Problem (1)**.** A game for a three move situation in which the first move is a choice between R and L, the second move is a choice between r and l, and the third move is a choice between ρ *and* λ.

Answer. (a) The game tree is drawn in Figure 1.

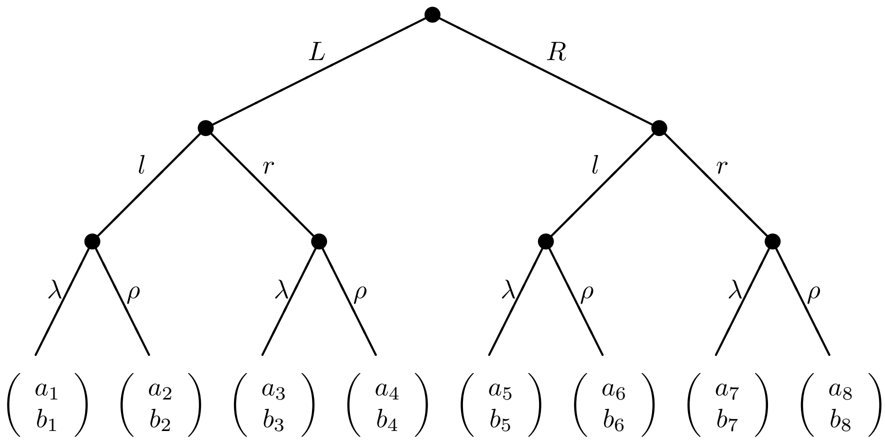

Figure 1: Tree of Game 1-a

(b) Suppose that the first move is owned by player 1, the second by player 2 and the third by player 1. Player 2 observes player 1's first move. At his last move, player 1 does not observe player 2's move, though he remembers his own initial move. Then the game tree is drawn as in Figure 2.

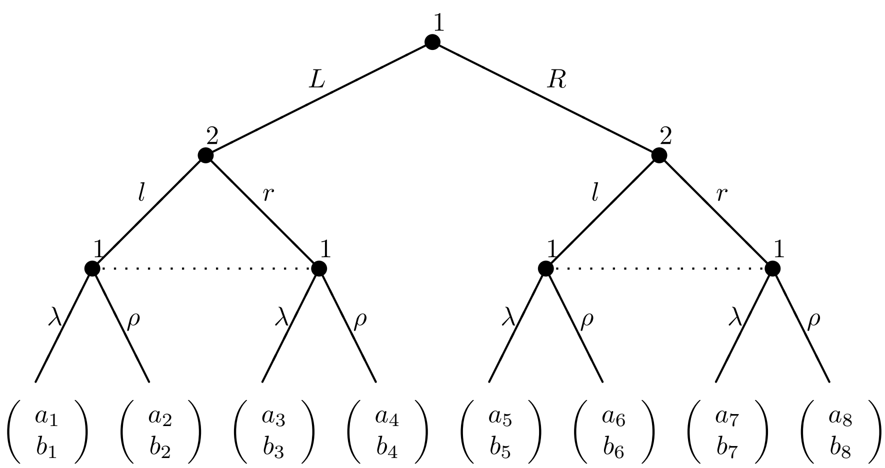

Figure 2: Tree of Game 1-b

(c) For the tree in (b), the complete set of pure strategies for each player are:

$$S_1 = \{L\lambda\lambda, L\lambda\rho, L\rho\lambda, L\rho\rho, R\lambda\lambda, R\lambda\rho, R\rho\lambda, R\rho\rho\}$$

and $S_2 = \{ll, lr, rl, rr\}$. Then the full normal form for the game is

Player 2

Player 1	ll	lr	rl	rr
$L\lambda\lambda$	a_1, b_1	a_1, b_1	a_3, b_3	a_3, b_3
$L\lambda\rho$	a_1, b_1	a_1, b_1	a_3, b_3	a_3, b_3
$L\rho\lambda$	a_2, b_2	a_2, b_2	a_4, b_4	a_4, b_4
$L\rho\rho$	a_2, b_2	a_2, b_2	a_4, b_4	a_4, b_4
$R\lambda\lambda$	a_5, b_5	a_7, b_7	a_5, b_5	a_7, b_7
$R\lambda\rho$	a_6, b_6	a_8, b_8	a_6, b_6	a_8, b_8
$R\rho\lambda$	a_5, b_5	a_7, b_7	a_5, b_5	a_7, b_7
$R\rho\rho$	a_6, b_6	a_8, b_8	a_6, b_6	a_8, b_8

,

while the reduced normal form is

player 2

player 1	l	r
$L\lambda\cdot$	a_1, b_1	a_3, b_3
$L\rho\cdot$	a_2, b_2	a_4, b_4
$R\cdot\lambda$	a_5, b_5	a_7, b_7
$R\cdot\rho$	a_6, b_6	a_8, b_8

.

(d) Alter the game in (b) as follows. At player 1's last move, he only knows whether or not the earlier moves matched. Given a mixed strategy σ_1 for player 1 and σ_2 for player 2, then, given the player 1 is in the information set for matched earlier moves, the probability of player 1 choosing $\rho$is

$$p = \frac{\sigma_1 (L\rho\cdot)\,\sigma_2 (l\cdot) + \sigma_1 (R\cdot\rho)\,\sigma_2 (\cdot r)}{[\sigma_1 (L\lambda\cdot) + \sigma_1 (L\rho\cdot)]\,\sigma_2 (l\cdot) + [\sigma_1 (R\cdot\lambda) + \sigma_1 (R\cdot\rho)]\,\sigma_2 (\cdot r)}.$$

(e) According to Kuhn's theorem, an arbitrary mixed strategy for player for the game in (b) induce a unique behavior strategy. However, it does not for the game in (d). □

Problem (2)**.** An absent-minded driver can either turn south or continue east through two junctions. His payoffs are 0 if he turns at the first junction, 4 at the second junction and 1 if he doesn't turn.

Answer. (a) Given perfect information, the best decision is to turn at the second junction and gain payoff 4, while the decision tree is drawn in Figure 1.

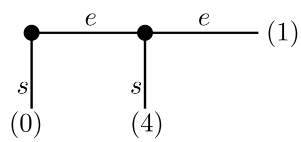

Figure 1: Driving Game with Perfect Information

(b) If the driver can recognize a junction but has no clue whether it is the first or the second, then the decision tree can be draw as in Figure 2.

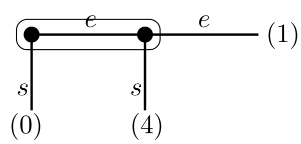

Figure 2: Driving Game with Imperfect Recall

(c) Denote the probability of turn south as q at either junction, and the probability of continuing east as $1-q$. Then with imperfect recall, the driver seeks to maximize his expected payoff

$$\max_q E(q) = q * 0 + (1-q)\, q * 4 + (1-q)^2 * 1,$$

and the FOC, $-6q+2=0$, implies $q^* = 1/3$. Therefore, the optimal behavior is to turn south whenever a junction appears with probability of $1/3$, and the corresponding expected payoff is $E(q^*) = 4/3$.

(d) The optimal behavior in (c) is calculated before the driver gets to some junction. If the driver sticks to this plan (i.e. with full commitment), then it is reasonable for him to believe when coming to one junction that with probability

$$\frac{1}{1+\frac{2}{3}} = \frac{3}{5} = 0.6$$

he is facing the first junction and with probability 0.4 he is facing the second one. So if he decides to turn south at this junction (i.e. $q=1$), then his expected payoff is $0.4 \times 4 = 1.6 > 4/3$. Hence here comes the paradox.

Denote p as the probability to turn at the junction, then it implies that the probability that the driver is facing the first junction is

$$\frac{1}{1+1-p} = \frac{1}{2-p},$$

and the probability of facing the second junction is

$$1 - \frac{1}{2-p} = \frac{1-p}{2-p}.$$

Therefore, the driver seeks to

$$\max_p \frac{1}{2-p}(1-p)(4p+1-p) + \frac{1-p}{2-p}(4p+1-p),$$

and the solution is $p^{**} = \frac{12-\sqrt{84}}{6} \approx 0.4725$ and $E(p^{**}) = 1.6697 > 4/3$. $\square$

3. Game settings

   - Players = {1, 2}
   - Actions $S_1 = \{H, T\}$, $S_2 = \{h, t\}$
   - Preference : Maximized expected payoff.

   (a) Verify that there no pure NE

      i. Best responses:

$$B_1(h) = H \qquad\qquad B_1(t) = T$$
$$B_2(H) = t \qquad\qquad B_2(T) = h$$

      We can see that there is no profile S^* such that for each player i and action s this is holds $s_i^* \in B_i(s_{-i}^*)$. So there is no pure strategy Nash Equilibrium in this game.

      ii. Denote by p the probability that player 1's mixed strategy assigns to H and by q the probability that player 2's mixed strategy assigns to h. Then, given player 2's mixing, player 1's expected payoff to the pure strategy H is

$$q \times a + (1-q) \times 0 = qa$$

      And player 1's expected payoff to the pure strategy T is

$$q \times 0 + (1-q) \times b = (1-q)b$$

      From these two, I solve for q, which is $q = \dfrac{b}{a+b}$

      Likewise, given player 1's mixing, player 2's expected payoff to the pure strategy h is

$$p \times 0 + (1-p) \times c = (1-p)c$$

      And player 1's expected payoff to the pure strategy T is

$$p \times d + (1-p) \times 0 = pd$$

      From these two, I solve for p, $p = \dfrac{c}{d+c}$

   So the unique mixed NE is $\big((\dfrac{c}{d+c}, \dfrac{d}{d+c}), (\dfrac{b}{a+b}, \dfrac{a}{a+b})\big)$

   (b) Comparative statics,

$$\frac{\partial p}{\partial d} = \frac{\partial}{\partial d}(\frac{c}{d+c}) = -\frac{c}{(d+c)^2}; \quad \text{and} \quad \frac{\partial p}{\partial d} = \frac{\partial}{\partial c}(\frac{c}{d+c}) = \frac{d}{(d+c)^2}$$
$$\frac{\partial q}{\partial a} = \frac{\partial}{\partial a}(\frac{b}{a+b}) = -\frac{b}{(a+b)^2}; \quad \text{and} \quad \frac{\partial q}{\partial b} = \frac{\partial}{\partial b}(\frac{b}{a+b}) = \frac{a}{(a+b)^2}$$

   - Player 1 will put a higher probability playing p when d increases. If c increases then player 1 will put higher probability by playing $1-p$.
   - Player 2 will put a higher probability playing q when b increases. If a increases then player 2 will put higher probability by playing $1-q$.

   (c) As we are in strategic settings and the findings in part b clearly shows that player i mixing is determined solely by what player $-i$'s value of the outcomes. If the valuation from the other player $-i$ change then player i should rationally change his mixing strategy.

4. (a) Game settings

   - Players = {1, 2}
   - Strategies $S_1 = \{T, M, B\}$, $S_2 = \{L, C, R\}$

   Since the strategy C for player 2 is dominated by L and R, then player 2 will remove this strategy. As player 1 knows that player 2 will remove strategy C then player 1 removes strategy B.

	L	R
T	$2, 0$	$4, 2$
M	$3, 4$	$2, 3$

   - Best responses in this new set up are as follow:

   $$B_1(L) = M \qquad B_1(R) = T$$
   $$B_2(T) = R \qquad B_2(L) = M$$

   So the pure strategy NE = $((M, L), (T, R))$

   - Now check whether there is a NE where player 1 play T vs player 2 mixing L, R: If player 1's strategy is T, then player 2's payoff to her two actions $(0, 2)$ which is different. For NE to exist, the payoff player 2 assigns positive probability must be the same.[1] Using the same logic, I could eliminate the possible pair between pure strategy and mixed.
   - Denote by p the probability that player 1's mixed strategy assigns to T and by q the probability that player 2's mixed strategy assigns to L. Then, given player 2's mixing, player 1's expected payoff to the pure strategy T is

   $$q \times 2 + (1 - q) \times 4 = 4 - 2q$$

   And player 1's expected payoff to the pure strategy M is

   $$q \times 3 + (1 - q) \times 2 = 2 + q$$

   From these two, I solve for q, which is $q = \dfrac{2}{3}$

   Likewise, given player 1's mixing, player 2's expected payoff to the pure strategy L is

   $$p \times 0 + (1 - p) \times 4 = (1 - p)4$$

   And player 1's expected payoff to the pure strategy R is

   $$p \times 2 + (1 - p) \times 3 = 3 - p$$

   From these two, I solve for p, $p = \dfrac{1}{3}$

   So the unique mixed NE is $\big((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})\big)$

   (b) (Looked at the separated paper).

   Using backward induction, I found the Sub Game Perfect Nash Eq, that is (T, R)

[1] I apply preposition 116.2 of Osborne's (2004)

Problem (5)**.** Software and Hardware form a joint venture in which they agree to split revenue evenly. Each can exert either high effort at cost 20 or low effort at cost 0. Hardware moves first but software does not observe her effort. Revenue is 100 if both exert low effort, or if parts are defective. If parts are not defective, revenue is 200 if both exerts high effort, and 200 with prob 0.1 (and 100 with prob 0.9) if only one partner exerts high effort. Both partners initially believe the prob of defective parts is 0.7. Hardware discovers the truth before she chooses effort level, but Software does not.

Answer. For the rest of the answer, I take $\Pr(defective) = 0.3$ *instead of 0.7 as sited in the question. Cause if* $\Pr(defective) = 0.7$*, then 0.7*30+0.3*80=45<50, which indicates that for Software giving low efforts always dominates giving high efforts, i.e. the only BNE is* (ll, l)*.*

(a) The game tree is illustrated in Figure 1.

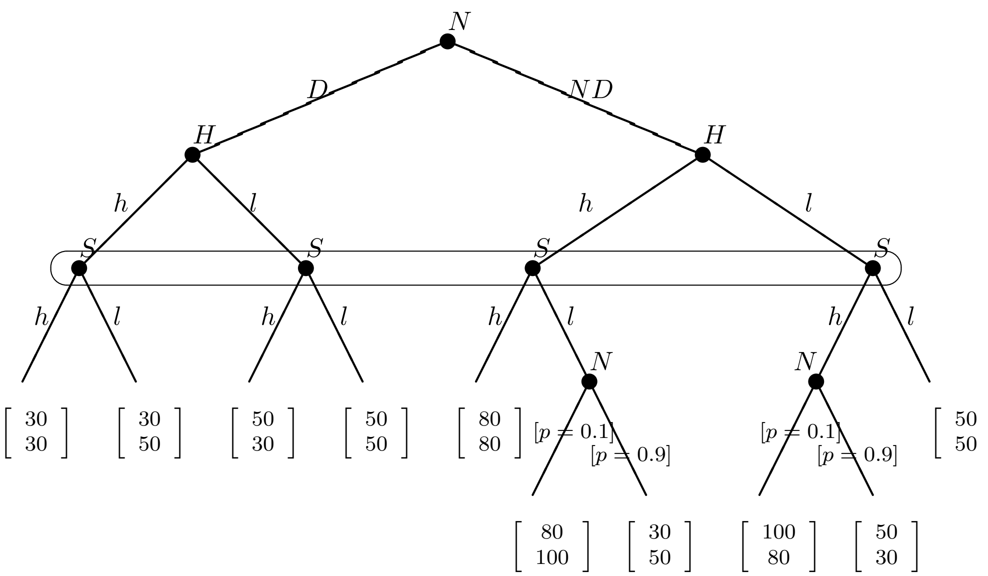

Figure 1: Game Tree of Game 5

(b) The bimatrix for the corresponding normal form is

Nature=Defective

		Software: h	Software: l
Hardware	h	30,30	30,50
	l	50,30	50,50

Nature=Not Defective

		Software: h	Software: l
Hardware	h	80,80	35,55
	l	55,35	50,50

(c) Apparently, the pure BNE are (ll, l) and (lh, h).

First, notice that the fact that Hardware knows the realization of nature (whether parts are defective or not) before making decision on effort is a common knowledge. And we know (so does Software) that when parts are defective, giving low effort is the dominant strategy that Hardware would play. Therefore, denote p as the probability that Hardware gives high effort when parts are not defective, and q as the probability that Software gives high effort regardless of the realization of the nature (since Software does not know the truth). Then in order to have a mix NE, we need to have

$$u_H(h, \sigma_S | N = D) = 80q + 35(1-q) = u_H(l, \sigma_S | N = D) = 55q + 50(1-q) \Rightarrow q = 0.375,$$

$$u_S(h, \sigma_H) = 0.3 * 30 + 0.7 * (80p + 35(1-p)) = u_S(l, \sigma_H) = 0.3 * 50 + 0.7 * (55p + 50(1-p))$$
$$\Rightarrow p = 0.5893.$$

Thus, the mix BNE is such that Software gives high effort with probability 0.375, and Hardware always gives low effort when parts are defective and gives high effort with probability 0.5893 when parts are not defective, i.e. $\sigma_H^*(lh) = 0.5893$, $\sigma_H^*(ll) = 0.4107$, and $\sigma_H^*(hh) = \sigma_H^*(hl) = 0$, while $\sigma_S^*(h) = 0.375$ and $\sigma_S^*(l) = 0.625$.

(d) In the mixed strategy BNE, the Software's belief about Hardware's choice is that Hardware always gives low effort when parts are defective and gives high effort with probability 0.5893 when parts are not defective.

(e) In a branch where Software chose high effort and sees revenue 100, then her belief that the parts were defective is

$$\begin{aligned}
\Pr(Defective|R = 100, S = h) &= \frac{\Pr(Defective, R = 100, S = h)}{\Pr(R = 100, S = h)} \\
&= \frac{0.3q}{0.3q + 0.7(1-p)q * 0.9} = 0.5369.
\end{aligned}$$

□

	LL	L	M	R
U	**100, 2**	-100, 1	0, 0	-100, -100
D	-100, -100	100, -49	1,0	**100, 2**

8.D.9 from MWG

a) I would choose M to avoid a large loss.

b) The pure NE are (U,LL) and (D,R), in bold in the matrix. For mixed NE, there are 11 possibilities:

- Mixes (L, M), (L, R), (M, R), and (L, M, R) will cause P1 to play D. This will cause P2 to play R, and (D, R) is already a pure NE

- For a mix of (LL, R) we get:
  $2p + (1\text{-}p)(-100) = 100p + (1\text{-}p)2 \implies p = 1/2$
  But this would imply that u(LL) = u(R) = -49, while u(M) = 0. So this cannot be part of a NE.

- This implies that (LL, M, R), (LL, L, R), and (LL, L, M, R) also cannot be part of a NE.

- For a mix of (LL, M), we need p = 50/51. This implies that u(LL) = u(M) = 0. But we know u(L) = 1/51 >0, so this cannot be part of a NE.

- This implies that (LL, L, M) cannot be part of a NE.

- This leaves only (LL, L)
  - For P2 to mix, we need $2p + (1\text{-}p)(-100) = p + (1\text{-}p)(-49)$, which gives p = 51/52
    - This gives utilities u(LL) = u(L) = 1/26, u(M) = 0, and u(R) < 0.
  - For P1 to mix, we need $100q + (1\text{-}q)(-100) = -100 \cdot q + 100 \cdot (1\text{-}q)$, which gives q = 1/2
    - This gives utilities u(U) = u(D) = 0
  - Therefore, (1/26, 1/2) is a mixed NE.

c) M is clearly not part of any NE, mixed or pure. But, if P1 plays (1/2, 1/2), then M is a unique best response, and is therefore rationalizable.

d) If we can talk beforehand, we can agree to play one of the pure NE, so I will play either LL or R (depending on the agreement).

8.E.1)

4 hours by myself and using MCWG's notes.

		Strong: Attack	Strong: Don't Attack	Weak: Attack	Weak: Don't Attack
Strong	Attack	$-S, -S$ (1)	$M, 0$ (2)	$M-S, -W$ (3)	$M, 0$ (4)
	Don't Attack	$0, M$ (5)	$0, 0$ (6)	$0, M$ (7)	$0, 0$ (8)
Weak	Attack	$-W, M-S$ (9)	$M, 0$ (10)	$-W, -W$ (11)	$M, 0$ (12)
	Don't Attack	$0, M$ (13)	$0, 0$ (14)	$0, M$ (15)	$0, 0$ (16)

Please look at next page

	AA	$A^S O^W$	$O^S A^W$	OO
Attack if strong or weak AA	* $\frac{M}{4} - \frac{S+W}{2}, \frac{M}{4} - \frac{S+W}{2}$	* $\frac{M}{2} - \frac{S+W}{4}, \frac{M}{4} - \frac{S}{2}$	* $\frac{3M}{4} - \frac{S+W}{4}, \frac{-W}{2}$	** $M, 0$
Attack if strong Don't Attack if weak $A^S D^W$	** $\frac{M}{4} - \frac{S}{2}, \frac{M}{2} - \frac{S+W}{2}$	** $\frac{M-S}{4}, \frac{M-S}{4}$	** $\frac{M}{2} - \frac{S}{4}, \frac{M-W}{4}$	*** $\frac{M}{2}, 0$
Attack if weak Don't Attack if strong $D^S A^W$	$\frac{-W}{2}, \frac{3M}{4} - \frac{S+W}{4}$	$\frac{M-W}{4}, \frac{M}{2} - \frac{S}{4}$	$\frac{M-W}{4}, \frac{M-W}{4}$	$\frac{M}{2}, 0$
Don't Attack if strong or weak NO	$0, M$	$0, \frac{M}{2}$	$0, \frac{M}{2}$	$0, 0$

* combination of box numbers : 1, 3, 9, 11 $\Rightarrow \frac{M-(2S+2W)}{4} = \frac{M}{4} - \frac{S+W}{2}$

explain more.

: $\frac{M-2S-2W}{4} = \frac{M}{4} - \frac{S+W}{2}$

* combination of box numbers : 1, 4, 9, 12

If $S < W$: NE . (AA, AD), (AD, AA) (if $\frac{M}{4} > \frac{S}{4} + \frac{W}{2}$) ✓

* Combination of 2, 3, 10, 11 $\frac{M+M-S+M-W}{4} = \frac{3M}{4} - \frac{S+W}{4}$, $\sim$

** " " 2, 4, 10, 12 $\frac{M+M+M+M}{4} = 4M/4 = M$, 0

** " " 1, 3, 13, 15 $\frac{M-2S}{4}$, $\frac{2M-S-W}{4}$

** " " 1, 4, 13, 16 $\frac{M-S}{4}$, $\frac{M-S}{4}$

** " " 2, 3, 14, 15 $\frac{M+M-S}{4}$, $\frac{M-W}{4}$, $\sim$

*** " 2, 4, 14, 16 $\frac{2M}{4}$, 0

Alireza Kharazi
PS2

Mas-Collel: 8.D.5

a) The boardwalk stretches from 0 to 1.

Define $\propto_i \in [0,1]$ to be distance of a vendor i from point 0.

The profits of the vendors (V1 and V2) are:

1) If V_2 *is to the right of* V_1:

$$\pi_1(\propto_1, \propto_2) = \alpha_1 + \frac{1}{2}(\alpha_2 - \alpha_1)$$
$$\pi_2(\propto_2, \propto_1) = (1 - \alpha_2) + \frac{1}{2}(\alpha_2 - \alpha_1)$$

2) 2) If V_1 *is to the right of* V_2:

$$\pi_1(\propto_1, \propto_2) = (1 - \alpha_1) + \frac{1}{2}(\alpha_2 - \alpha_1)$$
$$\pi_2(\propto_2, \propto_1) = \alpha_2 + \frac{1}{2}(\alpha_2 - \alpha_1)$$

The best response functions are:

For $0 < \varepsilon < \propto_2 - \propto_1$

For vendor 2:

$BR_2(\alpha_1) = \alpha_1 + \varepsilon$ If $\alpha_1 < \frac{1}{2}$
$BR_2(\alpha_1) = \alpha_1 - \varepsilon$ If $\alpha_1 > \frac{1}{2}$
$BR_2(\alpha_1) = (1 - \alpha_1)$ If $\alpha_1 = \frac{1}{2}$

For Vendor 1:

$BR_1(\alpha_2) = \alpha_2 + \varepsilon$ If $\alpha_2 < \frac{1}{2}$
$BR_1(\alpha_2) = \alpha_2 - \varepsilon$ If $\alpha_2 > \frac{1}{2}$
$BR_1(\alpha_2) = (1 - \alpha_2)$ If $\alpha_2 = \frac{1}{2}$

Nash Equilibrium Exists when a fixed point exists:

$$BR_i(\sigma^*) = \sigma^*$$

This happens when $\alpha_2 = \alpha_1 = \frac{1}{2}$

$$\Rightarrow BR_1(\alpha_2) = BR_2(\alpha_1)$$

b) Suppose an equilibrium $\alpha_1^*, \alpha_2^*, \alpha_3^*$ exists and $\alpha_1^* = \alpha_2^* = \alpha_3^*$

If $\alpha_1^*, \alpha_2^*, \alpha_3^* < \frac{1}{2} \Rightarrow BR_3(\propto_1, \propto_2) = \propto_1 + \varepsilon \; or \; \propto_2 + \varepsilon$
If $\alpha_1^*, \alpha_2^*, \alpha_3^* > \frac{1}{2} \Rightarrow BR_3(\propto_1, \propto_2) = \propto_1 - \varepsilon \; or \; \propto_2 - \varepsilon$

Therefore a contradiction.

Suppose $\alpha_1^* = \alpha_2^*$
If $\alpha_1^*, \alpha_2^* > \alpha_3^* \Rightarrow BR_3(\propto_1, \propto_2) = \propto_1 - \varepsilon \; or \; \propto_2 - \varepsilon$
If $\alpha_1^*, \alpha_2^* < \alpha_3^* \Rightarrow BR_3(\propto_1, \propto_2) = \propto_1 + \varepsilon \; or \; \propto_2 + \varepsilon$
Again a contradiction.

Suppose each firm is located at different places. Then the firm at the farthest right (left) and gain by moving left (right) therefore a contradiction.
$\Rightarrow$ This game does not have Pure Nash Equilibrium.