Chapter 0 Appetizer: using probability to cover geometric sets

Desh Raj

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Exercise 0.1.3.

Solution Part 1

We will first show that for independent random variables, variance of the sum is equal to sum of the variances.

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]^2 - \left[\mathbb{E}\sum_{i=1}^{n} X_i\right]^2 \tag{1}$$

Now consider each of the two terms on the RHS of this equation.

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right]^2 = \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbb{E}\, X_i X_j)$$

Similarly,

$$\left[\mathbb{E}\sum_{i=1}^{n} X_i\right]^2 = \left[\sum_{i=1}^{n} \mathbb{E}\, X_i\right]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\, X_i\, \mathbb{E}\, X_j$$

Using these in 1, we get

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbb{E}\, X_i X_j - \mathbb{E}\, X_i\, \mathbb{E}\, X_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{cov}(X_i, X_j)$$

If the variables are independent, we can write

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{cov}(X_i, X_j) = \sum_{i=1}^{n} \operatorname{cov}(X_i, X_i) = \sum_{i=1}^{n} \operatorname{var}(X_i)$$

Now consider the zero-mean independent random variables given. From the above proof, we have

$$\begin{aligned}
\operatorname{var}\left(\sum_{j=1}^{k} Z_j\right) &= \sum_{j=1}^{k} \operatorname{var}\left(Z_j\right) \\
\mathbb{E}\left\|\sum_{j=1}^{k} Z_j\right\|_2^2 - \left\|\mathbb{E}\sum_{j=1}^{k} Z_j\right\|_2^2 &= \sum_{j=1}^{k} \mathbb{E}\|Z_j\|_2^2 - \sum_{j=1}^{k}\|\mathbb{E}\, Z_j\|_2^2 && \text{(From the definition of variance)} \\
\mathbb{E}\left\|\sum_{j=1}^{k} Z_j\right\|_2^2 - \left\|\sum_{j=1}^{k}\mathbb{E}\, Z_j\right\|_2^2 &= \sum_{j=1}^{k} \mathbb{E}\|Z_j\|_2^2 - \sum_{j=1}^{k}\|\mathbb{E}\, Z_j\|_2^2 \\
& && \text{(Expectation of a sum equals sum of the expectations)} \\
\mathbb{E}\left\|\sum_{j=1}^{k} Z_j\right\|_2^2 &= \sum_{j=1}^{k} \mathbb{E}\|Z_j\|_2^2 && (\mathbb{E}\, Z_j = 0)
\end{aligned}$$

This is the required result in Part 1.

Part 2
This is derived easily from the linearity of expectations.

$$\begin{aligned}
\mathbb{E}\|Z - \mathbb{E}\, Z\|_2^2 &= \mathbb{E}\left(\|Z\|_2^2 - 2Z\,\mathbb{E}\, Z + \|\mathbb{E}\, Z\|_2^2\right) \\
&= \mathbb{E}\|Z\|_2^2 - 2\,\mathbb{E}\, Z\,\mathbb{E}\, Z + \|\mathbb{E}\, Z\|_2^2 && \text{(Expectation of an expectation is the same)} \\
&= \mathbb{E}\|Z\|_2^2 - 2\|\mathbb{E}\, Z\|_2^2 + \|\mathbb{E}\, Z\|_2^2 && \text{(Product of same expected values)} \\
&= \mathbb{E}\|Z\|_2^2 - \|\mathbb{E}\, Z\|_2^2
\end{aligned}$$

which is the required result.

Exercise 0.1.5.

Solution We will first show the lower bound. We have

$$\begin{aligned}
& m \le n \\
\Rightarrow\ & mk \le nk && \text{(for some } k > 0) \\
\Rightarrow\ & mn - mk \ge mn - nk \\
\Rightarrow\ & \frac{n-k}{n} \ge \frac{m-k}{m} && \text{(Dividing both sides by mn)} \\
\Rightarrow\ & \frac{n}{m} \le \frac{n-k}{m-k} \\
\Rightarrow\ & \left(\frac{n}{m}\right)^m \le \frac{n-1}{m-1} \cdots \frac{n-m+1}{1} = \binom{n}{m}
\end{aligned}$$

To show the upper bound, observe that

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\dots(n-m+1)}{m!} \leq \frac{n^m}{m!} \tag{2}$$

Consider the Taylor expansion of e^m:

$$e^m = \sum_{k=0}^{\infty} \frac{m^k}{k!}$$

If we take only the m^{th} term on the RHS, we get

$$e^m > \frac{m^m}{m!} \Rightarrow \frac{1}{m!} < \left(\frac{e}{m}\right)^m. \tag{3}$$

Using 3 in 2, we obtain the upper bound, i.e.,

$$\binom{n}{m} < \left(\frac{en}{m}\right)^m.$$

Exercise 0.1.6. (Improved covering)

Solution This proof follows the proof of Corollary 0.1.4 very closely, and differs only at the final step.

Given that P is a polytrope with N vertices. Let centers of the required balls be defined as

$$\mathcal{N} := \left(\frac{1}{k} \sum_{j=1}^{k} x_j : x_j \text{ are the vertices of } P \right),$$

where $k = \frac{1}{\epsilon^2}$.

For any point $x \in P$, x is within distance $\frac{1}{\sqrt{k}} \leq \epsilon$ from some point in $\mathcal{N}$. Now we just have to show the required bound on $|\mathcal{N}|$. Essentially, this quantity is equal to the number of ways of choosing k elements with replacement from a set of N elements.

$$\begin{aligned} |\mathcal{N}| = \binom{N+k-1}{k} \quad & \leq \left(\frac{e(N+k-1)}{k}\right)^k && \text{(Result from previous problem)} \\ = \left[e(1-\epsilon^2) + e\epsilon^2 N\right]^{\frac{1}{\epsilon^2}} & && \text{(Substituting value of k)} \\ < \left[e + e\epsilon^2 N\right]^{\frac{1}{\epsilon^2}}, & \end{aligned}$$

which is the desired result.