Harmonic waves

the assumption of linearity made in deriving the acoustic wave equation make it possible to treat any arbitrary disturbance as the sum of sinusoidal components.

$$p(\bar{x},t) = \underset{\text{space}}{P(\bar{x})}\ \underset{\text{Time}}{\cos(\omega t + \phi)}$$

$$= RP(\tilde{P}(\bar{x})e^{i\omega t}$$

RP = real part   Im = Imaginary Part

$\tilde{P}(\bar{x}) = P(\bar{x})e^{i\phi}$ => Spacial solution with respect to phase angle

$e^{i\phi} = \cos\theta + i\sin\theta$ => $RP(e^{i\theta}) = \cos\theta$

$Im(e^{i\theta}) = \sin\theta$

$$e^{i(\omega t+\phi)} = \cos(\omega t+\phi) + i\sin(\omega t+\phi)$$

$$RP(e^{i(\omega t+\phi)}) = \cos(\omega t+\phi) = RP(e^{i\omega t}e^{i\phi}) = RP(e^{i\omega t})\cdot RP(e^{i\phi})$$

$$p(x,t) = f_1(x - c_0 t) + f_2(x + c_0 t)$$

$$= A e^{i\omega(t-\frac{x}{c_0})} + B e^{i\omega(t+\frac{x}{c_0})}$$

From here on most sinusoids are represented as complex quantities and omit "RP" since it is understood in physical equations

If we define $\boxed{k = \frac{\omega}{c_0}}$ wave number

c_0 = Speed of wave propagation

$\omega = 2\pi f$ angular frequency $\left(\frac{\text{radians}}{\text{second}}\right)$

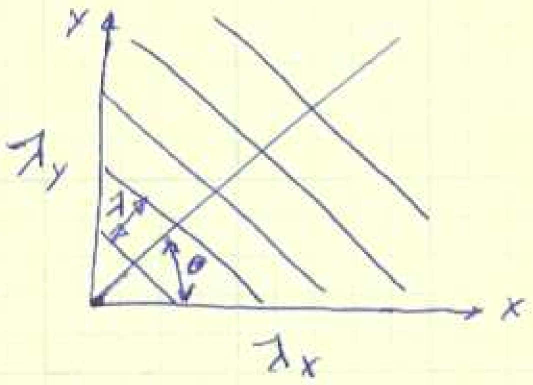
$$k = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} = \frac{2\pi}{\lambda}$$

$\therefore$ A solution of an Acoustic Wave Equation is

$$P(\bar{x},t) = Ae^{i\omega t - ikx} + Be^{i\omega t + ikx}$$

[complex form of the harmonic solution for the acoustic pressure of a plane wave]

Two-D waves



λ = wave length

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{1}{c_0}\frac{\partial^2 p}{\partial t^2} = 0$$

$$P(\bar{x},t) = \tilde{P}(\bar{x})e^{i\omega t} \qquad (\bar{x} = [x,y])$$

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = \tilde{P}(\bar{x})\frac{\partial^2}{\partial t^2}(e^{i\omega t})$$

$$= -\omega^2 \tilde{P}(\bar{x})e^{i\omega t}$$

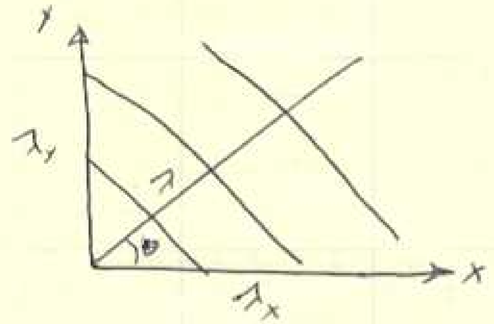
$$\left(\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{\omega^2 \tilde{p}}{c_0^2}\right)e^{i\omega t} = 0$$

IF $\tilde{P}(x) = Ae^{ik_1x + ik_2y} + Be^{\cdots} \ldots$

then $(-k_1^2\tilde{P} - k_2^2\tilde{P} + k^2\tilde{P})e^{i\omega t} = 0 \qquad k^2 = k_1^2 + k_2^2$

Recall $\overset{\text{frequency}}{\cancel{f}} = \frac{\omega}{2\pi}$, $\overset{\text{period}}{T} = \frac{2\pi}{\omega}$, $T = \frac{\lambda}{c} \Rightarrow k = \frac{2\pi}{\lambda}$

(Time for a wave to propagate through 1 wave length)



$k = \frac{2\pi}{\lambda}$

$k_1 = \cancel{\frac{2\pi}{\lambda\cos\theta}} = \frac{2\pi}{\lambda_x} = \frac{2\pi}{\frac{\lambda}{\cos\theta}} = \frac{2\pi\cos\theta}{\lambda} = k\cos\theta$

$k_2 = \cancel{\frac{2\pi}{\lambda\sin\theta}} = \frac{2\pi}{\lambda_y} = \frac{2\pi}{\frac{\lambda}{\sin\theta}} = \frac{2\pi\sin\theta}{\lambda} = k\sin\theta$

So $\tilde{P}(x) = Ae^{ikx\cos\theta + iky\sin\theta}$

or $\tilde{P}(\bar{x}, t) = Ae^{ikx\cos\theta \cancel{+} ik_y\sin\theta}e^{i\omega T}$

SUMMARY

Acoustic Wave Equation $\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0$

" Momentum Equation $-\nabla P = \rho_0 \frac{\partial \bar{u}}{\partial t}$

P = Pressure
c = speed of sound
t = time
ρ_0 = density
$\bar{u}$ = particle velocity

2D CARTESIAN HARMONIC Wave

$$P(\bar{x}, t) = Ae^{ik_x\cos\theta + ik_y\sin\theta}e^{i\omega t}$$

$$\nabla P = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y}$$

$$\nabla^2 P = \frac{\partial^2 P_x}{\partial x^2} + \frac{\partial^2 P_x}{\partial y^2}$$

A = amplitude
$\bar{x} = [x, y]$
k = wave number $= \frac{\omega}{c}$
θ = angle with x-axis

As before, the harmonic solution for the acoustic pressure of a plane wave is

$$\boxed{P = Ae^{i\omega t - ikx} + Be^{i\omega t + ikx}}$$

eqn 5.24
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Chapt 5 Handout

using the eqn $\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla P$

we get the associated particle velocity

$$U = \left[\frac{A}{\rho_0 c} e^{i\omega t - ikx} - \frac{B}{\rho_0 c} e^{i\omega t + ikx}\right]\hat{x}$$

(entirely in direction of propagation)

If $P_+ = Ae^{i\omega t - ikx}$

$P_- = Be^{i\omega t + ikx}$

<u>the particle speed:</u>

$$u_+ = +\frac{P_+}{\rho_0 c} \quad \text{and} \quad U_- = -\frac{P_-}{\rho_0 c}$$

∴ we ~~can~~ get

$$S_+ = +\frac{P_+}{\rho_0 c^2} \quad \text{and} \quad S_- = +\frac{P_-}{\rho_0 c^2}$$

and

$$\Phi_+ = -\frac{1}{i\omega\rho_0} P_+ \quad \text{and} \quad \Phi_- = -\frac{1}{i\omega\rho_0} P_-$$

Φ = velocity potential

"The harmonic approach to wave phenomena is used almost universally. this is because it is consistant with spectral analysis, and because there are cases for which the effective wave speed C is a function of frequency and for which the general wave equation is therefore invalid." Pg 21 Mechanics of Underwater Noise