

# Homography (Part I)

# What Transformation Really Happened To My DVD?



$T?$



# What Transformation Really Happened To My DVD?

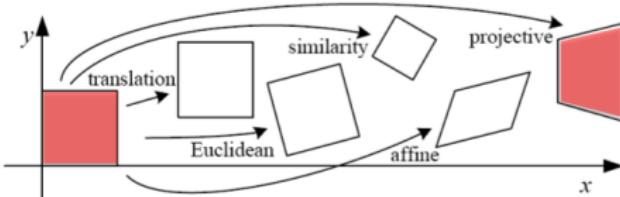
- A rectangle goes to **quadrilateral**



$T?$



# 2D Image Transformations



Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I & t \end{bmatrix}_{2 \times 3}$	2	orientation	□
rigid (Euclidean)	$\begin{bmatrix} R & t \end{bmatrix}_{2 \times 3}$	3	lengths	◇
similarity	$\begin{bmatrix} sR & t \end{bmatrix}_{2 \times 3}$	4	angles	◇
affine	$\begin{bmatrix} A \end{bmatrix}_{2 \times 3}$	6	parallelism	□/◇
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3 \times 3}$	8	straight lines	□/□

- These transformations are a nested set of groups
- Closed under composition and inverse is a member

[source: R. Szeliski]

# Projective Transformations

- Homography:

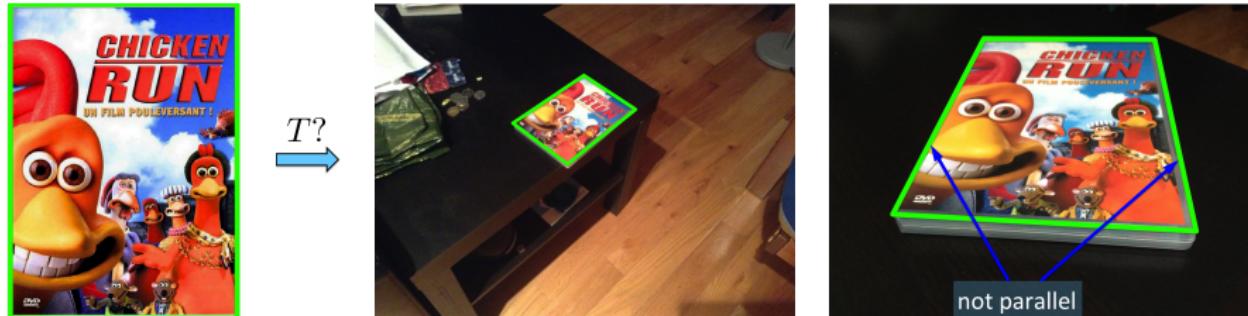
$$w \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Properties:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are **not** preserved
- Closed under composition
- Rectangle goes to quadrilateral
- Affine transformation is a special case, where  $g = h = 0$  and  $i = 1$

[Source: N. Snavely, slide credit: R. Urtasun]

# What Transformation Really Happened to My DVD?



For **planar** objects:

- Viewpoint change for planar objects is a **homography**
- Affine transformation **approximates** viewpoint change for planar objects that are far away from camera

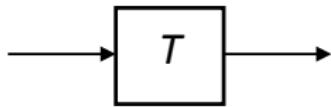
# What Transformation Happened to My DVD?

- Why should I care about homography?
- Now that I care, how should I estimate it?
- I want to understand the geometry behind homography. That is, why aren't parallel lines mapped to parallel lines in oblique viewpoints?  
How did we get that equation for computing the homography?

# Homography

- Why should I care about homography? **Let's answer this first**
- Now that I care, how should I estimate it?
- I want to understand the geometry behind homography. That is, why aren't parallel lines mapped to parallel lines in oblique viewpoints?  
How did we get that equation for computing the homography?

# Warping an Image with a Global Transformation



$$\mathbf{p} = (x, y)$$

$$\mathbf{p}' = (x', y')$$

- Transformation  $T$  is a coordinate-changing machine:

$$[x', y'] = T(x, y)$$

- What does it mean that  $T$  is global?
  - Is the same for any point  $p$
  - Can be described by just a few numbers (parameters)

[Source: N. Snavely, slide credit: R. Urtasun]

# Warping an Image with a Global Transformation

- Example of warping for different transformations:



translation



rotation



aspect

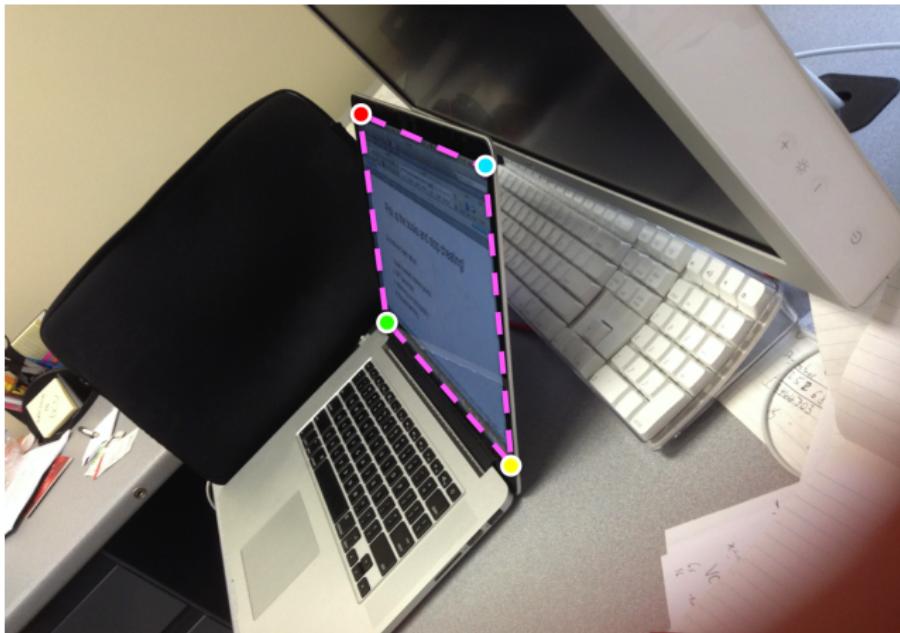


affine



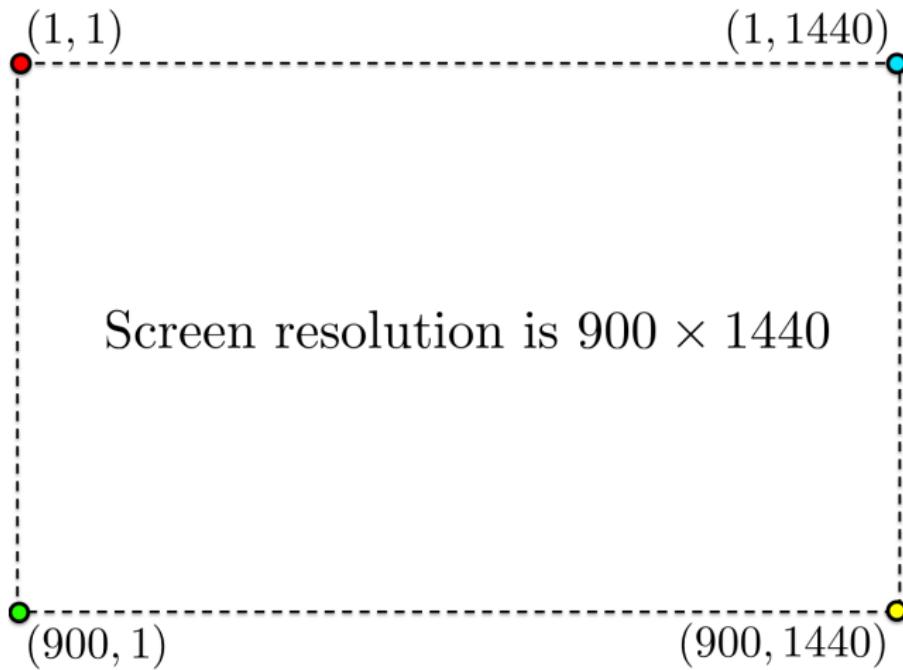
perspective

# Application 1: a Little Bit of CSI



- We want to transform the picture (plane) inside these 4 points into a rectangle (laptop screen)

## Application 1: a Little Bit of CSI

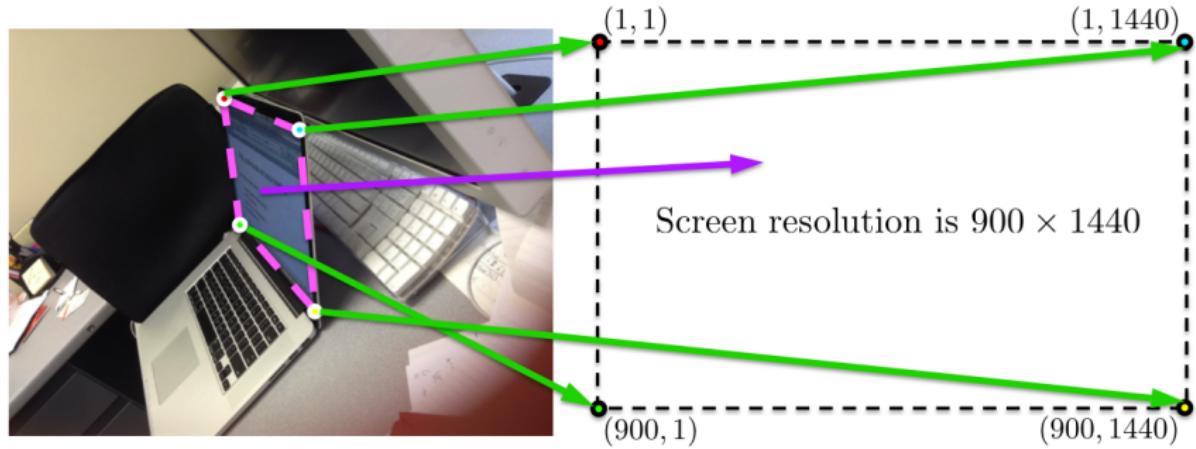


- We want it to look like this. How can we do this?

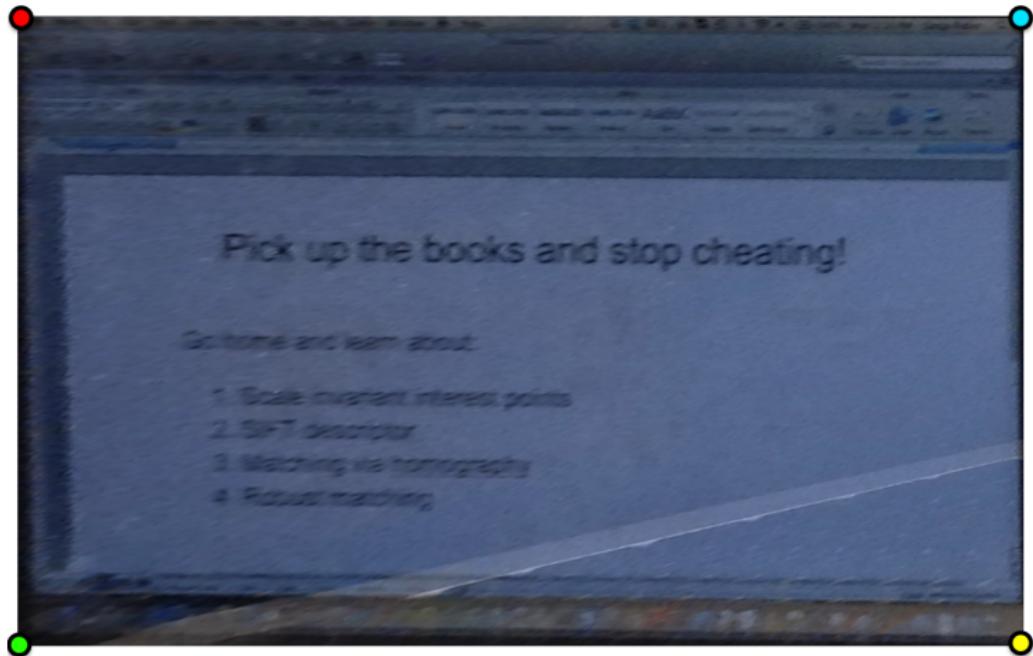
# Application 1: a Little Bit of CSI

- A transformation that maps a projective plane (a quadrilateral) to another projective plane (another quadrilateral, in this case a rectangle) is a homography

homography  $H$



# Application 1: a Little Bit of CSI



- If we compute the homography and warp the image according to it, we get this

## Application 2: How Much do Soccer Players Run?

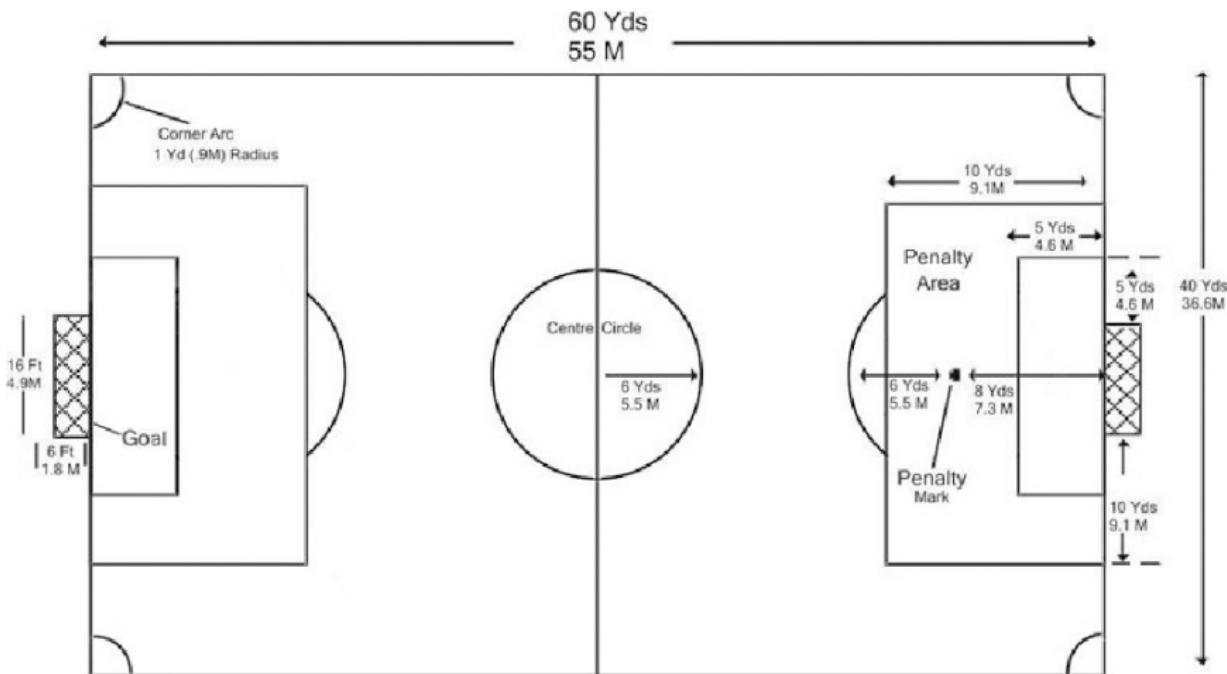


## Application 2: How Much do Soccer Players Run?



- How many meters did this player run?

## Application 2: How Much do Soccer Players Run?



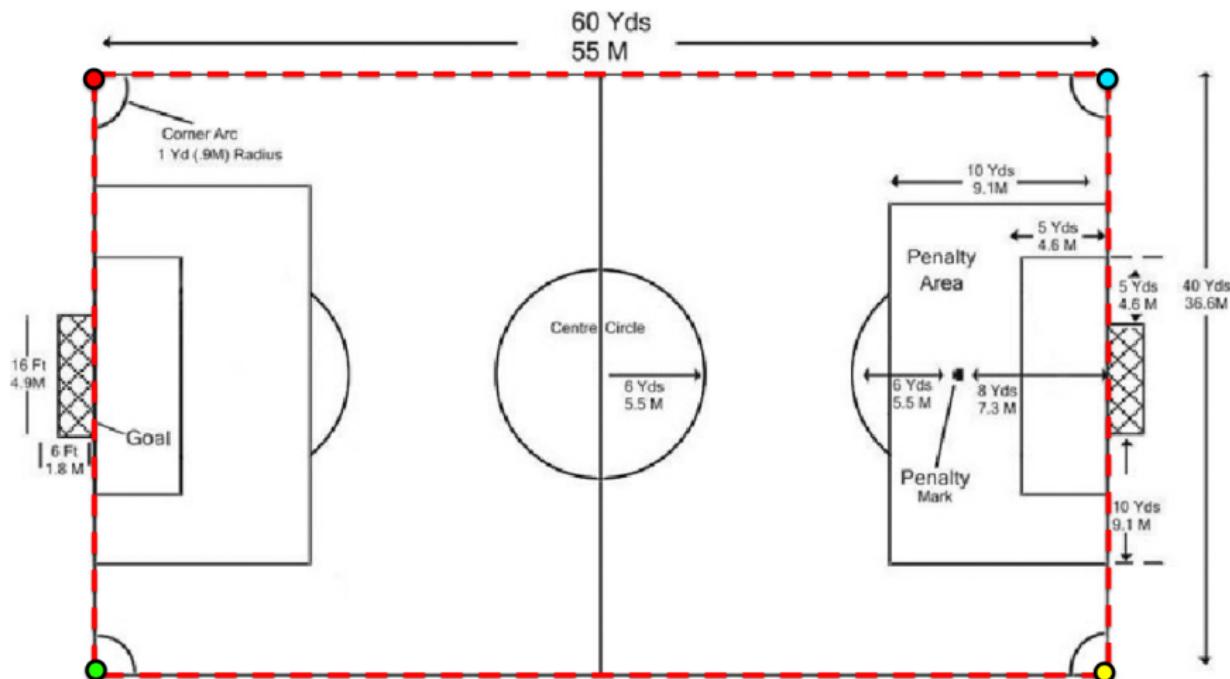
- Field is planar. We know its dimensions (look on Wikipedia).

## Application 2: How Much do Soccer Players Run?



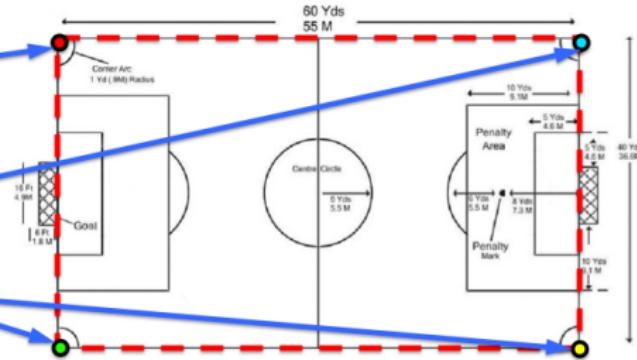
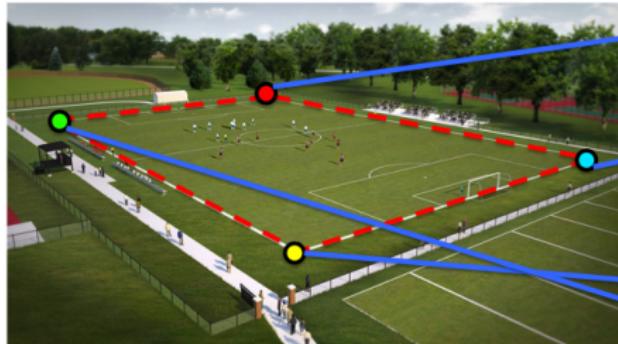
- Let's take the 4 corner points of the field

## Application 2: How Much do Soccer Players Run?



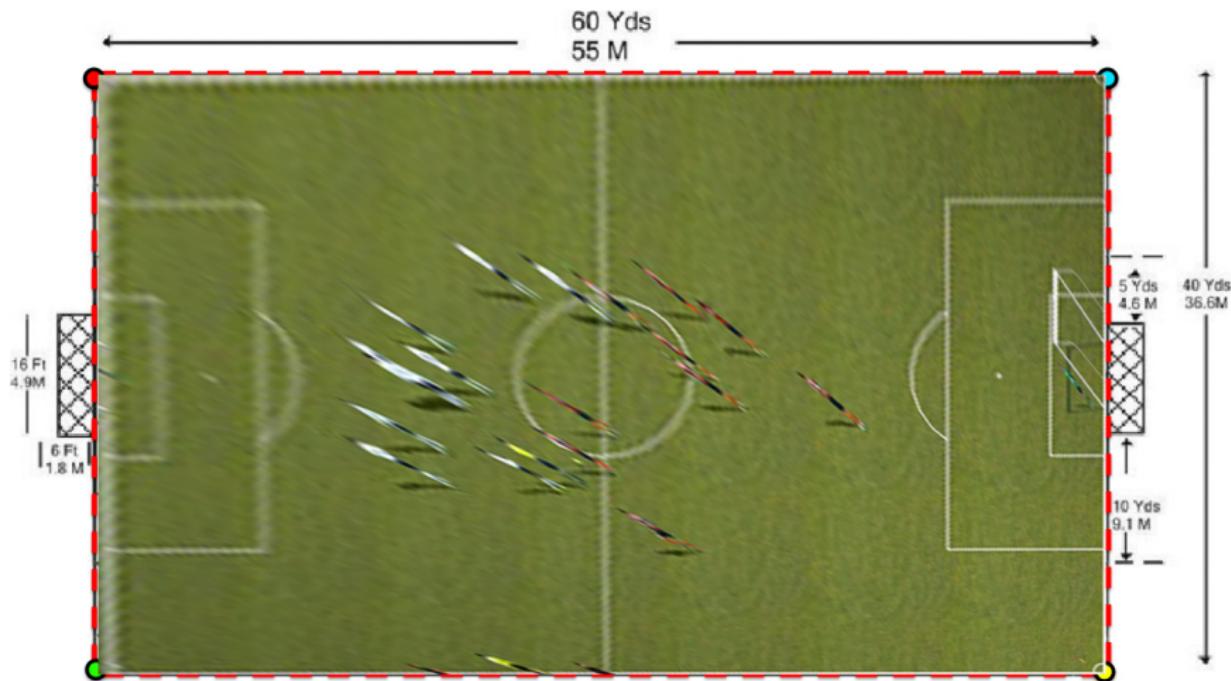
- We need to compute a homography that maps them to these 4 corners

## Application 2: How Much do Soccer Players Run?

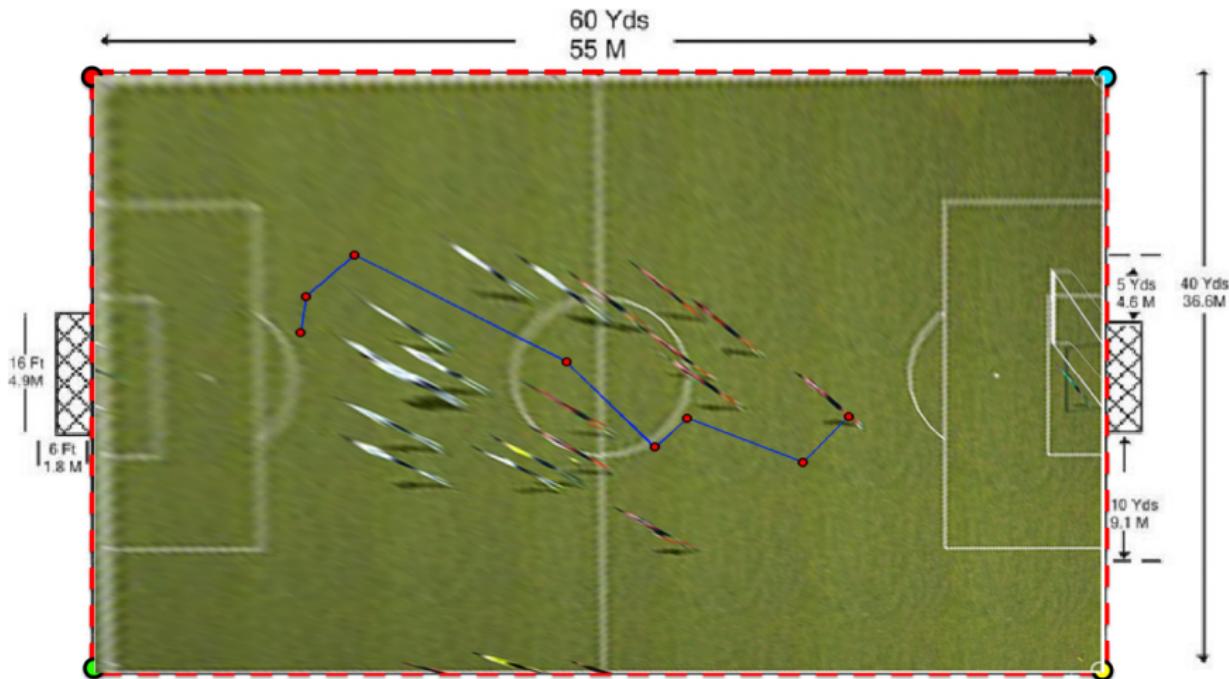


- We need to compute a homography that maps the 4 corners. Any other point from this plane (the field) also maps to the right with the same homography

## Application 2: How Much do Soccer Players Run?



## Application 2: How Much do Soccer Players Run?



- We can now also transform the player's trajectory → and we have it in meters!

# Application 3: Panorama Stitching



Take a tripod, rotate camera  
and take pictures

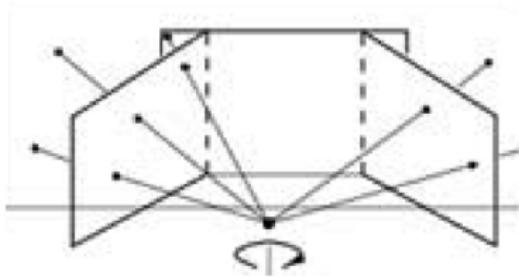
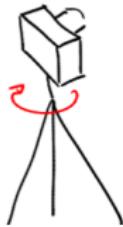
[Source: Fernando Flores-Mangas]

## Application 3: Panorama Stitching



[Source: Fernando Flores-Mangas]

## Application 3: Panorama Stitching



- Each pair of images is related by homography! **If we also moved the camera, this wouldn't be true** (next class)

[Source: Fernando Flores-Mangas]

## Application 3: Panorama Stitching

- To do panorama stitching, we need to:
  - Match points between pairs of images I and J
  - Compute a transformation between the matches in I and J : a homography
  - Do it robustly (RANSAC)
  - Warp the first image to the second using the estimated homography
- Apart from the last point, this is exactly the same procedure as for the problem of matching planar objects across viewpoints
- So this should motivate the **why do I care part** of the homographies

# Homography

- Why should I care about homography?
- Now that I care, how should I estimate it? **Let's do this now**
- I want to understand the geometry behind homography. That is, why aren't parallel lines mapped to parallel lines in oblique viewpoints?  
How did we get that equation for computing the homography?

# Solving for Homographies

- Projective mapping between any two projection planes with the same centre of projection
- Let  $(x_i, y_i)$  be a point on the reference (model) image, and  $(x'_i, y'_i)$  its match in the test image
- A homography  $H$  maps  $(x_i, y_i)$  to  $(x'_i, y'_i)$ :

$$\begin{bmatrix} ax'_i \\ ay'_i \\ a \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

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- We can get rid of that  $a$  on the left (we need a 2D image):

$$x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$
$$y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

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- Hmm... Can I still rewrite this into a linear system in  $h$ ?

# Solving for homographies

- From:

$$\begin{aligned}x'_i &= \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}} \\y'_i &= \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}\end{aligned}$$

- We can easily get this:

$$\begin{aligned}x'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= h_{00}x_i + h_{01}y_i + h_{02} \\y'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= h_{10}x_i + h_{11}y_i + h_{12}\end{aligned}$$

- Rewriting it a little:

$$\begin{aligned}h_{00}x_i + h_{01}y_i + h_{02} - x'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= 0 \\h_{10}x_i + h_{11}y_i + h_{12} - y'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= 0\end{aligned}$$

# Solving for homographies

- We can re-write these equations:

$$\begin{aligned} h_{00}x_i + h_{01}y_i + h_{02} - x'_i(h_{20}x_i - h_{21}y_i - h_{22}) &= 0 \\ h_{10}x_i + h_{11}y_i + h_{12} - y'_i(h_{20}x_i - h_{21}y_i - h_{22}) &= 0 \end{aligned}$$

- as a linear system!

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x'_i x_i & -x'_i y_i & -x'_i \\ 0 & 0 & 0 & x_i & y_i & 1 & -y'_i x_i & -y'_i y_i & -y'_i \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Solving for homographies

- Taking all our matches into account:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x'_1 x_1 & -x'_1 y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1 x_1 & -y'_1 y_1 & -y'_1 \\ & & & & & \vdots & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x'_n x_n & -x'_n y_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -y'_n x_n & -y'_n y_n & -y'_n \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

**A**  
 $2n \times 9$

**h**  
9

**0**  
 $2n$

# Solving for homographies

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- This defines a least squares problem:

$$\min_h ||Ah||_2^2$$

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 $2n \times 9$

**h**  
9

**0**  
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- Can we use Moore-Penrose like last time?

# Solving for homographies Pt.2

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & -x'_1x_1 & -x'_1y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1x_1 & -y'_1y_1 & -y'_1 \\ \vdots & & & & & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x'_nx_n & -x'_ny_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -y'_nx_n & -y'_ny_n & -y'_n \end{bmatrix}_{2n \times 9} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix}_{9 \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{2n \times 1}$$
$$\mathbf{A} \quad \mathbf{h} \quad \mathbf{0}$$

- This defines a constrained least squares problem:

$$\min_{\mathbf{h}} E = \|\mathbf{Ah}\|_2^2$$

$$s.t. \quad \|\mathbf{h}\|_2 = 1 \quad (1)$$

$$(2)$$

- Since  $\mathbf{h}$  is only defined up to scale, solve for unit vector

# Solving for homographies Pt.2

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & -x'_1x_1 & -x'_1y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1x_1 & -y'_1y_1 & -y'_1 \\ \vdots & & & & & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x'_nx_n & -x'_ny_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -y'_nx_n & -y'_ny_n & -y'_n \end{bmatrix}_{2n \times 9} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix}_{9 \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{2n \times 1}$$
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- This is an example of “Constrained” Optimization

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- Method of Lagrange Multipliers

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$$(2)$$

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- Solution:  $\hat{\mathbf{h}} = \text{eigenvector of } \mathbf{A}^T \mathbf{A} \text{ with smallest eigenvalue}$

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- How many matches do I need to estimate  $H$ ?

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- Method of Lagrange Multipliers
- Solution:  $\hat{\mathbf{h}} = \text{eigenvector of } \mathbf{A}^T \mathbf{A} \text{ with smallest eigenvalue}$
- How many matches do I need to estimate  $H$ ?
- Works with 4 or more matches, only 8 unknowns!

[Source: R. Urtasun]

# Image Alignment Algorithm: Homography

Given images  $I$  and  $J$

- ① Compute image features for  $I$  and  $J$
- ② Match features between  $I$  and  $J$

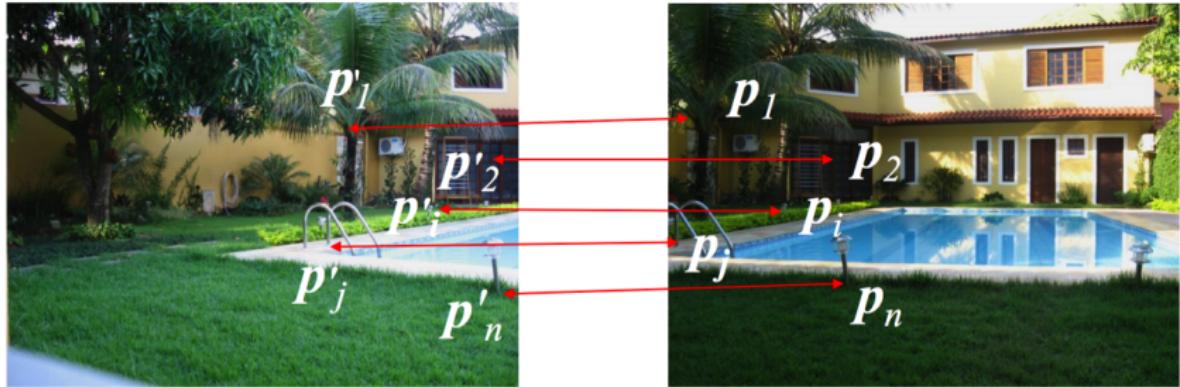
# Image Alignment Algorithm: Homography

Given images  $I$  and  $J$

- ① Compute image features for  $I$  and  $J$
- ② Match features between  $I$  and  $J$
- ③ Compute **homography** transformation  $A$  between  $I$  and  $J$  (with RANSAC)

[Source: N. Snavely]

# Panorama Stitching: Example 1



- Compute the matches

[Source: R. Queiroz Feitosa]

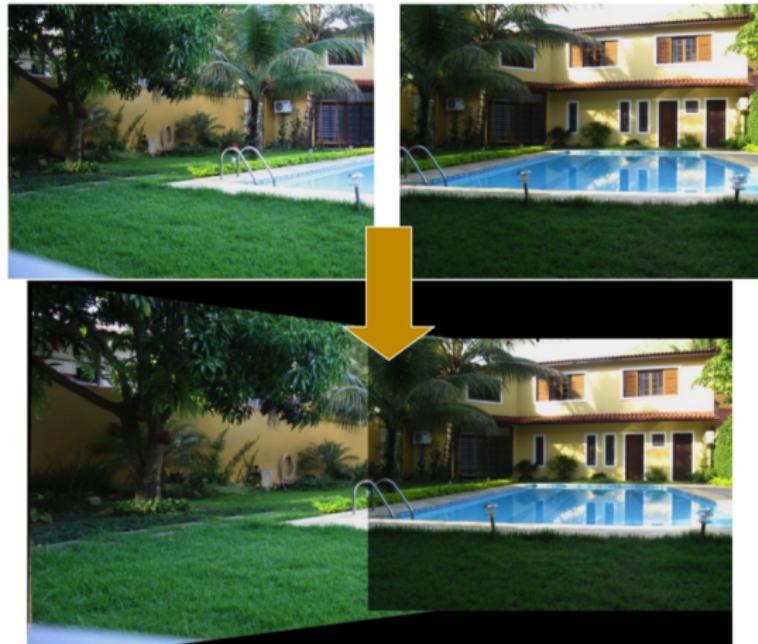
# Panorama Stitching: Example 1



- Estimate the homography and warp

[Source: R. Queiroz Feitosa]

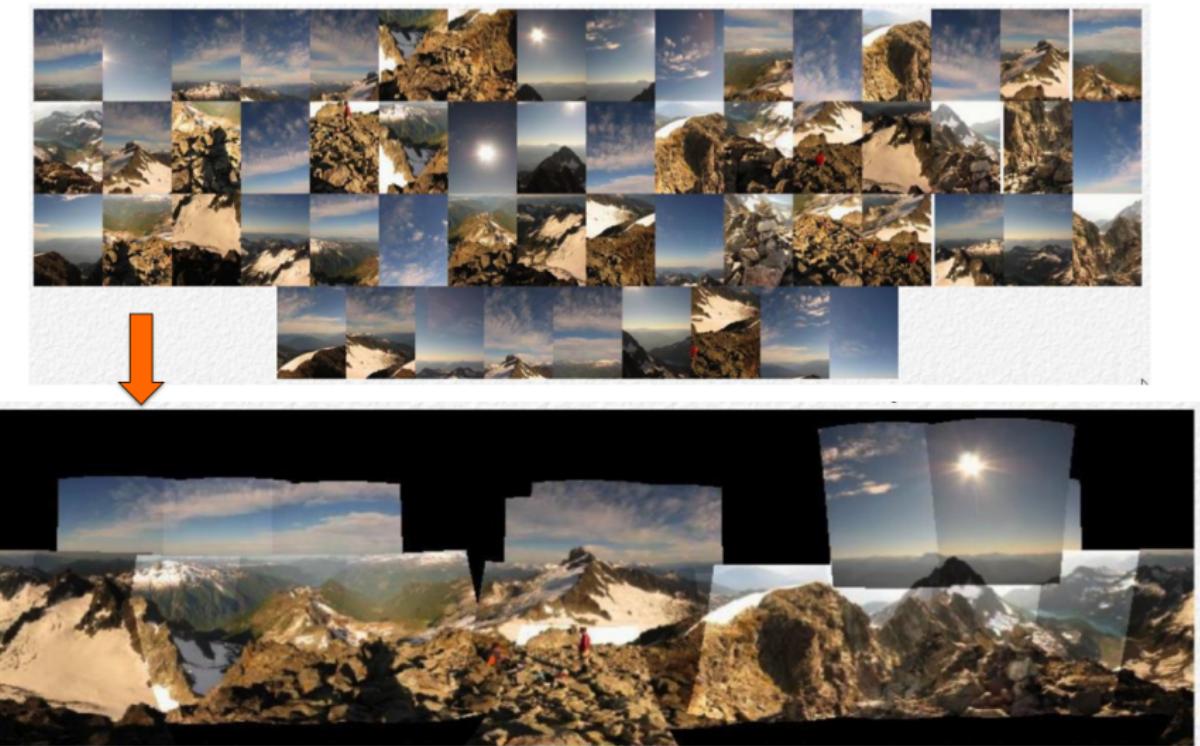
# Panorama Stitching: Example 1



- Stitch

[Source: R. Queiroz Feitosa]

## Panorama Stitching: Example 2



[Source: Fernando Flores-Mangas]

## Panorama Stitching: Example 2



[Source: Fernando Flores-Mangas]

## Panorama Stitching: Example 2



Laplacian Pyramid Blending  $\Downarrow$  seams not visible anymore



(Brown & Lowe; ICCV 2003)

google "Lowe Brown Autostitch"

[Source: Fernando Flores-Mangas]