Chapter 1 Preliminaries on random variables

Desh Raj

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Exercise 1.2.2.

Solution The derivation is similar to the one for Lemma 1.2.1, which is given in the book. We have,

$$\begin{aligned} x &= \int_0^x 1dt - \int_{-x}^0 1dt \\ &= \int_0^\infty \mathbb{1}_{t<x} dt - \int_{-\infty}^0 \mathbb{1}_{t>x} dt \end{aligned}$$

Replacing x with random variable X and taking expectation on both sides, we get

$$\begin{aligned} \mathbb{E}\, X &= \int_0^\infty \mathbb{1}_{t<X} dt - \int_{-\infty}^0 \mathbb{1}_{t>X} dt \\ &= \int_0^\infty \mathbb{P}(X > t) dt - \int_{-\infty}^0 \mathbb{P}(X < t) dt, \end{aligned}$$

which concludes our proof.

Exercise 1.2.3.

Solution Since $|X|^p$ is always non-negative, we can use the result in Lemma 1.2.1 to write

$$\mathbb{E}\, |X|^p = \int_0^\infty \mathbb{P}(|X|^p > t) dt. \tag{1}$$

Let $t = z^p$. Then, $dt = pz^{p-1}dz$. Substituting these values in 1, we get

$$\begin{aligned} \mathbb{E}\, |X|^p &= \int_0^\infty \mathbb{P}(|X|^p > z^p) pz^{p-1} dz \\ &= \int_0^\infty \mathbb{P}(|X| > z) pz^{p-1} dz && \text{(Using property of exponentiation)} \\ &= \int_0^\infty pt^{p-1} \mathbb{P}(|X| > t) dt, && \text{(Replacing } z \text{ with } t\text{)} \end{aligned}$$

which is the desired result.

Exercise 1.2.6. Chebyshev's inequality

Solution Given, X is a random variable with mean μ and variance σ^2. We define another random variable Z as $Z = \|X-\mu\|^2$. Then, using Markov inequality, we can write

$$\begin{aligned}
\mathbb{P}\{Z \geq t^2\} &\leq \frac{\mathbb{E}\, Z}{t^2} \\
\mathbb{P}\{\|X-\mu\|^2 \geq t^2\} &\leq \frac{\mathbb{E}\|X-\mu\|^2}{t^2} && \text{(Since } Z = \|X-\mu\|^2\text{)} \\
\mathbb{P}\{\|X-\mu\| \geq t\} &\leq \frac{\mathbb{E}\|X-\mu\|^2}{t^2} \\
\mathbb{P}\{\|X-\mu\| \geq t\} &\leq \frac{\sigma^2}{t^2} && \text{(Since } \mathbb{E}\|X-\mu\|^2 = \sigma^2\text{)}
\end{aligned}$$

,

which is the Chebyshev's inequality.

Exercise 1.3.3.

Solution Given, $X_1, \ldots, X_N$ are a sequence of i.i.d random variables with mean μ and some finite variance. W.l.o.g we can assume that all the variances are equal (say σ^2). We define a random variable $Z = \frac{1}{N}\sum_{i=1}^{N} X_i$. Then, we have

$$\mathbb{E}\, Z = \mu, \qquad \text{and} \qquad \text{var}(Z) = \frac{\sigma^2}{N} \qquad \text{(From earlier results)}$$

Using the definition of variance, we can then write

$$\begin{aligned}
\mathbb{E}\|Z-\mu\|^2 &= \frac{\sigma^2}{N} \\
\Rightarrow \mathbb{E}|Z-\mu| &= \frac{\sigma}{\sqrt{N}} && \text{(Taking positive square root on both sides)} \\
\Rightarrow \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right| &= \frac{\sigma}{\sqrt{N}} = \mathcal{O}(\frac{1}{\sqrt{N}}). && \text{(Using definition of } Z\text{)}
\end{aligned}$$