs of becoming impatient (discount factor less than $\frac{1}{1+r}$), then the investor will probably switch to $k = 0$ and leave.

Problem 3

(a)

	U	D
U	0, 0	$\underline{4}$, $\underline{1}$
D	$\underline{1}$, $\underline{4}$	2, 2

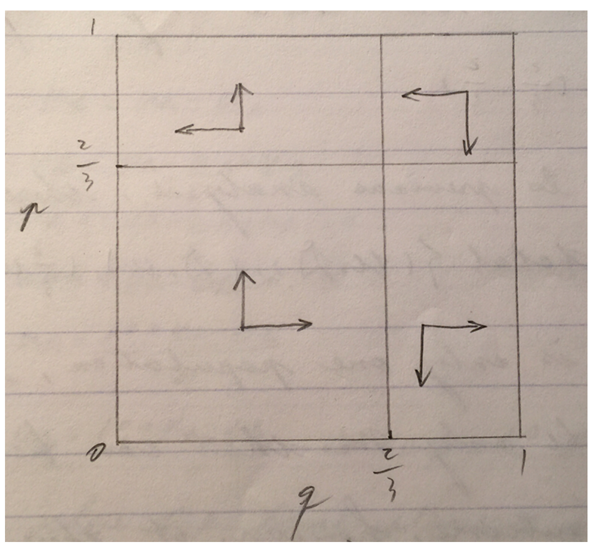
We can rewrite the game in streamlined form.

	p	1-p
U	0	4
D	1	2

$E\pi(U,p) = 0p + 4(1-p) = 4 - 4p$
$E\pi(D,p) = 1p + 2(1-p) = 2 - p$
$=> D(p) = 2 - 3p$
$D(p) = 0 => p^* = \frac{2}{3}$
Therefore, the unique evolutionary equilibrium is $p^* = \frac{2}{3}$, i.e. $(\frac{2}{3}U + \frac{1}{3}D, \frac{2}{3}U + \frac{1}{3}D)$

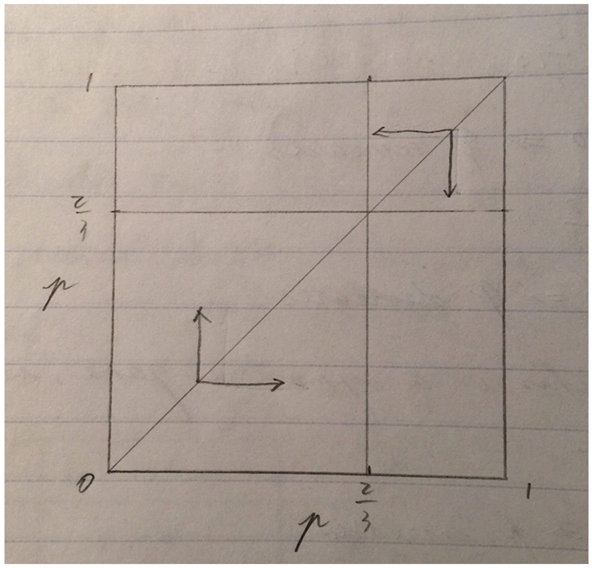
(b)

Set q to be the proportion of row player playing U.
If $p < \frac{2}{3}$
$D(p) > 0 =>$ q increases
if $p > \frac{2}{3}$
$D(p) < 0 =>$ q decreases
Since this is a symmetric game, we should have:
if $q < \frac{2}{3}$
$D(q) > 0 =>$ p increases;
if $q > \frac{2}{3}$
$D(q) < 0 =>$ p decreases.



As can be seen $(\frac{2}{3}, \frac{2}{3})$ is a saddle point which is not stable, and $(0,1),(1,0)$ are which are stable equilibria. Their basins of attraction are separated by the diagonal.

(c)



With only one population we must have $p = q$, so the feasible set in the square is the diagonal. As can be seen, there is a unique equilibrium on this set, which is $(\frac{2}{3}, \frac{2}{3})$, and it is stable.

(d)

According to previous analysis, there are three NE's in total $(U, D), (D, U), (\frac{2}{3}U + \frac{1}{3}D, \frac{2}{3}U + \frac{1}{3}D)$ If there is only one population, then the mixed NE is the only NE that can be a reasonable observed outcome. However, if there are two populations and no within population iteration, then one of the asymmetric pure NE's is the likely outcome after things settle down; and if the state initially is in one of the basins of attractions then its NE is the more likely eventual outcome.

Problem 4

(a)

Let u_1, u_2 be the gains for country A and country B from the trade, and $\underline{u}_1$, $\underline{u}_2$ are their threat points, so overall payoffs are $x_1 = u_1 + \underline{u}_1$ and $x_2 = u_2 + \underline{u}_2$. The problem is:

$$\max_{u_1,u_2}(x_1 - \underline{u}_1)(x_2 - \underline{u}_2) \quad \text{such that} \quad u_1 + u_2 = 10$$

A simple way to rewrite this is to set $u_i = x_i - \underline{u}_i$ and $u_2 = 10 - u_1$, so we want to solve

$$\max_{u_1} u_1(10 - u_1) = -{u_1}^2 + 10u_1$$

. Take the first order condition with respect to u_1:

$$-2u_1 + 10$$

so

$$u_1^* = u_2^* = 5 \quad \text{is the Nash Bargaining Solution}$$

Note that the SOC is $-2 < 0$ so we have a max.

(b)

The $Characteristic function$ for this three-player game is the mapping: $\nu : 2^{\{A,B,C\}} \to \mathcal{R}$ such that:

$$\nu(\varnothing) = \nu(A) = \nu(B) = \nu(C) = 0$$

$$\nu(\{A, B\}) = 10, \quad \nu(\{B, C\}) = 5, \quad \nu(\{A, C\}) = 15$$

$$\nu(\{A, B, C\}) = 15$$

(c)

According to MWG definition of convex: A game (I, ν) is convex if for every i the marginal contribution of i is larger to larger coalitions. Precisely, if $S \subset T$ and $i \in I \setminus T$, then:

$$\nu(S \cup i) - \nu(S) \leqslant \nu(T \cup i) - \nu(T)$$

In this three-player game defined in part b, take $S = \{A\}$, $T = \{A, B\}$, then $C \in I \setminus T$,

$$\nu(S \cup i) - \nu(S) = \nu(A \cup C) - \nu(A) = \nu(\{A, C\}) - \nu(A) = 15 - 0 = 15$$

$$\nu(T \cup i) - \nu(T) = \nu(\{A, B, C\}) - \nu(\{A, B\}) = 15 - 10 = 5$$

So in this case,

$$\nu(S \cup i) - \nu(S) > \nu(T \cup i) - \nu(T)$$

Thus the game is not convex.

(d)

Let u_3 be the gain for country C. The core is the set of all points satisfying:

$$u_1 + u_2 \geqslant 10 \quad \Rightarrow \quad 0 \leqslant u_3 \leqslant 5$$

$$u_1 + u_2 \geqslant 10 \quad \Rightarrow \quad 0 \leqslant u_2 \leqslant 0 \Rightarrow \quad u_2 = 0$$

$$u_2 + u_3 \geqslant 5 \quad \Rightarrow \quad 0 \leqslant u_1 \leqslant 10$$

$$u_1 + u_2 + u_3 = 15$$

So the core of the game is (10, 0, 5).

(e)

There are 8 different coalitions and 6 difference coalition formation sequences ρ: (A,B,C), (A,C,B), (B,A,C), (B,C,A), (C,A,B), (C,B,A). The Shapley value table is as follows:

Table 1: Shapley value

ρ	MC_1	MC_2	MC_3
ABC	0	10	5
ACB	0	0	15
BAC	10	0	5
BCA	10	0	5
CAB	15	0	0
CBA	10	5	0
$\sum$	45	15	30
ϕ_i	7.5	2.5	5

Problem 5

(a)

The payoff function for seller is:

$$\Pi_s(p, q) = (p - 10)q$$

The payoff function for buyer is:

$$\Pi_b(p, q) = \int_0^q (210 - p - q)dq = 210q - pq - \frac{q^2}{2}$$

(b)

To find the Pareto frontier in (p, q) space, taking the first order condition for payoff function of buyer, we have:

$$\frac{\partial \Pi_b(p,q)}{\partial q} = 210 - p - q = 0 \Rightarrow p = 210 - q$$

$$\frac{\partial \Pi_b(p,q)}{\partial p} = -q \leq 0$$

Similarly, for the seller, we have:

$$\frac{\partial \Pi_s(p,q)}{\partial q} = p - 10 \geq 0 \Rightarrow p \geq 10$$

$$\frac{\partial \Pi_s(p,q)}{\partial p} = q \geq 0$$

Based on part (c) and part (d), we learn that the two extreme points in Pareto frontier are $(110, 100), (10, 200)$, such that the Pareto frontier in (p,q) space is $p = 210 - q, 100 \leq q \leq 200$. Plugging this function into the payoff functions and we have:

$$(\Pi_b, \Pi_s) = (210q - pq - \frac{q^2}{2}, (p-10)q) = (\frac{q^2}{2}, 200q - q^2), \quad 100 \leq q \leq 200$$

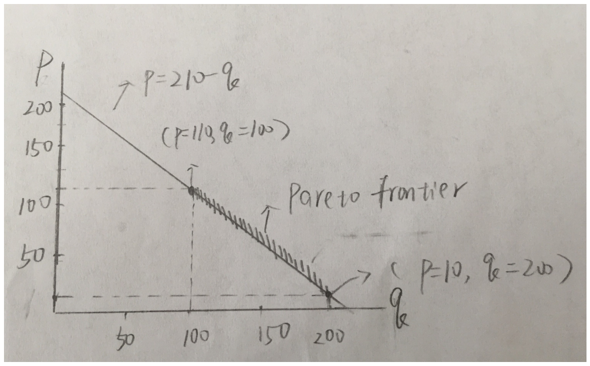
This implies the Pareto frontier in (Π_b, Π_s) space is:

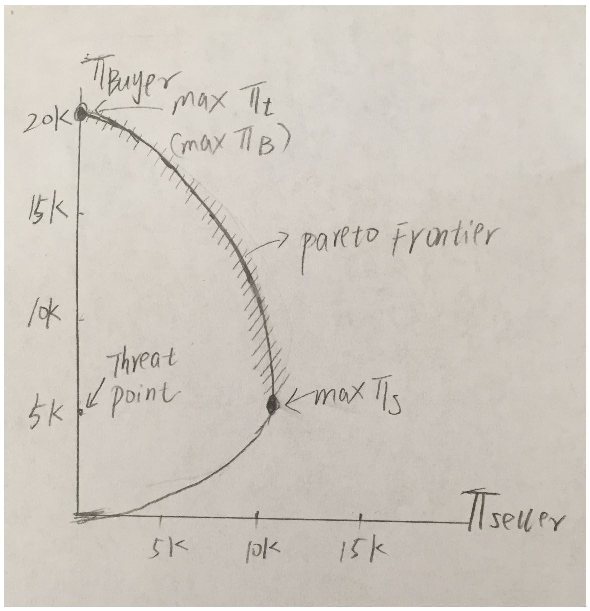
$$\Pi_s = 200\sqrt{2\Pi_b} - 2\Pi_b, \quad 5000 \leq \Pi_b \leq 20000$$

The sum of profits for the two player:

$$\Pi_t = 200q - \frac{q^2}{2}$$

F.O.C implies: $q^* = 200$
Such that, the point $(p^*, q^*) = (10, 200)$ (which corresponds to $(\Pi_b^*, \Pi_s^*) = (20000, 0)$) maximizes the sum of profits.
The following two diagrams show the feasible sets and Pareto frontiers in (p,q) space and in (Π_b, Π_s) space:





(c)

Suppose the seller chooses price, then for the buyer, given p, she maximizes her payoff so that marginal benefit equals to marginal cost, which implies $q = 210 - p$.
For the seller, knowing that the buyer will respond as $q = 210 - p$, she chooses p to maximize Π_s:

$$\max_p (p - 10)q = (p - 10)(210 - p)$$

F.O.C implies:

$$(210 - p) - (p - 10) = 0$$

which solves $p^* = 110$ and $q^* = 100$, the corresponding profits are $\Pi_b(110, 100) = 5000$ and $\Pi_s(110, 100) = 10000$

(d)

The buyer can maximize her utility by choosing the lowest price at which the seller is willing to sell. The seller will sell steel as long as $10 \leq p$. Such that, $p^* = 10$, then, buyer's problem becomes:

$$\max_q 200q - \frac{q^2}{2}$$

F.O.C implies: $q^* = 200$
Thus, $p^* = 10$ and $q^* = 200$,the corresponding profits are $\Pi_b(10, 200) = 20000$ and $\Pi_s(10, 200) = 0$

(e)

In this part, we use $(\underline{\Pi_b}, \underline{\Pi_s}) = (5000, 0)$ as the threat point because it is the minimum payoff they can guarantee players by not participating in a bargain. A rational buyer will never refuse buying from a rational seller, which leads to a minimum consumer surplus of 5000. Similarly, a rational seller will always sell and get non-negative profit.
With non-transferable utility (NTU):
Denote the threat point as$(\underline{\Pi_b}, \underline{\Pi_s}) = (5000, 0)$, the Nash bargaining problem is:

$$\max_{\Pi_b, \Pi_s} (\Pi_b - \underline{\Pi_b})(\Pi_s - \underline{\Pi_s})$$

$$s.t. \quad \Pi_s = 200\sqrt{2\Pi_b} - 2\Pi_b, \quad 5000 \leq \Pi_b \leq 20000$$

Rewrite the maximization problem as:

$$\max_{5000 \leq \Pi_b \leq 20000} (\Pi_b - 5000)(200\sqrt{2\Pi_b} - 2\Pi_b)$$

F.O.C implies:

$$(200\sqrt{2\Pi_b} - 2\Pi_b) + (\Pi_b - 5000)(100\sqrt{\frac{2}{\Pi_b}} - 2), \quad 5000 \leq \Pi_b \leq 20000$$

This is a cubic equation which could be solved analytically, but probably it is easier to use a spreadsheet. An approximate solution is $\Pi_b \approx 13090$, with corresponding $\Pi_s \approx 6180$.
With transferable utility (TU),the Nash bargaining problem is:

$$\max_{\Pi_b, \Pi_s} (\Pi_b - \underline{\Pi_b})(\Pi_s - \underline{\Pi_s})$$

$$s.t. \quad \Pi_s + \Pi_b = 20000$$

Rewrite the maximization problem as:

$$\max(\Pi_b - 5000)(20000 - \Pi_b)$$

F.O.C implies:

$$2\Pi_b = 25000$$

which solves $\Pi_b = 12500$ and the corresponding $\Pi_s = 7500$

Part II. Textbook problems

12.B.1

(a)

Denote $x(p^m)$ as the monopolist's demand function, $c(x(p^m))$ as the cost function. The problem of the monopolist is:

$$\max_{p^m}(p^m)(x(p^m)) - c(x(p^m))$$

The FOC with respect to p^m is:

$$x(p^m) + p^m x'(p^m) - c'(x(p^m))x'(p^m) = 0$$

$$\frac{p^m - c'(x(p^m))}{p^m} = -\frac{x(p^m)}{x'(p^m)p^m}.$$

The price elasticity of demand at p^m is: $-\frac{x'(p^m)p^m}{x(p^m)}$ and its inverse is just the monopolist's price-cost margin.

(b)

From (a), the price elasticity of demand at p^m is: $-\frac{x'(p^m)p^m}{x(p^m)}$. Then if $c'(x(p^m))$ is positive at every $x(p^m)$, $\frac{p^m}{p^m - c'(x(p^m))}$ must be greater than 1, i.e the demand must be elastic.

12.B.3

Assume the partial derivatives satisfy $x_1 < 0, x_{11} < 0, c_1 > 0$ and $c_{11} > 0$ (there are standard assumptions). The monopolist's problem is Max $x(p,\theta)p - c(x(p,\theta),\phi)$, which yields the FOC, $x_1(p,\theta)p + x(p,\theta) = c_1(x(p,\theta),\phi)x_1(p,\theta) = 0$.
Differentiating with respect to θ gives us:
$\frac{\partial p}{\partial \theta} = \frac{-(p-c_1)x_{12} + x_2(x_1 c_{11} - 1)}{2x_1 + (p-c_1)x_{11} - x_1^2 c_{11}}$
and under our assumptions above we will have $\frac{\partial p}{\partial \theta} > 0$, if $x_2 > 0$ and $x_{12} > 0$.
Differentiating the FOC with respect to ϕ gives us:
$\frac{\partial p}{\partial \theta} = \frac{x_1 c_{12}}{2x_1 + (p-c_1)x_{11} - x_1^2 c_{11}}$.
and under our assumptions above we will have $\frac{\partial p}{\partial \phi} > 0$ if $c_{12} > 0$.

12.B.6

From definition, $c'(q) > 0, c''(q) < 0$, Let t be a tax or subsidy. Then, the profit of the firm is

$$\pi = p(q)q - c(q) - tq$$

FOC is

$$p'(q)q + p(q) = c'(q) + t$$

On the other hand, in competitive markets, $p(q) = c'(q)$ is hold. Thus, if $t = p'(q)q$ — this is a subsidy, since $p'(q) < 0$ — the monopolist chooses the efficient level of output. (As a practical matter, subsidizing monopolists like this doesn't appeal to taxpayers.)

12.D.4

(a)

Let p^* be the most profitable price, and $p^m(c)$ to be monopolist price of firms. Then, the industry total profit is,

$$\pi^m = p^m(c)q(p^m(c)) - cq(p^m(c))$$

Let c_1 be the cost after change. If firms keep colluding, the present value of the payoff stream is,

$$PV(\tfrac{\pi^m(c_1)}{2}, \tfrac{\pi^m(c_1)}{2}\delta, ...) = \tfrac{\pi^m(c_1)}{2}\textstyle\sum_{t=0}^{\infty} \delta^t = \tfrac{\pi^m(c_1)}{2(1-\delta)}$$

On the other hand, if a firm were to deviate, then the present value of the payoff stream is,

$$PV(\pi^m(c_1), 0\delta, 0\delta^2, ...) = \pi^m(c_1)$$

Thus, collusion is rational if

$$\frac{\pi^m(c_1)}{2(1-\delta)} \geq \pi^m(c_1)$$
$$\Leftrightarrow \frac{1}{2(1-\delta)} \Leftrightarrow (1 - \delta \leq \frac{1}{2}) \Leftrightarrow \delta \geq \frac{1}{2}$$

Thus the incentive to collude does not impose conditions on c_1 itself. But the profitability of monopoly, or of collusion, does depend on c, and the temptation to defect will be greater when the firms think that the profitability will be less in the future.

(b)

Let c_1 be the cost at first period, and c_2 be the cost at second period. Then, the present value of the payoff stream for corporation is now,

$$
\begin{aligned}
&PV\left(\frac{\pi^m(c_1, p^m(c_1))}{2}, \frac{\pi^m(c_2, p^m(c_2))}{2}\delta, \frac{\pi^m(c_2, p^m(c_2))}{2}\delta^2, ...\right)\\
&= \frac{\pi^m(c_1, p^m(c_1))}{2} + \frac{\pi^m(c_2, p^m(c_2))}{2}\sum_{t=0}^{\infty}\delta^t\\
&= \frac{\pi^m(c_1, p^m(c_1))}{2} + \frac{\delta}{(1-\delta)}\frac{\pi^m(c_2, p^m(c_2))}{2}
\end{aligned}
$$

Thus, the firms keep corporation at period one if,

$$
\begin{aligned}
&\frac{1}{2}(\pi^m(c_1) + \frac{\delta}{(1-\delta)}\pi^m(c_2)) \geq \pi^m(c_2)\\
&\Leftrightarrow \frac{\delta}{(1-\delta)}\pi^m(c_2) \geq \pi^m(c_1)
\end{aligned}
$$

Let $\tilde{\delta}$ be a discount factor which satisfies,

$$\tfrac{\tilde{\delta}}{(1-\tilde{\delta})}\pi(c_2) = \pi(c_1) \Leftrightarrow \tilde{\delta} = \tfrac{\pi(c_1)}{\pi(c_1)+\pi(c_2)}$$

Also, $\tilde{\delta} > \frac{1}{2}$ since $\pi(c_1) > \pi(c_2)$.
Therefore, when $\delta > \tilde{\delta}$, the monopoly price is sustained at period 1, thus, $p^* = p^m(c_1)$. When $\delta < \tilde{\delta}$, deviation is rational and monopoly price is no longer sustained at period 1, thus, $p^* = c_1$.

12.E.4

Let n be the number of firms. Since the perfect cartel is formed, prices and quantities cannot be controlled by the social planner. The social planner can only achieve the optimal outcome by choosing n which satisfies

$$
\begin{aligned}
&\min_n nK\\
&s.t. n > 0
\end{aligned}
$$

Thus, $n = 1$
In case social planner cannot control entry, the number of firms is determined by the point where the firms' profit dissapears. The profit of each single firm is $\frac{\pi^m}{n}$. Firms can entry until $\frac{\pi^m}{n} - K = 0$.
Thus, $n = \frac{\pi^m}{K}$.

12.C.18

(a)

Firm 1 chooses q_1 knowing that given its choice, firm 2 will produce $b_2(q_1)$, where $b_2(q_1)$ is firm 2's best response function, firm 1's problem is:

$$\max_{q_1} \Pi^1(q_1, b_2(q_1))$$

F.O.C implies:

$$\Pi^1_1(q_1, b_2(q_1)) = -\Pi^1_2(q_1, b_2(q_1))b_2'(q_1) < 0$$

(since$\Pi^1_2(q_1, b_2(q_1)) < 0$ and $b_2'(q_1) < 0$.) If the firm instead choose quantities simultaneously, the F.O.C becomes:

$$\Pi^1_1(q_1, b_2(q_1)) = 0$$

since $\Pi^1_{11}(q_1, b_2(q_1)) < 0$, this implies that the Stackelberg leader picks a larger quantity in equilibrium than in the Cournot game. Since the best response function of firm 2 is downward sloping, this implies that the follower picks a smaller quantity and the aggregate output increase(and therefore price decreases). Since the leader could have chosen the Cournot quantity, we know that her payoff as a Stackelberg leader is higher. The follower produces less and obtains a lower price than in the Cournot outcome, which implies that her profits are lower.

(b)

The figure is as following. Denote N as Nash Equilibrium outcome and S is the equilibrium of the Stackelberg game.