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Hi Fractal People!

(Note: Because Stig has realised his third version of Cubic Parameterspace, that is CubicParameterspace3 included in his sp3-module, I've modified the formulas in the first group to this sub-module. They are called MilnorAB3, SorensenCubic3 etc. The resulting images are the same as those obtained by the older sets of sub-modules. I've put them in a new module ik3. Note there is no ik2-module, since the "2-versions" of the above formulas are included in my old ik-module. I named it "ik3 in order to be a parallel to sp3 and the third version of the cubic sub-modules. I strongly recommend everyone to use the new versions for new images. The attached parameter-files mentioned below use the new sub-modules).

In order to help you to get started with my ik- and ik3-modules, I've supplied a set of parameter-files, described below. In those I've made certain setups for magnification, period-check on/of etc. I've turned the period-check "of" where artifacts seem to appear. The sub-modules, all dealing with cubics, are divided into two groups. In the first group I have replaced the standard formula, z -> p(z) = z^3 - 3a^2 z + b with some other variants used by the great mathematicians. In order to get these variants I've modified Stig's sub-module *Cubic Parameterspace3.* In the second group the standard-formula is employed, but other methods than making regular slices of the four dimensional cubic monster are used. I don't know the mathematical meaning, but I suspect them to be some kind of "projection-methods". Most of them I have received them from *Stephen C Ferguson*, who has received them from Professor *Holger Jaenisch.* Not being a programmer I'm guilty many thanks to Stig Pettersson, who kindly have helped me with this module, especially by attaching Switch Julia to the second group. Thanks also to Stephen C Ferguson who sent me the source-code for the "projection-methods". Now a brief description of my sub-modules (Hint: The sign "^" means 'power of', for example z^3 means 'the third power of z', i e z*z*z. The sign "_" means 'index', and "sqrt(z)" means 'the square-rot of z)'.

First group. Variations of the cubic formula:

1) Cubic Parameterspace: First I have supplied a startup-file for Stig's "Cubic Parameterspace" just for reference. Here the standard formula, z -> z^3 - 3a^2 z + b is dealt with. In all startup-files two layers are used, named M+ and M- each having it's own color-gradient. M- is the set were "z" is initialized to the other critical point, in this case z = -a. In this startup-file (b_real, b_imag) is plotted and (a_real, a_imag) are fixed to zero, which gives the standard Cubic Mandelbrot set. If you fix a_real to 0.57735 by typing this value to "a-real" for both M+ and M- you will obtain the same figure as figure 7 (last Figure) in my "Cubic Tutorial":

http://user.tninet.se/~cim027f/CubTut/cubictut.html

If you don't have read this tutorial yet, it can be suitable to do so now, as all obscure terms in this GetStarted are explained there. Note: Every change done in dialog-boxes must be done for both M+ and M-. In the following startup-files (a_real, a_imag) are plotted as default.

2) MilnorAB: Here I've modified Stig's sub-module in order to study cubic parameter space in the way of Professor *John Milnor* in his paper "Remarks on iterated cubic maps" published in 1991. From the well-known standard-formula z -> p(z) = z^3 - 3a^2 z + b, Milnor have done the substitution A = a^2 and B = b^2 in order to study what he calls "the Moduli Space". That means that he iterates z -> z^3 -3Az + sqrt(B) with the critical points z = +sqrt(A) and z = -sqrt(A) instead of z = +a and z = -a. The critical points are obtained by putting the derivative to zero.

3) SorensenCubic: The Danish mathematician *Dan Sörensen* has written some small special applications for Macintosh for drawing fractal for scientific studies. For Cubics he, besides the standard formula (z -> z^3 - 3a^2 z + b), also uses the formula: z -> z^3 + az^2 + bz. Here the critical points becomes z = -a/3 + sqrt(a^2/9 - b/3) and z = -a/3 - sqrt(a^2/9 - b/3). The parent fractals have large range, so I've put the magnification to 0.5 in order to see the whole slices of the set. A strange thing when looking at slices of (a_real, a_imag) is that the whole of the right half of the plane belongs to M+, and the whole of the left half of the plane belongs to M-. Therefor it's suitable to use two layers, one for each subset, as been done in my parameter-files. In fact that's almost always suitable to do so when drawing 2D-slices of cubic parameter space.

4) KullbergCubic: Using the non-centered parametrization p(z) = z^3 + az^2 + b you get the critical points z = 0 and z = -2a/3. The story of this can be read at:

http://user.tninet.se/~cim027f/frholmes/cubic.html

Here, by modifying the submodule of Stig, all 6 perpendicular systems of planes can be studied and also with respect to the other critical point. When displaying (a_real, a_imag), when "b" is fixed to zero, and z is initialized to zero, you get a completely black screen. That's natural because for every "a" you get the orbit 0 -> 0 ->0. However if you initialize z to the other critical point "-2a/3" you obtain a "CCAP-shape" (however more than twice as big). Displaying (b_real, b_imag) gives the same images as (b_real, b_imag) in the standard-formula. The coordinates, however, are different. See the above URL.

5) DevaneyCubic and *DevaneyIICubic*: In his book "A First Course in Chaotic Dynamical Systems" Professor *Robert Devaney* makes the parametrization p(z) = z^3 + az + b, which gives the critical points z = +sqrt(-a/3) and z = -sqrt(-a/3) or z = +sqrt(a/3)i and z = -sqrt(a/3). Originally I used the last one because I had an obscure idea that UF would have problems calculating square roots of negative numbers. But negative numbers will occur sometimes in any case, and UF are written for calculating complex numbers. The two ways of initiating z will only make difference regarding some symmetries, the two subsets changing place etc. I encourage the dear fractalling reader to use DevaeyIICubic (using the first mentioned setup of critical points) for new images.

6) BrannerSpecialCubic: In a paper "The iteration of cubic polynomials. Part II: Patterns and parapatterns" (Acta Mathematica 69: 3 - 4, 229 - 325), the two great mathematicians *Bodil Branner* and *John Hamal Hubbard* in one place (page 237) make the alternative parametrization p(z) = (b/4 - a/4)(z^3 - 3z) + (a + b)/2 in order to prove a certain statement, which I don't understand a bit of. However I decided to include the function in my module. The two critical points turns out to be z = +1 and z = -1. Besides normal features of pictures of cubic parameter space, there are two slices, (a_real, a_imag) with "b" fixed to +1+0i, and (b_real, b_imag) with "a" fixed to -1+0i (M+ and M- coalesces in these slices) which have very interesting properties. I leave to the fractal-exploring people to make their own investigation.

7) EpsteinCubic (NEW): In the paper *GEOGRAPHY OF THE CUBIC CONNECTEDNESS LOCUS I: INTERTWINING SURGERY* the great mathematicians *Adam Epstein* and *Michael Yampolsky* besides the standard iteration-formula use the formula z -> a(z^3 – 3z) + b. Critical points are z = +1 and z = -1. In the special case if you plott the b-plane when "a" is fixed to zero, you obtain a completely black screen (natural since you for every initializing of "z" from the second iteration obtain the orbit b -> b -> b etc).

The second group. Other methods of displaying 2D images of the standard-formula:

1) CBAP and *2) CCAP:* These variants of cubics occur in some of the applications of Ferguson. He has obtained the formulas from a Professor Holger Jaenisch who, I suspect has obtained them from the very big mathematicians on the field of iteration of cubic polynomials. The reason for this assumption is that I've seen details of both in a slide series from Art Matrix in the early nineties. In one of the scenes in the two hours video-show "MANDELBROT SETS and JULIA SETS" from the above company, there is a deep zoom sequence in CBAP. Asking Ferguson for the source code of these variants, he displayed the CBAP code along with some of his images on abpf. Later he sent this and the below formulas to me by email (thanks Steve!). The quasi-code runs as:

init:
t=#pixel
a=(t^2+1)/3t
b=2a^3+(t^2-2)/3t

z=-a
loop:
z = z^3 - 3a^2 z + b

By an accident, when playing around with the above expression, I received CCAP. This was done simply by deleting "+1" and "-2" from the above! Being of a to large scale it received the correct size by deleting "3". NOTE: If you do that on the unmodified CBAP above, you get another fractal. After abbreviation the quasi-code of CCAP runs as:

init:
a=#pixel
b=2a^3+a
z=-a
loop:
z = z^3 - 3a^2 z + b

From “Lecture 12” (see 6 below) I’ve now learned that “b” is selected so that p(a) = a. This means that a^3 – 3a^3 + b = a which makes b = 2a^3 + a. The fact that the critical point z = +a always goes to a fix-point, and thus M+ covers the whole plane, explains why the secondary decorations attached to the copies of the Mandelbrot sets in M- have the shape of 1-periodic Julia sets with parameter values picked from the “center” of the Mandelbrot set.

The shape of CCAP also occurs in SörensenCubic and KullbergCubic in (a_real, a_imag) when “b” is fixed to zero. However the size in these two cases are between 2 – 3 times as big.

3) CFAP and *4) CGAP:* Also these formulas come from Professor Holger Jaenisch. The shape of CFAP also occurs in Milnor AB in (A_real, B_imag), SorensenCubic in (b_real, b_imag), and DevaneyCubic and EpsteinCubic in (a_real, a_imag) when intersecting origo. However the size in the two last cases are twice as big. There are also differences of the Julia sets, especially between CFAP and the other three. I leave to the diligent fractal explorers to make there own investigations. The quasi-code of CFAP runs as:

init:
a=#pixel
b=2a^3-2a
z=-a
loop:
z = z^3 - 3a^2 z + b

and the quasi-code of CGAP runs as:

init:

a=#pixel
b=2a^3+1
z=-a
loop:
z = z^3 - 3a^2 z + b

5) SteveCubic: This cubic formula occurs in Flarium24 as #31 and is created by Stephen Ferguson himself. The quasi-code runs as:

init:
t=#pixel
a=(t^3-1)/3t
b=2a^3-2a
z=-a

loop:
z = z^3 - 3a^2 z + b

The origin and meaning of these formulas I have no understanding of. I suspect they are some kind of "projections" rather than slices. An important thing regarding Julia sets from all the above methods as far as I've seen is that they all can be obtained from ordinary slices of cubic parameter space.

6) HomerCubic: This displaying variant I received from UNIVERSITY OF ROCHESTER *Mathematics* Lecture 12. Today the link is corrupt. The resulting fractal turned out to be the same as one in the scenes in the video-show "MANDELBROT SETS and JULIA SETS" from Art Matrix. I named it HomerCubic because Homer Wilson Smith together with Jane Elizabeth Staller are the editors of this video-show. The C-locus of the set is also displayed by Rudy Rucker, who named it "the Rudy set" in his site:

http://www.mathcs.sjsu.edu/faculty/rucker/cubic_mandel.htm

The quasi-code runs as:

init:
a=#pixel
b=a
z=+a (The layer M+)
z=-a (The layer M-)
loop:
z = z^3 - 3a^2 z + b

7) MysticCubic: This variant is also received from "Lecture 12" The parameter "b" is chosen so that p(-a) = +a. This means that (-a)^3 + 3a^3 + b = +a which makes b = a - 2a^3. This also means that only one critical orbit needs to be tested, and both subsets coalesce. The fractal has the same properties as the last mentioned slices in BrannerSpecialCubics. That's the reason for naming it " MysticCubic". The quasi-code runs as:

```
init:
a=#pixel
b=a-2a^3
z=a
loop:
z = z^3 - 3a^2 z + b
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Non-cubic formulas:

Article15: The iteration-formula is z -> z^2 + pz + c. You can vary "p" (z-coefficient) and the Starting point (the value to which "z" is initialized) and draw the c-plane. If you put the "Starting point" to minus half the z- coefficient, you always come up with the standard Mandelbrot set. Otherwise you come up with a so-called "Perturbed M set". All this is dealt with in Article 15 in *the Chaotic series*, uploaded May 2006 as pdf-dokumensts at:

http://klippan.seths.se/fractals/articles/index.html

NOTE: This sub-module is written only for pedagogical reasons and does not contribute any new forms that can not be brought out from the standard Mandelbrot formula in Ultra Fractal.

Multicorns: This is a generalizing of the tricorn-formula z -> conj(z)^2 + c to the general z -> conj(z)^d + c with free choice of the exponent "d". If z = x + iy, conj(z) = x – iy. I got the idea to this generalization from a note in a paper. In the attached startup-parameter file *Diff-Bailout* (see below) is enabled to denote 1-periodic component which otherwise would be black

Compasses: This module performs the iteration: z -> z^d – d a^(d-1) z, the critical point being z = a, and the "a-plane" the parameter plane. This formula is constructed in connection with Article 27, *Compasses* in the Chaotic series mentioned above, where the cases where the exponent "d" is settled to 2, 3, 4, ……etc is discussed. The title "Compasses" refers on the fact that for d = 3 and higher integers give rise to compass-like fractals. In fact the exponent "d" in this formula can be set to any complex number. Here also Diff-bailout is enabled in the attached startup-parameter file.

ExtendedCompasses: A parameter “b” is added to the above formula, so we actually iterate z -> z^d – d a^(d-1) z + b. As a result we obtain a 4D parameter space built up of the parameters (a, b) like in the cubic formulas in the first group. All systems of perpendicular slices can be studied. If “b” is fixed to zero, the forms of course are identical with those obtained with Compasses. When you move along the non-plotted axes, move very carefully. Start with 0,1. At some point, very big changes take place at some values. When you put the exponent to non-intigers the bailout makes big sense. I myself often use 100000.

Deformed Mandekbrot: Draws the four-dimensional parameter-space for quadratics parametrizised as z -> z^2 – 2az +b, “z” initiated to the critical point “a”. This sub-module is written for Article 29 in the *Chaotic series* in which it is shown that one parameter is enough for quadratics. The interface is the same as in Cubicparameterspace2.

Inverse Multibrots: Instead of drawing z -> z^d + c, this formula draws the inverse parameter plane (c-plane) for different exponents (d) that may be non-integers and complexes. That is the iteration formula is z -> z^d + 1/c. In the inverse sets, zero and infinity have changed place with each other, and the sets are turned inside out. In order to see the whole basin of attraction you have to zoom out a little bit and adjust the center for some degrees. If you unmark the inverse box, this formula will draw the ordinary multibrot sets, z -> z^d + c.

Multicorns2: This formula is the same as in Multicorns, the previous page. That is z -> conj(z)^2 + c. However in this formula you also can run the inverse form, that is z -> conj(z)^2 + 1/c. Just click the box “Inverse”. This is the same manner as in Inverse Multibrots above. Here, however, the non-inverse form is default. In exploring Multicorns you are strongly recommended to run this formula.

Diff-Bailout:

To the formulas in the second group (CBAP etc) and to the new non-cubic formulas the feature *Diff-Bailout* is adopted Cubic Parameterspace2. When “Diff Bailout” is enabled, 1-periodic components are shown with inside-coloring, the colors display the number of iterations required to take the variable “z” in close proximity to a fix point. NOTE: In order to avoid artifacts, turn the period check “off”, and set the bailout to at least 10 000 000. A high iteration-number, 1000 or so, is also recommendable.

The periods of hyperbolic components can even be seen using Stig’s new inside color routine “Period”. Components with different periods are colored in different solid colors. If one wish to now the period of a certain period, for example period

1, type "1" under "Period" and click "enable" under "Research" and components with other periodicity will show the ordinary set-color (usually black).

Fixed bug:

In the first versions based on the old module "Cubic Parameterspace" there is one bug of little interest. It's concerned with the first group as well as with original itself. If you have a parameter-plane flipped or rotated, the Switch Julias are the same as if the parameter-plane was not flipped or rotated. This bug is fixed in the newer versions. Suggestions and comments are always welcome. Enjoy!!!

About Julias:

You shall not take the default-Julias too seriously. I've changed the default seed from (0, 0) to other values in cases where (0, 0) would give rise to a completely black screen. In fact the best method to produce nice Julia sets is to use "Switch mode" in a parameter-plane (non-Julia fractal) and click when you see a nice shape in the little window situated lower right in the screen.

Note that in both the associated Julia to Inverse Multibrots (InverseMultibrotsJulia) and in Multicorns2 (MulitcornsJulia2) above, there is a bug in the fact as for c = 0, UF draws the same Julia set in both the inverse formula (1/c) as in the non inverse (c). That's because UF will prevent zero divides.

Regards,
Ingvar Kullberg

www.come.to/kullberg

And very welcome to my chaotic series of fractal articles:

http://klippan.seths.se/fractals/articles