Contents

To ease back into EFGs, we consider variants of the "cold war" game featured in the classic dark comedy movie Dr. Strangelove. Next we look more systematically at how to apply backward induction to EFGs, beginning with games of perfect information. Then we consider imperfect information, the role of beliefs, and Harsanyi's breakthrough idea on how to formalize incomplete information.

1 Dr. Strangelove

The plot of Stanley Kubrick's famous 1964 film centers on two players:

1. US Air Force General Jack D. Ripper (played by Sterling Hayden), who commands aircraft with nuclear bombs, capable of penetrating Soviet defenses, and
2. Soviet Premier Dmitri Kisov, who has a doomsday device (DDD) that can trigger global nuclear winter, making the surface of our planet uninhabitable for centuries.

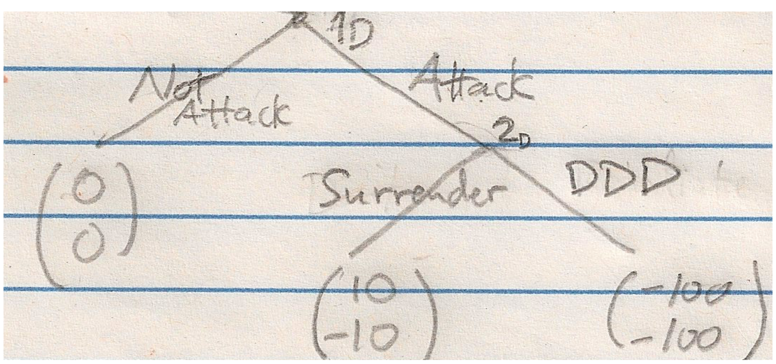

Figure 1: Dr Strangelove basic EFG.

- Player 1 has 2 strategies: Launch nuclear <u>A</u>ttack, or <u>N</u>ot Attack.
- Player 2 has 2 strategies. Set off <u>DDD</u>, or <u>S</u>urrender, if Player 1 chooses A.
- The basic NFG is:

	DDD	S
A	-100,-100	10,-10
N	0,0	0,0

-
  - There are two pure NE: {(N,DDD), (A,S)}.
  - In the first of these, player 1 is deterred from playing A because he fears that player 2 will choose DDD.
  - But it is not in player 2's interest ever to actually play DDD. The threat is helpful but actually carrying out that threat would be irrational.
  - This can be seen from the EFG in Figure 1. Applying BI (backwards induction) we get the other NE, (A, S). As these notes will explain, this NE is called Subgame Perfect, or SPNE for short.
  - In the movie, player 1 realizes this and plays A.

Of course, the SPNE outcome of this basic game is not very satisfactory for player 2. It would be in his interest to change the game so that the DDD threat is more credible, and the (N, DDD) equilibrium is played.

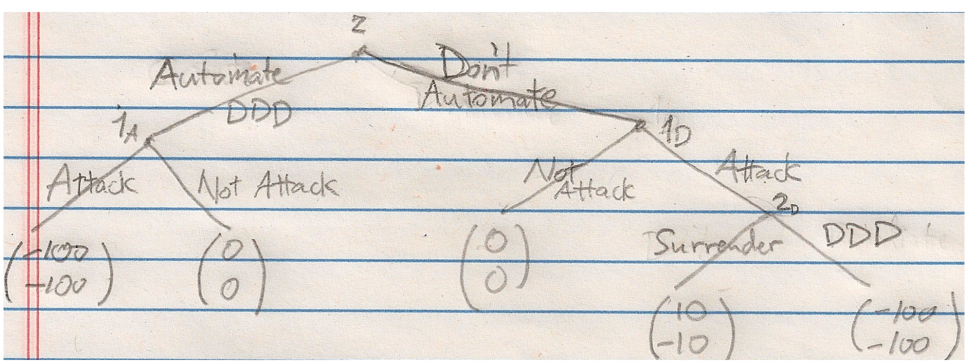

Figure 2: Dr Strangelove wider EFG.

- In the wider game, player 2 can Automate the DDD so that it would automatically set off when attacked or, alternatively, Don't automate and proceed with the basic game.

- As can be seen in the left branch of the wider game in Figure 2, player 1's best response to Au is N, with satisfactory (0) payoffs to both players in the NE for that branch.
- In the wider game shown in Figure 2, therefore, BI $\implies$ the NE (N, Au/S) with payoff vector (0,0).
- The idea is that now DDD is a credible threat, and deters A.

The twist in the movie was that player 2 had chosen Automate but (loving surprises) he had not yet told anyone. Player 1 had not considered the possibility that player 2 would make that choice, and had played A.

- How do we incorporate that twist?
- To the EFG in Figure 2, we put an information set bubble around the two nodes (1A and 1D) for player 1.
- We also must somehow account for **beliefs**. We might assign probability $p > 0$ to the left node 1A and probability $1 - p$ to the right hand node 1D. In the movie, p was minuscule, maybe 0.
- Now try to apply BI to this game of imperfect information. You can see that player 2 would play S at node 2D, but...
- It is not clear what BI tells player 1 to do at his info set.
- You can check that A is a best response (yielding highest expected payoff) if $p \leq 1/11$ and that N is a BR is $p \geq 1/11$, but...
- This begs the question of what is p, and where does it come from?

In this chapter, we will answer such questions, using classic decision theory, fortified by the insights that John Harsanyi and Reinhard Selten introduced around 1970.

2 Perfect Information and Backward Induction

Recall from Chapter 1.5 that backward induction (BI) is the only rational way to solve well-posed decision problems.

Recall that each terminal node is the utility of the corresponding final outcome. Each non-terminal node has at least 2 branches, and in a decision tree each such node is owned either by Nature or the decision maker (DM). The same is true of EFGs of perfect information, except that (a) different players may own different non-terminal nodes, and (b) the terminal nodes are payoff *vectors*, with a component for each (non-Nature) player specifying her utility of the corresponding outcome. Consequently, we can apply the BI cookbook with minor modifications to accommodate (a) and (b).

BI Cookbook for EFGs of perfect information.

1. Convert each penultimate node (those just before the terminal nodes) to a terminal node with payoff vector computed as follows:

   - If the node is owned by player i, then take the payoff vector on a branch that maximizes player i's payoff.

   - If it is a Nature node, then take the expected payoff vector using the branch probabilities.

2. Iterate the previous step at penultimate nodes of the converted (shorter) EFG until the initial node is a terminal node.

3. Reconstruct each player's strategy from the branch chosen at each node controlled by that player.

4. A strategy profile so reconstructed constitutes a subgame perfect Nash equilibrium (SPNE) of the original EFG, and the SPNE payoff vector is that obtained when

step 2 is completed.

A bit more terminology. (Should be a review from Ch02.)

Given an EFG,

- Take any nonterminal node ν.
- The tree with initial node ν and all its direct and indirect successor nodes (and corresponding probabilities and payoff vectors) is called a subgame of the original EFG.
- The subgame is called a *proper* subgame if ν is not the initial node of the original EFG, i.e., if the subgame is not the entire original EFG.
- The BI cookbook gives a solution of the original EFG that is also a solution (NE) when restricted to any subgame. That is why it is called subgame perfect or SPNE.

Proposition (Zermelo $\sim$ 1890). Every finite extensive form game (EFG) of perfect information has a SPNE solution. Except in borderline cases, the SPNE is unique.

Proof sketch: Employ the BI cookbook. In a finite EFG of perfect information, nothing will prevent the completion of step 2, yielding a SPNE. The solution of N-nodes is always unique in a well-specified EFG, and the same is usually true of the other player nodes. The "borderline" exceptions are when a node owned by i has two or more branches with the same payoff for that player.

Caveats:

- BI may not be feasible.

  For example, chess is a finite game of perfect information (win$\succ$draw$\succ$loss). But if you try BI, the astronomical number of branches makes your task impossible. Deep Blue: compute x moves ahead and *estimate* the expected payoff from the continuation. This is a "brute force" approach, not BI. Zermelo's theorem (plus a symmetry argument) implies that in SPNE either (a) Player 1 (white) wins for sure, or (b) player 2 (black) can always force a draw. But nobody knows whether (a) or (b) is correct.

- SPNE may not reasonable or empirically valid in extreme cases.

  The centipede game shown in Figure 3 is a finite EFG of perfect information. Its unique SPNE is for both players always to choose "pocket," resulting in the very inefficient payoff vector (2,0).

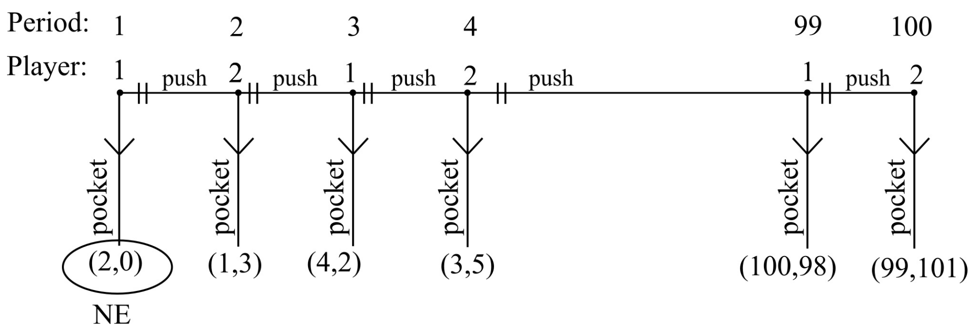

Figure 3: The Centipede game. There are two piles. Initially one pile has two coins, the other zero. Two actions: either pocket a pile, leaving the other pile for the other player and ending the game, or push the two piles across the table to the other player, in which case a coin is added to each pile. The game stops after 99 pushes, if neither player has already pocketed a pile.

  Reasonable people might want to "push" for a while in hopes of getting to higher payoff nodes. And, indeed, this often happens when the game is played in the lab. For further discussion, see Section 7 below.

3 Information sets and subgames

Things get even more interesting when information is imperfect.

- Games of imperfect information have at least one player that does not know the full history of play at some move.

- For games of imperfect information we have the concept of an information set.
  - This summarizes what you know when it is time for you to move.
- *Definition.* An information set for player i in an EFG is a set of nodes satisfying:
  - Player i has the move at every node in the information set.
  - When reaching the info set, player i does not know which of its nodes she is are at.
    - * This implies that player i has the same action set at each of those nodes.
    - * Otherwise the set of allowable actions would yield additional information that one or more nodes are not in the information set.
- Therefore, a game of imperfect information can be defined as an EFG with at least one non-singleton information set.

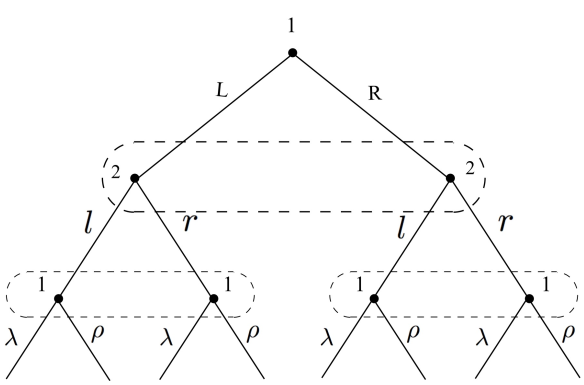

Figure 4: This perfect recall game form has no proper subgames.

- What now is a subgame? It's a little trickier than for games with perfect information, but makes lots of sense once you have the idea.

- A subgame of an EFG:

  1. Begins at a decision node ν that is a *singleton* information set, i.e., not in a bubble with other nodes.
  2. As before, includes all decision nodes and terminal nodes following ν in the game tree and no nodes that precede ν.
  3. Does not break any information sets.

- The point is that breaking an information set means that somebody would know something that she didn't know in the original game, so it wouldn't be a subgame of that game.

- Refer back to Ch02 for more examples of finding subgames.

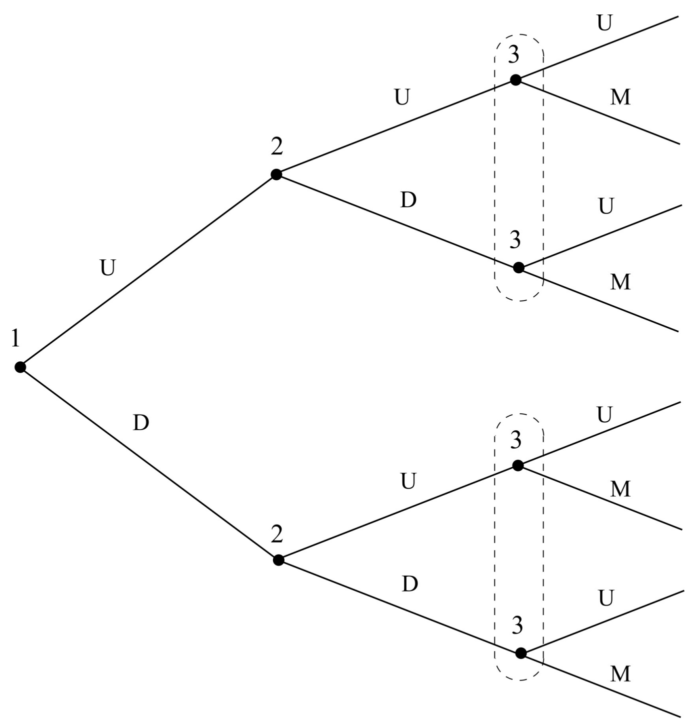

Figure 5: The two proper subgames in this example each start with a node owned by player 2.

Now we are ready for the general definition of subgame perfection.

- A strategy profile for an EFG is a subgame perfect Nash equilibrium (SPNE) if the restriction of the profile to every subgame constitutes a NE of that subgame.

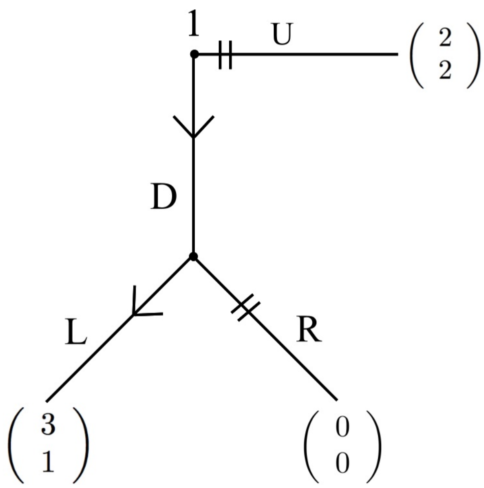

Figure 6: Here BI gives us the unique SPNE where player 2 chooses L and consequently player 1 chooses D.

- Figure 6 provides another example.
  - There is one proper subgame beginning at player 2's move.
  - The NFG representation is:

	L	R
U	$2,\underline{2}$	$\underline{2},\underline{2}$
D	$\underline{3},\underline{1}$	$0,0$

  - Note that R is weakly dominated
  - There are two pure NE: {(U,R), (D,L)}.
  - There is also a mixed NE: As usual, let p and q denote the probabilities that row and column choose their first pure strategy. Then:

$$\begin{aligned} 0 = E\pi_1(U) - E\pi_1(D) &= [2q + 2(1-q)] - [3q + 0(1-q)] \\ &= 2 - 3q \implies q = q^* = \frac{2}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned} 0 = E\pi_2(L) - E\pi_2(R) &= [2p + 1(1-p)] - [2p + 0(1-p)] \\ &= 1 - p = 0 \implies p = p^* = 1. \end{aligned}$$

- Thus, the mixed NE has row playing U with probability 1 and column playing L with probability $\frac{2}{3}$.

- However, R is dominated in the proper subgame, so it is not credible in the original game.

- Now look for SPNE. Recall the definition: the players' strategies yield a NE in every subgame.

- Thus, subgame perfection eliminates non-credible threats.

To find SPNE use BI.

- Find the smallest subgames that contain terminal nodes.

- Then replace each subgame with the payoffs from one of its NE.

- Work backwards this way to the root, which implies a NE in every subgame.

A more detailed example.

Figure 7 is the extensive form for a key strategic interaction in Tom Clancy's bestselling 1984 novel "Hunt for Red October," later a 1990 movie. Red October is a new submarine developed by the USSR. Its captain, Marko Ramius (player 1, played by Sean Connery) believes that the new technology will destabilize the Cold War. As the submarine departs on its maiden voyage, Ramius chooses whether or not to send his boss a letter revealing his intention to defect to the US. His chief officers, Borodin (player 2) and Melekhin (player 3), have agreed to Defect with him, but they might change their minds (Renege). Their choice is not coordinated, so is effectively simultaneous. Payoffs are as indicated.

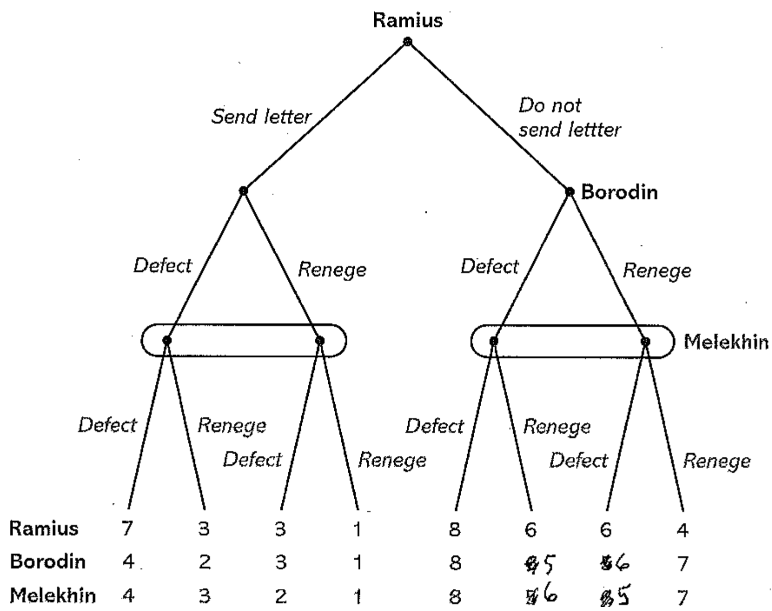

Figure 7: Hunt for Red October Game, slightly modified from an exercise in Harrington's textbook.

- What are the NE of this game? To find them write out the NF.
- This can be done via 2 trimatrices, each 4x4. (Try it out!)
- Solving that NFG gives 10 pure strategy NE, plus numerous mixed NE. But which of these is subgame perfect?
- In Figure 7, there are two proper subgames, in which players 2 and 3 simultaneously choose whether to defect (D or d) or to renege (R or r).
- Now consider the subgame following Player 1's choice Send. Since Player 1 is no longer active in the subgame, we drop (for the moment) his payoff and write out the subgame's NF. It is an otherwise routine 2x2 bimatrix where player 2 is the row player and player 3 is the column player:

	d	r
D	**4,4**	**2**,3
R	3,**2**	1,1

- This subgame has a DS solution, (D, d), hence this is the unique NE.

- Now consider the other proper subgame, following Player 1's choice Do Not Send.

It has bimatrix:

	d	r
D	**8,8**	5,6
R	6,5	**7,7**

- This subgame has two pure NE — (D, d) and (R, r), in bold — and one mixed NE, $(\frac{1}{2}, \frac{1}{2})$.

- Proceeding with the generalized BI algorithm, we replace the first subgame by its unique NE payoff vector $(7, 4, 4)$ — note that we now include the first player's payoff, as we must — and replace the second subgame by the payoff vector $(8, 8, 8)$ from the first of its NE.

- In this reduced game, player 1 chooses N (Do Not Send) since that gives him payoff 8 versus 7 from his alternative strategy.

- We conclude that one SPNE is (N, DD, dd) with payoff payoff vector $(8, 8, 8)$.

- But there are other SPNE since some subgames have other NE. As the next step in gerneralized BI, we replace the second subgame by the payoff vector (4,7,7) from its risk dominant NE.

- Now player 1's best response is S (send the letter), so we have a new SPNE (S, DR, dr) with payoff vector (7,4,4).

- Finally, replace the second subgame by the payoff vector (6,6.5,6.5) from its mixed NE.

- Again player 1's best response is S (send the letter), so we have a third SPNE $(S, D[.5], d[.5])$ with payoff vector (7,4,4).

- That is all the SPNE in this game. If more than one subgame had multiple equilibria, we would have to include all combinations.

- Which of the three SPNE in this game seems most reasonable? If Ramius is not completely confident that his officers will play the payoff dominant NE in the subgame following N, then he is better off sending the letter. He can feel confident that, with the indicated payoffs, his officers will play their dominant strategies, yielding almost as good a payoff to him (7).

- So what happened in Tom Clancy's story? Is the plot consistent with the SPNE (S, DR, dr)?

4 Beliefs

1. To find BRs in games of incomplete info, we need to assign a probability distribution over nodes in each info set. These represent the beliefs of the player who owns the info set.

2. Beliefs shouldn't be completely arbitrary. We might ask that they are consistent with probability theory (e.g., obey Bayes theorem) and that they are in some sense consistent with rationality.

3. The definition of Nash equilibrium can be unpacked to pin down the probabilities.

To expand on point 2, note that iteration in BI is justified if the player is rational and believes that [downstream players believe that]^n downstream players are rational.

To expand on point 3, recall that a NE by definition is a profile $s^* = (s_1^*, ..., s_n^*)$ such that $s_i^* \in B(s_{-i}^*), \quad i = 1, ..., n$. This can be broken down into two pieces:

i. s_i^* is a best response to player i's beliefs about the profile s_{-i} she faces.

ii. Her beliefs are precise and correct, that is her subjective probability is 1.0 for $s_{-i} = s_{-i}^*$ and is 0 for $s_{-i} \neq s_{-i}^*$.

Examples from NFGs. Consider the following bimatrix and the set of rational (permissible) beliefs by player 2.

$$
\begin{array}{ll|c|c|}
 & & (q) & (1-q) \\
 & & t_1 & t_2 \\
\hline
s_1 & (p) & 2, - & 3, - \\
\hline
s_2 & (1-p) & \underline{5}, - & \underline{6}, - \\
\hline
\end{array}
$$

- – Strategy s_2 strictly dominates s_1 so the only rational belief that player 2 could have is $p = 0$.

- However, with the following bimatrix any belief $p \in [0, 1]$ by player 2 is permissible.

$$
\begin{array}{ll|c|c|}
 & & t_1\ (q) & t_2\ (1-q) \\
\hline
p & s_1 = U & 2, - & \underline{4}, - \\
\hline
(1-p) & s_2 = D & \underline{5}, - & 3, - \\
\hline
\end{array}
$$

- – Since:
    - $E\pi_1(U) = 2q + 4(1-q) = 4 - 2q$
    - $E\pi_1(D) = 5q + 3(1-q) = 2q + 3$
    - $E\pi_1(U) - E\pi_1(D) = 1 - 4q$
  - Thus, if $q < \frac{1}{4}$ then U is a best-response, and if $q > \frac{1}{4}$ then D is a best-response.
  - Since there is no dominant strategy for player 1, player 2 could have any beliefs regarding p.

5 Harsanyi conversion

- Recall "games of incomplete information" originally referred to situations where a player didn't know something crucial about the game. The most important case (which is very common in applications) is that a player does not know someone else's payoffs. A more subtle case is that you don't know some other player's beliefs (e.g., about your intended strategy, or a third player's strategy).

In jargon, a player doesn't know someone else's "type."

- However, you have a belief about someone else's type. Implicitly, everyone has such beliefs.
  - Over time, information may be revealed that causes you to update your belief about their type (ex: car salesperson problem).
  - The only rational way to update beliefs is via Bayes' rule.
  - Thus, games of incomplete information are also called Bayesian games.
  - A static Bayesian game has incomplete information and simultaneous moves

Harsanyi (1967) showed how to convert ill-defined games of incomplete information into well-defined games of imperfect information.

The Harsanyi conversion has Nature moving first "choosing" the players' types.

Ex: War. You (player 2) don't know if your opponent (player 1) is strong (WMD) or weak (no WMD) and you must consider this uncertainty when deciding to attack or not. Player 2's payoff for a given action depend on the opponent's type.

Nature moves first choosing player 1's type, strong with probability p, weak with probability $(1-p)$. Player 2 does not observe Nature's move.

The expected payoff for player 2 from choosing attack is:

$E\pi_2(attack) = -10p + 10(1-p) > 0$ if $p < \frac{1}{2}$

$E\pi_2(not\ attack) = 0$

Thus, we are left with a single strategy that is a decision rule: $(attack|p < \frac{1}{2}$, not $attack|p > \frac{1}{2})$. This strategy is in terms of a "cutoff value" for the probability at which the optimal decision changes.

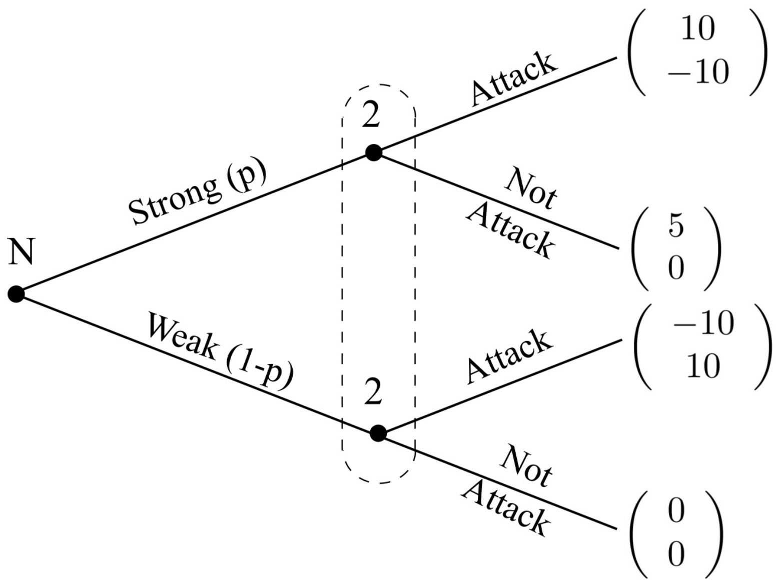

Figure 8: Example of Harsanyi conversion. Each possible type of player 1 is present in one subtree (not subgame!) following Nature's initial move.

- Now the game has been converted into a game of imperfect information, since player 2 does not know if they are on the upper branch or the lower branch.

- Given prior probability p that player 1 is strong player 2 can choose whether to attack or not based on player 2's expected payoff from each alternative.

- Player 2's expected payoffs from attack and not are:

$$E[\pi^2(attack)] = p(-10) + (1-p)10 = 10 - 20p$$

$$E[\pi^2(not)] = p(0) + (1-p)0 = 0$$

- Thus, player 2's optimal choice is to attack if $p < \frac{1}{2}$, not attack if $p > \frac{1}{2}$, and indifferent if $p = \frac{1}{2}$.

We have the following Harsanyi "cookbook" for games of incomplete information.

- Use this when you are confronted with a game where all the players payoffs are not completely known.

1. Encapsulate the incomplete information as a set of types (like strong or weak in the above example) for one or more players.
    (a) These types must cover all the relevant possibilities regarding payoffs and beliefs.
2. Specify type contingent games, one for each type combination.
3. Tie the type contingent games together by an initial Nature move and information set.
4. Assign probs of Nature's initial move. This was called a common prior in Chapter 3.
5. Solve the "big" game for NE and subgame perfect NE, keeping track of all the relevant probabilities via Bayes' rule.

6 Equilibrium in Bayesian Games

- Formal definition of BNE from MCWG pg. 255. In addition to the usual sets of players (I), strategy sets S_i and payoff functions u_i, it also explicitly keeps track of possible types ($\theta \in \Theta$) and their prior distribution $F(\theta)$.
- A strategy profile $(s_1, ..., s_I)$ is a BNE in a Bayesian game $[I, \{S_i\}, \{u_i(.)\}, \theta, F(\theta)]$ if and only if, for all i, and all $\bar{\theta}_i \in \theta_i$ occurring with positive probability,

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s'_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\bar{\theta}_i] \forall s'_i \in S_i \tag{1}$$

    - Note that $\bar{\theta}_i$ is a type realization for player i, θ is the set of all types for all players, u_i is a utility function, $F(\theta)$ is a probability distribution across the set of types, and the expectations are taken over the realizations of all other players' random variables, conditional on player $i's$ realization of their own type $\bar{\theta}_i$.

Refinements of Bayesian games are expressed in terms of beliefs μ as well as strategies. Here are the key properties.

A strategy profile $x = (x_1, x_2, \ldots, x_I)$, possibly mixed, and beliefs μ at each information set such that:

1. Beliefs for each player are consistent with initial Nature move probabilities, the equilibrium strategy profile x, and Bayes theorem.

2. Strategies for each player i maximize her expected payoffs conditioned on her beliefs μ_i.

3. (1) and (2) hold in every subgame.

4. Robust to sufficiently small trembles in x. (The given equilibrium is the limit of equilibria of trembled games as tremble amplitude $\to 0$.)

- (1) and (2) constitute a Bayesian Nash equilibrium (BNE).

- (1), (2) and (3) constitute a perfect Bayesian Nash equilibrium (PBE).

- (1), (2), (3) and (4) constitute a sequential equilibrium.

Need some examples here...

7 Behavioral Considerations

Social preferences, and vengefulness.

- Take another look at the game in Figure 6. Imagine that payoffs are tens of dollars, and you are player 2. Player 1 could have split the \$40 evenly, keeping \$20 and letting you have \$20. Instead, he is trying to keep \$30, leaving you only \$10. Would that make you mad? You can retaliate by choosing R, so the greedy guy gets \$0. Of course, you would also get \$0, but maybe it is worth it.

- This exemplifies social preferences, where you care about the entire allocation, not just your own piece of it.

- Of course, a vengeful personality as above has utilities that are quite different from the payoffs in Figure 6. Suppose for example, that player 2's utility of the unequal split is -2 and of the zero payoff is 0.

- The SPNE of this transformed game is (U,R).

- Here is a unifying exercise. Player 2 has two possible types, Normal (with payoffs as in the original Figure 6) or Vengeful (payoffs as in the previous bullet point). Write out the Harsanyi conversion for this situation, and find the PBE.

Mention Bob Rosenthal and his reason for inventing the Centipede game.

McKelvey and Palfrey (1992) Econometrica and a slew of subsequent studies of the centipede game in the lab show that NE play is rarely observed. Usually players ignore the SPNE strategies and keep passing for quite a while, though not until the very end.

BI is much more reliable when there are only a few steps in the iteration, when the situation is transparent, and when the players are experienced. (cite Larry Samuelson article of around 2000)

8 Further Readings and Resources

Samuelson article of around 2000

McKelvey and Palfrey (1992) Econometrica

Cox, Friedman and Gjerstad (200x) and Cox, Friedman and Sadiraj (200x) contribute to the literature on social preferences by examining two player EFGs like the one in Figure 6.

Rosenthal, R. (1981). "Games of Perfect Information, Predatory Pricing, and the Chain Store". Journal of Economic Theory. 25 (1): 92–100.

Don't forget to mention Gambit.