# Contents

# List of symbols

| Symbol | Description |
|---|---|
| $\mathbb{N}$ | set of nonnegative integers $\{0, 1, 2, 3, \dots\}$ |
| $\mathbb{Z}$ | set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ |
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{C}$ | set of complex numbers |
| $\mathbb{Q}$ | set of rational numbers |
| $\mathbb{Q}(\beta)$ | the smallest field containing the set $\mathbb{Q}$ and algebraic number $\beta$ |
| $\#S$ | number of elements of the finite set $S$ |
| $C^*$ | complex conjugation and transposition of the complex matrix $C$ |
| | |
| $m_\beta$ | monic minimal polynomial of the algebraic number $\beta$ |
| $\deg \beta$ | degree of the algebraic number $\beta$ (over $\mathbb{Q}$) |
| | |
| $(\beta, \mathcal{A})$ | numeration system with the base $\beta$ and the alphabet $\mathcal{A}$ |
| $(x)_{\beta, \mathcal{A}}$ | $(\beta, \mathcal{A})$-representation of the number $x$ |
| $\mathrm{Fin}_\mathcal{A}(\beta)$ | set of all complex numbers with a finite $(\beta, \mathcal{A})$-representation |
| $\mathcal{A}^{\mathbb{Z}}$ | set of all bi-infinite sequences of digits in $\mathcal{A}$ |
| $\mathbb{Z}[\omega]$ | the smallest ring containing $\mathbb{Z}$ and $\omega$ |
| $\pi$ | isomorphism from $\mathbb{Z}[\omega]$ to $\mathbb{Z}^d$ ($d = \deg \omega$) |
| | |
| $\mathcal{B}$ | alphabet of input digits |
| $q_j$ | weight coefficient for the $j$-th position |
| $\mathcal{Q}$ | weight coefficients set |
| $\mathcal{Q}_{[w_j, \dots, w_{j-m+1}]}$ | set of possible weight coefficients for the input digits $w_j, \dots, w_{j-m+1}$ |
| | |
| $\lfloor x \rfloor$ | floor function of the number $x$ |
| $\mathrm{Re}\, x$ | real part of the complex number $x$ |
| $\mathrm{Im}\, x$ | imaginary part of the complex number $x$ |

# Chapter 1

# Preliminaries

## 1.1 Numeration systems

Firstly, we give a general definition of numeration system.

**Definition 1.1.** Let $\beta \in \mathbb{C}, |\beta| > 1$ and $\mathcal{A} \subset \mathbb{C}$ be a finite set containing $0$. A pair $(\beta, \mathcal{A})$ is called a *positional numeration system* with *base* $\beta$ and *digit set* $\mathcal{A}$, usually called *alphabet*.

So-called standard numeration systems have an integer base $\beta$ and an alphabet $\mathcal{A}$ which is a set of contiguous integers. We restrict ourselves to the base $\beta$ which is an algebraic integer and possibly non-integer alphabet $\mathcal{A}$.

**Definition 1.2.** Let $(\beta, \mathcal{A})$ be a positional numeration system. We say that a complex number $x$ has a $(\beta, \mathcal{A})$-*representation* if there exist digits $x_n, x_{n-1}, x_{n-2}, \cdots \in \mathcal{A}, n \geq 0$ such that $x = \sum_{j=-\infty}^{n} x_j \beta^j$.

We write briefly a *representation* instead of a $(\beta, \mathcal{A})$-representation if the base $\beta$ and the alphabet $\mathcal{A}$ follow from context.

**Definition 1.3.** Let $(\beta, \mathcal{A})$ be a positional numeration system. The set of all complex numbers with a finite $(\beta, \mathcal{A})$-representation is defined by

$$\mathrm{Fin}_{\mathcal{A}}(\beta) := \left\{ \sum_{j=-m}^{n} x_j \beta^j : n, m \in \mathbb{N}, x_j \in \mathcal{A} \right\}.$$

For $x \in \mathrm{Fin}_{\mathcal{A}}(\beta)$, we write

$$(x)_{\beta, \mathcal{A}} = 0^{\omega} x_n x_{n-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-m} 0^{\omega},$$

where $0^{\omega}$ denotes right, respectively left-infinite sequence of zeros. Notice that indices are decreasing from left to right as it is usual to write the most significant digits first. In what follows, we omit the starting and ending $0^{\omega}$ when we work with numbers in $\mathrm{Fin}_{\mathcal{A}}(\beta)$. We remark

that existence of an algorithm (standard or parallel) producing a finite $(\beta, \mathcal{A})$-representation of $x + y$ where $x, y \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ implies that the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition, i.e.,

$$\mathrm{Fin}_{\mathcal{A}}(\beta) + \mathrm{Fin}_{\mathcal{A}}(\beta) \subset \mathrm{Fin}_{\mathcal{A}}(\beta) \,.$$

Designing an algorithm for parallel addition requires some redundancy in numeration system. According to [8], a numeration system $(\beta, \mathcal{A})$ is called *redundant* if there exists $x \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ which has two different $(\beta, \mathcal{A})$-representations. For instance, the number 1 has $(2, \{-1, 0, 1\})$-representations $1\bullet$ and $1(-1)\bullet$. Redundant numeration system can enable us to avoid carry propagation in addition. On the other hand, there are some disadvantages. For example, comparison is problematic.

## 1.2 Parallel addition

A local function, which is also often called a sliding block code, is used to mathematically formalize parallelism.

**Definition 1.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be alphabets. A function $\varphi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is said to be *p-local* if there exist $r, t \in \mathbb{N}$ satisfying $p = r + t + 1$ and a function $\phi : \mathcal{B}^p \to \mathcal{A}$ such that, for any $w = (w_j)_{j \in \mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$ and its image $z = \varphi(w) = (z_j)_{j \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, we have $z_j = \phi(w_{j+t}, \cdots, w_{j-r})$ for every $j \in \mathbb{Z}$. The parameter $t$, resp. $r$, is called *anticipation*, resp. *memory*.

This means that each digit of the image $\varphi(w)$ is computed from $p$ digits of $w$ in a sliding window. Suppose that there is a processor on each position with access to $t$ input digits on the left and $r$ input digits on the right. Then computation of $\varphi(w)$, where $w$ is a finite sequence, can be done in constant time independent on the length of $w$.

**Definition 1.5.** Let $\beta$ be a base and $\mathcal{A}$ and $\mathcal{B}$ two alphabets containing 0. A function $\varphi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that

1. for any $w = (w_j)_{j \in \mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$ with finitely many non-zero digits, $z = \varphi(w) = (z_j)_{j \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ has only finite number of non-zero digits, and

2. $\sum_{j \in \mathbb{Z}} w_j \beta^j = \sum_{j \in \mathbb{Z}} z_j \beta^j$

is called *digit set conversion* in base $\beta$ from $\mathcal{B}$ to $\mathcal{A}$. Such a conversion $\varphi$ is said to be *computable in parallel* if $\varphi$ is a $p$-local function for some $p \in \mathbb{N}$.

In fact, addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel if there is a digit set conversion from $\mathcal{A} + \mathcal{A}$ to $\mathcal{A}$ computable in parallel as we can easily output digitwise sum of two $(\beta, \mathcal{A})$-representations in parallel.

We recall few results about parallel addition in a numeration system with an integer alphabet. C. Frougny, E. Pelantová and M. Svobodová proved the following sufficient condition of existence of an algorithm for parallel addition in [4].

**Theorem 1.1.** *Let $\beta \in \mathbb{C}$ be an algebraic number such that $|\beta| > 1$ and all its conjugates in modulus differ from 1. There exists an alphabet $\mathcal{A}$ of contiguous integers containing 0 such that addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel.*

The proof of the theorem provides the algorithm for the alphabet of the form $\{-a, -a + 1, \ldots, 0, \ldots, a - 1, a\}$. But in general, $a$ is not minimal.

The same authors showed in [3] that the condition on the conjugates of the base $\beta$ is also necessary:

**Theorem 1.2.** *Let the base $\beta \in \mathbb{C}, |\beta| > 1$, be an algebraic number with a conjugate $\beta'$ such that $|\beta'| = 1$. Let $\mathcal{A} \subset \mathbb{Z}$ be an alphabet of contiguous integers containing 0. Then addition on $Fin_{\mathcal{A}}(\beta)$ cannot be computable in parallel.*

The question of minimality of the alphabet is studied in [5]. The following lower bound for the size of the alphabet is provided:

**Theorem 1.3.** *SJEDNOTIT ZNACENI MIN POLY*
*Let $\beta \in \mathbb{C}, |\beta| > 1$, be an algebraic integer with the minimal polynomial $p$. Let $\mathcal{A} \subset \mathbb{Z}$ be an alphabet of contiguous integers containing 0 and 1. If addition on $Fin_{\mathcal{A}}(\beta)$ is computable in parallel, then $\#\mathcal{A} \geq |p(1)|$. Moreover, if $\beta$ is a positive real number, $\beta > 1$, then $\#\mathcal{A} \geq |p(1)| + 2$.*

In this thesis, we work in a more general concept as we consider also non-integer alphabets. First, we recall the following definition.

**Definition 1.6.** Let $\omega$ be a complex number. The set of values of all polynomials with integer coefficients evaluated in $\omega$ is denoted by

$$\mathbb{Z}[\omega] = \left\{ \sum_{i=0}^{n} a_i \omega^i \colon n \in \mathbb{N}, a_i \in \mathbb{Z} \right\} \subset \mathbb{Q}(\omega).$$

Notice that $\mathbb{Z}[\omega]$ is a commutative ring (for our purposes, a ring is associative under multiplication and there is a multiplicative identity).

From now on, let $\omega$ be an algebraic integer which generates the set $\mathbb{Z}[\omega]$ and let the base $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$. We remark that $\beta$ is also an algebraic integer as all elements of $\mathbb{Z}[\omega]$ are algebraic integers. Finally, let the alphabet $\mathcal{A}$ be a finite subset of $\mathbb{Z}[\omega]$ such that $0 \in \mathcal{A}$.

Few parallel addition algorithms for such numeration system with a non-integer alphabet were found ad hoc. We introduce the method for construction of the parallel addition algorithm for a given numeration system $(\beta, \mathcal{A})$ in Chapter **??**.

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# Chapter 2

# Design of extending window method

We recall the general concept of addition at the beginning of this chapter and then we describe a so-called *extending window method* which is due to M. Svobodová [9].

From now on, let $\omega$ be an algebraic integer and $(\beta, \mathcal{A})$ be a numeration system such that the base $\beta \in \mathbb{Z}[\omega]$ and the alphabet $\mathcal{A} \ni 0$ is a finite subset of $\mathbb{Z}[\omega]$.

The general concept of addition (standard or parallel) in any numeration system $(\beta, \mathcal{A})$, such that $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition, is the following: we add numbers digitwise and then we convert the result into the alphabet $\mathcal{A}$. Obviously, digitwise addition is computable in parallel, thus the crucial point is the digit set conversion of the obtained result. It can be easily done in a standard way but a parallel digit set conversion is nontrivial. However, formulas are basically the same but the choice of coefficients differs.

Now we go step by step more precisely. Let $x = \sum_{-m'}^{n'} x_i \beta^i, y = \sum_{-m'}^{n'} y_i \beta^i \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ with $(\beta, \mathcal{A})$-representations padded by zeros to have the same length. We set

$$w = x + y = \sum_{-m'}^{n'} x_i \beta^i + \sum_{-m'}^{n'} y_i \beta^i = \sum_{-m'}^{n'} (x_i + y_i) \beta^i$$

$$= \sum_{-m'}^{n'} w_i \beta^i \, ,$$

where $w_j = x_j + y_j \in \mathcal{A} + \mathcal{A}$. Thus, $w_{n'} w_{n'-1} \cdots w_1 w_0 \bullet w_{-1} w_{-2} \cdots w_{-m'}$ is a $(\beta, \mathcal{A} + \mathcal{A})$-representation of $w \in \mathrm{Fin}_{\mathcal{A}+\mathcal{A}}(\beta)$.

We also use column notation of addition in what follows, e.g.,

$$
\begin{array}{l}
x_{n'} \; x_{n'-1} \cdots x_1 \; x_0 \bullet x_{-1} \; x_{-2} \; \cdots x_{-m'} \\
y_{n'} \; y_{n'-1} \cdots y_1 \; y_0 \bullet y_{-1} \; y_{-2} \; \cdots y_{-m'} \\
\hline
w_{n'} w_{n'-1} \cdots w_1 w_0 \bullet w_{-1} w_{-2} \cdots w_{-m'} \, .
\end{array}
$$

As we want to obtain a $(\beta, \mathcal{A})$-representation of $w$, we search for a sequence

$$z_n z_{n-1} \cdots z_1 z_0 z_{-1} z_{-2} \cdots z_{-m}$$

such that $z_j \in \mathcal{A}$ and

$$z_n z_{n-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots z_{-m} = (w)_{\beta,\mathcal{A}}.$$

From now on, we consider without lost of generality only $\beta$-integers since modification for representations with rational part is obvious:

$$\beta^m \cdot z_n z_{n-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots z_{-m} = z_n z_{n-1} \cdots z_1 z_0 z_{-1} z_{-2} \cdots z_{-m} \bullet$$

Particularly, let $(w)_{\beta,\mathcal{A}+\mathcal{A}} = w_{n'} w_{n'-1} \cdots w_1 w_0 \bullet$. We search for a number $n \in \mathbb{N}$ and $z_n, z_{n-1}, \ldots, z_1, z_0 \in \mathcal{A}$ such that $(w)_{\beta,\mathcal{A}} = z_n z_{n-1} \cdots z_1 z_0 \bullet$.

We use a suitable representation of zero to convert digits $w_j$ into the alphabet $\mathcal{A}$. For our purpose, we use the simplest possible representation deduced from the polynomial

$$x - \beta \in (\mathbb{Z}[\omega])[x].$$

We remark that any polynomial $R(x) = r_s x^s + r_{s-1} x^{s-1} + \cdots + r_1 x + r_0$ with coefficients $r_i \in \mathbb{Z}[\omega]$, such that $R(\beta) = 0$ gives us a possible representation of zero. The polynomial $R$ is called a *rewriting rule*.

Within a digit set conversion with an arbitrary rewriting rule $R$, one of the coefficients of $R$ which is greatest in modulus (so-called *core coefficient*) is used for the conversion of a digit $w_j$. But using of an arbitrary rewriting rule $R$ is out of scope of this thesis, so we focus on the simplest possible rewriting rule $R(x) = x - \beta$.

As $0 = \beta^j \cdot R(\beta) = 1 \cdot \beta^{j+1} - \beta \cdot \beta^j$, we have a representation of zero

$$1(-\beta) \underbrace{0 \cdots 0}_{j} \bullet = (0)_\beta.$$

for all $j \in \mathbb{N}$. We multiply this representation by $q_j \in \mathbb{Z}[\omega]$, which is called a *weight coefficient*, to obtain another representation of zero

$$q_j(-q_j\beta) \underbrace{0 \cdots 0}_{j} \bullet = (0)_\beta.$$

This is digitwise added to $w_n w_{n-1} \cdots w_1 w_0 \bullet$ to convert the digit $w_j$ into the alphabet $\mathcal{A}$. The conversion of $j$-th digit causes a *carry* $q_j$ on the $(j+1)$-th position. The digit set conversion runs from the right $(j = 0)$ to the left until all digits and carries are converted into the alphabet $\mathcal{A}$:

$$
\begin{array}{ccccccccc}
w_n w_{n-1} & \cdots & w_{j+1} & w_j & w_{j-1} & \cdots & w_1 w_0 \bullet \\
& & & & q_{j-2} & \cdot^{\displaystyle\cdot} & \\
& & & q_{j-1} & -\beta q_{j-1} & & \\
& & q_j & -\beta q_j & & & \\
& \cdot^{\displaystyle\cdot} & -\beta q_{j+1} & & & & \\
\hline
z_{n+s} \cdots z_n z_{n-1} & \cdots & z_{j+1} & z_j & z_{j-1} & \cdots & z_1 \; z_0 \bullet
\end{array}
\tag{2.1}
$$

Hence, the desired formula for conversion on the $j$-th position is

$$z_j = w_j + q_{j-1} - q_j\beta$$

for $j \in \mathbb{N}$. We set $q_{-1} = 0$ as there is no carry from the right on the 0-th position.

Clearly, the value of $w$ is preserved:

$$
\begin{aligned}
\sum_{j \geq 0} z_j \beta^j &= w_0 - \beta q_0 + \sum_{j > 0} (w_j + q_{j-1} - q_j \beta) \beta^j \\
&= \sum_{j \geq 0} w_j \beta^j + \sum_{j > 0} q_{j-1} \beta^j - \sum_{j \geq 0} q_j \cdot \beta^{j+1} \\
&= \sum_{j \geq 0} w_j \beta^j + \sum_{j > 0} q_{j-1} \beta^j - \sum_{j > 0} q_{j-1} \cdot \beta^j \\
&= \sum_{j \geq 0} w_j \beta^j = w \,.
\end{aligned}
\tag{2.2}
$$

The weight coefficient $q_j$ must be chosen so that the converted digit is in the alphabet $\mathcal{A}$, i.e.,

$$
z_j = w_j + q_{j-1} - q_j \beta \in \mathcal{A} \,.
\tag{2.3}
$$

The choice of weight coefficients is a crucial part of construction of addition algorithms which are computable in parallel. The extending window method determining weight coefficients for a given input is described in Section 2.1.

On the other hand, the following example shows that determining weight coefficients is trivial for standard numeration systems.

**Example 2.1.** Assume now a standard numeration system $(\beta, \mathcal{A})$, where

$$
\beta \in \mathbb{N} \,, \beta \geq 2 \,, \mathcal{A} = \{0, 1, 2, \ldots, \beta - 1\} \,.
$$

Notice that

$$
z_j \equiv w_j + q_{j-1} \mod \beta \,.
$$

There is only one representative of each class modulo $\beta$ in the standard numeration system $(\beta, \mathcal{A})$. Therefore, the digit $z_j$ is uniquely determined for a given digit $w_j \in \mathcal{A} + \mathcal{A}$ and carry $q_{j-1}$ and thus so is the weight coefficient $q_j$. This means that $q_j = q_j(w_j, q_{j-1})$ for all $j \geq 0$. Generally,

$$
q_j = q_j(w_j, q_{j-1}(w_{j-1}, q_{j-2})) = \cdots = q_j(w_j, \ldots, w_1, w_0)
$$

and

$$
z_j = z_j(w_j, \ldots, w_1, w_0) \,,
$$

which implies that addition runs in linear time.

We require that the digit set conversion from $\mathcal{A} + \mathcal{A}$ into $\mathcal{A}$ is computable in parallel, i.e., there exist constants $r, t \in \mathbb{N}_0$ such that for all $j \geq 0$ is $z_j = z_j(w_{j+r}, \ldots, w_{j-t})$. To avoid the dependency on all less, respectively more, significant digits, we need variety in the choice of weight coefficient $q_j$. This implies that the used numeration system must be redundant.

## 2.1 Extending window method

In order to construct a digit set conversion in numeration system $(\beta, \mathcal{A})$ which is computable in parallel, we consider a more general case of digit set conversion from an *input alphabet $\mathcal{B}$*

such that $\mathcal{A} \subsetneq \mathcal{B} \subset \mathcal{A} + \mathcal{A}$ instead of the alphabet $\mathcal{A} + \mathcal{A}$. As mentioned above, the key problem is to find for every $j \geq 0$ a weight coefficient $q_j$ such that

$$z_j = \underbrace{w_j}_{\in \mathcal{B}} + q_{j-1} - q_j\beta \in \mathcal{A}$$

for any input $w_{n'}w_{n'-1} \ldots w_1 w_0 \bullet = (w)_{\beta,\mathcal{B}}, w \in \mathrm{Fin}_\mathcal{B}(\beta)$. We remark that the weight coefficient $q_{j-1}$ is determined by the input $w$. For a digit set conversion to be computable in parallel the digit $z_j$ is required to satisfy $z_j = z_j(w_{j+r}, \ldots, w_{j-t})$ for a fixed anticipation $r$ and memory $t$ in $\mathbb{N}$.

We introduce following definitions.

**Definition 2.1.** Let $\mathcal{B}$ be a set such that $\mathcal{A} \subsetneq \mathcal{B} \subset \mathcal{A} + \mathcal{A}$. Then any finite set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ containing 0 such that

$$\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}$$

is called a *weight coefficients set*.

We see that if $\mathcal{Q}$ is a weight coefficients set, then

$$(\forall w_j \in \mathcal{B})(\forall q_{j-1} \in \mathcal{Q})(\exists q_j \in \mathcal{Q})(w_j + q_{j-1} - q_j\beta \in \mathcal{A}).$$

In other words, there is a weight coefficient $q_j \in \mathcal{Q}$ for a carry from the right $q_{j-1} \in \mathcal{Q}$ and digit $w_j$ in the input alphabet $\mathcal{B}$. I.e., we satisfy the basic digit set conversion formula (2.3). Notice that $q_{-1} = 0$ is in $\mathcal{Q}$ by the definition. Thus, all weight coefficients may be chosen from $\mathcal{Q}$.

**Definition 2.2.** Let $M$ be an integer and $q : \mathcal{B}^M \to \mathcal{Q}$ be a mapping such that

$$w_j + q(w_{j-1}, \ldots, w_{j-M}) - \beta q(w_j, \ldots, w_{j-M+1}) \in \mathcal{A}$$

for all $w_j, w_{j-1}, \ldots, w_{j-M} \in \mathcal{B}$, and $q(0, 0, \ldots, 0) = 0$. Then $q$ is called a *weight function* and $M$ is called a *length of window*.

Having a weight function $q$, we define a function $\phi : \mathcal{B}^{M+1} \to \mathcal{A}$ by

$$\phi(w_j, \ldots, w_{j-M}) = w_j + \underbrace{q(w_{j-1}, \ldots, w_{j-M})}_{=q_{j-1}} - \beta \underbrace{q(w_j, \ldots, w_{j-M+1})}_{=q_j} =: z_j, \qquad (2.4)$$

which verifies that the digit set conversion is indeed a $(M+1)$-local function with anticipation $r = 0$ and memory $t = M$. The requirement of zero output of the weight function $q$ for the input of $M$ zeros guarantees that $\phi(0, 0, \ldots, 0) = 0$. Thus the first condition of Definition 1.5 is satisfied. The second one follows from the equation (2.2).

Let us summarize the construction of the digit set conversion by the rewriting rule $x - \beta$. We need to find weight coefficients for all possible combinations of digits of the input alphabet $\mathcal{B}$. Their multiples of the rewriting rules are digitwise added to the input sequence. In fact, it means that the equation (2.3) is applied on each position. If the digit set conversion is computable in parallel, the weight coefficients are determined as the outputs of the weight function $q$ with some fixed length of window $M$.

We search for the weight function $q$ for a given base $\beta$ and input alphabet $\mathcal{B}$ by the extending window method. It consists of two phases. First, we find a minimal possible weight

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coefficients set $\mathcal{Q}$. We know that it is possible to convert an input sequence by choosing the weight coefficients from the set $\mathcal{Q}$. The set $\mathcal{Q}$ serves as the starting point for the second phase in which we increment the expected length of the window $M$ until the weight function $q$ is uniquely defined for each $(w_j, w_{j-1}, \ldots, w_{j-M+1}) \in \mathcal{B}^M$. Then, the local conversion is determined – we use the weight function outputs as weight coefficients in the formula (2.3).

Note that the convergence of both phases is discussed separately in Chapter 4.

## 2.2    Phase 1 – Weight coefficients set

The goal of the first phase is to compute a weight coefficients set $\mathcal{Q}$, i.e., to find a set $\mathcal{Q} \ni 0$ such that
$$\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}\,.$$

We build the sequence $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ iteratively so that we extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a way to cover all elements of the set $\mathcal{B} + \mathcal{Q}_k$ by elements of the extended set $\mathcal{Q}_{k+1}$, i.e.,

$$\mathcal{B} + \mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}\,.$$

This procedure is repeated until the extended weight coefficients set $\mathcal{Q}_{k+1}$ is the same as the original set $\mathcal{Q}_k$. We remark that the expression "a weight coefficient $q$ covers an element $x$" means that there is $a \in \mathcal{A}$ such that $x = a + \beta q$.

In other words, we start with $\mathcal{Q}_0 = \{0\}$ meaning that we search all weight coefficients $q_j$ necessary for digit set conversion for the case where there is no carry from the right, i.e., $q_{j-1} = 0$. We add them to the weight coefficients set $\mathcal{Q}_0$ to obtain the set $\mathcal{Q}_1$. Assume now that we have the set $\mathcal{Q}_k$ for some $k \geq 1$. The weight coefficients in $\mathcal{Q}_k$ now may appear as a carry $q_{j-1}$. If there are no suitable weight coefficients $q_j$ in the weight coefficients set $\mathcal{Q}_k$ to cover all sums of added coefficients and digits of the input alphabet $\mathcal{B}$, we extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ by suitable coefficients using Algorithm 2. And so on until there is no need to add more elements, i.e., the extended set $\mathcal{Q}_{k+1}$ equals $\mathcal{Q}_k$. Then the weight coefficients set $\mathcal{Q} := \mathcal{Q}_{k+1}$ satisfies Definition 2.1.

The precise description of the algorithm in a pseudocode is in Algorithm 1. For better understanding, see Figures ??–?? in Appendix ?? which illustrate the construction of the weight coefficients set $\mathcal{Q}$ for the Eisenstein base and a complex alphabet (see Example ?? for its description).

Section ?? discusses the convergence of Phase 1, i.e. whether it happens that $\mathcal{Q}_{k+1} = \mathcal{Q}_k$ for some $k$.

There may be more possible weight coefficients which cover some element of the set $\mathcal{B}+\mathcal{Q}_k$. Let us suppose that we have the list which contains the lists of these candidates for each element of the set $\mathcal{B}+\mathcal{Q}_k$. This list of lists is saved in the variable `candidates` in Algorithm 2. Now, for each element, we check the list of candidates which cover this element and if there is none of them contained in the set $\mathcal{Q}_k$, the smallest (in absolute value) weight coefficient from the list of candidates is added to the set $\mathcal{Q}_k$. If there are more elements with the same absolute value, we deterministically choose one of them. The extension $\mathcal{Q}_{k+1}$ of the set $\mathcal{Q}_k$ is obtained in this manner.

We may slightly improve this procedure: for example we may first extend $\mathcal{Q}_k$ by all single-element lists of `candidates`. These elements may be enough to cover also other elements of $\mathcal{B}+\mathcal{Q}_k$. It implies that the resulting $\mathcal{Q}$ is dependent on the way of selection from `candidates`.

---
**Algorithm 1** Search for weight coefficients set (Phase 1)
---
1: $k := 0$
2: $Q_0 := \{0\}$
3: **repeat**
4:     By Algorithm 2, extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a minimal possible way so that

$$\mathcal{B} + \mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}$$

5:     $k := k + 1$
6: **until** $\mathcal{Q}_k = \mathcal{Q}_{k+1}$
7: $\mathcal{Q} := \mathcal{Q}_k$
8: **return** $\mathcal{Q}$
---

---
**Algorithm 2** Extending intermediate weight coefficients set
---
**Input:** `candidates` from Algorithm 3, previous weight coefficients set $\mathcal{Q}_k$
1: $\mathcal{Q}_{k+1} := \mathcal{Q}_k$
2: **for all** `cand_for_x` in `candidates` **do**
3:     **if** no element of `cand_for_x` in $\mathcal{Q}_k$ **then**
4:         Add the smallest element (in absolute value) of `cand_for_x` to $\mathcal{Q}_{k+1}$
5:     **end if**
6: **end for**
7: **return** $\mathcal{Q}_{k+1}$
---

Algorithm 3 describes the search for the list of lists of candidates. For each element $x \in \mathcal{B} + \mathcal{Q}_k$ we build the list of candidates (in the variable `cand_for_x`) so that we test the divisibility of $x - a$ by the base $\beta$ for all letters $a \in \mathcal{A}$. In the positive case, the result of division is appended to `cand_for_x` as a possible weight coefficient. We remark that Theorem 3.2 is used to check the divisibility.

---
**Algorithm 3** Search for candidates
---
**Input:** the previous weight coefficients set $\mathcal{Q}_k$, alternatively also the set $\mathcal{Q}_{k-1}$
1: `candidates`:= empty list of lists
2: **for all** $x \in \mathcal{B} + \mathcal{Q}_k$ **do** {Alternatively, $x \in (\mathcal{B} + \mathcal{Q}_k) \setminus (\mathcal{B} + \mathcal{Q}_{k-1})$}
3:     `cand_for_x`:= empty list
4:     **for all** $a \in \mathcal{A}$ **do**
5:         **if** $(x - a)$ is divisible by $\beta$ in $\mathbb{Z}[\omega]$ (using Theorem 3.2) **then**
6:             Append $\frac{x-a}{\beta}$ to `cand_for_x`
7:         **end if**
8:     **end for**
9:     Append `cand_for_x` to `candidates`
10: **end for**
11: **return** `candidates`
---

We can improve the performance of Algorithm 3 by substituting the set $\mathcal{B} + \mathcal{Q}_k$ by $(\mathcal{B} + \mathcal{Q}_k) \setminus (\mathcal{B} + \mathcal{Q}_{k-1})$ on the line 2 because

$$\mathcal{B} + \mathcal{Q}_{k-1} \subset \mathcal{A} + \beta\mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}$$

for any $\mathcal{Q}_{k+1} \supset \mathcal{Q}_k$. Thus there is no need to check whether the elements of $\mathcal{B} + \mathcal{Q}_{k-1}$ are covered by some weight coefficient from $\mathcal{Q}_k$ in Algorithm 2.

## 2.3   Phase 2 – Weight function

We want to find a length of the window $M$ and a weight function $q : \mathcal{B}^M \to \mathcal{Q}$. We start with the weight coefficients set $\mathcal{Q}$ obtained in Phase 1. The idea is to reduce necessary weight coefficients for the conversion of a given digit up to single value. This is done by enlarging the number of considered input digits (extending the length of window) – there might be less possible carries from the right if we know which digits on the right are converted.

We introduce the following notation. Let $\mathcal{Q}$ be a weight coefficients set and $w_j \in \mathcal{B}$. Denote by $\mathcal{Q}_{[w_j]}$ any set such that

$$(\forall q_{j-1} \in \mathcal{Q})(\exists q_j \in \mathcal{Q}_{[w_j]})(w_j + q_{j-1} - q_j\beta \in \mathcal{A}).$$

By induction with respect to $m \in \mathbb{N}, m > 1$, for all $(w_j, \ldots, w_{j-m+1}) \in \mathcal{B}^m$ denote by $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ any subset of $\mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ such that

$$(\forall q_{j-1} \in \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]})(\exists q_j \in \mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]})(w_j + q_{j-1} - q_j\beta \in \mathcal{A}).$$

Recall the scheme (2.1) of the digit set conversion for better understanding of the notation and method:

$$
\begin{array}{cccccc}
\cdots w_{j+1} & w_j & w_{j-1} & \cdots w_{j-M+1} w_{j-M} \cdots \\
 & & q_{j-2} & \\
 & q_{j-1} & -\beta q_{j-1} & \\
 q_j & -\beta q_j & \\
 -\beta q_{j+1} & \\
\hline
\cdots z_{j+1} & z_j & z_{j-1} & \cdots z_{j-M+1}\ z_{j-M} \cdots
\end{array}
$$

The idea is to check all possible right carries $q_{j-1} \in \mathcal{Q}$ and determine values $q_j \in \mathcal{Q}$ such that

$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}.$$

So we obtain a set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ of weight coefficients which are necessary to convert the digit $w_j$ with any carry $q_{j-1} \in \mathcal{Q}$. Assuming that we know the input digit $w_{j-1}$, the set of possible carries from the right is also reduced to $\mathcal{Q}_{[w_{j-1}]}$. Thus we may reduce the set $\mathcal{Q}_{[w_j]}$ to a set $\mathcal{Q}_{[w_j,w_{j-1}]} \subset \mathcal{Q}_{[w_j]}$ which is necessary to cover all elements of $w_j + \mathcal{Q}_{[w_{j-1}]}$. Prolonging the length of window in this manner may lead to a unique weight coefficient $q_j$ for enough given input digits.

Accordingly, the weight function $q$ is found if there is $M \in \mathbb{N}$ such that

$$\#\mathcal{Q}_{[w_j,\ldots,w_{j-M+1}]} = 1$$

for all $w_j, \ldots, w_{j-M+1} \in \mathcal{B}^M$. The precise description of the construction of the weight function is in Algorithm 4. Figures ??–?? in Appendix ?? illustrate the construction of the set $\mathcal{Q}_{[\omega,1,2]}$ for the Eisenstein numeration system.

**Algorithm 4** Search for weight function (Phase 2)

**Input:** weight coefficients set $\mathcal{Q}$

1: $m := 1$
2: **for all** $w_j \in \mathcal{B}$ **do**
3:       By Algorithm 5, find set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ such that

$$w_j + \mathcal{Q} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_j]}$$

4: **end for**
5: **while** $\max\{\#\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} : (w_j,\ldots,w_{j-m+1}) \in \mathcal{B}^m\} > 1$ **do**
6:       $m := m + 1$
7:       **for all** $(w_j,\ldots,w_{j-m+1}) \in \mathcal{B}^m$ **do**
8:           By Algorithm 5, find set $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} \subset \mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ such that

$$w_j + \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]},$$

9:       **end for**
10: **end while**
11: $M := m$
12: **for all** $(w_j,\ldots,w_{j-M+1}) \in \mathcal{B}^M$ **do**
13:       $q(w_j,\ldots,w_{j-M+1}) :=$ only element of $\mathcal{Q}_{[w_j,\ldots,w_{j-M+1}]}$
14: **end for**
15: **return** $q$

There is space to improve Phase 2 by a modification of Algorithm 5. It is possible that the effort to reduce the size of $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ as much as possible is not the best for convergence of Phase 2.

Unfortunately, we do not know when Phase 2 converges. But we may reveal the nonconvergence of Phase 2 with deterministic Algorithm 5 for some numeration systems by Algorithm 6, which is described in Section **??**.

Notice that for a given length of window $M$, the number of calls of Algorithm 5 within Algorithm 4 is

$$\sum_{m=1}^{M} \#\mathcal{B}^m = \#\mathcal{B} \sum_{m=0}^{M-1} \#\mathcal{B}^m = \#\mathcal{B} \frac{\#\mathcal{B}^M - 1}{\#\mathcal{B} - 1}.$$

It implies that the time complexity grows exponentially as about $\#\mathcal{B}^M$. The required memory is also exponential because we have to store sets $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ at least for $m \in \{M-1, M\}$ for all $w_j,\ldots,w_{j-m+1} \in \mathcal{B}$.

We may reduce the number of the combinations of the input digits so that if for some $(w_j,\ldots,w_{j-m+1}) \in \mathcal{B}^m, m < M$ we have $\#\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} = 1$, we do not extend the window for these digits but we set the ouput of $q(w_j,\ldots,w_{j-m+1},w_{j-m},\ldots w_{j-M+1})$ to the single element of $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ for all $(w_{j-m},\ldots w_{j-M+1}) \in \mathcal{B}^{M-m}$. <span style="color:red">KONEC VLOZENI Z VYZKUMAKU</span>

# Chapter 3

# Properties of $\mathbb{Z}[\omega]$

## 3.1 Isomorphism of $\mathbb{Z}[\omega]$ and $\mathbb{Z}^d$

In this section, we recall that the ring $\mathbb{Z}[\omega]$ is isomorphic to the set $\mathbb{Z}^d$ equiped with multiplication, where $d$ is the degree of an algebraic integer $\omega$. This structure is useful as it allows to handle elements of $\mathbb{Z}[\omega]$ as vectors. For example, division in $\mathbb{Z}[\omega]$ can be replaced by searching for an integer solution of a linear system (Theorem 3.2) which is used in our implementation of the extending window method.

For defining multiplication in $\mathbb{Z}^d$, we recall the notion of companion matrix.

**Definition 3.1.** Let $p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1x + p_0 \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients, $d \geq 1$. The matrix

$$S := \begin{pmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & \cdots & 0 & -p_1 \\ 0 & 1 & \cdots & 0 & -p_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -p_{d-1} \end{pmatrix} \in \mathbb{Z}^{d \times d}$$

is called *companion matrix* of the polynomial $p$.

Let $S$ be the companion matrix of a polynomial $p$. It is well known (see for instance [6]) that the characteristic polynomial of the companion matrix $S$ is $p$. The matrix $S$ is also root of the polynomial $p$.

We remark that the minimal polynomial of an algebraic integer $\omega$ is denoted by $m_\omega$ and is always meant to be monic. Multiplication in $\mathbb{Z}^d$ is defined in the following way.

**Definition 3.2.** Let $\omega$ be an algebraic integer of degree $d \geq 1$ and let $S$ be the companion matrix of $m_\omega$. We define a mapping $\odot_\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$u \odot_\omega v := \left( \sum_{i=0}^{d-1} u_i S^i \right) \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} \quad \text{for all } u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{pmatrix}, v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} \in \mathbb{Z}^d.$$

The technical proof of Theorem 3.1, which shows that $\mathbb{Z}^d$ with multiplication $\odot_\omega$ is a ring isomorphic to $\mathbb{Z}[\omega]$, can be found in [7].

**Theorem 3.1.** *Let $\omega$ be an algebraic integer of degree d. Then*

$$\mathbb{Z}[\omega] = \left\{ \sum_{i=0}^{d-1} a_i \omega^i : a_i \in \mathbb{Z} \right\},$$

$(\mathbb{Z}^d, +, \odot_\omega)$ *is a commutative ring and the mapping $\pi : \mathbb{Z}[\omega] \to \mathbb{Z}^d$ defined by*

$$\pi(u) = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{pmatrix} \qquad \text{for every } u = \sum_{i=0}^{d-1} u_i \omega^i \in \mathbb{Z}[\omega]$$

*is a ring isomorphism.*

This theorem provides simple representation of elements of $\mathbb{Z}[\omega]$ in computer – they are represented by integer vectors and multiplication in $\mathbb{Z}[\omega]$ is replaced by multiplying by an appropriate matrix.

Divisibility in $\mathbb{Z}[\omega]$ can be also transformed into $\mathbb{Z}^d$. To check whether an element of $\mathbb{Z}[\omega]$ is divisible by another element, we look for an integer solution of a linear system given by Theorem 3.2. Moreover, this solution provides the result of division in the positive case.

**Theorem 3.2.** *Let $\omega$ be an algebraic integer of degree d and let S be the companion matrix of its minimal polynomial. Let $\beta = \sum_{i=0}^{d-1} b_i \omega^i$ be a nonzero element of $\mathbb{Z}[\omega]$. Then for every $u \in \mathbb{Z}[\omega]$*

$$u \in \beta\mathbb{Z}[\omega] \iff S_\beta^{-1} \cdot \pi(u) \in \mathbb{Z}^d,$$

*where $S_\beta = \sum_{i=0}^{d-1} b_i S^i$.*

The proof can be found in [7].

## 3.2   $\beta$-norm

The goal of this section is to construct a norm in $\mathbb{Z}[\omega]$. We use the isomorphism with $\mathbb{Z}^d$ and some results from matrix theory. This norm is used for the proof of convergence of Phase 1 in Chapter 4.

First, we recall a simple way how a new norm can be constructed from another one.

**Lemma 3.3.** *Let $\nu$ be a norm of the vector space $\mathbb{C}^d$ and P be a nonsingular matrix in $\mathbb{C}^d$. Then the mapping $\mu : \mathbb{C}^d \to \mathbb{R}_0^+$ defined by $\mu(x) = \nu(Px)$ is also a norm of the vector space $\mathbb{C}^d$.*

*Proof.* Let $x$ and $y$ be vectors in $\mathbb{C}^d$ and $\alpha \in \mathbb{C}$. We use linearity of matrix multiplication, nonsingularity of matrix $P$ and the fact that $\nu$ is a norm to prove the following statements:

1. $\mu(x) = \nu(Px) \geq 0$,

2. $\mu(x) = 0 \iff \nu(Px) = 0 \iff Px = 0 \iff x = 0$,

3. $\mu(\alpha x) = \nu(P(\alpha x)) = \nu(\alpha Px) = |\alpha|\nu(Px) = |\alpha|\mu(x)$,

4. $\mu(x + y) = \nu(P(x + y)) = \nu(Px + Py) \leq \nu(Px) + \nu(Py) = \mu(x) + \mu(y)$.

This verifies that $\mu$ is a norm. $\qquad\square$

Now we use Lemma 3.3 to define a new norm for a given diagonalizable matrix.

**Definition 3.3.** Let $M \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix and $P \in \mathbb{C}^{n \times n}$ be a nonsingular matrix which diagonalizes $M$, i.e., $M = P^{-1}DP$ for some diagonal matrix $D \in \mathbb{C}^{n \times n}$. We define a vector norm $\|\cdot\|_M$ by

$$\|x\|_M := \|Px\|_2 \tag{3.1}$$

for all $x \in \mathbb{C}^n$, where $\|\cdot\|_2$ is Euclidean norm. A matrix norm $\|\|\cdot\|\|_M$ is induced by the norm $\|\cdot\|_M$, i.e,

$$\|\|A\|\|_M = \sup_{\|x\|_M = 1} \|Ax\|_M$$

for all $A \in \mathbb{C}^{n \times n}$.

The following theorem is a known result from matrix theory – for a given diagonalizable matrix, there is a matrix norm such that the norm of the matrix equals its spectral radius. <span style="color:red">JE POTREBA NEJAKOU CITACI?</span>

**Theorem 3.4.** *Let $M \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix. Then*

$$\rho(M) = \|\|M\|\|_M\,,$$

*where $\rho(M)$ is the spectral radius of the matrix $M$.*

*Proof.* First, we prove that $\|\|M\|\|_M \geq \rho(M)$. For all eigenvalues $\lambda$ in the spectrum $\sigma(M)$ with a respective eigenvector $u$ such that $\|u\|_M = 1$, we have

$$\|\|M\|\|_M = \sup_{\|x\|_M = 1} \|Mx\|_M \geq \|Mu\|_M = \|\lambda u\|_M = |\lambda| \cdot \|u\|_M = |\lambda|\,.$$

Secondly, we show that $\|\|M\|\|_M \leq \rho(M)$. Following Definition 3.3, let $P \in \mathbb{C}^{n \times n}$ be a nonsingular matrix and $D \in \mathbb{C}^{n \times n}$ a diagonal matrix with the eigenvalues of $M$ on the diagonal such that $PMP^{-1} = D$.

Let $y$ be a vector such that $\|y\|_M = 1$ and set $z = Py$. Notice that

$$\sqrt{z^*z} = \|z\|_2 = \|Py\|_2 = \|y\|_M = 1\,.$$

Consider

$$\|My\|_M = \|PMy\|_2 = \|DPy\|_2 = \|Dz\|_2 = \sqrt{z^*D^*Dz}$$
$$\leq \sqrt{\max_{\lambda \in \sigma(M)} |\lambda|^2 z^*z} = \max_{\lambda \in \sigma(M)} |\lambda| = \rho(M)\,.$$

which implies the statement. $\qquad\square$

Before we define a norm in $\mathbb{Z}[\omega]$, we verify that a specific matrix given by an algebraic number $\beta \in \mathbb{Z}[\omega]$ is diagonalizable. Lemma 3.5 summarizes also some other properties of this matrix and the norm which it induces according to Theorem 3.4.

**Lemma 3.5.** *Let $\omega$ be an algebraic integer of degree $d$ and let $S$ be the companion matrix of its minimal polynomial $m_\omega$. Let $\beta = \sum_{i=0}^{d-1} b_i \omega^i$, where $b_i \in \mathbb{Z}$, be a nonzero element of $\mathbb{Z}[\omega]$. Set $S_\beta = \sum_{i=0}^{d-1} b_i S^i$. Then*

*i) The matrix $S_\beta$ is diagonalizable.*

*ii) The characteristic polynomial of $S_\beta$ is $m_\beta^k$ with $k = d/\deg \beta$.*

*iii) $|\det S_\beta| = |m_\beta(0)|^k$.*

*iv) $\|x\|_{S_\beta} = \|x\|_{S_\beta^{-1}}$ for all $x \in \mathbb{C}^d$ and $\|\|X\|\|_{S_\beta} = \|\|X\|\|_{S_\beta^{-1}}$ for all $X \in \mathbb{C}^{d \times d}$.*

*v) $\|\|S_\beta\|\|_{S_\beta} = \max\{|\beta'| : \beta'$ is conjugate of $\beta\}$ and $\|\|S_\beta^{-1}\|\|_{S_\beta} = \max\{\frac{1}{|\beta'|} : \beta'$ is conjugate of $\beta\}$.*

*Proof.* The characteristic polynomial of the companion matrix $S$ is the same as minimal polynomial of $\omega$ which has no multiple roots. Hence, $S$ is diagonalizable, i.e., $S = P^{-1}DP$ where $D$ is diagonal matrix with the conjugates of $\omega$ on the diagonal and $P$ is a nonsingular complex matrix. The matrix $S_\beta$ is also diagonalized by $P$:

$$S_\beta = \sum_{i=0}^{d-1} b_i S^i = \sum_{i=0}^{d-1} b_i \left(P^{-1}DP\right)^i = P^{-1} \underbrace{\left(\sum_{i=0}^{d-1} b_i D^i\right)}_{D_\beta} P.$$

There is a result in theory of algebraic numbers about conjugates under a field isomorphism – let $\alpha$ be an algebraic number with a conjugate $\alpha'$ and $\sigma : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha')$ be a field isomorphism. If $\gamma \in \mathbb{Q}(\alpha)$, then $\sigma(\gamma)$ is a conjugate of $\gamma$. <span style="color:red">JE POTREBA NEJAKOU CITACI?</span>

Therefore, the diagonal elements of the diagonal matrix $D_\beta$ are conjugates of $\beta$. Since $S_\beta \in \mathbb{Z}^{d \times d}$, its characteristic polynomial $p_{S_\beta}$ has integer coefficients. There exists $k \in \mathbb{N}, k \geq 1$ such that $p_{S_\beta} = m_\beta^k$ as all roots of $p_{S_\beta}$ must be conjugates of $\beta$. The value $k$ follows from the equality $d = \deg(m_\beta^k) = k \deg m_\beta$.

The modulus of the determinant of $S_\beta$ equals the modulus of the absolute coefficient of the characteristic polynomial $p_{S_\beta}$ which is $|m_\beta(0)|^k$.

The matrix $S_\beta^{-1}$ is also diagonalized by $P$ since $S_\beta^{-1} = (P^{-1}D_\beta P)^{-1} = P^{-1}D_\beta^{-1}P$. Thus, the norms $\|\cdot\|_{S_\beta}$ and $\|\cdot\|_{S_\beta^{-1}}$ are the same and so are the induced matrix norms $\|\|\cdot\|\|_{S_\beta}$ and $\|\|\cdot\|\|_{S_\beta^{-1}}$.

The matrix $S_\beta$ is diagonalizable and its eigenvalues are the conjugates of $\beta$. Theorem 3.4 implies that

$$\|\|S_\beta\|\|_{S_\beta} = \rho(S_\beta) = \max\{|\beta'| : \beta' \text{ is conjugate of } \beta\}.$$

For the second part of the last statement, we use the part *iv)*, Theorem 3.4 and the fact that the eigenvalues of $S_\beta^{-1}$ are reciprocal of the conjugates of $\beta$. $\qquad\square$

Finally, we may define a norm in $\mathbb{Z}[\omega]$.

**Definition 3.4.** Let $\pi$ be the isomorphism between $\mathbb{Z}[\omega]$ and $(\mathbb{Z}^d, +, \odot_\omega)$. Using the notation of the previous lemma, we define $\beta$-*norm* $\|\cdot\|_\beta : \mathbb{Z}[\omega] \to \mathbb{R}_0^+$ by

$$\|x\|_\beta = \|\pi(x)\|_{S_\beta}$$

for all $x \in \mathbb{Z}[\omega]$.

We can easily verify that $\beta$-norm is a norm in $\mathbb{Z}[\omega]$:

1. $\|x\|_\beta = \|\pi(x)\|_{S_\beta} \geq 0$,

2. $\|x\|_\beta = 0 \iff \|\pi(x)\|_{S_\beta} = 0 \iff \pi(x) = 0 \iff x = 0$,

3. $\|\alpha x\|_\beta = \|\pi(\alpha x)\|_{S_\beta} = |\alpha| \|\pi(x)\|_{S_\beta} = |\alpha| \|x\|_\beta$,

4. $\|x + y\|_\beta = \|\pi(x+y)\|_{S_\beta} = \|\pi(x) + \pi(y)\|_{S_\beta} \leq \|\pi(x)\|_{S_\beta} + \|\pi(y)\|_{S_\beta} = \|x\|_\beta + \|y\|_\beta$,

for all $x, y \in \mathbb{Z}[\omega]$ and $\alpha \in \mathbb{Z}[\omega]$.

The important property of $\beta$-norm is that there is only finitely many elements of $\mathbb{Z}[\omega]$ which are bounded in this norm by a given constant. The explanation is following – images of elements of $\mathbb{Z}[\omega]$ under the isomorphism $\pi$ are integer vectors and there are only finitely many integer vectors in any finite dimensional vector space bounded by any norm. It follows from equivalence of all norms a finite dimensional vector space.

## 3.3 Number of congruence classes

Congruence classes play important role in the structure of an alphabet which allows parallel addition. We have seen that the isomorphism with $\mathbb{Z}^d$ is an efficient tool for handling elements of $\mathbb{Z}[\omega]$. We use it also for counting number of congruence classes. The definition of congruence in $\mathbb{Z}^d$ is following.

**Definition 3.5.** Let $M \in \mathbb{Z}^{d \times d}$ be a nonsingular integer matrix. Vectors $x, y \in \mathbb{Z}^d$ are *congruent modulo $M$ in $\mathbb{Z}^d$* if $x - y \in M\mathbb{Z}^d$.

Let $x, y, z \in \mathbb{Z}^d$. We verify that congruence modulo $M$ is an equivalence.

i) reflexivity: $x - x = 0 = M \cdot 0$,

ii) symetry: if $\exists v \in \mathbb{Z}^d$ such that $x - y = M \cdot v$, then $y - x = M \cdot (-v)$,

iii) transitivity: if $\exists v, v' \in \mathbb{Z}^d$ such that $x - y = M \cdot v$ and $y - z = M \cdot v'$, then $z - x = (z - y) + (y - x) = M \cdot (v' + v)$.

<span style="color:red">JE POTREBA TEN PRIKLAD?</span>

In Theorem 3.7, we will see that a congruence class modulo $\beta$ in $\mathbb{Z}[\omega]$ corresponds to a congruence class modulo $S_\beta$ in $\mathbb{Z}^d$, where we use the notation from the previous section. Therefore, we count number of congruence classes modulo a matrix $M$ in Lemma 3.6.

**Lemma 3.6.** *Let $M \in \mathbb{Z}^{d \times d}$ be a nonsingular integer matrix. The number of congruence classes modulo $M$ in $\mathbb{Z}^d$ is $|\det M|$.*

*Proof.* Set $y_i := Me_i$ for $i \in \{0, \ldots, d-1\}$ and

$$B_{(\alpha_0, \ldots, \alpha_{d-1})} := \left\{ \sum_{i=0}^{d-1} (\alpha_i + t_i)y_i \colon t_i \in [0,1) \right\}$$

for $(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{Z}^d$. Obviously,

$$\mathbb{R}^d = \bigcup_{(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{Z}^d} B_{(\alpha_0, \ldots, \alpha_{d-1})} \cdot$$

For fixed $(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{Z}^d$, the number of points of $\mathbb{Z}^d$ in $B_{(\alpha_0, \ldots, \alpha_{d-1})}$ is the volume of $B_{(\alpha_0, \ldots, \alpha_{d-1})}$ which is $|\det M|$. Hence, it is enough to prove that there is exactly one representative of each congruence class in $B_{(\alpha_0, \ldots, \alpha_{d-1})}$.

To show that there are representatives of all classes, assume an arbitrary vector $x \in \mathbb{Z}^d$. Since $(y_0, \ldots, y_{d-1})$ is a basis of $\mathbb{R}^d$, there are scalars $s_0, \ldots, s_{d-1} \in \mathbb{R}$ such that $x = \sum_{i=0}^{d-1} s_i y_i$. Set $\gamma_i := \lfloor s_i \rfloor$ and $t_i := s_i - \gamma_i$. Now

$$x = \sum_{i=0}^{d-1}(\gamma_i + t_i)y_i = \sum_{i=0}^{d-1} t_i y_i + \sum_{i=0}^{d-1}(\gamma_i - \alpha_i)y_i + \sum_{i=0}^{d-1} \alpha_i y_i = \underbrace{\sum_{i=0}^{d-1}(\alpha_i + t_i)y_i}_{\in B_{(\alpha_0, \ldots, \alpha_{d-1})}} + M \underbrace{(\gamma - \alpha)}_{\in \mathbb{Z}^d},$$

where $\alpha = (\alpha_0, \ldots, \alpha_{d-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{d-1})^T$. Hence, there is an integer vector $\sum_{i=0}^{d-1}(\alpha_i + t_i)y_i$ in $B_{(\alpha_0, \ldots, \alpha_{d-1})}$ which is congruent to $x$ modulo $M$.

Let $x' = \sum_{i=0}^{d-1} s_i' y_i \in \mathbb{Z}^d$ and $x'' = \sum_{i=0}^{d-1} s_i'' y_i \in \mathbb{Z}^d$ be distinct elements of $B_{(\alpha_0, \ldots, \alpha_{d-1})}$ which are congruent modulo $M$, i.e., there exists $z = (z_0, \ldots, z_{d-1})^T \in \mathbb{Z}^d$ such that $x' = x'' + Mz$. There is $i_0 \in \{0, \ldots, d-1\}$ such that $|z_{i_0}| \geq 1$ as $x' \neq x''$. Thus, $|s_{i_0}' - s_{i_0}''| = |z_{i_0}| \geq 1$ which contradicts that $x', x'' \in B_{(\alpha_0, \ldots, \alpha_{d-1})}$. $\qquad \square$

Now we compute number of congruence classes modulo $\beta$ in $\mathbb{Z}[\omega]$ since two elements of $\mathbb{Z}[\omega]$ are congruent modulo $\beta$ if and only if the corresponding vectors in $\mathbb{Z}^d$ are congruent modulo $S_\beta$.

**Theorem 3.7.** *Let $\omega$ be an algebraic integer of degree $d$ and $\beta = \sum_{i=0}^d b_i \omega^i$, where $b_i \in \mathbb{Z}$, be such that $\deg \omega = \deg \beta$. The number of congruence classes modulo $\beta$ in $\mathbb{Z}[\omega]$ is $|m_\beta(0)|$.*

*Proof.* Let $x, y \in \mathbb{Z}[\omega]$ and let $S$ be the companion matrix of the minimal polynomial $m_\omega$. Set $S_\beta = \sum_{i=0}^{d-1} b_i S^i$. Then

$$
\begin{aligned}
x \equiv y \mod \beta &\iff \exists z \in \mathbb{Z}[\omega] \colon x - y = \beta z \\
&\iff \exists z \in \mathbb{Z}[\omega] \colon \pi(x - y) = S_\beta \pi(z) \\
&\iff \pi(x) \equiv \pi(y) \mod S_\beta.
\end{aligned}
$$

Thus, the number of congruence classes modulo $\beta$ is $|\det S_\beta|$ by Lemma 3.6. The statement follows from Lemma 3.5. $\qquad \square$

# Chapter 4

# Convergence

## 4.1  Convergence of Phase 1

In this section, we show that if the extending window method converges, then the base $\beta$ must be expanding, i.e., all its conjugates are greater than one in modulus. Then we prove that it is also sufficient condition for convergence of Phase 1.

We use the following notation:

**Definition 4.1.** Let $\omega$ be a complex number and $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$. Let $\mathcal{A} \subset \mathbb{Z}[\omega]$ be an alphabet. Set

$$\mathcal{A}[\beta] := \left\{ \sum_{i=0}^{N} a_i \beta^i \colon a_i \in \mathcal{A}, N \in \mathbb{N} \right\} .$$

The essential part of the proof that $\beta$ must be expanding is Theorem 4.1 which is based on the papers of Akiyama, Thuswaldner and Zäimi [1, 2].

**Theorem 4.1.** *Let $\omega$ be a complex number and $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$. Let $\mathcal{A} \subset \mathbb{Z}[\omega]$ be an alphabet. If $\mathbb{N} \subset \mathcal{A}[\beta]$, then the number $\beta$ is expanding.*

*Proof.* For all $n \in \mathbb{N}$ we may write

$$n = \sum_{i=0}^{N} a_i \beta^i ,$$

where $N \in \mathbb{N}$, $a_i \in \mathcal{A}$ and $a_N \neq 0$.

Set $m := \max\{|a| \colon a \in \mathcal{A}\}$. We take $n \in \mathbb{N}$ such that $n > m$. Since $|a_0| \leq m < n$, we have $N \geq 1$ and there is $i_0 \in \{1, 2, \ldots, N\}$ such that $a_{i_0} \neq 0$. Thus, $\omega$ is an algebraic number as $a_i \in \mathcal{A} \subset \mathbb{Z}[\omega]$ and $\beta$ can be expressed as an integer combination of powers of $\omega$. Therefore, $\beta$ is also an algebraic number.

Let $\beta'$ be an algebraic conjugate of $\beta$. Since $\beta \in \mathbb{Z}[\omega] \subset \mathbb{Q}(\omega)$, there is an algebraic conjugate $\omega'$ of $\omega$ and an isomorphism $\sigma \colon \mathbb{Q}(\omega) \to \mathbb{Q}(\omega')$ such that $\sigma(\beta) = \beta'$. Now

$$n = \sigma(n) = \sum_{i=0}^{N} \sigma(a_i)(\beta')^i .$$

Set $\tilde{m} := \max\{|\sigma(a)| \colon a \in \mathcal{A}\}$. For all $n \in \mathbb{N}$, we have

$$n = |n| \leq \sum_{i=0}^{N} |\sigma(a_i)| \cdot |\beta'|^i \leq \sum_{i=0}^{\infty} |\sigma(a_i)| \cdot |\beta'|^i \leq \tilde{m} \sum_{i=0}^{\infty} |\beta'|^i .$$

Hence, the sum on the right side diverges which implies that $|\beta'| \geq 1$. Thus, all conjugates of $\beta$ are at least one in modulus.

If the degree of $\beta$ is one, the statement is obvious. Therefore, we may assume that $\deg \beta \geq 2$.

Suppose for contradiction that $|\beta'| = 1$ for an algebraic conjugate $\beta'$ of $\beta$. The complex conjugate $\overline{\beta'}$ is also an algebraic conjugate of $\beta$. Take any algebraic conjugate $\gamma$ of $\beta$ and the isomorphism $\sigma' : \mathbb{Q}(\beta') \to \mathbb{Q}(\gamma)$ given by $\sigma'(\beta') = \gamma$. Now

$$\frac{1}{\gamma} = \frac{1}{\sigma'(\beta')} = \sigma'\left(\frac{1}{\beta'}\right) = \sigma'\left(\frac{\overline{\beta'}}{\beta'\overline{\beta'}}\right) = \sigma'\left(\frac{\overline{\beta'}}{|\beta'|^2}\right) = \sigma'(\overline{\beta'}).$$

Hence, $\frac{1}{\gamma}$ is also an algebraic conjugate of $\beta$. Moreover, $\left|\frac{1}{\gamma}\right| \geq 1$ and $|\gamma| \geq 1$ which implies that $|\gamma| = 1$. We may set $\gamma = \beta$ which contradicts $|\beta| > 1$. Thus all conjugates of $\beta$ are greater than one in modulus, i.e., $\beta$ is an expanding algebraic number. $\qquad\square$

Now we can easily prove that existence of a parallel addition algorithm with rewriting rule $x - \beta$ implies that $\beta$ is expanding.

**Theorem 4.2.** *Let $\mathcal{A} \subset \mathbb{Z}[\omega]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If the extending window method with the rewriting rule $x - \beta$ converges for the numeration system $(\beta, \mathcal{A})$, then the base $\beta$ is expanding.*

*Proof.* The existence of an algorithm for addition which is computable in parallel implies that the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition. Moreover, the set $\mathcal{A}[\beta]$ is closed under addition since there is no carry to the right when the rewriting rule $x - \beta$ is used. For any $n \in \mathbb{N}$, the sum $1 + 1 + \cdots + 1 = n$ is in $\mathcal{A}[\beta]$ by the assumption $1 \in \mathcal{A}[\beta]$. Therefore, $\mathbb{N} \subset \mathcal{A}[\beta]$ and thus the base $\beta$ is expanding by Theorem 4.1. $\qquad\square$

Since we know that it makes sense to consider only expanding base, we may ask if Phase 1 converges for such a base. The answer is positive, with some natural assumption on the alphabet $\mathcal{A}$. The following lemma provides a finite set of weight coefficients $\mathcal{Q}$.

**Lemma 4.3.** *Let $\omega$ be an algebraic integer, $\deg \omega = d$, and $\beta$ be an expanding algebraic integer in $\mathbb{Z}[\omega]$. Let $\mathcal{A}$ and $\mathcal{B}$ be finite subsets of $\mathbb{Z}[\omega]$ such that $\mathcal{A}$ contains at least one representative of each congruence class modulo $\beta$ in $\mathbb{Z}[\omega]$. There exists a finite set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ such that $\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}$.*

*Proof.* We use the isomorphism $\pi : \mathbb{Z}[\omega] \to \mathbb{Z}^d$ and $\beta$-norm $\|\cdot\|_\beta$ to give a bound on the elements of $\mathbb{Z}[\omega]$. Let $\gamma$ be the smallest conjugate of $\beta$ in modulus. Denote $C := \max\{\|b - a\|_\beta : a \in \mathcal{A}, b \in \mathcal{B}\}$. Consequently, set $R := \frac{C}{|\gamma|-1}$ and $\mathcal{Q} := \{q \in \mathbb{Z}[\omega] : \|q\|_\beta \leq R\}$. By Lemma 3.5, we have

$$\left\|\left|S_\beta^{-1}\right|\right\|_{S_\beta} = \max\{\frac{1}{|\beta'|} : \beta' \text{ is conjugate of } \beta\} = \frac{1}{|\gamma|}.$$

Also, $|\gamma| > 1$ as $\beta$ is an expanding integer. Since $C > 0$, the set $\mathcal{Q}$ is nonempty. Any element $x = b + q \in \mathbb{Z}[\omega]$ with $b \in \mathcal{B}$ and $q \in \mathcal{Q}$ can be written as $x = a + \beta q'$ for some $a \in \mathcal{A}$ and $q' \in \mathbb{Z}[\omega]$ due to existence of a representative of each congruence class in $\mathcal{A}$. Using the

isomorphism $\pi$, we may write $\pi(q') = S_\beta^{-1} \cdot \pi(b - a + q)$. We prove that $q'$ is in $\mathcal{Q}$:

$$\left\| q' \right\|_\beta = \left\| \pi(q') \right\|_{S_\beta} = \left\| S_\beta^{-1} \cdot \pi(b - a + q) \right\|_{S_\beta} \leq \left\| \left\| S_\beta^{-1} \right\| \right\|_{S_\beta} \left\| b - a + q \right\|_\beta$$

$$\leq \frac{1}{|\gamma|}(\|b - a\|_\beta + \|q\|_\beta) = \frac{1}{|\gamma|}(C + R) = \frac{C}{|\gamma|}(1 + \frac{1}{|\gamma| - 1}) = R\,.$$

Hence $q' \in \mathcal{Q}$ and thus $x = b + q \in \mathcal{A} + \beta\mathcal{Q}$.

Since there are only finitely many elements of $\mathbb{Z}^d$ bounded by the constant $R$, the set $\mathcal{Q}$ is finite. □

The way how candidates for the weight coefficients are chosen in Algorithm 3 is the same as in the proof of Lemma 4.3. Therefore, the convergence of Phase 1 is guaranteed by the following theorem.

**Theorem 4.4.** *Let $\omega$ be an algebraic integer and $\beta \in \mathbb{Z}[\omega]$. Let the alphabet $\mathcal{A} \subset \mathbb{Z}[\omega]$ be such that $\mathcal{A}$ contains at least one representative of each congruence class modulo $\beta$ in $\mathbb{Z}[\omega]$. Let $\mathcal{B} \subset \mathbb{Z}[\omega]$ be the input alphabet.*

*If $\beta$ is expanding, then Phase 1 of the extending window method converges.*

*Proof.* Let $R$ be a constant and $\mathcal{Q}$ be a finite set from Lemma 4.3 for the alphabet $\mathcal{A}$ and input alphabet $\mathcal{B}$. We prove by induction that all intermediate weight coefficient sets $\mathcal{Q}_k$ in Algorithm 1 are subsets of the finite set $\mathcal{Q}$.

We start with $\mathcal{Q}_0 = \{0\}$ whose elements are bounded by any positive constant. Suppose that the intermediate weight coefficients set $\mathcal{Q}_k$ has elements bounded by the constant $R$. We see from the previous proof that the candidates obtained by Algorithm 3 for the set $\mathcal{Q}_k$ are also bounded by $R$. Thus, the next intermediate weight coefficients set $\mathcal{Q}_{k+1}$ has elements bounded by the constant $R$, i.e., $\mathcal{Q}_{k+1} \subset \mathcal{Q}$.

Since $\#\mathcal{Q}$ is finite and $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}$, Phase 1 successfully ends. □

## 4.2 Convergence of Phase 2

We have no simple sufficient or necessary condition of convergence of Phase 2 in the sense of properties of a base $\beta$ or an alphabet $\mathcal{A}$. Nevertheless, the convergence can be controled during a run of algorithm. An easy check of non-convergence can be done by searching $\mathcal{Q}_{[b,...,b]}$ for each $b \in \mathcal{B}$ separately. This was already described in [7], but we recall it with a simpler proof. For its purposes, we introduce a notion of stable Phase 2, which is used also in the main result of this section – the control of convergence during Phase 2 is transformed into searching for a cycle in an oriented graph.

Firstly, we menshion some equivalent conditions of non-convergence of Phase 2. It enables us to handle easier with non-convergence in later proofs.

**Lemma 4.5.** *The following statements are equivalent:*

*i) Phase 2 does not converge,*

*ii) $\forall\, k \in \mathbb{N}\, \exists\, (w_0, \ldots, w_{-k}) \in \mathcal{B}^{k+1}\colon \#\mathcal{Q}_{[w_0,...,w_{-k}]} \geq 2,$*

*iii) $\exists\, (w_{-k})_{k \geq 0},\, w_{-k} \in \mathcal{B}\, \exists\, k_0 \forall k \geq k_0\colon \#\mathcal{Q}_{[w_0,...,w_{-k}]} = \#\mathcal{Q}_{[w_0,...,w_{-(k-1)}]} \geq 2.$*

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*Proof.* $i) \iff ii)$: The while loop in Algorithm 4 ends if and only if there exist $k \in \mathbb{N}$ such that $\#\mathcal{Q}_{[w_0,\dots,w_{-k}]} = 1$ for all $(w_0,\dots,w_{-k}) \in \mathcal{B}^{k+1}$.

$ii) \iff iii)$: There is an infinite sequence $(w_{-k})_{k \geq 0}$ such that $\#\mathcal{Q}_{[w_0,\dots,w_{-k}]} \geq 2$ for all $k \in \mathbb{N}$ since $\mathcal{Q}_{[w_0,\dots,w_{-k}]} \supset \mathcal{Q}_{[w_0,\dots,w_{-(k+1)}]}$. Hence, the sequence of integers $(\#\mathcal{Q}_{[w_0,\dots,w_{-k}]})_{k \geq 0}$ is eventually constant. The opposite implication is trivial. $\qquad\square$

We need to ensure that choice of a possible weight coefficient set $\mathcal{Q}_{[w_0,\dots,w_{-k}]} \subset \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is determined by an input digit $w_0$ and a set $\mathcal{Q}_{[w_{-1},\dots,w_{-0}]}$, while the influence of the set $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is limited. It is formalized in the following definition.

**Definition 4.2.** Let $\mathcal{B}$ be an alphabet of input digits. We say that Phase 2 is *stable*, if the following condition on the sets of possible weight coefficients which are produced by the algorithm holds:

$$\mathcal{Q}_{[w_{-1},\dots,w_{-k}]} = \mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]} \implies \mathcal{Q}_{[w_0,\dots,w_{-k}]} = \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$$

for all $k \in \mathbb{N}, k \geq 2$ and for all $(w_0,\dots,w_{-(k+1)}) \in \mathcal{B}^{k+1}$.

<span style="color:red">II)$\#\mathcal{Q}_{[w_{-1},\dots,w_{-K}]} = 1 \implies \#\mathcal{Q}_{[w_0,\dots,w_{-K}]} = 1$, NENI POTREBA POKUD NEBUDE OPACNA IMPLIKACE</span>

The definition may seem too restrictive, but note that $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ if fully determined by $\mathcal{Q}_{[w_{-1},\dots,w_{-k}]}, \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ and $w_0$ for a fixed deterministic way of choice of possible weight coefficients sets. Therefore, the assumption $\mathcal{Q}_{[w_{-1},\dots,w_{-k}]} = \mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]}$ implies that the only difference in the choice of $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ an $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is that $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ is a subset of $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$, while $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is chosen as a subset of $\mathcal{Q}_{[w_0,\dots,w_{-(k-2)}]}$. At the same time, $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is a subset of $\mathcal{Q}_{[w_0,\dots,w_{-(k-2)}]}$. Thus, the requirement that Phase 2 is stable means that the same possible weight coefficients set is found even if it is searched as a subset of smaller set. This is natural way how an algorithm of choice should be constructed – the set $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ is searched such that

$$\mathcal{B} + \mathcal{Q}_{[w_{-1},\dots,w_{-k}]} \subset \mathcal{A} + \beta\mathcal{Q}_{[w_0,\dots,w_{-k}]},$$

i.e., there is no reason to choose the set $\mathcal{Q}_{[w_0,\dots,w_{-k}]} \subsetneq \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ as we know that

$$\mathcal{B} + \underbrace{\mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]}}_{=\mathcal{Q}_{[w_{-1},\dots,w_{-k}]}} \subset \mathcal{A} + \beta\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}.$$

In other word, if $\mathcal{Q}_{[w_0,\dots,w_{-k}]} \subsetneq \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$, the set $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ might have been chosen smaller.

Now we use that finiteness of Phase 2 implies that there exists a length of window $m$ such that the set $\mathcal{Q}_{[b]}^m$ contains only one element for all $b \in \mathcal{B}$, where $\mathcal{Q}_{[b]}^m$ is a shorter notation for

$$\mathcal{Q}_{[\underbrace{b,\dots,b}_{m}]}.$$

The following theorem was proved in [7] with the assumption that Phase 2 is deterministic. Briefly, it says that $\#\mathcal{Q}_{[b]}^m$ must decrease every time we increase $m$, otherwise Phase 2 does not converge. When we consider only inputs of the form $bb\dots b$ for some $b \in \mathcal{B}$, determinism implies that Phase 2 is stable. The given proof with Phase 2 being stable is slightly shorter.

**Theorem 4.6.** *Let $m_0 \in \mathbb{N}$ and $b \in \mathcal{B}$ be such that sets $\mathcal{Q}_{[b]}^{m_0}$ and $\mathcal{Q}_{[b]}^{m_0-1}$ produced by stable Phase 2 have the same size. Then*

$$\#\mathcal{Q}_{[b]}^{m} = \#\mathcal{Q}_{[b]}^{m_0} \qquad \forall m \geq m_0 - 1 \, .$$

*Particularly, if $\#\mathcal{Q}_{[b]}^{m_0} \geq 2$, Phase 2 does not converge.*

*Proof.* As $\mathcal{Q}_{[b]}^{m_0} \subset \mathcal{Q}_{[b]}^{m_0-1}$, the assumption of the same size implies

$$\mathcal{Q}_{[b]}^{m_0} = \mathcal{Q}_{[b]}^{m_0-1} \, .$$

By the assumption that Phase 2 is stable, we have

$$
\begin{aligned}
\mathcal{Q}_{[b]}^{m_0} = \mathcal{Q}_{[b]}^{m_0-1} &\implies \mathcal{Q}_{[b]}^{m_0+1} = \mathcal{Q}_{[b]}^{m_0} \\
&\implies \mathcal{Q}_{[b]}^{m_0+2} = \mathcal{Q}_{[b]}^{m_0+1} \\
&\quad\vdots
\end{aligned}
$$

This implies the statement.

If $\#\mathcal{Q}_{[b]}^{m_0} \geq 2$, then the statement *iii)* in Lemma 4.5 holds for the sequence $(b)_{k \geq 0}$. □

The condition of convergence during Phase 2 is formulated as a searching for an infinite path in a so-called Rauzy graph. This term comes from combinatorics on words. The vertices of our graph are combinations of input digits for which the size of their possible weight coefficients sets did not decreased with an increment of lenght of the window. Whereas in combinatorics on words, vertices are given as factors of some language. But the edges are placed identically – if some combination of digits without the first one equals another one without the last digit.

**Definition 4.3.** Let $\mathcal{B}$ be an alphabet of input digits. Let Phase 2 is stable. Let $k \in \mathbb{N}, k \geq 2$. We set

$$V := \left\{ (w_{-1}, \ldots, w_{-k}) \in \mathcal{B}^k \colon \#\mathcal{Q}_{[w_{-1}, \ldots, w_{-k}]} = \#\mathcal{Q}_{[w_{-1}, \ldots, w_{-(k-1)}]} \right\}$$

and

$$E := \left\{ (w_{-1}, \ldots, w_{-k}) \to (w'_{-1}, \ldots, w'_{-k}) \in V \times V \colon (w_{-2}, \ldots, w_{-k}) = (w'_{-1}, \ldots, w'_{-(k-1)}) \right\} \, .$$

The oriented graph $G_k = (V, E)$ is called *Rauzy graph of Phase 2 (for the window of length $k$)*.

The structure of Rauzy graph $G_k$ signifies whether the non-decreasing combinations are such that they cause nonconvergence of Phase 2. Existence of an infinite walk in $G_k$ implies that Phase 2 does not converge:

**Theorem 4.7.** *Let Phase 2 is stable. If there exists $k_0 \in \mathbb{N}, k_0 \geq 2$, and $(w_0, \ldots, w_{-k_0}) \in \mathcal{B}^{k_0+1}$ such that*

*i)* $\#\mathcal{Q}_{[w_0, \ldots, w_{-(k_0-1)}]} > 1$ *and*

*ii) there exist an infinite walk $((w_{-1}^{(i)}, \ldots, w_{-k_0}^{(i)}))_{i \geq 1}$ in $G_{k_0}$ which starts in the vertex*

$$(w_{-1}^{(1)}, \ldots, w_{-k_0}^{(1)}) = (w_{-1}, \ldots, w_{-k_0}) \, ,$$

*then Phase 2 does not converge.*

*Proof.* Set
$$(w_k)_{k \geq 0} := w_0, w_1^{(1)}, \ldots, w_{k_0-1}^{(1)}, w_{k_0}^{(1)}, w_{k_0}^{(2)}, w_{k_0}^{(3)}, w_{k_0}^{(4)}, \ldots$$
We prove that $\#\mathcal{Q}_{[w_0,\ldots,w_{-k}]} = \#\mathcal{Q}_{[w_0,\ldots,w_{-(k_0-1)}]} > 1$ for all $k \geq k_0 - 1$, i.e., the condition *iii)* in Lemma 4.5 is satisfied.

Let $l \in \mathbb{N}$. Since $(w_{-(1+l)}, \ldots, w_{-(k_0+l)})$ is a vertex of $G_{k_0}$, the set $\mathcal{Q}_{[w_{-l},\ldots,w_{-(k_0+l)}]}$ equals $\mathcal{Q}_{[w_{-l},\ldots,w_{-(k_0+l-1)}]}$. As Phase 2 is stable, we have

$$\mathcal{Q}_{[w_{-l},\ldots,w_{-(k_0+l)}]} = \mathcal{Q}_{[w_{-l},\ldots,w_{-(k_0+l-1)}]}$$
$$\implies \mathcal{Q}_{[w_{-(l-1)},\ldots,w_{-(k_0+l)}]} = \mathcal{Q}_{[w_{-(l-1)},\ldots,w_{-(k_0+l-1)}]}$$
$$\vdots$$
$$\implies \mathcal{Q}_{[w_{-1},\ldots,w_{-(k_0+l)}]} = \mathcal{Q}_{[w_{-1},\ldots,w_{-(k_0+l-1)}]}$$
$$\implies \mathcal{Q}_{[w_0,\ldots,w_{-(k_0+l)}]} = \mathcal{Q}_{[w_0,\ldots,w_{-(k_0+l-1)}]}.$$

Hence, $\#\mathcal{Q}_{[w_0,\ldots,w_{-k}]} = \#\mathcal{Q}_{[w_0,\ldots,w_{-(k_0-1)}]} > 1$ for all $k \geq k_0 - 1$. $\qquad\square$

<span style="color:red">BYLO BY FAJN DOKAZAT I OPACNY SMER, JEDINE, PRES CO SE NEUMIM DOSTAT JE KDYBY EXISTOVALA JEN APERIODICKA POSLOUPNOUST, KVULI KTERE TO NEKONVERGUJE</span>

We remark that existence of an infinite walk in a finite graph is equivalent to existence of a cycle in the graph. Thus, if there is an infinite walk, we may find another one whose sequence of vertices is eventually periodic. We use this fact in Section 5.3 which describes modified algorithm for Phase 2 which implements the result of Theorem 4.7.

## 4.3   Minimal alphabet $\mathcal{A}$

Frougny, Pelantová and Svodová [5] proved a lower bound on the size of an alphabet $0 \in \mathcal{A} \subset \mathbb{Z}$ of consecutive integers which enables parallel addition. In this sextion, we prove the same bound for an arbitrary alphabet $\mathcal{A} \in \mathbb{Z}[\beta]$. We recall their auxiliary results in Theorem 4.8 an Lemma 4.9, but only for a parallel digit set conversion without anticipation as our rewriting rule $x - \beta$ does not require memory. The following theorem says that all classes modulo $\beta$ which are contained in $\mathcal{A} + \mathcal{A}$ must have their representatives in $\mathcal{A}$.

**Theorem 4.8.** *Let $\omega$ be an algebraic integer. Let the base $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$ and the alphabet $\mathcal{A} \subset \mathbb{Z}[\omega]$ be such that $0 \in \mathcal{A}$. If there exists a p-local digit set conversion defined by the function $\phi \colon (\mathcal{A} + \mathcal{A})^p \to \mathcal{A}$ and $p = r + 1$, then the number $\phi(b, \ldots, b) - b$ belongs to the set $(\beta - 1)\mathbb{Z}[\omega]$ for any $b \in \mathcal{A} + \mathcal{A}$.*

*Proof.* Let $b \in \mathcal{A} + \mathcal{A}$ and $a = \phi(b, \ldots, b)$. For $n \in \mathbb{N}, n \geq 1$, we denote $S_n$ the number represented by

$$^\omega 0 \underbrace{b \ldots b}_{n} \bullet \underbrace{b \ldots b}_{r} 0^\omega.$$

The representation of $S_n$ after the digit set conversion is of the form

$$^\omega 0 \underbrace{w_r \ldots w_1}_{\beta^n W} \underbrace{a \ldots a}_{n} \bullet \underbrace{\widetilde{w_1} \ldots \widetilde{w_r}}_{\beta^{-r}\widetilde{W}} 0^\omega \,,$$

where

$$W = \sum_{j=1}^{r} w_j \beta^{j-1} \qquad \text{and} \qquad \widetilde{W} = \sum_{j=1}^{r} \widetilde{w_j} \beta^{r-j} \,.$$

Since both representations have same value, we have

$$b \sum_{j=-r}^{n-1} \beta^j = W\beta^n + a \sum_{j=0}^{n-1} \beta^j + \beta^{-r}\widetilde{W}$$

$$b \sum_{j=-r}^{-1} \beta^j + b\frac{\beta^n - 1}{\beta - 1} = W\beta^n + a\frac{\beta^n - 1}{\beta - 1} + \beta^{-r}\widetilde{W} \,, \tag{4.1}$$

for all $n \geq 1$. We substract this equation for $n$ and $n-1$ to obtain

$$b\frac{\beta^n - \beta^{n-1}}{\beta - 1} = W(\beta^n - \beta^{n-1}) + a\frac{\beta^n - \beta^{n-1}}{\beta - 1} \,.$$

We simplify it to

$$b = W(\beta - 1) + a \,. \tag{4.2}$$

Hence, $a = \phi(b, \ldots, b) \equiv b$ modulo $\beta - 1$. $\qquad\square$

If a base $\beta$ has a real conjugate greater than one, then there are some extra requirements on the alphabet $\mathcal{A}$. For simplicity, we assume that the base $\beta$ itself is real and greater than one. We show later that this assumption is without loss of generality.

**Lemma 4.9.** *Let $\omega$ be a real algebraic integer and the base $\beta \in \mathbb{Z}[\omega]$ be such that $\beta > 1$. Let the alphabet $\mathcal{A} \subset \mathbb{Z}[\omega]$ be such that $0 \in \mathcal{A}$ and denote $\lambda = \min \mathcal{A}$ and $\Lambda = \max \mathcal{A}$. If there exists a p-local digit set conversion defined by the function $\phi \colon (\mathcal{A} + \mathcal{A})^p \to \mathcal{A}$ and $p = r + 1$, then:*

*i)* $\phi(b, \ldots, b) \neq \lambda$ *for all* $b \in \mathcal{A} + \mathcal{A}$ *such that* $b > \lambda$.

*ii)* $\phi(b, \ldots, b) \neq \Lambda$ *for all* $b \in \mathcal{A} + \mathcal{A}$ *such that* $b < \Lambda$.

*iii) If* $\Lambda \neq 0$, *then* $\phi(\Lambda, \ldots, \Lambda) \neq \Lambda$.

*iv) If* $\lambda \neq 0$, *then* $\phi(\lambda, \ldots, \lambda) \neq \lambda$.

*Proof.* To prove *i)*, assume in contradiction that $\phi(b, \ldots, b) = \lambda$. We proceed in the same manner as in Theorem 4.8, the equation (4.1) implies

$$b \sum_{j=-r}^{-1} \beta^j + b\frac{\beta^n - 1}{\beta - 1} = \beta^n W + \lambda\frac{\beta^n - 1}{\beta - 1} + \beta^{-r}\widetilde{W} \,.$$

We apply also the equation c to obtain

$$b \sum_{j=-r}^{-1} \beta^j + b\frac{\beta^n - 1}{\beta - 1} = \beta^n \frac{b - \lambda}{\beta - 1} + \lambda \frac{\beta^n - 1}{\beta - 1} + \beta^{-r}\widetilde{W}.$$

Now we may simplify and estimate

$$b \sum_{j=-r}^{-1} \beta^j + \frac{-b}{\beta - 1} = \frac{-\lambda}{\beta - 1} + \beta^{-r} \sum_{j=1}^{r} \widetilde{w_j}\beta^{r-j}$$

$$b \underbrace{\left( \sum_{j=1}^{r} \frac{1}{\beta^j} - \frac{1}{\beta - 1} \right)}_{-\sum_{j=r+1}^{\infty} \frac{1}{\beta^j}} = -\lambda \frac{1}{\beta - 1} + \sum_{j=1}^{r} \widetilde{w_j}\beta^{-j} \geq \lambda \underbrace{\left( -\frac{1}{\beta - 1} + \sum_{j=1}^{r} \frac{1}{\beta^j} \right)}_{-\sum_{j=r+1}^{\infty} \frac{1}{\beta^j}}.$$

Hence $b \leq \lambda$ which is a contradiction. The proof of *ii)* can be done in the same way.

For *iii)*, assume that $\phi(\Lambda, \dots, \Lambda) = \Lambda$. Now consider a number $T_q$ represented by

$$^\omega 0 \bullet \underbrace{\Lambda \dots \Lambda}_{r} \underbrace{(2\Lambda) \dots (2\Lambda)}_{q} 0^\omega.$$

After the digit set conversion, a representation is

$$^\omega 0 \underbrace{w_r \dots w_1}_{W} \bullet z_1 \dots z_{r+q} 0^\omega.$$

The value $T_q$ preserves, thus,

$$\Lambda \sum_{j=1}^{r} \beta^{-j} + 2\Lambda \sum_{j=r+1}^{r+q} \beta^{-j} = W + \sum_{j=1}^{r+q} z_j \beta^{-j}.$$

But $W = 0$ from the equation (4.2). We estimate

$$\Lambda \sum_{j=1}^{r+q} \beta^{-j} + \Lambda \sum_{j=r+1}^{r+q} \beta^{-j} = \sum_{j=1}^{r+q} z_j \beta^{-j} \leq \Lambda \sum_{j=1}^{r+q} \beta^{-j}$$

$$\Lambda \sum_{j=r+1}^{r+q} \beta^{-j} \leq 0.$$

This contradicts that $\Lambda$ is positive as it is a nonzero, maximal element of the alphabet $\mathcal{A}$ which contains 0. The proof of *iv)* is analogous. $\qquad\square$

In order to prove the lower bound, we need to show that the alphabet $\mathcal{A}$ must contain all representatives modulo $\beta$ and $\beta - 1$.

**Theorem 4.10.** *Let $\beta$ be an algebraic integer such that $|\beta| > 1$. Let $0 \in \mathcal{A} \subset \mathbb{Z}[\beta]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If addition in the numeration system $(\beta, \mathcal{A})$ which uses the rewriting rule $x - \beta$ is computable in parallel, then the alphabet $\mathcal{A}$ contains at least one representative of each congruence class modulo $\beta$ and $\beta - 1$ in $\mathbb{Z}[\beta]$.*

*Proof.* The existence of an algorithm for addition with the rewriting rule $x - \beta$ implies that the set $\mathcal{A}[\beta]$ is closed under addition. By the assumption $1 \in \mathcal{A}[\beta]$, the set $\mathbb{N}$ is subset of $\mathcal{A}[\beta]$. Since $0 \in \mathcal{A}$, we have $\beta \cdot \mathcal{A}[\beta] \subset \mathcal{A}[\beta]$. Hence, $\mathbb{N}[\beta] \subset \mathcal{A}[\beta]$.

For any element $x = \sum_{i=0}^{N} x_i \beta^i \in \mathbb{Z}[\beta]$ there is an element $x' = \sum_{i=0}^{N} x_i' \beta^i \in \mathbb{N}[\beta]$ such that $x \equiv_\beta x'$ since $m_\beta(0) \equiv_\beta 0$ and $\beta^i \equiv_\beta 0$. As $x' \in \mathbb{N}[\beta] \subset \mathcal{A}[\beta]$, we have

$$x \equiv_\beta x' = \sum_{i=0}^{n} a_i \beta^i \equiv_\beta a_0\,,$$

where $a_i \in \mathcal{A}$. Hence, for any element $x \in \mathbb{Z}[\omega]$, there is a letter $a_0 \in \mathcal{A}$ such that $x \equiv_\beta a_0$.

In order to prove that there is at least one representative of each congruence class modulo $\beta - 1$ in the alphabet $\mathcal{A}$, we consider again an element $x = \sum_{i=0}^{N} x_i \beta^i \in \mathbb{Z}[\beta]$. Similarly, there is an element $x' = \sum_{i=0}^{N} x_i' \beta^i \in \mathbb{N}[\beta]$ such that $x \equiv_{\beta-1} x'$ since $m_{\beta-1}(0) \equiv_{\beta-1} 0$ and $(\beta - 1)^i \equiv_{\beta-1} 0$.

Since $x' \in \mathbb{N}[\beta] \subset \mathcal{A}[\beta]$,

$$x' = \sum_{i=0}^{n} a_i \beta^i\,,$$

where $a_i \in \mathcal{A}$. We prove by induction with respect to $n$ that $x' \equiv_{\beta-1} a$ for some $a \in \mathcal{A}$. If $n = 0$, $x' = a_0$. Now we use the fact that if there is a parallel addition algorithm, for each letter $b \in \mathcal{A} + \mathcal{A}$, there is $a \in \mathcal{A}$ such that $b \equiv_{\beta-1} a$ (Theorem 4.8). For $n + 1$, we have

$$
\begin{aligned}
x' = \sum_{i=0}^{n+1} a_i \beta^i &= a_0 + \sum_{i=1}^{n+1} a_i \beta^i \\
&= a_0 + \beta \sum_{i=0}^{n} a_{i+1} \beta^i - \sum_{i=0}^{n} a_{i+1} \beta^i + \sum_{i=0}^{n} a_{i+1} \beta^i \\
&\equiv_{\beta-1} a_0 + (\beta - 1) \sum_{i=0}^{n} a_{i+1} \beta^i + a \equiv_{\beta-1} a_0 + a \equiv_{\beta-1} a' \in \mathcal{A}\,,
\end{aligned}
$$

where we use the induction assumption

$$\sum_{i=0}^{n} a_{i+1} \beta^i \equiv_{\beta-1} a\,.$$

$\square$

The following lemma summarizes that if we have a parallel addition algorithm for a base $\beta$, then we easily obtain an algorithm also for conjugates of $\beta$.

**Lemma 4.11.** *Let $\omega$ be an algebraic integer with a conjugate $\omega'$. Let $\beta \in \mathbb{Z}[\omega], |\beta| > 1$ and let $\sigma : \mathbb{Q}(\omega) \to \mathbb{Q}(\omega')$ be an isomorphism such that $|\sigma(\beta)| > 1$. Let $\varphi$ is a digit set conversion in the base $\beta$ from $\mathcal{A} + \mathcal{A}$ to $\mathcal{A}$. There exists is a digit set conversion $\varphi'$ in the base $\beta'$ from $\mathcal{A}' + \mathcal{A}'$ to $\mathcal{A}'$ where $\beta' = \sigma(\beta)$ and $\mathcal{A}' = \{\sigma(a) \colon a \in \mathcal{A}\}$.*

*Proof.* Let $\phi : \mathcal{A}^p \to \mathcal{A}$ be a mapping which defines $\varphi$ with $p = r + t + 1$. We define a mapping $\phi' : \mathcal{A}^p \to \mathcal{A}$ by

$$\phi'(w_{j+t}', \dots, w_{j-r}') = \sigma\left(\phi\left(\sigma^{-1}(w_{j+t}'), \dots, \sigma^{-1}(w_{j-r}')\right)\right)\,.$$

Next, we define a digit set conversion $\varphi' : (\mathcal{A}' + \mathcal{A}') \to \mathcal{A}'$ by $\varphi'(w') = (z'_j)_{j \in \mathbb{Z}}$ where $w' = (w'_j)_{j \in \mathbb{Z}}$ and $z'_j = \phi'(w'_{j+t}, \ldots, w'_{j-r})$. Obviously, if $w'$ has only finitely many nonzero entries, then there is only finitely many nonzeros in $(z'_j)_{j \in \mathbb{Z}}$ since

$$\phi'(0, \ldots, 0) = \sigma\left(\phi\left(\sigma^{-1}(0), \ldots, \sigma^{-1}(0)\right)\right) = \sigma\left(\phi\left(0, \ldots, 0\right)\right) = \sigma\left(0\right) = 0\,.$$

The value of the number represented by $w'$ is also preserved:

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} w'_j \beta'^j &= \sum_{j \in \mathbb{Z}} \sigma(w_j) \sigma(\beta)^j = \sigma(\sum_{j \in \mathbb{Z}} w_j \beta^j) \\
&= \sigma(\sum_{j \in \mathbb{Z}} z_j \beta^j) = \sigma(\sum_{j \in \mathbb{Z}} \phi\left(w_{j+t}, \ldots, w_{j-r}\right) \beta^j) \\
&= \sum_{j \in \mathbb{Z}} \sigma(\phi\left(w_{j+t}, \ldots, w_{j-r}\right)) \beta'^j = \sum_{j \in \mathbb{Z}} z'_j \beta'^j
\end{aligned}
$$

where $w_j = \sigma^{-1}(w'_j)$ for $j \in \mathbb{Z}$ and $\varphi((w_j)_{j \in \mathbb{Z}}) = (z_j)_{j \in \mathbb{Z}}$. $\qquad\square$

Finally, we put together that the alphabet $\mathcal{A}$ contains all representative modulo $\beta$ and $\beta - 1$, number of congruence classes and restrictions on the alphabet for a base with a real conjugate greater than one.

**Theorem 4.12.** *Let $\beta$ be an algebraic integer such that $|\beta| > 1$. Let $0 \in \mathcal{A} \subset \mathbb{Z}[\beta]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If addition in the numeration system $(\beta, \mathcal{A})$ which uses the rewriting rule $x - \beta$ is computable in parallel, then*

$$\#\mathcal{A} \geq \max\{|m_\beta(0)|, |m_\beta(1)|\}\,.$$

*Moreover, if $\beta$ is such that it has a real conjugate greater than 1, then*

$$\#\mathcal{A} \geq \max\{|m_\beta(0)|, |m_\beta(1)| + 2\}\,.$$

*Proof.* By Theorem 4.10, there are all representatives modulo $\beta$ and modulo $\beta - 1$ in the alphabet $\mathcal{A}$. The numbers of congruence classes are $|m_\beta(0)|$ and $|m_{\beta-1}(0)|$ by Theorem 3.7. Obviously, $m_{\beta-1}(x) = m_\beta(x+1)$. Thus $m_{\beta-1}(0) = m_\beta(1)$.

Let $\phi$ be a mapping which defines the parallel addition. According to Lemma 4.11, we may asssume that $\beta$ is real and greater than 1 in the proof of the second part. The assumption $1 \in \mathcal{A}[\beta]$ implies that $\Lambda > 0$. Thus, there are at least three elements in the alphabet $\mathcal{A}$, because $\mathcal{A} \ni \phi(\Lambda, \ldots, \Lambda) \neq \lambda$ and $\mathcal{A} \ni \phi(\Lambda, \ldots, \Lambda) \neq \Lambda$ by Lemma 4.9. It also implies that there are at least two representatives modulo $\beta - 1$ in the alphabet in the class which contains $\Lambda$, since $\phi(\Lambda, \ldots, \Lambda) \equiv_{\beta-1} \Lambda$.

If $\lambda \equiv_{\beta-1} \Lambda$, there must be one more element of the alphabet $\mathcal{A}$ in this class, since $\lambda \neq \Lambda$. Therefore, $\#\mathcal{A} \geq |m_\beta(1)| + 2$.

The case that $\lambda \not\equiv_{\beta-1} \Lambda$ is divided into two subcases. Suppose now that $\lambda \neq 0$. Then $\phi(\lambda, \ldots, \lambda) \neq \lambda$ and hence there is one more element in the alphabet in the class containing $\lambda$. Thus, there are at least two congruence classes which contain at least two elements of the alphabet $\mathcal{A}$. Therefore, $\#\mathcal{A} \geq |m_\beta(1)| + 2$.

If $\lambda = 0$, then all elements of $\mathcal{A} + \mathcal{A}$ are nonnegative and $\phi(b, \ldots, b) \neq 0$ for all $b \in (\mathcal{A} + \mathcal{A}) \setminus 0$. Suppose for contradiction, that there is no nonzero element of the alphabet $\mathcal{A}$ congruent to 0. We know that there is at least one represenative of each congruence class class modulo

$\beta - 1$ in $\mathcal{A}$ and at least two representatives in the congruence class which contains $\Lambda$. Let $k \in \mathbb{N}$ denote the number of elements which are in $\mathcal{A}$ extra to one element in each congruence class, i.e., $\#A = |m_\beta(1)| + k$. For $d \in \Lambda + \mathcal{A}$, the value $\phi(d, \ldots, d) \in \mathcal{A}$ is not congruent to 0 as it is nonzero and the class containing zero has only one element by the assumption. Therefore, the values $\phi(d, \ldots, d) \in \mathcal{A}$ for $|m_\beta(1)| + k$ distinct letters $d \in \Lambda + \mathcal{A}$ belong to only $|m_\beta(1)| - 1$ congruence classes. Hence, there exists $e_1, \ldots e_k, e_{k+1} \in \mathcal{A}$, pairwise distinct, and $f_1, \ldots f_k, f_{k+1} \in \mathcal{A}$ such that $e_i \neq f_i$ and $\phi(e_i + \Lambda, \ldots, e_i + \Lambda) \equiv_{\beta-1} \phi(f_i + \Lambda, \ldots, f_i + \Lambda)$ for all $i, 1 \leq i \leq k + 1$. Since

$$e_i + \Lambda \equiv_{\beta-1} \phi(e_i + \Lambda, \ldots, e_i + \Lambda) \equiv_{\beta-1} \phi(f_i + \Lambda, \ldots, f_i + \Lambda) \equiv_{\beta-1} f_i + \Lambda .$$

also $e_i \equiv_{\beta-1} f_i$ for all $i, 1 \leq i \leq k + 1$. This is a contradiction since it implies that $\#\mathcal{A} = |m_\beta(1)| + k + 1$. Hence, classes containing $\lambda$ and $\Lambda$ have both at least one more element of the alphabet $\mathcal{A}$, i.e., $\#\mathcal{A} \geq |m_\beta(1)| + 2$. $\qquad\square$

Note that the larger necessary alphabet for a base with a real conjugate greater than one is caused by the fact that minimal and maximal element of $\mathcal{A}$ both require another element in $\mathcal{A}$ which is in the same congruence class. This should be considered when an alphabet for such a base is generated.

<span style="color:red">BYLO BY FAJN TO JESTE ZOBECNIT NA ABECEDU ZE Z[OMEGA] NEJAKY POPIS GENEROVANI?</span>

# Chapter 5

# Modifications of the extending window method

## 5.1 Different choices in Phase 1

TADY POPSAT RUZNE METODY VYBERU? NEBO TO DAT NEKAM JINAM?

## 5.2 Different choices in Phase 2

TADY POPSAT RUZNE METODY VYBERU? NEBO TO DAT NEKAM JINAM?
SHRNOUT PHASE 2 S KONTROLOU BBB, STABLE PODMINKOU A KONTROLOU
ZACYKLENI

For construction of the set $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ we first choose such elements of $\mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ which are the only possible to cover some value $x \in w_0 + \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]}$. Other elements from $\mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ which cover an uncovered value are added one by one to $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ until each $x$ equals $a + \beta q_j$ for some $q_j$ in $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$ and $a \in \mathcal{A}$. The pseudocode is in Algorithm 5.

Notice that the result of Algorithm 5 is influenced by the way how we pick an element on line 8. It can be done deterministically or non-deterministically. We use the following deterministic choice – suppose that we want to choose from elements $x_1 = \sum_{i=0}^{d-1} x_{1,i}\omega^i, x_2 = \sum_{i=0}^{d-1} x_{2,i}\omega^i, \ldots, x_n = \sum_{i=0}^{d-1} x_{n,i}\omega^i \in \mathbb{Z}[\omega]$, where $d$ is the degree of $\omega$. Let $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ be such that

$$\sum_{i=0}^{d-1} a_i\omega^i = \sum_{j=1}^{n} x_j\,.$$

Set $c := \sum_{i=0}^{d-1} c_i\omega^i \in \mathbb{Z}[\omega]$ with $c_i = [\frac{a_i}{n}]$ where $[\cdot]$ denotes rounding to the nearest integer. Let the index set $I_0 \subset \{1, \ldots, n\}$ be such that

$$|x_{j,0} - c_0| = \min\{|x_{k,0} - c_0| \colon k \in 1, \ldots, n\}$$

for all $j \in I_0$. For all $i \in \{1, \ldots, d-1\}$, let the index set $I_i \subset I_{i-1}$ be such that

$$|x_{j,i} - c_i| = \min\{|x_{k,i} - c_i| \colon k \in I_{i-1}\}$$

---
**Algorithm 5** Search for set $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$
---
**Input:** Input digit $w_j$, set of possible carries $\mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]}$, previous set of possible weight
    coefficients $\mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$
1: `list_of_coverings`:=empty list of lists
2: **for all** $x \in w_j + \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]}$ **do**
3:      Build a list `x_covered_by` of weight coefficients $q_j \in \mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ such that

$$x = a + \beta q_j \qquad \text{for some } a \in \mathcal{A}\,.$$

4:      Append `x_covered_by` to `list_of_coverings`
5: **end for**
6: $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$:= empty set
7: **while** `list_of_coverings` is nonempty **do**
8:      Pick an element $q$ of one of the shortest lists of `list_of_coverings`
9:      Add the element $q$ to $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$
10:      Remove lists of `list_of_coverings` containing the element $q$ from
        `list_of_coverings`
11: **end while**
12: **return** $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$
---

for all $j \in I_i$. If there is only one element in the index set $I_{d-1} = \{j_0\}$, choose the element $x_{j_0}$. Otherwise choose $j_0 \in I_{d-1}$ such that $x_{j_0,0}$ is the smallest from all $x_{j,0}$ such that $j \in I_{d-1}$. If there are more such elements, then choose from them according to the value $x_{j,1}$ etc.

In other words, we take the elements which are the "closest" ones to the rounded center of gravity $c$ of the values $x_1,\ldots,x_n$ where "closest" is measured by absolute value of the first coordinate of $\pi(x_j) - \pi(c)$. In case of equality, according to the second coordinate etc. If there is more than one such element, we choose the element $x_{j_0}$ with the smallest first, resp. second, etc. coordinate of $\pi(x_{j_0})$.

## 5.3 Modified Phase 2

In this section, we introduce algorithms based on the theorems proven in Chapter 4. We recall an algorithm for checking convergence of Phase 2 for $bb\ldots b$ inputs. Next, we show that it is possible to make Phase 2 stable by wrapping the choice of a set of possible weight coefficients into a simple while loop. Finally, we present an algorithm for Phase 2 which includes all modifications – the menshioned check for $bb\ldots b$ inputs and control of convergence by searching for an infinite walk in Rauzy graph $G_k$.

**Checking $bb\ldots b$ inputs**

Algorithm 6 was proposed in [7]. It checks whether Phase 2 stops when it processes input digits $bb\ldots b$. Sets $\mathcal{Q}_{[b]}^m$ can be easily constructed separately for each $b \in \mathcal{B}$ for given $m$. We build the set $\mathcal{Q}_{[b]}^m$ for input digits $bb\ldots b$ in the same way as in Phase 2. This means that we first search for $\mathcal{Q}_{[b]}^1$ such that

$$b + \mathcal{Q} \subset \mathcal{A} + \beta \mathcal{Q}_{[b]}^1\,.$$

Until the set $\mathcal{Q}^m_{[b]}$ contains only one element, we increment the length of window $m$ and, using Algorithm 7, we find a subset $\mathcal{Q}^{m+1}_{[b]}$ of the set $\mathcal{Q}^m_{[b]}$ such that

$$b + \mathcal{Q}^m_{[b]} \subset \mathcal{A} + \beta \mathcal{Q}^{m+1}_{[b]}.$$

In each iteration, we check whether the set $\mathcal{Q}^{m+1}_{[b]}$ is strictly smaller than the set $\mathcal{Q}^m_{[b]}$. If not, we know by Theorem 4.6 that Phase 2 does not converge because $\#\mathcal{Q}^m_{[b]}$ is eventually a constant greater than 2.

Hence, non-finiteness of Phase 2 can be revealed by running Algorithm 6 for each input digit $b \in \mathcal{B}$.

---

**Algorithm 6** Check the input $bb \dots b$

---

**Input:** Weight coefficient set $\mathcal{Q}$, digit $b \in \mathcal{B}$
**Ouput:** TRUE if there is a unique weight coefficient for input $bb \dots b$, FALSE otherwise
 1: Find minimal set $\mathcal{Q}^1_{[b]} \subset \mathcal{Q}$ such that

$$b + \mathcal{Q} \subset \mathcal{A} + \beta \mathcal{Q}^1_{[b]}.$$

 2: $m := 1$
 3: **while** $\#\mathcal{Q}^m_{[b]} > 1$ **do**
 4: $\quad m := m + 1$
 5: $\quad$ By Algorithm 7, find minimal set $\mathcal{Q}^m_{[b]} \subset \mathcal{Q}^{m-1}_{[b]}$ such that

$$b + \mathcal{Q}^{m-1}_{[b]} \subset \mathcal{A} + \beta \mathcal{Q}^m_{[b]}.$$

 6: $\quad$ **if** $\#\mathcal{Q}^m_{[b]} = \#\mathcal{Q}^{m-1}_{[b]}$ **then**
 7: $\quad\quad$ **return** FALSE
 8: $\quad$ **end if**
 9: **end while**
10: **return** TRUE

---

## Stable Phase 2

We remind that for checking whether Phase 2 converges we assume that it is stable. An algorithm of choice of possible weight coefficients set can be easily modified to ensure that

$$\mathcal{Q}_{[w_{-1},\dots,w_{-k}]} = \mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]} \implies \mathcal{Q}_{[w_0,\dots,w_{-k}]} = \mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$$

for all $(w_{-1},\dots,w_{-k}) \in \mathcal{B}^k$. If the algorithm is deterministic, then the set $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ is determined by the set $\mathcal{Q}_{[w_{-1},\dots,w_{-k}]}$, $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ and $w_0$. Similarly, the set $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is determined by the set $\mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]}$, $\mathcal{Q}_{[w_0,\dots,w_{-(k-2)}]}$ and $w_0$.

Suppose that $\mathcal{Q}_{[w_{-1},\dots,w_{-k}]} = \mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]}$. Now, the only difference between $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ and $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ is that $\mathcal{Q}_{[w_0,\dots,w_{-k}]}$ is searched as a subset of $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$, whereas $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ as a subset of $\mathcal{Q}_{[w_0,\dots,w_{-(k-2)}]}$. In orded to find $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$, we use Algorithm 5 repeatedly instead of once. In each iteration, the input digit $w_0$ and the set $\mathcal{Q}_{[w_{-1},\dots,w_{-(k-1)}]}$ remains the same but we search for $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]}$ as a subset of $\mathcal{Q}'_w$, where $\mathcal{Q}'_w$ is the output of the previous iteration or $\mathcal{Q}_{[w_0,\dots,w_{-(k-2)}]}$ in the first iteration. The algorithm stops when $\mathcal{Q}_{[w_0,\dots,w_{-(k-1)}]} = \mathcal{Q}'_w$. This loop is desribed in Algorithm 7.

Now, when the set $\mathcal{Q}_{[w_0,\ldots,w_{-k}]}$ is searched as a subset of $\mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$ in the same manner, it runs with the same inputs as the last iteration of the search for $\mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$. Hence, $\mathcal{Q}_{[w_0,\ldots,w_{-k}]} = \mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$.

---

**Algorithm 7** Stable search for possible weight coefficient set $\mathcal{Q}_{[w_0,\ldots,w_{-k}]}$

---

**Input:** Input digit $w_0$, set of possible carries $\mathcal{Q}_{[w_{-1},\ldots,w_{-k}]}$, previous set of possible weight coefficients $\mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$

1: $\mathcal{Q}'_w := \mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$
2: **while** TRUE **do**
3:     By Algorithm 5, find the set of possible weight coefficients $\mathcal{Q}_{[w_0,\ldots,w_{-k}]}$ as a subset of $\mathcal{Q}'_w$ instead of the previous set of weight coefficients $\mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$.
4:     **if** $\mathcal{Q}_{[w_0,\ldots,w_{-k}]} = \mathcal{Q}'_w$ **then**
5:         **return** $\mathcal{Q}_{[w_0,\ldots,w_{-k}]}$
6:     **end if**
7:     $\mathcal{Q}'_w := \mathcal{Q}_{[w_0,\ldots,w_{-k}]}$
8: **end while**

---

## Infinite walk in Rauzy graph $G_k$

As the Rauzy graph $G_k$ is finite, there exist an infinite walk in $G_k$ if and only if there exists an oriented cycle. Algorithm 8 checks whether some walk starting in $(w_{-1},\ldots,w_{-k})$ enters such a cycle eventually. Firts, it checks if the vertex $(w_{-1},\ldots,w_{-k})$ is in the graph $G_k$. If yes, all vertices which are accessible by appropriately oriented edge from $(w_{-1},\ldots,w_{-k})$ are entered by Algorithm 9. It is called recursively to enter all accessible vertices from the given one. The already traversed path is passed in each call and if an vertex is visited second time, then the cycle is found and algorithm ends. If all branches of the recursion ends up without visiting some vertex twice, then there is no cycle and thus no infinite path.

Note that saving of the traversed path requires only the label of the first vertex and last digits of the labels of next visited vertices due to the construction of Rauzy graph.

---

**Algorithm 8** Check if there is in an infinite walk in $G_k$ starting in $(w_{-1},\ldots,w_{-k})$

---

**Input:** Rauzy graph $G_k$, combination of input digits $(w_{-1},\ldots,w_{-k})$
**Ouput:** Return TRUE if TRUE is returned in any step of the recursion, that is when a walk $w_1, w_2,\ldots$ enters an oriented cycle in $G_k$. Otherwise return FALSE.

1: **if** $(w_{-1},\ldots,w_{-k}) \in G_k$ **then**
2:     By Algorithm 9, enter next vertices from $(w_{-1},\ldots,w_{-k})$ with the traversed path $(w_{-1},\ldots,w_{-k})$.
3: **else**
4:     **return** FALSE
5: **end if**
6: **return** FALSE

---

## Phase 2 with convergence control

Algorithm 10 modifies the basic proposal of Phase 2 to reveal its possible non-convergence. First, the necessary condition given by Theorem 4.6 is checked, i.e., the convergence of Phase 2

---

**Algorithm 9** Enter vertices from $(w'_{-1}, \ldots, w'_{-k})$

---

**Input:** Rauzy graph $G_k$, vertex $(w'_{-1}, \ldots, w'_{-k})$, traversed path $(w_1, \ldots, w_l)$.

1: **for all** $w'_{k+1} \in \mathcal{B}$ such that $(w'_{-1}, \ldots, w'_{-k}) \to (w'_{-2}, \ldots, w'_{-(k+1)}) \in G_k$ **do**

2:      **if** $(w'_{-2}, \ldots, w'_{-(k+1)})$ is in the traversed path $(w_1, \ldots, w_l)$ **then**

3:          **return** TRUE

4:      **else**

5:          By Algorithm 9, enter next vertices from $(w'_{-2}, \ldots, w'_{-(k+1)})$ with the traversed path $(w_1, \ldots, w_l, w'_{(k+1)})$.

6:      **end if**

7: **end for**

---

for inputs $bb \ldots b$ is verified by Algorithm 6 for all $b \in \mathcal{B}$.

Then we proceed in the same way as in Algorithm 4 with the following modifications: it is sufficient to process only $(w_0, \ldots, w_{-k}) \in \mathcal{B}^{k+1}$ such that $\#\mathcal{Q}_{[w_0, \ldots, w_{-(k-1)}]} > 1$ since if $\#\mathcal{Q}_{[w_0, \ldots, w_{-(k-1)}]} = 1$, then $\mathcal{Q}_{[w_0, \ldots, w_{-k'}]} = \mathcal{Q}_{[w_0, \ldots, w_{-(k-1)}]}$ for all $k' > k$.

Moreover, we check possible non-convergence according to Theorem 4.7. If $\mathcal{Q}_{[w_0, \ldots, w_{-k}]} = \mathcal{Q}_{[w_0, \ldots, w_{-(k-1)}]}$, then the vertex $\mathcal{Q}_{[w_0, \ldots, w_{-k}]}$ is added into the Rauzy graph $G_{k+1}$ and the Rauzy graph $G_k$ is examined by Algorithm 8 whether it contains an infinite walk starting in $(w_{-1}, \ldots, w_{-k})$. Note that $\mathcal{Q}_{[w_0, \ldots, w_{-k}]} = \mathcal{Q}_{[w_0, \ldots, w_{-(k-1)}]}$ is a necessary condition for $(w_{-1}, \ldots, w_{-k})$ be a vertex of $G_k$. It also implies that $\#\mathcal{Q}_{[w_0, \ldots, w_{-k}]} > 1$.

We remark that the non-convergence caused by an input $bb \ldots b$ for some $b \in \mathcal{B}$ would be revealed also as a cycle in Rauzy graph. Nevertheless, the number of calls of Algorithm 5 during Algorithm 6 is at most $\#Q$ for each $b \in \mathcal{B}$, while it grows exponentially with the length of window in search for a weight function. Thus, we save a lot of computational time in cases which fail already on $bb \ldots b$ inputs. At the same time, the costs of this check are low in other cases.

**Algorithm 10** Modified search for a weight function (Phase 2)

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**Input:** weight coefficients set $\mathcal{Q}$

1: **for all** $b \in \mathcal{B}$ **do**
2:      **if not** Check the input $bb \ldots b$ by Algorithm 6 **then**
3:          **return** Phase 2 does not converge for input $bb \ldots b$.
4:      **end if**
5: **end for**
6: **for all** $w_0 \in \mathcal{B}$ **do**
7:      By Algorithm 5, find set $\mathcal{Q}_{[w_0]} \subset \mathcal{Q}$ such that

$$w_0 + \mathcal{Q} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_0]}$$

8: **end for**
9: $k := 0$
10: **while** $\max\{\#\mathcal{Q}_{[w_0,\ldots,w_{-k}]} : (w_0,\ldots,w_{-k}) \in \mathcal{B}^{k+1}\} > 1$ **do**
11:      $k := k + 1$
12:      **for all** $\{(w_0,\ldots,w_{-k}) \in \mathcal{B}^{k+1} : \#\mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]} > 1\}$ **do**
13:          By Algorithm 7, find set $\mathcal{Q}_{[w_0,\ldots,w_{-k}]} \subset \mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$ such that

$$w_j + \mathcal{Q}_{[w_{-1},\ldots,w_{-k}]} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_0,\ldots,w_{-k}]}.$$

14:          **if** $\mathcal{Q}_{[w_0,\ldots,w_{-k}]} = \mathcal{Q}_{[w_0,\ldots,w_{-(k-1)}]}$ **and** $k \geq 2$ **then**
15:             Add the vertex $(w_0,\ldots,w_{-k})$ to the Rauzy graph $G_{k+1}$.
16:             **if** Check infinite walks in $G_k$ starting in $(w_{-1},\ldots,w_{-k})$ by Alg. 8 **then**
17:                 **return** Phase 2 does not converge.
18:             **end if**
19:          **end if**
20:      **end for**
21: **end while**
22: $r := k$
23: **for all** $l \in \mathbb{N}, l \leq r$ **do**
24:      **for all** $\{(w_0,\ldots,w_{-l}) \in \mathcal{B}^{l+1} : \mathcal{Q}_{[w_0,\ldots,w_{-l}]}$ was found, $\#\mathcal{Q}_{[w_0,\ldots,w_{-l}]} = 1\}$ **do**
25:          **for all** $(w_{-(l+1)},\ldots,w_{-r}) \in \mathcal{B}^{r-l}$ **do**
26:             $q(w_0,\ldots,w_{-r}) :=$ only element of $\mathcal{Q}_{[w_0,\ldots,w_{-l}]}$
27:          **end for**
28:      **end for**
29: **end for**
30: **return** $q$

---

# Chapter 6

# Design and implementation

CASTECNE VYUZIT Z VYZKUMAKU??
PRIDAT POZNAMKU O GOOGLE TABULCE, HROMADNE TESTOVANI

# Chapter 7

# Testing

<span style="color:red">TABULKA VSECH USPESNYCH PLUS NEJAKE NEUSPESNE A OKOMENTOVAT</span>

## 7.1 Comparing different choices in Phase 1 and 2

<span style="color:red">VSECHNY MILENINY RICNE SPOSTENE, TAKY BY TADY MOHLY BYT I NEJAKE NEUSPESNE</span>

## 7.2 Quadratic bases with integer alphabet

## 7.3 Quadratic bases with non-integer alphabet

## 7.4 Cubic bases

| Ex. | $\omega$ | $\beta$ | $m_\beta$ | conj. | $\#\mathcal{A}$ | min. | Phase 1 | No. | $\#\mathcal{Q}$ | $bb\ldots b$ | Phase 2 | No. | $m$ |
|-----|----------|---------|-----------|-------|------|------|---------|-----|------|--------|---------|-----|-----|
| D.1 | $\frac{1}{2}i\sqrt{7}-\frac{1}{2}$ | $\omega$ | $x^2+x+2$ | no | 4 | yes | ✓ | 3 | 29 | ✓ | ✓ | 15 | 8 |
| D.2 | $\frac{1}{2}i\sqrt{7}-\frac{1}{2}$ | $\omega$ | $x^2+x+2$ | no | 4 | yes | ✓ | 3 | 29 | ✓ | ✓ | 15 | 8 |
| D.3 | $i\sqrt{2}$ | $\omega$ | $x^2+2$ | no | 3 | yes | ✓ | 3 | 9 | ✓ | ✓ | 15 | 4 |
| D.4 | $i\sqrt{2}$ | $\omega$ | $x^2+2$ | no | 3 | yes | ✓ | 1 | 9 | ✓ | ✓ | 4 | 4 |

Table 7.1: xxx

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# Appendices

## A    Illustration of Phase 1

STEJNE JAK VE VYZKUMAKU?

## B    Illustration of Phase 2

STEJNE JAK VE VYZKUMAKU?

## C    GUI

CLOUD, GOOGLE TABULKA?

## D    Tested examples

**Quadratic bases with integer alphabet**

**Example D.1.** Parameters:

- Minimal polynomial of $\omega$: $t^2 + t + 2$

- Base $\beta = \omega$

- Minimal polynomial of base: $x^2 + x + 2$

- Alphabet $\mathcal{A} = \{0, \omega + 1, 1, -1\}$

- Input alphabet $\mathcal{B} = \mathcal{A} + \mathcal{A}$

The result of the extending window method is:

1. Phase 1 was successful. The number of elements in the weight coefficient set $\mathcal{Q}$ is 29.

2. There is a unique weight coefficient for input $b, b, \ldots, b$ for all $b \in \mathcal{B}$.

3. Phase 2 was successful. The length of window $m$ of the weight function $q$ is 8.

**Example D.2.** Parameters:

- Minimal polynomial of $\omega$: $t^2 + t + 2$

- Base $\beta = \omega$

- Minimal polynomial of base: $x^2 + x + 2$

- Alphabet $\mathcal{A} = \{0, \omega + 1, 1, -1\}$

- Input alphabet $\mathcal{B} = \mathcal{A} + \mathcal{A}$

The result of the extending window method is:

1. Phase 1 was successful. The number of elements in the weight coefficient set $\mathcal{Q}$ is 29.

2. There is a unique weight coefficient for input $b, b, \ldots, b$ for all $b \in \mathcal{B}$.

3. Phase 2 was successful. The length of window $m$ of the weight function $q$ is 8.

**Example D.3.** Parameters:

- Minimal polynomial of $\omega$: $t^2 + 2$

- Base $\beta = \omega$

- Minimal polynomial of base: $x^2 + 2$

- Alphabet $\mathcal{A} = \{0, \omega + 1, -\omega - 1\}$

- Input alphabet $\mathcal{B} = \mathcal{A} + \mathcal{A}$

The result of the extending window method is:

1. Phase 1 was successful. The number of elements in the weight coefficient set $\mathcal{Q}$ is 9.

2. There is a unique weight coefficient for input $b, b, \ldots, b$ for all $b \in \mathcal{B}$.

3. Phase 2 was successful. The length of window $m$ of the weight function $q$ is 4.

**Example D.4.** Parameters:

- Minimal polynomial of $\omega$: $t^2 + 2$

- Base $\beta = \omega$

- Minimal polynomial of base: $x^2 + 2$

- Alphabet $\mathcal{A} = \{0, 1, -1\}$

- Input alphabet $\mathcal{B} = \mathcal{A} + \mathcal{A}$

The result of the extending window method is:

1. Phase 1 was successful. The number of elements in the weight coefficient set $\mathcal{Q}$ is 9.

2. There is a unique weight coefficient for input $b, b, \ldots, b$ for all $b \in \mathcal{B}$.

3. Phase 2 was successful. The length of window $m$ of the weight function $q$ is 4.

**Quadratic bases with non-integer alphabet**

sdgv

**Cubic bases**

dfb