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1. Course Outline

A course on water wave mechanics at the undergraduate level poses many challenges. The first of which is keep students excited while introducing very complicated topics and ideas. At the engineering level, this text seeks to provide the introductory level concepts and details that will enable the reader to gain an understanding of the physics and mechanisms without diving too deep into the waters of higher order theory. This text will prepare students to enter the workplace with the tools necessary to succeed. For those who choose to advance their academic degree, this text provides the foundation for water wave mechanics that can be advanced, built upon and even molded as higher order solutions and ideas are learned.

This text is written in both narrative and in bullet / equation form. Narratives are provided to offer background and insight into the subject matter. The bullet and equation format help the student to focus on the pertinent information and serve as a typed notes format for the course.

1.1. Course topics

- Introduction
- Mathematics review
- Hydrodynamics
- BVP Solutions for water waves
- Engineering Properties Derivable from Linear Theory
- Wave Statistics
- Extra Topics
    - Long Waves
    - Wave Forces
    - Wavemaker Theory
    - Nonlinear Properties for Small-amplitude Waves

1.2. Learning Objectives

We will be learning the fundamentals of water wave theory backed up by the fluid principals learned previously and reinforced during the first potion of the course. It is important to note that there is not exactly a single unique solution to this problem as assumptions must be made to arrive at the analytical solutions.

- You will learn to pay attention to detail
    - **READ THE PROBLEMS THOROUGHLY**
- You will learn MatLab
- Bring Calculators to class, I expect you to work along at your desk as I am working on the board and will call on the class for solutions
- I expect that you want to be here
- No eating, phones, computers, games, puzzles…period!
- No cheating

2. Introduction to Waves

Waves are the driving force behind shoreline evolution. The energy that waves release as they end their lives on the beaches of the world, drive the evolution of those shorelines. In shallow and intermediate depth ecosystems, the movement of water due to the presence of the waves provides critical circulation, cycling nutrients and oxygen, providing the mixing needed to reproduce life in the seas.

For the engineer, the design wave condition is one of the major factors for the coastal and ocean environment. The second major design factor is the water level. Together the water elevation and the wave climate will drive any engineering project in the coastal zone. As we will learn, these two design factors are closely coupled. Relationships will be developed between water depth and wave characteristics such as wave height, wave speed or celerity, and wavelength.

2.1. What are waves?

- A wave is a disturbance that propagates through a medium:
  1) some source of disturbance
  2) a medium that can be disturbed
  3) some physical mechanism through which particles of the medium can influence each other.

- Gravity and surface tension act to keep water surface flat
- Remains flat until acted upon by a force
  - Wind, rock impacts, ship moving in water, gravitational pull of moon and sun, low pressure system
- Wave size depends on magnitude of force acting on the water body
- Wave length
  - Long waves can be as long as the ocean basin (tides)
    - Gravity is the important restoring force
  - Short waves can be less than a centimeter (capillary waves)
    - Surface tension the important restoring force
- What are waves responsible for?
  - drive sediment motion along the coast
  - flushing of lagoons (estuaries)
  - forces on offshore structures (oil Platforms)
  - resistance to ship motion (wake)
  - forces on ships

2.2. Wave Characteristics

- Height =H(crest to trough), Period=T (time between consecutive crests), Length=L(crest to crest), water depth=h
    - From these we can calculate other parameters
- The wave diagram, Figure 1, helps to identify the major variables that we will be dealing with.

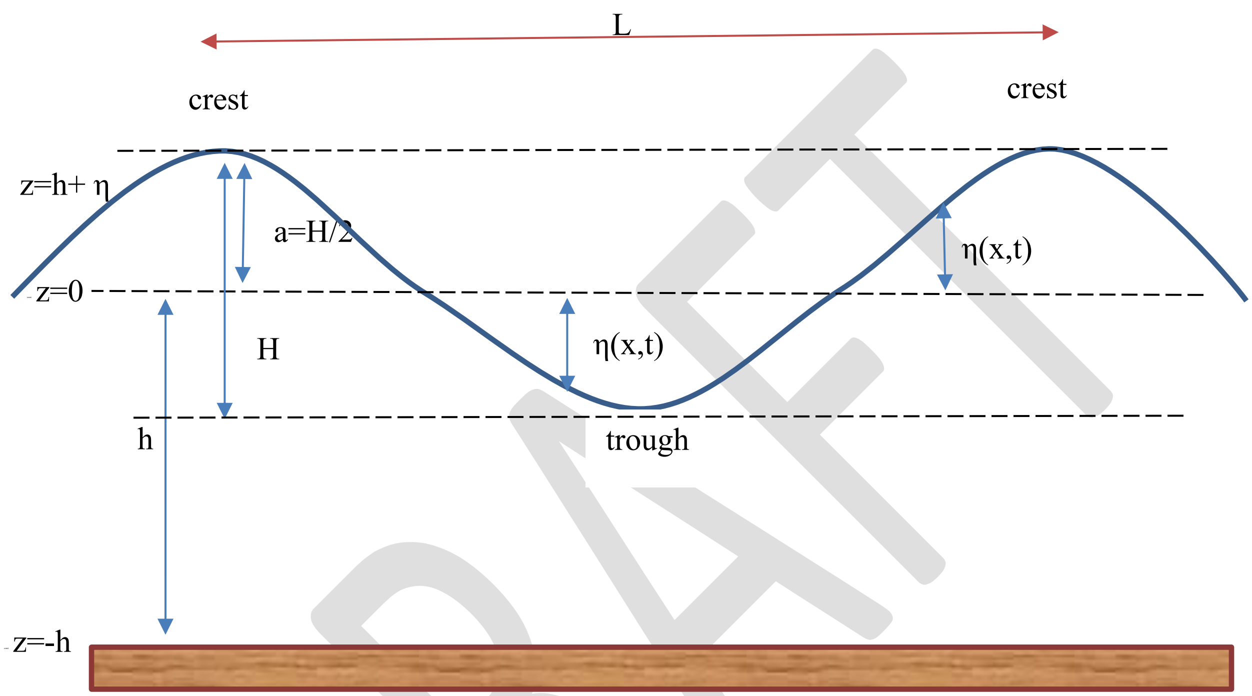


Figure 1: Diagram of a wave

h=still water depth

Relative water depth $\frac{h}{L}$ $or\ kh = 2\pi\frac{h}{L}$

Wave Steepness $\frac{H}{L}$

Relative Wave Height $\frac{H}{h}$

Ursell Number $U_r = \left(\frac{L}{h}\right)^2\frac{H}{h} = \frac{L^2H}{h^3}$ (measure of nonlinearity)

- Airy- $(linear\ wave)$
- Stokes- $(higher\ order\ nonlinear)$
- Cnoidal $\frac{h}{L} < \frac{1}{8}, U_r > 20\ (Long\ Wavelength)$
- Solitary $(Infinite\ Wavelength)$ limiting case of cnoidal

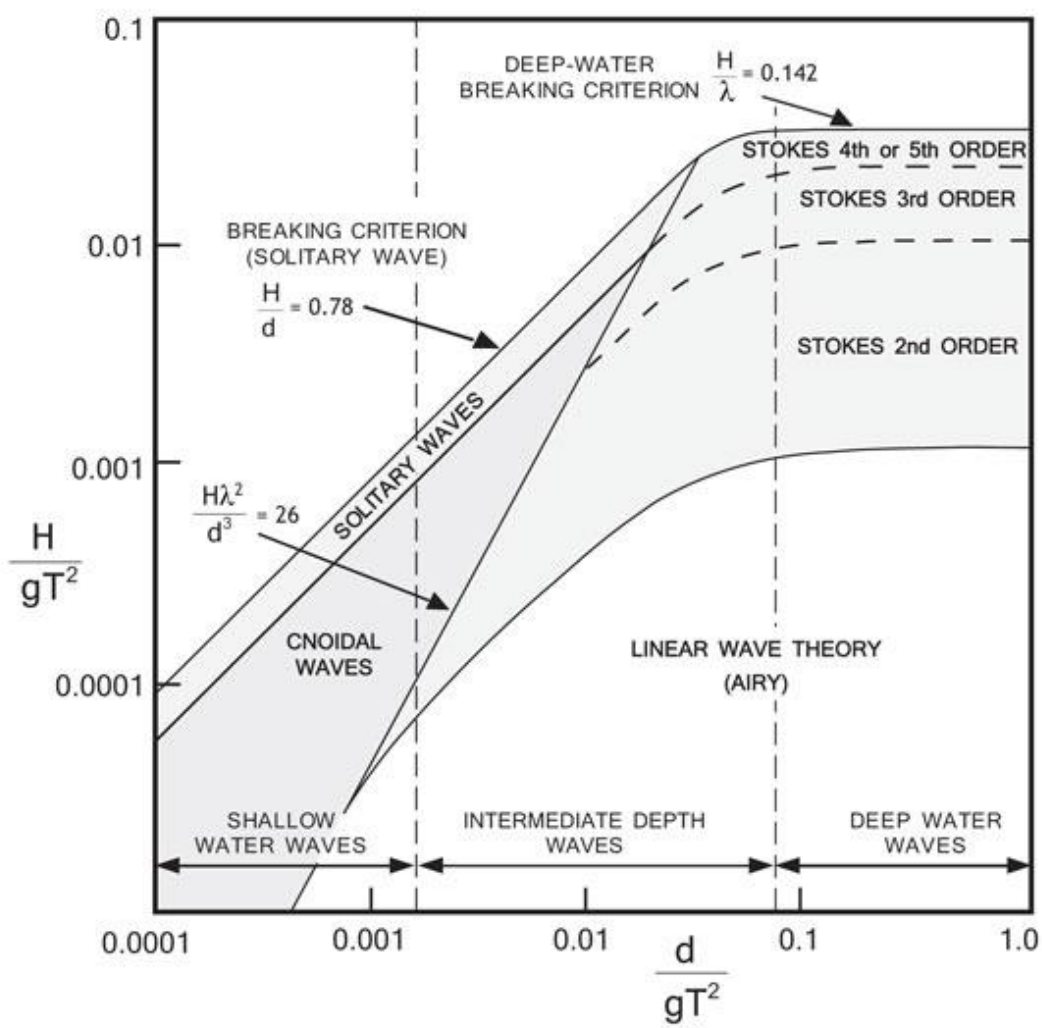
2.3. Waves in nature

Rarely look like what we have drawn

- Vary in height, period and direction
- Water surface (free surface elevation (eta)) is the superposition of multiple waves

2.4. Wave theories

- Airy (1845) linear wave theory
- Stokes (1847) higher order theory
- Boussinesq (1872) long wave theory



2.5. Interests in Water waves

- World War II
- Military
- Petroleum Industry
- Coastal Erosion
- Shipping
- Ocean Energy

2.6. Reading:

Varying levels of complexity

- Butt and Russell, *Surf Science*, University of Hawaii Press, 2002
- CEM (Coastal Engineering Manual) US Army Corps of Engineers Part II.
- Holthuijsen, *Waves in Oceanic and Coastal Waters,* Cambridge Press, 2007
- Ochi, *Ocean Waves the Stochastic Approach*, Cambridge Press, 1998

Exercise 1: Define in your own words Freak Wave or Rogue Wave. Find an example of a freak wave or rogue wave, reference the source and summarize your findings. Include any interesting information on the circumstances including wave height and damage.

3. Introduction & Mathematics Review

3.1. Introduction to Mathematics and Fluids Review

This course is focused on the physical characteristics of water waves; however in order to understand wave mechanics, we must first understand the physical characteristics of fluid flows. As physics is applied mathematics, we must be proficient in the mathematics if we are to learn the physics of fluids and waves. This section of the course is aimed at laying out the essential tools needed to be successful.

3.2. Taylor Series

If a continuous function $f(x)$ of the independent variable x, is known at some location, $x = x_0$, then it can be approximated at another location on the x-axis, $x = x_0 + \Delta x$, by the Taylor series:

$$f(x_0 + \Delta x) = f(x_0) + \frac{\partial f(x_0)}{\partial x}\Delta x + \frac{\partial^2 f(x_0)}{\partial x^2}\frac{\Delta x^2}{2!} + \cdots + \frac{\partial^n f(x_0)}{\partial x^n}\frac{\Delta x^n}{n!}$$

Where all the derivatives are taken at $x = x_0$, and "n", is the desired order of accuracy. For very small values of Δx, the terms involving $(\Delta x)^n$, where $n > 1$, are much smaller than the first two terms on the right hand side. We "linearize" the Taylor series by assuming $\Delta x \ll 1$, keeping the first two terms on the right hand side and neglecting the higher order terms (these become our error). The Taylor series is a ***very*** useful and handy mathematical tool (for developing relationships between fluid properties at two closely spaced locations).

3.3. Useful Derivative and Integration Techniques

- **Chain Rule:**

$$\frac{d}{dx}\cos(u) = (-\sin u)\frac{du}{dx}$$

- **Integration by Parts:**

$$\int u\,dv = uv - \int v\,du$$

- **Even Odd functions**
- Even Function: $f(-x) = f(x)$

  Cosine is an even function

  Function is symmetric about an axis
- Odd Function: $f(-x) = -f(x)$

  Sine is an odd function

  Function is asymmetric about an axis (opposite responses on either side of an axis)

- **Liebnitz Rule**

The Liebnitz Rule allows us to differentiate integrals in which the limits of integration are also functions of one of the independent variables. Thus, for a function $f(x,t)$:

$$\frac{d}{dt}\int_{A(t)}^{B(t)} f(x,t)\,\mathrm{dx} = \int_{A(t)}^{B(t)} \frac{d}{dt} f(x,t)\,\mathrm{dx} + \mathrm{B}'(\mathrm{t})\, f(\mathrm{B(t)},t) - \mathrm{A}'(\mathrm{t}) f(\mathrm{A(t)},t)$$

Where the primes denote differentiation.

Example: Let $I(t) = \int_{-t}^{t^2} \cos(tx)\,\mathrm{dx}$

Then by the Liebnitz Rule,

$$\frac{d}{dt} I(t) = \int_{-t}^{t^2} \frac{d}{dt}\cos(tx)\,\mathrm{dx} + \frac{d}{dt} t^2 \cos((t^2)t) - \frac{d}{dt}(-t)\cos((-t)t)$$

$$\frac{d}{dt} I(t) = -\int_{-t}^{t^2} x \sin tx\,\mathrm{dx} + 2t\cos((t^2)t) - (-1)\cos((-t)t)$$

$$\cos is\ an\ even\ function:$$

$$\frac{d}{dt}I(t) = -\int_{-t}^{t^2} x \sin tx \,\mathrm{dx} + 2t\cos(t^3) + \cos(t^2)$$

$$= -\int_{-t}^{t^2} x \sin tx \,\mathrm{dx} + 2t\cos(t^3) + \cos(t^2)$$

Using integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

$$u = x$$

$$du = 1\,dx$$

$$dv = -\sin(tx)\,dx$$

$$v = \frac{1}{t}\cos(tx)$$

$$-\int_{-t}^{t^2} x\sin(tx)\,\mathrm{dx} = x\frac{1}{t}\cos(tx) - \int \frac{1}{t}\cos(tx)\,(1)dx$$

$$\therefore -\int_{-t}^{t^2} x \sin tx \,\mathrm{dx} + 2t\cos(t^3) + \cos(t^2) =$$

$$x\frac{1}{t}\cos(tx) - \int \frac{1}{t}\cos\,(tx)(1)dx + 2t\cos(t^3) + \cos(t^2)$$

$$= \frac{\mathrm{x}}{t}\cos(tx) - \frac{1}{t^2}\sin\,(tx)|_{-t}^{t^2} + 2t\cos(t^3) + \cos(t^2)$$

$$= \frac{\mathrm{x}}{t}\cos(tx) - \left[\frac{1}{t^2}\sin(t^3) - \frac{1}{t^2}\sin(-t^2)\right] + 2t\cos(t^3) + \cos(t^2)$$

$$= \frac{\mathrm{x}}{t}\cos(tx) - \left[\frac{1}{t^2}\sin(t^3) + \frac{1}{t^2}\sin(t^2)\right] + 2t\cos(t^3) + \cos(t^2)$$

$$= \frac{\mathrm{x}}{t}\cos(tx) - \left[\frac{1}{t^2}\sin(t^3) + \frac{1}{t^2}\sin(t^2)\right] + 2t\cos(t^3) + \cos(t^2)$$

3.4. Complex Numbers and Relationships to Trigonometric Functions

- **Circular**

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{e^{iz} - e^{-iz}}{2i} = \frac{\sinh(iz)}{i}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz)$$

- **Hyperbolic**

- Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$for\ x \ll 1,\ \ \sinh x = x\,; for\ x \gg 1,\ \ \sinh x = \frac{e^x}{2}$$

$$\sinh x = -\mathrm{i}\sin \mathrm{i}x$$

- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$for\ x \ll 1,\ \ \cosh x = 1\,; for\ x \gg 1,\ \ \cosh x = \frac{e^x}{2}$$

$$\cosh x = \cos \mathrm{i}x$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$for\ x \ll 1,\ \ \tanh x = x\,; for\ x \gg 1, \tanh x = 1$$

$$\tanh x = -\mathrm{i}\tan \mathrm{i}x$$

3.5. Vector Analysis:

We will also be referencing Vector Algebra throughout the course. A vector in three dimensions is shown below along with the x, y, and z unit vectors, **i, j**, and **k** respectively:

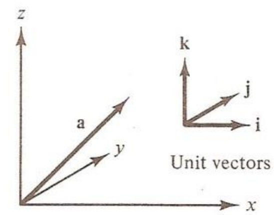


Figure 2.7 Unit vectors in a Cartesian coordinate system.

In a coordinate system we can describe a vector in terms of its components. In the 3-Dimensional Cartesian Coordinate system, the unit vector (a vector of unit length 1) is denoted, $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$,or $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ in the (x,y,z) directions. The vector $\boldsymbol{a}$ can be written:

$$\boldsymbol{a} = a_x\hat{\mathbf{\imath}} + a_y\hat{\mathbf{\jmath}} + a_z\widehat{\boldsymbol{k}}$$

Where a_x, a_y, a_z, are projections of the vector $\boldsymbol{a}$**,** along the (x,y,z) axes.

The velocity vector **u** can be written:

$$\boldsymbol{u} = u\mathbf{i} + v\mathbf{j} + w\boldsymbol{k}$$

- ### Dot Product (orthogonality)

  A measure of the projection of one vector onto another, if the vectors are co-directional then the cosine function is equal to one and the dot product is the product of the magnitudes of the vectors.
  If the vectors are perpendicular then the dot product will be zero since there is no component of one vector projected onto the other.

  The dot product is a scalar product of vector arguments, and is defined as:

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |a|\,|b| \cos\theta$$

  Where θ, is the angle between vectors a and b.

  If $\theta = (2n + 1)90°$ (or $(2n + 1)\frac{\pi}{2}$ $radians$) then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular (orthogonal). (Neither vector projects onto the other)

  The following identities for the unit vectors are readily derived:

$\mathbf{i} \cdot \mathbf{i} = 1$ $\mathbf{j} \cdot \mathbf{j} = 1$ $\mathbf{k} \cdot \mathbf{k} = 1$

$\mathbf{i} \cdot \mathbf{j} = 0$ $\mathbf{j} \cdot \mathbf{k} = 0$ $\mathbf{i} \cdot \mathbf{k} = 0$

The dot product is commutative ($\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i}$)

$$\boldsymbol{a} \cdot \boldsymbol{b} = |a|\,|b| \cos\theta = a_x b_x + a_y b_y + a_z b_z$$

The dot product is used to determine the projection of one vector onto another (or the orientation of one vector with respect to another).

In general, the projection of $\boldsymbol{a}$ onto $\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{|a|\,|b| \cos\theta}{\sqrt{\boldsymbol{b} \cdot \boldsymbol{b}}}$

And the angle, θ, between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is found as:

$$\boldsymbol{\theta} = \boldsymbol{arccos} \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

- **Cross Product**

The magnitude of the cross product is equal to the ***area of the parallelogram described by the two vectors***. Zero when the vectors are collinear and at a maximum when the vectors are perpendicular.

The cross product is a vector product of vector arguments. Also called the vector product, the magnitude is defined as:

$$\boldsymbol{a} \times \boldsymbol{b} = |a|\,|b| \sin\theta$$

The direction of the resultant is defined according to the "right hand rule" in a direction perpendicular to the plane defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
For unit vectors

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

The Cross product is not commutative,

$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

A useful method for determining the cross product is to use a determinant form:

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z}\\ \boldsymbol{b_x} & \boldsymbol{b_y} & \boldsymbol{b_z}\end{vmatrix}=(\boldsymbol{a_y b_z}-\boldsymbol{a_z b_y})\mathbf{i}+(\boldsymbol{a_z b_x}-\boldsymbol{a_x b_z})\mathbf{j}+(\boldsymbol{a_x b_y}-\boldsymbol{a_y b_x})\mathbf{k}$$

If the cross product is zero and the vectors ***a*** and ***b*** are nonzero then the two vectors must be parallel or collinear.
Note that both the dot product and the cross product operate on vectors with arbitrary orientation, and are referenced to our Cartesian coordinate system.

3.6. Vector Differential Operators and the Gradient

In this section we introduce the vector differential operations, these allow derivatives of scalars and vectors to be taken in three dimensional space.

$$\boldsymbol{\nabla}\equiv\frac{d}{dx}\boldsymbol{i}+\frac{d}{dy}\boldsymbol{j}+\frac{d}{dz}\boldsymbol{k}$$

Where $\boldsymbol{\nabla}$ is referred to as the gradient or del operator.

3.6.1 Gradient

The Gradient or gradient vector operating on a scalar field, S, is written as grad S, or $\boldsymbol{\nabla}$S, and is given by:

$$\boldsymbol{\nabla}\mathrm{S}=\left(\frac{d}{dx}\mathbf{i}+\frac{d}{dy}\mathbf{j}+\frac{d}{dz}\mathbf{k}\right)S(x,y,z)=\left(\frac{\partial S}{\partial x}\mathbf{i}+\frac{\partial S}{\partial y}\mathbf{j}+\frac{\partial S}{\partial z}\mathbf{k}\right)$$

The Gradient of a scalar is a vector, and it indicates the direction of maximum change of the scalar field.

$$\left(\frac{\partial S}{\partial x}\mathbf{i}+\frac{\partial S}{\partial y}\mathbf{j}+\frac{\partial S}{\partial z}\mathbf{k}\right)=\vec{\boldsymbol{S}}$$

If we know the scalar value at one point in the domain, then by using the Taylor series we can estimate the value at some nearby location.

$$f(x+\Delta x,y+\Delta y,z+\Delta z)=f(x,y,z)+\frac{\partial f(x,y,z)}{\partial x}\Delta x+\frac{\partial f(x,y,z)}{\partial y}\Delta y+\frac{\partial f(x,y,z)}{\partial z}\Delta z$$

We can rewrite the right hand side of the equation as:

$$= f(x,y,z) + \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot (\Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k})$$

The second term is the gradient of the function *f*. and the third term is the differential vector $\Delta\mathbf{r}$.

$$= f(x,y,z) + \boldsymbol{\nabla} f(x,y,z) \cdot \Delta\boldsymbol{r}(x,y,z)$$

3.6.2 Divergence

By applying the differential operator to a vector using the dot product we obtain the divergence of that vector, a scalar value:

The divergence of a vector field $\boldsymbol{u}(x,y,z)$is given by:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\boldsymbol{k}\right) = \left(\frac{du_x}{dx} + \frac{du_y}{dy} + \frac{du_z}{dz}\right)$$

$$= \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

The divergence of a gradient, del squared, ∇^2, is a special operator known as the Laplacian.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\mathrm{S} = \left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right) = \nabla^2\mathrm{S}$$

This operator will show up again later when we set up the boundary value problem for water waves.

3.6.3 Curl

If instead we apply the differential operator to a vector using the cross product, we obtain the curl of the vector.

$$\boldsymbol{\nabla} \times \mathbf{u} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\boldsymbol{k}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}$$

The curl is a measure of the rotation of in a vector field.

The curl of a vector is a vector.
The Divergence of the curl is zero:

$$\nabla \cdot (\nabla \times \mathbf{u}) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] = 0$$

3.7 Line Integrals

If we have two points in a plane, say the x-y plane, and a vector exists over this plane, ***a***(x,y). If we integrate the vector ***a*** along a contour line that connects the two points, we would anticipate that the value of that integral will depend on that path the contour line takes from one of the points to the other.

$$F = \oint_{P0}^{P1} \boldsymbol{u} \cdot d\boldsymbol{l}$$

We would like to be able to integrate the vector between the two points and have the integral be independent of the path we take.

It turns out that if the integrand is an exact differential than the solution will only be a function of the end points and not the contour path taken between those end points:

$$F = \oint_{P0}^{P1} dF$$

If we now impose that:

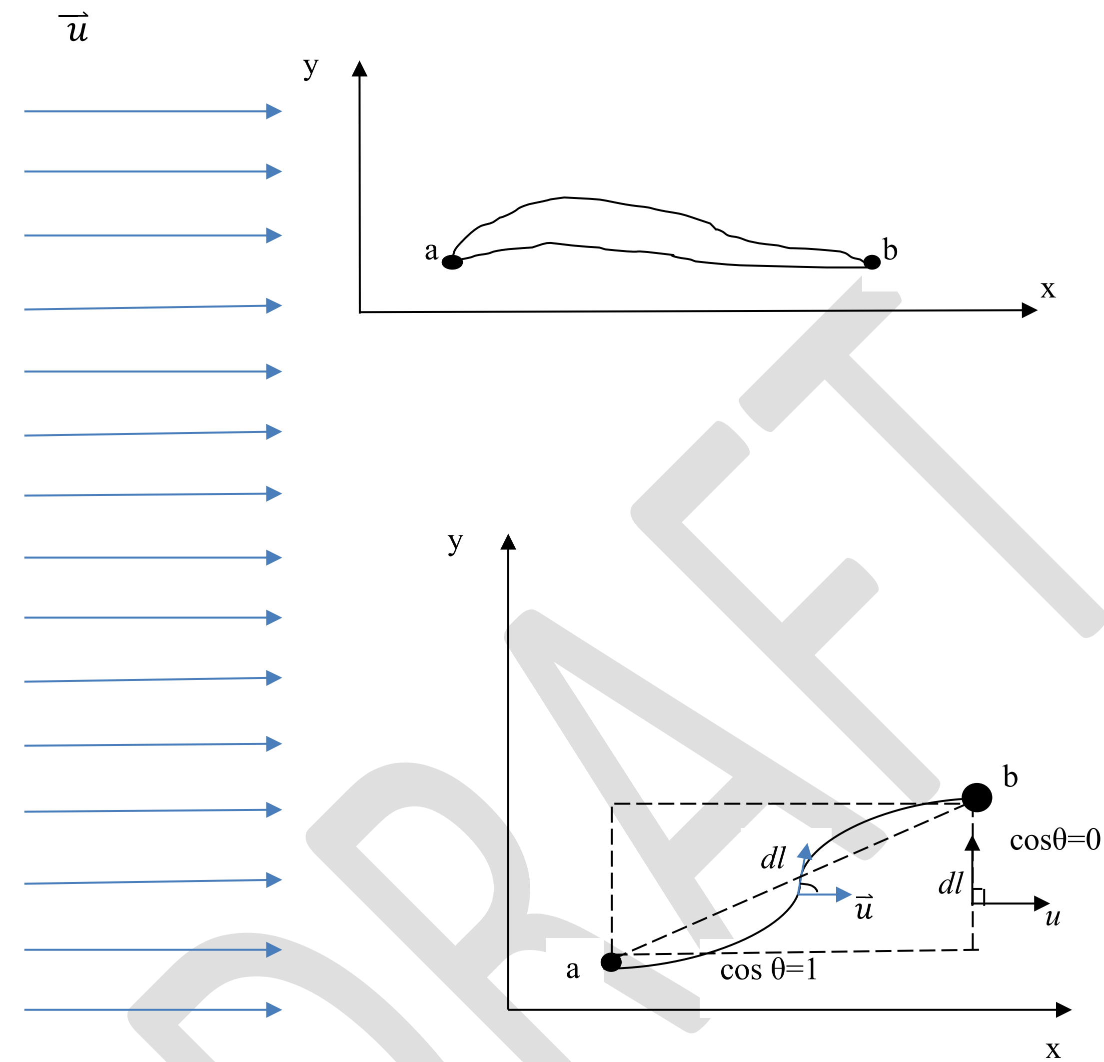
$$\boldsymbol{u} \cdot d\boldsymbol{l} = dF$$

(or the velocity component in the along contour direction equals dF)

Then we should have independence of path.

Conceptually, if we only add the component of the velocity that is parallel to the contour then the total sum can only be the integrated velocity from point a to the line perpendicular to the velocity passing through point b.

Line integral: Looking down (Plan view)



Integrating (sum) $\vec{u} \cdot d\vec{l}$ = |u||dl|cosθ, the projection of u onto dl

If Mean flow is || to contour then, cos θ=1

If Mean flow is perpendicular to contour then, cos θ=0

Only the along contour flow component is counted

Mathematically we express this dot product as:

$$\boldsymbol{u} \cdot d\boldsymbol{l} = u_x dx + u_y dy = dF \ and$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \nabla F \cdot d\boldsymbol{l}$$

By inspection we see that independence of path requires:

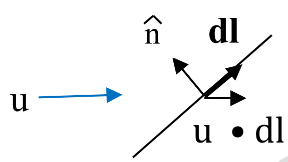
$$u_x = \frac{\partial F}{\partial x} \text{ and } u_y = \frac{\partial F}{\partial y} \text{ or } \boldsymbol{u} = \nabla F$$

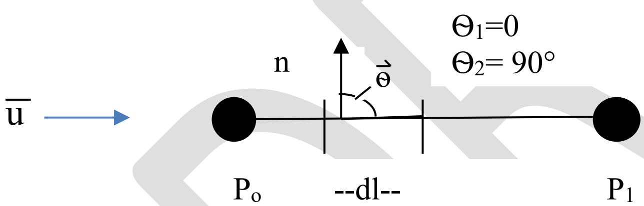
Therefore, since

$$\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} - \frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial x \partial y} = 0$$

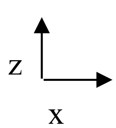
We see that in order for independence of path the curl of $\boldsymbol{u}$ must be zero.





No fluid crosses the contour P_0-P_1 all flow is along the contour

$$u \bullet dl = |u||dl| \cos\theta_1 = u$$
$$u \bullet ndl = |u||ndl| \cos\theta_2 = 0$$
since n is $\perp$ to path of integration



in an x-z preferred frame, $n = \frac{-dzi + dxk}{dl}$

3.8 Fourier Analysis

Any piecewise continuous function can be represented over a given time interval as a sum of sines and cosines. The time t is arbitrary and it is assumed that the function periodic over the time period T.

The Fourier series is given as:

$$f(t) = \sum_{n=0}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\sigma t + b_n \sin n\sigma t)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos (nt) dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin (nt) dt$$

Or in Complex notation:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

$$C_n = \frac{1}{2}(a_n - ib_n)$$

4. Basic Hydrodynamics

4.1. Def Fluid

A fluid is a substance that constantly deforms under shear stresses

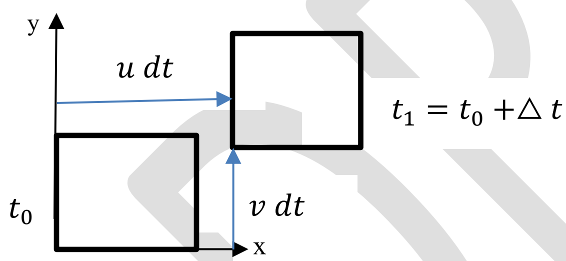
4.2. Mass conservation

- Unit parcel of fluid

a) Derive net rate of mass accumulation inside control volume
b) Derive accumulation of mass over time
c) These must be equal
d) Continuity equation
e) Incompressibility of water
f) Conservation of mass equation for an incompressible fluid

4.3 Types of Fluid motion

a) Translation: movement without strain or rotation (velocity gradients are zero)

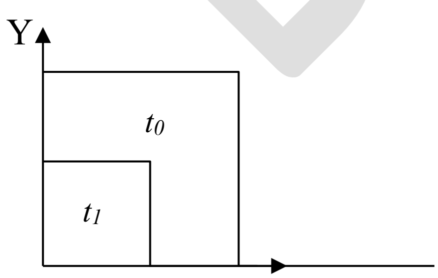


Relative motion depends on velocity

Velocity gradients = 0

$$\frac{\partial u}{\partial x} = 0;\ \frac{\partial u}{\partial y} = 0 \qquad etc$$

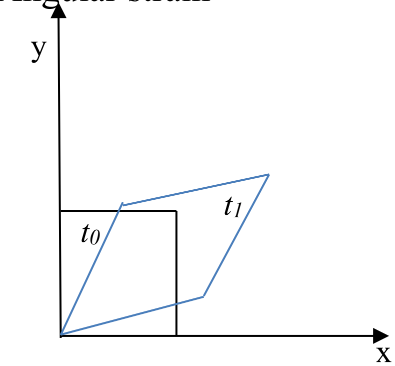
b) Linear Strain
(volume expansion/contraction)



If $\nabla \cdot \mathbf{u} = 0$, volume is conserved
Fluid is incompressible
Non-divergent flow

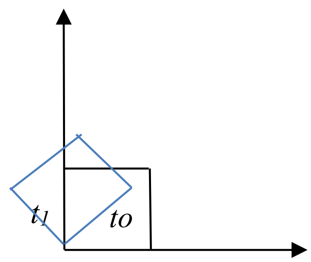
$$\frac{1}{\partial \forall}\frac{\partial}{\partial t}(\partial \forall) = \nabla \cdot \mathrm{u}$$

c) Angular strain



d) Rotation

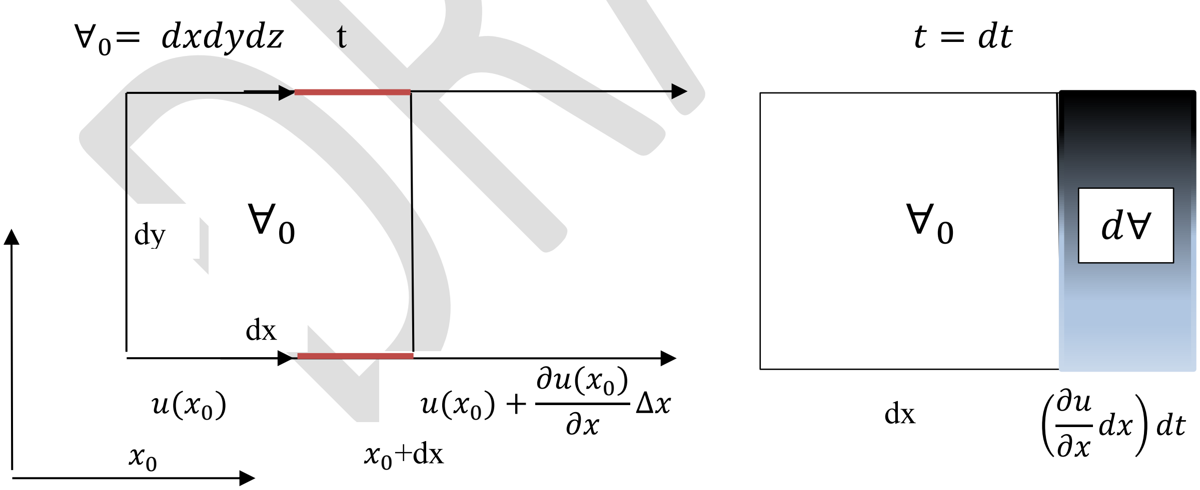
e)



4.3.1 Linear Strain

(Beginnings of Mass Continuity)

Consider a purely x directed flow



The volume increases as the change in velocity times the time change, dt

Change in Volume, $d\forall = \left(\left(\frac{\partial u}{\partial x}dx\right)dt\right)(dydz)$

Look at time rate of change per unit volume:

$$\frac{1}{\Delta\forall}\frac{\partial}{\partial t}(d\forall) = \frac{1}{\Delta\forall}\frac{\partial u}{\partial x}\Delta x\Delta y\Delta z$$

$$= \frac{1}{\Delta x\Delta y\Delta z}\frac{\partial u}{dx}\Delta x\Delta y\Delta z = \frac{\partial u}{\partial x} for\ x\ direction$$

Strain is the increase in the length of a side of the fluid as a fraction of the original length, so the change in the length divides by the original length and we are left with,

$\frac{\partial u}{\partial x}dt$. Now convert this into a rate, divide by dt.

For the x direction, the change in volume is $\frac{\partial u}{\partial x}$. Volume change is equal to the velocity gradient and the velocity gradient is also equal to the linear strain rate in the x direction.

Similarly for y and z directions

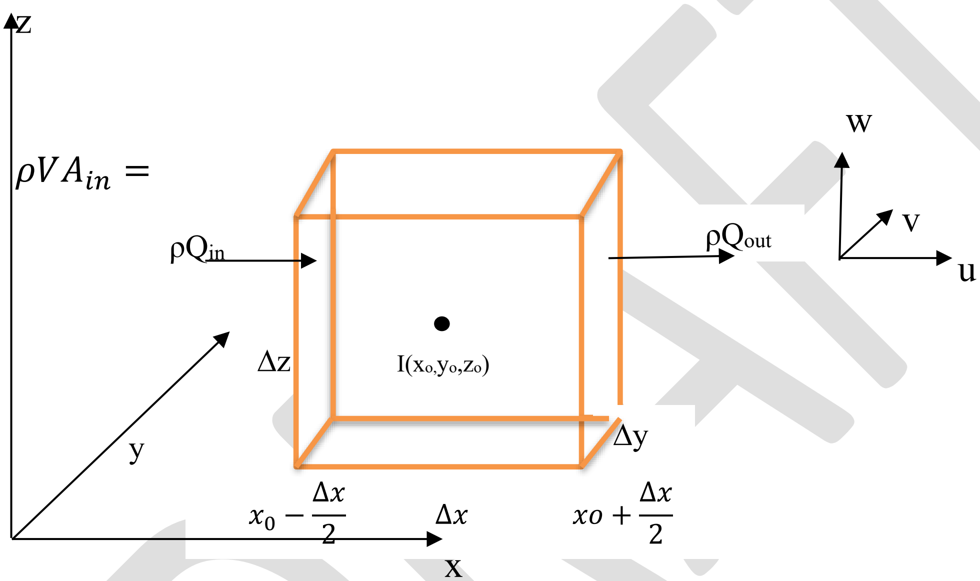
4.4 Conservation of Mass

-we rely on principal that mass is neither created nor destroyed. There is continuity in the amount of mass in the system

-We can track the changes of mass within a particular system by accounting for influx and efflux of mass through the boundary

Assume that we know the fluid properties at point (x_0, y_0, z_0).

Let us examine a reference parcel of fluid about point (x_0, y_0, z_0); a cube oriented along the x, y and z axes with side lengths $\Delta x, \Delta y, \Delta z$.



The rate at which fluid mass flows in is equal to the velocity components in the x-directions multiplied by the area of the face of the cube ⊥ to flow, times the density of the fluid (ρ)

Solving for the mass inflow at x= $x_o - \frac{\Delta x}{2}$:

$$\rho\left(x_o - \frac{\Delta x}{2}, y_o, z_o\right) \underbrace{u\left(x_o - \frac{\Delta x}{2}, y_o, z_o\right) \Delta y \Delta z}_{}$$

[kg/m³] Q=discharge [m³/s] = [kg/s]

Mass rate

Using Taylor series we can write the mass flux on the face of the cube as a function of the fluid properties at the center of the cube

In: $\rho\left(x_o - \frac{\Delta x}{2}, y_o, z_o\right)\, u\left(x_o - \frac{\Delta x}{2}, y_o, z_o\right)\Delta y \Delta z =$

$$\left[\rho(x_o, y_o, z_o)u(x_o, y_o, z_o) - \frac{\partial(\rho(x_o, y_o, z_o)u(x_o, y_o, z_o)}{\partial x}\frac{\Delta x}{2} + \frac{\partial^2\left(\rho(x_o, y_o, z_o)u(x_o, y_o, z_o)\right)}{\partial x^2}\frac{(\frac{\Delta x}{2})^2}{2!} + H.O.T.\right]\Delta y \Delta z$$

Set,

$$O(\Delta x^2) = \frac{\partial^2(\rho u)}{\partial x^2}\frac{(\frac{\Delta x}{2})^2}{2!}$$

$$= \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y \Delta z$$

Out: at x= $x_o + \frac{\Delta x}{2}$ we obtain;

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2} + O(\Delta x^2)\right]\Delta y \Delta z$$

By subtracting the mass flow rate *out* from mass flow rate *in* we obtain the ***net*** flux in the x direction

$$-\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z + higher\ order\ terms$$

Assuming $\Delta x, \Delta y, \Delta z\ are\ of\ same\ order$; H.O.T. = $O(\Delta x^2)\Delta y \Delta z$ are $O(\Delta x^4)$

Following same procedure for y and z directions we obtain the net flux across all six faces:

Mass flux - $\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\Delta x \Delta y \Delta z + H.O.T.O(\Delta x^4)$

Now consider accumulation of mass over the small time increment Δt:

Mass in cube at time $t_{o:}$

$\rho(t_o)\ \Delta x\Delta y\Delta z$

Mass at time $t_o+\Delta t$:

$\rho(t_o+\Delta t)\ \Delta x\Delta y\Delta z$

The net mass accumulation in cube over time Δt:

$$[\rho(t_o+\Delta t)-\rho(t_o)]\Delta x\Delta y\Delta z =$$

{Expand by Taylor series}

$$\cancel{\rho(t)}+\frac{\partial\rho}{\partial t}\Delta t-\cancel{\rho(t)} = [\frac{\partial\rho}{\partial t}\Delta t + H.O.T.]\Delta x\Delta y\Delta z$$

The time accumulation (increase/decrease) in mass ***must*** equal net mass inflow accumulation over time Δt

$$\left[\frac{\partial\rho}{\partial t}\Delta t + HOT\right]\Delta x\Delta y\Delta z = -\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial t}\right]\Delta x\Delta y\Delta z\Delta t + H.O.T.$$

$$\left[\frac{kg}{\cancel{m^3}}\frac{\cancel{s}}{\cancel{s}}\cancel{mm}m\right] \qquad \left[\frac{kg}{\cancel{m^3}}\frac{\cancel{m}}{\cancel{s}}\frac{1}{\cancel{m}}\right]\left[\cancel{mmms}\right]$$

Divide by ΔxΔyΔzΔt, neglect H.O.T., and allow the time increments and the size of the cube to approach zero, we obtain the *exact* equation:

$$\frac{\partial\rho}{dt}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}=0$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0$$

To better understand:

Now expand product terms

$$\frac{\partial\rho}{\partial t}+\left(\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}\right)+\left(\rho\frac{dv}{dy}+v\frac{\partial\rho}{\partial y}\right)+\left(\rho\frac{\partial w}{\partial z}+w\frac{\partial\rho}{\partial z}\right)=0$$

Group terms with ρ

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Note: $\left(\frac{\partial\rho}{\partial t}+\frac{\partial x}{\partial t}\frac{\partial\rho}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial\rho}{\partial y}+\frac{\partial z}{\partial t}\frac{\partial\rho}{\partial z}\right)+\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\vec{u}=0$

$= \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u}$ *total or material derivitive*

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$1/_{s} + \left[m/_{s} \ \frac{1}{m}\right] = [\tfrac{1}{s}] = \text{Rate of Strain}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{u} = 0$$

We have shrunk cube to zero, every particle entering the cube is the cube, and as such our results found using the cube approach apply everywhere, this is the continuum hypothesis.

The first term, $\frac{1}{\rho}\frac{D\rho}{\partial t}$ governs the change in density of the fluid.
The second term $\nabla \cdot \vec{u}$ governs the flow of the fluid into control volume

Continuity says that the total flow into a fixed volume must be matched by a change in density. This can only happen if the fluid is compressible.
The Bulk Modulus of water, $E = \frac{\partial P}{\partial \rho/\rho}$ is large

$E_{water} = 2.07\ x\ 10^9$ N/m^2

We assume that water in incompressible therefore ρ is constant and $\frac{D\rho}{Dt} = 0$
Leaving us with:
$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ in order for mass conservation, flow must be ; $\nabla \cdot \vec{u} = 0$ non divergent

Divergence is zero, divergenceless

We have **not** eliminated stratified flow or density variations; however we assume that *each fluid particle has a constant density* as it moves through the flow.

Density variations are a result of mixing heavier/lighter fluids in the flow.

In this course we do not consider density driven flows.

Example: Show that water is essentially incompressible.
At what depth is there sufficient pressure to compress H_2O such that the density of the water has changed by 1%?

Given: $E_{v_{water}} = 2 * 10^9 Pa,$
$\frac{\Delta\rho}{\rho} = 1\% = 0.01$

Find: depth (h)

Sol'n:

$$\mathrm{E_v} = \frac{\partial p}{\partial\rho/\rho} \rightarrow \Delta p = E_v \frac{\Delta\rho}{\rho}$$
$$\Delta p = 2 * 10^9\ Pa\ (0.01)$$
$$\Delta p = 2 * 10^7\ Pa$$
$$p_h - p_0 = 2 * 10^7 Pa$$

Using gage pressures, $p_0 = p_{ATM} = 0_{gage}$ $\therefore p_0 = 0$
And knowing that, $p = \rho g\ h$, the formula for hydrostatic pressure (we will learn this soon!!!)

$$p_h - 0 = 2 * 10^7 Pa$$
$$\rho g\ h = 2 * 10^7 Pa$$

Taking $\rho = 1000 \frac{kg}{m^3}; g = 10\ m/s^2$

$$1 * 10^4 \frac{kg}{m^2 s^2}\ \ h = 2 * 10^7 Pa$$
$$h = 2 * 10^3 \frac{Pa\ m^2 s^2}{kg}$$

PA =pressure=force per unit area=$\frac{N}{m^2} = kg\ m/s^2 m^2$

$$h = 2 * 10^3 \frac{kg \frac{m\ m^2 s^2}{s^2 m^2}}{kg} = 2 * 10^3\ m$$

4.5 Conservation of momentum

Mass conservation governs mass accumulation and depletion, but offers no information on the flow itself. For this we need conservation of momentum, Newton's Second Law of Motion:
Total force equals the time rate of change of momentum

$$\vec{\boldsymbol{F}} = \frac{d\rho\vec{\boldsymbol{u}}}{dt} = m\vec{\boldsymbol{a}}$$

Where the acceleration ***a***, is given as:

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{dt}$$

Where ***u*** is a function of x, y, z, and t. Also x, y, and z are functions of t as well, so we must use the chain rule to accurately represent ***a***.

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{dt}$$

$$= \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\partial z}{\partial t}$$

$$where\ \frac{\partial x}{\partial t} = u, \frac{\partial y}{\partial t} = v, \frac{\partial z}{\partial t} = w$$

$$= \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} + w\frac{\partial \boldsymbol{u}}{\partial z}$$

The acceleration is a combination of the local acceleration and advection.

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

So we say:

$$\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt}$$

Where:

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + u\frac{\partial(\,)}{\partial x} + v\frac{\partial(\,)}{\partial y} + w\frac{\partial(\,)}{\partial z} = \frac{\partial(\,)}{\partial t} + (\boldsymbol{u}\cdot\nabla)(\,)$$

Is the *Material Derivative*

We have developed equation for the RHS of the eq. but need the LHS

$$\vec{\boldsymbol{F}} = \frac{d\rho\vec{\boldsymbol{u}}}{dt} = m\vec{\boldsymbol{a}}$$

To better understand this we examine the forces on a fluid particle.

4.6 Body forces and Buoyancy

Body Forces: act through the center or mass of fluid particle cube:

Gravity, Electro-Magnetic, radiative heat transfer

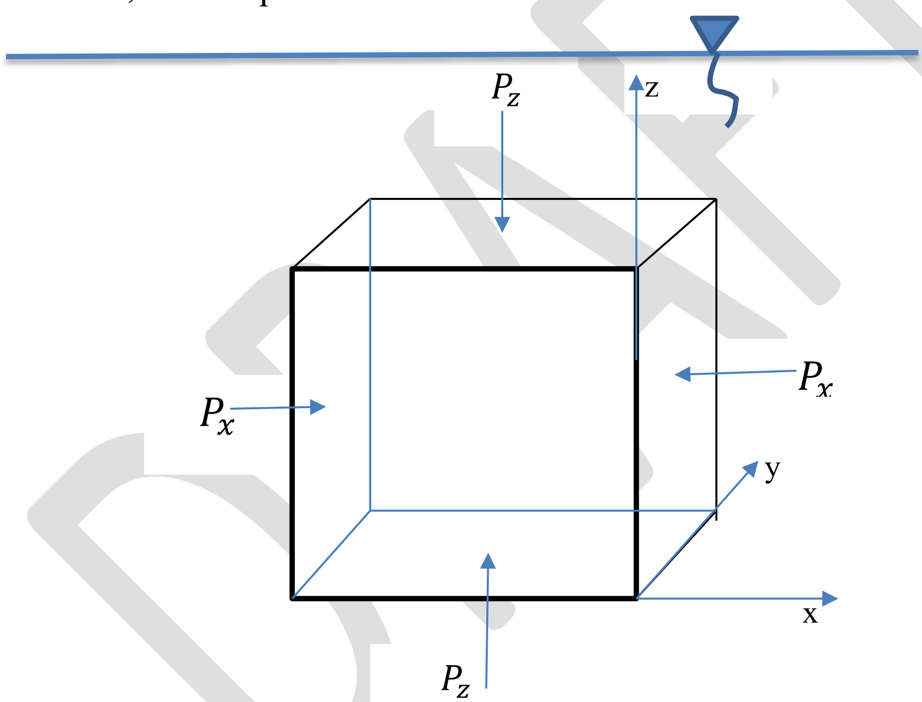
Inertial body forces: (Coriolis, centrifugal)

Surface forces: act on the surface of the cube:

Pressure and shear stress

Consider a fluid without motion: static, no shear stress

$\Sigma\vec{F} = m\vec{a} = 0$,Static equilibrium



We find that the pressure acting on the cube… P=-ρgz
Hydrostatic pressure varies linearly with depth, z

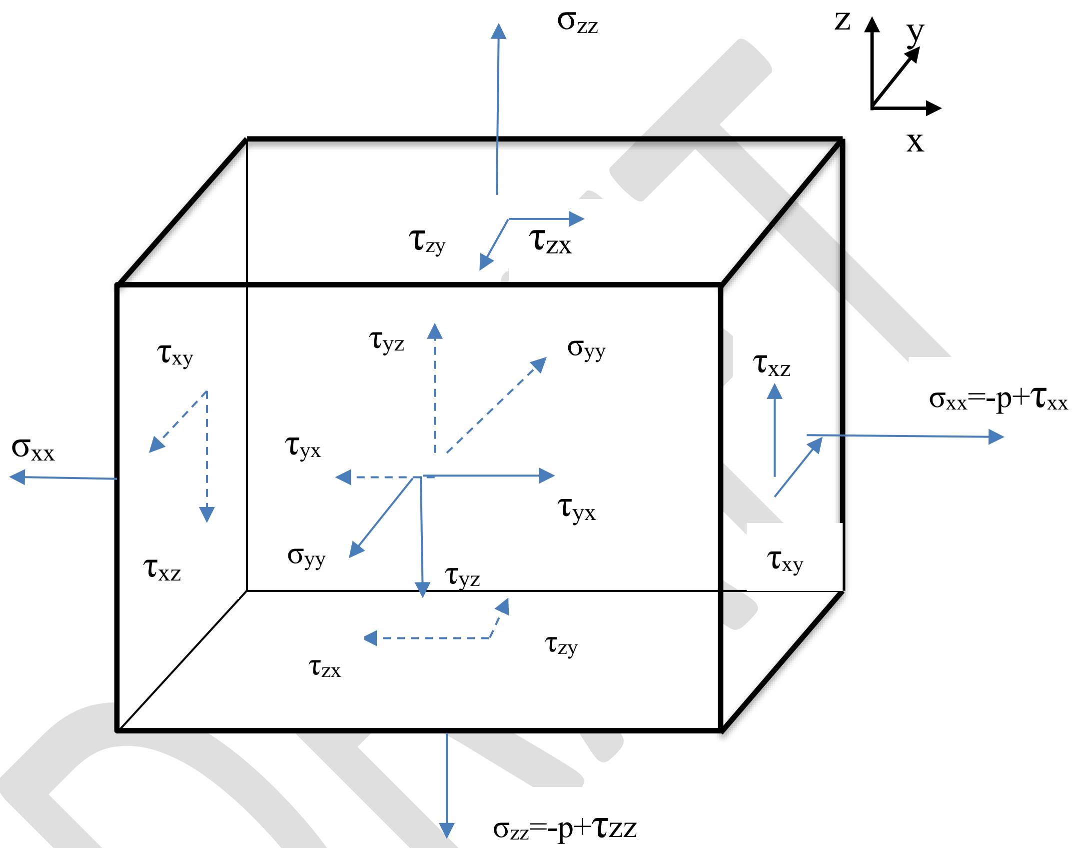
Exercise 2:

a) **Using your own diagram of a unit cube submerged in a fluid, Show that $P = -\rho g z$**
b) **Show that the equation for the Buoyant force is; Buoyant force:**

$$F_B = \rho_{fluid} g V$$

4.7 Shear Stresses

Let's examine the fluid cube again, this time with both normal and shear forces. Convention is that the 1st subscript refers to the axis normal to the plane on which the stress is acting, and the 2nd refers to the direction of the stress.



Each surface has 2 shear stresses, (τ_{ij}) and a normal stress (σ_{ii})

It can be shown that as the cube is shrunk to toward a point taken to be smaller and smaller, we end up with six stresses to consider.

$\sigma_{xx}=-p+\tau_{xx}$

τ_{xy}

τ_{xz}

τ_{yz}

$\sigma_{yy}=-p+\tau_{yy}$

$\sigma_{zz}=-p+\tau_{zz}$

Where:

$$p \equiv \frac{-(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3}$$

Where:

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -p + \tau_{xx} - \mathrm{p} + \tau_{yy} - \mathrm{p} + \tau_{zz}$$
$$p = -\left(\frac{-3p + \tau_{xx} + \tau_{yy} + \tau_{zz}}{3}\right)$$

Therefore: $(\tau_{xx} + \tau_{yy} + \tau_{zz}) = 0$, though each term does not need to be zero Conceptually, the volume of the cube cannot change, so and stretching in one direction must be accompanied by a compression in the other direction

p ≡isotropic $p_x=p_y=p_z=p$
τ≡ non-isotropic

Additionally it can be shown through the conservation of angular momentum that:

$$\tau_{xy} = \tau_{yx}; \tau_{xz} = \tau_{zx}; \tau_{yz} = \tau_{zy};$$

We can now sum Forces $\Sigma\vec{\boldsymbol{F}}$ in the x-direction and we have the acceleration $\vec{\boldsymbol{a}}$ so we can examine momentum in the x-direction.

$$\Sigma\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$
$$\Sigma F_x = m\frac{Du}{Dt}$$

4.8 Equations of Motion

We now are familiar with the forces and can sum forces in the x-direction

$$\Sigma F_x = m\frac{Du}{Dt}$$

We sum the surface forces and body forces using Taylor Series to apply forces at surfaces in terms of the value at the center of the cube $\left(x_0 + \frac{\Delta x}{2}, x_0 - \frac{\Delta x}{2}\right)$

$$\begin{aligned}
&\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z \\
&+\left(\tau_{yx} + \frac{\partial \tau_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\Delta z - \left(\tau_{yx} - \frac{\partial \tau_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\Delta z \\
&+\left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\Delta y + \left(\tau_{xz} - \frac{\partial \tau_{xz}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\Delta y \\
&+\rho\Delta x\Delta y\Delta z\boldsymbol{X} = \rho\Delta x\Delta y\Delta z\frac{Du}{Dt}
\end{aligned}$$

X refers to any and all body forces which may be present

Cancel and divide by ΔxΔyΔz to obtain:

$$\rho\frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho\boldsymbol{X}$$

$$\sigma_{xx} = -P + \tau_{xx}$$

SO, in all three coordinate directions we obtain:

$$\frac{Du}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) + \boldsymbol{X}$$

$$\frac{Dv}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + \boldsymbol{Y}$$

$$\frac{Dw}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \boldsymbol{Z}$$

Most general form of momentum equation; however, also unusable since we do not know the stresses…

We assume:

1. Assume stresses are zero, we know that only Body force is gravity (g) acting in the –z direction, so *Z*= -g

We obtain the Euler Equations

$$\frac{Du}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial x}; \qquad \frac{Dv}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial y}; \qquad \frac{Dw}{Dt} = \frac{-1}{\rho}\frac{\partial P}{\partial z} - g$$

$$or \ \frac{D\vec{\boldsymbol{u}}}{Dt} = \frac{-1}{\rho}\nabla P + \vec{\boldsymbol{F}}_{\boldsymbol{B}}$$

$$where \quad \overrightarrow{\boldsymbol{F}_{\boldsymbol{B}}} = -g\widehat{\boldsymbol{k}}$$

These equations are okay when we can assume μ=0 or inviscid flow

If not, we must use Navier-Stokes Equation, denoted by assuming a relationship between stresses, velocity gradient and viscosity.

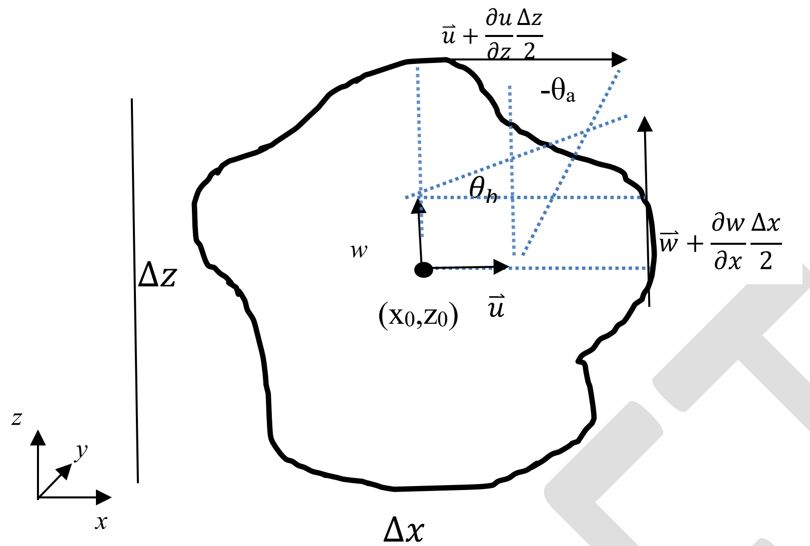
$$\tau_{xx} = \mu\frac{\partial u}{\partial x}; \qquad \tau_{xy} = \mu\frac{\partial u}{\partial y}; \qquad \tau_{xz} = \mu\frac{\partial u}{\partial z}$$

Stress= rate of strain

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

N.-S. equation requires viscous, laminar flow.
If turbulent, must use Reynolds averages N-S eq.

4.9 Vorticity



If we know the velocities at point (x_0, z_0), then at a small distance away, $\pm\frac{\Delta x}{2}, \pm\frac{\Delta z}{2}$ we find the velocities at the particle edge using the Taylor Series expansion.

Except in this case we are looking at the u velocity change not in the x direction but in the z-direction, and the w velocity change not in the z direction but in the x direction

$$u\left(x_0, z_0+\frac{\Delta z}{2}\right) = u(x_0,z_0) + \frac{\partial u(x_0,z_0)}{\partial z}\frac{\Delta z}{2}$$

$$w\left(x_0+\frac{\Delta x}{2}, z_0\right) = w(x_0,z_0) + \frac{\partial w(x_0,z_0)}{\partial x}\frac{\Delta x}{2}$$

The Angular Velocity θ_a is equal to the difference in the u velocity between the edge of the particle and the center divided by that distance between the center and the edge.

or $\vec{\theta}_a = -\frac{\vec{u}+\frac{\partial \vec{u}}{\partial z}\frac{\Delta z}{2}-\vec{u}}{\frac{\Delta z}{2}} = -\frac{\partial u}{\partial z}$

$\theta_b = \frac{\partial w}{\partial x}$

The average rate of Rotation is the average of $\theta_a + \theta_b$ or

$$\theta = \frac{\theta_a + \theta_b}{2} \ or \ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$$

This is $\frac{1}{2}$ times the curl for the $\hat{\jmath}$ component

In general

$$\nabla \times \vec{u} = 2\theta = \overrightarrow{\omega}$$

$\vec{\omega}$ is the vorticity, defined as twice the angular velocity of a rotating fluid particle.

Explore vorticity:

So we take the curl of the Navier-Stokes equation.

$$\nabla \times \left\{\frac{D\boldsymbol{u}}{Dt} = \frac{-1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{\boldsymbol{u}} + g\hat{\boldsymbol{k}}\right\}$$

Substitution $\nabla \mathrm{x}\, \vec{u} = \vec{\omega}$

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{\boldsymbol{u}} + \frac{1}{\rho^2}\nabla\rho \times \nabla \mathrm{p} + \frac{\mu}{\rho}\nabla^2\overrightarrow{\boldsymbol{\omega}}$$

Change in vorticity due to flow (vortex stretching)

Vorticity generation due to baroclinic effects

Vorticity generation due to viscous effects

Exercise 2.5: Graduate only: show the above relationship (do the math)

If $\vec{\omega} = 0 \ \ then \ \ \nabla \times \vec{U} = 0$, we say the fluid is irrotational

$\frac{D\omega}{Dt} = (\vec{\omega} \cdot \nabla)\vec{\mathrm{U}}$ for *inviscid* fluid with no change in density

An inviscid incompressible fluid is incapable of generating vorticity.

If $\vec{\omega} = 0$ it will remain 0 → irrotational

Particles can follow a circular or elliptical path, that is OK, since the particles themselves do not rotate. (think Ferris wheel)

4.10 Velocity Potential and Stream Function

We have arrived at two fundamental statements for inviscid flows:

1. $\nabla \cdot \boldsymbol{u} = 0$ (Continuity Eq for incompressible flows) (non-divergent)
2. $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ (Vorticity = 0) (Irrotational)

4.10.1 Velocity Potential

The second statement GUARANTEES that the integral between two points above a plane is independent of path

recall from line integrals:

$$\boldsymbol{u} \cdot d\boldsymbol{l} = u_x dx + u_y dy = dF \; and$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \nabla F \cdot d\boldsymbol{l}$$

$$\boldsymbol{u} \cdot d\boldsymbol{l} = \nabla F \cdot d\boldsymbol{l}$$

$$\boldsymbol{and\ this\ holds\ if}\ \nabla \times \boldsymbol{u} = \boldsymbol{0}$$

We can specify the velocity $\boldsymbol{u}$ as the gradient of a scalar, -$\nabla\Phi$. This is because the curl of a gradient is identically zero.

$$\nabla \times -\nabla\Phi = \boldsymbol{0}$$

This scalar function is the VELOCITY POTENTIAL, Φ, and we can conveniently represent the vector $\boldsymbol{u}$ in terms of a scalar function, $-\nabla\Phi$, (it is easier to work with scalars that with vectors).

Since 2. We know that there exists a scalar function $\boldsymbol{\Phi}$ such that

$$-\Phi = \oint -d\Phi = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \oint u dx + v dy + w dz$$

$$\boldsymbol{u} = -d\Phi \;=\; -\nabla\Phi$$

$$u = \frac{-\partial\Phi}{\partial x}; v = \frac{-\partial\Phi}{\partial y}; w = \frac{-\partial\Phi}{\partial z}$$

$$\boldsymbol{u} \cdot \partial \boldsymbol{l} = u\partial x + v\partial y + w\partial z = -\partial \Phi$$

$$-d\Phi = udx + vdy + wdz$$

$$= \frac{-\partial \Phi}{\partial x}dx + \frac{-\partial \Phi}{\partial y}dy + \frac{-\partial \Phi}{\partial z}dz$$

Taking the Divergence of $\vec{\boldsymbol{u}}$, we get:

Recall $\nabla \cdot \vec{\boldsymbol{u}} = 0$; incompressible continuity

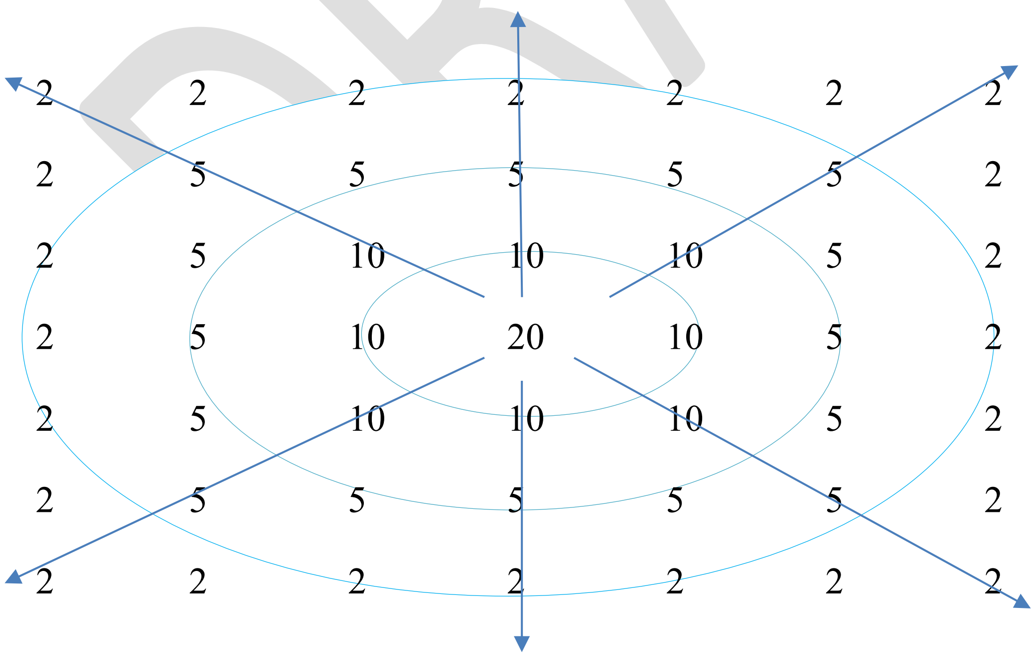
$$\nabla \cdot \vec{\boldsymbol{u}} = 0 = \nabla \cdot (-\nabla \Phi) = -\nabla^2 \Phi = 0$$

This the <u>Laplace Equation</u>, which governs many phenomena (tidal motion, magnetic fields, electrostatics, ground water)

Φ is defined for inviscid, irrotational flows. The flow can be compressible, though we will not address compressible flows.

Φ, a scalar field whose gradient is the velocity. We represent the velocity vector field with a scalar field.

At every location we have a scalar value



Think of a topographic map. You could define topography/ bathymetry by vectors that (in 3 space) specify how steep and in what direction the land/sea floor is or we can give a scalar value a height (reference to some datum).

4.3.1 Stream Function

A 2D concept can be used for 3D flows that are uniform in 1D

A function perpendicular to the velocity potential

$$\Psi = \oint_{P_0}^{P_1} \vec{u} \cdot \vec{n}\, dl \qquad\qquad dl = |d\vec{\boldsymbol{l}}|$$

Ψ is the amount of fluid crossing the contour (perpendicular) between points P_0 and P_1

solving for normal vector $\vec{n}$ and substituting into our equation:

$$\Psi = \oint_{P_0}^{P_1} (-udz + wdx)$$

Again, for independence of path, the integrand must be exact differential

$$w = \frac{\partial \Psi}{\partial x} \;\; ; \;\; u = -\frac{\partial \Psi}{\partial z}$$

Recall; for independence of path we need the curl to be zero;

$$\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} = 0$$

But in our case $f_x = w$, and $f_z = -u$

So,

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0$$

which we remember from continuity

4.4 Bernoulli Equation

The Laplace Equation, $-\nabla^2\Phi$, and the Euler Equations govern particle velocity (u,v,w) and pressure (p). Now that we have collapsed the 3 velocity components into a scalar function, Φ, we have only 2 unknowns and 4 equations.

We will show that the Euler Equations can be collapsed into one fundamental equation and 2 redundant equations.

Start with the Euler Equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} = -g$$

Neglect the y terms for simplicity

Substitute $\vec{\boldsymbol{u}} = \nabla\Phi$ into Euler Equation….

and from irrotationality $\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$;

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

In X:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(u) + \left(u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)(u) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial x}\right) + \left(-\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial x} + -\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial z}\right)\left(-\frac{\partial \Phi}{\partial x}\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial x}\right) + \left(-\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x} + -\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial z} - \frac{\partial \Phi}{\partial x}\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial x}\right) + \left(-\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x} + -\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial z}\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\left(\frac{\partial}{\partial t}(u) + \left(u\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x}\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0\right)$$

Need chain rule:

$$\frac{1}{2}\frac{\partial}{\partial x}u^2 = \frac{2}{2}u\frac{\partial u}{\partial x} \; and \; \frac{1}{2}\frac{\partial}{\partial x}w^2 = \frac{2}{2}w\frac{\partial w}{\partial x}$$

Or in terms of Velocity potential:

$$\frac{1}{2}\frac{\partial}{\partial x}(-\frac{\partial \Phi}{\partial x})^2 = \frac{2}{2}\frac{\partial \Phi}{\partial x}\left(\frac{\partial^2 \Phi}{\partial x^2}\right)$$

$$\left(\frac{\partial}{\partial t}(u) + \left(\frac{1}{2}\frac{\partial}{\partial x}u^2 + \frac{1}{2}\frac{\partial}{\partial x}w^2\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0\right)$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial z}\right)^2 + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{-\partial \Phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\left(-\frac{\partial \Phi}{\partial x}\right)^2 + \left(-\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

IN X:

$$\frac{\partial}{\partial x}\left[-\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\left(-\frac{\partial \Phi}{\partial x}\right)^2 + \left(-\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{P}{\rho}\right] = 0$$

IN Z :

$$\frac{\partial}{\partial z}\left[-\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\left(-\frac{\partial \Phi}{\partial x}\right)^2 + \left(-\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{P}{\rho}\right] = -g$$

Now use the inverse chain rule to modify the product

Integrate over x and z

And substitute in $u = \frac{-\partial \Phi}{\partial x} \qquad and \; w = \frac{-\partial \Phi}{\partial z}$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{P}{\rho} = C'(z,t)$$

where $C'(z,t)$ is a function of z and t only (not a function of x)

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{P}{\rho} = -gz + C(x,t)$$

where $C(x,t)$ is a function of x and t only (not a function of z)

$$C'(z,t) = -gz + C(t)$$

No x in either of the constants of integration only time and z

$$\boxed{-\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{P}{\rho} + gz = C(t)}$$

This is the unsteady form, C(t), of the Bernoulli Equation. For steady flow, C(t) = Constant.

Bernoulli Equation is a relationship between the fluid pressure, particle elevation and velocity potential at one location in the flow fluid pressure, particle elevation and velocity potential at another location in the flow.

We have :
Continuity + Euler Equations (u,v,w,p)
And
Laplace Equation and Bernoulli Equation (Φ,p)

Both sets are equally *comprehensive* description of the flow field for *irrotational, inviscid flows.*

Homework #1
Dean&Dalrymple: 2.2, 2.3 plot in Matlab, 2.6, 2.7, 2.8 plot in Matlab, 2.10 plot in Matlab , 3.6

5 Small Amplitude Wave Theory

5.1 Introduction to Wave Theory

Waves come in various sizes, shapes and speeds depending on the force generating them.

Wind waves, waves generated by rocks or boats, will all have a different shape during generation.

Once the waves have been generated and leave the region of forcing, the generation zone, they are free to propagate at the speed dictated by their physics and they share some of the same characteristics. For every wave, whether generated by wind, rock, boat, or other impulse process, the restorative force is gravity. These are called “Gravity Waves”.

We will talk about two types of waves: “wind sea” and “swell”

Wind Sea is the term used to describe waves that are still under the influence of the winds. The size of the wind generated waves (seas) is dependent on the distance over which the wind is blowing or “fetch” and the time or “duration” of the wind.

We say that the seas are “fetch limited”, “duration limited” or “fully developed”

In the region of generation, the wave speed and direction are closely coupled to the wind.

As these waves travel, propagate, out of the fetch /generation region they become swell seas. The waves disperse both circumferentially and radially.

The radial dispersion is a function of the speed of propagation, as waves move away radially from the zone of generation. Waves segregate themselves into waves of similar wavelengths and periods.

As the waves leave the generation zone the crest length grows as the circumference of the disturbance increases.

We know that speed is equal to distance traveled divided by the time it takes to travel that distance.

The speed of travel is: $\frac{\Delta distance}{\Delta time} = \frac{wave\ length}{wave\ period}$

$$C = L/T$$

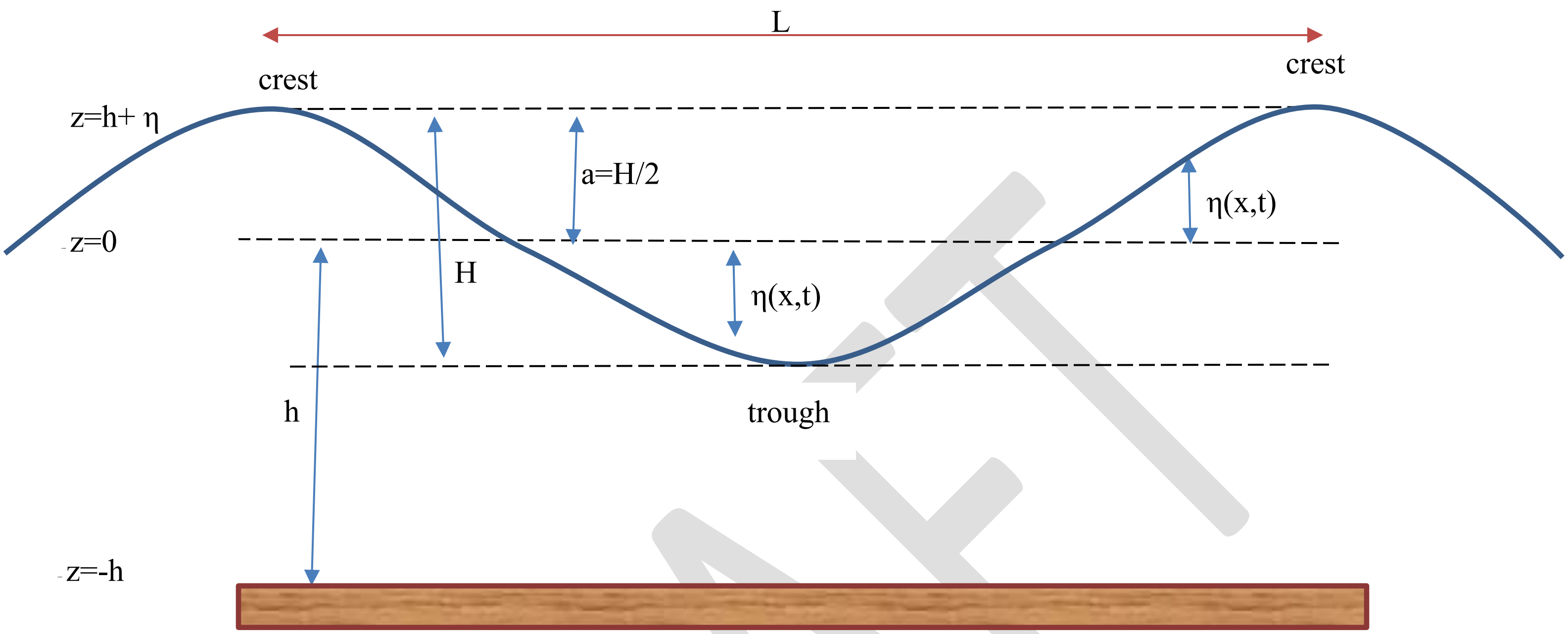
For waves, the longer waves (waves with a larger wavelength L) will travel faster.

Some coasts are more subject to swell than others: West coast USA vs. GOMEX or East Coast (why?)

The reason is the distance from the coast to the generating zone can be larger for the west coast, thereby allowing more dispersion. For the east coast, the distance between the generating storms and the coast is usually closer and the waves do not have time to sort themselves before reaching land.

In this class we will be focused on the swell waves, or waves that are no longer affected by generating forcing. We will be concentrating on waves in water depths shallow enough to affect the characteristics. Measuring the deep water wave characteristics, we will learn how to predict what the wave climate will be at the coast.

Below is a figure identifying the important features of a wave:



L: wavelength = distance between successive crests (or troughs)
H: wave height = distance from crest to trough
A: wave amplitude, ½ wave height
T: wave period = time required for two successive wave crests to pass a stationary point
C: celerity or phase speed = C=L/T
h: water depth
η(x,t): water level elevation
z: vertical reference dimension

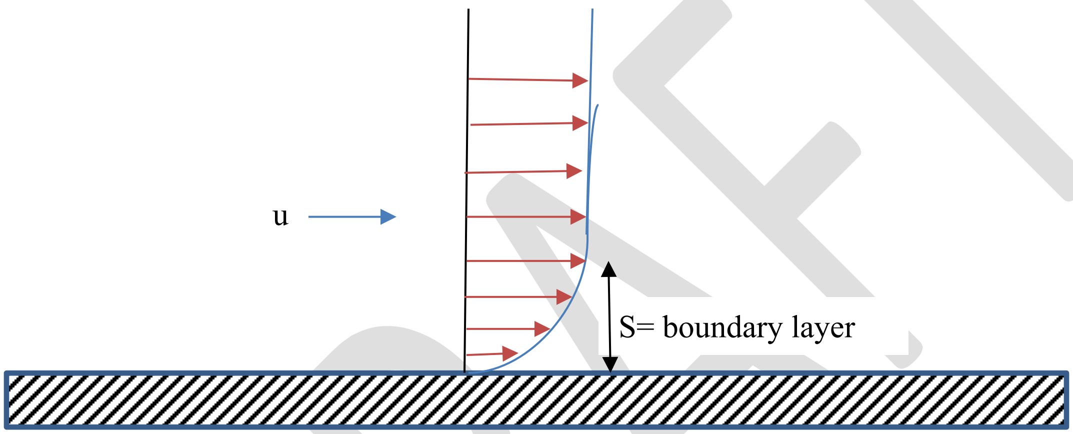
5.2 Intro to linear wave Boundary Value Problem (BVP)

Development of Water Wave theory

In Nature, as water waves propagate, they travel through a medium of viscous fluids and over bottoms with variable permeability and roughness. Our simplifications of the nature of fluid flow however tend to be surprisingly successful for the majority of wave conditions.

The reason is due to the fact that the effects of frictional shear stresses at the bottom are limited to a thin layer near the bottom, ‘bottom boundary layer.’

For typical water waves the boundary layer is θ(3mm)



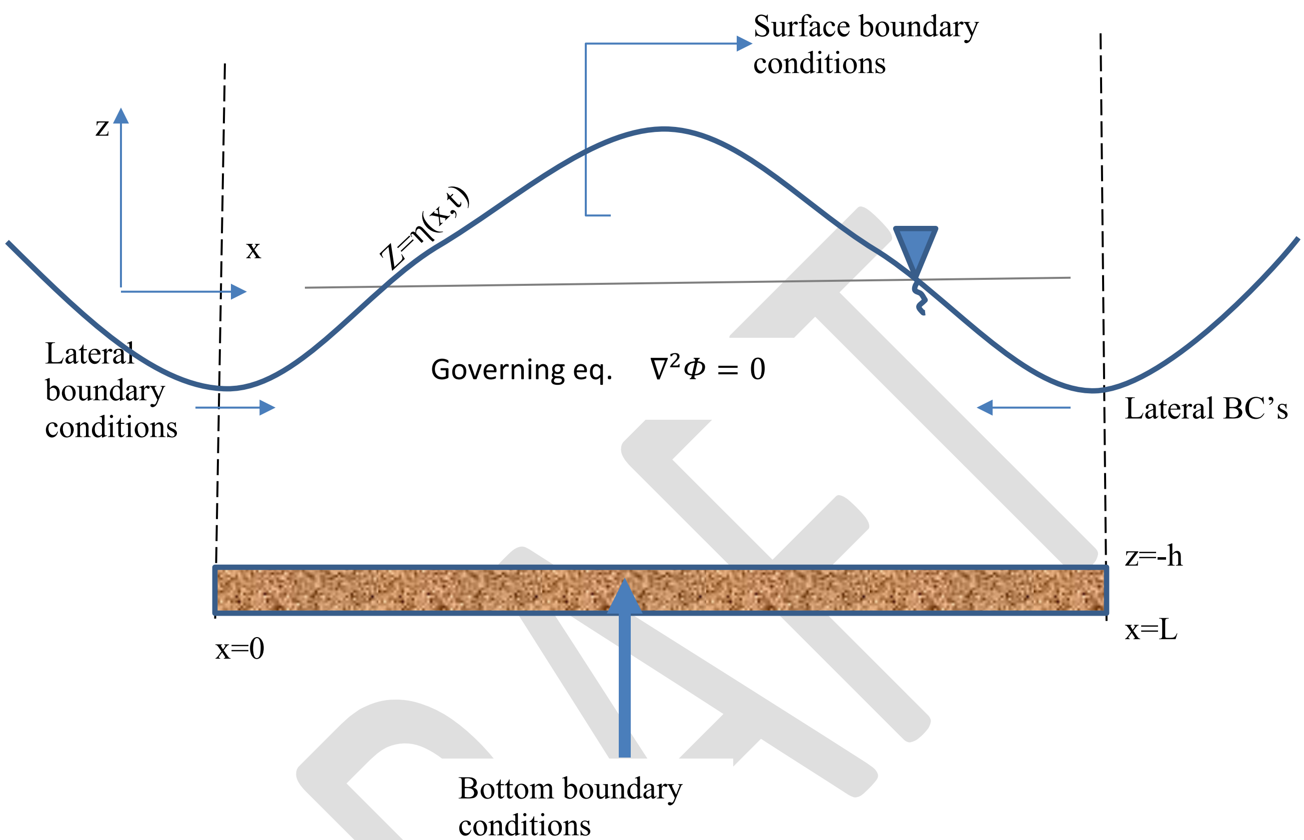
In fact, energy losses due to friction with the bottom (for most cohesionless sediment) (sand) are very small relative to energy lost due to breaking. On the other hand, bottom friction effects are cumulative, effects over 100 km of continental shelf will affect the near shore wave climate.

For now we make use of our assumptions and results developed previously to develop the theory of small amplitude waves.

In order to pose our Boundary Value problem we need a governing equation and associated boundary conditions describing our domain.

We could use either the Euler equation approach or the approach of the Laplace Equation. We will use the Laplace Equation as it is one equation from which other properties can be described.

Limiting ourselves to (x,z,t) for simplicity (we will later expand to x,y,z,t) we can define our domain as:



One wavelength x[0 L]
By assuming irrotational motion $\vec{u} = \nabla\emptyset$ and incompressibility $\nabla \cdot \vec{u} = 0$
Our governing equation is:

$$\nabla^2\emptyset = 0 \qquad -h \leq z \leq \eta(x,t)$$

$$\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0$$

$for\ our\ boundary\ conditions\ we\ need$:

- Kinematic BC's: these relate fluid motion to the motion of the boundary surface
- Dynamic BC's: these relate the pressure on the 'free' surface of the fluid to the deformation of the boundary in response to the pressure
- Lateral BC's: these define the fluid behavior at the lateral boundaries

5.2.1 Kinematic and Dynamic BC's

- Kinematic BC: we first define our boundary as a function F_B such that $F_B(x, z, t) = 0$, we are neglecting y for now

(***it can be shown that any equation can be rewritten as a sum (or difference) equal to zero***)

The KBC is given as…

$$\boldsymbol{u} \cdot \boldsymbol{n} = \frac{-\frac{\partial F}{\partial t}}{|\nabla F|} \text{ on surface } F(x, z, t) = 0$$

$$|\nabla F| = \sqrt{\nabla F \cdot \nabla F} = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}$$

There are 2 surfaces where the KBC apply. The 'free surface' (KFSBC) and the bottom (KBBC or BBC)

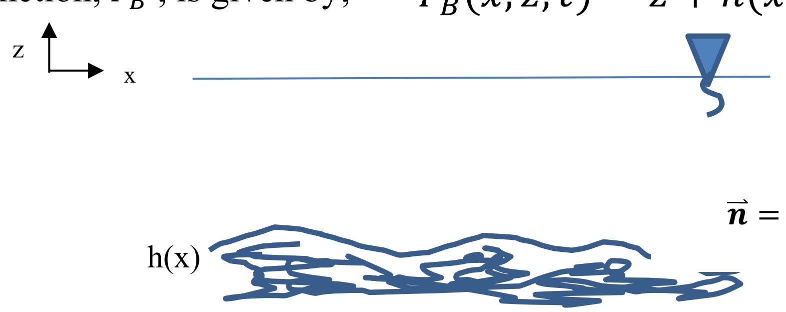
At the bottom, since we assume inviscid flow, we have what is termed a 'free slip' boundary condition, as opposed to a 'no slip' condition for viscous flows.

The Free slip condition states that
-all velocities at the bottom are tangential to the bottom or
-<u>No normal flow</u> occurs at the bottom

So if we find the unit normal vector to boundary

$$\vec{\boldsymbol{n}} = \frac{\nabla F}{|\nabla F|}, \qquad then \qquad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{n}} = 0 \; at\ the\ bottom$$

Since we have defined our domain such that $z = -h(x)$, at the bottom our function, F_B , is given by; $F_B(x, z, t) = z + h(x)$



$$\vec{\boldsymbol{n}} = \frac{\nabla F}{|\nabla F|}$$

Applying our KBBC to $F_B(x, z, t) = z + h(x)$

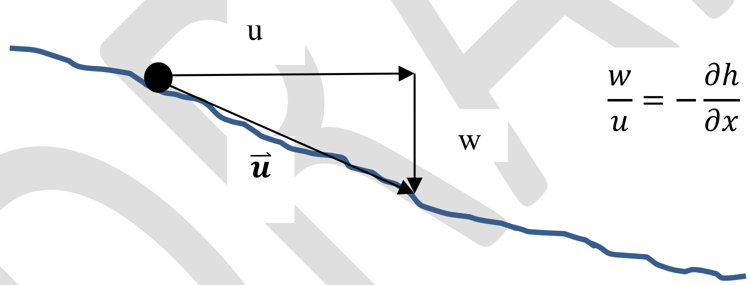
$$\boldsymbol{u} \cdot \boldsymbol{n} = \vec{\boldsymbol{u}} \cdot \frac{\frac{\partial h}{\partial x}\hat{\boldsymbol{\imath}} + 1\widehat{\boldsymbol{k}}}{\sqrt{\frac{\partial h}{\partial x}^2 + 1}} = 0$$

$$u\frac{\partial h}{\partial x} + w = 0\,;\ on\ z = -h(x)$$

$$w = -u\frac{\partial h}{\partial x}$$

For a flat bottom $\frac{\partial h}{\partial x} = 0$ and we obtain $w = 0$ as expected.

For a sloping bottom, the velocity is allowed to be tangential.



Alternatively,

If we choose a point, $p(\vec{r}, t)$, on a surface, $F_B(\vec{r}, t)$, its motion is described by bottom velocity $\overrightarrow{\boldsymbol{u_b}}$.

If we take F_B at some later time, t+Δt, p will have moved $\vec{r} + \overrightarrow{\boldsymbol{u_b}}\Delta t$

The equation for the surface at the new location then becomes:

$F_B(\vec{r} + \overrightarrow{\boldsymbol{u_b}}\Delta t, t + \Delta t) = 0$ and using Taylor series….

$$0 = F_B(\vec{r}, t) + \frac{dF_B}{dt}\Delta t + \emptyset(\Delta t)^2$$

Expand derivative $\frac{dF}{dt}$ using product (chain rule)

$$= F_B(\vec{r},t) + \left[\frac{\partial F_B}{\partial t} + (\vec{\boldsymbol{u}}_B \cdot \nabla F_B)\right]\Delta t + \cdots + \emptyset(\Delta t)^2$$

If we expand the derivatives in brackets,

$$\frac{\partial F_B}{\partial t} + (\vec{\boldsymbol{u}}_B \cdot \nabla F_B) = \frac{\partial F_B}{\partial t} + u\frac{\partial F_B}{\partial x} + v\frac{\partial F_B}{\partial y} + w\frac{\partial F_B}{\partial z}$$

Since $F_B(\vec{r},t)$ is defined to be zero, and neglecting Higher Order Terms, then we must have

$$\frac{\partial F_B}{\partial t} + (\vec{\boldsymbol{u}}_B \cdot \nabla F_B) = 0 \qquad on\ z = -h(x)$$

Goes to zero for our case of solid nonmoving bottom, no change in F with time

And for the case of the bottom boundary:

$$F_B(x,z,t) = z + h(x)$$

$$u\frac{\partial h}{\partial x} + w = 0\ at\ z = -h(x)$$

$$-\frac{\partial \emptyset}{\partial x}\frac{\partial h}{\partial x} + -\frac{\partial \emptyset}{\partial z} = 0\ at\ z = -h(x)$$

And for a flat bottom $\frac{\partial h}{\partial x} = 0$

$$-\frac{\partial \emptyset}{\partial z} = 0\ or\ w = 0\ ; z = -h(x)$$

At the surface boundary, we have the Kinematic Free surface BC=KFSBC

$$Recall: \boldsymbol{u}\cdot\boldsymbol{n} = \frac{-\frac{\partial F}{\partial t}}{|\nabla F|} \text{ on surface } F(x,z,t) \;=\; 0$$

Again we define the surface $F_s(x,z,t) = 0$ for our domain sketch: $z = \eta(x,t)$ so:

$$F_s(x,z,t) = \; z - \eta(x,t)$$

$$Note\ that\ on\ the\ surface\ \frac{-dF_s}{dt} = \frac{d\eta}{dt}$$

$$\nabla F_s = \frac{d\eta}{dx} + 1 \qquad ; \boldsymbol{n} = \frac{\nabla F_s}{|\nabla F_s|} = \frac{-\frac{d\eta}{dx}\hat{\boldsymbol{\imath}} + \; 1\widehat{\boldsymbol{k}}}{\sqrt{\left(\frac{d\eta}{dx}\right)^2 + 1}}$$

$$\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = \frac{\frac{d\eta}{dt}}{\sqrt{\left(\frac{d\eta}{dx}\right)^2 + 1}}$$

$$\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = \frac{-u\frac{d\eta}{dx} + w}{\sqrt{\left(\frac{d\eta}{dx}\right)^2 + 1}} = \frac{\frac{d\eta}{dt}}{\sqrt{\left(\frac{d\eta}{dx}\right)^2 + 1}}$$

$$\therefore \frac{d\eta}{dt} + u\frac{d\eta}{dx} - w = 0$$

$$or\ w = \frac{d\eta}{dt} + u\frac{d\eta}{dx} \quad on\ z = \eta(x,t)$$

Again, we can derive this using the Taylor series expansion on F_s for a short time interval Δt.

$$\frac{\partial F_B}{\partial t} + u\frac{\partial F_B}{\partial x} + v\frac{\partial F_B}{\partial y} + w\frac{\partial F_B}{\partial z} = 0$$
$$F_B(x,z,t) = \; z - \eta(x,t)$$
$$-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + w(1) = 0$$

$$or\ w = \frac{d\eta}{dt} + u\frac{d\eta}{dx} \quad on\ z = \eta(x,t)$$

The expansion shows more clearly that the surface itself F_s does not change $\frac{dF}{dt} = 0$ (if we move with the surface).

On the 'free surface' we have a second BC due to the pressure affects, this is the Dynamic free surface BC, or, DFSBC.

- Dynamic BC

The KFSBC is applied on η, which is initially unknown. We must develop another BC to position the surface.

For weaker waves the pressure on the free surface must be equal to atmospheric pressure, which is the same over the surface due to our assumption of small amplitude, small waves not affected by wind or transitioning pressure variations.

We will use Bernoulli eq. to derive DFSBC

$$-\frac{d\emptyset}{dt} + \frac{P_n}{\rho} + \frac{1}{2}(u^2 + w^2) + gz = C(t)$$

We can simplify by taking P_n as a gage pressure over the surface such that $P_n = 0$.

$$-\frac{d\emptyset}{dt} + \frac{1}{2}(u^2 + w^2) + gz = C(t) \qquad z = \eta(x,t)$$

$$or -\frac{d\emptyset}{dt} + \frac{1}{2}\left[\left(-\frac{d\emptyset}{dx}\right)^2 + \left(-\frac{d\emptyset}{dz}\right)^2\right] + gz = C(t)$$

To this point we have defined over governing eq. and 3 boundary conditions:

-2 kinematic BC's

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{n}} = \frac{-\frac{dF}{dt}}{|\nabla F|}$$

One for the bottom and one for the 'free' surface, and one Dynamic BC for the 'free' surface…

-Bernoulli's Eq. $-\frac{d\emptyset}{dt} + \frac{1}{2}(u^2 + w^2) + g\eta = 0 \qquad z = \eta$

It remains to establish Lateral BC's.

- Lateral BC

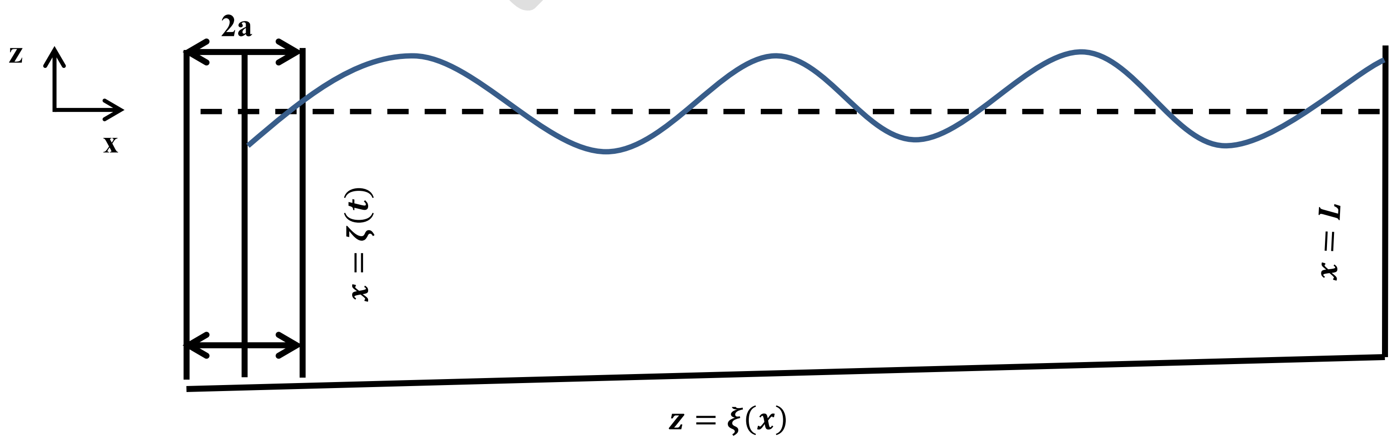
We must derive the Lateral BC's from the problem specification.

The BC will depend on if we have a closed box, or flow in one direction or a moving lateral boundary (i.e. wavemaker, or inflow and outflow).

It is possible that a Kinematic BC must be employed if the lateral boundary is a surface where we have an interaction.

At that point we define the Lateral surface S(x,z,t) as a function of z and t and then implement our KBC formula.

$$\vec{u} \cdot \vec{n} = \frac{\frac{dS}{dt}}{|\nabla S|}$$



Assume the equation $\boldsymbol{\zeta(t) = a \sin \sigma t}$ describes the position of the boundary at the piston.

Then, $S(x, z, t) = a \sin \sigma t - x = 0$, at the Left Boundary and $S(x, z, t) = x - L = 0$, at the Right Boundary.

For our problem as defined in our BVP, we have a wave, periodic in space and time. Therefore, our lateral LB will be,

$$\emptyset(x, t) = \emptyset(x + L, z, t)$$

$$\emptyset(x, t) = \emptyset(x, z, t + T)$$

Make L: wavelength, and T: wave Period

Similarly the free surface is periodic and we can put Lateral BC's on the surface position.

$$\eta(x, z, t) = \eta(x + L, z, t)$$

$$\eta(x, z, t) = \eta(x, z, t + T)$$

Clearly, the assumption of periodicity does not hold in the presence of a wall or structure

- Our complete BVP on (x,z,t):

Gov. Eq.:

$$-\nabla^2\emptyset = 0 \qquad -h \le z \le \eta$$

Linear → apply superposition

KFSBC:

$$\frac{\partial\eta}{\partial t} + \frac{-\partial\emptyset}{\partial x}\frac{\partial\eta}{\partial x} = \frac{-\partial\emptyset}{\partial z} \quad ; \quad z = \eta \quad (nonlinear)$$

BBC:

$$-\frac{\partial\emptyset}{\partial z} = 0\,;\ z = -h \quad (linear)$$

DFSBC:

$$-\frac{\partial\emptyset}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\emptyset}{\partial x}\right)^2 + \left(\frac{\partial\emptyset}{\partial z}\right)^2\right] + g\eta = 0 \quad ;\ z = \eta\ (nonlinear)$$

Lateral BC:

$$\emptyset(x,z,t) = \emptyset(x+L,z,t)$$

$$\eta\,(x,z,t) = \eta(x+L,z,t)$$

$$\emptyset(x,z,t) = \emptyset(x,z,t+T)$$

$$\eta(x,z,t) = \eta(x,z,t+T)$$

Now solve for $\emptyset$ & η

For simplicity, to solve analytically we need to linearize our BC,What do we mean by linear and nonlinear?

-an equation containing products of unknowns $(u\frac{du}{dx})$ is referenced as a nonlinear equation.

5.3 Linearization

We need a way to reduce the problem such that…

a) Regular physical arguments can be made to the 'large' or 'smallness' of terms relative to other terms
b) Resulting solution returns at least the gross characteristics of waves as observed; simplification cannot render the solution physically meaningless.

If we assume η is some small deviation from $z = 0$ then we can use Taylor series to take BC's to the $z = 0$ line ($\Delta z = \eta$)

For the DFSBC:

$\Delta z = \eta$ *is small*

$$\frac{-\partial \emptyset}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + gz \Big|_{z=\eta}$$
$$= \frac{-\partial \emptyset}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + gz\Big|_{z=0} + \frac{\partial}{\partial z}[(-\frac{\partial \emptyset}{\partial t} + \frac{1}{2}(u^2 + w^2) + gz]\eta\Big|_{z=0} + H.O.T$$

η is small, the velocities are small.

Products of $\eta \& u, \eta \& w$ as well $as\ \ u^2\ and\ w^2$ are smaller.

So keeping terms linear in η or Ø at z=0

$$= \frac{-\partial \emptyset}{\partial t} + \frac{1}{2}\left(u^2 + w^2\right) + gz\Big|_{z=0} + -\eta\frac{\partial^2 \emptyset}{\partial z \partial t} + \frac{\eta}{2}\left(\frac{\partial u^2}{\partial z} + \frac{\partial w^2}{\partial z}\right) + \eta g\frac{\partial z}{\partial z}\Big|_{z=0}$$

$$\frac{-\partial \emptyset}{\partial t} + gz + \eta g = C, \quad z = 0$$

Datum C≡0 (pressure at surface)

$$\frac{-\partial \emptyset}{\partial t} + g\eta = 0 \quad ; \quad z = 0$$

We now have the Linearized DFSBC, LDFSBC

Bring the Boundary from $z = \eta$ to $z = 0$

We do the same for the Kinematic BC

$\Delta z = \eta \; is\; small$

$$\left.\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - w\right|_{z=\eta} = \left.\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - w\right|_{z=0} + \underbrace{\eta\frac{\partial}{\partial z}\left.\left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - w\right)\right|_{z=0}}_{\text{All small}} + \cdots = 0$$

Applying same assumptions as before we have:

$$\frac{\partial \eta}{\partial t} = w \qquad on\; z = 0$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \emptyset}{\partial z} \;\; ; z = 0$$

Linearized KFSBC, LKFSBC, Other BC's are linear and remain unchanged.

Our complete Linearized BVP on (x,z,t):

Gov. Eq.:

$$\nabla^2 \emptyset = 0 \;\; ; \quad -h \le z \le \eta \quad (linear)$$

LKFSBC:

$$\frac{d\eta}{dt} = \frac{-d\emptyset}{dz} \;;\; z = 0 \;\; (linear)$$

BBC:

$$-\frac{d\emptyset}{dz} = 0 \;;\; z = -h \quad (linear)$$

LDFSBC:

$$\frac{d\emptyset}{dt} = g\eta; \; z = 0 \; (linear)$$

Lateral BC: $(linear)$

$$\emptyset(x,z,t) = \emptyset(x+L,z,t)$$
$$\eta\,(x,z,t) = \eta(x+L,z,t)$$
$$\emptyset(x,z,t) = \emptyset(x,z,t+T)$$
$$\eta(x,z,t) = \eta(x,z,t+T)$$

Now solve for ∅ & η

5.4 Solution of the Linearized BVP for small amplitude water waves

- Step 1:

Using the method of separation of variables, we assume that the solution can be written as a product of functions of each of the variables of interest

$$\emptyset(x,z,t) = X(x)Z(z)T(t)$$

T is not period.

$$product\ solution$$

- Step 2:

For $T(t)$, we know that we need a periodic solution, so pick a simple periodic function, try:

$$T(t) = \ sin(at)$$

TO test we first use our temporal LBC :

T is period.

$$\emptyset(t) = \emptyset(t+T)$$

$$X(x)Z(z)sin(at) = \ X(x)Z(z)sin[a(t+T)]$$

Is this true, or better, when is this true?

When : $\sin(at) = \sin(at + aT)$

Using trig identity for $\sin(a+b)$

$$sin(at+aT) = sin(at)cos(aT) + cos(at)sin(aT)$$

When is $\sin(at) = \sin(at + aT)$ true? When $aT = 2\pi$

or $\boxed{a = m\frac{2\pi}{T}}$ … this is angular frequency, σ

$$\sigma = \frac{2\pi}{T}$$

So: $\emptyset(x,z,t) = \ X(x)Z(z)sin(\sigma t);$ where $\sigma = \frac{2\pi}{T}$

- Step 3:

Substitute into Laplace Eqn:

$$\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0$$

Linear: superposition OK

$$X''Zsin(\sigma t) + XZ''sin(\sigma t) = 0$$

Cancel sin fcn and multiplying by $\frac{1}{XZ}$

$$\frac{X''}{X} + \frac{Z''}{Z} = 0$$

Depends only on x

Depends only on z

$$\frac{X''}{X} = \frac{-Z''}{Z} = -\lambda$$

Arbitrary separation constant

So,

$$X'' + \lambda X = 0$$
$$Z'' - \lambda Z = 0$$

A set of ODE's!
In Sturm-Liouville form.

What can λ be? Find λ for the nontrivial solution

$\lambda = 0$

$\lambda < 0: \lambda = -k^2$

$\lambda > 0: \lambda = k^2$

Try λ =0

$$X'' = 0 \rightarrow X(x) = Ax + B$$

$$Z'' = 0 \rightarrow Z(z) = Cz + D$$

Now we have a ***possible solution*** to the Laplace Eq.

Apply a spatial periodic BC to obtain constants:

$$\emptyset(x) = \emptyset(x + L)$$

$$(Ax + B)\cancel{(Cz + D)}\cancel{sin(\sigma t)} = (A(x + L) + B)\cancel{(Cz + D)}\cancel{sin(\sigma t)}$$

$$\cancel{Ax} + \cancel{B} = \cancel{Ax} + AL + \cancel{B}$$

$$0 = AL \rightarrow A = 0$$

If A=0, then

$$(0(x) + B)\cancel{(Cz + D)}\cancel{\sin(\sigma t)} = (0(x + L) + B)\cancel{(Cz + D)}\cancel{\sin}(\sigma t)$$

$$B = B$$

$$0 = 0 \rightarrow trivial\ solution!$$

$\lambda = 0$, yields a trivial solution, no help in understanding the problem.

Now Try $\lambda < 0$: $\lambda = -k^2$

This also leads a trivial solution.

$$\lambda < 0 \rightarrow \lambda = -k^2 \qquad n = some\ integer$$

$$X'' - k^2 X = 0$$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Z'' - (-k^2)Z = 0 \quad therefore \quad Z'' + k^2 Z = 0$$

$$Z(z) = C\ cos(kz) + D\ sin(kz)$$

Then

$$\emptyset \ = XZ\sin(\sigma t)$$

Apply Spatial LBC:

$$\emptyset(x) = \emptyset(x+L)$$

$$(Ae^{kx} \ + Be^{-kx})(Ccos(kz) + Dsin(kz))\sin(\sigma t)$$
$$= (Ae^{k(x+L)} + Be^{-k(x+L)})(Ccos(kz) + Dsin(kz))\sin(\sigma t)$$

$$Ae^{kx} + Be^{-kx} = Ae^{kx}e^{kL} + Be^{-kx}e^{-kL}$$

$$0 = -Ae^{kx} + Ae^{kx}e^{kL} - Be^{-kx} + Be^{-kx}e^{-kL}$$

$$0 = A(e^{kx}e^{kL} - e^{kx}) + B(e^{-kx}e^{-kL} - e^{-kx})$$

First Try factoring

$$0 = Ae^{kx}(e^{kL} - 1) + Be^{-kx}(e^{-kL} - 1)$$

$$0 = Ae^{kx} + Be^{-kx} \ \frac{(\frac{1}{e^{kL}}-1)}{(e^{kL}-1)}$$

multiply by $e^{kx}e^{kL}$

$$0 = Ae^{2kx}e^{kL} + B\ \frac{(1-e^{kL})}{(e^{kL}-1)}$$

$$0 = Ae^{k(2x+L)} - B\frac{(e^{kL}-1)}{(e^{kL}-1)}$$

$$B = Ae^{k(2x+L)}$$

no good start over from two lines ago

$$Ae^{kx} + Be^{-kx} = Ae^{kx}e^{kL} + Be^{-kx}e^{-kL}$$

$$Ae^{kx} + B\frac{1}{e^{kx}} = Ae^{kx}e^{kL} + B\frac{1}{e^{kx}}\ \frac{1}{e^{kL}}$$

Multiply through by e^{kx}

$$Ae^{kx}e^{kx} + B = Ae^{kx}e^{kx}e^{kL} + B\frac{1}{e^{kL}}$$

Multiply through by e^{kL}

$$Ae^{kx}e^{kx}e^{kL} + Be^{kL} = Ae^{kx}e^{kx}e^{kL}e^{kL} + B$$

$$Ae^{kx}e^{kx}e^{kL}(1 - e^{kL}) = B(1 - e^{kL})$$

$$B = Ae^{k(2x+L)}$$

Then, substitute back in for B

$$Ae^{kx} + \left[Ae^{k(2x+L)}\right]e^{-kx} = Ae^{k(x+L)} + \left[Ae^{k(2x+L)}\right]e^{-k(x+L)}$$

$$Ae^{kx} + Ae^{kx}e^{kx}e^{kL}e^{-kx} = Ae^{kx}e^{kL} + Ae^{kx}e^{kx}e^{kL}e^{-kx}e^{-kL}$$

Cancel some terms,

$$Ae^{kx} + Ae^{kx}e^{kL} = Ae^{kx}e^{kL} + Ae^{kx}$$

Multiply by e^{-kx}

$$A + Ae^{kL} = Ae^{kL} + A$$

Trivial!

Again a trivial solution, one more eigenvalue to test.

$$\underline{Try\ \ \lambda > 0: \lambda = k^2}$$

$$X'' + k^2X = 0 \rightarrow X(x) = Acos(kx) + Bsin(kx)$$
$$Z'' - k^2Z = 0 \rightarrow Z(z) = Ce^{kz} + De^{-kz}$$

final solution to try, $\emptyset(x) = \emptyset(x + L)$

$$[Acos(kx) + Bsin(kx)][Ce^{kz} + De^{-kz}]sin(\sigma t)$$
$$= [Acos(k[x + L]) + Bsin(k[x + L])[Ce^{kz} + De^{-kz}]sin(\sigma t)$$

$$A\cos kx + B\sin kx = A\cos(kx + kL) + B\sin(kx + kL)$$

Aside:Trig Identity

$$cos(kx + kL) = cos(kx)cos(kL) - sin(kx)sin(kL)$$

$$sin(kx + kL) = sin(kx)cos(kL) + cos(kx)sin(kL)$$

$$A\cos kx + B\sin kx$$
$$= A(cos(kx)cos(kL) - sin(kx)sin(kL))$$
$$+ B(sin(kx)cos(kL) + cos(kx)sin(kL))$$

The above is true when

$cos(kL) = 1$ & $sin(kL) = 0$

$kL = 2n\pi$

$k = \frac{2n\pi}{L}$ $n = 0,1,2 \ldots$

Focus on 1st harmonic (n=1)

$$\boxed{k = \frac{2\pi}{L}} \leftarrow wave\ number$$

So the spatially periodic BC's are satisfied when $k = \frac{2\pi}{L}$. Thus the general solution is:

$$\emptyset = \underbrace{[Acos(kx) + Bsin(kx)]}_{X}\underbrace{[Ce^{kz} + De^{-kz}]}_{Z}\underbrace{\sin(\sigma t)}_{T}$$

Since linear in $\emptyset$, superposition of solutions is allowed.

$$\emptyset = [Acos(kx) + Bsin(kx)][Ce^{kz} + De^{-kz}]\sin(\sigma t) =$$

$$\emptyset = [Acos(kx)[Ce^{kz} + De^{-kz}]\sin(\sigma t) + Bsin(kx)[Ce^{kz} + De^{-kz}]\sin(\sigma t)] =$$

$$\emptyset_I + \emptyset_{II}$$

We will focus on $\emptyset_I$, for now and add back in $\emptyset_{II}$ later.

- Step 4:

Solve for constants via BC's

First start with simplest BC, BBC ($w = 0$)

(i) Apply BBC

$$w = -\frac{\partial \emptyset}{\partial z} = 0 \quad at\ z = -h$$

$$w = \frac{-\partial}{\partial z}\{[Acos(kx)][Ce^{kz} + De^{-kz}]\sin\sigma t\} = 0$$
$$= -Acos(kx)[kCe^{kz} - kDe^{-kz}]\sin\sigma t = 0$$

Now evaluate at $z = -h$,

$$w(-h) = -Akcos(kx)[Ce^{-kh} - De^{kh}]\sin\sigma t = 0$$

The above BC must hold for any and all x and t.

We know cos is not zero everywhere and sin is also not zero everywhere, therefore, the only way for this to be true is if:

$$Ce^{-kh} - De^{kh} = 0$$
$$Ce^{-kh} = De^{kh}$$

$$C = De^{2kh}$$

$$now\ substitute\ into\ general\ eq\ for\ vel.pot.(\Phi)\ ...$$

$$\emptyset = Acos(kx)[De^{2kh}e^{kz} + De^{-kz}]\sin\sigma t$$

now Force the solution,

$$\emptyset = Acos(kx)\left[De^{2kh}e^{kz} + \frac{e^{kh}}{e^{kh}}De^{-kz}\right]\sin\sigma t$$

Factor out De^{kh},

$$\emptyset = ADe^{kh}\cos(kx)\,[e^{kh}e^{kz} + e^{-kh}e^{-kz}]\sin\sigma t$$
$$\emptyset = ADe^{kh}\cos(kx)\left[e^{k(h+z)} + e^{-k(h+z)}\right]\sin\sigma t$$

recall: $\boxed{\cosh[k(h+z)] = \frac{e^{k(h+z)}+e^{-k(h+z)}}{2}}$

$$\emptyset_I = 2ADe^{kh}\cos kx\,[\cosh(k(h+z))]\sin\sigma t \quad \leftarrow(**)$$
$$\emptyset_{II} = 2BDe^{kh}\sin kx\,[\cosh(k(h+z))]\sin\sigma t$$

constant G So now we have, general form of vel.pot. in terms of constant G

$$\emptyset_I = G_I cos(kx)\cosh[k(h+z)]\sin\sigma t$$
$$\emptyset_{II} = G_{II} sin(kx)\cosh[k(h+z)]\sin\sigma t$$

Type equation here.

(ii) Apply LDFSBC to (**) to obtain G.

$$At\ z = 0:\ -\frac{\partial \emptyset}{\partial t} + g\eta = C(t)$$

Re-arrange to solve for η (free surface)

P=0 at surface

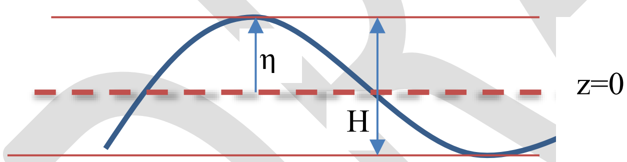
$$\eta = \frac{1}{g}\frac{\partial \emptyset}{\partial t} + \frac{C(t)}{g};\ at\ z = 0$$

Substitute (**) into BC,

$$\eta = \frac{1}{g}\frac{\partial}{\partial t}[Gcos(kx)\cosh[k(h+z)]\sin\sigma t]\Big|_{z=0}$$

$$\eta = \frac{G\sigma}{g}\cosh(kh)\cos(kx)\cos(\sigma t) \qquad at\ z = 0$$

We need η, by looking at the **<u>spatial</u>** and **<u>temporal</u>** BC's,



By inspection we can choose

$$\eta = \frac{H}{2}\cos(kx)\cos(\sigma t)$$

(other choices are available)

Equate η's

$$\frac{H}{2}\cos(kx)\cos(\sigma t) = \frac{G\sigma}{g}\cosh(kh)\cos(kx)\cos(\sigma t)$$

$$\therefore G = \frac{Hg}{2\,\sigma\cosh(kh)}$$

Standing wave solution!

Substitute G into (**)..

$$\boxed{\emptyset = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh[k(h+z)]}{\cosh(kh)}\cos(kx)\sin(\sigma t)} \qquad (***)$$

Recall the Linearization of the KFSBC: application of this BC will yield a relationship between σ and k

$$w = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} \quad at\ z = \eta(x,t)$$

$$w - \frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x} = 0 \quad (*)$$

Apply Taylor Series to (*) about z=0,

$$w - \frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x}\bigg|_{z=\eta} = w - \frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x}\bigg|_{z=0} + \eta\frac{\partial}{\partial z}\left(w - \frac{\partial\eta}{\partial t} - u\frac{\partial\eta}{\partial x}\right)\bigg|_{z=0} + \cdots = 0$$

Retaining only the linear terms,

$$w - \frac{\partial\eta}{\partial t}\bigg|_{z=0} = 0$$

$$or, w = \frac{\partial\eta}{\partial t}$$

$$or\ -\frac{\partial\emptyset}{\partial z} = \frac{\partial\eta}{\partial t}$$

Now, using LKFSBC:

a.k.a. $w = \frac{\partial\eta}{\partial t}\Big|_{z=0}$ ← speed of η moving up and down

take our $\emptyset$ and η solutions and apply LKFSBC,

$$-\frac{\partial\emptyset}{\partial z} = \frac{\partial\eta}{\partial t}, at\ z = 0$$

$$\frac{-\partial}{\partial z}\left[\frac{Hg\cosh[k(h+z)]}{2\,\sigma\cosh(kh)}\cos(kx)\sin(\sigma t)\right] = \frac{\partial}{\partial t}\left[\frac{H}{2}\cos(kx)\cos(\sigma t)\right]\ ;\ at\ z=0$$

$$-\frac{Hgk}{2\,\sigma\cosh(kh)}\cos(kx)\sin(\sigma t)\sinh[k(h+z)] = -\sigma\frac{H}{2}\cos(kx)\sin(\sigma t)\quad ;\ at\ z=0$$

Cancel and evaluate at $z = 0$;

$$-\frac{gk}{\sigma\cosh(kh)}\sinh kh = -\sigma$$

$$gk\frac{\sinh(kh)}{\cosh(kh)} = \sigma^2$$

Waterdepth (h)
Wavelength (k,L)
Period (σ, T)

$$\boxed{\sigma^2 = gk\tanh(kh)}$$

Dispersion Equation!
$\sigma = 2\pi/T$
$k = 2\pi/L$

5.5 Parameters Derivable from dispersion equation

$$\boldsymbol{\sigma^2 = gk\tanh(kh)}$$

1) Knowing $\sigma = 2\pi/T$ and $k = 2\pi/L$

$$\left(\frac{2\pi}{T}\right)^2 = g\left(\frac{2\pi}{L}\right)\tanh\left(\frac{2\pi h}{L}\right)$$

$$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi h}{L}\right)$$

2) Wave speed (celerity)

$[\frac{\sigma}{k}]$

$$C \equiv \boxed{\frac{L}{T}} = \left[\frac{gT^2}{2\pi}\tanh(kh)\right]\left[\frac{1}{T}\right]$$

L_o= deep water wave length.
Subscript '0' = deepwater

$$L_0 = L = \frac{gT^2}{2\pi}$$

$$C \equiv \frac{L}{T} = \left[\frac{gT}{2\pi}\tanh(kh)\right]$$

$$C = \frac{L_o}{T}\tanh(kh)$$

Or

$$C_o = \frac{L_o}{T}$$

T does not change

Dispersion equation → $\sigma^2 = gk \tanh kh$

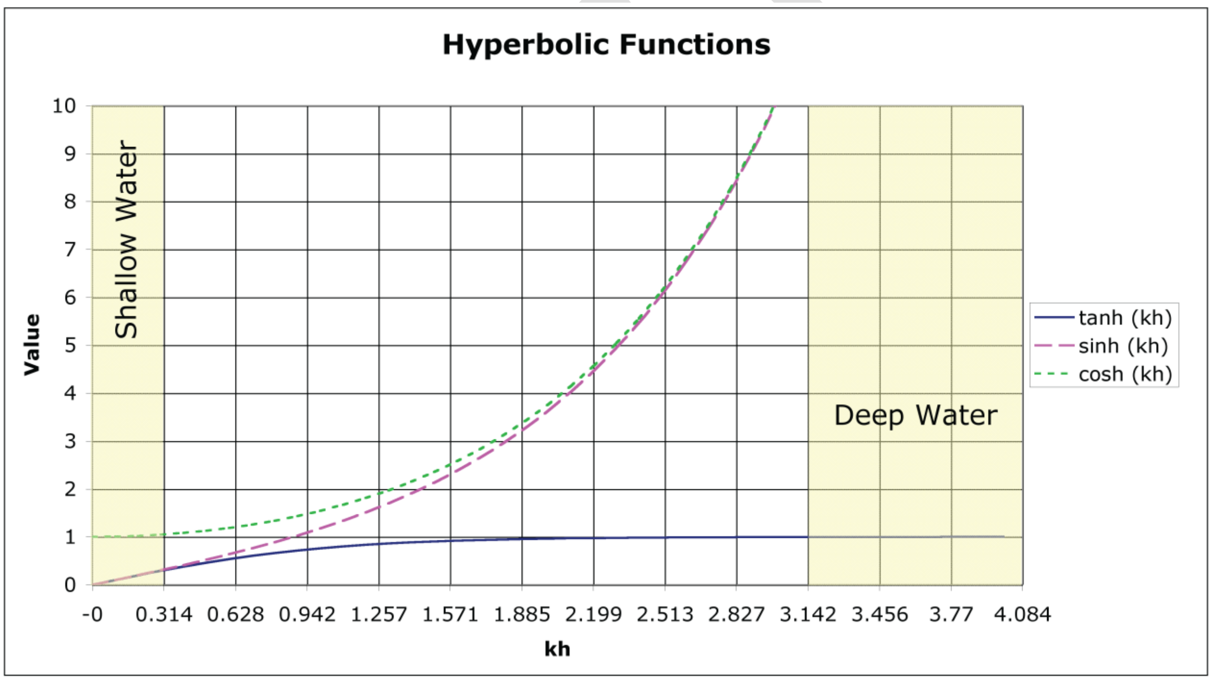
Most useful relationship for costal engineering

Only one value of k for a given σ and h

for the remainder of the class we will explore the dispersion equation and see what can be gleaned

Let's look at limits of dispersion

Plot $tanh(kh)$



$$For\ kh \gg 1\ \ \tanh(kh) = 1$$
$$For\ kh \ll 1\ \tanh(kh) = kh$$
$$kh \gg 1\ \text{“}deep\text{”}\ water$$
$$kh \ll 1\ \ \text{“}shallow\text{”}\ water$$

in deep water, dispersion is:

$$\sigma^2 = \ gk(1)$$
$$\frac{2\pi^2}{T^2} = g\frac{2\pi}{L}$$
$$L_0 = L = \frac{gT^2}{2\pi};$$

in deepwater the deepwater wavelength

In shallow water, dispersion reduces to:

$$\sigma^2 = gk(kh)$$

$$\frac{2\pi^2}{T^2} = g\frac{2\pi^2}{L^2}h$$

$$L=\sqrt{gh}\,T$$

$$\frac{L}{T} = \sqrt{gh} = C_{SW}$$

shallow water celerity

Let's go back to wavelength

We have found that $L_0 = \frac{gT^2}{2\pi}$ or $1.56T^2$m

$L = L_o \tanh(kh)$ where $\tanh(kh)$ gets smaller as kh gets smaller.
As the wave propagates into shallow water, the wavelength gets shorter.
L is at its maximum in deep water

Looking back at C in shallow water, $C_{SW} = \sqrt{gh}$

$$C = \frac{L}{T} = \frac{L_0}{T} tanh\, kh$$

In deep water $C_0 = \frac{gT}{2\pi}$ or $1.56T$ m/s

We can see that the wave will ***slow down*** as depth decreases. So the waves slow down and the ***wavelengths become shorter*** as depth decreases.

What is 'deep water' and 'shallow water' when we are talking about waves?
We define deep and shallow basted on the values of kh and the limits of the hyperbolic functions.

Water is 'deep' to a wave when either $h \gg 1$ or $k \gg 1$. Water is 'shallow' to a wave when either $h \ll 1$ or $k \ll 1$.

$$k = 2\pi/L \; so \; k \ll 1 \; when \; L \gg 1$$
$$and \; k \gg 1 \; when \; L \ll 1$$

really long wave: Tsunami, Tide, surge,
always shallow water
really short wave: ripples capillary
always deep water

Formally.
(Remember h is water depth)

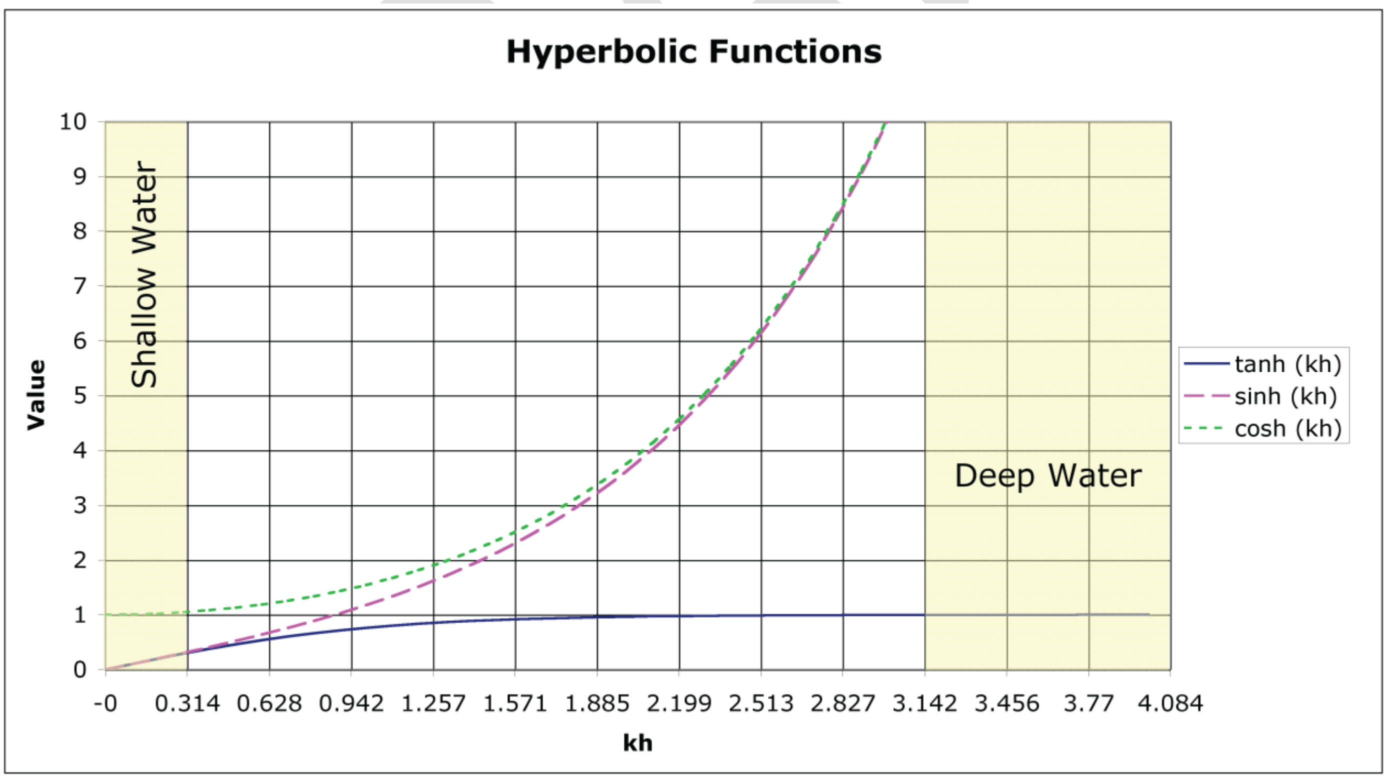
Shallow water $kh \leq \frac{\pi}{10}$ (0.3141)

$$\frac{2\pi h}{L} \leq \frac{\pi}{10} \quad or \quad \frac{h}{L} \leq \frac{1}{20} \quad or \quad h \leq \frac{L}{20}$$

Deep water $kh \geq \pi$

$$\frac{2\pi h}{L} \geq \pi \ (3.141)$$

$h/L \geq 1/2 \quad or \quad h \ \geq \ L/2$ h is greater than ½ the wavelength



Exercise: At what depth is a 9 second wave considered in shallow water?

$$kh \leq \ \pi/10 \cong 0.314$$

$$\frac{2\pi}{L}h = 0.314$$

$$h = 0.314\frac{L}{2\pi} = 0.05\ L$$

Solve for h once we get L

$$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi h}{L}\right)$$

$$L = \frac{g9^2}{2\pi}\tanh\left(\frac{\pi}{10}\right)\ \textit{or use SW approx.}\ L = \frac{g9^2}{2\pi}\frac{\pi}{10}$$

$$L = 126.47(0.304) = 38.47\text{m} \qquad\qquad L = 39.73\text{m}$$

$$\sim 3.3\%\ difference$$

$$h = 0.05 * 38.47\text{m} = 1.92\text{m} \qquad or \qquad h = 0.05 * 39.73\text{m} = 1.98\text{m}$$

or Plug this back into eq and solve for h

$$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi h}{L}\right)$$

$$38.47\text{m} = 126.47\text{m}\tanh\left(\frac{2\pi h}{38.47\text{m}}\right)$$

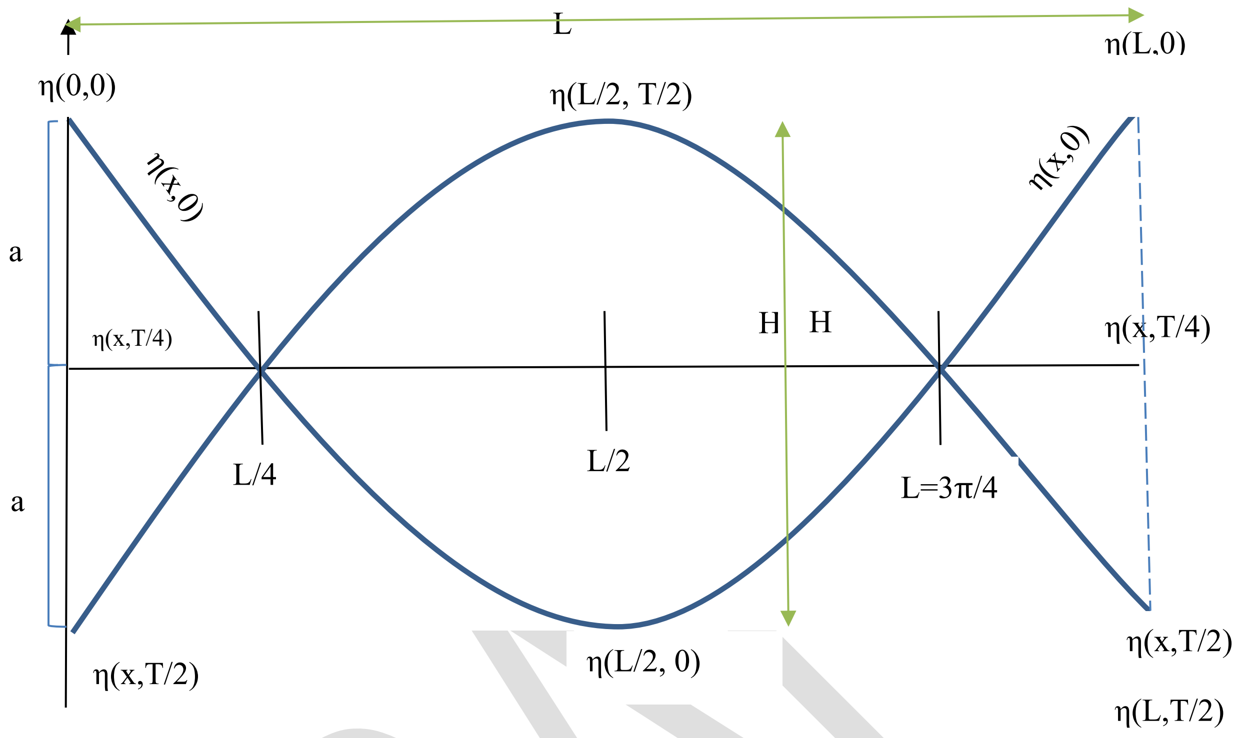
$$0.304 = \tanh\left(\frac{2\pi h}{38.47\text{m}}\right)$$

$$0.314 = \frac{2\pi h}{38.47\text{m}}$$

$$h = 1.92\text{m} \sim 6\text{ft}$$

5.6 Standing wave

(Draw diagram first)



$$\Phi = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos kx \sin \sigma t$$

$$\eta = \frac{H}{2}\cos kx \ \cos \sigma t$$

What happens in time first?

$$\sigma t = \frac{2\pi}{T}t = \frac{2\pi}{T}\frac{T}{4} = \frac{\pi}{2}$$

At time $t = 0, cos\ 0 = 1$

At time $t = \ T/4, cos\ \frac{\pi}{2} = 0$

At time $t = T/2, cos\ \pi \ = -1$

$$\sigma t = \frac{2\pi}{T}t = \frac{2\pi}{T}\frac{T}{2} = \pi$$

What about x?

$$kx = \frac{2\pi}{L}x = \frac{2\pi}{L}L = 2\pi$$

$$kx = \frac{2\pi}{L}x = \frac{2\pi}{L}\frac{L}{2} = \pi$$

$$kx = \frac{2\pi}{L}x = \frac{2\pi}{L}\frac{L}{4} = \frac{\pi}{2}$$

At x=0, L cos0=cos 2π=1

At x=L/2 cosπ=-1

At x=L/4 cos(π/2)=0

Develop surface displacement for standing wave:

A standing wave is formed when a wave is reflected off of a barrier.

With a standing wave the initial wave form reflects back onto itself off a vertical boundary.

Suppose we have a +x direction wave and a –x direction wave

+x: $\phi^{I} = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh(k(h+z))}{\cosh(kh)}\cos(kx)\sin(\sigma t)$

$\eta^{I} = \frac{H}{2}\cos(kx)\cos(\sigma t)$

-x: $\phi^{II} = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh(k(h+z))}{\cosh(kh)}\cos(-kx)\sin(\sigma t)$

$\eta^{II} = \frac{H}{2}\cos(-kx)\cos(\sigma t)$

Since ϕ is linear…superposition is allowed

$$\phi = \phi^{I} + \phi^{II}$$

$$\phi = \left[\frac{H}{2}\frac{g}{\sigma}\frac{\cosh\big(k(h+z)\big)}{\cosh(kh)}\sin\sigma t\right][\cos kx + \cos(-kx)]$$

EVEN FCN $cos(-kx) = cos(kx)$ ∴ $[\cos kx + \cos(-kx)] = 2\,cos(kx)$

$$\phi_{\text{stand}} = H\frac{g}{\sigma}\frac{\cosh\big(k(h+z)\big)}{\cosh(kh)}\sin\sigma t\cos kx$$

$$\eta_{stand} = \eta^{I} + \eta^{II}$$

$$\eta_{stand} = \left[\frac{H}{2}\cos(kx)\cos(\sigma t)\right] + \left[\frac{H}{2}\cos(-kx)\cos(\sigma t)\right]$$

$$\eta_{stand} = \left[\frac{H}{2}\cos(\sigma t)\right]\ [\cos(kx) + \cos(-kx)]$$

$$[\cos kx + \cos(-kx)] = 2\,cos(kx)$$

$$\eta_{stand} = Hcos(\sigma t)\,cos(kx)$$

$$\eta = \left.\frac{\partial \Phi}{\partial t}\right|_{z=0}$$

Solution is a standing wave with twice the amplitude!

5.7 Progressive Wave :

Look again at the Standing wave

$$\phi_I = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}\cos(kx)\sin(\sigma t)$$
$$\eta_I = \frac{H}{2}\cos(kx)\cos(\sigma t)$$
$$and\ \ \sigma^2 = gk\tanh(kh)$$

Now let's go back and grab the Φ_{II} , sin(kx) solution

This is a second solution to $-\nabla^2\Phi=0$

$$\phi_{II} = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}\sin kx\cos\sigma t$$
$$\eta_{\rm II} = -\frac{H}{2}\sin kx\sin\sigma t$$

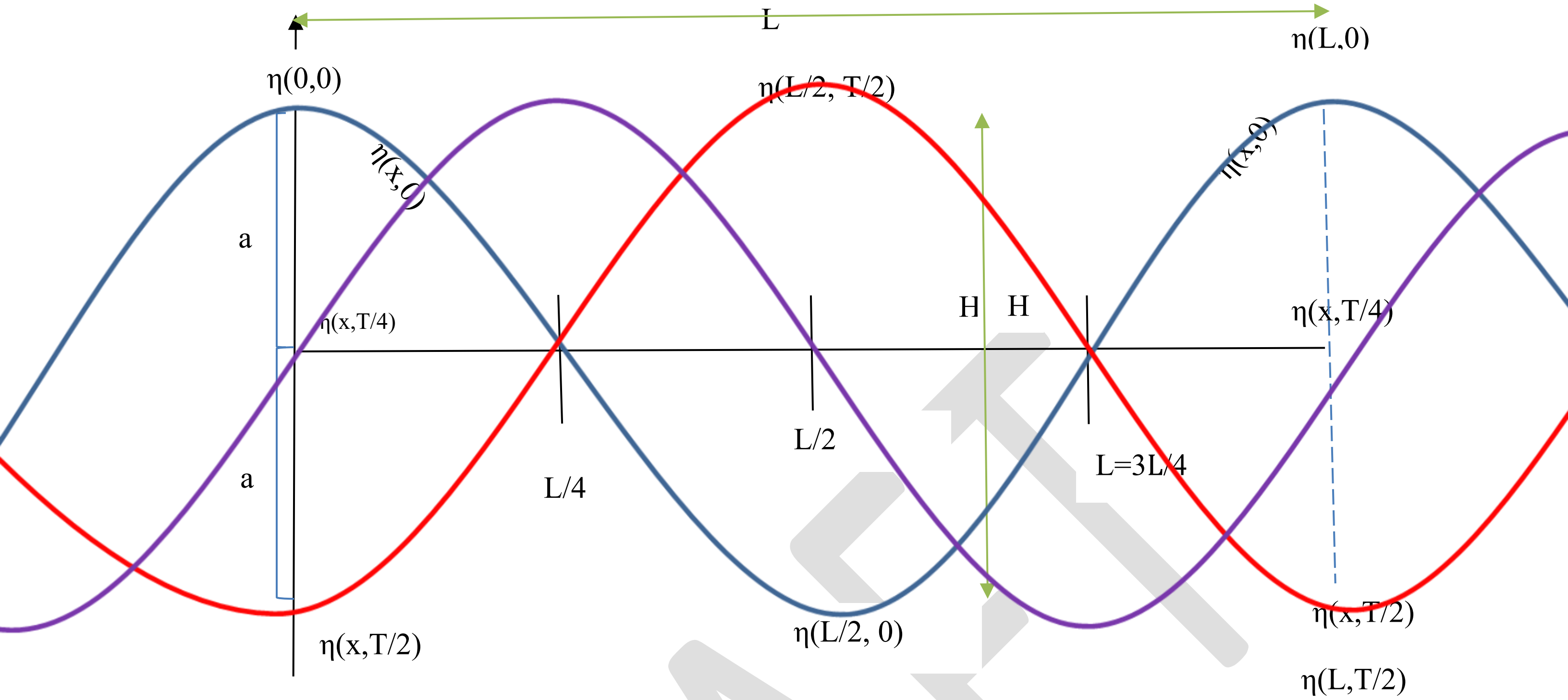
Superposition allows us to add or subtract this is from our standing wave solution

$$\phi = \phi_I - \phi_{II} = \frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}(\cos kx\sin\sigma t - \sin kx\cos\sigma t)$$
$$= -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}(\sin kx\cos\sigma t - \cos kx\sin\sigma t)$$
$$= -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}\sin(kx-\sigma t)$$

And for surface elevation,

$$\eta = \eta_{\rm I} - \eta_{\rm II} = \frac{H}{2}\cos(kx)\cos(\sigma t) + \frac{H}{2}\sin kx\sin\sigma t$$
$$= \frac{\rm H}{2}\cos(\rm kx - \sigma t)$$
$$\text{or} \equiv$$
$$\eta = \frac{1}{g}\frac{\partial\Phi}{\partial t}\ ;\text{at z} = 0$$
$$\eta = -\frac{H}{2}\cos(kx-\sigma t)$$

Let's explore surface elevation…



$$\eta = -\frac{H}{2}\cos(kx - \sigma t)$$

What happens to surface elevation in time first?

At time $t = 0, \cos(kx - \sigma t) = cos\ kx$

At time $t = \frac{T}{4}, \cos(kx - \sigma t) = cos\ (kx - \frac{\pi}{2})$

At time $t = \frac{T}{2}, \cos(kx - \sigma t) = \cos\ (kx - \ \pi)$

$$\sigma t = \frac{2\pi}{T} t = \frac{2\pi}{T}\frac{T}{2} = \pi$$

Now let's see how wave form responds in x. What about x for t=0?

Blue Line

At $x = 0, L$; $\cos 0 = \cos\ 2\pi = 1$

$$\text{kx} = \frac{2\pi}{L}x = \frac{2\pi}{L}L = 2\pi$$

At $x = \frac{L}{2}$; $\quad \cos\pi = -1$

$$\text{kx} = \frac{2\pi}{L}x = \frac{2\pi}{L}\frac{L}{2} = \pi$$

At $x = \frac{L}{4}$; $\quad \cos\frac{\pi}{2} = 0$

$$\text{kx} = \frac{2\pi}{L}x = \frac{2\pi}{L}\frac{L}{4} = \frac{\pi}{2}$$

At $x = \frac{3L}{4}$; $\quad \cos\frac{3\pi}{2} = 0$

What about x for $t = \frac{T}{4}$; $\cos(kx - \frac{\pi}{2})$?

Purple Line

At $x = 0, L$; $\cos\ 2\pi - \frac{\pi}{2} = \cos\ \frac{3\pi}{2} = 0$

At $x = \frac{L}{2}$; $\cos\ \pi - \frac{\pi}{2} = \cos\ \frac{\pi}{2} = 0$

At $x = \frac{L}{4}$; $\cos\ \frac{\pi}{2} - \frac{\pi}{2} = \cos 0 = 1$

At $x = \frac{3L}{4}$; $\cos\ \frac{3\pi}{2} - \frac{\pi}{2} = \cos\pi = -1$

What about x for $t = \frac{T}{2}$; $\cos(kx - \pi)$?

Red Line

At $x = 0, L$; $\cos\ 2\pi - \pi = \cos\pi = -1$

At $x = \frac{L}{2}$; $\cos\ \pi - \pi = \cos\ 0 = 1$

At $x = \frac{L}{4}$; $\cos\ \frac{\pi}{2} - \pi = \cos - \frac{\pi}{2} = 0$

At $x = \frac{3L}{4}$; $\cos\ \frac{3\pi}{2} - \pi = \cos\frac{\pi}{2} = 0$

The wave form moves horizontally (to the right) in time!!

5.8 EXAMPLES:

1) A wave is measured by bottom mounted pressure sensor at a depth of 20 m, and the wavelength is found to be L=60 m

a) Is the wave in deep water at h=20m?
   At h=20m , L = 60m, L/2= 30 m, so no not in deep water at 20m.
b) What is the wavelength at water depth of 8m?

Use dispersion, solve for wave period.

$$\sigma^2 = gk \tanh kh$$

$$\left[\frac{2\pi}{T}\right]^2 = \frac{g2\pi}{L}\tanh\left(\frac{2\pi h}{L}\right)$$

$$\frac{1}{T} = \left[\frac{g}{L2\pi}\tanh\left(\frac{2\pi h}{L}\right)\right]^{\frac{1}{2}}$$

$$T = \left[\frac{2\pi L}{g}\tanh\left(\frac{2\pi h}{L}\right)\right]^{1/2} = 6.29\text{s}$$

Get deep water wavelength using period

$$L_o = \frac{gT^2}{2\pi} \quad = \ 61.85\, m$$

Now use modified dispersion to complete L at 8m.

$$L_{8m} = L_o \tanh\left(\frac{2\pi}{L}h\right) = 61.85m \tanh\left(\frac{2\pi}{L}\ 8m\right)$$

Iterate for solution

Starting value for L enter deep water or some reduced value (could try to guess shallow water, $L_{shallow,} \frac{L}{T} = \sqrt{gh_{sw}},\ \ L = T\sqrt{gh}_{sw};\ \ \sqrt{gh_{sw}} = \ do\ not\ know\ h_{sw}$, so use h=8m to get $C\ = 8.86\frac{m}{s}$, L_guess=6.29*8.86=55.72m

$$L_{8m}{\sim}48.187m$$

IS it deep water? Is h/L > 1/2 → 8/48.187=0.166 < 1/2 =0.5 not deep water
Is h/L < 1/20 ? → 0.166 > 0.05 NO (not shallow water either)

kh = 1.04313, tanh(kh) = 0.779 > 0.314 Not shallow water!

2) A peak wave period is measured at the end of a pier.

T=12.2 s

A breakwater is planned to be built 200 ft from the pier (this distance does not matter in computation just to say that the site is adjacent to the location of the measurement of the wave period) in 10 m water depth. What is the wavelength expected at the site associated with the peak period?

Use period to obtain L_o

$$L = L_o \tanh kh$$

$$L_o = \frac{gT^2}{2\pi} = 232.4\, m$$

$$L_{10} = 232.4m \tanh \frac{2\pi}{L_{10}} h$$

=115.4m in 10m water depth

----

3) 2011 Tohoku Earthquake; ON March 11, 2011 an earth quake struck off the coast of Japan. The time of the 9.0 magnitude quake was 05:46 UTC (GMT) or 2:46pm local time or 7:46pm Hawaii Standard Time.
What was the approx. speed of Tsunami was as it traveled away from the source?

Average depth of Pacific Ocean: Depth=15000 ft, or 4570m

Shallow water wave: $C = \sqrt{gh} = 214\ m/s$ keep in mind that 1 m/s is 2.237 mi/hr

= 478 mi/hr

x≅4000 mi at 478 mi/hr ≅ 8.4 hrs

Let's check our answer

5:46 GMT quake+8.4 hrs (time to reach Hawaii as calculated above)

At Hawaii, we have 13:37 UTC (GMT) or 3:37 am Pac/Honolulu time

4) Semi-diurnal Tide in Atlantic: how fast does the tidal wave propagate?
   Atlantic is ~4000 miles from North America to Europe
   Semi-diurnal tides peak every 12 hours so 333 mi/hr
   Apply what we learned from dispersion eq:
   C= $\sqrt{gh}$
   If h is 3000m then C= $\sqrt{9.8 * 3000} = 171\frac{m}{s} = 384\ mi/hr$

Homework #2:

D&D

3.3

3.4

3.5

3.7

3.9

3.11

3.12

3.13

3.15; in MatLab code up a function for dispersion.

6. EXAM I

7. Engineering Properties Derivable from Linear Theory

a. Kinematics and Pressure field for progressive waves
b. Kinematics and Pressure field for standing waves
c. Energy and Energy propagation
d. Conservation of waves
e. Refraction, Snells Law, conservation of energy flux
f. Wave Breaking
g. Diffraction

7.1 Progressive Waves

$$\eta = \frac{H}{2}\cos(kx - \sigma t)$$

$$\phi = -\frac{H}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)}\sin(kx - \sigma t)$$

Properties of Waves $\quad \sigma^2 = gk \tanh kh \rightarrow$ Dispersion Eqn.

$$\vec{u} = -\nabla\phi$$

We can rewrite ϕ by forcing the substitution of the dispersion equation:

Let's look again at Dispersion

$$\sigma^2 = gk \tanh kh$$

$$\sigma^2 = gk\,\frac{\sinh(kh)}{\cosh(kh)}$$

$$\frac{\sigma^2}{gk}\cosh kh = \sinh kh$$

$$\frac{gk}{\sigma^2}\frac{1}{\cosh kh} = \frac{1}{\sinh kh}$$

$$\frac{g}{\sigma}\frac{k}{\sigma}\frac{1}{\cosh kh} = \frac{1}{\sinh kh}$$

$$C = \frac{\sigma}{k} \quad \therefore \frac{g}{\sigma}\frac{1}{\cosh kh} = \frac{\sigma}{k}\frac{1}{\sinh kh} = C\frac{1}{\sinh kh}$$

If we multiply ϕ by $\frac{\sigma}{\sigma} = 1$; and then substitute $\sigma^2 = gk \tanh kh$

$$\begin{aligned}\phi &= -\frac{H}{2}\frac{\sigma g}{\sigma^2}\frac{\cosh k(h+z)}{\cosh kh}\sin\,(kx - \sigma t)\\ &= \frac{-Hg\sigma}{2gk\tanh kh}\frac{\cosh k(h+z)}{\cosh kh}\sin(kx-\sigma t)\\ &= \frac{-H\sigma}{2k}\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\sigma t)\\ &= -\frac{H}{2}C\frac{\cosh k(h+z)}{\sinh kh}\sin(kx-\sigma t)\end{aligned}$$

7.2 Uniqueness of solutions

The question of uniqueness is not 'simple' at all.

One subscriber to the coastal list notes:

“I am trying to calculate the horizontal particle velocity for standing waves. I have gone through a few references but got different equations.

1) Basic Coastal Engineering (Sorensen 3rd edition 2010)
u = pi*H/T*cosh(ks)/sinh(kd)*sin(kx)*sin(wt)

2) Mechanics of Wave Forces on Offshore Structures (Sarpkaya/Isaacson 1981)
u = 2*pi*H/T*cosh(ks)/sinh(kd)*sin(kx)*sin(wt)

3) Introduction to Coastal Engineering and Management (Kamphuis 2000)
u = 2*pi*H/T*cosh(ks)/cosh(kd)*sin(kx)*sin(wt)

4) Waves in Oceanic and Coastal Waters (Holthuijsen 2007)
u = -2*pi*H/T*cosh(ks)/sinh(kd)*sin(kx)*cos(wt)

s = d + z

Equation 1 and Equation 2 are different by a factor of 2.
Equation 2 and Equation 3 are different on the denominator -> sinh(kd) instead of cosh(kd) Equation 4 is quite different than others.

I have set up an Excel spreadsheet and confirmed that none of the equations give the same results.

Also, for Equations 1 to 3, the term sin(wt) = 0 when t = 0. Thus, u = 0 for all x when t = 0. I am somewhat puzzled by this. It would be greatly appreciated if someone can point me to any discussion on this. Thanks.”

And Professor Reeve at Swansea responds

… is necessary to retrace the derivation of linear wave theory. My copy of Lamb's 'Hydrodynamics' from 1906 has a preface in which Lamb states "I have persisted in the use of the reversed sign of the velocity potential, writing u = -dphi/dx, etc., in place of u = dphi/dx, etc. The physical interpretation of the function is thereby greatly improved, and the far-reaching analogy with the magnetic potential is at the same time made more complete. It should be remembered that the prevalent usage had a purely analytical origin, in connection with the notion of 'perfect

differentials.' From a physical point of view it is so much more natural to regard the state of motion of a dynamical system, in any given configuration, as specified by the impulses which would start it, rather than those which would stop it, that the change here advocated would seem to require no further justification.!
" {In the above I have typed phi for the velocity potential}. Whether one agrees with this or not is rather irrelevant. However, the convention is important in determining the form of the final solution."

And he continues;

"variations in solutions due to differing conventions have a long history. Indeed, the definition of the velocity potential is simply the beginning of a trail of conventions, (which, if justified at all are often done so in a rather offhand manner), that lead to a confusing plethora of possible solutions. {If you are able to find a copy of Lamb's book, Chapter IX covers surface waves.}

Most of the variations can be traced to one or more of:
1) Writing the amplitude of the wave as half the wave height (H/2);
2) Assuming the surface purturbation to be of the form cos(..) rather than sin(..);
3) Assuming the argument of the (co)sinusoidal variation to be wt-kx rather than kx-wt;
4) Taking z = 0 at the seabed rather than at the level of the undisturbed fluid;
5) Noting that L/T = c and using the substitution c = (g/w)tanh(kd).

This already gives 2^5 = 32 variations before including more exotic variants! I would not expect any of them to agree with any of the others for specific values of x, z and t. However, they should all be equivalent in that they should describe the propagation of a linear wave of sinusoidal shape. The waves may be cosines or sines, travelling from right to left or left to right etc etc.

If you do not wish to work through the solution from first principles to obtain a self-consistent set of formulae for velocity potential, surface elevation, horizontal velocity etc. then I suggest you select a set of formulae from one of your sources and at least check that they are self-consistent, perhaps with your Excel spreadsheet.

All your four solutions will have u = 0 for all x for certain specific values of t, (for 1-3 this is when t = 0+n2pi, and for 4 it is when t = pi/2 + n2pi, where n is a positive integer). This is correct as in a standing wave there will be instants when the water surface is momentarily in its undisturbed position, although it will be in motion. [A standing wave is simply a fixed cosine waveform in space modulated in a sinusoidal manner with time. Some of the time the modulation is positive, some of the time the modulation is negative, and occasionally it is zero.]

The usual derivation of standing waves uses the superposition of a left-going and a right-going wave. For sake of argument let us say we have two waves of unit amplitude. The superposition of the waves may be written as cos(kx-wt) + cos(kx+wt). Using standard trigonometric relations this can be simplified to 2cos(kx)cos(wt). If on the other hand we have sinusoidal waves then sin(kx-wt) + sin(kx+wt) = 2sin(kx)sin(wt). The solution you quote that does not begin with a 2 looks odd from this perspective - but you will need to check the author's conventions to verify this.

I leave the final word to Horace Lamb who notes, with a certain amount of glee I suspect, that the corresponding solution of the streamfunction is given, (for the conventions he adopts), by

-tanh(ks)tan(kx).phi
"

I include this conversation to illustrate the need to understand and accept the concept of progressive and standing waves and their solutions, and the assumptions and conditions under which those solutions were derived.

7.3 Particle Velocities for a Progressive Wave

Horizontal velocity:

$$u = -\frac{\partial \Phi}{\partial x} = +\frac{H}{2}\frac{gk}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\sigma t)$$

$$= +\frac{H}{2}\sigma\frac{\cosh k(h+z)}{\sinh(kh)}\cos(kx-\sigma t)$$

And acceleration:

$$\frac{\partial u}{\partial t} = +\frac{H}{2}\sigma^2\frac{\cosh k(h+z)}{\sinh(kh)}\sin(kx-\sigma t)$$

Vertical velocity:

$$w = -\frac{\partial \Phi}{\partial z} = \frac{H}{2}\sigma\frac{\sinh k(h+z)}{\sinh(kh)}\sin(kx-\sigma t)$$

And acceleration

$$\frac{\partial w}{\partial t} = -\frac{H}{2}\sigma^2\frac{\sinh k(h+z)}{\sinh kh}\cos(kx-\sigma t)$$

At the bottom, $z = -h, k(h+z) = 0$

$$\cosh(0) = 1$$

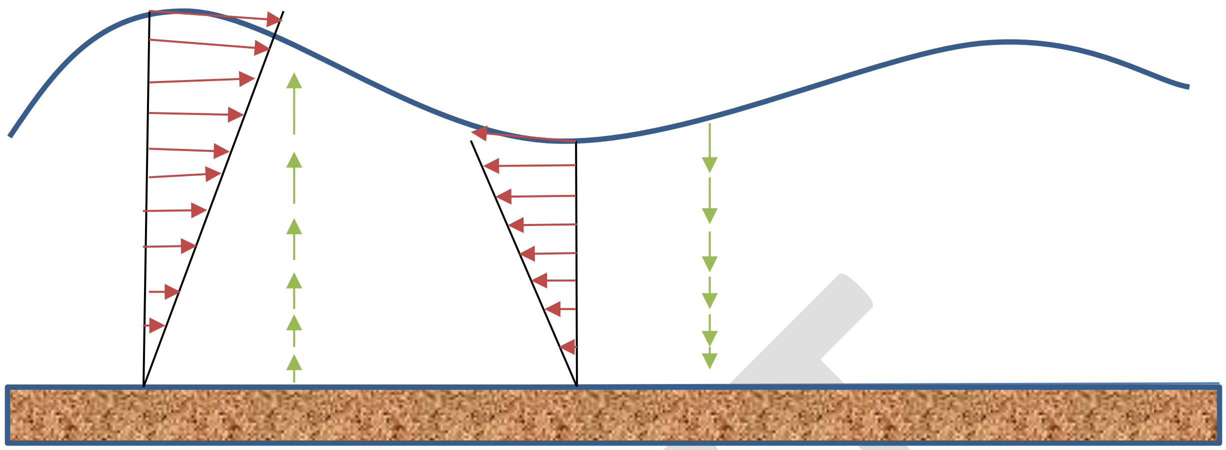
$$\sinh(0) = 0$$

$$w = 0 \text{ and } u = -\frac{H}{2}\sigma\frac{1}{\sinh\ kh}\cos(kx-\sigma t)$$

Both at minimum value. (see figure of cosh / sinh)

w and u increase as z→0 from the bottom to the surface

Particle Velocity



$$(\zeta, \xi)$$

$$(\zeta, \xi) = (zeta\,, xi\ )$$

7.4 Particle Displacement

To find particle position, integrate velocity

$$\zeta(x, z, t) = \int u(x_1 + \zeta\,, z_1 + \xi)\, dt$$

$$\xi(x, z, t) = \int w\,(x_1 + \zeta, z_1 + \xi)\, dt$$

Using Taylor Series and linearizing, this reduces to,

$$\zeta(x, z, t) = \int u\,(x_1, z_1) dt$$

$$\xi(x, z, t) = \int w(x_1, z_1)\, dt$$

7.4.1 Horizontal Displacement

$$\zeta = \int u\,(x_1, z_1) dt = \int \frac{H}{2} \frac{gk}{\sigma} \frac{\cosh k(h + z_1)}{\cosh kh} \cos(kx_1 - \sigma t)$$

$$= -\frac{H}{2} \frac{gk}{\sigma^2} \frac{\cosh k(h + z_1)}{\cosh kh} \sin(kx_1 - \sigma t)$$

$$= -\frac{H}{2}\frac{\cos k(h+z_1)}{\sinh kh}\sin(kx_1 - \sigma t)$$

7.4.2 Vertical Displacement

$$\xi = \frac{H}{2}\frac{\sinh k(h+z_1)}{\sinh kh}\cos(kx_1 - \sigma t)$$

$$\zeta = \left[-\frac{H}{2}\frac{\cos k(h+z_1)}{\sinh kh}\right]\sin(kx_1 - \sigma t) = -A\sin(kx_1 - \sigma t)$$

$$\xi = \frac{H}{2}\frac{\sinh k(h+z_1)}{\sinh kh}\cos(kx_1 - \sigma t) = B\ \cos(kx_1 - \sigma t)$$

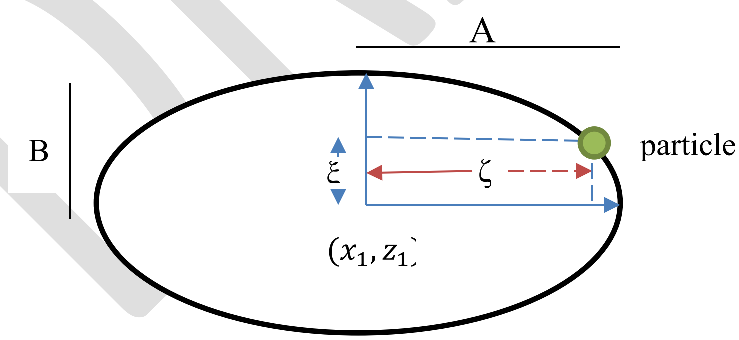
$$\sin^2 \quad + \cos^2 \quad = 1$$

$$\left(\frac{\zeta}{A}\right)^2 + \left(\frac{\xi}{B}\right)^2 = 1$$

The equation of an ellipse

We can learn more by looking at the limits of how these equations behave

Looking at the shallow $kh < \frac{\pi}{10}$ and deep



A=major axis

B=minor axis

"Show matlab video"

-in Deep Water $kh > \pi$; $\frac{h}{L} > \frac{1}{2}$

$$\sinh z \to \frac{1}{2} e^{z}$$
$$\cosh z \to \frac{1}{2} e^{z}$$

$$A = \frac{H}{2} \frac{\cos k(h + z_1)}{\sinh kh} = \frac{H}{2} \frac{e^{kh} e^{kz_1}}{e^{kh}} = \frac{H}{2} e^{kz_1}$$
$$B = \frac{H}{2} \frac{\sinh k(h + z_1)}{\sinh kh} = \frac{H}{2} \frac{e^{kh} e^{kz_1}}{e^{kh}} = \frac{H}{2} e^{kz_1} = \boldsymbol{A}$$

$$A = B$$

This defines circular displacement in deep water and further more at depths deeper than $z \le -\frac{L}{2}$ visibly (≤4%) no movement

-waves do not 'feel' the bottom. Presence at bottom does not affect any wave whose wavelength is less than twice the waver depth.

L/T = $\sqrt{gh}$

-Shallow water

$$A = \frac{H}{2} \frac{\cos k(h + z_1)}{\sinh kh} = \frac{H}{2} \frac{1}{kh} = \frac{HL}{4\pi h} = \frac{HT}{4\pi} \sqrt{\frac{g}{h}}$$
$$B = \frac{H}{2} \frac{\sinh k(h + z_1)}{\sinh kh} = \frac{H}{2} \left(1 + \frac{z_1}{h}\right)$$

Only B (the vertical displacement) is dependent on z

At the bottom;

$$z = -h;\ B = 0\,, A = \frac{HT}{4\pi} \sqrt{\frac{g}{h}}$$

At the surface:

$$z = 0;\ B = \frac{H}{2}\,, A = \frac{HT}{4\pi} \sqrt{\frac{g}{h}}$$

7.5 Standing Waves

$$\eta = \frac{H_S}{2} \cos kx \cos \sigma t$$

$$\Phi = \frac{H_S}{2}\frac{g}{\sigma}\frac{\cosh k(h+z)}{\cosh(kh)} \cos kx \sin \sigma t$$

$$\sigma^2 = gk \tanh kh \rightarrow \text{Dispersion Eqn.}$$

7.5.1 Particle Velocities under a Standing Wave

$$u = -\frac{\partial \Phi}{\partial x} = \frac{H_S}{2}\frac{gk}{\sigma}\frac{cosh\, k(h+z)}{cosh\, kh} sin\, kx\, sin\, \sigma t$$

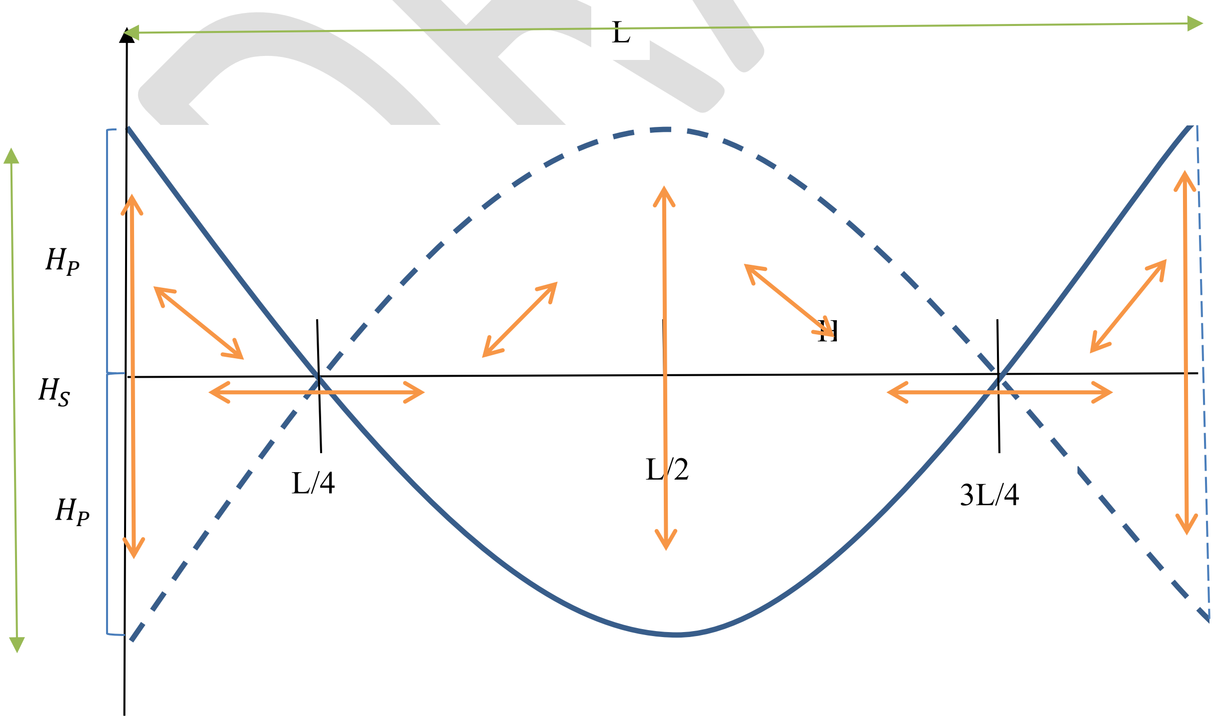
$$w = -\frac{\partial \Phi}{\partial z} = -\frac{H_S}{2}\frac{gk}{\sigma}\frac{sinh\, k(h+z)}{cosh\, kh} cos\, kx\, sin\, \sigma t$$

Where u and w, are "in phase" and the maximums are at the nodes (horizontal velocity only) and antinodes (vertical velocity only), and $H_S = 2 * H_P$.

7.2.2 Particle Displacement under a standing wave

Particle displacements are linear under a standing wave (back and forth motion)

(Draw diagram first)



Particles travel a straight line

$$\zeta = -\frac{H}{2}\frac{\cos k(h+z_1)}{\sinh kh}\sin(kx_1)\cos(\sigma t) = -A\cos(\sigma t)$$
$$\xi = \frac{H}{2}\frac{\sinh k(h+z_1)}{\sinh kh}\cos(kx_1)\cos(\sigma t) = B\ \cos(\sigma t)$$

$$\left(\frac{\zeta}{A}\right)^2 + \left(\frac{\xi}{B}\right)^2 = 2\cos^2(\sigma t) \ \neq 1$$

Since ζ is the x-component of the position and ξ is the z-component, we can write the position vector $\boldsymbol{r}$ **as:**

$$\boldsymbol{r} = \zeta\hat{\boldsymbol{\imath}} + \xi\hat{\boldsymbol{k}}$$

$$with\ magnitude\ of:$$

$$|\boldsymbol{r}| = \sqrt{A^2 + B^2}\ \cos\sigma t$$

The particles move back and forth along a straight line and the line of motion has the slope:

$$\frac{\zeta}{\xi} = -\frac{\tanh k(h+z)}{\tan kx}$$

At the bottom, $z = -h$, $\tanh k(h+z) = 0$, and the trajectories are horizontal as expected and required.

7.6 Partial Standing Waves

As the name implies, when waves are reflected off irregular objects, as is most often the case, the irregularities prevent pure reflection, and we end up with a partial standing wave.

Reflection- Waves are partially reflected by obstacles, bridge waters, and beaches. Reflected waves are smaller than the incident wave due to dissipation of wave energy.

$$\eta_t = \frac{H_i}{2}\cos(kx - \sigma t) + \frac{H_r}{2}\cos(kx + \sigma t + \epsilon)$$

$$H_i > H_r$$

$$\Omega_i = \; kx - \sigma t \; (wave\ propogating\ to\ hte\ right)$$
$$\Omega_r = kx + \sigma t + \; \epsilon; \;\; (wave\ propogating\ to\ the\ left)$$
$$where\ \epsilon\ is\ the\ phase\ lag$$

$$T_i = T_r$$

Using Trigonometric identities, the total surface displacement is:

$$\eta_t = \left[\frac{H_i}{2}\cos kx + \frac{H_r}{2}\cos(kx + \epsilon)\right]\cos\sigma t + \; \left[\frac{H_i}{2}\sin kx - \frac{H_r}{2}\sin\,(kx + \epsilon)\right]\sin\sigma t$$

$$\eta_t = I(x)\cos\sigma t + F(x)\sin\sigma t$$

To find the total surface displacement of η_x, take derivatives and set to 0.

$$\frac{\partial\eta_t}{\partial t} = -I(x)\sigma\sin\sigma t + F(x)\sigma\cos\sigma t = 0$$

$$\tan(\sigma t)_{max} = \frac{F(x)}{I(x)}$$

If we look at the extreme values for insight, we note:

$$\eta_t(x) = -\sqrt{\left(\frac{H_i}{2}\right)^2 + \left(\frac{H_r}{2}\right)^2 + \frac{H_i\,H_r}{2}\cos(2x + \epsilon)}$$

$$(\eta_t)_{max} = -\sqrt{\left(\frac{H_i}{2} + \frac{H_r}{2}\right)^2}$$

$$(\eta_t)_{max} = \frac{1}{2}(H_i + H_r) \;\;\; quasi\ antinodes$$

$$(\eta_t)_{min} = \frac{1}{2}(H_i - H_r) \;\;\; quasi\ nodes$$

$$H_i = (\eta_t)_{max} + (\eta_t)_{min}$$

$$H_r = (\eta_t)_{max} - (\eta_t)_{min}$$

Reflection coefficient:

$$k_r = \frac{H_r}{H_i}$$

7.7 Pressure

The pressure under a wave is the sum of two components, the hydrostatic component and dynamic component.

7.7.1 Fluid pressures under a progressive wave

In order to invoke pressure we need to use the Bernoulli equation which relates pressures in our flow, the Bernoulli equation can be written in a variety of forms depending on the units of each of the terms that make up the equations (head, m; pressure, $\frac{N}{m^2}$, velocity squared).

$$-\frac{d\emptyset}{dt}+\frac{P}{\rho}+\frac{1}{2}(u^2+w^2)+gz=0$$

From the linear Bernoulli Equation:

$$P=-\rho gz+\rho\frac{\partial\Phi}{\partial t}$$

$$(Nonlinear: P=-\rho gz+\frac{\rho}{2}(u^2+w^2)+\rho\frac{\partial\Phi}{\partial t})$$

The first term of the linear equation is the hydrostatic pressure due to the still fluid. The second term is the dynamic pressure due to the motion of the wave.

➔ Substituting in the expression for Φ:

$$P=-\rho gz+\rho g\frac{H}{2}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\sigma t)$$

Often the factor $\frac{\cosh k(h+z)}{\cosh kh}$ is referred to as the Pressure response factor $K_p(z)$.

Thus:

$$K_p(z)=\frac{\cosh k(h+z)}{\cosh kh}$$

$$\mathrm{P(z)}=\rho g\big(\eta K_p(z)-z\big)\ or\ P(z)=-\rho g(z-\eta K_p(z))$$

The dynamic contribution of the wave increases the pressure under a crest and reduces it under a trough. However, the dynamic contribution is reduced at the bottom by:

$$\eta K_p(-h)=\frac{1}{\cosh kh}\eta$$

The attenuation of the dynamic pressure is a limiting factor in using bottom mounted pressure gages for wave measurement. For any given depth h, wave number k increases with increasing σ. Thus $\cosh(kh)$ increases and K_p decreases. Pressure gages usually have a high frequency limit above which waves go undetected.

$$k = \frac{2\pi}{L} \text{ gets big as L gets small}$$
$$\sigma = \frac{2\pi}{T} \text{ gets big as T gets small}$$
$$and\ as\ \sigma\ \uparrow; k \uparrow\ or\ as\ T \downarrow; L \downarrow$$
$$as\ \sigma\ \uparrow; kh \uparrow\ relative\ depth\ gets\ deeper$$

Short period waves have small K_p at the bottom; researcher may not be able to pick up the wave using pressure gauges. For Long period waves, $K_p \to 1$

7.7.2 Fluid pressures under a standing wave

$$P = -\rho g z + \rho \frac{\partial \Phi_s}{\partial t}$$

We then substitute in the velocity potential for a standing wave and solve to get;

$$P(z) = -\rho g (z - \eta_s K_p(z))$$

Where $K_p(z) = \frac{\cosh k(h+z)}{\cosh kh}$, is the same as for progressive wave.

At nodes $P = -\rho g z$

At antinodes the force on the wall is found by integrating the $P(z)$ over depth:

$$F = \int_{-h}^{\eta} P(z) dz$$

7.8 Energy and Energy Propagation in progressive waves

Energy is important as it allows us to determine how waves change, determine the power for energy extraction and wave generation.

We remember from physics that there are 2 kinds of energy:

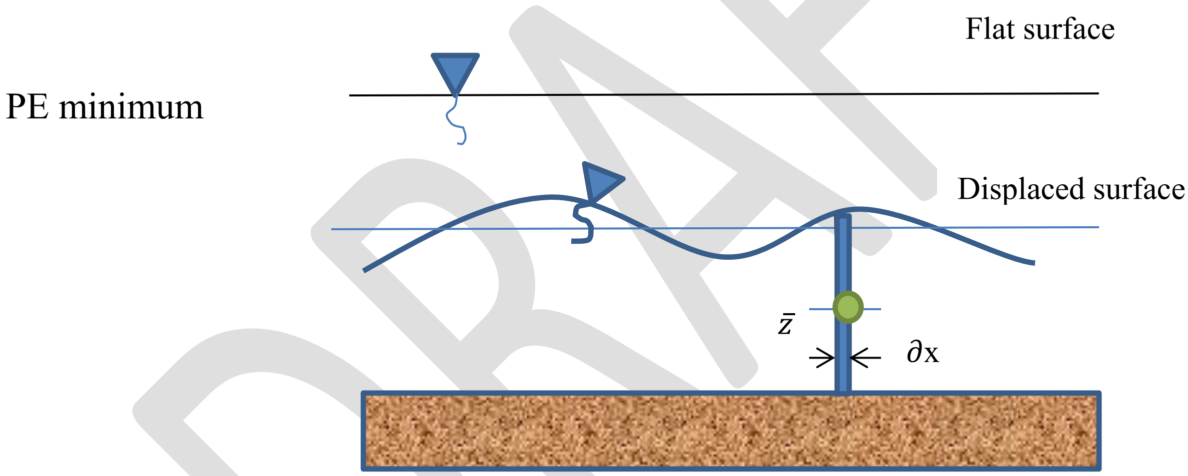
- Potential Energy: displacement of free surface against gravity

$$PE = mgz$$

- Kinetic Energy: water particles in fluid that are moving

$$KE = \frac{1}{2}mv^2$$

7.8.1 Potential Energy :



The differential mass associated with the column of fluid is:

$$dm = \rho(h + \eta)dx$$

And the potential energy associated with that differential column is:

$$d(PE) = dm(g\bar{z});\ where\ \bar{z} = \frac{h + \eta}{2}$$

$$d(PE) = \rho g \frac{(h + \eta)^2}{2} dx$$

Now if we integrate this energy over one wavelength L, for a wave with height H, and then average the energy over that same wavelength, L, we obtain the wave averaged potential energy.

$$\text{'}wave\ averaging\text{'}:\ \frac{1}{L}\int_{x}^{x+L}\ dx$$

Instead of averaging over one wavelength, we could have chosen to average over one wave period, this is termed, phase averaging:

$$\text{'}phase\ averaging\text{'}:\ \frac{1}{T}\int_{t}^{t+T}\ dt$$

$$wave\ averaged\ PE =$$

$$PE = \frac{1}{L}\int_{x}^{x+L} \rho g\ \frac{(h+\eta)^2}{2}dx$$

$$= \frac{\rho g}{L}\int_{x}^{x+L}\left(\frac{h^2}{2} + h\eta + \frac{\eta^2}{2}\right)dx$$

$$= \frac{\rho g}{L}\left(\frac{h^2}{2}L + h\int_{x}^{x+L}\eta dx + \int_{x}^{x+L}\frac{1}{2}\eta^2 dx\right)$$

(PLBC tells us that $\eta(x) = \eta(x+L)$)

$$\eta = \frac{H}{2}\cos(kx - \sigma t)$$

So $\int_{x}^{x+L}\eta dx = 0$

$$\eta^2 = \frac{H^2}{4}cos^2(kx - \sigma t)$$

$$Let\ \alpha = kx - \sigma t$$

$$= \frac{H^2}{4}\left(\frac{1}{2}(1 + \cos 2\alpha)\right)$$

$$= \frac{\rho g h^2}{2} + \frac{\rho g}{L} \int_x^{x+L} \left(\frac{1}{2}\left(\frac{H^2}{8}\right) + \frac{1}{2}\cos 2\alpha \right) dx$$

Again the integral over a complete wavelength (or period), $\int_x^{x+L} (\cos 2\alpha)\, dx = 0$; since cos is periodic.

$$PE = \underbrace{\frac{\rho g h^2}{2}}_{\text{PE due to depth w/o waves}} + \underbrace{\rho g \frac{H^2}{16}}_{\text{PE waves}}$$

$$PE_{waves} = \rho g \frac{H^2}{16}$$

Dependent only on wave height

7.8.2 Kinetic Energy

We can obtain the equation for the Kinetic energy in much the same fashion.

$$KE = \frac{1}{2}\, mv^2$$

The differential Kinetic Energy is then:

$$d(KE) = \frac{1}{2} dm(u^2 + w^2)$$

$$d(KE) = \ dm\frac{(u^2 + w^2)}{2} = \rho\, dx\, dz\, \frac{(u^2 + w^2)}{2} = \ \rho\, \frac{(u^2 + w^2)}{2} dx\, dz$$

Substituting in for u & w

$$u = \frac{H}{2}\sigma\frac{\cosh k(h + z)}{\sinh kh}\cos(kx - \sigma t)$$

$$w = \frac{H}{2}\sigma\frac{\sinh\ k(h + z)}{\sinh kh}\sin(kx - \sigma t)$$

$$d(KE) = \frac{1}{2}\rho\frac{H^2\sigma^2}{4(\sinh kh)^2}[\cosh^2 k(h + z)\cos^2(kx - \sigma t) + \sinh^2 k(h + z)\sin^2(kx - \sigma t)]dxdz$$

Useful Identities:

$$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$$

$$\sinh^2 x = \frac{1}{2}\cosh 2x\ - \frac{1}{2}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$let\ \ q = k(h + z), \Omega = kx - \sigma t$$

$$\cosh^2 q \cos^2 \Omega + \sinh^2 q \sin^2 \Omega = \left[\frac{1}{2}\cosh 2q + \frac{1}{2}\right]\left[\frac{1}{2} + \frac{1}{2}\cos 2\Omega\right] + \left[\frac{1}{2}\cosh 2q - \frac{1}{2}\right]\left[\frac{1}{2} - \frac{1}{2}\cos 2\Omega\right]$$

$$= \left(\frac{1}{2}\frac{1}{2}\cosh 2q + \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\cos 2\Omega + \frac{1}{2}\frac{1}{2}\cosh 2q \cos 2\Omega\right) + \left(\frac{1}{2}\frac{1}{2}\cosh 2q - \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\cos 2\Omega - \frac{1}{2}\frac{1}{2}\cosh 2q \cos 2\Omega\right)$$
$$= \left(\frac{1}{2}\frac{1}{2}\cosh 2q + \frac{1}{2}\frac{1}{2}\cos 2\Omega\right) + \left(\frac{1}{2}\frac{1}{2}\cosh 2q + \frac{1}{2}\frac{1}{2}\cos 2\Omega\right)$$
$$= \left(\frac{1}{2}\cosh 2q + \frac{1}{2}\cos 2\Omega\right)$$

Substitute back in for q and Ω and into our equation for KE:

$$= \frac{1}{2}\rho \frac{H^2\sigma^2}{4(\sinh kh)^2}\left[\frac{1}{2}\left[\cosh 2\big(k(h+z)\big) + \cos 2(kx - \sigma t)\right]\right] dxdz$$

We have rid ourselves of the pesky squared terms and now can integrate.

Integrate over depth and average over time or wavelength

$$\overline{KE} = \frac{1}{16}\rho \frac{H^2\sigma^2}{\sinh^2 kh}\frac{1}{L}\int_x^{x+L}\int_{-h}^{\eta} \cosh 2k(h+z) + \cos 2(kx - \sigma t) dz\, dx$$

$$let\ X = \frac{1}{16}\frac{\rho H^2\sigma^2}{\sinh^2 kh}$$

Derive KE:

$$\overline{KE} = \frac{X}{L}\int_x^{x+L}\int_{-h}^{\eta} \cosh 2k(h+z) + \cos 2(kx - \sigma t)\, dz\, dx$$

First integrate over depth

$$= \frac{X}{L}\int_x^{x+L}\left[\int_{-h}^{\eta\approx 0}\cosh 2k(h+z)dz + \int_{-h}^{\eta\approx 0}\cos 2(kx - \sigma t)\, dz\right] dx$$
$$= \frac{X}{L}\int_x^{x+L}\left[\frac{1}{2k}\sinh 2k(h+z)\,\Big|_{-h}^{0} + \cos 2(kx - \sigma t)(z)\Big|_{-h}^{0}\right] dx$$

$$= \frac{X}{L}\left[\int_{x}^{x+L} \frac{1}{2k}\ (\sinh 2k(h+0)\ - \sinh 2k(h-h)) \right.$$
$$\left. + (\cos 2(kx-\sigma t)(0) - \cos 2(kx-\sigma t)(-h))dx \right]$$
$$= \frac{X}{L}\left[\int_{x}^{x+L} \frac{1}{2k}\ \sinh 2kh\ dx\ + \int_{x}^{x+L} h \cos 2(kx-\sigma t)\ \ dx\right]$$

Second integrate over wavelength

$$= \frac{X}{L}\left[\frac{1}{2k}\sinh 2kh\ (x)|\begin{matrix} x+L \\ x \end{matrix} + h\int_{x}^{x+L} \cos 2(kx-\sigma t)\ dx\right]$$
$$= \frac{X}{L}\left[\frac{1}{2k}\sinh 2kh\ (x+L-x) +\ 0\right]$$
$$= \frac{X}{L}\frac{1}{2k}(\sinh 2kh)\ L$$
$$= X\ \frac{\sinh 2kh}{2k}$$
$$= \frac{1}{16}\frac{\rho H^2\sigma^2}{\sinh^2 kh}\ \frac{\sinh 2kh}{2k}$$
$$= \frac{\rho H^2}{16}\ \frac{gk \tanh kh}{2k \sinh^2 kh}\sinh 2kh$$
$$= \frac{\rho g H^2}{16}\frac{1}{2}\frac{\frac{\sinh kh}{\cosh kh}}{\sinh^2 kh}\ (sin2kh)$$
$$= \frac{\rho g H^2}{16}\ \frac{\frac{1}{2}\sinh 2kh}{\cosh kh \sinh kh}$$
$$= \frac{\rho g H^2}{16}$$

$$KE = \frac{\rho g H^2}{16}$$

Alternate derivation:$\frac{KE}{unit\ volume} = \frac{1}{2}p(u^2+w^2)$ $\qquad Q = k(h+z), q = kh$

$$= \frac{1}{2}\ \rho\left[\frac{H^2\sigma^2}{4\sinh^2(q)}\right]\left[\frac{1}{2}\cosh 2Q + \cos 2(kx-\sigma t)\right]$$

Integrate over depth, then average over time for average KE per unit surface area

$$KE = \frac{1}{16}\rho\frac{H^2\sigma^2}{\sinh^2 q}\ \frac{1}{T}\int_0^T\int_{-h}^{\eta}(\cosh 2Q + \cos 2(kx - \sigma t))\,dz\,dt$$

$$= \frac{\rho H^2\sigma^2}{16\sinh^2 q}\frac{1}{T}\int_0^T \frac{1}{2k}\sinh 2q\,dT$$

Constant in time

Use σ^2= gk tanh(kh)

$$= \frac{\rho g H^2}{16}\left[\frac{\sinh 2q}{2\sinh q\cosh q}\right]$$

$$\frac{KE}{unit\ volume} = \frac{\rho g H^2}{16}$$

7.8.3 Total Average Energy

Energy in wave is equally divided between energy contained in difference in water elevation and still water level→energy due to water movement

We find that PE=KE for our system, therefore the total average energy per unit surface are of a wave is given by

$$E = PE + KE = \frac{\rho g H^2}{16} + \frac{\rho g H^2}{16} = \frac{1}{8}\rho g H^2$$

for all linear waves.

7.9 Energy and Energy flux

$$E = \frac{1}{8}\rho g H^2$$

$$\left[\frac{kg}{m^3}\frac{m}{s^2}m^2\right] \equiv \left[\frac{N}{m}\right] \equiv \left[\frac{J}{m^2}\right]$$

For all linear waves.

And the total energy per unit crest width is

$$E_L = \frac{1}{8}\rho g H^2 L$$

$$\left[\frac{kg}{m^3}\frac{m}{s^2}m^2 m\right] \equiv [N] \equiv \left[\frac{J}{m}\right]$$

Example: for a wave L=50m, H=1m, calculate the Energy:

$$\mathrm{E} = 1263.0\ N/m \equiv \left[\frac{J}{m^2}\right]$$

Or

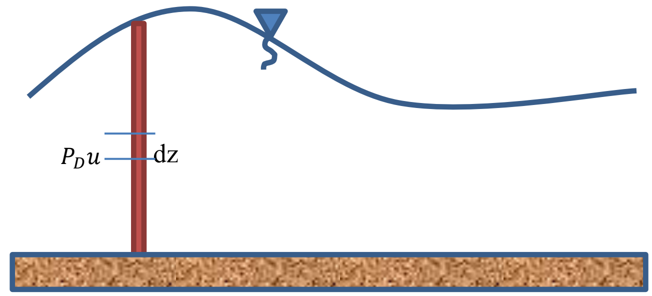
$$\mathrm{E_L} = 63{,}152\ N = 63kN = 6.3\ ton\ force$$

Relate this to a value that we can understand:

NAME	DEFINITION	UNIT
Force	push or pull = mass × acceleration	$N = kg\frac{m}{s^2}$
		$10kN \approx 1\ ton$
Energy	Capacity to do work = force × distance	$J = N * m$
Avg. Energy Per unit wave area		$J/m^2 = N/m$
Avg. Energy Per unit wave crest		$J/m = N$
Power	energy ÷ time	$J/s = N * m/s$
Wave energy flux Or Power / unit crest length	Average energy × speed of propagation per crest length	$\frac{J}{s}/m = N/s$

Energy flux: rate of work done due to wave's rate of transport of wave energy. The rate at which the dynamic pressure is doing work on neighboring column of water

$d\mathfrak{I} = \int_{-h}^{\eta} P_D u\, dz$



Integrate $P_D u$ over the column of water, with $P_D u$ is the dynamic pressure acting in the direction of wave propagation.

The average energy flux (pressure times area times velocity) is:

$$\mathfrak{I} = \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} P_D u\, dzdt$$

'Phase averaged' average over one period

$$\mathfrak{I} = \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} [\,\rho g \eta \frac{\cosh k(h+z)}{\cosh kh}\,] \left[\frac{g}{\sigma}\frac{Hk}{2}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx-\sigma t)\right] dzdt$$

Use dispersion eq. to rewrite u

$$= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} \rho g \eta \frac{\cosh k(h+z)}{\cosh kh}\frac{H}{2}\sigma\frac{\cosh k(h+z)}{\sinh kh}\cos(kx-\sigma t) dzdt$$

use definition of η to rewrite u

$$= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} \rho g \eta \frac{\cosh k(h+z)}{\cosh kh}\sigma\eta\frac{\cosh k(h+z)}{\sinh kh} dzdt$$

$$= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} \rho g \eta^2 \sigma \frac{\cosh^2 k(h+z)}{\cosh kh \sinh kh}\, dzdt$$

Recall that $\cosh kh \sinh kh = \frac{1}{2}\sinh 2kh$

$$= \frac{1}{T}\int_{t}^{t+T}\int_{-h}^{\eta} \frac{2\rho g \eta^2 \sigma}{\sinh 2kh}\ \cosh^2 k(h+z)\ dzdt$$

Recall that $\cosh^2 k(h+z) = \frac{1}{2}(1+\cosh 2k(h+z))$

$$= \frac{1}{T}\int_{t}^{t+T} \frac{2\rho g\eta^2\sigma}{\sinh 2kh} \int_{-h}^{\eta} \frac{1}{2}(\cosh 2k(h+z)+1)\ dzdt$$

for small η, $\eta \approx 0$

$$= \frac{1}{T}\int_{t}^{t+T} \frac{2\rho g\eta^2\sigma}{\sinh 2kh} \frac{1}{2}\left(\frac{1}{2k}\sinh 2k(h+z)+z\right)\Big|_{-h}^{0} dt$$

$$where \quad \frac{1}{2}\left[\frac{1}{2k}\sinh 2k(h+z)+z\right]\Big|_{-h}^{0}$$

Factor out $\frac{1}{2k}$

$$= \frac{1}{2}\frac{1}{2k}[\sinh 2k(h+z)+2kz]\Big|_{-h}^{0}$$

$$= \frac{1}{4k}\ [(\sinh 2k\ \ (h+0)+2k0)-\left(\sinh 2k\left(h+(-h)\right)+\left(2k(-h)\right)\right]$$

$$= \frac{1}{4k}\ [\sinh 2kh+0-0+2kh]$$

$$= \frac{1}{4k}\ [\sinh 2kh+2kh\,]$$

$$= \frac{1}{T}\int_{t}^{t+T} \frac{2\rho g\eta^2\sigma}{\sinh 2kh}\frac{1}{4k}[\sinh 2kh+2kh]\ dt$$

η^2 *is the only variable that is a function of time.*

$$= \frac{1}{T}\frac{2\rho g\sigma}{\sinh 2kh}\frac{1}{4k}[\sinh 2kh+2kh]\int_{t}^{t+T}\left(\frac{H}{2}\right)^2\ \cos^2(kx-\sigma t)\ dt$$

Recall trig identity $\cos^2(kx-\sigma t) = \frac{1}{2}\ (1+\cos 2(kx-\sigma t))$

$$= \frac{1}{T}\ \frac{2\rho g\sigma}{4k}\ \frac{\sinh 2kh+2kh}{\sinh 2kh}\ \frac{H^2}{4}\int_{t}^{t+T}\frac{1}{2}\ (1+\cos 2(kx-\sigma t))\ dt$$

$$= \frac{1}{T}\ \frac{\rho g}{2}\ C\ \left(1+\frac{2kh}{\sinh 2kh}\right)\frac{H^2}{4}\left(\frac{1}{2}T\ \right)$$

$$= \frac{\rho g H^2}{8} \frac{1}{2} \ C \ \left(1 + \frac{2kh}{\sinh 2kh}\right)$$
$$= EC \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

Define:

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

$$= ECn$$
$$\mathfrak{J} = ECn$$

Cn is the speed at which energy is transported in a wave.

Let's have a look at n:

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

For big kh $\quad \sinh 2kh \rightarrow \frac{1}{2}\ e^{2kh}$

$\frac{2kh}{\frac{1}{2}e^{2kh}} \rightarrow 0, and\ n \rightarrow \frac{1}{2}$ deepwater

For small kh $\quad \sinh 2kh \rightarrow 2kh$

$\frac{2kh}{2kh} = 1, and\ n = 1$ shallow water

Energy moves slower than individual waves.

Let's examine wave travel a bit more and see if we can shed more light on the speed of energy travel Cn

Take 2 waves travelling in the same direction with slightly different wave #'s, k, and frequencies σ.

$$k_1 = k - \frac{\Delta k}{2} \qquad \sigma_1 = \sigma - \frac{\Delta \sigma}{2}$$

$$k_2 = k + \frac{\Delta k}{2} \qquad \sigma_2 = \sigma + \frac{\Delta \sigma}{2}$$

By superposition we obtain

$$\eta = \eta_1 + \eta_2 = \frac{H_1}{2}\cos(k_1 x - \sigma_1 t) + \frac{H_2}{2}\cos(k_2 x - \sigma_2 t)$$

If $H1 = H2 = H$ and Recall trig identity for $\cos(k_1 x - \sigma_1 t) + \cos(k_2 x - \sigma_2 t)$

$$= \frac{H}{2}\left[2\cos\frac{1}{2}((k_1 x - \sigma_1 t) + (k_2 x - \sigma_2 t))\cos\frac{1}{2}((k_1 x - \sigma_1 t) - (k_2 x - \sigma_2 t))\right]$$

$$k_1 + k_2 = 2k \;\; ; \;\; k_1 - k_2 = -\Delta k \, ; \;\; -\sigma_1 + -\sigma_2 = -2\sigma \, ; \;\; -\sigma_1 + \sigma_2 = \Delta\sigma$$

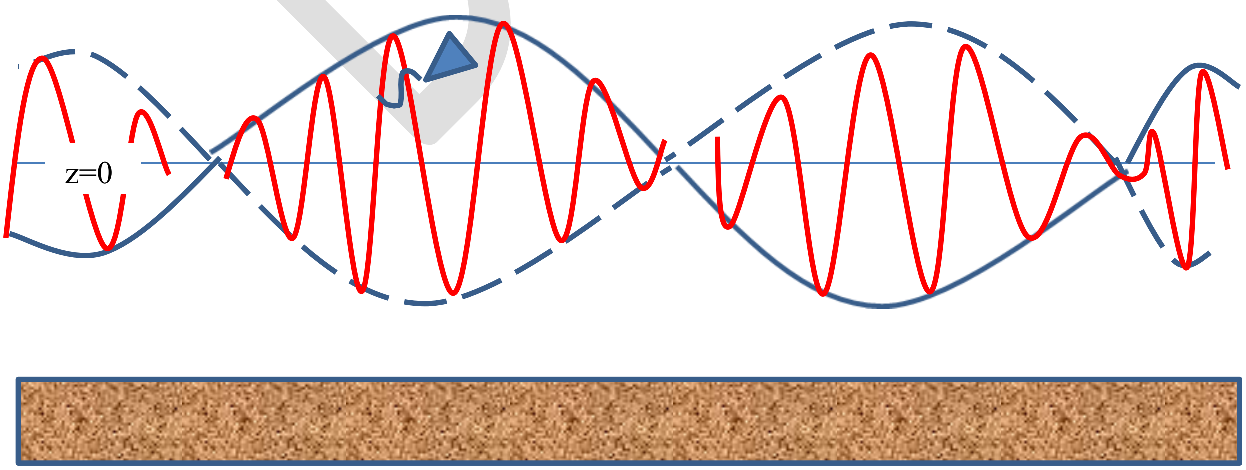
$$= \frac{H}{2}\; 2\cos\frac{1}{2}(2kx - 2\sigma t)\cos\frac{1}{2}(-\Delta kx + \Delta\sigma t)$$

Reduce and factor out Δk

$$\eta = \frac{H}{2} 2\cos(kx - \sigma t)\cos\left(\frac{\Delta k}{2}\left(-x + \frac{\Delta\sigma}{\Delta k}t\right)\right)$$

$$\eta = H\cos(kx - \sigma t)\cos\left(\frac{\Delta k}{2}\left(x - \frac{\Delta\sigma}{\Delta k}t\right)\right)$$

Our new wave form, $Hcos(kx - \sigma t)$ consists of waves moving with velocity $C = \sigma/k$, these are contained (modulated) in an envelope moving with speed $\Delta\sigma/\Delta k$.



The envelope is our wave group & it moves with celerity of

$$C_g \equiv \ group\ velocity = \ \Delta\sigma/\Delta k$$

If we take derivatives of dispersion equation with respect to k

$$\sigma^2 = gk \tanh kh$$
$$\frac{d}{dk}\sigma^2 = \frac{d}{dk} gk \tanh kh$$
$$\frac{d}{dk}\sigma^2 = \left(\frac{d}{dk} gk\right) \tanh kh + gk\left(\frac{d}{dk} \tanh kh\right)$$
$$2\sigma\frac{d\sigma}{dk} = g \tanh kh + gkh \operatorname{sech}^2 kh$$
$$\frac{d\sigma}{dk} = \frac{g}{2\sigma}\ (\tanh kh + kh \operatorname{sech}^2 kh)$$

Substitute dispersion equation into denominator

$$\frac{d\sigma}{dk} = \frac{g\sigma}{2gk}\left(\frac{\tanh kh + kh \operatorname{sech}^2 kh}{\tanh kh}\right)$$
$$\frac{d\sigma}{dk} = \frac{1}{2}C\left(1 + \frac{kh \operatorname{sech}^2 kh}{\tanh kh}\right)$$
$$\frac{kh \operatorname{sech}^2 kh}{\tanh kh} = \left(\frac{kh \cosh kh}{\sinh kh\ \cosh^2 kh}\right)$$

$$\frac{d\sigma}{dk} = \frac{1}{2}C\ \left(1 + \frac{kh}{\sinh kh\ \cosh kh}\right) = C\frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

$$\boxed{\sinh kh \cosh kh\ = \frac{1}{2}\sinh 2kh}$$

$$\frac{d\sigma}{dk} = \frac{\Delta\sigma}{\Delta k} = nC$$

$$\frac{\Delta\sigma}{\Delta k} = C_g \quad \therefore\ C_g = nC$$

Relates group velocity to
wave phase velocity

$$n = \frac{C_g}{C}$$

And energy travels at

$$\boxed{F = EnC = ECg \quad = Group\ velocity!}$$

As kh changes, wave properties change.

Wave Length L ↓ decreases

Phase speed C ↓ decreases

Group velocity → phase velocity $C_g = C$

Wave height increases shoaling = H ∝ h break

Wave angle decreases to Perpendicular

Youtube wave – phase vs group

https://www.youtube.com/watch?v=EIqKG5TiSYs

7.10 Waves entering shallow water

As waves propagate from deep water towards shore, they undergo transformations, as the water depth gets shallower. This transformation process is of particular interest to the coastal engineer. Using our knowledge of wave transformations enables the engineer to predict the wave conditions at the project site using wave information collected long distances away at some deep-water buoy location.

We first convince ourselves that the wave period remains constant as waves propagate into shallow waters. The conceptual argument rests in the fact that the both the wavelength and wave speed decrease as waves move into shallower waters. In order for both to decrease, the period must remain constant or change only slightly. We will develop a more rigorous argument establishing that for steady waves the wave period remains essentially constant.

7.10.1 Conservation of waves

Let's look at 2d waves

First, define Ω such that,

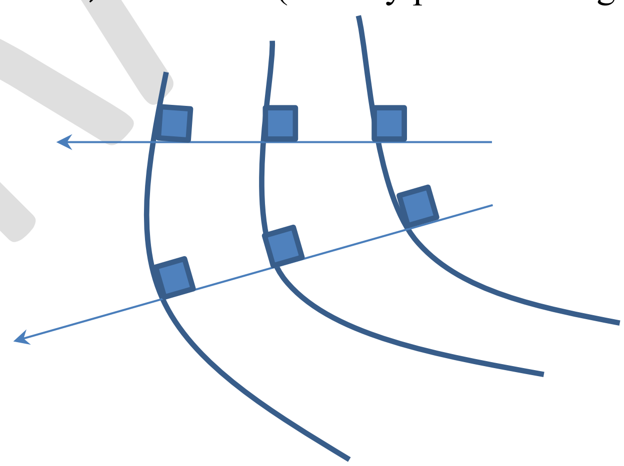
$$\eta = \frac{H}{2}\cos\Omega$$

for waves travelling in the x direction,

$\Omega = (kt - \sigma t)$ and crests occur when $\Omega = 0, 2\pi \text{ or } 2n\pi$ (n is any positive integer)

And $\eta = \frac{H}{2} \equiv max\ value\ \equiv crest$

If we look at waves traveling, the direction of propagation is perpendicular to the wave crest



We call this the wave ray

The unit normal, $\vec{\boldsymbol{n}}$, to the direction of travel is equal to the normal vector $\vec{\boldsymbol{N}}$ divided by the magnitude of the gradient of Ω

$$\vec{\boldsymbol{N}} = \boldsymbol{n}|\nabla\Omega| = \vec{\boldsymbol{k}}$$

Define wave number, $\vec{\boldsymbol{k}}$, as the vector oriented to the direction of wave propagation:

$$\vec{\boldsymbol{k}} = \nabla\boldsymbol{\Omega} = \boldsymbol{n}|\nabla\Omega|$$

Where $\nabla = \nabla_h = \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}}$ is the horizontal gradient operator

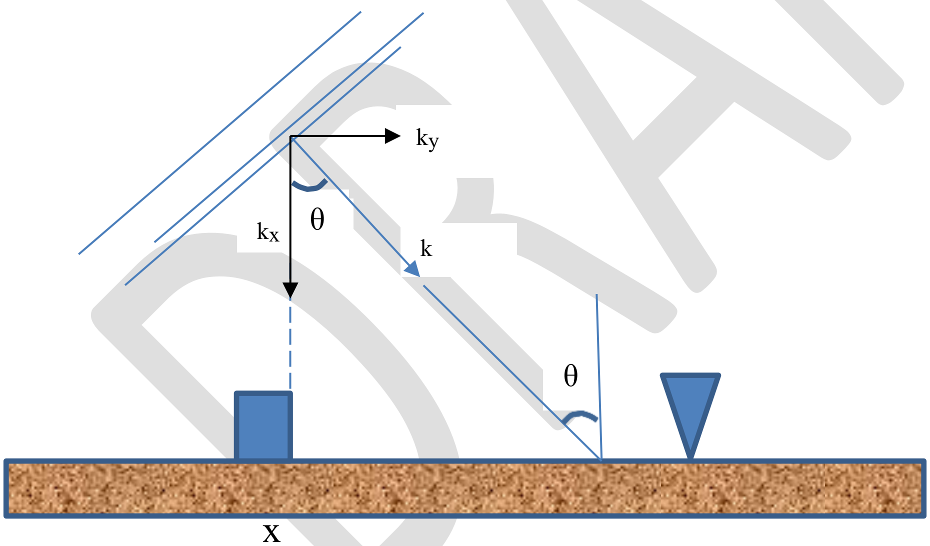
In 2D, $\vec{\boldsymbol{k}} = k_x\hat{\boldsymbol{\imath}} + k_y\hat{\boldsymbol{\jmath}}$

$$|\vec{\boldsymbol{k}}| = k = \sqrt{{k_x}^2 + {k_y}^2}$$

Where k is the wave number and

$$k_x = |\vec{\boldsymbol{k}}|cos\theta \qquad k_y = |\vec{\boldsymbol{k}}|sin\theta$$

Where θ is the wave angle, angle of incidence, the angle made between the beach normal and the wave direction



So, we obtain, $\boldsymbol{\Omega}(x, y, t) = \vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{x}} - \sigma t$

$$= k_x x + k_y y - \sigma t$$

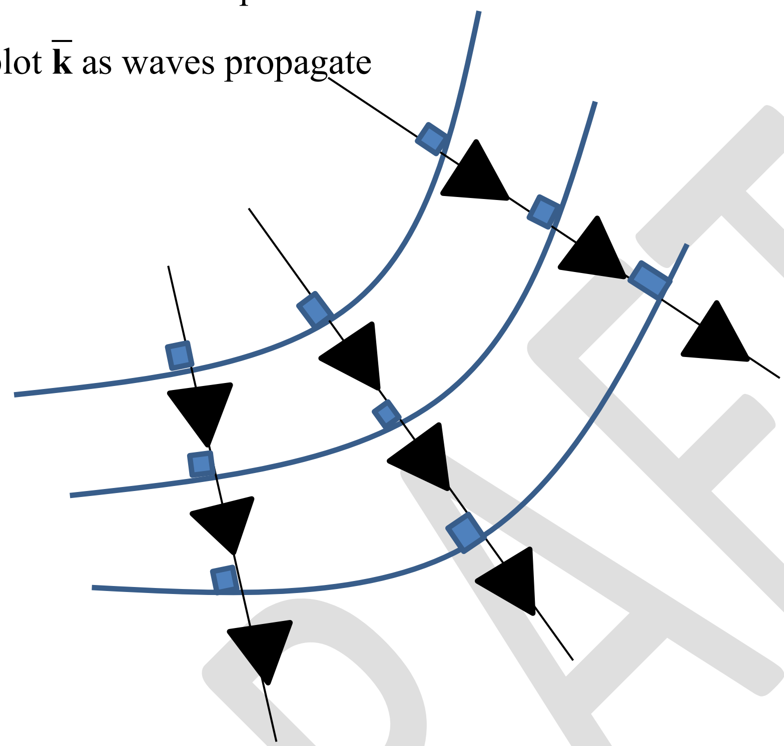
$$= k\cos\theta\ \ x + k\sin\theta\ y\ - \sigma t$$
$$= kx\cos\theta + ky\sin\theta - \sigma t$$

For the case of waves propagating onshore (shore normal), θ=0 and only the x-term remains

$$\boldsymbol{\Omega}(x, y, t) = kx - \sigma t$$

Okay, reduces to 1D equation.

If we plot $\bar{\mathbf{k}}$ as waves propagate



The curved horizontal line everywhere tangent to the wave number, **k**, is called the wave '**ray**.'

<u>Energy travels along the wave ray</u>

We determine wave frequency, σ from the phase

$$\boldsymbol{\Omega} = kx\cos\theta + ky\sin\theta - \sigma t$$

Taking the time derivative of $\boldsymbol{\Omega}$ we get:

$$\sigma = \frac{-\partial\boldsymbol{\Omega}}{\partial t}$$

It follows that

$$\nabla\sigma = \nabla\left(-\frac{\partial\boldsymbol{\Omega}}{\partial t}\right)$$

$$\nabla\sigma + \frac{\partial \nabla \boldsymbol{\Omega}}{\partial t} = 0$$

recall, $\vec{\boldsymbol{k}} = \nabla\boldsymbol{\Omega}$

then:

$$\nabla\sigma + \frac{\partial \vec{\mathbf{k}}}{\partial t} = 0$$

So a spatial change in frequency must be balanced by the time change in k.

$$and\ recall\ that\ \frac{\partial \nabla \boldsymbol{\Omega}}{\partial t} = -\frac{\partial \vec{\boldsymbol{k}}}{\partial t} = 0\ ; for\ steady\ waves$$

Since steady waves do not change in time,

$$\nabla\sigma = 0$$

∴ σ does not change in space even as water depth changes

Frequency and therefore period is constant in space for steady waves

$$-\nabla\sigma = \frac{\partial \boldsymbol{k}}{\partial t}$$

$$\boxed{\frac{\partial \boldsymbol{k}}{\partial t} + \nabla\sigma = 0 \qquad conservation\ of\ waves}$$

Just as we know conservation of mass, we have conservation of waves. Any ***time*** change in L must be accompanied by a ***spatial*** change in T.

7.10.2 Wave refraction

To determine how wave θ changes in space let's look at the curl of k.

$$\vec{\boldsymbol{k}} = \nabla\boldsymbol{\Omega} \rightarrow \nabla\times\vec{\boldsymbol{k}} = \nabla\times\nabla\boldsymbol{\Omega} = 0 \text{ curl of gradient is always zero}$$

Therefore, Curl of wave number vector field ***must*** be zero

$$\nabla\times\nabla\boldsymbol{\Omega} = \nabla\times\vec{\boldsymbol{k}} = \frac{\partial}{\partial x}k_y - \frac{\partial}{\partial y}k_x = 0$$

$$\frac{\partial}{\partial x}k\sin\theta - \frac{\partial}{\partial y}k\cos\theta = 0$$

Chain rule yields;

$$\left[\sin\theta\frac{\partial k}{\partial x} + k\frac{\partial\sin\theta}{\partial x}\right] - \left[\cos\theta\frac{\partial\theta}{\partial y} + k\frac{\partial\cos\theta}{\partial y}\right] = 0$$

$$\sin\theta\frac{\partial k}{\partial x} - \cos\theta\frac{\partial k}{\partial y} + k\cos\theta\frac{\partial\theta}{\partial x} + k\sin\theta\frac{\partial\theta}{\partial y} = 0$$

This equation for θ can be simplified for the unique case of ***straight and parallel offshore contours,*** in that case

$$\frac{\partial}{\partial y} = 0$$

$$\frac{d}{dx}(k\sin\theta) = 0 \quad \therefore \quad k\sin\theta = constant\ in\ x$$

For steady waves σ is also constant so we use a trick and divide through σ

$$\frac{k}{\sigma}\sin\theta = constant$$

$$\frac{\sin\theta}{C} = constant$$

In deep water $C = C_o$ and $\theta = \theta_o$ so,

$$\boxed{\frac{\sin\theta}{C} = \frac{\sin\theta_o}{C_o}}$$

Snell's Law, governing wave refraction

We know that C decreases in shallow water in order for $\frac{\sin\theta}{C}$ to remain constant as C decreases $\sin(\theta)$ must also decrease or θ must decrease.

The result is that waves become more shore normal as they approach the coast.

Waves Refract as they move into shallow water.

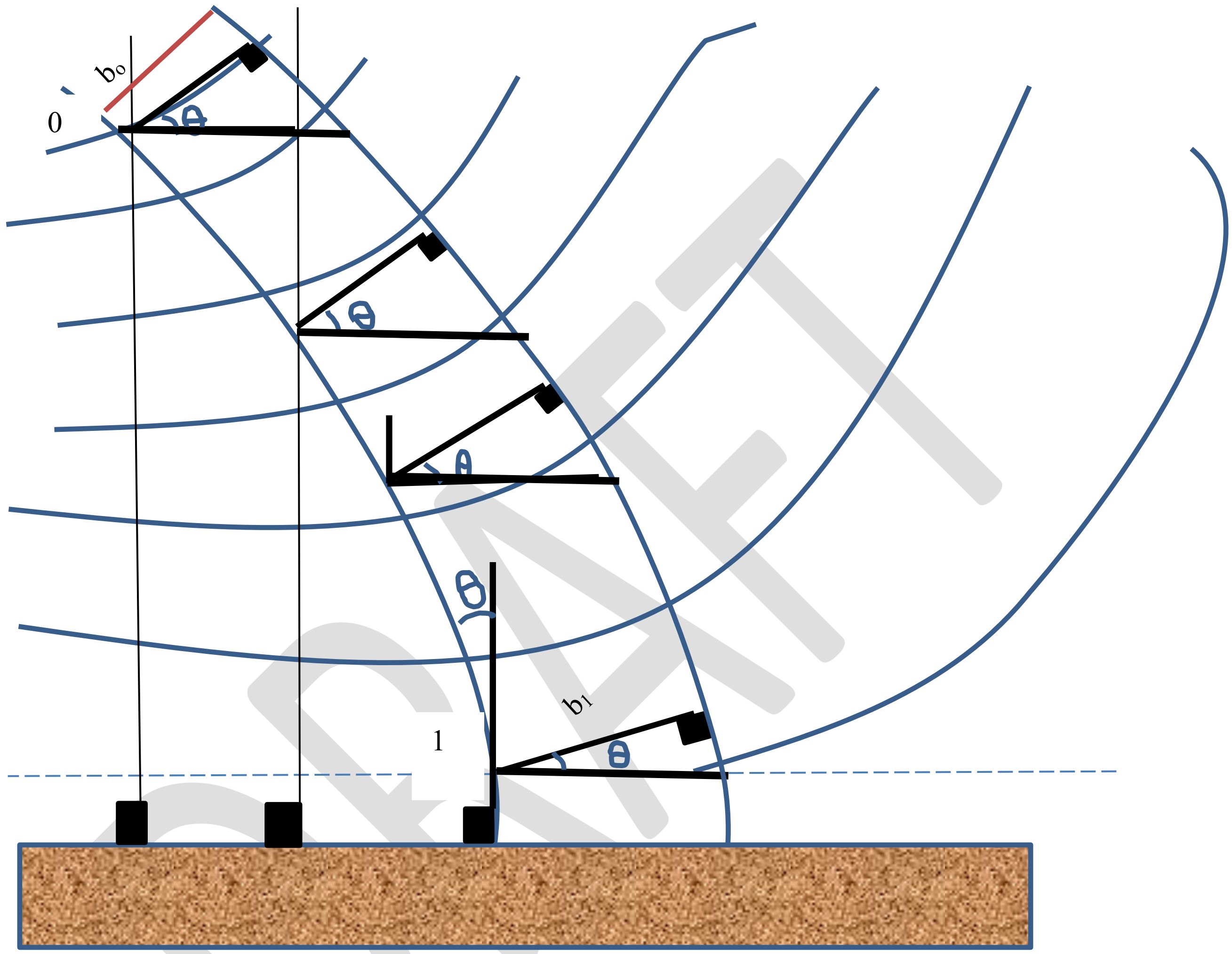
Conceptual discussion:

The reason for the change in wave direction is related to the bathymetric profile of the seafloor. We know that the wavelength shortens as waves move into shallow water, and in response to this shortening of the wavelength, the wave slows down. As waves approach the shore at some oblique angle, the landward portion of the wave crest will reach the shallower waters first. This part of the wave crest will begin to slow down before the rest of the wave that is still travelling over deeper waters. So as one section of the wave begins to move slower than the rest, the wave begins to turn toward the shore.

We can understand conceptually why this occurs and now we can use the equations given above to calculate the change.

7.11 Conservation of Energy

- ***Energy travels along wave Rays***
- ***There is no energy flux between/across wave Rays***



Let's look at 2 wave rays separated by an along crest distance of b_o in deep water.

The total average energy flux along that crest length, b_o *is* $E_oC_on_ob_o$ as the wave refracts and moves toward shore.

At location 1 the total average energy flux along the crest length b_1 *is* $E_1C_1n_1b_1$

Conservation of energy dictates that in absence of dissipation, Energy flux is constant,

$$\boldsymbol{F_o}b_o = \boldsymbol{F}_1b_1$$

$$E_oC_on_ob_o = E_1C_1n_1b_1$$

Our total average energy flux must be conserved though energy density can change.

Thus, we can solve for the change in wave height:

$$E = \frac{1}{8}\rho g H^2$$

$$\frac{1}{8}\rho g {H_o}^2 C_o n_o b_o = \frac{1}{8}\rho g H_1^2 C_1 n_1 b_1$$

$$H_1 = H_o\sqrt{\frac{C_{g_0}}{C_{g_1}}\frac{b_o}{b_1}}$$

$$H_1 = H_o\sqrt{\frac{C_{g_0}}{C_{g_1}}}\sqrt{\frac{b_o}{b_1}}$$

Shoaling Coef.
K_s due to speed

Refraction Coeff
K_r due to direction change

b_1, perpendicular distance always increases as waves shoal

$$b_o = l \quad \cos\theta_o$$

The perpendicular distance b equals the horizontal distance, l, times cos of the angle, b approaches l as waves approach coast.

$$\frac{b_o}{b_1} = \frac{l\cos\theta_o}{l\cos\theta_1} \quad \therefore \quad K_r = \sqrt{\frac{\cos\theta_o}{\cos\theta_1}}$$

$$H_1 = H_o\sqrt{\frac{C_{g_0}}{C_{g_1}}}\sqrt{\frac{\cos\theta_o}{\cos\theta_1}}$$

General Form: $H_2 = H_1 K_r K_s$

$$H_2 = H_1\sqrt{\frac{C_{g_1}}{C_{g_2}}}\sqrt{\frac{\cos\theta_1}{\cos\theta_2}}$$

We can solve by directly calculating C_{g_1} and C_{g_2} and $\cos\theta_2$ using dispersion and Snell's Law or,

Figure 4.18 relates $\theta, \frac{h}{gT^2}$ $to\ K_r$

Figure 4.19 relates $\theta, \frac{h}{gT^2}$ $to\ K_r K_s$

7.11.1 Shoaling:

$$K_s \quad = \quad \sqrt{\frac{C_{g_0}}{C_{g_1}}}$$

Change in wave height due to the slowing down of the wave

As wave slows, L decreases; HOWEVER, Energy = total average energy per unit crest length must be the same!

(Recall: in shallow water, $C = C_g, n = 1$)

At zeroth order:

$$\frac{1}{8}\rho g H_1^2 L_1 = \frac{1}{8}\rho g H_2^2 L_2$$

$$H_1^2 L_1 = H_2^2 L_2$$

Conservation of Average Energy per unit crest width

H increases as L decreases

$E_1 C_1 n_1 = E_2 C_2 n_2$

$E_1 C_1 = E_2 C_2$

$$\frac{1}{8}\rho g H_1^2 \frac{L_1}{T_1} = \frac{1}{8}\rho g H_2^2 \frac{L_2}{T_2}$$

$$H_1^2 L_1 = H_2^2 L_2$$

Conservation of energy flux

Because energy travels at group velocity in intermediate depths we must balance the interplay of n with C in order to find ΔH

From deep to shallow water we find as $n_o = ½$ and $n_{sw} = 1$

$2H_{sw}^2 L_{sw} = H_0^2 L_o$ we see a factor of 2 introduced

$$\frac{H_{sw}}{H_o} = \frac{1}{\sqrt{2}}\sqrt{\frac{L_o}{L_{sw}}}$$

$$H_{sw} = \frac{H_o}{\sqrt{2}}\sqrt{\frac{L_o}{L_{sw}}}$$

7.11.2 Example

Wave H=2m in deep water, propagates toward shore at 30° angle with period of 15 s

Assume straight and parallel contours.

What is the height and direction of 8m water?

<u>Method 1</u>: Solve

$$\theta_o = 30°\,, \qquad H_o = 2m, \quad T = 15s$$

Solve for L_o

$$L_o = \frac{gT^2}{2\pi} = 351\, m \qquad k_o = 0.0179 \frac{rad}{m}$$

$$C_o = \frac{\sigma}{k_o} = 23.4 \frac{m}{s} \qquad \sigma = 0.149 \frac{rad}{s}$$

$$C_g = nC = 11.7 \frac{m}{s}$$

In 8m:

$$L_8 = 129.71\, m \qquad k_o = 0.0484 \frac{rad}{m}$$

$$C_8 = 8.65 \frac{m}{s}$$

$$C_{g8} = 8.24 \frac{m}{s}$$

<u>Snell's Law</u> $\quad k_o \sin\theta_o = k_1 \sin\theta_1$

$$\sin\theta_1 = \frac{k_o}{k_1} \sin\theta_o = 0.370 \sin\theta_o$$

$$\sin\theta_1 = 0.185$$

$$\theta_1 = 10.70$$

<u>Energy Cons</u> $\quad E_o C_{go} \cos\theta_o = E_1 C_{g1} \cos\theta_1$

$$H_o{}^2 C_{go} \cos\theta_o = H_1 C_{g1} \cos\theta_1$$

$$H_1 = H_o \sqrt{\frac{Cg_o}{Cg_1}} \sqrt{\frac{\cos\theta_o}{\cos\theta_1}} 156$$

$$= 2m \sqrt{\frac{11.7}{8.24}} \sqrt{\frac{\cos 30}{\cos 10.7}} = 2m\sqrt{1.42}\,\sqrt{0.881}$$

$$= 2.24\,m$$

<u>Method 2</u>: Use figure 4.19

$$\frac{h}{gT^2} = \frac{8m}{9.81\frac{m}{s^2}(15s)^2} = 0.0036$$

$$\theta_o = 30°$$

$$\theta_1 \approx 10.5°, \quad k_r k_s \approx 1.13$$
$$H_1 = 2m(1.13) = 2.26m$$

<u>Method 3</u>: Wave refFree app

H=2.0 L_o= 351.29 m

T=15 C_o= 23.42 m/s

Θ= 30.0 C_{go}=11.71 m/s

h=8

GO L_8=129.71m

C_8=8.65 m/s

C_{g8}= 8.24 m/s

kr= 0.94

ks=1.19 H_8=2.24m

$Θ_8$=10.6°

7.12 Diffraction

→ Spreading of wave energy laterally to direction of wave propagation.

Important in structures and where waves come to together-caustics

Waves ‘bend’ around objects as energy spreads (moves with crest along a ray.)

The energy between the rays is stretched along the wave crest as one wave ray follows the structure.

Refraction and Diffraction <u>often</u> occur together

Direction and height changes due to Δh	Direction and height changes due to structures

Most structures are in water depths that are shallow enough for wave-bottom interactions. It is important to calculate together, but difficult.

Typically, diffraction is estimation using nomograms from CEM.
Or more modernly computed:

- REF/ DIF,
- STWAVE,
- SWAN, etc.

8 Wave Breaking

2 kinds of Breaking: ***depth limited*** and ***steepness limited***. Waves break when the wave height is approximately equal to the wave depth.

As the wave feels the bottom, friction slows the wave speed; however, the particles in the crest do not feel the friction to the same extent as the particles closer to the bottom.

The wave 'trips' over the bottom and breaks.

$$u_{crest} > C$$

It is of interest to find the water depth and wave height at the point of breaking.

A waves shoal, we have shown that height increases. This becomes unstable and eventually waves break.

$M = \tan\beta =$ beach slope where $\beta =$ angle of the beach slope

8.1 Surf Similarity number – Iribarren Number

In order to determine the extent to which a wave breaks or is reflected off the beach face back offshore, we refer to the Iribarren Number,

$$\xi_0 = \frac{\tan\beta}{\sqrt{\frac{H_o}{L_o}}}$$

or surf similarly parameter based on the deep water wave height.
Can also base parameter on Breaking wave height just set: $H=H_b$

$$\xi_b = \frac{\tan\beta}{\sqrt{\frac{H_b}{L_o}}}$$

$$\xi_0 = \frac{m}{\sqrt{\frac{2\pi H_o}{gT^2}}} \rightarrow \xi_0^2 = \frac{m^2 gT^2}{2\pi H_o}$$

$$\xi_0 = \frac{m}{\sqrt{\frac{H}{L_o}}}$$

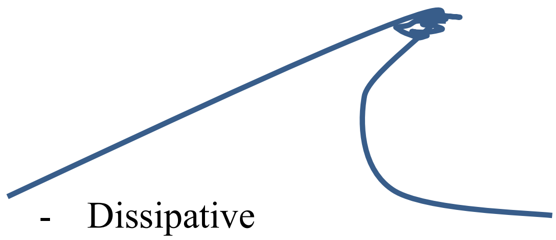
$$\boldsymbol{\xi_o < 0.5 < \xi_o < 3.3 < \xi_o}$$

$$\boldsymbol{\xi_b < 0.4 < \xi_b < 2.0 < \xi_b < 4 < \xi_b}$$

Spilling plunging surging reflection
(no breaking)

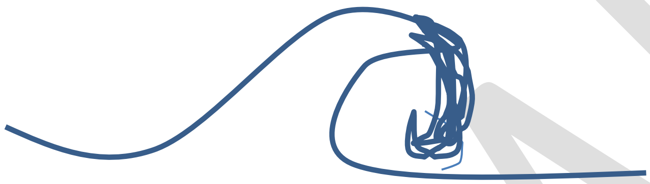
3 kinds of depth limited breaking:

1. Spilling: $\xi_o < 0.5 \; or \; \xi_b < 0.4$ $\frac{H}{mgT^2} > 0.08$



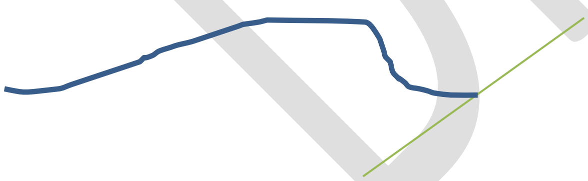
- Dissipative
- Mild sloping beach
- Numerous waves in surf zone
- Gentle breaking, tends to be further from shoreline

2. Plunging: $0.5 < \xi_o < 3.3 \; or \; 0.4 < \xi_b < 2.0$ $0.005 < \frac{H}{mgT^2} < 0.08$



- Dissipative
- Steeper beaches
- Violent breaking
- Aerated in tube

3. Collapsing / Surging: $\xi_o > 3.3 \; or \; 2 < \xi_b < 4$ $\frac{H}{mgT^2} < 0.005$



- Small waves on steep beach
- Breaks weakly on beach face
- Reflection of wave

For, $4 < \xi_b$, Little or no breaking, wave energy is reflected

8.2 Depth limited breaking

Waves break in water depth approximately equal to the wave height.

$$H_b = \kappa h_b$$

The proportion $\kappa \cong 0.78$ (McCowan 1894)
κ can range from 0.4 to 1.2 depending on the dissipation.

Lab studies showed a dependency of κ on Beach slope m:

$$\kappa = b(m) - a(m)\frac{H_b}{gT^2}$$
$$where:$$
$$a(m) = 43.8(1.0 - e^{-19m})$$
$$b(m) = 1.56\,(1.0 + e^{-19.5m})^{-1}$$

As m approaches zero, $\kappa \rightarrow 0.78$

From energy conservation shoaling and refraction

$$H_b = H_o\sqrt{\frac{\frac{1}{2}C_o}{n_b C_b}}\sqrt{\frac{\cos\theta_0}{\cos\theta_b}}$$

We can solve this for independent breaking depth. First assume breaking is a depth limited and wave is in shallow water.

$$H_b = H_o\sqrt{\frac{\frac{1}{2}C_o}{2\sqrt{gh}}}\sqrt{\frac{\cos\theta_0}{1}}$$

Second assume McCowan's Breaking condition,

$$H_b = \kappa h_b$$
$$\kappa \approx 0.78$$
$$\kappa h_b = H_o\sqrt{\frac{C_o\cos\theta_0}{2\sqrt{gh}}}$$

Now solve for h_b

$$\kappa^2 h_b^2 = H_o^2 \frac{C_o \cos\theta_0}{2\sqrt{gh}}$$

$$2\sqrt{gh_b}h_b^2 = \frac{H_o^2}{\kappa^2}\ C_o \cos\theta$$

→

$$h_b^5 = \frac{H_o^4}{4g\kappa^4} C_o^2 \cos\theta_0^2$$

$$h_b = \frac{1}{g^{\frac{1}{5}}\kappa^{\frac{4}{5}}} \frac{(H_o^2 C_o \cos\theta_0)^{\frac{2}{5}}}{4^{\frac{1}{5}}}$$

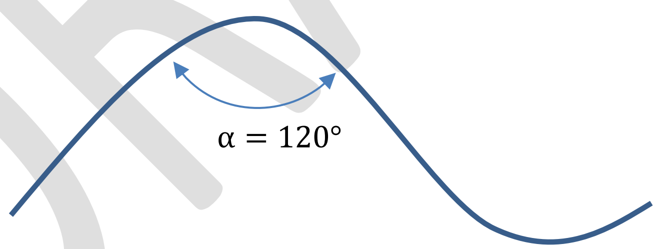
8.3 Steepness limited Breaking

In deep water, limiting wave steepness is:

$$\left(\frac{H_o}{L_o}\right)_{max} \approx 0.17 - 0.14 \ \cong \frac{1}{7}$$

$H\ is \dfrac{1}{7}\ h, L\ is\ 7\ times\ H$

Steepness limited breaking
Minimum sustainable crest angle is 120 degrees



9 Summary of Linear (Airy) Wave theory- Properties

L=wavelength; k=wave number; h= water depth; σ=Angular wave frequency; T=period; C=wave celerity; Cg=group velocity

Relative Depth	Shallow water h/L <1/20	Transitional Water 1/20< h/L <1/2	Deep Water h/L <1/2
Wave Profile	Same as →	$\eta = \frac{H}{2}\cos[kx-\sigma t] = \frac{H}{2}\cos[\Omega]$	← Same as
Wave Length	$L = T\sqrt{gh}$	$L = \frac{gT^2}{2\pi}\tanh(kh)$	$L = L_0 = \frac{gT^2}{2\pi}$
Wave Celerity	$C = \frac{L}{T} = \sqrt{gh}$	$C = \frac{\sigma}{k} = \frac{L}{T} = \frac{gT}{2\pi}\tanh(kh)$	$C = C_0 = \frac{L_0}{T} = \frac{gT}{2\pi} = \frac{g}{\sigma}$
Group Velocity	$C_g = C = \frac{L}{T} = \sqrt{gh}$	$C_g = nC = \frac{1}{2}\left[1+\frac{2kh}{\sinh(2kh)}\right]C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
a) Horizontal particle Velocity	$u = \frac{H}{2}\sqrt{\frac{g}{h}}\cos(\Omega)$	$u = \frac{H}{2}\frac{gk}{\sigma}\frac{\cosh[k(h+z)]}{\cosh(kh)}\cos\Omega$	$u = \frac{H\sigma}{2}e^{(kz)}\cos(\Omega)$
b) Vertical particle Velocity	$w = \frac{H}{2}\sigma\left(1+\frac{z}{h}\right)\sin(\Omega)$	$w = \frac{H}{2}\frac{gk}{\sigma}\frac{\sinh[k(h+z)]}{\cosh(kh)}\sin\Omega$	$w = \frac{H\sigma}{2}e^{(kz)}\sin(\Omega)$
a) Horizontal Particle acceleration	$a_x = \frac{\pi H}{T}\sqrt{\frac{g}{h}}\sin(\Omega)$	$a_x = \frac{H}{2}gk\frac{\cosh k(h+z)}{\cosh(kh)}\sin(\Omega)$	$a_x = \frac{H}{2}\sigma^2 e^{(kz)}\sin(\Omega)$
b) Vertical Particle acceleration	$a_z = -\frac{H}{2}\sigma^2\left(1+\frac{z}{h}\right)\cos(\Omega)$	$a_z = -\frac{H}{2}gk\frac{\sinh k(h+z)}{\cosh(kh)}\cos(\Omega)$	$a_z = -\frac{H}{2}\sigma^2 e^{(kz)}\cos(\Omega)$
a) Horizontal particle displacement	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{h}}\sin(\Omega)$	$\xi = -\frac{H}{2}\frac{\cosh k(h+z)}{\sinh(kh)}\sin(\Omega)$	$\xi = -\frac{H}{2}e^{(kz)}\sin(\Omega)$
b) Vertical particle displacement	$\zeta = \frac{H}{2}\left(1+\frac{z}{h}\right)\cos(\Omega)$	$\zeta = \frac{H}{2}\frac{\sinh k(h+z)}{\sinh(kh)}\cos(\Omega)$	$\zeta = -\frac{H}{2}e^{(kz)}\cos(\Omega)$
Sub-surface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\frac{H}{2}\frac{\cosh k(h+z)}{\cosh(kh)}\cos(\Omega) - \rho gz$	$p = \rho g\eta e^{(kz)} - \rho gz$

10 Wave Statistics and Spectra:

Single frequency waves are called monochromatic referencing light waves.

Up until now we have only been looking at monochromatic waves. We have derived solutions based on single frequency waves.

In reality, the sea surface is a superposition of many waves each having its own frequency, direction and amplitude.

In order to represent the sea surface we must superimpose many waves.

And to understand a sea surface we must decompose the raw elevation signal into the constituent components.

How do we extract the wave properties from a raw signal?

- Wave-by-wave analysis
    - Counting individual waves in a time series signal
    - Analog time series/Plot on paper
- Spectral analysis
    - This decomposition is performed by applying Fourier Transforms to the signal.
    - Digital time series/Numerical calculation

For our wave record the sea surface, $\eta(t)$ is given by:

$$\eta(t) = \sum_{n=0}^{\infty} a_n \cos(\sigma_n t - \epsilon_n)$$

It is useful to arrange the data such that the researcher can gain insight into the characteristic frequencies of a signal.

The ability to plot the representative energy, a^2, vs frequency allows us to determine which frequencies are carrying the most energy in our signal.

Such a plot a_n^2 vs σ_n is called the energy spectrum. In reality we have waves of different σ and θ. When interested in direction we will calculate the directional wave spectrum.

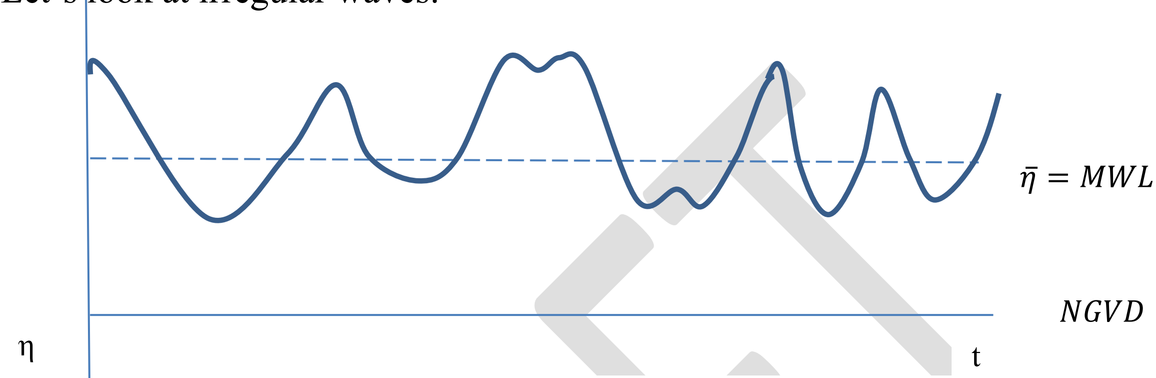
10.1 Wave-by-wave analysis

Divide up signal into individual ‘waves’

Zero up crossing and zero down crossing

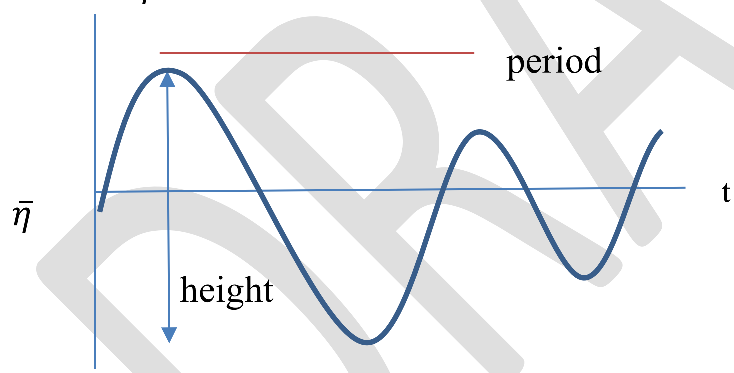
Need heights and periods for each wave

Let’s look at irregular waves:



Superposition of many waves with different heights, frequency, phases, and directions

If we examine $\bar{\eta}$ *in time*



We define a wave by height and period. From the signal we can pick out ‘waves’ by marking successive zero-upcrossings.

So let’s go back to η above and we see that at this location, for this time duration we count 5 waves. Each has its own height, period. We can then collect this discrete data (height and period) and plot as bins (histograms).

In practice we record data from an analog plot then rank order (sort) the waves from largest to smallest, and finally bin the data into histograms.

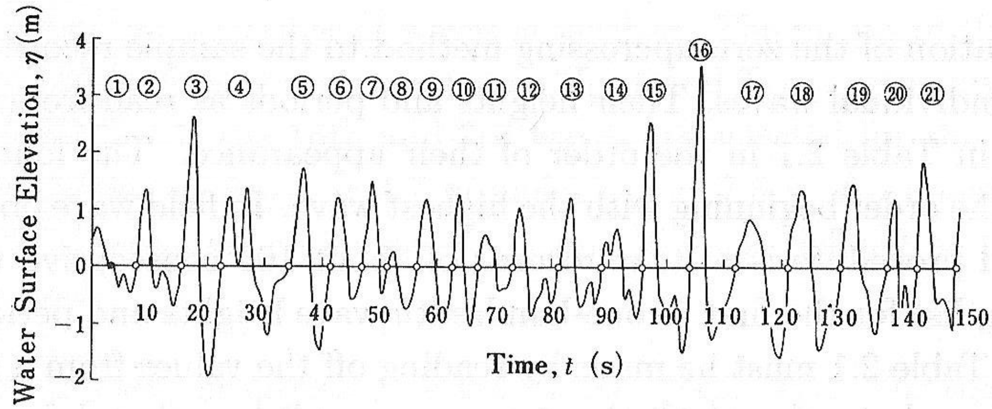
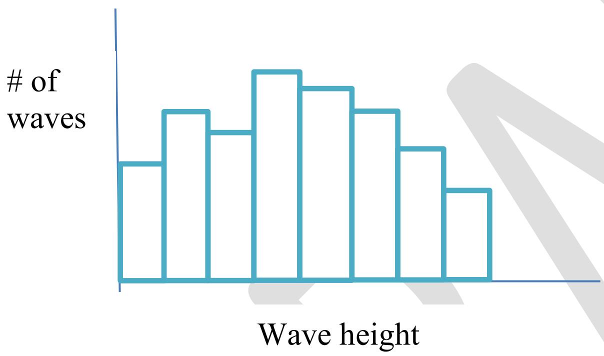


Fig. 2.2. Example of a wave record.



The histogram can be binned by height

Statistical measures can now be calculated.

Mean $\overline{H} = \frac{1}{n}\sum_i H_i$ (average height)

Root-mean squared wave height $H_{rms} = \sqrt{\frac{1}{n}\sum_i H_i^2}$ Measure of the magnitude of the varying wave height

H_{rms} is always larger than $\overline{H}$

Max

Min

Mean → average value

Mode → most common value

Median → half greater, half less

$Rms \equiv H_{rms} = \sqrt{\frac{1}{n}\sum_i H_i^2}$

Gives wave height with **equivalent energy** to the average energy in signal

Special statistics we like:

$H_{\frac{1}{n}} =$ the mean of the heighest 1/n waves

$H_{\frac{1}{3}} = H_s =$ significant wave height = the mean height of the highest 1/3 waves

$H_{\frac{1}{10}} =$ the mean height of the highest 1/10 waves

$H_1 = \overline{H} =$ the mean height of all waves

If N ≡ total number of waves,
The wave height distribution is defined by the probability.
Probability that a wave height is greater or equal to some value is given by:

$$P(H > \widehat{H}) = \frac{n_{H>\widehat{H}}}{N} \qquad or\ P(H \le \widehat{H}) = 1 - \frac{n_{H>\widehat{H}}}{N}$$

$n_{H>\widehat{H}}$ = number of waves for which H>$\widehat{H}$

10.2 Discrete Wave Statistics:

$\overline{H}, H_{rms}, H_{\frac{1}{3}}$ → $btw H_{\frac{1}{3}}$ ≡significant wave height

Kinsman (1965) determined a strong correlation between significant wave height and the visual wave height determined by the experienced observer.

Period can be defined as time between consecutive crests or zero crossing period (up or down). Now that the signal has been analyzed we can calculate probabilities. [Longuette-Higgins 1952 determined that the random distribution of waves followed known probability laws].

For a single, monochromatic wave signal

$\eta(t) = \left(\frac{H}{2}\right) cos\ \sigma t$; all waves have the same height

∴ $H_p=H_o$ for all P and $H_{rms}=H_o$

If we now add another monochromatic wave of the slightly different phase but the same wave height (as we did for constructing the wave group), we obtain a new signal of waves modulated by an envelope.

To determine H_p. since wave height decreases monotonically from max to min,

We average the wave height average from t=0 to P(π/Δσ)

$H_p = 4\frac{H_o}{p\pi}\sin\frac{p\pi}{2}$ and $H_{rms} = \sqrt{2}H_o$

So $H_p = \frac{2\sqrt{2}H_{rms}}{p\pi}\sin\frac{p\pi}{2}$ as p→ 0, $\sin(p)$ →p

$$\therefore \sin(P\pi/2) \rightarrow P\pi/2$$

As P→0, H_p→ H_{max}

$$H_{max} = \sqrt{2}H_{rms}$$

And recall $H_{rms} = \sqrt{2}H_0$ $\therefore H_{max} = 2H_0$ as expected!

10.2.1 EXAMPLE:

What are $H_{max}, H_{\frac{1}{10}}, and\ H_1\ in\ terms\ of\ H_{rms}$

$$H_{max} = \sqrt{2}H_{rms} = 1.414H_{rms}$$

$$H_{\frac{1}{10}} = \frac{20\sqrt{2}H_{rms}}{\pi}sin\frac{\pi}{20} = 1.408H_{rms}$$

$$H_{\frac{1}{3}} = \frac{6\sqrt{2}}{\pi}H_{rms}\,sin\frac{\pi}{6} = 1.350H_{rms}$$

$$H_1 = \frac{2\sqrt{2}}{\pi}H_{rms}\,sin\frac{\pi}{2} = .9H_{rms}$$

Example 2: If you are an experienced observer and observe average wave height at the beach of 1.219m≈4 ft, what is the H_{max} and H_1?

Well it has been found that observed heights are correlated with $H_{\frac{1}{3}}$

$$\therefore 1.22m = H_{\frac{1}{3}} = 1.350 H_{rms}$$

$$H_{rms} = 0.904m = 2.97ft$$

$$H_1 = .813m, 2.67ft$$

$$H_{max} = 1.28m, 4.2ft$$

10.3 Probability

Probability distribution (Probability mass function) P(x)= This function gives the frequency of occurrence of a possible outcome or value of a random variable.

It is important to note that:

$$0 \leq P(x)\ and \sum P(x) = 1,$$

- Probability is always greater than or equal to zero (non-negative)
- Sum of all probabilities is one

Another way to express probabilities is by the cumulative Probability C(x), this is the fraction of total population of events that a particular event is not exceeded.

$$C(x) = P\left(H \leq \hat{H}\right) = 1 - \frac{n}{N}$$

Probability Density, p(x): this is the fraction of events in a population that is made up of a particular event. It represents the rate of change of a particular distribution. Or p(x)= $\frac{dC(x)}{dx}$. The two most common probabilities are Gaussian and Rayleigh Densities.

The Gaussian Probability Density is

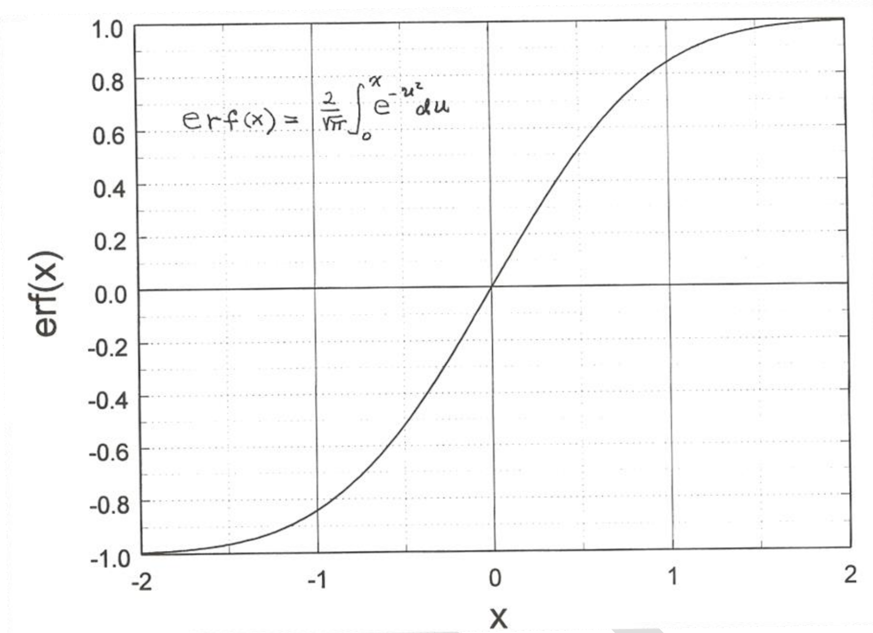
$p(x)=\frac{1}{\sigma_x\sqrt{2\pi}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}$ σ is stand dev of x
μ is mean of x

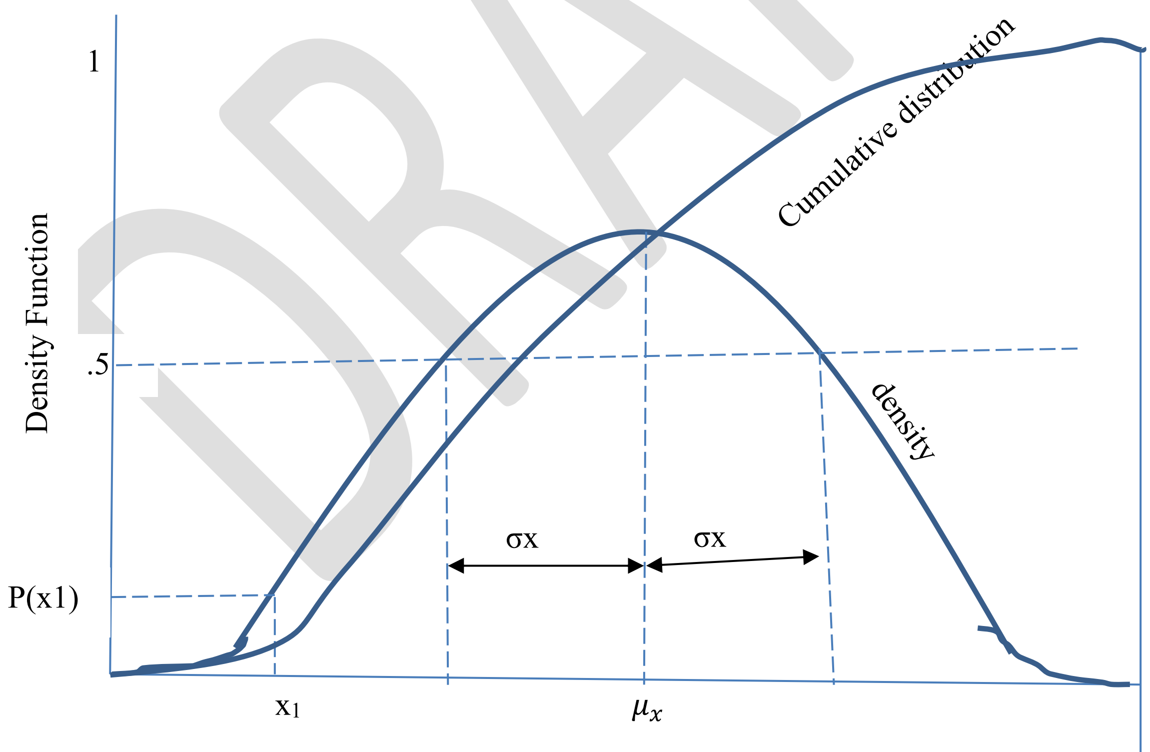
The integral of P(x) is the probability distribution *P*(x) for a zero mean, μ=0, and unit σ=1,

$p(x)=\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$P(x)=\Phi\left[\frac{x-\mu_x}{\sigma_x}\right] = \Phi = \int_0^x p(x)dx$

Where the integral $\Phi=\int_0^x \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$ is known as the error function, erf(x)





The probability of exceedance Q(x) can be found as $Q(x(t)>x_1)=1-P\,(x(t)<x_1)=$

1-$\Phi\left[\frac{x-\mu_x}{\sigma_x}\right]$ which is the probability that x will exceed x_1 over time t.

Gaussian distribution is most useful for short term probabilities of free surface η(x,t).

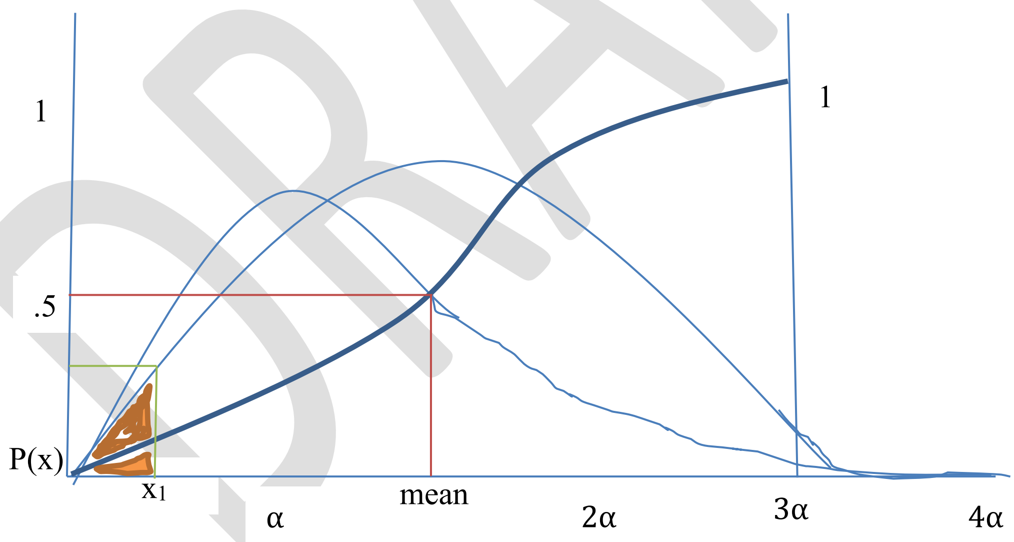
We are more concerned with wave height H rather than η. Longuet-Higgins (1952) examined statistically η(t) and found that both H and a followed the Rayleigh distribution.

Narrow Banded Spectra: **The Rayleigh Distribution**

$$p(x)=\frac{\pi x}{2\mu_x^2}e^{-\frac{\pi}{4}\left(\frac{x}{\mu_x}\right)^2} \quad \text{for } x\geq 0$$

$$P(x)=1-e^{-\frac{\pi}{4}\left(\frac{x}{\mu_x}\right)^2} \quad \text{for } x\geq 0$$

Rayleigh Dist. is asymmetric about mean



If we assume most wave energy falls under a small band of wave periods (narrow banded) the wave height probability density can be determined by

$p(H) = \frac{2H}{H_{rms}^2} e^{\left(\frac{-H}{H_{rms}}\right)^2}$ or $P(H) = 1 - e^{-H^2/Hrms^2}$ or P(H>$\widehat{H}$)

$$= e^{-\left(\frac{H}{H_{rms}}\right)^2}$$

Previously, P(H>$\widehat{H}$)= n/N for rank ordered grouping

$\therefore \frac{n}{N} = e^{-\left(\frac{\widehat{H}}{Hrms}\right)^2}$

N is the number of waves onto N total waves that have a height greater than or equal to $\widehat{H}$

We can rearrange this eq to find the height exceeded by the n largest waves in the group

$\widehat{H} = H_{rms}\sqrt{\ln\frac{n}{N}}$ $\quad H_p = H_{rms}\sqrt{\ln\frac{1}{p}} \equiv$ height exceeding pN of the waves

Example: 400 waves in record

N=400

H_{rms}=$\sqrt{\frac{1}{N}\sum H_i^2}$

a) How many waves are expected to exceed H=2H_{rms}

$$n = Ne^{-\left(\frac{\widehat{H}}{H_{rms}}\right)^2} = 400e^{\left(-\frac{2H_{rms}}{H_{rms}}\right)^2} = 400e^{(-2)^2} = 400e^{-4} = 7.3, 7\ waves$$

$$\frac{n}{N} = \frac{7}{400} = 0.0175\ x\ 100\% = 1.75\%\ of\ the\ total\ \#\ of\ waves\ exceeded\ 2H_{rms}$$

b) Height $\widehat{H}$ exceeded by ½ waves (n=N/2 or P=1/2)

$$H_{1/2} = H_{rms}\sqrt{\ln 2} = 0.833H_{rms}$$

c) Height exceeded by one wave? For $H_{1/N}$ we have P=1/N=1/400

$$H_{1/400} = H_{rms}\sqrt{\ln 400} = 2.45\ H_{rms}$$

As N ↑, Hmax ↑
Since PDF never actually reaches zero, rather decays asymptotically.
For a Rayleigh Distributed wave field (Table 7.1 D&D):

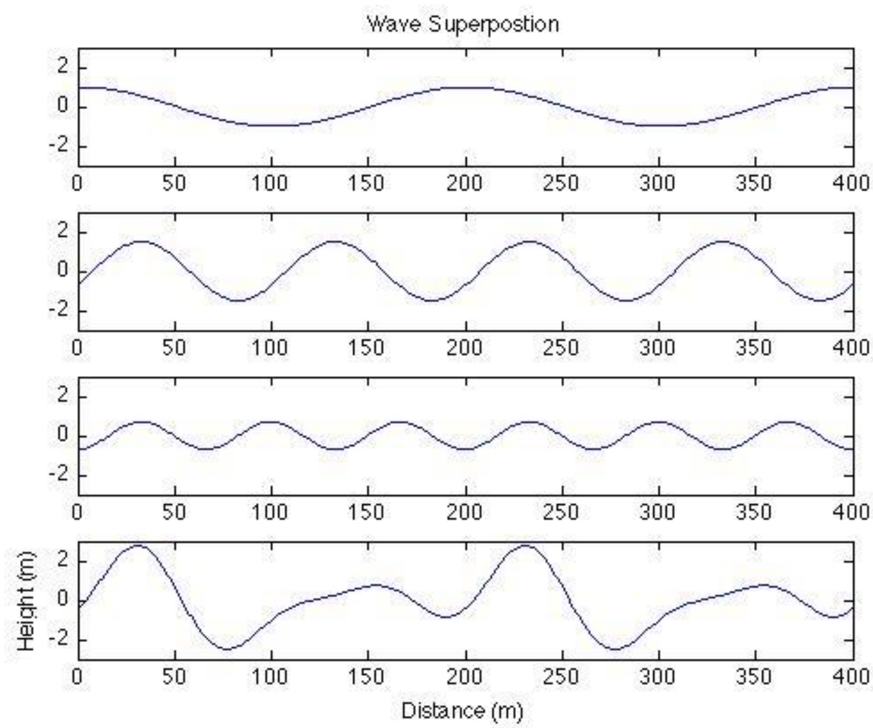
$$H_{\frac{1}{10}} = 1.80 H_{rms}$$
$$H_{\frac{1}{3}} = 1.416\, H_{rms}$$
$$H_{1} = 0.886 H_{rms}$$

Or in general

$$\frac{H_p}{H_{rms}} = \sqrt{\ln\frac{1}{p}} + \frac{\sqrt{\pi}}{2p} erfc(\sqrt{\ln\frac{1}{p}}$$

$$WHere\ erfc\ is\ the\ complentary\ error\ function$$

Time series and Statistics (DD 7.1-7.2)



Histograms and Distributions (DD7.1-7.2)

10.4 Spectral Analysis

Any piecewise continuous function can be expressed by a sum of sines and cosines over some interval $t_o \rightarrow t_o + \tau$

Wave record sampled at regular discrete interval (Δt~0.5s-2s) and finite in length: 20 min – 2 hr

Spectral method will bin record into frequencies 'automatically'. No need to sort the wave periods first.

Any periodic signal can be represented by an infinite sum of Fourier components:

$$\eta(t) = \sum_{n=0}^{\infty} a_n \cos n\sigma t + b_n \sin n\sigma t$$

In reality we have a limited series not an infinite, N not ∞, so for a finite record

$$\eta(t) = a_o + \sum_{n=1}^{N} a_n \cos n\sigma t + b_n \sin n\sigma t$$

where $a_n = \frac{2}{T}\int_t^{t+\tau} \eta(t) \cos n\sigma t \, dt$ even component

$b_n = \frac{2}{T}\int_t^{t+\tau} \eta(t) \sin n\sigma t \, dt$ odd component

and $a_o = \frac{1}{T}\int_t^{t+\tau} \eta(t) \, dt$ = average for mean value in record

a_o (Mean) usually subtracted out

Or in Complex notation:

$$\eta(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \eta(t) e^{-int} dt$$

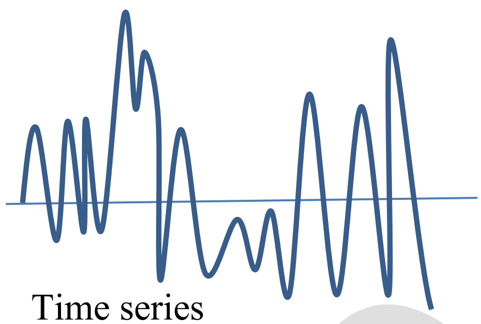
$$C_n = \frac{1}{2}(a_n - ib_n)$$

We typically use FFT (Cooley and Tukey,1965) which exploits symmetries to streamline calculations, to obtain a_n and b_n, 'fast former transform'

We will not get into theory behind Fourier sums and transforms for this class.

The FFT is a routine available in most canned mathematical programs like MatLab and Mathematica also in 'numerical recipes' (Press et al 1986)

FFT is most efficient when N is some power of 2
N= 512, 1024, 2084 or 4096 points
We treat FFT as a 'black box' so for us



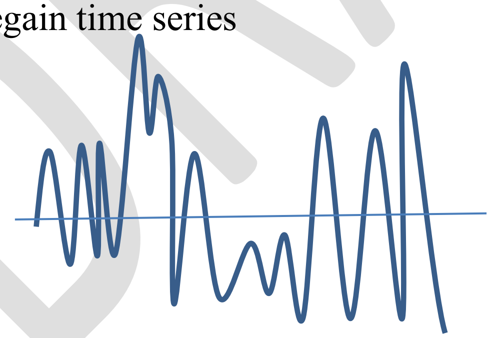
→ FFT→ N, a_n,b_n

Time series

Where N is the number of frequencies returned, a_n *and* b_n, are the coefficients for each of the N frequencies.

And the inverse to regain time series

N, a_n b_n → FFT^{-1}→



Time series

Let's describe a time series at Δt intervals

So if we have a total record length of τ seconds and it is split or digitized into N evenly spaced, Δt, points:

$$N\Delta t = \tau$$

Or the total number of samples is,

$$N = \frac{\tau}{\Delta t}$$

Our continuous time record becomes a discreet number of points. In a finite discrete form (only N points):

$$\eta(t) \rightarrow \eta_j, j = 1, 2, 3, \ldots, N$$

$$\sigma = \frac{2\pi}{\tau}$$

where τ is the total time interval

$$\sigma = \frac{2\pi}{N\Delta t}$$

We can write

$$\eta(t) = \sum_{n=-N}^{N} F_n e^{in\sigma t}$$

$$F_n = \left(\frac{a_n - ib_n}{2}\right) \ and \ \sigma = \frac{2\pi}{\tau} = \frac{2\pi}{N\Delta t}$$

$$\eta(t) = \eta_j = \sum_{n=0}^{N-1} F_n e^{i\left(2\pi\left(\frac{nt_j}{N\Delta t}\right)\right)}$$

So if we know ALL F_n from n=0 to N-1 we can find the time record at time j.
We can solve for the n^{th} coefficient F_n by:

$$F_n = \frac{1}{\tau}\int_t^{t+\tau} \eta(t)\, e^{-in\sigma t} dt = \frac{1}{\tau}\int_0^{\tau} \eta(t) e^{-2\pi i\left(\frac{nt}{N\Delta t}\right)} dt = \frac{a_n}{2} - i\frac{b_n}{2}$$

$$or\ for\ F_{-n} = F_n^* = \frac{a_n}{2} + i\frac{b_n}{2}$$

These are the finite discrete forms of the former transform pairs

We can use the equation above for F_n, to calculate a_n & b_n, however typically we will use FFT to do this.

Inputting a time series (N values) we get back a_n , b_n , one for each N frequency.

Our frequency resolution for the lowest frequency that can be resolved (fundamental frequency or Bandwith), Δf or f_0, is determined by the Δt and number of records.

$$\Delta f = f_0 = \frac{1}{\tau} = \frac{1}{N\Delta t}$$

For a fixed N and a finer sampling rate will lead to a larger Δf, or higher frequency resolution.

For a 'good' representation of infinite series using a finite record a_n & b_n, we need a large enough record length, N.

But how big is big enough?

How do we know if we have a long enough wavelength record?

If we take the average mean square value of our recorded time series function :

$$\frac{1}{\tau}\int_0^{t+\tau} \eta^2(t)dt$$

(average mean square value)

½ the sum of the squares of the Fourier coefficient should equal approx.:

$$a_o^2 + \frac{1}{2}\sum_n (a_n^2 + b_n^2) \approx \frac{1}{\tau}\int_t^{t+\tau} \left[a_o + \sum_n (a_n \cos(n\sigma t) + b_n \sin(n\sigma t)\right]^2 dt$$
$$= \frac{1}{\tau}\int_0^{t+\tau} \eta^2(t)dt$$

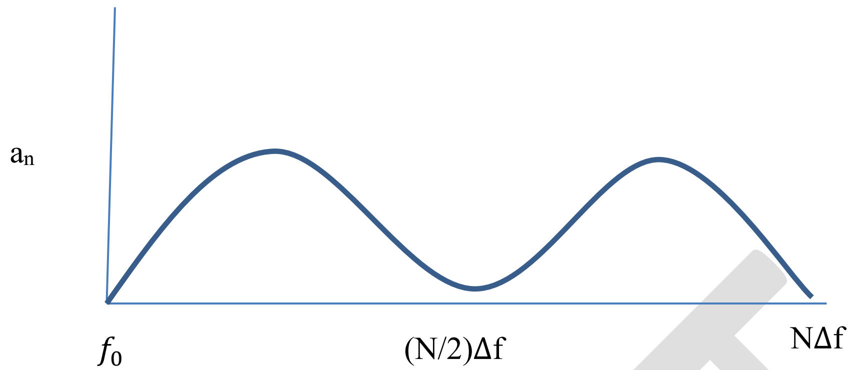
Parseval's Theorm

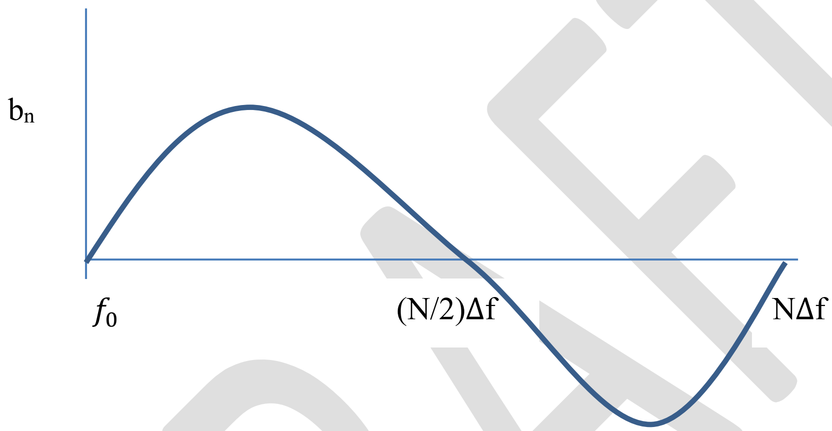
(Often de-mean the signal to get rid of a_o)

If not, then we need to add more numbers to the series, make N bigger.

What do a_n & b_n look like?

Most output is ordered from longest frequency to NΔf



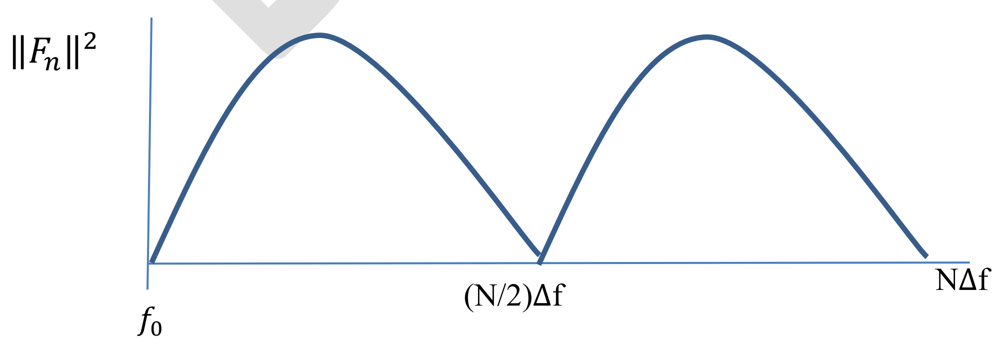


a's are exactly symmetric about (N/2)Δf, while the b_n are skew-symmetric about (N/2)Δf.

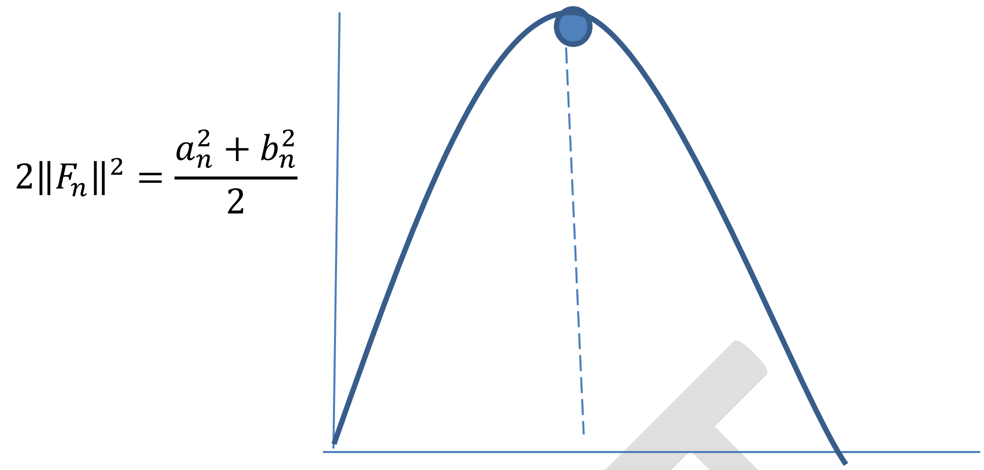
The results are typically presented by plots

$$\|F_n\|^2 = \left(\frac{{a_n}^2}{4} + \frac{{b_n}^2}{4}\right)$$

Results in a two sided spectrum



Which is then typically 'folded' about (N/2)Δf.



Results in a spectrum - a distribution of energy with frequency.
Most energy is contained in the spectral peak.

We can calculate the spectral density (or energy density).

$$S(f) = \frac{2\|F_n\|^2}{\Delta f}$$

For a free surface time series, in meters S(f) has units of m^2s or m^2/Hz

The variance $\sigma^2 = \overline{[\eta(t)]^2}$ (mean square of the surface elevation)

Let's connect the energy density spectrum to the energy of a wave.
Assume: $\eta(t) = a \sin\sigma t$

Calculating variance over 2π

$$\sigma^2 = \overline{[\eta(t)]^2} = \frac{1}{2\pi}\int_0^{2\pi} a^2 sin^2\sigma t \ \ d(\sigma t) = \frac{a^2}{2}$$

$\frac{a^2}{2} = 2\int_0^\infty S(f)df$ for a monochromatic wave.

The variance, wave energy and wave energy spectrum are related.
Given an energy spectrum S(f), the wave amplitude for a given frequency is

$$a(f) = \sqrt{2S(f)\Delta f}$$

For monochromatic wave show fig 6.11 pg 175
For a random wave show fig 6.12 pg 176
Directional spectra, 6.13, 6.14

The limiting frequency at which a wave can be adequately resolved is called the Nyquist frequency, or folding frequency

$$f_{Nyq} = \frac{1}{2\Delta t}$$

The shortest measurable limit is $T = 2\Delta t,\ \sigma = \pi/\Delta t$

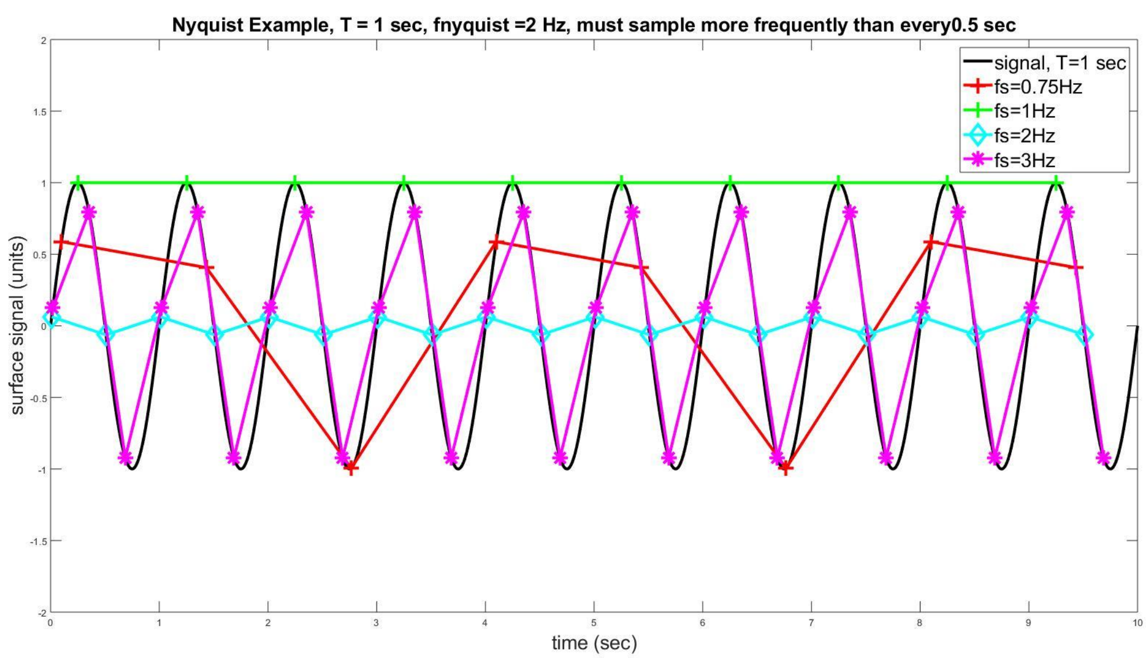
For any higher frequency waves, (T smaller, f (or σ) bigger), the sampling rate is inadequate. Researcher must design Δt to avoid aliasing.

Sampling frequency must be greater than twice the highest frequency in the signal.

Nyquist frequency = smallest wave or maximum frequency can be represented given a sampling rate

Nyquist Rate = sampling rate needed to prevent aliasing

The sampling rate should be more than double the maximum frequency that we are trying to capture.
All frequencies less than (larger waves) the maximum frequency we are trying to measure will not be aliased. However the max. frequency could still be aliased:



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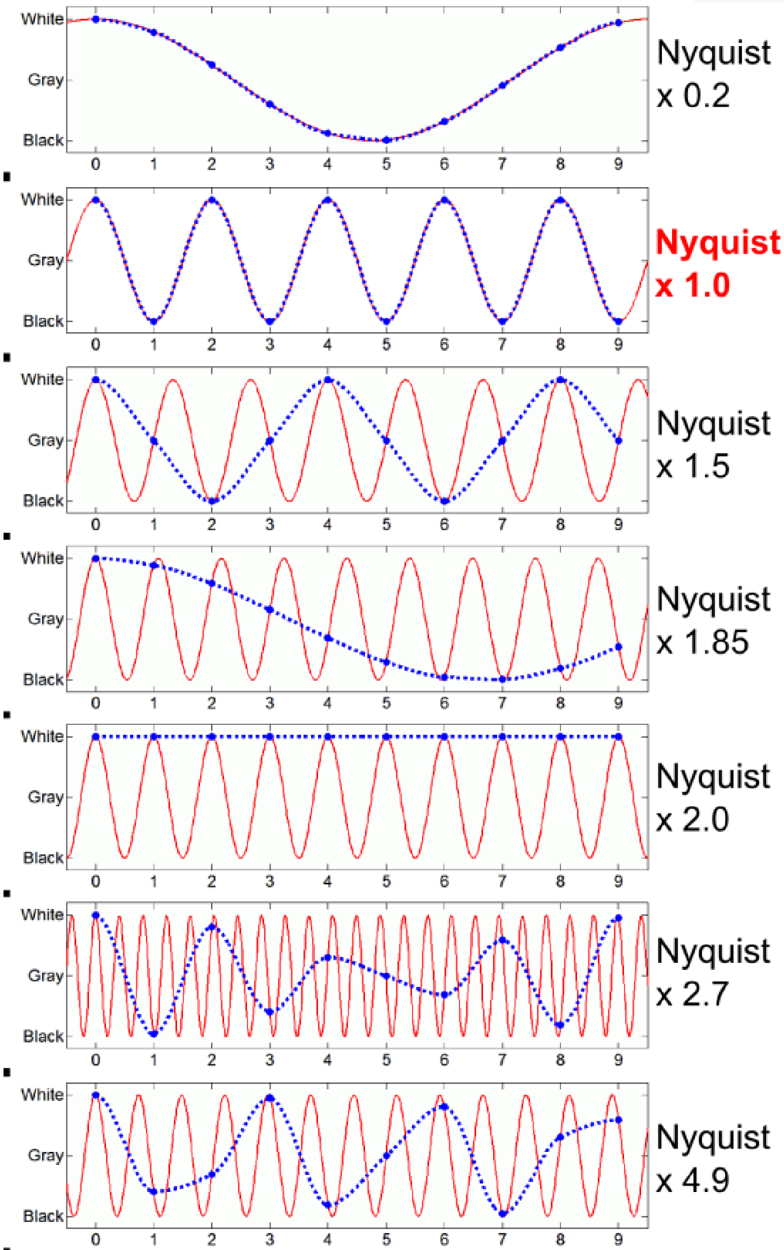

10.5 Directional Spectra

In nature waves travel in all directions, as they are generated all over the planet. In order to better understand which direction sends the most wave energy to a location of interest, we can look at the directional spectrum. Peaks in the directional spectrum indicate the dominant location from which the most energy is coming.

$$S(f,\theta) = S(f) \cdot D(\theta)$$

$$S(f) = \sum_{\theta=\theta min}^{\theta max} S(f,\theta)\Delta\theta$$

$$D(\theta) = \sum_{N=1}^{N} Sn(F,\theta)\Delta f$$

For directional spectrum (wave energy as a function of frequency and direction)

$$a(f,\theta) = \sqrt{2S(f,\theta)\Delta f\Delta\theta}$$

In depth details about directional spectra are left for a more advanced course on wave mechanics.

10.6 Spectral moments

The power density plot has a lot of information contained within. We have the Energy in the form of wave amplitude squared and period in the form of frequency. By analyzing the spectral density we should be able to tease out pertinent characteristics.
One method of achieving this goal, is through the calculation of spectral moments. Given the formula for spectral moments:

$$m_i = \int_0^\infty f^i S(f)df = \sum_{n=0}^{N} f_n^i S(f_n)\Delta f \qquad i = 0,1,2 \ldots$$

For Rayleigh Dist Waveheights (L-H 1992) determined

$$H_{\frac{1}{3}} = H_s \sim H_{m0} = 4\sqrt{m_o} = 4\sqrt{\sum_{n=1}^{N} S(f_n)\Delta f}$$

Or 4 times the sqrt of the area under the curve S(f)

For Directional spectra:

$$H_{1/3} = \boldsymbol{H_s \sim 4\sqrt{m_o}} = 4\sqrt{\sum_{n=1}^{N}\sum_{p=1}^{P} S(f_n,\theta_p)\Delta\theta\Delta f}$$

We can also calculate mean wave period T_{mo1}

$$\bar{T} = 2\pi\left(\frac{m_0}{m_1}\right)$$

CHECK THIS

$$\bar{f} = \left(\frac{m_1}{m_0}\right)$$

$$T_{Mo1} = \left[\frac{\sum_1^N fS(f)\Delta f}{\sum_1^N S(f)\Delta f}\right]^{-1}$$

$$T_{mo2} = \left[\frac{\sum_1^N f^2 S(f)\Delta f}{\sum_1^N S(f)\Delta f}\right]^{-1}$$

Peak Period= $1/f_p$ → frequency of spectral plate

$m_o \equiv$ area under spectral curve (power in signal) $so\ \sqrt{m_0} \equiv (H)$

Signal with f G has

$2\sqrt{\frac{m_2}{m_o}}$ zero crossings/sec

So for $T_{mean} = T_{mo2} \equiv \sqrt{\frac{m_o}{m_2}}$

Some other parameters that may be found about a sea state though its energy spectrum include number of zero up-crossings (N_0), the average zero crossing period ($\bar{T}_z$), The average apparent wavelength ($\bar{L}_{wz}$) and the average crest to crest period ($\bar{T}_c$).

No of zero upcrossings	$N_0 = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}$ (s^{-1})
Average zero crossing period	$\bar{T}_z = \frac{1}{N_0} = \sqrt{\frac{m_0}{m_2}}$ (s)
Average apparent wave length	$\bar{L}_{wz} = g\sqrt{\frac{m_0}{m_4}}$ (m)

Average crest to crest period $\bar{T}_c = \sqrt{\dfrac{m_2}{m_4}}$ (s)

(Mean) Characteristic Period $T_1 = \left(\dfrac{m_0}{m_1}\right)$ (s)

FFT and Wave Spectra II (DD7.3-7.4)

Nomenclature

$H_{1/10}$	Average of the highest 1/10 waves
$H_{1/100}$	Average of the highest 1/100 waves
H_{max}	Maximum wave height
H_s $H_{1/3}$	Significant wave height
U_{w10}	Wind speed at 10m above SWL.
$U_{w19.5}$	Wind speed at 19.5m above SWL
X	Fetch length.
η	Sea surface elevation
ε	Spectral bandwidth
m_0	Zero Moment (area under spectrum).
m_n	n^{th} moment of area under the spectrum.
B	Dimensionless constant used for the Neumann spectrum
$\bar{T}_z$	Average Crossing period

ABBREVIATIONS

CF	Correction Factor
FFT	Fast Fourier Transform
ISSC	International Ship Structures Congress
ITTC	International Towing Tank Conference
JONSWAP	Joint North Sea Wave Project
P-M	Pierson – Moskowitz spectrum
SWL	Still Water Level

Wave Energy Spectrum

If the ocean is observed at a particular point on a short term basis it may be assumed to consist of numerous waves of different periods, propagation directions and height. Due to this idealization to analyze random seas in terms of a regular sinusoidal waveforms spectral analysis may be employed. A wave spectrum will analyze a sea state in the frequency domain and describe the energy transmitted by the wave conditions at a given frequency. However, when the wave data is collected it will usually be done so in terms of the surface elevation (η) in the time domain and therefore must be converted into the frequency domain. Also, the continuous wave system must be calculated at discrete intervals. The method used to perform this conversion is the Fast Fourier Transform (FFT). For a more detailed discussion on FFT refer to Chakrabarti (1987).

Ocean waves are primarily influenced by the speed and directionality of the wind and as such these are also the determining factors when describing spectra. The wind speed used to calculate the spectra will be taken as $U_{w19.5}$ however U_{w10} may also be used in some cases. The directionality component of the wind will also encapsulate the fetch length (X), which is the length that the wind will blow to propagate a wave. As the wind blows over the surface of the sea waves will continue to grow until the point at which they break, thus limiting their height. At this point, ignoring the effect of swells, the sea is said to be fully developed and thus will be in a relatively steady condition. A large area and propagation time are required to produce fully developed seas. In many parts of the world it is more common to have either fetch limited or duration limited conditions. Fetch limited seas are seen when fetch is the limiting factor in producing fully developed seas. Duration limited seas are evident when the wind does not blow for long enough to produce the fully developed conditions.

Once a spectrum has been established it must be smoothed to eliminate the effects of noise (Chakrabarti 1987). This is done by averaging it over frequency ranges longer than the finite elements described in the Fourier method (Kamphuis 2000). The effect of smoothing the spectra will not change its total energy but it will eliminate the numerous peaks either side of the main peak which are referred to above as noise.

It is often important to refer to moments of a wave spectrum, which are defined as:

$$m_n = \int_0^{\alpha} \omega_n^n S_\zeta(\omega_w) d\omega_w$$

This is because they enable easy interpretation of the energy spectrum and easy manipulation to give meaningful outputs. For example the zero moment (m_0) is representative of the area under the spectrum and once it is known the wave heights may be determined as follows;

(note: The units are dependent on the input data)

Significant wave height, $H_s = 4.0\sqrt{m_0}$ (m)

Average of the highest 1/10th, $H_{1/10} = 5.09\sqrt{m_0}$ (m)

Average of the highest 1/100th, $H_{1/100} = 6.67\sqrt{m_0}$ (m)

Root mean square of wave height $H_{rms} = 2\sqrt{2m_0}$

Since the formulas above are derived assuming Rayleigh distribution which is for narrow frequency spectrum, they will produce erroneous results when applied to broad banded ocean wave spectra. This problem can be overcome through the use of a CF;

$$CF = \sqrt{1-\varepsilon^2}$$

Where; $$\varepsilon^2 = \frac{m_0 m_4 - m_2^2}{m_0 m_4}$$

ε is also known as spectral bandwidth and for a narrow bandwidth where ε approaches 0 the wave train is said to remain almost the same. Now that the CF is known it can be applied to the formula for wave height as well as the following formula to provide accurate results for real ocean conditions.

Significant wave height, $H_s = CF \times 4.0\sqrt{m_0}$ (m)

Average of the highest 1/10th, $H_{1/10} = CF \times 5.09\sqrt{m_0}$ (m)

Average of the highest 1/100th, $H_{1/100} = CF \times 6.67\sqrt{m_0}$ (m)

Root mean square of wave height $H_{rms} = CF \times 2\sqrt{2m_0}$

Some other parameters that may be found about a sea state though its energy spectrum include number of zero up-crossings (N_0), the average zero crossing period ($\bar{T}_z$), The average apparent wavelength ($\bar{L}_{wz}$) and the average crest to crest period ($\bar{T}_c$).

No of zero upcrossings $\quad N_0 = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}$ (s^{-1})

Average zero crossing period $\quad \bar{T}_z = \frac{1}{N_0} = 2\pi\sqrt{\frac{m_0}{m_2}}$ (s)

Average apparent wave length $\quad \bar{L}_{wz} = 2\pi g\sqrt{\frac{m_0}{m_4}}$ (m)

Average crest to crest period $\quad \bar{T}_c = 2\pi\sqrt{\frac{m_2}{m_4}}$ (s)

(Mean) Characteristic Period $\quad T_1 = 2\pi\left(\frac{m_0}{m_1}\right)$ (s)

It is also possible to manipulate the data in the wave energy spectrum to produce useful information relating to ships and other floating structures that are present at that point in the ocean.

Details on Six Commonly Used Spectrum

Neumann Spectrum

The first and simplest analytical model of ocean wave spectrum is the Neumann spectrum (1953). It is a single parameter spectrum based on wind speed, Uw, which can be defined as;

$$S(\omega) = 1.466 H_s^2 \frac{\omega_0^5}{\omega^6} e^{\left[-3\left(\frac{\omega}{\omega_0}\right)^{-2}\right]}$$

Where;

H_s = Significant wave height

$$\omega_0 = \sqrt{\frac{2}{3}}\frac{g}{U_w}$$

Pierson – Moskowitz Spectrum

In 1964 Pierson and Moskowitz proposed a new type of spectrum. The P-M spectrum is defined as a single parameter spectrum that can be applied to fully developed seas, where the peak frequency occurs somewhere in the middle of the sampling time (CEM 2002). To use the P-M spectrum the wind fetch and duration are assumed to be infinite. The wind must also be assumed to be blowing over a large area, greater than 5000 wave lengths on either side, and from a nearly constant direction. If all these conditions apply then a P-M model may be used to generate the spectrum where $U_{w19.5}$ will be the only variable. Due to the conditions that must be applied to the P-M spectrum its is usually limited to offshore and severe storm applications. The P-M spectrum is described as follows;

$$S(\omega)=\frac{\alpha g^2}{\omega^5}e^{\left[-0.74\left(\frac{\omega U_{w19.5}}{g}\right)^{-4}\right]}$$

Where; $\alpha = 0.0081$

JONSWAP Spectrum

This spectral model was developed by Hasselman et al (1973). It gets it name from the Joint North Sea Wave Project, hence JONSWAP. For development of the constants used in the spectrum wave data was collected over a large area extending into the North Sea from Sylt Island (Delft University Notes). The spectrum was developed as a result of the need to model seas produced by winds of limited fetch and duration. It is a two parameter spectrum based on the P-M spectrum. In reality there are five parameters that define the JONSWAP spectrum, these are γ, τ, ω_0, X and U_w. However it is more usual that only two of the parameters, U_w and X, are left as variables. The following formula defines the JONSWAP spectrum;

$$S(\omega)=\frac{\alpha g^2}{\omega^5}e^{\left[-1.25\left(\frac{\omega}{\omega_0}\right)^{-4}\right]}\gamma^{e^{\left[-\frac{(\omega-\omega_0)^2}{2\tau^2\omega_0^2}\right]}}$$

Where; γ = peakedness parameter

This is described as the ratio between the peak frequencies of the JONSWAP and P-M spectrum (Figure 2) and can range between 1 and 7. However it is more commonly given as γ=3.3.

τ = shape parameter. This varies from $\tau = 0.07$ for $\omega \le \omega_0$ and $\tau = 0.09$ for $\omega > \omega_0$.

X = Fetch. This is one of the variable parameters determined by the operator.

U_w = Wind speed, also determined by the operator.

$\alpha = 0.076(X_0)^{-0.22}$ If the fetch (X) is unknown then $\alpha = 0.0081$ may be used instead.

$$\omega_0 = 2\pi\frac{g}{U_w}X^{-0.33}$$

$$X_0 = \frac{gX}{U_w}$$

It is important to note that the moments (m_n) of the JONSWAP spectrum cannot be calculated analytically but may be estimated by numerical integration (Reeve 2004).

Bretschneider Spectrum

The Bretschneider spectrum is another example of a two parameter spectrum, its parameter relate to wave height and period. The development of the Bretschneider spectrum is based on the assumption that the spectrum is "narrow banded and the individual wave heights and wave periods follow the Rayleigh distribution" (Chakrabarti 1987). The Bretschneider spectrum may be applied to fully developed seas,

seas that are 90% developed or seas that are 80% developed. This report, and the accompanying Microsoft Excel spreadsheet, will only deal with the Bretschneider spectrum for fully developed seas. The formula for fully developed seas is;

$$S(\omega)=0.1687H_s^2\frac{\omega_s^2}{\omega^5}e^{\left[-0.675\left(\frac{\omega_s}{\omega}\right)^2\right]}$$

Where; $\omega_s = \frac{2\pi}{T_s}$

T_s = Significant wave period, defined as the average period of the significant waves.

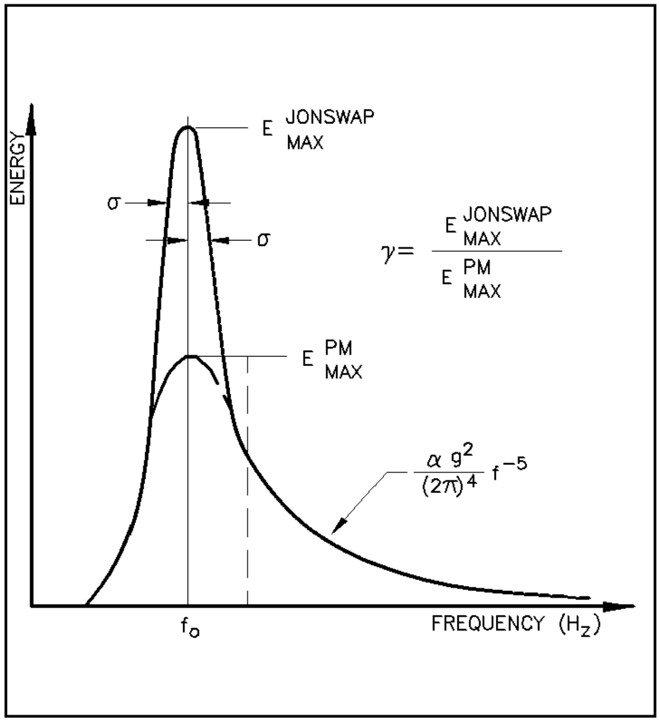

Figure 2: Peakedness Parameter (CEM 2002)

Also, it can be shown that $T_s = 0.946T_0$ where T_0 is the peak period. This makes the Bretschneider and P-M spectral models equivalent.

11 Long Waves

11.1 Long Wave Theory

11.2 Long Wave Equations of Motion

11.3 Hydrostatic approximation

11.4 Seiching

11.5 Storm Surge

12 Nonlinear Properties for Small-amplitude Waves

12.1 Mass Transport

12.2 Mean Water Level

12.3 Momentum Flux

12.4 Wave Forcing

13 Wave Forces

13.1 Intro and Flow around a Cylinder

13.2 Morison Equation and total force calculation

14 Nonlinear Waves

14.1 Stokes Theory

15 Extreme value analysis

15.1 Extreme value:

Extreme value: the largest value expected to occur in a given or specified number of observations.

Establish how many observations that you would expect to have over a given time period, and then determine the largest predicted value for that number of observations.

16 FINAL EXAM