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OCE3521

Homework 1

Problem 1 – DD 2.2

Part A

Starting with the material derivative for **u**

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad 1.1$$

Most of the terms go to zero, leaving

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} \qquad 1.2$$

The equation for the horizontal flow velocity is

$$u = \frac{Q}{A(x)}\ \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \qquad 1.3$$

Where Q is the volumetric flow rate and A(x) is the change in cross-sectional area along the x-axis. The equation for the changing cross-sectional area is

$$A(x) = 0.1(-0.2x + 0.4)\ [\mathrm{m}^2] \qquad 1.4$$

Plugging Equation 1.4 into 1.3

$$u = \frac{0.1}{0.1(-0.2x + 0.4)} = (-0.2x + 0.4)^{-1}\ \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \qquad 1.5$$

The acceleration of the fluid throughout the pipe is therefore

$$\frac{du}{dx} = \frac{0.2}{(-0.2x + 0.4)^2}\ \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \qquad 1.6$$

Plugging Equation 1.6 into 1.2

$$\frac{Du}{Dt} = \frac{1}{-0.2x + 0.4} \times \frac{0.2}{(-0.2x + 0.4)^2} = \frac{0.2}{(-0.2x + 0.4)^3} \qquad 1.7$$

Solving for acceleration at x=0.5 m using Equation 1.7

$$\frac{Du}{Dt} = \frac{0.2}{(-0.2(0.5) + 0.4)^3} = 7.4\ \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \qquad 1.8$$

Part B

Beginning with Equation 1.3 and knowing the value of Q(t)

$$u = \frac{Q}{A(x)} = \frac{t^2}{100 \times 0.1(0.2 + 0.2)} = \frac{t^2}{2x + 2} = t^2(2x + 2)^{-1} \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \qquad 1.9$$

Calculating the derivative of Equation 1.9

$$\frac{du}{dx} = -\frac{2t^2}{(2x + 2)^2} \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \qquad 1.10$$

Using Equation 1.10 to get acceleration at t=4.48 s and x=0.5 m

$$\frac{du}{dx}_{t=4.48,\ \ x=0.5} = -\frac{2(4.48)^2}{(2(0.5) + 2)^2} = -4.46 \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \qquad 1.11$$

Problem 2 – DD 2.3

Part A

The velocity potential function can be broken down into its vector components

$$u = -\frac{\partial \phi}{\partial x} = 3\cos\left(\frac{2\pi t}{T}\right) \quad 2.1$$

$$v = -\frac{\partial \phi}{\partial y} = 0 \quad 2.2$$

$$w = -\frac{\partial \phi}{\partial z} = -5\cos\left(\frac{2\pi t}{T}\right) \quad 2.3$$

To find rotationality of the field, take the cross-product of the del operator and the velocity potential function

$$\nabla \times \vec{u} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ u & w \end{bmatrix} = \langle \frac{\partial w}{\partial y}, -\frac{\partial u}{\partial x} \rangle = 0 \quad 2.4$$

Since neither u or w have x or z variables, the cross-product is 0 and therefore the flow is irrotational

Part B

Using the Equations 2.1-2.3, the flow divergence can be determined by

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad 2.5$$

Since neither u or w have x or z variables, the dot-product is 0 and therefore the flow is non-divergent

Part C

The stream function is defined by

$$\Psi = \oint -u dz + w dx \quad 2.6$$

Plugging in Equations 2.1 and 2.3 to 2.6

$$\Psi = \oint -3\cos\left(\frac{2\pi t}{T}\right) dz + 5\cos\left(\frac{2\pi t}{T}\right) dx \quad 2.7$$

If the integration constant is assumed to be 0, the integral can be solved to

$$\Psi = -3z\cos\left(\frac{2\pi t}{T}\right) + 5x\cos\left(\frac{2\pi t}{T}\right) = -\cos\left(\frac{2\pi t}{T}\right)(3z + 5x) \quad 2.8$$

At t= T/8, the stream function is

$$\Psi_{t=\frac{T}{8}} = -\cos\frac{\pi}{4}(3z + 5x) \qquad 2.9$$

For plotting the stream functions, two arbitrary constants, 0 and 1, were the value of $\Psi_{t=\frac{T}{8}}$

$$\Psi = 0 = -\cos\frac{\pi}{4}(3z + 5x) \qquad 2.10$$

$$\Psi = 1 = -\cos\frac{\pi}{4}(3z + 5x) \qquad 2.11$$

Rearranging Equations 2.10 and 2.11 in terms of z respectively yields

$$z_{\Psi=0} = -\frac{5}{3}x \qquad 2.12$$

$$z_{\Psi=1} = -\frac{1}{3}\left(5x + \sec\left(\frac{\pi}{4}\right)\right) \qquad 2.13$$

Which produces the following plot from x = [-10, 10]

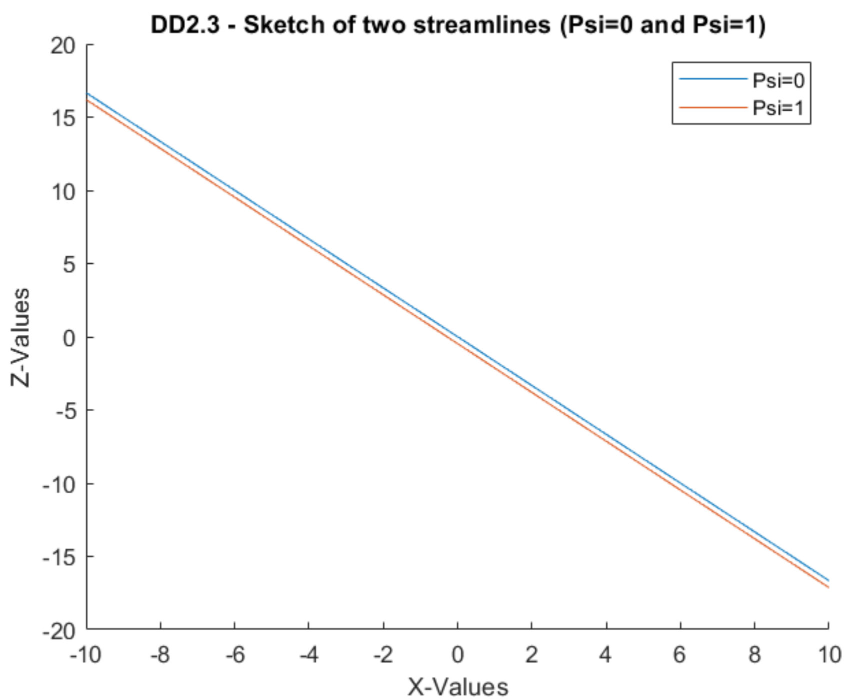


Figure 1: Plot of two streamlines (Psi=0 and Psi=1)

Problem 3 – DD 2.6

From momentum conservation in the z-axis, the following is known

$$\vec{a} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 \qquad 3.1$$

Since the flow is in a steady state condition $\frac{\partial w}{\partial t}$ goes to zero. There are two additional accelerations acting on an inviscid, non-divergent flow: gravity and the pressure gradient. These can be derived from the body forces acting on the fluid

$$F = m(\vec{a}) + mg + \frac{m}{\rho}\frac{\partial P}{\partial z} = 0 \qquad 3.2$$

Removing m and substituting in Equation 3.1 yields

$$\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + g + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0 \qquad 3.3$$

Rearranging Equation 3.3 yields the expected result

$$\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -g - \frac{1}{\rho}\frac{\partial P}{\partial z} \qquad 3.4$$

Problem 4 – DD 2.7

The expansion begins by applying the del operator to the product of the three scalar functions

$$\nabla(\phi\Psi f) = \frac{\partial(\phi\Psi f)}{\partial x} + \frac{\partial(\phi\Psi f)}{\partial z} + \frac{\partial(\phi\Psi f)}{\partial z} \qquad 4.1$$

The Product Rule is used to take the respective derivative of each function product

$$\Psi f\frac{\partial\phi}{\partial x} + \phi f\frac{\partial\Psi}{\partial x} + \phi\Psi\frac{\partial f}{\partial x} + \Psi f\frac{\partial\phi}{\partial y} + \phi f\frac{\partial\Psi}{\partial y} + \phi\Psi\frac{\partial f}{\partial y} + \Psi f\frac{\partial\phi}{\partial z} + \phi f\frac{\partial\Psi}{\partial z} + \phi\Psi\frac{\partial f}{\partial z} \qquad 4.2$$

Rearranging Equation 4.2 with common factors

$$\Psi f\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\right) + \phi f\left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y} + \frac{\partial\Psi}{\partial z}\right) + \phi\Psi\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) \qquad 4.3$$

The terms in parentheses can be simplified using the gradient operator yielding

$$\Psi f\nabla\phi + \phi f\nabla\Psi + \phi\Psi\nabla f \qquad 4.4$$

Problem 5 – DD 2.8

The velocity vector for the two-dimensional flow is given by

$$\vec{u} = \langle \frac{kx}{x^2 + z^2}, \frac{kz}{x^2 + z^2} \rangle \qquad 5.1$$

Part A

The divergence of the flow is determined by the dot-product of the del operator and $\vec{u}$

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{k(x^2 - z^2)}{(x^2 + z^2)^2} - \frac{k(z^2 - x^2)}{(x^2 + z^2)^2} = 0 \qquad 5.2$$

Since the dot-product equals 0, the flow is non-divergent

Part B

The rotationality of the flow is determined the cross-product of the del operator and $\vec{u}$

$$\nabla \times \vec{u} = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ \dfrac{kx}{x^2 + z^2} & \dfrac{kz}{x^2 + z^2} \end{bmatrix} = \frac{\partial}{\partial x}\left(\frac{kx}{x^2 + z^2}\right) - \frac{\partial}{\partial z}\left(\frac{kz}{x^2 + z^2}\right) = 0\hat{j} \qquad 5.3$$

Since the resultant vector is 0, the flow is irrotational

Part C

The streamline function is defined by Equation 2.6. When taken for this problem, the integral yields

$$\Psi = -\frac{k}{x}\arctan\left(\frac{z}{x}\right) + \frac{k}{z}\arctan\left(\frac{z}{x}\right) \qquad 5.4$$

When plotted, the streamline through points (1, 1) and (1, 2) produces the following graph

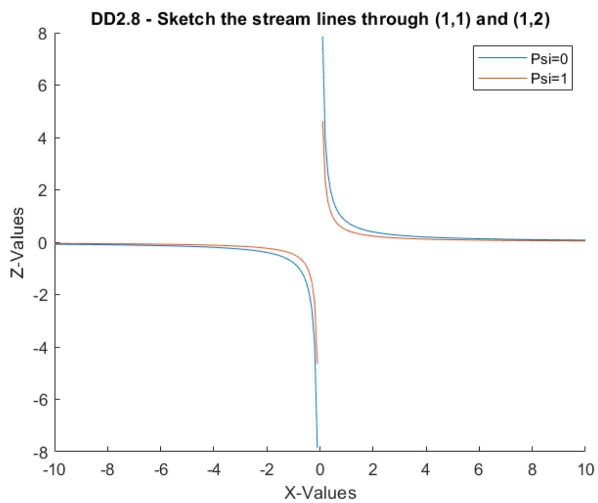


Figure 2: plot of Psi=0 and Psi=1 from Equation 5.4 through the points (1,1) and (1,2)

Problem 6 – DD 2.10

Part A

The first streamline for $\Psi = 0$ is

$$\Psi = Ax^2zt = 0 \quad 6.1$$

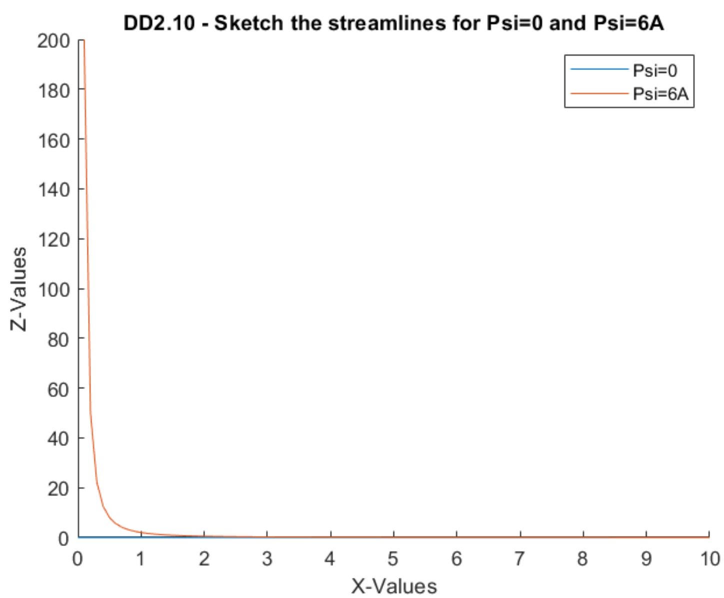
The second streamline for $\Psi = 6A$ is

$$\Psi = Ax^2zt = 6A \quad 6.2$$

Which yields

$$\Psi = x^2zt = 6 \quad 6.3$$

When plotted, Equations 6.1 and 6.3 generate the following graph. Note, $\Psi = 0$ is a straight line at z = 0.



Part B

When the streamline is $\Psi = 100A$ and $t = 5$ s, the streamline equation reduces to

$$\Psi = 20 = x^2z \quad 6.4$$

Rearranging to solve for z yields

$$z = \frac{20}{x^2} \quad 6.5$$

To find where the derivative equals -5, the first derivative must be taken and solved for the x-value

$$\frac{dz}{dx} = -\frac{40}{x^3} = 5 \xrightarrow{yields} x = 2 \quad 6.6$$

Plugging in the x-value found from Equation 6.6 into 6.5 produces

$$z = \frac{20}{2^2} = 5 \tag{6.7}$$

Therefore, $\frac{dz}{dx} = -5$ at (2, 5)

Part C

From Equation 2.97 in *Water Wave Mechanics for Engineers and Scientists* by Dean and Dalrymple, the pressure gradient in a streamline can be determined by

$$-\frac{1}{\rho} u_s \frac{\partial u_s}{\partial s} = \frac{\partial P}{\partial s} \tag{6.8}$$

By the definition of a streamline, u_s and $\frac{\partial u_s}{\partial x}$ can be found with

$$u_s = -\frac{\partial \Psi}{\partial x} = -Ax^2 t \tag{6.9}$$

$$\frac{\partial u_s}{\partial x} = -2Axt \tag{6.10}$$

Plugging in x=2, z=5, t=3, A=1, ρ=1 and simplifying Equations 6.9 and 6.10 into Equation 6.8 yields

$$\frac{\partial P}{\partial s} = -2Ax^3 t^2 \rho = -2(1)(2)^3(3)^2(1) = -144 \tag{6.11}$$

Problem 7 – DD 3.6

Starting with the definitions for the u and v components of the velocity potential, it is known that

$$u = -\frac{\partial \phi}{\partial x} = -20x \tag{7.1}$$

$$v = -\frac{\partial \phi}{\partial y} = 20y \tag{7.2}$$

Bernoulli's equation is required to solve the problem and the u and v terms within can be replaced by Equations 7.1 and 7.2

$$-\frac{d\phi}{dt} + \frac{1}{2}(u^2 + v^2) + \frac{P}{\rho} + gz = \frac{1}{2}((-20x)^2 + (20y)^2) + \frac{P}{\rho} = C(t) \tag{7.3}$$

Equation 7.3 can be simplified down to

$$200(x^2 + y^2) + \frac{P}{\rho} = C(t) \tag{7.4}$$

The initial value problem for (1, 1) allows C(t) to be determined

$$C(t)_{x=1,\ y=1,\ P=0} = 200(1^2 + 1^2) + 0 = 400 \tag{7.5}$$

Rearranging Equation 7.4 to solve for P yields

$$P = -200\rho[(x^2 + y^2) - 2] \tag{7.6}$$

To find the location of local maxima and minima, the first derivative test must be performed

$$\frac{\partial P}{\partial x} = -400\rho x = 0 \xrightarrow{yields} x = 0 \tag{7.7}$$

$$\frac{\partial P}{\partial y} = -400\rho y = 0 \xrightarrow{yields} y = 0 \tag{7.8}$$

Since there is only one set of values, the absolute maximum value can be assumed to be at (0, 0). Therefore, plugging in (0, 0) to Equation 7.6 yields

$$P_{max} = -200\rho[(0 + 0) - 2] = -400\rho \tag{7.9}$$

Appendix

OCE3521_Homework1_Duffy.m

```
% OCE3521 - Homework 1
% Braidan Duffy
% Due: 02/11/21

%% Problem 2 - DD2.3
% Sketch two streamlines for t=T/8

% psi(t=T/8) = -cos(pi/4) * (3z+5x)

x = -10:0.1:10;
z_0 = -5/3 .* x; % Z function when psi = 0
z_1 = -1/3 * (5.*x+sec(pi/4)); % z function when psi = 1

% Plotting
figure(1)
title("DD2.3 - Sketch of two streamlines (Psi=0 and Psi=1)")
hold on
plot(x, z_0)
plot(x, z_1)
xlabel("X-Values")
ylabel("Z-Values")
legend("Psi=0", "Psi=1")
hold off

%% Problem 5 - DD2.8
% Sketch the two streamlines through (1, 1) and (1, 2)
x = -10:0.1:10;

z1 = atan(1) ./ x;
z2 = atan(0.5) ./ x;

% Plotting
figure(2)
title("DD2.8 - Sketch the stream lines through (1,1) and (1,2)")
hold on
plot(x, z1)
plot(x, z2)
xlabel("X-Values")
ylabel("Z-Values")
legend("Psi=0", "Psi=1")
hold off

%% Problem 6 - DD2.10
% Sketch the streamlines for psi=0 and psi=6A

x = 0:0.1:10; % Domain x = [0, 10]
z_2 = 2 ./ x .^ 2; % Implicit z function for

% Plotting
figure(3)
title("DD2.10 - Sketch the streamlines for Psi=0 and Psi=6A")
```

```
hold on
plot(x, zeros(1, length(x))) % first function of z is implicitly 0
plot(x, z_2)
xlabel("X-Values")
ylabel("Z-Values")
legend("Psi=0", "Psi=6A")
hold off
```