Q.N 3.1 a)

Defⁿ

$X_1 \times \ldots \times X_n = \{(x_1, \ldots x_n) \mid \forall i \; 1 \le i \le n \Rightarrow x_i \in X_i\}$

To prove:

a) $(A \times B) \cap (C \times D) = (A \times C) \cap (B \times D)$

Taking L.H.S

$\exists x, y \in (A \times B) \cap (C \times D)$

$\Leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$

$\Leftrightarrow (x \in A, y \in B)$ and $(x \in C, y \in D)$ [∵ from defⁿ]

$\Leftrightarrow x \in A$ and $y \in B$ and $x \in C$ and $y \in D$

$\Leftrightarrow x \in A$ and $x \in C$ and $y \in B$ and $y \in D$ [rearrangement]

$\Leftrightarrow x \in (A \cap C)$ and $y \in (B \cap D)$

$\Leftrightarrow x \in (A \cap C), y \in (B \cap D)$

$\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D)$

Hence,

$(A \times B) \cap (C \times D) = (A \times C) \cap (B \times D)$

QN 3.1 b)

We can disprove $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$ by a counter example,

Let's take

$A = \{1, 2\}$, $B = \{3, 4\}$

$C = \{4, 6\}$, $D = \{2, 8\}$

Computing L.H.S

$(A \cup B) \times (C \cup D)$

$= \{1, 2, 3, 4\} \times \{2, 4, 6, 8\}$

$= \{(1,2), (1,4), (1,6), (1,8)$
$(2,2), (2,4), (2,6), (2,8)$
$(3,2), (3,4), (3,6), (3,8)$
$(4,2), (4,4), (4,6), (4,8)\}$

Now,

Computing R.H.S

$(A \times C) \cup (B \times D)$

$= (\{1,2\} \times \{4,6\}) \cup (\{3,4\} \times \{2,8\})$

$= \{(1,4), (1,6), (2,4), (2,6)\} \cup \{(3,2), (3,8), (4,2), (4,8)\}$

$= \{(1,4), (1,6), (2,4), (2,6) (3,2), (3,8), (4,2), (4,8)\}$

As L.H.S $\neq$ R.H.S

We can easily say that $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$ in general

Q: N 3.2a)

$R = \{(a,b) \mid a,b \in \mathbb{Z} \wedge |a-b| \leq 3\}$

Checking if R is reflexive,

for being reflexive

$\forall a \in \mathbb{Z}$ s.t. $(a,a) \in R$

If $(a,a) \in R$ then it must satisfy,

$|a-a| \leq 3$

$\Rightarrow |0| \leq 3$

$\Rightarrow 0 \leq 3$ (True)

So, we can conclude that R is reflexive

Now, checking if R is symmetric,

for being

$\forall a,b \in \mathbb{Z}.\ (a,b) \in R \Rightarrow (b,a) \in R$

Then $|b-a| \leq 3$ must be true,

Taking, $|a-b| \leq 3$

$\Rightarrow |b-a| \leq 3$ ($\because |a-b| = |b-a|$)

(True)

So, we can conclude that R is symmetric.

Lastly, Checking if R is transitive,

for being transitive

$$\forall a,b,c \in \mathbb{Z}. ((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$$

For the relation R to be transitive the following must hold,

$$(|a-b| \leq 3) \wedge (|b-c| \leq 3) \Rightarrow (|a-c| \leq 3) \; \forall a,b,c \in \mathbb{Z}$$

We can disprove the above statement by following example,

lets take $a = 10$, $b = 7$ and $c = 5$

$|a-b| \leq 3 \Rightarrow (10-7) \leq 3 \Rightarrow 3 \leq 3$ (True)

and,

$|b-c| \leq 3 \Rightarrow (7-5) \leq 3 \Rightarrow 2 \leq 3$ (True)

Now,

$|a-c| \leq 3 \Rightarrow (10-5) \leq 3 \Rightarrow 5 \leq 3$ (False)

Hence,

By giving a counter example we can conclude that R is non-transitive.

Q.N 3.2 b)

$R = \{(a,b) \mid a,b \in Z \wedge (a \bmod 10) = (b \bmod 10)\}$

Checking if R is reflexive,

for being reflexive

$\forall a \in Z : (a,a) \in R$

If $(a,a) \in R$ then it must satisfy,

$(a \bmod 10) = (a \bmod 10)$ (True)

So, it is reflexive

Checking if R is symmetric,

for being symmetric,

$\forall a,b \in Z . (a,b) \in R \Rightarrow (b,a) \in R$

If $(a,b) \in R$ then it must hold that,

$(a \bmod 10) = (b \bmod 10) \Rightarrow (b \bmod 10) = (a \bmod 10)$

As it is equality the above statement is true.

So, it is symmetric.

Checking if R is transitive,

for being transitive,

$\forall a,b,c \in \mathbb{Z}.((a \bmod 10) = (b \bmod 10)) \wedge ((b \bmod 10) = (c \bmod 10))$
$\Rightarrow ((a \bmod 10) = (c \bmod 10))$

Taking $(a \bmod 10) = (c \bmod 10)$

$\Rightarrow (b \bmod 10) = (c \bmod 10)$ (As $(a \bmod 10) = (b \bmod 10)$)

(True)

Hence, we can say that R is transitive.