List of symbols

Symbol	Description
$\mathbb{N}$	set of nonnegative integers $\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}$	set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Q}(\beta)$	the smallest field containing the set $\mathbb{Q}$ and algebraic number β
$\#S$	number of elements of the finite set S
C^*	complex conjugation and transposition of the complex matrix C
m_β	monic minimal polynomial of the algebraic number β
$\deg \beta$	degree of the algebraic number β
$(\beta, \mathcal{A})$	numeration system with the base β and the alphabet $\mathcal{A}$
$(x)_{\beta,\mathcal{A}}$	$(\beta, \mathcal{A})$-representation of the number x
$\mathrm{Fin}_{\mathcal{A}}(\beta)$	set of all complex numbers with a finite $(\beta, \mathcal{A})$-representation
$\mathcal{A}^{\mathbb{Z}}$	set of all bi-infinite sequences of digits in $\mathcal{A}$
$\mathbb{Z}[\omega]$	set of values of all polynomials with integer coefficients evaluated in ω
π	isomorphism from $\mathbb{Z}[\omega]$ to $\mathbb{Z}^d$
$\mathcal{B}$	alphabet of input digits
q_j	weight coefficient for the j-th position
$\mathcal{Q}$	weight coefficients set
$\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$	set of possible weight coefficients for the input digits $w_j, \dots, w_{j-m+1}$
$\lfloor x \rfloor$	floor function of the number x
$\mathrm{Re}\, x$	real part of the complex number x
$\mathrm{Im}\, x$	imaginary part of the complex number x

Chapter 1

Preliminaries

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In this chapter, we recall few definitions and results connected to numeration systems and parallelism. We define the set $\mathbb{Z}[\omega]$ for an algebraic integer ω and we prove that $\mathbb{Z}[\omega]$ is isomorphic to $\mathbb{Z}^d$. This property is used in Theorem **??** which is an important tool for divisibility in $\mathbb{Z}[\omega]$. Division in $\mathbb{Z}[\omega]$ is necessary for the extending window method described in Chapter **??**.

1.1 Numeration systems

Firstly, we give a general definition of numeration system.

Definition 1.1. Let $\beta \in \mathbb{C}, |\beta| > 1$ and $\mathcal{A} \subset \mathbb{C}$ be a finite set containing 0. A pair $(\beta, \mathcal{A})$ is called a *positional numeration system* with *base* β and *digit set* $\mathcal{A}$, usually called *alphabet*.

So-called standard numeration systems have an integer base β and an alphabet $\mathcal{A}$ which is a set of contiguous integers. We restrict ourselves to the base β which is an algebraic integer and possibly non-integer alphabet $\mathcal{A}$.

Definition 1.2. Let $(\beta, \mathcal{A})$ be a positional numeration system. We say that a complex number x has a $(\beta, \mathcal{A})$-*representation* if there exist digits $x_n, x_{n-1}, x_{n-2}, \cdots \in \mathcal{A}, n \geq 0$ such that $x = \sum_{j=-\infty}^{n} x_j \beta^j$.

We write briefly a *representation* instead of a $(\beta, \mathcal{A})$-representation if the base β and the alphabet $\mathcal{A}$ follow from context.

Definition 1.3. Let $(\beta, \mathcal{A})$ be a positional numeration system. The set of all complex numbers with a finite $(\beta, \mathcal{A})$-representation is defined by

$$\mathrm{Fin}_{\mathcal{A}}(\beta) := \left\{ \sum_{j=-m}^{n} x_j \beta^j : n, m \in \mathbb{N}, x_j \in \mathcal{A} \right\}.$$

For $x \in \mathrm{Fin}_{\mathcal{A}}(\beta)$, we write

$$(x)_{\beta,\mathcal{A}} = 0^\omega x_n x_{n-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-m} 0^\omega ,$$

where 0^ω denotes right, respectively left-infinite sequence of zeros. Notice that indices are decreasing from left to right as it is usual to write the most significant digits first. In what follows, we omit the starting and ending 0^ω when we work with numbers in $\mathrm{Fin}_{\mathcal{A}}(\beta)$. We remark that existence of an algorithm (standard or parallel) producing a finite $(\beta, \mathcal{A})$-representation of $x + y$ where $x, y \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ implies that the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition, i.e.,

$$\mathrm{Fin}_{\mathcal{A}}(\beta) + \mathrm{Fin}_{\mathcal{A}}(\beta) \subset \mathrm{Fin}_{\mathcal{A}}(\beta)\,.$$

Designing an algorithm for parallel addition requires some redundancy in numeration system. According to [4], a numeration system $(\beta, \mathcal{A})$ is called *redundant* if there exists $x \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ which has two different $(\beta, \mathcal{A})$-representations. For instance, the number 1 has $(2, \{-1, 0, 1\})$-representations $1\bullet$ and $1(-1)\bullet$. Redundant numeration system can enable us to avoid carry propagation in addition. On the other hand, there are some disadvantages. For example, comparison is problematic.

1.2 Parallel addition

A local function, which is also often called a sliding block code, is used to mathematically formalize parallelism.

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be alphabets. A function $\varphi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is said to be *p-local* if there exist $r, t \in \mathbb{N}$ satisfying $p = r + t + 1$ and a function $\phi : \mathcal{B}^p \to \mathcal{A}$ such that, for any $w = (w_j)_{j\in\mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$ and its image $z = \varphi(w) = (z_j)_{j\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, we have $z_j = \phi(w_{j+t}, \cdots, w_{j-r})$ for every $j \in \mathbb{Z}$. The parameter t, resp. r, is called *anticipation*, resp. *memory*.

This means that each digit of the image $\varphi(w)$ is computed from p digits of w in a sliding window. Suppose that there is a processor on each position with access to t input digits on the left and r input digits on the right. Then computation of $\varphi(w)$, where w is a finite sequence, can be done in constant time independent on the length of w.

Definition 1.5. Let β be a base and $\mathcal{A}$ and $\mathcal{B}$ two alphabets containing 0. A function $\varphi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that

1. for any $w = (w_j)_{j\in\mathbb{Z}} \in \mathcal{B}^{\mathbb{Z}}$ with finitely many non-zero digits, $z = \varphi(w) = (z_j)_{j\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ has only finite number of non-zero digits, and
2. $\sum_{j\in\mathbb{Z}} w_j\beta^j = \sum_{j\in\mathbb{Z}} z_j\beta^j$

is called *digit set conversion* in base β from $\mathcal{B}$ to $\mathcal{A}$. Such a conversion φ is said to be *computable in parallel* if φ is a p-local function for some $p \in \mathbb{N}$.

In fact, addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel if there is a digit set conversion from $\mathcal{A} + \mathcal{A}$ to $\mathcal{A}$ computable in parallel as we can easily output digitwise sum of two $(\beta, \mathcal{A})$-representations in parallel.

We recall few results about parallel addition in a numeration system with an integer alphabet. C. Frougny, E. Pelantová and M. Svobodová proved the following sufficient condition of existence of an algorithm for parallel addition in [2].

Theorem 1.1. *Let $\beta \in \mathbb{C}$ be an algebraic number such that $|\beta| > 1$ and all its conjugates in modulus differ from 1. There exists an alphabet $\mathcal{A}$ of contiguous integers containing 0 such that addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel.*

The proof of the theorem provides the algorithm for the alphabet of the form $\{-a, -a+1, \ldots, 0, \ldots, a-1, a\}$. But in general, a is not minimal.

The same authors showed in [1] that the condition on the conjugates of the base β is also necessary:

Theorem 1.2. *Let the base $\beta \in \mathbb{C}, |\beta| > 1$, be an algebraic number with a conjugate β' such that $|\beta'| = 1$. Let $\mathcal{A} \subset \mathbb{Z}$ be an alphabet of contiguous integers containing 0. Then addition on $Fin_{\mathcal{A}}(\beta)$ cannot be computable in parallel.*

The question of minimality of the alphabet is studied in [3]. The following lower bound for the size of the alphabet is provided:

Theorem 1.3. *Let $\beta \in \mathbb{C}, |\beta| > 1$, be an algebraic integer with the minimal polynomial p. Let $\mathcal{A} \subset \mathbb{Z}$ be an alphabet of contiguous integers containing 0 and 1. If addition on $Fin_{\mathcal{A}}(\beta)$ is computable in parallel, then $\#\mathcal{A} \geq |p(1)|$. Moreover, if β is a positive real number, $\beta > 1$, then $\#\mathcal{A} \geq |p(1)| + 2$.*

In this thesis, we work in a more general concept as we consider also non-integer alphabets. First, we recall the following definition.

Definition 1.6. Let ω be a complex number. The set of values of all polynomials with integer coefficients evaluated in ω is denoted by

$$\mathbb{Z}[\omega] = \left\{ \sum_{i=0}^{n} a_i \omega^i \colon n \in \mathbb{N}, a_i \in \mathbb{Z} \right\} \subset \mathbb{Q}(\omega)\,.$$

Notice that $\mathbb{Z}[\omega]$ is a commutative ring (for our purposes, a ring is associative under multiplication and there is a multiplicative identity).

From now on, let ω be an algebraic integer which generates the set $\mathbb{Z}[\omega]$ and let the base $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$. We remark that β is also an algebraic integer as all elements of $\mathbb{Z}[\omega]$ are algebraic integers. Finally, let the alphabet $\mathcal{A}$ be a finite subset of $\mathbb{Z}[\omega]$ such that $0 \in \mathcal{A}$.

Few parallel addition algorithms for such numeration system with a non-integer alphabet were found ad hoc. We introduce the method for construction of the parallel addition algorithm for a given numeration system $(\beta, \mathcal{A})$ in Chapter **??**.

1.3 Isomorphism of $\mathbb{Z}[\omega]$ and $\mathbb{Z}^d$

The goal of this section is to show a connection between the ring $\mathbb{Z}[\omega]$ and the set $\mathbb{Z}^d$. Using Theorem **??**, division in $\mathbb{Z}[\omega]$ can be replaced by searching for an integer solution of a linear system. This is used for the implementation of the extending window method.

First we recall the notion of companion matrix which we use to define multiplication in $\mathbb{Z}^d$. By the minimal polynomial of an algebraic integer, we always mean the monic minimal polynomial.

Definition 1.7. Let ω be an algebraic integer of degree $d \geq 1$ with the minimal polynomial

$p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$. The matrix

$$S := \begin{pmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & \cdots & 0 & -p_1 \\ 0 & 1 & \cdots & 0 & -p_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -p_{d-1} \end{pmatrix} \in \mathbb{Z}^{d\times d}$$

is called *companion matrix* of the minimal polynomial of ω.

In what follows, the standard basis vectors of $\mathbb{Z}^d$ are denoted by

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_{d-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Definition 1.8. Let ω be an algebraic integer of degree $d \geq 1$, let p be its minimal polynomial and let S be its companion matrix. We define the mapping $\odot_\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$u \odot_\omega v := \left(\sum_{i=0}^{d-1} u_i S^i\right) \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} \quad \text{for all } u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{pmatrix}, v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} \in \mathbb{Z}^d.$$

and we define powers of $u \in \mathbb{Z}^d$ by

$$\begin{aligned} u^0 &= e_0, \\ u^i &= u^{i-1} \odot_\omega u \text{ for } i \in \mathbb{N}. \end{aligned}$$

We will see later that $\mathbb{Z}^d$ equipped with elementwise addition and multiplication $\odot_\omega$ builds a commutative ring. Let us first recall an important property of a companion matrix – it is a root of its defining polynomial.

Lemma 1.4. *Let ω be an algebraic integer with a minimal polynomial p and let S be its companion matrix. Then*

$$p(S) = 0.$$

Now we can prove that there is a correspondence between elements of $\mathbb{Z}[\omega]$ and $\mathbb{Z}^d$.

Theorem 1.5. *Let ω be an algebraic integer of degree d. Then*

$$\mathbb{Z}[\omega] = \left\{ \sum_{i=0}^{d-1} a_i \omega^i \colon a_i \in \mathbb{Z} \right\},$$

$(\mathbb{Z}^d, +, \odot_\omega)$ is a commutative ring and the mapping $\pi : \mathbb{Z}[\omega] \to \mathbb{Z}^d$ defined by

$$\pi(u) = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{pmatrix} \quad \textit{for every } u = \sum_{i=0}^{d-1} u_i \omega^i \in \mathbb{Z}[\omega]$$

is a ring isomorphism.

Due to this theorem we may work with integer vectors instead of elements of $\mathbb{Z}[\omega]$ and multiplication in $\mathbb{Z}[\omega]$ is replaced by multiplying by an appropriate matrix.

The last theorem of this section is a practical tool for divisibility in $\mathbb{Z}[\omega]$. To check whether an element of $\mathbb{Z}[\omega]$ is divisible by another element, we look for an integer solution of a linear system. Moreover, this solution provides the result of division in the positive case.

Theorem 1.6. *Let ω be an algebraic integer of degree d and let S be the companion matrix of its minimal polynomial. Let $\beta = \sum_{i=0}^{d-1} b_i \omega^i$ be a nonzero element of $\mathbb{Z}[\omega]$. Then for every $u \in \mathbb{Z}[\omega]$*

$$u \in \beta\mathbb{Z}[\omega] \iff S_\beta^{-1} \cdot \pi(u) \in \mathbb{Z}^d\,,$$

where $S_\beta = \sum_{i=0}^{d-1} b_i S^i$.

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1.4 MRIZKOVA NORMA

Lemma 1.7. *Let ν be a norm of the vector space $\mathbb{C}^d$ and P be a nonsingular matrix in $\mathbb{C}^d$. Then the mapping $\mu : \mathbb{C}^d \to \mathbb{R}_0^+$ defined by $\mu(x) = \nu(Px)$ is also a norm of the vector space $\mathbb{C}^d$.*

Proof. Let x and y be vectors in $\mathbb{C}^d$ and $\alpha \in \mathbb{C}$. We use linearity of matrix multiplication, nonsingularity of matrix P and the fact that ν is a norm to prove the following statements:

1. $\mu(x) = \nu(Px) \geq 0\,,$
2. $\mu(x) = 0 \iff \nu(Px) = 0 \iff Px = 0 \iff x = 0\,,$
3. $\mu(\alpha x) = \nu(P(\alpha x)) = \nu(\alpha Px) = |\alpha|\nu(Px) = |\alpha|\mu(x)\,,$
4. $\mu(x+y) = \nu(P(x+y)) = \nu(Px+Py) \leq \nu(Px) + \nu(Py) = \mu(x) + \mu(y)\,.$

This verifies that μ is a norm. □

Lemma 1.1 enables us to define a new norm.

Definition 1.9. Let $M \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix and $P \in \mathbb{C}^{n\times n}$ be a nonsingular matrix which diagonalizes M, i.e., $M = P^{-1}DP$ for some diagonal matrix $D \in \mathbb{C}^{n\times n}$. Then we define a vector norm $\|\cdot\|_M$ by

$$\|x\|_M := \|Px\|_2 \tag{1.1}$$

for all $x \in \mathbb{C}^n$, where $\|\cdot\|_2$ is Euclidean norm. A matrix norm $|||\cdot|||_M$ is induced by the norm $\|\cdot\|_M$.

Theorem 1.8. *Let $M \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix. Then*

$$\rho(M) = |||M|||_M\,,$$

where $\rho(M)$ is the spectral radius of the matrix M.

Proof. First, we prove that $|||M||| \geq \rho(M)$ for every natural matrix norm induced by $\|\cdot\|$. For all eigenvalues λ in the spectrum $\sigma(M)$ with a respective eigenvector u such that $\|u\| = 1$, we have

$$|||M||| = \max_{\|x\|=1} \|Mx\| \geq \|Mu\| = \|\lambda u\| = |\lambda| \cdot \|u\| = |\lambda| \,.$$

Now, we construct the natural matrix norm $\|\cdot\|_M$ such that $|||M|||_M \leq \rho(M)$. Since M is diagonalizable, there exist nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and diagonal matrix $C \in \mathbb{C}^{n\times n}$ with the eigenvalues of M on the diagonal such that

$$PMP^{-1} = C \,.$$

Now, the natural matrix norm $\|\cdot\|_M$ is induced by the vector norm $\|\cdot\|_M$, i.e.,

$$|||M|||_M = \max_{\|y\|_M=1} \|My\|_M \,.$$

Let y be a vector such that $\|y\|_M = 1$ and set $z = Py$. Notice that

$$\sqrt{z^*z} = \|z\|_2 = \|Py\|_2 = \|y\|_M = 1 \,.$$

Consider

$$\begin{aligned} \|My\|_M = \|PMy\|_2 = \|CPy\|_2 = \|Cz\|_2 = \sqrt{z^*C^*Cz} \\ \leq \sqrt{\max_{\lambda\in\sigma(M)} |\lambda|^2 z^*z} = \max_{\lambda\in\sigma(M)} |\lambda| = \rho(M) \,. \end{aligned}$$

which implies the statement. □

Lemma 1.9. *Let ω be an algebraic integer of degree d and let S be the companion matrix of its minimal polynomial. Let $\beta = \sum_{i=0}^{d-1} b_i\omega^i$ be a nonzero element of $\mathbb{Z}[\omega]$. Set $S_\beta = \sum_{i=0}^{d-1} b_i S^i$. Then*

i) The matrix S_β is diagonalizable.

ii) The characteristic polynomial of S_β is m_β^k with $k = d/\deg\beta$.

iii) $|\det S_\beta| = |m_\beta(0)|^k$.

iv) $\|x\|_{S_\beta} = \|x\|_{S_\beta^{-1}}$ *for all* $x \in \mathbb{C}^d$ *and* $|||X|||_{S_\beta} = |||X|||_{S_\beta^{-1}}$ *for all* $X \in \mathbb{C}^{d\times d}$.

v) $|||S_\beta|||_{S_\beta} = \max\{|\beta'| \colon \beta' \text{ is conjugate of } \beta\}$ *and* $\left|\left|\left|S_\beta^{-1}\right|\right|\right|_{S_\beta} = \max\{\frac{1}{|\beta'|} \colon \beta' \text{ is conjugate of } \beta\}$.

Proof. The characteristic polynomial of the companion matrix S is the same as minimal polynomial of ω which has no multiple roots. JE TO ZREJME??
Hence, S is diagonalizable, i.e., $S = P^{-1}DP$ where D is diagonal matrix with the conjugates of ω on the diagonal and P is a nonsingular complex matrix. The matrix S_β is also diagonalized by P:

$$S_\beta = \sum_{i=0}^{d-1} b_i S^i = \sum_{i=0}^{d-1} b_i \left(P^{-1}DP\right)^i = P^{-1} \underbrace{\left(\sum_{i=0}^{d-1} b_i D^i\right)}_{D_\beta} P \,.$$

By Theorem CONJUGATES SE ZOBRAZUJI NA CONJUGATES, the diagonal elements of the diagonal matrix D_β are conjugates of β. Since $S_\beta \in \mathbb{Z}^{d\times d}$, its characteristic polynomial has integer coefficients. Thus it is k-th power of the minimal polynomial m_β. The value k follows from the equality $d = \deg(m_\beta^k) = k \deg m_\beta$.

The modulus of the determinant of S_β equals the modulus of the absolute coefficient of the characteristic polynomial which is $|m_\beta(0)|^k$.

The matrix S_β^{-1} is also diagonalized by P since $S_\beta^{-1} = (P^{-1}D_\beta P)^{-1} = P^{-1}D_\beta^{-1}P$. Thus, the norms $\|\cdot\|_{S_\beta}$ and $\|\cdot\|_{S_\beta^{-1}}$ are same and so the induced matrix norms $|||\cdot|||_{S_\beta}$ and $|||\cdot|||_{S_\beta^{-1}}$ are.

The matrix S_β is diagonalizable and its eigenvalues are the conjugates of β. Theorem 1.2 implies that

$$|||S_\beta|||_{S_\beta} = \rho(S_\beta) = \max\{|\beta'| \colon \beta' \text{ is conjugate of } \beta\}\,.$$

For the second part of the last statement, we use the part *iv)*, Theorem 1.2 and the fact that the eigenvalues of S_β^{-1} are reciprocal for the conjugates of β. □

Definition 1.10. Using the notation of the previous lemma, we define a *β-norm* $\|\cdot\|_\beta : \mathbb{Z}[\omega] \to \mathbb{R}_0^+$ by

$$\|x\|_\beta = \|\pi(x)\|_{S_\beta}$$

for all $x \in \mathbb{Z}[\omega]$.BUDEME TOMU RIKAT TAKHLE???

We can easily verify that β-norm is a norm in $\mathbb{Z}[\omega]$:

1. $\|x\|_\beta = \|\pi(x)\|_{S_\beta} \geq 0\,,$
2. $\|x\|_\beta = 0 \iff \|\pi(x)\|_{S_\beta} = 0 \iff \pi(x) = 0 \iff x = 0\,,$
3. $\|\alpha x\|_\beta = \|\pi(\alpha x)\|_{S_\beta} = |\alpha|\,\|\pi(x)\|_{S_\beta} = |\alpha|\,\|x\|_\beta\,,$
4. $\|x+y\|_\beta = \|\pi(x+y)\|_{S_\beta} = \|\pi(x)+\pi(y)\|_{S_\beta} \leq \|\pi(x)\|_{S_\beta} + \|\pi(y)\|_{S_\beta} = \|x\|_\beta + \|y\|_\beta\,,$

for all $x, y \in \mathbb{Z}[\omega]$ and $\alpha \in \mathbb{Z}[\omega]$.

1.5 Number of congruence classes

Definition 1.11. Let $M \in \mathbb{Z}^{d\times d}$ be a nonsingular integer matrix. Vectors $x, y \in \mathbb{Z}^d$ are *congruent modulo M in $\mathbb{Z}^d$*, if $x - y \in M\mathbb{Z}^d$.

Lemma 1.10. *Let $M \in \mathbb{Z}^{d\times d}$ be a nonsingular integer matrix. The number of congruence classes modulo M in $\mathbb{Z}^d$ is $|\det M|$.*

Proof. Set $y_i := Me_i$ for $i \in \{0, \ldots, d-1\}$ and

$$B_{(\alpha_0,\ldots,\alpha_{d-1})} := \left\{ \sum_{i=0}^{d-1} (\alpha_i + t_i) y_i \colon t_i \in [0,1) \right\}$$

for $(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{Z}^d$. Obviously,

$$\mathbb{R}^d = \bigcup_{(\alpha_0,\ldots,\alpha_{d-1})\in\mathbb{Z}^d} B_{(\alpha_0,\ldots,\alpha_{d-1})}\,.$$

For fixed $(\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{Z}^d$, the number of points of $\mathbb{Z}^d$ in $B_{(\alpha_0,\ldots,\alpha_{d-1})}$ is volume of $B_{(\alpha_0,\ldots,\alpha_{d-1})}$ which is $|\det M|$. Hence, it is enough to prove that there is exactly one representative of each congruence class in $B_{(\alpha_0,\ldots,\alpha_{d-1})}$.

To show that there are representatives of all classes, assume $x \in \mathbb{Z}^d$. Since $(y_0, \ldots, y_{d-1})$ is a basis of $\mathbb{R}^d$, there are scalars $s_0, \ldots, s_{d-1} \in \mathbb{R}$ such that $x = \sum_{i=0}^{d-1} s_i y_i$. Set $\gamma_i := \lfloor s_i \rfloor$ and $t_i := s_i - \gamma_i$. Now

$$x = \sum_{i=0}^{d-1} (\gamma_i + t_i) y_i = \sum_{i=0}^{d-1} t_i y_i + \sum_{i=0}^{d-1} (\gamma_i - \alpha_i) y_i + \sum_{i=0}^{d-1} \alpha_i y_i = \underbrace{\sum_{i=0}^{d-1} (\alpha_i + t_i) y_i}_{\in B_{(\alpha_0,\ldots,\alpha_{d-1})}} + M \underbrace{(\gamma - \alpha)}_{\in \mathbb{Z}^d},$$

where $\alpha = (\alpha_0, \ldots, \alpha_{d-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{d-1})^T$.

Let $x = \sum_{i=0}^{d-1} s_i y_i \in \mathbb{Z}^d$ and $x' = \sum_{i=0}^{d-1} s'_i y_i \in \mathbb{Z}^d$ be distinct elements of $B_{(\alpha_0,\ldots,\alpha_{d-1})}$ which are congruent modulo M, i.e., there exists $z = (z_0, \ldots, z_{d-1})^T \in \mathbb{Z}^d$ such that $x = x' + Mz$. There is $i_0 \in \{0, \ldots, d-1\}$ such that $|z_{i_0}| \geq 1$ as $x \neq x'$. Thus, $|s_{i_0} - s'_{i_0}| = |z_{i_0}| \geq 1$ which contradicts that $x, x' \in B_{(\alpha_0,\ldots,\alpha_{d-1})}$. □

Theorem 1.11. *Let ω be an algebraic integer of degree d and β be an element of $\mathbb{Z}[\omega]$ such that $\deg \omega = \deg \beta$. The number of congruence classes modulo β in $\mathbb{Z}[\omega]$ is $|m_\beta(0)|$.*

Proof. Let $x, y \in \mathbb{Z}[\omega]$ and let S be the companion matrix of its minimal polynomial. Set $S_\beta = \sum_{i=0}^{d-1} b_i S^i$. Then

$$\begin{aligned} x \equiv y \mod \beta &\exists z \in \mathbb{Z}[\omega] \colon x - y = \beta z \\ &\iff \exists z \in \mathbb{Z}[\omega] \colon \pi(x - y) = S_\beta \pi(z) \\ &\iff \pi(x) \equiv \pi(y) \mod S_\beta \,. \end{aligned}$$

Thus, the number or congruence classes modulo β is $|\det S_\beta|$ by Lemma 1.4. The statement follows from Lemma 1.3. □

Chapter 2

Design of extending window method

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We recall the general concept of addition at the beginning of this chapter and then we describe a so-called *extending window method* which is due to M. Svobodová [5].

From now on, let ω be an algebraic integer and $(\beta, \mathcal{A})$ be a numeration system such that the base $\beta \in \mathbb{Z}[\omega]$ and the alphabet $\mathcal{A} \ni 0$ is a finite subset of $\mathbb{Z}[\omega]$.

The general concept of addition (standard or parallel) in any numeration system $(\beta, \mathcal{A})$, such that $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition, is the following: we add numbers digitwise and then we convert the result into the alphabet $\mathcal{A}$. Obviously, digitwise addition is computable in parallel, thus the crucial point is the digit set conversion of the obtained result. It can be easily done in a standard way but a parallel digit set conversion is nontrivial. However, formulas are basically the same but the choice of coefficients differs.

Now we go step by step more precisely. Let $x = \sum_{-m'}^{n'} x_i\beta^i, y = \sum_{-m'}^{n'} y_i\beta^i \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ with $(\beta, \mathcal{A})$-representations padded by zeros to have the same length. We set

$$\begin{aligned} w = x + y &= \sum_{-m'}^{n'} x_i\beta^i + \sum_{-m'}^{n'} y_i\beta^i = \sum_{-m'}^{n'} (x_i + y_i)\beta^i \\ &= \sum_{-m'}^{n'} w_i\beta^i\,, \end{aligned}$$

where $w_j = x_j + y_j \in \mathcal{A} + \mathcal{A}$. Thus, $w_{n'}w_{n'-1}\cdots w_1w_0 \bullet w_{-1}w_{-2}\cdots w_{-m'}$ is a $(\beta, \mathcal{A}+\mathcal{A})$-representation of $w \in \mathrm{Fin}_{\mathcal{A}+\mathcal{A}}(\beta)$.

We also use column notation of addition in what follows, e.g.,

$$\begin{array}{l} x_{n'}\ x_{n'-1}\cdots x_1\ x_0 \bullet x_{-1}\ x_{-2}\ \cdots x_{-m'} \\ y_{n'}\ y_{n'-1}\cdots y_1\ \ y_0 \bullet y_{-1}\ y_{-2}\ \cdots y_{-m'} \\ \hline w_{n'}w_{n'-1}\cdots w_1w_0 \bullet w_{-1}w_{-2}\cdots w_{-m'}\,. \end{array}$$

As we want to obtain a $(\beta, \mathcal{A})$-representation of w, we search for a sequence

$$z_nz_{n-1}\cdots z_1z_0z_{-1}z_{-2}\cdots z_{-m}$$

such that $z_j \in \mathcal{A}$ and

$$z_n z_{n-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots z_{-m} = (w)_{\beta,\mathcal{A}} \,.$$

From now on, we consider without lost of generality only β-integers since modification for representations with rational part is obvious:

$$\beta^m \cdot z_n z_{n-1} \cdots z_1 z_0 \bullet z_{-1} z_{-2} \cdots z_{-m} = z_n z_{n-1} \cdots z_1 z_0 z_{-1} z_{-2} \cdots z_{-m} \bullet$$

Particularly, let $(w)_{\beta,\mathcal{A}+\mathcal{A}} = w_{n'} w_{n'-1} \cdots w_1 w_0 \bullet$. We search for a number $n \in \mathbb{N}$ and $z_n, z_{n-1}, \ldots, z_1, z_0 \in \mathcal{A}$ such that $(w)_{\beta,\mathcal{A}} = z_n z_{n-1} \cdots z_1 z_0 \bullet$.

We use a suitable representation of zero to convert digits w_j into the alphabet $\mathcal{A}$. For our purpose, we use the simplest possible representation deduced from the polynomial

$$x - \beta \in (\mathbb{Z}[\omega])\,[x] \,.$$

We remark that any polynomial $R(x) = r_s x^s + r_{s-1} x^{s-1} + \cdots + r_1 x + r_0$ with coefficients $r_i \in \mathbb{Z}[\omega]$, such that $R(\beta) = 0$ gives us a possible representation of zero. The polynomial R is called a *rewriting rule*.

Within a digit set conversion with an arbitrary rewriting rule R, one of the coefficients of R which is greatest in modulus (so-called *core coefficient*) is used for the conversion of a digit w_j. But using of an arbitrary rewriting rule R is out of scope of this thesis, so we focus on the simplest possible rewriting rule $R(x) = x - \beta$.

As $0 = \beta^j \cdot R(\beta) = 1 \cdot \beta^{j+1} - \beta \cdot \beta^j$, we have a representation of zero

$$1(-\beta) \underbrace{0 \cdots 0}_{j} \bullet = (0)_\beta \,.$$

for all $j \in \mathbb{N}$. We multiply this representation by $q_j \in \mathbb{Z}[\omega]$, which is called a *weight coefficient*, to obtain another representation of zero

$$q_j(-q_j\beta) \underbrace{0 \cdots 0}_{j} \bullet = (0)_\beta \,.$$

This is digitwise added to $w_n w_{n-1} \cdots w_1 w_0 \bullet$ to convert the digit w_j into the alphabet $\mathcal{A}$. The conversion of j-th digit causes a *carry* q_j on the $(j+1)$-th position. The digit set conversion runs from the right ($j = 0$) to the left until all digits and carries are converted into the alphabet $\mathcal{A}$:

$$\begin{array}{ccccccccc}
w_n w_{n-1} & \cdots & w_{j+1} & w_j & w_{j-1} & \cdots & w_1 w_0 \bullet \\
 & & & & q_{j-2} & \iddots & \\
 & & & q_{j-1} & -\beta q_{j-1} & & \\
 & & q_j & -\beta q_j & & & \\
 & \iddots & -\beta q_{j+1} & & & & \\
\hline
z_{n+s} \cdots z_n z_{n-1} & \cdots & z_{j+1} & z_j & z_{j-1} & \cdots & z_1\ z_0 \bullet
\end{array} \tag{2.1}$$

Hence, the desired formula for conversion on the j-th position is

$$z_j = w_j + q_{j-1} - q_j \beta$$

for $j \in \mathbb{N}$. We set $q_{-1} = 0$ as there is no carry from the right on the 0-th position.

Clearly, the value of w is preserved:

$$\begin{aligned}\sum_{j\geq 0} z_j\beta^j &= w_0 - \beta q_0 + \sum_{j>0}(w_j + q_{j-1} - q_j\beta)\beta^j \\ &= \sum_{j\geq 0} w_j\beta^j + \sum_{j>0} q_{j-1}\beta^j - \sum_{j\geq 0} q_j \cdot \beta^{j+1} \\ &= \sum_{j\geq 0} w_j\beta^j + \sum_{j>0} q_{j-1}\beta^j - \sum_{j>0} q_{j-1} \cdot \beta^j \\ &= \sum_{j\geq 0} w_j\beta^j = w\,.\end{aligned} \tag{2.2}$$

The weight coefficient q_j must be chosen so that the converted digit is in the alphabet $\mathcal{A}$, i.e.,

$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}\,. \tag{2.3}$$

The choice of weight coefficients is a crucial part of construction of addition algorithms which are computable in parallel. The extending window method determining weight coefficients for a given input is described in Section **??**.

On the other hand, the following example shows that determining weight coefficients is trivial for standard numeration systems.

Example 2.1. Assume now a standard numeration system $(\beta, \mathcal{A})$, where

$$\beta \in \mathbb{N}\,, \beta \geq 2\,, \mathcal{A} = \{0, 1, 2, \ldots, \beta - 1\}\,.$$

Notice that

$$z_j \equiv w_j + q_{j-1} \mod \beta\,.$$

There is only one representative of each class modulo β in the standard numeration system $(\beta, \mathcal{A})$. Therefore, the digit z_j is uniquely determined for a given digit $w_j \in \mathcal{A} + \mathcal{A}$ and carry q_{j-1} and thus so is the weight coefficient q_j. This means that $q_j = q_j(w_j, q_{j-1})$ for all $j \geq 0$. Generally,

$$q_j = q_j(w_j, q_{j-1}(w_{j-1}, q_{j-2})) = \cdots = q_j(w_j, \ldots, w_1, w_0)$$

and

$$z_j = z_j(w_j, \ldots, w_1, w_0)\,,$$

which implies that addition runs in linear time.

We require that the digit set conversion from $\mathcal{A} + \mathcal{A}$ into $\mathcal{A}$ is computable in parallel, i.e., there exist constants $r, t \in \mathbb{N}_0$ such that for all $j \geq 0$ is $z_j = z_j(w_{j+r}, \ldots, w_{j-t})$. To avoid the dependency on all less, respectively more, significant digits, we need variety in the choice of weight coefficient q_j. This implies that the used numeration system must be redundant.

2.1 Extending window method

In order to construct a digit set conversion in numeration system $(\beta, \mathcal{A})$ which is computable in parallel, we consider a more general case of digit set conversion from an *input alphabet* $\mathcal{B}$

such that $\mathcal{A} \subsetneq \mathcal{B} \subset \mathcal{A} + \mathcal{A}$ instead of the alphabet $\mathcal{A} + \mathcal{A}$. As mentioned above, the key problem is to find for every $j \geq 0$ a weight coefficient q_j such that

$$z_j = \underbrace{w_j}_{\in \mathcal{B}} + q_{j-1} - q_j \beta \in \mathcal{A}$$

for any input $w_{n'} w_{n'-1} \dots w_1 w_0 \bullet = (w)_{\beta,\mathcal{B}}, w \in \mathrm{Fin}_{\mathcal{B}}(\beta)$. We remark that the weight coefficient q_{j-1} is determined by the input w. For a digit set conversion to be computable in parallel the digit z_j is required to satisfy $z_j = z_j(w_{j+r}, \dots, w_{j-t})$ for a fixed anticipation r and memory t in $\mathbb{N}$.

We introduce following definitions.

Definition 2.1. Let $\mathcal{B}$ be a set such that $\mathcal{A} \subsetneq \mathcal{B} \subset \mathcal{A} + \mathcal{A}$. Then any finite set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ containing 0 such that

$$\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}$$

is called a *weight coefficients set*.

We see that if $\mathcal{Q}$ is a weight coefficients set, then

$$(\forall w_j \in \mathcal{B})(\forall q_{j-1} \in \mathcal{Q})(\exists q_j \in \mathcal{Q})(w_j + q_{j-1} - q_j\beta \in \mathcal{A})\,.$$

In other words, there is a weight coefficient $q_j \in \mathcal{Q}$ for a carry from the right $q_{j-1} \in \mathcal{Q}$ and digit w_j in the input alphabet $\mathcal{B}$. I.e., we satisfy the basic digit set conversion formula (**??**). Notice that $q_{-1} = 0$ is in $\mathcal{Q}$ by the definition. Thus, all weight coefficients may be chosen from $\mathcal{Q}$.

Definition 2.2. Let M be an integer and $q : \mathcal{B}^M \to \mathcal{Q}$ be a mapping such that

$$w_j + q(w_{j-1}, \dots, w_{j-M}) - \beta q(w_j, \dots, w_{j-M+1}) \in \mathcal{A}$$

for all $w_j, w_{j-1}, \dots, w_{j-M} \in \mathcal{B}$, and $q(0, 0, \dots, 0) = 0$. Then q is called a *weight function* and M is called a *length of window*.

Having a weight function q, we define a function $\phi : \mathcal{B}^{M+1} \to \mathcal{A}$ by

$$\phi(w_j, \dots, w_{j-M}) = w_j + \underbrace{q(w_{j-1}, \dots, w_{j-M})}_{=q_{j-1}} - \beta \underbrace{q(w_j, \dots, w_{j-M+1})}_{=q_j} =: z_j\,, \tag{2.4}$$

which verifies that the digit set conversion is indeed a $(M+1)$-local function with anticipation $r = 0$ and memory $t = M$. The requirement of zero output of the weight function q for the input of M zeros guarantees that $\phi(0, 0, \dots, 0) = 0$. Thus the first condition of Definition **??** is satisfied. The second one follows from the equation (**??**).

Let us summarize the construction of the digit set conversion by the rewriting rule $x - \beta$. We need to find weight coefficients for all possible combinations of digits of the input alphabet $\mathcal{B}$. Their multiples of the rewriting rules are digitwise added to the input sequence. In fact, it means that the equation (**??**) is applied on each position. If the digit set conversion is computable in parallel, the weight coefficients are determined as the outputs of the weight function q with some fixed length of window M.

We search for the weight function q for a given base β and input alphabet $\mathcal{B}$ by the extending window method. It consists of two phases. First, we find a minimal possible weight

coefficients set $\mathcal{Q}$. We know that it is possible to convert an input sequence by choosing the weight coefficients from the set $\mathcal{Q}$. The set $\mathcal{Q}$ serves as the starting point for the second phase in which we increment the expected length of the window M until the weight function q is uniquely defined for each $(w_j, w_{j-1}, \ldots, w_{j-M+1}) \in \mathcal{B}^M$. Then, the local conversion is determined – we use the weight function outputs as weight coefficients in the formula (??).

Note that the convergence of both phases is discussed separately in Chapter ??.

2.2 Phase 1 – Weight coefficients set

The goal of the first phase is to compute a weight coefficients set $\mathcal{Q}$, i.e., to find a set $\mathcal{Q} \ni 0$ such that

$$\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}\,.$$

We build the sequence $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ iteratively so that we extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a way to cover all elements of the set $\mathcal{B} + \mathcal{Q}_k$ by elements of the extended set $\mathcal{Q}_{k+1}$, i.e.,

$$\mathcal{B} + \mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}\,.$$

This procedure is repeated until the extended weight coefficients set $\mathcal{Q}_{k+1}$ is the same as the original set $\mathcal{Q}_k$. We remark that the expression "a weight coefficient q covers an element x" means that there is $a \in \mathcal{A}$ such that $x = a + \beta q$.

In other words, we start with $\mathcal{Q}_0 = \{0\}$ meaning that we search all weight coefficients q_j necessary for digit set conversion for the case where there is no carry from the right, i.e., $q_{j-1} = 0$. We add them to the weight coefficients set $\mathcal{Q}_0$ to obtain the set $\mathcal{Q}_1$. Assume now that we have the set $\mathcal{Q}_k$ for some $k \geq 1$. The weight coefficients in $\mathcal{Q}_k$ now may appear as a carry q_{j-1}. If there are no suitable weight coefficients q_j in the weight coefficients set $\mathcal{Q}_k$ to cover all sums of added coefficients and digits of the input alphabet $\mathcal{B}$, we extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ by suitable coefficients using Algorithm ??. And so on until there is no need to add more elements, i.e., the extended set $\mathcal{Q}_{k+1}$ equals $\mathcal{Q}_k$. Then the weight coefficients set $\mathcal{Q} := \mathcal{Q}_{k+1}$ satisfies Definition ??.

The precise description of the algorithm in a pseudocode is in Algorithm ??. For better understanding, see Figures ??–?? in Appendix ?? which illustrate the construction of the weight coefficients set $\mathcal{Q}$ for the Eisenstein base and a complex alphabet (see Example ?? for its description).

Section ?? discusses the convergence of Phase 1, i.e. whether it happens that $\mathcal{Q}_{k+1} = \mathcal{Q}_k$ for some k.

There may be more possible weight coefficients which cover some element of the set $\mathcal{B}+\mathcal{Q}_k$. Let us suppose that we have the list which contains the lists of these candidates for each element of the set $\mathcal{B}+\mathcal{Q}_k$. This list of lists is saved in the variable `candidates` in Algorithm ??. Now, for each element, we check the list of candidates which cover this element and if there is none of them contained in the set $\mathcal{Q}_k$, the smallest (in absolute value) weight coefficient from the list of candidates is added to the set $\mathcal{Q}_k$. If there are more elements with the same absolute value, we deterministically choose one of them. The extension $\mathcal{Q}_{k+1}$ of the set $\mathcal{Q}_k$ is obtained in this manner.

We may slightly improve this procedure: for example we may first extend $\mathcal{Q}_k$ by all single-element lists of `candidates`. These elements may be enough to cover also other elements of $\mathcal{B}+\mathcal{Q}_k$. It implies that the resulting $\mathcal{Q}$ is dependent on the way of selection from `candidates`.

Algorithm 1 Search for weight coefficients set (Phase 1)

1: $k := 0$
2: $Q_0 := \{0\}$
3: **repeat**
4: By Algorithm **??**, extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a minimal possible way so that

$$\mathcal{B} + \mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}$$

5: $k := k + 1$
6: **until** $\mathcal{Q}_k = \mathcal{Q}_{k+1}$
7: $\mathcal{Q} := \mathcal{Q}_k$
8: **return** $\mathcal{Q}$

Algorithm 2 Extending intermediate weight coefficients set

Input: `candidates` from Algorithm **??**, previous weight coefficients set $\mathcal{Q}_k$
1: $\mathcal{Q}_{k+1} := \mathcal{Q}_k$
2: **for all** `cand_for_x` in `candidates` **do**
3: **if** no element of `cand_for_x` in $\mathcal{Q}_k$ **then**
4: Add the smallest element (in absolute value) of `cand_for_x` to $\mathcal{Q}_{k+1}$
5: **end if**
6: **end for**
7: **return** $\mathcal{Q}_{k+1}$

Algorithm **??** describes the search for the list of lists of candidates. For each element $x \in \mathcal{B} + \mathcal{Q}_k$ we build the list of candidates (in the variable `cand_for_x`) so that we test the divisibility of $x - a$ by the base β for all letters $a \in \mathcal{A}$. In the positive case, the result of division is appended to `cand_for_x` as a possible weight coefficient. We remark that Theorem **??** is used to check the divisibility.

Algorithm 3 Search for candidates

Input: the previous weight coefficients set $\mathcal{Q}_k$, alternatively also the set $\mathcal{Q}_{k-1}$
1: `candidates`:= empty list of lists
2: **for all** $x \in \mathcal{B} + \mathcal{Q}_k$ **do** {Alternatively, $x \in (\mathcal{B} + \mathcal{Q}_k) \setminus (\mathcal{B} + \mathcal{Q}_{k-1})$}
3: `cand_for_x`:= empty list
4: **for all** $a \in \mathcal{A}$ **do**
5: **if** $(x - a)$ is divisible by β in $\mathbb{Z}[\omega]$ (using Theorem **??**) **then**
6: Append $\frac{x-a}{\beta}$ to `cand_for_x`
7: **end if**
8: **end for**
9: Append `cand_for_x` to `candidates`
10: **end for**
11: **return** `candidates`

We can improve the performance of Algorithm **??** by substituting the set $\mathcal{B} + \mathcal{Q}_k$ by $(\mathcal{B} + \mathcal{Q}_k) \setminus (\mathcal{B} + \mathcal{Q}_{k-1})$ on the line **??** because

$$\mathcal{B} + \mathcal{Q}_{k-1} \subset \mathcal{A} + \beta\mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}$$

for any $\mathcal{Q}_{k+1} \supset \mathcal{Q}_k$. Thus there is no need to check whether the elements of $\mathcal{B} + \mathcal{Q}_{k-1}$ are covered by some weight coefficient from $\mathcal{Q}_k$ in Algorithm **??**.

2.3 Phase 2 – Weight function

We want to find a length of the window M and a weight function $q : \mathcal{B}^M \to \mathcal{Q}$. We start with the weight coefficients set $\mathcal{Q}$ obtained in Phase 1. The idea is to reduce necessary weight coefficients for the conversion of a given digit up to single value. This is done by enlarging the number of considered input digits (extending the length of window) – there might be less possible carries from the right if we know which digits on the right are converted.

We introduce the following notation. Let $\mathcal{Q}$ be a weight coefficients set and $w_j \in \mathcal{B}$. Denote by $\mathcal{Q}_{[w_j]}$ any set such that

$$(\forall q_{j-1} \in \mathcal{Q})(\exists q_j \in \mathcal{Q}_{[w_j]})(w_j + q_{j-1} - q_j\beta \in \mathcal{A})\,.$$

By induction with respect to $m \in \mathbb{N}, m > 1$, for all $(w_j, \dots, w_{j-m+1}) \in \mathcal{B}^m$ denote by $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ any subset of $\mathcal{Q}_{[w_j,\dots,w_{j-m+2}]}$ such that

$$(\forall q_{j-1} \in \mathcal{Q}_{[w_{j-1},\dots,w_{j-m+1}]})(\exists q_j \in \mathcal{Q}_{[w_j,\dots,w_{j-m+1}]})(w_j + q_{j-1} - q_j\beta \in \mathcal{A})\,.$$

Recall the scheme (**??**) of the digit set conversion for better understanding of the notation and method:

$$\begin{array}{cccc}
\cdots w_{j+1} & w_j & w_{j-1} & \cdots w_{j-M+1} w_{j-M} \cdots \\
 & & q_{j-2} & \\
 & q_{j-1} & -\beta q_{j-1} & \\
q_j & -\beta q_j & & \\
-\beta q_{j+1} & & & \\
\hline
\cdots z_{j+1} & z_j & z_{j-1} & \cdots z_{j-M+1}\ z_{j-M} \cdots
\end{array}$$

The idea is to check all possible right carries $q_{j-1} \in \mathcal{Q}$ and determine values $q_j \in \mathcal{Q}$ such that

$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}\,.$$

So we obtain a set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ of weight coefficients which are necessary to convert the digit w_j with any carry $q_{j-1} \in \mathcal{Q}$. Assuming that we know the input digit w_{j-1}, the set of possible carries from the right is also reduced to $\mathcal{Q}_{[w_{j-1}]}$. Thus we may reduce the set $\mathcal{Q}_{[w_j]}$ to a set $\mathcal{Q}_{[w_j,w_{j-1}]} \subset \mathcal{Q}_{[w_j]}$ which is necessary to cover all elements of $w_j + \mathcal{Q}_{[w_{j-1}]}$. Prolonging the length of window in this manner may lead to a unique weight coefficient q_j for enough given input digits.

Accordingly, the weight function q is found if there is $M \in \mathbb{N}$ such that

$$\#\mathcal{Q}_{[w_j,\dots,w_{j-M+1}]} = 1$$

for all $w_j, \dots, w_{j-M+1} \in \mathcal{B}^M$. The precise description of the construction of the weight function is in Algorithm **??**. Figures **??**–**??** in Appendix **??** illustrate the construction of the set $\mathcal{Q}_{[\omega,1,2]}$ for the Eisenstein numeration system.

For construction of the set $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ we first choose such elements of $\mathcal{Q}_{[w_j,\dots,w_{j-m+2}]}$ which are the only possible to cover some value $x \in w_0 + \mathcal{Q}_{[w_{j-1},\dots,w_{j-m+1}]}$. Other elements

Algorithm 4 Search for weight function (Phase 2)

Input: weight coefficients set $\mathcal{Q}$

$m := 1$
for all $w_j \in \mathcal{B}$ **do**
    By Algorithm **??**, find set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ such that

$$w_j + \mathcal{Q} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_j]}$$

end for
while $\max\{\#\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} : (w_j, \ldots, w_{j-m+1}) \in \mathcal{B}^m\} > 1$ **do**
    $m := m + 1$
    for all $(w_j, \ldots, w_{j-m+1}) \in \mathcal{B}^m$ **do**
        By Algorithm **??**, find set $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]} \subset \mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ such that

$$w_j + \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]} \subset \mathcal{A} + \beta \mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}\,,$$

    end for
end while
$M := m$
for all $(w_j, \ldots, w_{j-M+1}) \in \mathcal{B}^M$ **do**
    $q(w_j, \ldots, w_{j-M+1}) :=$ only element of $\mathcal{Q}_{[w_j,\ldots,w_{j-M+1}]}$
end for
return q

Algorithm 5 Search for set $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$

Input: Input digit w_j, set of possible carries $\mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]}$, previous set of possible weight coefficients $\mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$

`list_of_coverings`:=empty list of lists
for all $x \in w_j + \mathcal{Q}_{[w_{j-1},\ldots,w_{j-m+1}]}$ **do**
    Build a list `x_covered_by` of weight coefficients $q_j \in \mathcal{Q}_{[w_j,\ldots,w_{j-m+2}]}$ such that

$$x = a + \beta q_j \qquad \text{for some } a \in \mathcal{A}\,.$$

    Append `x_covered_by` to `list_of_coverings`
end for
$\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$:= empty set
while `list_of_coverings` is nonempty **do**
    Pick an element q of one of the shortest lists of `list_of_coverings`
    Add the element q to $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$
    Remove lists of `list_of_coverings` containing the element q from `list_of_coverings`
end while
return $\mathcal{Q}_{[w_j,\ldots,w_{j-m+1}]}$

from $\mathcal{Q}_{[w_j,\dots,w_{j-m+2}]}$ which cover an uncovered value are added one by one to $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ until each x equals $a+\beta q_j$ for some q_j in $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ and $a \in \mathcal{A}$. The pseudocode is in Algorithm **??**.

Notice that the result of Algorithm **??** is influenced by the way how we pick an element on line **??**. It can be done deterministically or non-deterministically. We use the following deterministic choice – suppose that we want to choose from elements $x_1 = \sum_{i=0}^{d-1} x_{1,i}\omega^i, x_2 = \sum_{i=0}^{d-1} x_{2,i}\omega^i, \dots, x_n = \sum_{i=0}^{d-1} x_{n,i}\omega^i \in \mathbb{Z}[\omega]$, where d is the degree of ω. Let $a_0, \dots, a_{d-1} \in \mathbb{Z}$ be such that

$$\sum_{i=0}^{d-1} a_i\omega^i = \sum_{j=1}^{n} x_j\,.$$

Set $c := \sum_{i=0}^{d-1} c_i\omega^i \in \mathbb{Z}[\omega]$ with $c_i = [\frac{a_i}{n}]$ where $[\cdot]$ denotes rounding to the nearest integer. Let the index set $I_0 \subset \{1,\dots,n\}$ be such that

$$|x_{j,0} - c_0| = \min\{|x_{k,0} - c_0| \colon k \in 1,\dots,n\}$$

for all $j \in I_0$. For all $i \in \{1,\dots,d-1\}$, let the index set $I_i \subset I_{i-1}$ be such that

$$|x_{j,i} - c_i| = \min\{|x_{k,i} - c_i| \colon k \in I_{i-1}\}$$

for all $j \in I_i$. If there is only one element in the index set $I_{d-1} = \{j_0\}$, choose the element x_{j_0}. Otherwise choose $j_0 \in I_{d-1}$ such that $x_{j_0,0}$ is the smallest from all $x_{j,0}$ such that $j \in I_{d-1}$. If there are more such elements, then choose from them according to the value $x_{j,1}$ etc.

In other words, we take the elements which are the "closest" ones to the rounded center of gravity c of the values $x_1,\dots,x_n$ where "closest" is measured by absolute value of the first coordinate of $\pi(x_j) - \pi(c)$. In case of equality, according to the second coordinate etc. If there is more than one such element, we choose the element x_{j_0} with the smallest first, resp. second, etc. coordinate of $\pi(x_{j_0})$.

There is space to improve Phase 2 by a modification of Algorithm **??**. It is possible that the effort to reduce the size of $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ as much as possible is not the best for convergence of Phase 2.

Unfortunately, we do not know when Phase 2 converges. But we may reveal the non-convergence of Phase 2 with deterministic Algorithm **??** for some numeration systems by Algorithm **??**, which is described in Section **??**.

Notice that for a given length of window M, the number of calls of Algorithm **??** within Algorithm **??** is

$$\sum_{m=1}^{M} \#\mathcal{B}^m = \#\mathcal{B}\sum_{m=0}^{M-1} \#\mathcal{B}^m = \#\mathcal{B}\frac{\#\mathcal{B}^M - 1}{\#\mathcal{B} - 1}\,.$$

It implies that the time complexity grows exponentially as about $\#\mathcal{B}^M$. The required memory is also exponential because we have to store sets $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ at least for $m \in \{M-1, M\}$ for all $w_j,\dots,w_{j-m+1} \in \mathcal{B}$.

We may reduce the number of the combinations of the input digits so that if for some $(w_j,\dots,w_{j-m+1}) \in \mathcal{B}^m, m < M$ we have $\#\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]} = 1$, we do not extend the window for these digits but we set the ouput of $q(w_j,\dots,w_{j-m+1},w_{j-m},\dots w_{j-M+1})$ to the single element of $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}$ for all $(w_{j-m},\dots w_{j-M+1}) \in \mathcal{B}^{M-m}$. KONEC VLOZENI Z VYZKUMAKU

Chapter 3

Convergence

3.1 FAZE 1 IFF BETA EXPANDING

Theorem 3.1. *A[BETA] NENI DEFINOVANE*
Let ω be a complex number and $\beta \in \mathbb{Z}[\omega]$ be such that $|\beta| > 1$. Let $\mathcal{A} \subset \mathbb{Z}[\omega]$ be an alphabet. If $\mathbb{N} \subset \mathcal{A}[\beta]$, the number β is expanding.

Proof. For all $n \in \mathbb{N}$ we may write

$$n = \sum_{i=0}^{N} a_i \beta^i \,,$$

where $N \in \mathbb{N}$, $a_i \in \mathcal{A}$ and $a_N \neq 0$.

Set $m := \max\{|a| \colon a \in \mathcal{A}\}$. We take $n \in \mathbb{N}$ such that $n > m$. Since $|a_0| \leq m < n$, we have $N \geq 1$ and there is $i_0 \in \{1, 2, \ldots, N\}$ such that $a_{i_0} \neq 0$. Thus, ω is an algebraic number as $a_i \in \mathcal{A} \subset \mathbb{Z}[\omega]$ and β can be expressed as an integer combination of powers of ω. Therefore, β is also an algebraic number.

Let β' be an algebraic conjugate of β. Since $\beta \in \mathbb{Z}[\omega] \subset \mathbb{Q}(\omega)$, there is an algebraic conjugate ω' of ω and an isomorphism $\sigma : \mathbb{Q}(\omega) \to \mathbb{Q}(\omega')$ such that $\sigma(\beta) = \beta'$. Now

$$n = \sigma(n) = \sum_{i=0}^{N} \sigma(a_i)(\beta')^i \,.$$

Set $\tilde{m} := \max\{|\sigma(a)| \colon a \in \mathcal{A}\}$. For all $n \in \mathbb{N}$, we have

$$n = |n| \leq \sum_{i=0}^{N} |\sigma(a_i)| \cdot |\beta'|^i \leq \sum_{i=0}^{\infty} |\sigma(a_i)| \cdot |\beta'|^i \leq \tilde{m} \sum_{i=0}^{\infty} |\beta'|^i \,.$$

Hence, the sum on the right side diverges which implies that $|\beta'| \geq 1$. Thus, all conjugates of β are at least one in modulus.

If the degree of β is one, the statement is obvious. Therefore, we may assume that $\deg \beta \geq 2$.

Suppose for contradiction that $|\beta'| = 1$ for an algebraic conjugate β' of β. The complex conjugate $\overline{\beta'}$ is also an algebraic conjugate of β. Take any algebraic conjugate γ of β and the isomorphism $\sigma' : \mathbb{Q}(\beta') \to \mathbb{Q}(\gamma)$ given by $\sigma'(\beta') = \gamma$. Now

$$\frac{1}{\gamma} = \frac{1}{\sigma'(\beta')} = \sigma'\left(\frac{1}{\beta'}\right) = \sigma'\left(\frac{\overline{\beta'}}{\beta'\overline{\beta'}}\right) = \sigma'\left(\frac{\overline{\beta'}}{|\beta'|^2}\right) = \sigma'(\overline{\beta'}) \,.$$

Hence, $\frac{1}{\gamma}$ is also an algebraic conjugate of β. From the previous, $\left|\frac{1}{\gamma}\right| \geq 1$ and $|\gamma| \geq 1$ which implies that $|\gamma| = 1$. We may set $\gamma = \beta$ which contradicts $|\beta| > 1$. Thus all conjugates of β are greater than one in modulus, i.e., β is an expanding algebraic number. □

Theorem 3.2. *Let $\mathcal{A} \subset \mathbb{Z}[\omega]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If the extending window method with the rewriting rule $x - \beta$ converges for the numeration system $(\beta, \mathcal{A})$, then the base β is expanding.*

Proof. The existence of an algorithm for addition which is computable in parallel implies that the set $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is closed under addition. Moreover, the set $\mathcal{A}[\beta]$ is closed under addition since there is no carry to the right when the rewriting rule $x - \beta$ is used. For any $n \in \mathbb{N}$, the sum $1 + 1 + \cdots + 1 = n$ is in $\mathcal{A}[\beta]$ by the assumption $1 \in \mathcal{A}[\beta]$. Therefore, $\mathbb{N} \subset \mathcal{A}[\beta]$ and thus the base β is expanding by Theorem 3.1. □

Lemma 3.3. *Let ω be an algebraic integer, $\deg \omega = d$, and β be an expanding algebraic integer in $\mathbb{Z}[\omega]$. Let $\mathcal{A}$ and $\mathcal{B}$ be finite subsets of $\mathbb{Z}[\omega]$ such that $\mathcal{A}$ contains at least one representative of each congruence class modulo β in $\mathbb{Z}[\omega]$. Then there exists a finite set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ such that $\mathcal{B} + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}$.*

Proof. We use the isomorphism $\pi : \mathbb{Z}[\omega] \to \mathbb{Z}^d$ and β-norm $\|\cdot\|_\beta$ to bound the elements of $\mathbb{Z}[\omega]$. Let γ be the smallest conjugate of β in modulus. Denote $C := \max\{\|b - a\|_\beta : a \in \mathcal{A}, b \in \mathcal{B}\}$. Consequently, set $R := \frac{C}{|\gamma|-1}$ and $\mathcal{Q} := \{q \in \mathbb{Z}[\omega] \colon \|q\|_\beta \leq R\}$. By Lemma 1.3, we have

$$\left|\left|\left|S_\beta^{-1}\right|\right|\right|_{S_\beta} = \max\{\frac{1}{|\beta'|} \colon \beta' \text{ is conjugate of } \beta\} = \frac{1}{|\gamma|}\,.$$

Also, $|\gamma| > 1$ as β is an expanding integer. Since $C > 0$, the set $\mathcal{Q}$ is nonempty. Any element $x = b + q \in \mathbb{Z}[\omega]$ with $b \in \mathcal{B}$ and $q \in \mathcal{Q}$ can be written as $x = a + \beta q'$ for some $a \in \mathcal{A}$ and $q' \in \mathbb{Z}[\omega]$ due to existence of a representative of each congruence class in $\mathcal{A}$. Using the isomorphism π, we may write $\pi(q') = S_\beta^{-1} \cdot \pi(b - a + q)$. We prove that q' is in Q:

$$\begin{aligned}\|q'\|_\beta = \|\pi(q')\|_{S_\beta} = \left\|S_\beta^{-1} \cdot \pi(b - a + q)\right\|_{S_\beta} &\leq \left|\left|\left|S_\beta^{-1}\right|\right|\right|_{S_\beta} \|b - a + q\|_\beta \\ &\leq \frac{1}{|\gamma|}(\|b - a\|_\beta + \|q\|_\beta) = \frac{1}{|\gamma|}(C + R) = \frac{C}{|\gamma|}(1 + \frac{1}{|\gamma| - 1}) = R\,.\end{aligned}$$

Hence $q' \in \mathcal{Q}$ and thus $x = b + q \in \mathcal{A} + \beta\mathcal{Q}$.

Since there are only finitely many elements of $\mathbb{Z}^d$ bounded by the constant R, the set Q is finite. □

Theorem 3.4. *Let ω be an algebraic integer and $\beta \in \mathbb{Z}[\omega]$. Let the alphabet $\mathcal{A} \subset \mathbb{Z}[\omega]$ be such that $\mathcal{A}$ contains at least one representative of each congruence class modulo β in $\mathbb{Z}[\omega]$. Let $\mathcal{B} \subset \mathbb{Z}[\omega]$ be the input alphabet.*

If β is expanding, Phase 1 of the extending window method converges.

Proof. We have the constant R and finite set $\mathcal{Q}$ from Lemma 3.3 for the alphabet $\mathcal{A}$ and input alphabet $\mathcal{B}$. We prove by induction that all intermediate weight coefficient sets $\mathcal{Q}_k$ in Algorithm **??** are subsets of the finite set $\mathcal{Q}$.

We start with $\mathcal{Q}_0 = \{0\}$ which is bounded by any positive constant. Suppose that the intermediate weight coefficients set $\mathcal{Q}_k$ has elements bounded by the constant R. We see

from the previous proof that the candidates obtained by Algorithm **??** for the set $\mathcal{Q}_k$ are also bounded by R. Thus, the next intermediate weight coefficients set $\mathcal{Q}_{k+1}$ has elements bounded by the constant R, i.e., $\mathcal{Q}_{k+1} \subset \mathcal{Q}$.

Since $\#\mathcal{Q}$ is finite and $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \mathcal{Q}$, Phase 1 successfully ends. □

SHRNUJICI EKVIVALENCE

3.2 Convergence Phase 2

VSTUPY BBB – ODKAZ NA VYZKUMAK
TADY JE POTREBA ZABRAT!!!!

3.3 NUTNOST REPREZENTANTU + DOLNI ODHAD VELIKOSTI ABECEDY

Theorem 3.5. *Let β be an algebraic integer such that $|\beta| > 1$. Let $0 \in \mathcal{A} \subset \mathbb{Z}[\beta]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If addition in the numeration system $(\beta, \mathcal{A})$ which uses the rewriting rule $x - \beta$ is computable in parallel, the alphabet $\mathcal{A}$ contains at least one representative of each congruence class modulo β and $\beta - 1$ in $\mathbb{Z}[\beta]$.*

Proof. The existence of an algorithm for addition with the rewriting rule $x - \beta$ implies that the set $\mathcal{A}[\beta]$ is closed under addition. By the assumption $1 \in \mathcal{A}[\beta]$, the set $\mathbb{N}$ is subset of $\mathcal{A}[\beta]$. Since $0 \in \mathcal{A}$, we have $\beta \cdot \mathcal{A}[\beta] \subset \mathcal{A}[\beta]$. Hence, $\mathbb{N}[\beta] \subset \mathcal{A}[\beta]$.

For any element $x = \sum_{i=0}^{N} x_i \beta^i \in \mathbb{Z}[\beta]$ there is an element $x' = \sum_{i=0}^{N} x'_i \beta^i \in \mathbb{N}[\beta]$ such that $x \equiv_\beta x'$ since $m_\beta(0) \equiv_\beta 0$ and $\beta^i \equiv_\beta 0$. As $x' \in \mathbb{N}[\beta] \subset \mathcal{A}[\beta]$, we have

$$x \equiv_\beta x' = \sum_{i=0}^{n} a_i \beta^i \equiv_\beta a_0 \,,$$

where $a_i \in \mathcal{A}$. Hence, for any element $x \in \mathbb{Z}[\omega]$, there is a letter $a_0 \in \mathcal{A}$ such that $x \equiv_\beta a_0$.

In order to prove that there is at least one representative of each congruence class modulo $\beta - 1$ in the alphabet $\mathcal{A}$, we consider again an element $x = \sum_{i=0}^{N} x_i \beta^i \in \mathbb{Z}[\beta]$. Similarly, there is an element $x' = \sum_{i=0}^{N} x'_i \beta^i \in \mathbb{N}[\beta]$ such that $x \equiv_{\beta-1} x'$ since $m_{\beta-1}(0) \equiv_{\beta-1} 0$ and $(\beta - 1)^i \equiv_{\beta-1} 0$.

Since $x' \in \mathbb{N}[\beta] \subset \mathcal{A}[\beta]$,

$$x' = \sum_{i=0}^{n} a_i \beta^i \,,$$

where $a_i \in \mathcal{A}$. We prove by induction with respect to n that $x' \equiv_{\beta-1} a$ for some $a \in \mathcal{A}$. If $n = 0$, $x' = a_0$. Now we use the fact, that if there is a parallel addition algorithm, for each letter $b \in \mathcal{A} + \mathcal{A}$, there is $a \in \mathcal{A}$ such that $b \equiv_{\beta-1} a$ ODKAZ NA PRISLUSNOU VETU. For

$n+1$, we have

$$\begin{aligned} x' &= \sum_{i=0}^{n+1} a_i\beta^i = a_0 + \sum_{i=1}^{n+1} a_i\beta^i \\ &= a_0 + \beta\sum_{i=0}^{n} a_{i+1}\beta^i - \sum_{i=0}^{n} a_{i+1}\beta^i + \sum_{i=0}^{n} a_{i+1}\beta^i \\ &\equiv_{\beta-1} a_0 + (\beta-1)\sum_{i=0}^{n} a_{i+1}\beta^i + a \equiv_{\beta-1} a_0 + a \equiv_{\beta-1} a' \in \mathcal{A}\,, \end{aligned}$$

where we use the induction assumption

$$\sum_{i=0}^{n} a_{i+1}\beta^i \equiv_{\beta-1} a\,.$$

□

Theorem 3.6. *Let β be an algebraic integer such that $|\beta| > 1$. Let $0 \in \mathcal{A} \subset \mathbb{Z}[\beta]$ be an alphabet such that $1 \in \mathcal{A}[\beta]$. If addition in the numeration system $(\beta, \mathcal{A})$ which uses the rewriting rule $x - \beta$ is computable in parallel, then*

$$\#\mathcal{A} \geq \max\{|m_\beta(0)|, |m_\beta(1)|\}\,.$$

Proof. By Theorem 3.5, there are all representatives modulo β and modulo $\beta - 1$ in the alphabet $\mathcal{A}$. The numbers of congruence classes are $|m_\beta(0)|$ and $|m_{\beta-1}(0)|$ by Theorem 1.5. Obviously, $m_{\beta-1}(x) = m_\beta(x+1)$. Thus $m_{\beta-1}(0) = m_\beta(1)$. □

BYLO BY FAJN TO JESTE ZOBECNIT NA ABECEDU ZE Z[OMEGA]

Chapter 4

Design and implementation

CASTECNE VYUZIT Z VYZKUMAKU??
PRIDAT POZNAMKU O GOOGLE TABULCE, HROMADNE TESTOVANI

Chapter 5

Testing

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