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VÝZKUMNÝ ÚKOL

Konstrukce algoritmů pro paralelní sčítání

Construction of algorithms for parallel addition

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Čestné prohlášení

Prohlašuji na tomto místě, že jsem předloženou práci vypracoval samostatně a že jsem uvedl veškerou použitou literaturu.

V Praze dne ???

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Poděkování

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Obsah

Introduction

Chapter 1

Preliminaries

Preliminaries *Positional numeration system* $(\beta, \mathcal{A})$ is defined by

- *Base* $\beta \in \mathbb{C}, |\beta| > 1$.
- Finite *digit set* $\mathcal{A} \subset \mathbb{Z}$ containing 0, usually called *alphabet*.

A complex number x has a finite $(\beta, \mathcal{A})$-representation if $x = \sum_{j=-m}^{n} x_j \beta^j$ with coefficients x_j in $\mathcal{A}$.

$$x = x_n x_{n-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-m}$$

Chapter 2

Method for construction of algorithms for parallel addition

2.1 Method deduction

The general concept of addition (standard or parallel) in any numeration system $(\beta, \mathcal{A})$ is following: we add numbers digitwise and then we convert the result into the alphabet $\mathcal{A}$. Obviously, digitwise addition can be run in parallel, thus the crucial point is the conversion of the obtained result. It can be easily done in a standard way but a parallel conversion is nontrivial. However, formulas are basically same but the choice of coefficients differs.

Now we go step by step more precisely. Let $x = \sum_{-m'}^{n'} x_i\beta^i, y = \sum_{-m'}^{n'} y_i\beta^i \in \mathrm{Fin}_\beta(\mathcal{A})$ with $(\beta, \mathcal{A})$-representantions padded by zeros to have the same length. We add

$$\begin{aligned} w = x + y = \sum_{-m'}^{n'} x_i\beta^i + \sum_{-m'}^{n'} y_i\beta^i = \sum_{-m'}^{n'} (x_i + y_i)\beta^i \\ = \sum_{-m'}^{n'} w_i\beta^i \,, \end{aligned}$$

where $w_j = x_j + y_j \in \mathcal{A} + \mathcal{A}$. Thus, $w_{n'}w_{n'-1} \cdots w_1w_0 \bullet w_{-1}w_{-2} \cdots w_{-m'}$ is a $(\beta, \mathcal{A} + \mathcal{A})$-representation of $w \in \mathrm{Fin}_\beta(\mathcal{A} + \mathcal{A})$.

We also use column notation of addition in the following, e.g.,

$$\begin{array}{c} x_{n'}\ x_{n'-1} \cdots x_1\ x_0 \bullet x_{-1}\ x_{-2}\ \cdots x_{-m'} \\ y_{n'}\ y_{n'-1} \cdots y_1\ \ y_0 \bullet y_{-1}\ y_{-2}\ \cdots y_{-m'} \\ \hline w_{n'}w_{n'-1} \cdots w_1w_0 \bullet w_{-1}w_{-2} \cdots w_{-m'} \,, \end{array}$$

Goal – find $(\beta, \mathcal{A})$-representation of w, i.e. a sequence

$$z_{n'}z_{n'-1} \cdots z_1z_0z_{-1}z_{-2} \cdots z_{-m'}$$

such that $z_j \in \mathcal{A}$ and

$$z_{n'}z_{n'-1} \cdots z_1z_0 \bullet z_{-1}z_{-2} \cdots z_{-m'} = w$$

We have $0 = 1 \cdot \beta^j - \beta \cdot \beta^{j-1} = 1(-\beta)0 \cdots 0\bullet$

→ We add $q_j \cdot 0$ for each j:
$0 = q_j \cdot \beta^j - \beta q_j \cdot \beta^{j-1}$

$$\begin{array}{ccccccc}
w_n w_{n-1} & \cdots & w_{j+1} & w_j & w_{j-1} & \cdots & w_1 w_0 \bullet \\
 & & & & q_{j-2} & \iddots & \\
 & & & q_{j-1} & -\beta q_{j-1} & & \\
 & & q_j & -\beta q_j & & & \\
 & \iddots & -\beta q_{j+1} & & & & \\
\hline
z_{n+1} z_n z_{n-1} & \cdots & z_{j+1} & z_j & z_{j-1} & \cdots & z_1\ z_0 \bullet
\end{array}$$

$\Longrightarrow$

$$z_j = w_j + q_{j-1} - q_j \beta \in \mathcal{A}$$

Assume now a standard numeration system $(\beta, \mathcal{A})$:

$$\beta \in \mathbb{N}\,, \beta \geq 2\,, \mathcal{A} = \{0, 1, 2, \ldots, \beta - 1\}$$

Conversion runs from right to left:

$$\begin{array}{rl}
w_n w_{n-1} \cdots w_{j+1} w_j w_{j-1} \cdots w_1 w_0 \bullet & , w_i \in \mathcal{A} + \mathcal{A}\,, \\
\longrightarrow z_{n+1}\ z_n\ z_{n-1}\ \cdots z_{j+1} z_j\ z_{j-1}\ \cdots\ z_1\ z_0 \bullet & , z_i \in \mathcal{A}\,.
\end{array}$$

$$z_j = w_j + q_{j-1} - q_j \beta$$

The weight coefficients q_j and digits z_j are unique in the standard numeration system

$$\Longrightarrow\ z_j = z_j(w_j, \ldots, w1, w_0)$$

Thus, addition is linear in length of inputs.
Parallel Addition Introduced by Avizienis in 1961:

$$\begin{array}{rl}
\cdots w_{j+t+1} w_{j+t} \cdots w_{j+1} w_j w_{j-1} \cdots w_{j-r} w_{j-r-1} \cdots & , w_i \in \mathcal{A} + \mathcal{A}\,, \\
\longrightarrow \cdots z_{j+t+1}\ z_{j+t}\ \cdots\ z_{j+1} z_j\ z_{j-1}\ \cdots z_{j-r}\ z_{j-r-1}\ \cdots & , z_i \in \mathcal{A}\,.
\end{array}$$

Digit conversion is a mapping $(\mathcal{A} + \mathcal{A})^{t+r+1} \to \mathcal{A}$:

$$z_j = z_j(w_{j+t} \cdots w_{j+1} w_j w_{j-1} \cdots w_{j-r})\,.$$

Thus the parallel addition is done in constant time but the numeration system must be redundant – a number has more than one admissible representation.

For example:

$$\beta \in \mathbb{N}, \beta \geq 3, \mathcal{A} = \{-a, \ldots, 0, \ldots a\}, b/2 < a \leq b - 1\,.$$

2.1.1 Non-standard numeration systems

Non-standard numeration systems

Integer alphabets:

- Base $\beta \in \mathbb{C}, |\beta| > 1$.
- Addition is computable in parallel if and only if β is an algebraic number with no conjugate of modulus 1 [Frougny, Heller, Pelantová, S.].
- Algorithms are known but with large alphabets.

Non-integer alphabets:

- Base $\beta \in \mathbb{Z}[\omega] = \left\{ \sum_{j=0}^{d-1} a_j \omega^j \ : a_j \in \mathbb{Z} \right\}$, where $\omega \in \mathbb{C}$ is an algebraic integer of degree d.
- Alphabet $\mathcal{A} \subset \mathbb{Z}[\omega], 0 \in \mathcal{A}$.
- Only few manually found algorithms.

How do we find systematically algorithms with minimal alphabets?

2.2 Method

Method for construction of parallel addition algorithms Key problem – find weight coefficients q_j such that

$$z_j = \underbrace{w_j}_{\in \mathcal{A}+\mathcal{A}} + q_{j-1} - q_j \beta \in \mathcal{A}$$

for any input w and every j.

In order to do that we need to find $M \in \mathbb{N}$ and $q : (\mathcal{A}+\mathcal{A})^M \to \mathcal{Q} \subset \mathbb{Z}[\omega]$ such that $q_j = q(w_j, \ldots, w_{j-M+1})$.

The mapping q is called the *weight function*.

Our method:

1. Find set $\mathcal{Q} \subset \mathbb{Z}[\omega]$ of possible weight coefficients.
2. Increment M and for each $w_j, w_{j-1}, \ldots, w_{j-M+1} \in (\mathcal{A}+\mathcal{A})^M$ try to find a weight coefficient from $\mathcal{Q}$ to define q.

2.2.1 Phase 1 – Set of weight coefficients

Phase 1 – searching for the set of weight coefficients We want to find set of weight coefficients $\mathcal{Q} \subset \mathbb{Z}[\omega]$ such that

$$\underbrace{(\mathcal{A}+\mathcal{A})}_{w_j \in} + \underbrace{\mathcal{Q}}_{q_{j-1} \in} \subset \underbrace{\mathcal{A}}_{z_j \in} + \underbrace{\beta \mathcal{Q}}_{\beta q_j \in}$$

It implies that there is $q_j \in \mathcal{Q}$ such that

$$z_j = w_j + q_{j-1} - q_j \beta \in \mathcal{A}\,.$$

for all $q_{j-1} \in \mathcal{Q}$ and $w_j \in \mathcal{A}+\mathcal{A}$.

We build $\mathcal{Q}$ iteratively:

Phase 1 $k := 0$

Set $\mathcal{Q}_0 := \{0\}$

Repeat:

- extend $\mathcal{Q}_k$ to $\mathcal{Q}_{k+1}$ in a minimal possible way so that
$$(\mathcal{A}+\mathcal{A}) + \mathcal{Q}_k \subset \mathcal{A} + \beta\mathcal{Q}_{k+1}\,,$$
- $k := k+1$

until $\mathcal{Q}_k = \mathcal{Q}_{k+1}$.

$\mathcal{Q} := \mathcal{Q}_k$

2.2.2 Phase 2 – Weight function

Phase 2 – searching for a weight function

We want to find a length of the window M and a weight function $q : (\mathcal{A}+\mathcal{A})^M \to \mathcal{Q}$.

Suppose that the length of the window is m.

The idea is to check all possible right carries q_{j-1} and determine values q_j such that
$$z_j = w_j + q_{j-1} - q_j\beta \in \mathcal{A}\,.$$
The set of all such needed values of q_j is denoted by $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]} \subset \mathcal{Q}$

The length M and weight function q is found when
$$\#\mathcal{Q}_{[w_j,\dots,w_{j-M+1}]} = 1$$
for all $w_j,\dots,w_{j-M+1} \in (\mathcal{A}+\mathcal{A})^M$.

Phase 2 $m := 1$

For each $w_j \in \mathcal{A}+\mathcal{A}$ find minimal set $\mathcal{Q}_{[w_j]} \subset \mathcal{Q}$ such that
$$w_j + \mathcal{Q} \subset \mathcal{A} + \beta\mathcal{Q}_{[w_j]}$$
While $(\max\{\#\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]} : (w_j,\dots,w_{j-m+1}) \in (\mathcal{A}+\mathcal{A})^m\} > 1)$ do:

- $m := m+1$
- For each $(w_j,\dots,w_{j-m+1}) \in (\mathcal{A}+\mathcal{A})^m$ find minimal set $\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]} \subset \mathcal{Q}_{[w_j,\dots,w_{j-m+2}]}$ such that
$$w_j + \mathcal{Q}_{[w_{j-1},\dots,w_{j-m+1}]} \subset \mathcal{A} + \beta\mathcal{Q}_{[w_j,\dots,w_{j-m+1}]}\,,$$

$M := m$

$q(w_j,\dots,w_{j-M+1}) :=$ only element of $\mathcal{Q}_{[w_j,\dots,w_{j-M+1}]}$

Now we have parallel conversion algorithm:
$$\begin{aligned} z_j &= w_j + q_{j-1} - q_j\beta = \\ &= w_j + q(w_{j-1},w_{j-2},\dots,w_{j-M}) - \beta q(w_j,w_{j-1},\dots,w_{j-M+1}) = \\ &= z_j(w_j,w_{j-1},\dots,w_{j-M})\,. \end{aligned}$$

Chapter 3

Implementation

Chapter 4

Examples

Summary