CHAPTER 5

THE ACOUSTIC WAVE EQUATION AND SIMPLE SOLUTIONS

5.1 INTRODUCTION. Acoustic waves that produce the sensation of sound are one of a variety of pressure disturbances that can propagate through a compressible fluid. There are also ultrasonic and infrasonic waves whose frequencies are beyond the audible limits, high-intensity waves (such as those present near jet engines and missiles) which may produce a sensation of pain rather than of sound, and shock waves generated by explosions and supersonic aircraft.

Acoustic waves in inviscid fluids are longitudinal waves: the molecules move back and forth in the direction of propagation of the wave, producing adjacent regions of compression and rarefaction similar to those produced by longitudinal waves in a bar. Fluids exhibit fewer constraints to deformations than do solids. As a result, the restoring force responsible for propagating a wave is simply the pressure change that occurs when a fluid is compressed or expanded.

Previously, we have been able to limit the difficulty of the mathematics by restricting our waves to one or two dimensions. It is now appropriate to discuss the behavior of waves in three dimensions. Once the three-dimensional wave equation has been developed, it will be convenient to look at some simple examples, starting with the simplest case of plane waves.

The characteristic property of plane waves is that each acoustic variable (particle displacement, density, pressure, etc.) has constant amplitude on any given plane perpendicular to the direction of wave propagation. Plane waves may be produced readily in a fluid that is confined in a rigid pipe through the action of a low-frequency vibrating piston located at one end of the pipe. Since wave fronts of any type of divergent wave in a homogeneous medium become nearly planar far from their source, we may expect that the properties of divergent waves will, at large distances, also become very similar to those of plane waves.

The following symbols* will be used:

$\vec{r}$ = equilibrium position of a particle of the fluid at (x, y, z)

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

* The symbols $\hat{x}$, $\hat{y}$, and $\hat{z}$ represent the *unit vectors* in the x, y, and z directions, respectively.

$\vec{\xi}$ = particle displacement from the equilibrium position

$$\vec{\xi} = \xi_x \hat{x} + \xi_y \hat{y} + \xi_z \hat{z}$$

$\vec{u}$ = *particle velocity*

$$\vec{u} = \frac{\partial \vec{\xi}}{\partial t} = u_x \hat{x} + u_y \hat{y} + u_z \hat{z}$$

ρ = *instantaneous density* at any point
ρ_0 = constant equilibrium density of the fluid
s = *condensation* at any point

$$s = (\rho - \rho_0)/\rho_0$$

$\mathscr{P}$ = instantaneous pressure at any point
$\mathscr{P}_0$ = constant equilibrium pressure in the fluid
p = excess pressure or *acoustic pressure* at any point

$$p = \mathscr{P} - \mathscr{P}_0$$

c = phase speed of the wave
Φ = *velocity potential*

$$\vec{u} = \nabla\Phi$$

T_K = temperature in kelvins (K)
T = temperature in degrees Celsius (or centigrade) (°C)

$$T + 273.15 = T_K$$

The term *particle of the fluid* means a volume element large enough to contain millions of molecules so that the fluid may be thought of as a continuous medium, yet small enough that all acoustic variables may be considered nearly constant throughout the volume element. The molecules of a fluid do not have fixed mean positions in the medium; even without the presence of a wave, they are in constant motion, with average velocities far in excess of any particle velocity associated with the wave motion. However, a small volume may be treated as an unchanging unit since those molecules leaving its confines are replaced by an equal number possessing (on the average) identical properties, so that the macroscopic properties of the element remain unchanged. As a consequence, it is possible to speak of particle displacements and velocities when discussing acoustic waves in fluids, as was done for elastic waves in solids. In the following analysis the effects of gravitational forces will be neglected, so that ρ_0 and $\mathscr{P}_0$ have uniform values throughout the fluid. The fluid is also assumed to be homogeneous, isotropic, and perfectly elastic; no dissipative effects such as those arising from viscosity or heat conduction are present. Finally, the analysis will be limited to waves of relatively small amplitude so that changes in density of the medium will be small compared with its equilibrium value, $|s| \ll 1$. These assumptions are necessary, to arrive at the simplest theory for sound in fluids. We are indeed fortunate that experiments have shown that this simplest theory adequately describes most common acoustical phenomena. However, one must remember that there are interesting situations where these assumptions are violated and the theory must be modified.

5.2 THE EQUATION OF STATE. The equation of state for a fluid relates the internal restoring forces to the corresponding deformations, as was done for oscillators, strings, and bars. As before, we will search for a linear relationship which, while simplifying the development, restricts the amount of allowed deformation. For fluid media, the equation of state must relate three physical quantities describing the thermodynamic behavior of the fluid. For example, the equation of state for a *perfect gas*

$$\mathscr{P} = \rho r T_K \tag{5.1}$$

gives the relationship between the total pressure $\mathscr{P}$ in pascals (Pa), the density in kilograms per cubic meter (kg/m³), and the absolute temperature T_K in kelvins. The quantity r is a constant whose value depends on the particular gas involved. This equation is general and describes any thermodynamic process for a perfect gas (see Appendix A9).

Greater simplification can be achieved if the thermodynamic process is *restricted*. For example, if the fluid is contained within a vessel whose walls are highly thermally conductive, then slow variations in the volume of the vessel will result in thermal energy being transferred between the walls and the fluid. If the walls have sufficient thermal capacity, they and the fluid will remain at a constant temperature. In this case, the perfect gas is described by the *isothermal* equation of state

$$\frac{\mathscr{P}}{\mathscr{P}_0} = \frac{\rho}{\rho_0} \qquad \text{(perfect gas)}$$

On the other hand, it is found experimentally that *acoustic* processes are nearly *adiabatic*: there is insignificant exchange of thermal energy from one particle of fluid to another. Under these conditions, the entropy (and not the temperature) of the fluid remains nearly constant. The behavior of the perfect gas under these conditions is described by the *adiabatic* equation of state

$$\frac{\mathscr{P}}{\mathscr{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \qquad \text{(perfect gas)} \tag{5.2}$$

where γ is the ratio of the specific heats (see Appendix A9, Eq. (A9.12)). For the acoustic disturbance of the fluid to be adiabatic, neighboring elements of the fluid must not exchange thermal energy. This means that the thermal conductivity of the fluid and the temperature gradients of the disturbance must be small enough that no significant thermal flux occurs during the time of the disturbance. For the frequencies and amplitudes usually of interest in acoustics, this is the case. The major effect of finite thermal conductivity is to dissipate very small fractions of the acoustic energy so that the disturbance attenuates slowly with time or distance. These effects will be considered in Chapter 7.

For fluids other than a perfect gas, the adiabatic equation of state is more

complicated. In these cases it is preferable to determine experimentally the isentropic relationship between pressure and density fluctuations. Given this relationship, we write a Taylor's expansion

$$\mathscr{P} = \mathscr{P}_0 + \left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_{\rho_0} (\rho - \rho_0) + \frac{1}{2}\left(\frac{\partial^2 \mathscr{P}}{\partial \rho^2}\right)_{\rho_0} (\rho - \rho_0)^2 + \cdots \tag{5.3}$$

where the partial derivatives are constants determined for the adiabatic compression and expansion of the fluid about its equilibrium density. If the fluctuations are small, only the lowest order term in $(\rho - \rho_0)$ need be retained. This gives a linear relationship between the pressure fluctuation and the change in density

$$\mathscr{P} - \mathscr{P}_0 \doteq \mathscr{B}\left(\frac{\rho - \rho_0}{\rho_0}\right) \tag{5.4}$$

where $\mathscr{B} = \rho_0(\partial \mathscr{P}/\partial \rho)_{\rho_0}$ is the *adiabatic bulk modulus*. In terms of the acoustic pressure p and the condensation s, (5.4) can be reexpressed as

$$\boxed{p \doteq \mathscr{B} s} \tag{5.5}$$

The essential restriction is that the condensation must be small, $|s| \ll 1$.

5.3 THE EQUATION OF CONTINUITY. To relate the motion of the fluid to its compression or dilatation, we need a functional relationship between the particle velocity $\vec{u}$ and the instantaneous density ρ. Consider a small rectangular-parallelepiped volume element $dV = dx\,dy\,dz$ which is *fixed* in space and through which elements of the fluid travel. The net rate with which mass flows into the volume through its surface must equal the rate with which the mass within the volume increases. Referring to Fig. 5.1, we see that the net influx of mass into this *spatially fixed* volume, resulting from flow in the x direction, is

$$\left\{\rho u_x - \left[\rho u_x + \frac{\partial(\rho u_x)}{\partial x}\,dx\right]\right\} dy\,dz = -\frac{\partial(\rho u_x)}{\partial x}\,dV$$

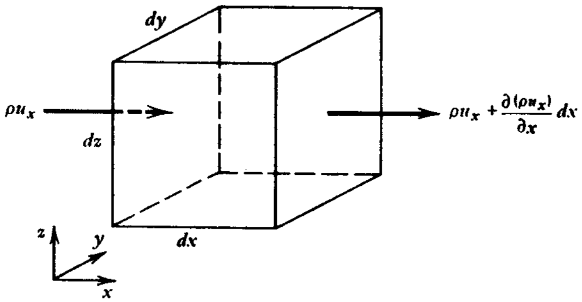

Fig. 5.1. Mass flow in the x direction through a fixed volume dV.

Similar expressions give the net influx for the y and z directions, so that the total influx must be

$$-\left[\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z}\right] dV \equiv -[\nabla\cdot(\rho\vec{u})]\, dV$$

where $\nabla\cdot$ is the *divergence* operator.* The rate with which the mass increases in the volume is $(\partial\rho/\partial t)\, dV$. Since the net influx must equal the rate of increase, we obtain

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \tag{5.6}$$

the *equation of continuity*. Notice that it is nonlinear; the second term involves the product of particle velocity and instantaneous density, both of which are acoustic variables. However, if we write $\rho = \rho_0(1 + s)$, use the fact that ρ_0 is a constant in both space and time, and assume that s is very small, (5.6) becomes

$$\boxed{\frac{\partial s}{\partial t} + \nabla\cdot\vec{u} \doteq 0} \tag{5.7}$$

the *linearized continuity equation*.

Before obtaining a force equation, let us combine the equations of state and continuity. If (5.7) is integrated with respect to time, we have

$$\int\left(\frac{\partial s}{\partial t} + \nabla\cdot\vec{u}\right) dt = \text{constant}$$

The integration constant must be zero since acoustic quantities vanish if there is no acoustic disturbance. Furthermore, since $\int\nabla\cdot\vec{u}\, dt = \nabla\cdot\int\vec{u}\, dt = \nabla\cdot\int(\partial\vec{\xi}/\partial t)\, dt = \nabla\cdot\vec{\xi}$, this becomes

$$s = -\nabla\cdot\vec{\xi} \tag{5.8}$$

and combination with the equation of state (5.5) gives

$$p \doteq -\mathscr{B}\nabla\cdot\vec{\xi}$$

Notice that if we consider one-dimensional waves (as was done in Chapter 3), the compressive force is $f = pS$, $\nabla\cdot\vec{\xi}$ reduces to $\partial\xi_x/\partial x$, and the above equation becomes $f = -S\mathscr{B}\,\partial\xi_x/\partial x$, which is equivalent to (3.5) with $Y = \mathscr{B}$. Thus, $p = -\mathscr{B}\nabla\cdot\vec{\xi}$ is the three-dimensional analog of $f = -SY(\partial\xi/\partial x)$, written in terms of pressure rather than compressive force.

5.4 THE SIMPLE FORCE EQUATION: EULER'S EQUATION.

In real fluids, the existence of viscosity and the failure of acoustic processes to be perfectly adiabatic introduce dissipative terms. However, since we have already neglected the effects of thermal conductivity in the equation of state, we also ignore the effects of viscosity and consider the fluid to be *inviscid*.

* If readers are unfamiliar with the operator ∇, they should refer to Appendix A7.

Consider a fluid element $dV = dx\, dy\, dz$ which *moves with the fluid*, containing a specified mass dm of fluid. The net force $d\vec{f}$ on the element will accelerate it according to Newton's second law $d\vec{f} = \vec{a}\, dm$. In the absence of viscosity, the net force experienced by the element in the x direction is

$$df_x = \left[\mathscr{P} - \left(\mathscr{P} + \frac{\partial\mathscr{P}}{\partial x} dx\right)\right] dy\, dz = -\frac{\partial\mathscr{P}}{\partial x} dV$$

Analogous expressions for df_y and df_z allow us to write the complete vector force $d\vec{f} = df_x\hat{x} + df_y\hat{y} + df_z\hat{z}$ as

$$d\vec{f} = -\nabla\mathscr{P}\, dV$$

The acceleration of the fluid is a little more complicated. The particle velocity $\vec{u}$ is a function of both time and space. When the fluid element with velocity $\vec{u}(x, y, z, t)$ at (x, y, z) and time t moves to a new location $(x + dx, y + dy, z + dz)$ at a later time $t + dt$, its new velocity is $\vec{u}(x + dx, y + dy, z + dz, t + dt)$. Thus the acceleration is

$$\vec{a} = \lim_{dt\to 0}\frac{\vec{u}(x + u_x\, dt, y + u_y\, dt, z + u_z\, dt, t + dt) - \vec{u}(x, y, z, t)}{dt}$$

The move from the former position to the new one allows us to relate the incrementals through the velocity components of the element,

$$dx = u_x\, dt \qquad dy = u_y\, dt \qquad dz = u_z\, dt$$

Since we are assuming all incrementals very small, the new velocity can be expressed by the first terms of its Taylor's expansion

$$\vec{u}(x + u_x\, dt, y + u_y\, dt, z + u_z\, dt, t + dt) =$$

$$\vec{u}(x, y, z, t) + \frac{\partial\vec{u}}{\partial x}u_x\, dt + \frac{\partial\vec{u}}{\partial y}u_y\, dt + \frac{\partial\vec{u}}{\partial z}u_z\, dt + \frac{\partial\vec{u}}{\partial t} dt$$

and the acceleration of the chosen fluid element is

$$\vec{a} = \frac{\partial\vec{u}}{\partial t} + u_x\frac{\partial\vec{u}}{\partial x} + u_y\frac{\partial\vec{u}}{\partial y} + u_z\frac{\partial\vec{u}}{\partial z}$$

If we define the vector operator $(\vec{u}\cdot\nabla)$ as

$$(\vec{u}\cdot\nabla) = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$$

then $\vec{a}$ can be written more succinctly

$$\vec{a} = \frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} \tag{5.9}$$

Now, since the mass dm of the element is $\rho\, dV$, substitution into $d\vec{f} = \vec{a}\, dm$ gives

$$-\nabla \mathscr{P} = \rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right] \tag{5.10}$$

This nonlinear, inviscid force equation is *Euler's equation*. It can be simplified if we require $|s| \ll 1$ and $|(\vec{u}\cdot\nabla)\vec{u}| \ll |\partial\vec{u}/\partial t|$. Then ρ can be replaced with ρ_0, and the term $(\vec{u}\cdot\nabla)\vec{u}$ can be dropped in (5.10). Finally, we can substitute $\nabla\mathscr{P} = \nabla p$, since $\mathscr{P}_0$ is a constant, and obtain

$$\boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} \doteq -\nabla p} \tag{5.11}$$

This is the *linear inviscid force equation*, valid for acoustic processes of small amplitude.

5.5 THE LINEARIZED WAVE EQUATION.

The three equations (5.5), (5.7), and (5.11) must be combined to yield a single differential equation with one dependent variable. The particle velocity can be eliminated between (5.7) and (5.11). Take the divergence of (5.11)

$$\rho_0 \nabla\cdot\frac{\partial \vec{u}}{\partial t} = -\nabla\cdot(\nabla p) = -\nabla^2 p$$

where ∇^2 is the three-dimensional Laplacian operator. Next, take the time derivative of (5.7) and use $\partial(\nabla\cdot\vec{u})/\partial t = \nabla\cdot(\partial\vec{u}/\partial t)$,

$$\frac{\partial^2 s}{\partial t^2} + \nabla\cdot\frac{\partial \vec{u}}{\partial t} = 0$$

Combination gives

$$\nabla^2 p = \rho_0 \frac{\partial^2 s}{\partial t^2}$$

Use of the equation of state (5.5) to eliminate s yields

$$\boxed{\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}} \tag{5.12}$$

where c is defined by

$$\boxed{c = \sqrt{\mathscr{B}/\rho_0}} \tag{5.13}$$

Equation (5.12) is the linearized, lossless wave equation for the propagation of sound in fluids. As should be obvious from the discussion of longitudinal waves in bars, c is the phase speed for acoustic waves in fluids.

Use of (5.13) allows the equation of state to be written in a more convenient form

$$\boxed{p = \rho_0 c^2 s} \tag{5.14}$$

Because p and s are proportional, the condensation satisfies the wave equation. Since the density ρ and the condensation are linearly related, the instantaneous density also satisfies the wave equation.

Since the curl of the gradient of a function must vanish, $\nabla\times\nabla f = 0$, from (5.11) the particle velocity must be irrotational, $\nabla\times\vec{u} = 0$. This means that it can be expressed as the gradient of a scalar function Φ,

$$\boxed{\vec{u} = \nabla\Phi} \tag{5.15}$$

where Φ is defined as the *velocity potential*. The physical meaning of this important result is that the acoustical excitation of an inviscid fluid involves no rotational flow; there are no effects such as boundary layers, shear waves, or turbulence. In real fluids, for which there is finite viscosity, the particle velocity is not curl-free everywhere, but for most acoustic processes the presence of small rotational effects is confined to the vicinity of boundaries and exerts little influence on the propagation of sound.

If we substitute (5.15) into (5.11) to obtain

$$\rho_0 \frac{\partial}{\partial t}\nabla\Phi = -\nabla p$$

or

$$\nabla\left(\rho_0\frac{\partial\Phi}{\partial t} + p\right) = 0$$

and notice that the quantity in parentheses can be chosen to vanish identically if there is no acoustic excitation, then

$$\boxed{p = -\rho_0\frac{\partial\Phi}{\partial t}} \tag{5.16}$$

Substitution of this equation into (5.12) and integrating with respect to time will show that Φ also satisfies the wave equation.

5.6 SPEED OF SOUND IN FLUIDS.

By combining (5.4) and (5.13), we get a thermodynamic expression for the speed of sound

$$c = \sqrt{\left(\frac{\partial\mathscr{P}}{\partial\rho}\right)_{adiabatic}} \tag{5.17}$$

where the partial derivative is evaluated at equilibrium conditions of pressure and density. It is a characteristic property of the fluid dependent on the thermodynamic variables of temperature, pressure, and density. For ordinary acoustic waves corresponding to those normally audible to the human ear it is independent of frequency.

When a sound wave propagates through a *perfect gas*, the adiabatic gas law relating pressure and density may be utilized to derive an important special form of (5.17). Direct differentiation of (5.2) leads to

$$\left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_{adiabatic} = \gamma \frac{\mathscr{P}}{\rho}$$

If this expression is now evaluated at ρ_0 and substituted into (5.17),

$$c = \sqrt{\gamma \mathscr{P}_0/\rho_0} \tag{5.18}$$

is obtained.

Included in the appendix are values of γ and ρ_0 for various gases at 0°C and standard pressure $\mathscr{P}_0 = 1$ atm $= 1.013 \times 10^5$ Pa. Substitution of the appropriate values for air gives

$$c_0 = \sqrt{\frac{1.402 \times 1.013 \times 10^5}{1.293}} = 331.6 \text{ m/s}$$

as the theoretical value for the speed of sound in air at 0°C. This is in excellent agreement with measured values and thereby supports our earlier assumption that acoustic processes in a fluid are adiabatic.

For most gases at constant temperature the ratio of $\mathscr{P}_0/\rho_0$ is nearly independent of pressure: a doubling of pressure is accompanied by a doubling of the density of the gas, so that the speed of sound in a gas does not change with variations in the barometric pressure.

An alternate expression for the speed of sound in a perfect gas is found from (5.1) and (5.18) to be

$$c = \sqrt{\gamma r T_K} \tag{5.19}$$

The speed is proportional to the square root of the absolute temperature. In terms of the speed c_0 at 0°C, this becomes

$$c = c_0\sqrt{T_K/273} = c_0\sqrt{1 + T/273} \tag{5.20}$$

Theoretical prediction of the speed of sound for liquids is considerably more difficult than for gases. However, it is possible to show theoretically that $\mathscr{B} = \gamma \mathscr{B}_T$ where $\mathscr{B}_T$ is the isothermal bulk modulus. Since $\mathscr{B}_T$ is much easier to measure experimentally than $\mathscr{B}$, a convenient expression for the speed of sound in liquids is then obtained from (5.13) and the above,

$$c = \sqrt{\gamma \mathscr{B}_T/\rho_0} \tag{5.21}$$

where γ, $\mathscr{B}_T$, and ρ_0 all vary with the temperature and pressure of the liquid. Since no simple theory is available for predicting these variations, they must be measured experimentally and the resulting speed of sound expressed as a numerical formula.

For example, in distilled water a simplified formula is for c in m/s is

$$c(\mathscr{P}, t) = 1402.7 + 488t - 482t^2 + 135t^3 + (15.9 + 2.8t + 2.4t^2)(\mathscr{P}_G/100) \tag{5.22}$$

where $\mathscr{P}_G$ is the gauge pressure in bars and $t = T/100$, with T in degrees Celsius. This equation is accurate to within 0.05 percent for $0 \le T \le 100$°C and $0 \le \mathscr{P}_G \le 200$ bar (1 bar $= 10^5$ Pa).

5.7 HARMONIC PLANE WAVES. If all the acoustic variables are functions of only one spatial coordinate, the phase of any variable is a constant on any plane perpendicular to this coordinate. Such a wave is called a *plane wave*. If the coordinate system is chosen so that this plane wave propagates along the x axis, the wave equation reduces to

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.23}$$

where $p = p(x, t)$. By direct comparison with (2.5), we see that all the mathematical discussion of the solutions for transverse waves in Sects. 2.4 and 2.5 can be applied here and need not be repeated. Let us therefore proceed directly to harmonic plane waves and the interrelationships among the acoustic variables.

The complex form of the harmonic solution for the acoustic pressure of a plane wave is

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - kx)} + \mathbf{B}e^{j(\omega t + kx)} \tag{5.24}$$

and the associated particle velocity, from (5.11),

$$\vec{\mathbf{u}} = \left[\frac{\mathbf{A}}{\rho_0 c} e^{j(\omega t - kx)} - \frac{\mathbf{B}}{\rho_0 c} e^{j(\omega t + kx)}\right]\hat{x} \tag{5.25}$$

is entirely in the direction of propagation.

If we write

$$\mathbf{p}_+ = \mathbf{A}e^{j(\omega t - kx)}$$
$$\mathbf{p}_- = \mathbf{B}e^{j(\omega t + kx)} \tag{5.26}$$

then (5.24) and (5.25) yield the particle speeds

$$\mathbf{u}_+ = +\frac{\mathbf{p}_+}{\rho_0 c} \quad \text{and} \quad \mathbf{u}_- = -\frac{\mathbf{p}_-}{\rho_0 c} \tag{5.27}$$

Use of (5.14) and (5.16) shows that

$$\mathbf{s}_+ = +\frac{\mathbf{p}_+}{\rho_0 c^2} \quad \text{and} \quad \mathbf{s}_- = +\frac{\mathbf{p}_-}{\rho_0 c^2} \tag{5.28}$$

and

$$\mathbf{\Phi}_+ = -\frac{1}{j\omega\rho_0}\mathbf{p}_+ \quad \text{and} \quad \mathbf{\Phi}_- = -\frac{1}{j\omega\rho_0}\mathbf{p}_- \tag{5.29}$$

For a plane wave travelling in some arbitrary direction, it is plausible to try a solution of the form

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - k_x x - k_y y - k_z z)} \tag{5.30}$$

Substitution into (5.12) shows that this is acceptable if

$$\sqrt{k_x^2 + k_y^2 + k_z^2} = \omega/c \tag{5.31a}$$

If we define the *propagation vector*

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \tag{5.31b}$$

which has magnitude ω/c and a position vector

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

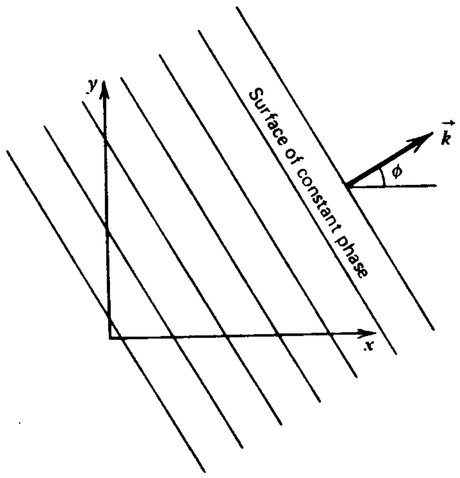

Fig. 5.2. An oblique plane wave.

which gives the location of the point (x, y, z) with respect to the origin $(0, 0, 0)$ of the coordinate system, then our trial solution (5.30) can be expressed by

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{5.32}$$

and the surfaces of constant phase are given by $\vec{k}\cdot\vec{r} = constant$. Since $\vec{k} = \nabla(\vec{k}\cdot\vec{r})$ is a vector perpendicular to the surfaces of constant phase, $\vec{k}$ points in the direction of propagation. The magnitude of $\vec{k}$ is the *wave number* k and k_x/k, k_y/k, and k_z/k are the *direction cosines* of $\vec{k}$ with respect to the x, y, and z axes. As a special case, for a plane wave with the propagation vector $\vec{k}$ parallel to the $z = 0$ plane, we have $k_z = 0$, and the direction cosines become $k_x/k = \cos\phi$ and $k_y/k = \sin\phi$ where ϕ is the angle of elevation above the x axis, as suggested by Fig. 5.2.

5.8 ENERGY DENSITY.

The energy transported by acoustic waves through a fluid medium is of two forms: the *kinetic energy* of the moving particles and the

potential energy of the compressed fluid. Consider a small fluid element that moves with the fluid and occupies volume V_0 of the undisturbed fluid. The kinetic energy of this element is

$$E_k = \tfrac{1}{2}\rho_0 V_0 u^2 \tag{5.33}$$

where $\rho_0 V_0$, the mass of the element, is calculated using the density and volume of the undisturbed fluid. The change in potential energy associated with a volume change from V_0 to V is

$$E_p = -\int_{V_0}^{V} p\, dV \tag{5.34a}$$

where the negative sign indicates that the potential energy will increase as work is done on the element when its volume is decreased by action of a positive acoustic pressure p. To carry out this integration it is necessary to express all variables under the integral sign in terms of one variable, p for example. From conservation of mass we have $\rho V = \rho_0 V_0$ so that

$$dV = -\frac{V}{\rho}\,d\rho \doteq -\frac{V_0}{\rho_0}\,d\rho$$

Now, use of $p = \rho_0 c^2 s$ and $s = (\rho - \rho_0)/\rho_0$ gives

$$dV = -\frac{V_0}{\rho_0 c^2}\,dp$$

Substitution of this into (5.34a) and integration of the acoustic pressure from 0 to p gives

$$E_p = \frac{1}{2}\frac{p^2}{\rho_0 c^2}V_0 \tag{5.34b}$$

The total acoustic energy of the volume element is

$$E = E_k + E_p = \tfrac{1}{2}\rho_0\left(u^2 + \frac{p^2}{\rho_0^2 c^2}\right)V_0$$

and the *instantaneous energy density* $\mathscr{E}_i = E/V_0$ in joules per cubic meter (J/m^3) is

$$\mathscr{E}_i = \tfrac{1}{2}\rho_0\left(u^2 + \frac{p^2}{\rho_0^2 c^2}\right) \tag{5.35}$$

The instantaneous particle speed and acoustic pressure are functions of both position and time, and consequently the instantaneous energy density $\mathscr{E}_i$ is not constant throughout the fluid. The time average of $\mathscr{E}_i$ gives the *energy density* $\mathscr{E}$ at any point in the fluid

$$\mathscr{E} = \langle\mathscr{E}_i\rangle_t = \frac{1}{t}\int_0^t \mathscr{E}_i\,dt \tag{5.36}$$

where the time interval is one period T of a harmonic wave.

These expressions apply to any acoustic wave. However, to proceed further, it is necessary to know the relationship between p and u. For a plane harmonic wave traveling in the $\pm x$ direction, reference to (5.27) shows that $p = \pm\rho_0 cu$ so that (5.35) gives

$$\mathscr{E}_i = \rho_0 u^2 = pu/c$$

and if P and U are the amplitudes of the acoustic pressure and particle speed,

$$\mathscr{E} = \tfrac{1}{2}PU/c = \tfrac{1}{2}P^2/(\rho_0 c^2) = \tfrac{1}{2}\rho_0 U^2 \tag{5.37}$$

It must be noticed that in any other case (for example, spherical or cylindrical waves, or standing waves in a room), the pressure and particle speed in (5.35) must be the *real* quantities obtained from the *superposition* of all waves present. In these more complicated cases, there is no guarantee that $p = \pm\rho_0 cu$ nor that the energy density is given by $\mathscr{E} = \tfrac{1}{2}PU/c$. It is true, however, that $\mathscr{E} = \tfrac{1}{2}PU/c$ is approximately correct for progressive waves when the surfaces of constant phase become so close to planar that the radius of curvature is much greater than a wavelength. This occurs, for example, for spherical and cylindrical waves at great distances (many wavelengths) from their sources.

5.9 ACOUSTIC INTENSITY.

The *acoustic intensity* I of a sound wave is defined as the *average* rate of flow of energy through a unit area normal to the direction of propagation. Its fundamental units are watts per square meter (W/m^2). The instantaneous rate at which work is done per unit area by one element of fluid on an adjacent element is pu. The intensity is the time average of this rate,

$$I = \langle pu \rangle_t = \frac{1}{t}\int_0^t pu\, dt \tag{5.38}$$

where the integration is taken over a time corresponding to the period of one complete cycle. To evaluate this integral for any particular wave, it is necessary to know the relationship between p and u.

For a plane harmonic wave traveling in the positive x direction, $p = \rho_0 cu$ so that

$$I = \tfrac{1}{2}P_+ U_+ = \tfrac{1}{2}P_+^2/(\rho_0 c) \tag{5.39a}$$

On the other hand, for a plane harmonic wave traveling in the negative x direction, we have $p = -\rho_0 cu$ and

$$I = -\tfrac{1}{2}P_- U_- = -\tfrac{1}{2}P_-^2/(\rho_0 c) \tag{5.39b}$$

To emphasize the similarity of (5.39) with corresponding equations for electromagnetic waves and voltage waves on transmission lines, as well as to write them in a more practical form, let us express them in terms of effective (root-mean-square) amplitudes. If we define F_e as the *effective amplitude* of a *periodic* quantity $f(t)$, then

$$F_e = \sqrt{\frac{1}{T}\int_0^T f^2(t)\, dt}$$

where T is the period of the motion. For harmonic waves this yields

$$P_e = P/\sqrt{2} \qquad \text{and} \qquad U_e = U/\sqrt{2} \tag{5.40}$$

so that

$$I_\pm = \pm P_e U_e = \pm P_e^2/(\rho_0 c) \tag{5.41}$$

for a plane wave traveling in either the $+x$ or $-x$ direction.

Once more it must be emphasized that, while (5.38) is completely general, $I_\pm = \pm P_e U_e$ are exact only for plane harmonic waves and approximate for diverging waves at great distances from their sources.

5.10 SPECIFIC ACOUSTIC IMPEDANCE.

The ratio of acoustic pressure in a medium to the associated particle speed is the *specific acoustic impedance*

$$\boxed{\mathbf{z} = \frac{\mathbf{p}}{\mathbf{u}}} \tag{5.42}$$

For plane waves this ratio is

$$\mathbf{z} = \pm\rho_0 c \tag{5.43}$$

the plus or minus sign depending on whether propagation is in the plus or minus direction. The MKS unit of specific acoustic impedance is Pa · s/*m*.* In later sections it will become apparent that the product $\rho_0 c$ often has greater significance as a characteristic property of the medium than does either ρ_0 or c individually. For this reason $\rho_0 c$ is called the *characteristic impedance* (*resistance*) of the medium.

Although the specific acoustic impedance of the medium is a real quantity for progressive plane waves, this is not true for standing plane waves or for diverging waves. In general, $\mathbf{z}$ will be found to be complex

$$\mathbf{z} = r + jx \tag{5.44}$$

where r is called the *specific acoustic resistance* and x the *specific acoustic reactance* of the medium for the particular wave being considered.

The characteristic impedance of a medium for acoustic waves is analogous to the index of refraction n of a transparent medium for light waves, to the wave impedance $\sqrt{\mu/\varepsilon}$ of a dielectric medium for electromagnetic waves, and to the characteristic impedance Z_0 of an electric transmission line.

Numerical values of $\rho_0 c$ for various fluids, and some solids, are given in Appendix A10.

At a temperature of 20°C and atmospheric pressure the density of air is 1.21 kg/m^3 and the speed of sound is 343 m/s, giving the standard characteristic impedance of *air*

$$(\rho_0 c)_{20} = 415 \text{ Pa}\cdot\text{s/m} \tag{5.45a}$$

* The unit of specific acoustic impedance is often given as *rayl*, where 1 MKS rayl = 1 Pa · s/m, established in honor of John William Strutt, Baron Rayleigh.

At 20 C and one atmosphere, the speed of sound in *distilled water*, as computed from (5.22), is 1482.3 m/s and its density is 998.2 kg/m^3, resulting in a characteristic impedance of

$$(\rho_0 c)_{20} = 1.48 \times 10^6 \text{ Pa}\cdot\text{s/m} \tag{5.45b}$$

5.11 SPHERICAL WAVES. Expressed in spherical coordinates the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, as shown in Appendix A7. If the waves have spherical symmetry, the acoustic pressure p is a function of radial distance and time but not of the angular coordinates θ and ϕ. Then this equation simplifies to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

and the wave equation for spherically symmetric pressure fields is

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.46}$$

(Conservation of energy and the relationship $I = P^2/(2\rho_0 c)$ lead us to expect that the pressure amplitude should fall off as $1/r$, so that the quantity rp might have amplitude independent of r.) Rewriting (5.46) with rp treated as the dependent variable results in

$$\frac{\partial^2(rp)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(rp)}{\partial t^2} \tag{5.47}$$

If the product rp in this equation is considered as a single variable, the equation is of the same form as the plane wave equation with the general solution

$$rp = f_1(ct - r) + f_2(ct + r)$$

or

$$p = \frac{1}{r}f_1(ct - r) + \frac{1}{r}f_2(ct + r)$$

The first term represents a spherical wave diverging from a point source at the origin with speed c; the second term represents a wave converging on the origin.

The most important diverging spherical waves are harmonic. Such waves are represented in complex form by

$$\mathbf{p} = \frac{\mathbf{A}}{r} e^{j(\omega t - kr)} \tag{5.48}$$

Use of the relationships developed in Sect. 5.5 for a general wave allows the other acoustic variables to be expressed in terms of the pressure

$$\mathbf{\Phi} = -\frac{1}{j\omega\rho_0}\mathbf{p} \tag{5.49}$$

$$\hat{\mathbf{u}} = \nabla\mathbf{\Phi} = \left(1 - \frac{j}{kr}\right)\frac{\mathbf{p}}{\rho_0 c}\hat{r} \tag{5.50}$$

The observed acoustic variables are obtained by taking the real parts of (5.48) through (5.50).

It is apparent from (5.50) that, in contrast with plane waves, the particle speed is *not* in phase with the pressure. The specific acoustic impedance is not $\rho_0 c$, but rather

$$\mathbf{z} = \rho_0 c \frac{kr}{\sqrt{1 + (kr)^2}} e^{j\theta} \tag{5.51}$$

or

$$\mathbf{z} = \rho_0 c \cos\theta\, e^{j\theta} \tag{5.52}$$

where

$$\cot\theta = kr \tag{5.53}$$

A geometrical representation of θ is given in Fig. 5.3. As is true with many other acoustic phenomena, the product of k and r is the determining factor, rather than the magnitude of either. Since $kr = 2\pi r/\lambda$, the phase angle θ is a function of the ratio of the source distance to the wavelength. When the distance from the source is only a small fraction of a wavelength, the phase difference between the complex pressure and particle speed is large. On the other hand, at distances corresponding to a considerable number of wavelengths, $\mathbf{p}$ and $\mathbf{u}$ are very nearly in phase, and the spherical wave then assumes the characteristics of a plane wave. This behavior is to be expected, since the wave fronts of all spherical waves become essentially plane at great distances from their source.

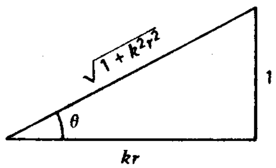

Fig. 5.3. The relationship between θ and kr.

Separating (5.51) into real and imaginary parts, we have

$$\mathbf{z} = \rho_0 c \frac{(kr)^2}{1 + (kr)^2} + j\rho_0 c \frac{kr}{1 + (kr)^2} \tag{5.54}$$

The first term is the *specific acoustic resistance*, and the second term contains the *specific acoustic reactance*. Both terms approach zero for very small values of kr, but for very large values of kr the resistive term approaches $\rho_0 c$, while the reactive term approaches zero. When $kr = 1$, both the specific acoustic resistance and reactance are equal to $\rho_0 c/2$and the specific acoustic reactance has its maximum value.

The absolute magnitude z of the specific acoustic impedance is equal to the ratio of the pressure amplitude P of the wave to its speed amplitude U,

$$z = \frac{P}{U} = \rho_0 c \cos \theta \tag{5.55}$$

and the relationship between pressure and speed amplitude may be written as

$$P = \rho_0 c\, U \cos \theta \tag{5.56}$$

For large values of kr, $\cos \theta$ approaches unity, and the relationship between pressure and speed is then the same as that given for a plane wave. As the distance from the source of a spherical acoustic wave to the point of observation is decreased, both kr and $\cos \theta$ decrease, so that larger and larger particle speeds are associated with a given pressure amplitude. For very small distances from a point source of sound, the particle speed corresponding to even very low acoustic pressures becomes impossibly large: a source small compared to a wavelength is inherently incapable of generating waves of large intensity.

Let us rewrite (5.48) as

$$\mathbf{p} = \frac{A}{r} e^{j(\omega t - kr)} \tag{5.57}$$

where without any loss in generality we have chosen a new origin of time such that the complex amplitude $\mathbf{A}$ becomes a real constant A. Then A/r is the *pressure amplitude* of the wave. It should be noted that the pressure amplitude in an undamped spherical wave is not constant, as it is for an undamped plane wave, but decreases inversely with the distance from the source. The actual pressure is the real part of (5.57),

$$p = \frac{A}{r} \cos(\omega t - kr) \tag{5.58}$$

Since $\mathbf{u} = \mathbf{p}/\mathbf{z}$, the corresponding complex expression for the particle speed is

$$\mathbf{u} = \frac{A}{r\mathbf{z}} e^{j(\omega t - kr)} \tag{5.59}$$

In replacing $\mathbf{z}$ by (5.52) and then taking the real part of the resulting expression, we have for the actual particle speed

$$u = \frac{1}{\rho_0 c} \frac{A}{r} \frac{1}{\cos \theta} \cos(\omega t - kr - \theta) \tag{5.60}$$

It is apparent that, since θ is a function of kr, the speed amplitude

$$U = \frac{1}{\rho_0 c} \frac{A}{r} \frac{1}{\cos \theta} \tag{5.61}$$

is not inversely proportional to the distance from the source. As a result it is usually advantageous to treat problems involving spherical waves in terms of pressure amplitude rather than speed amplitude.

For a harmonic spherical wave (5.38) yields

$$I = \frac{1}{T}\int_0^T P \cos(\omega t - kr) U \cos(\omega t - kr - \theta)\, dt = \frac{PU \cos \theta}{2} = \frac{P^2}{2\rho_0 c} \tag{5.62}$$

where the factor $\cos \theta$ is analogous to the power factor of an alternating-current circuit. Notice that the formula $I = P^2/(2\rho_0 c)$ has been found to be *exactly* true for both plane and spherical waves. In the case of the spherical wave, this is consistent with the argument following (5.46), and in fact verifies that P falls off *exactly* as $1/r$ for a lossless medium.

The average rate at which energy flows through a closed spherical surface of radius r surrounding a source of symmetrical spherical waves is

$$\Pi = 4\pi r^2 I = 4\pi r^2 P^2/(2\rho_0 c) \tag{5.63}$$

or since $P^2 = A^2/r^2$

$$\Pi = 2\pi A^2/(\rho_0 c) \tag{5.64}$$

The average rate of energy flow through any spherical surface surrounding the origin is independent of the radius of the surface, a conclusion that is consistent with conservation of energy in a lossless medium.

5.12. DECIBEL SCALES. It is customary to describe sound pressures and intensities through the use of logarithmic scales known as *sound levels*. One reason for doing this is the very wide range of sound pressures and intensities encountered in our acoustic environment; audible intensities range from approximately 10^{-12} to $10\ \mathrm{W/m^2}$. The use of a logarithmic scale compresses the range of numbers required to describe this wide range of intensities. A second reason is that humans judge the relative loudnesses of two sounds by the ratio of their intensities, a logarithmic behavior.

The most generally used logarithmic scale for describing sound levels is the decibel scale. The *intensity level* IL of a sound of intensity I is defined by

$$IL = 10 \log(I/I_{ref}) \tag{5.65}$$

where I_{ref} is a reference intensity, IL is expressed in *decibels referenced to* I_{ref} (*dB re* I_{ref}), and "log" represents logarithm to the base 10.

We have shown in Sects. 5.9 and 5.11 that intensity and effective pressure of progressive plane and spherical waves are related by $I = P_e^2/(\rho_0 c)$. Consequently, the intensities in (5.65) may be replaced by expressions for pressure, leading to the *sound pressure level*

$$SPL = 20 \log (P_e/P_{ref}) \tag{5.66}$$

where SPL is expressed in dB *re* P_{ref}; P_e is the measured effective pressure of the sound wave, and P_{ref} is the reference effective pressure. If we choose $I_{ref} = P_{ref}^2/(\rho_0 c)$, then IL *re* $I_{ref} = SPL$ *re* P_{ref}.

There is a multiplicity of units used to specify pressures throughout the various scientific disciplines and many of these are found in acoustics. In addition, a number of reference levels, of various degrees of antiquity, are encountered. Let us first catalog a few units:

CGS units
 1 dyne/cm², also called the microbar (μbar) (The microbar was originally 10^{-6} atmospheres but is now defined to be *identically* 1 dyne/cm².)

MKS units
 1 pascal (Pa), identical to 1 N/m²

Others
 1 atmosphere = 1.013×10^5 Pa = 1.013×10^6 μbar
 1 kilogram/cm² (kgf/cm²) = 0.968 atm = 0.981×10^5 Pa

Equivalents
 1 μbar = 0.1 N/m² = 10^5 μPa

The reference standard for airborne sounds is 10^{-12} W/m², which is approximately the intensity of a 1000 Hz pure tone that is just barely audible to a person with unimpaired hearing. Substitution of this intensity into (5.39) shows that it corresponds to a peak pressure amplitude of

$$P = \sqrt{2\rho_0 c I} = 2.89 \times 10^{-5} \text{ Pa}$$

or a corresponding effective (root-mean-square) pressure of

$$P_e = P/\sqrt{2} = 20.4\ \mu\text{Pa} \tag{5.67}$$

The latter pressure, rounded off to 20 μPa, is often used as a reference for sound pressure levels in air. Since this pressure is almost exactly equivalent to the effective pressure corresponding to a reference intensity of 10^{-12} W/m² used in (5.65), essentially identical numerical results are obtained by use of either (5.65) or (5.66) when plane or spherical progressive waves are being measured in air. However, in certain more complex sound fields, such as standing wave patterns, intensity and pressure are no longer simply related by (5.41) and (5.62); as a consequence, (5.65) and (5.66) will not yield identical results. Since the voltage outputs of microphones and hydrophones commonly used in acoustic measurements are proportional to pressure, acoustic pressure is the most readily measured variable in a sound field. For this reason, sound pressure levels are more widely used in specifying sound levels than are intensity levels.

Three different pressures are encountered as reference pressures for specifying sound pressure levels in underwater acoustics. One is an effective pressure of 20 μPa (the same as the reference pressure in air). The second reference pressure is 1 μbar and the third is 1 μPa. Of these three, the last is now preferred.

This plurality of reference pressures can lead to confusion unless care is taken to always specify the reference pressure being used: *SPL re* 20 μPa, *re* 1 μPa, or *re* 1 μbar. Table 5.1 summarizes the various methods of expressing the level of a sound in decibels.

From the above discussion, note that a given acoustic pressure in air corresponds to a much higher intensity than does the same acoustic pressure in water.

Table 5.1. References and conversions

Medium	*Reference*	*Nearly equivalent to*
Air	10^{-12} W/m²	20 μPa
	20 μPa = 0.0002 μbar	10^{-12} W/m²
Water	1 μbar = 10^5 μPa	6.76×10^{-9} W/m²
	0.0002 μbar = 20 μPa	2.70×10^{-16} W/m²
	1 μPa	6.76×10^{-19} W/m²

SPL re 1 μbar + 100 = *SPL re* 1 μPa
SPL re 0.0002 μbar − 74 = *SPL re* 1 μbar
SPL re 0.0002 μbar + 25 = *SPL re* 1 μPa

Since (5.41) or (5.62) shows that for a given pressure amplitude, intensity is inversely proportional to the characteristic impedance of the medium, the ratio of the intensities in air to that in water for the *same acoustic pressure* is $1.48 \times 10^6/415 = 3570$. On the other hand, if we compare two acoustic waves of the *same frequency and particle displacement*, the intensity of the one in water is 3570 times that in air.

5.13 RAYS AND WAVES.

Plane waves of infinite spatial extent do not exist in nature. The nearest thing to a plane wave that can be produced in the laboratory is the wave that travels down a rigid-walled tube when the frequency is low enough that the wavelength is much greater than the cross-sectional dimensions of the tube. Even here the influence of viscosity at the walls will introduce small but measureable deviations. In the real world, instead of plane waves, we find beams of sound whose cross-sectional areas and directions of propagation may change as the beams traverse the medium. In such circumstances, we frequently find it useful to think of *rays* rather than *waves*. A *ray* can be defined as a line everywhere perpendicular to the surfaces of constant phase. Its usefulness lies in the intuitive feeling, mathematically justified under certain conditions, that energy is carried along a ray. In many cases, especially where c is a function of space or where the wave is restricted to a limited solid angle (such as the beam of sound from a highly directional source), description in terms of rays is much easier than in terms of wave fronts. However, rays are not exact replacements for waves but only approximations that are valid under certain rather restrictive conditions.

Let us begin by first considering a plausible solution to the wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{5.68}$$

where $c = c(x, y, z)$. For a beam of finite aperture traversing either a homogeneous fluid (c = *constant*) or an inhomogeneous fluid (c = a function of position), we must expect that the amplitude of this wave will vary with position and that the surfaces of constant phase can be complicated functions of space. As a trial solution try

$$\mathbf{p}(x, y, z, t) = A(x, y, z)e^{j\omega[t - \Gamma(x, y, z)/c_0]} \tag{5.69}$$

where A has units of pressure, Γ has units of length, and c_0 is a constant value of the phase speed to be defined later.

The values of (x, y, z) satisfying Γ = *constant* define the surfaces of constant phase and, as noted in the discussion of Sect. 5.7, $\nabla\Gamma$ is everywhere perpendicular to each surface of constant phase. For example, notice that if $A = constant$ and $\Gamma = x$, (5.69) becomes $\mathbf{p} = A\ \exp[j\omega(t - x/c_0)]$, a plane wave solution of (5.68) if $c = constant = c_0$. Furthermore, notice that $\nabla\Gamma = \hat{x}$ has unit magnitude and points in the direction of propagation (always in the x direction in this simple example).

Substitution of the trial solution into (5.68) gives

$$\frac{\nabla^2 A}{A} - \left(\frac{\omega}{c_0}\right)^2 \nabla\Gamma \cdot \nabla\Gamma + \left(\frac{\omega}{c}\right)^2 - j\frac{\omega}{c_0}\left(2\frac{\nabla A}{A} \cdot \nabla\Gamma + \nabla^2\Gamma\right) = 0 \qquad (5.70a)$$

If A and $\nabla\Gamma$ vary slowly enough that $|A^{-1}\nabla^2 A| \ll (\omega/c)^2$, $|\nabla^2\Gamma| \ll \omega/c$, and $|A^{-1}\nabla A \cdot \nabla\Gamma| \ll \omega/c$ all terms in (5.70a) except the second and third can be considered small. Then (5.70a) simplifies to

$$\boxed{\nabla\Gamma \cdot \nabla\Gamma \doteq n^2} \qquad (5.70b)$$

where

$$n(x, y, z) = \frac{c_0}{c(x, y, z)} \qquad (5.70c)$$

is the *refractive index* and the constant c_0 is an arbitrary reference speed. Equation (5.70b) is called the *Eikonal equation.*

Sufficient conditions for the inequalities required to reduce (5.70a) to (5.70b) are that (1) the amplitude of the wave must not change appreciably in distances comparable to a wavelength, and (2) the speed of sound must not change appreciably in distances comparable to a wavelength. If we consider a beam of sound traveling through a fluid, the first condition states that the Eikonal equation can be applied to the central portion of the beam where A is not rapidly varying. At the edges of the beam, however, A may rapidly decay to zero over distances of a wavelength and then (5.70b) is no longer valid. The failure of the Eikonal equation reveals itself in the *diffraction* of sound at the edges of the beam. This is the acoustic analog of the familiar diffraction of light through a slit or pinhole. The second condition requires that the speed of sound be so slowly varying with space that the local direction of propagation of the wavefronts does not change significantly over distances of a wavelength; the *refraction* of the sound beam must not be too rapid.*

This means that (5.70b) is in general an accurate description of acoustic propagation only in the limit of high frequencies (short wavelength); how high the frequency must be depends on the rapidity of spatial variations of c and A.

If A and Γ are slowly varying functions of space, then the waves described by (5.69) are similar to plane waves over small regions of space, but over larger distances these waves may be observably different from plane waves. Furthermore, from (5.70b) $\nabla\Gamma$ defines, point by point, the direction of travel of each ray so that solution of the Eikonal equation gives the trajectories of the ray paths traversed by the acoustic energy.

For example, assume that $\nabla\Gamma$ can be written as

$$\nabla\Gamma = n(\cos\phi\ \hat{x} + \sin\phi\ \hat{y}) \qquad (5.71)$$

* For simplicity, we have given *sufficient* conditions for the validity of (5.70b). More rigorous, *necessary* conditions can be stated, but their physical meanings are less direct. Indeed, there are examples of propagating waves that do not satisfy the sufficient conditions, but for which (5.70b) is valid.

so that we are dealing with propagation in the x, y plane. The angle of elevation ϕ above the x axis will be a function of position; therefore, the local direction of propagation of the wave will be different at different locations. At some instant of time, $\nabla\Gamma$ evaluated at a point on a surface of constant phase indicates the direction in which that portion of the surface will advance (see Fig. 5.4). As time advances, this surface element advances and the magnitude and direction of $\nabla\Gamma$ for this portion of the surface will change. In this way, each portion of the wave can be seen to trace out a trajectory in space. Each of these trajectories is a ray path, and the portion of the wave traveling along that path defines a ray. By labeling the *distance*

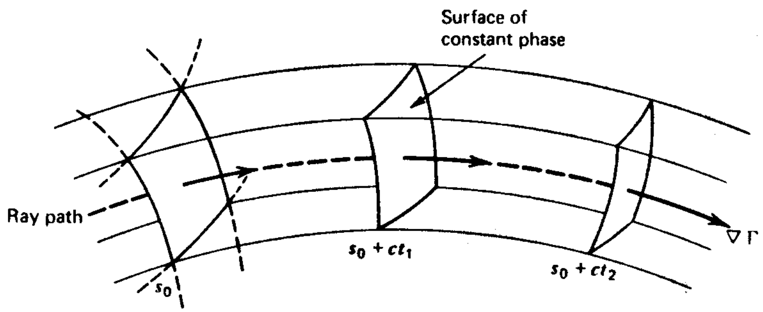

Fig. 5.4. A time history of a portion of a surface of constant phase.

along the ray path by a coordinate s, we see that the way $\nabla\Gamma$ changes along a ray path can be determined from the vector differential $d(\nabla\Gamma)/ds$. Examining this, component by component, in the x direction we have

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial\Gamma}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial\Gamma}{\partial x}\right)\frac{dx}{ds} + \frac{\partial}{\partial y}\left(\frac{\partial\Gamma}{\partial x}\right)\frac{dy}{ds}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial\Gamma}{\partial x}\right)\frac{dx}{ds} + \frac{\partial}{\partial x}\left(\frac{\partial\Gamma}{\partial y}\right)\frac{dy}{ds}\\ &= \cos\phi\frac{\partial}{\partial x}(n\cos\phi) + \sin\phi\frac{\partial}{\partial x}(n\sin\phi) = \frac{\partial n}{\partial x}\end{aligned}$$

Similarly, in the y direction,

$$\frac{d}{ds}\left(\frac{\partial\Gamma}{\partial y}\right) = \frac{\partial n}{\partial y}$$

so that the behavior of $\nabla\Gamma$ is found from

$$\frac{d}{ds}(\nabla\Gamma) = \nabla n \qquad (5.72)$$

Although this derivation was carried out in two dimensions, the same argument can be applied in three dimensions, except that the cosine and sine must be replaced by direction cosines. This complicates the development, but leads to the same vector equation (5.72).

A very powerful consequence of (5.72) is *Snell's law*. A simple statement of this law can be obtained if we let the speed of sound be a function only of x. Then $n = n(x)$ and the components of (5.72) become

$$\frac{d}{ds}\left(\frac{c_0}{c}\sin\phi\right) = 0 \tag{5.73a}$$

and

$$\frac{d}{ds}\left(\frac{c_0}{c}\cos\phi\right) = -\frac{c_0}{c^2}\frac{dc}{dx} \tag{5.73b}$$

It is important to notice that, since ∇c points in the x direction, our choice of ϕ is such that when $\phi = 0$, $\hat{k}$ is *parallel* to ∇c. Integration of the first equation gives

$$\boxed{\frac{\sin\phi}{c(x)} = constant} \tag{5.74}$$

which is a form of Snell's law. It states that the direction of propagation given by ϕ for a ray is uniquely determined. If the ray has some angle ϕ_0 at a value $x = x_0$ where the speed of sound is c_0, then the angle of this same ray when it reaches any other value x where the speed of sound is $c(x)$ is determined by

$$\frac{\sin\phi}{c(x)} = \frac{\sin\phi_0}{c_0}$$

[The importance of (5.74) will be revealed in Chapter 15, which is devoted to underwater acoustics.]

5.14 THE INHOMOGENEOUS WAVE EQUATION. In previous sections we developed a wave equation that applied to regions of space not containing any sources of acoustic energy. However, a source must be present to generate any acoustic disturbance. If the source is *external* to the region of interest, it can be taken into account by introducing time-dependent boundary conditions, as we did in Chapters 2, 3, and 4. Alternately, the hydrodynamic equations can be modified to include *source terms*. There are many possible types of sources, but only two will be considered here.

(1) If mass is being injected into the space at a rate per unit volume $G(\vec{r}, t)$, the linearized equation of continuity becomes

$$\rho_0\frac{\partial s}{\partial t} + \rho_0\nabla\cdot\vec{u} = G(\vec{r}, t) \tag{5.75}$$

This $G(\vec{r}, t)$ is generated by any closed surface that changes volume, such as a loudspeaker in an enclosed cabinet.

(2) If there are *body forces* present in the fluid, then a body force per unit volume $\vec{F}(\vec{r}, t)$ must be included in Euler's equation. The linearized equation of motion then becomes

$$\rho_0\frac{\partial\vec{u}}{\partial t} + \nabla p = \vec{F}(\vec{r}, t) \tag{5.76}$$

An example of this kind of force is that produced by a body that oscillates back and forth without any change in volume, such as the cone of an unbaffled loudspeaker.

If these two modifications are combined with the linearized equation of state, an *inhomogeneous* wave equation is obtained,

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial G}{\partial t} + \nabla\cdot\vec{F} \tag{5.77}$$

For all regions of space without sources, the right-hand side of (5.77) vanishes, leaving the homogeneous wave equation.

5.15 THE POINT SOURCE. The monofrequency spherical wave given by (5.57) is a solution to the homogeneous wave equation (5.12) everywhere *except* at $r = 0$. (This is consistent with the fact that there must be a source at $r = 0$ to generate the wave.) However, (5.57) does satisfy the inhomogeneous wave equation

$$\nabla^2\mathbf{p} - \frac{1}{c^2}\frac{\partial^2\mathbf{p}}{\partial t^2} = -4\pi A\,\delta(\vec{r})e^{j\omega t} \tag{5.78}$$

for *all* r: the three-dimensional delta function $\delta(\vec{r})$ is defined by

$$\int_V \delta(\vec{r})\,dV = \begin{cases} 1 & \text{if } \vec{r} = 0 \text{ is inside } V \\ 0 & \text{if } \vec{r} = 0 \text{ is outside } V \end{cases} \tag{5.79}$$

To prove this, multiply both sides of (5.78) by dV, integrate over a volume V that includes $\vec{r} = 0$, and then use Gauss' law (see Appendix A8) to reduce the volume integral to a surface integral and (5.79) to evaluate the delta function integral:

$$\int_S \nabla\mathbf{p}\cdot\hat{n}\,dS - \frac{1}{c^2}\int_V \frac{\partial^2\mathbf{p}}{\partial t^2}\,dV = -4\pi A e^{j\omega t}$$

where $\hat{n}$ is the unit outward normal to the surface S of V. By substituting (5.57) for $\mathbf{p}$ and carrying out the surface integration over a sphere centered on $\vec{r} = 0$ with vanishingly small radius (so that the volume integration vanishes), we can complete the proof.

It is trivial to generalize to a point source located at $\vec{r} = \vec{r}_0$: Making the appropriate change of variable in (5.48) results in

$$\mathbf{p} = \frac{A}{|\vec{r} - \vec{r}_0|}\exp[j(\omega t - k|\vec{r} - \vec{r}_0|)] \tag{5.80a}$$

which is a solution to

$$\nabla^2\mathbf{p} - \frac{1}{c^2}\frac{\partial^2\mathbf{p}}{\partial t^2} = -4\pi A\,\delta(\vec{r} - \vec{r}_0)e^{j\omega t} \tag{5.80b}$$

This may appear to be a difficult way to account for sources, but it will be seen that the incorporation of a point source directly into the wave equation is, in the proper circumstances, a considerable mathematical simplification. (See Sects. 15.13 to 15.15 as one example.) We will, however, use this formalism only when necessary, utilizing in most cases methods that are more closely related to elementary physical intuition.

PROBLEMS

5.1. (*a*) Linearize (5.2) by assuming $s \ll 1$. Then, by comparing this result with (5.4), obtain the adiabatic bulk modulus of a perfect gas in terms of $\mathscr{P}_0$ and γ. (*b*) With the help of (5.1) applied to equilibrium conditions, obtain the temperature dependence of $\mathscr{B}$ at constant volume.

5.2 Another form of the perfect gas law is $\mathscr{P}V = n\mathscr{R}T_K$ where n is the number of modes and $\mathscr{R}$ is the universal gas constant. Obtain a relationship between r and $\mathscr{R}$.

Solved Problems

WAVE EQUATION

3.1. Derive the general three-dimensional acoustic wave equation.

The derivation of the general acoustic wave equation in a form valid for discussing any three-dimensional type of nondissipative progressive wave is based on the following assumptions and procedure.

(1) The medium is assumed to be continuous and homogeneous, (2) the process is adiabatic, (3) a completely elastic medium, and (4) small amplitudes of particle displacements and velocities, as well as small changes in pressure and density.

(*a*) Develop the equation of continuity, (*b*) derive the dynamic equations from elastic properties and force equations, and (*c*) combine the three dynamic equations to form the general wave equation.

Consider a small element $dx\,dy\,dz$ of the fluid as having equilibrium coordinates x, y, z as shown in Fig. 3-1. Let u, v, w be the components of the particle velocity in the x, y, z directions respectively and ρ the density of the element. Then the mass flow of fluid through the left surface of this element will be

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{dx}{2}\right] dy\,dz$$

while the mass flow through the right surface is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{dx}{2}\right] dy\,dz$$

The resultant flow in the x direction is therefore equal to the difference of these two flows,

$$\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz$$

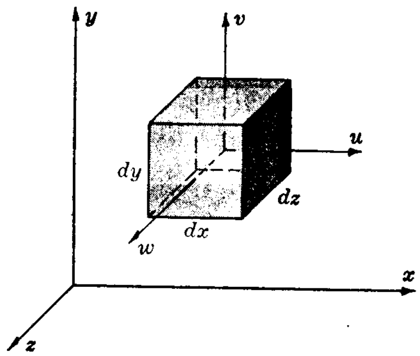

Fig. 3-1

Similarly, the resultant flows in the y and z directions are

$$\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz, \qquad \frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz$$

so that the net flow through the entire element is

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\,dy\,dz$$

Thus the equation of continuity is given by equating the net flow per unit mass to the time rate change of density

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial \rho}{\partial t}$$

To obtain the dynamic equation in the x direction, let p be the pressure at the center of the element. Then the pressures at the left and right faces of the element are respectively

$$p - \frac{\partial p}{\partial x}\left(\frac{dx}{2}\right), \qquad p + \frac{\partial p}{\partial x}\left(\frac{dx}{2}\right)$$

Hence the net force acting on the element in the x direction is

$$\left\{\left[p - \frac{\partial p}{\partial x}(dx/2)\right] - \left[p + \frac{\partial p}{\partial x}(dx/2)\right]\right\} dy\,dz = -(\partial p/\partial x)\,dx\,dy\,dz$$

For small amplitudes of particle displacement and velocity, the mass of this element can be expressed as $\rho\,dx\,dy\,dz$, and the velocity throughout the element in the x direction is u. From Newton's second law $\sum F = \frac{d}{dt}(mv)$, we have

$$-\frac{\partial p}{\partial x}\,dx\,dy\,dz = \frac{\partial}{\partial t}(\rho u\,dx\,dy\,dz)$$

or

$$-\frac{\partial p}{\partial x} = \frac{\partial}{\partial t}(\rho u) \qquad (1)$$

Similar dynamic equations in the y and z directions are

$$-\frac{\partial p}{\partial y} = \frac{\partial}{\partial t}(\rho v) \tag{2}$$

$$-\frac{\partial p}{\partial z} = \frac{\partial}{\partial t}(\rho w) \tag{3}$$

Now differentiate equations (1), (2), (3) respectively with respect to x, y, z:

$$-\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2}{\partial t\,\partial x}(\rho u) \tag{4}$$

$$-\frac{\partial^2 p}{\partial y^2} = \frac{\partial^2}{\partial t\,\partial y}(\rho v) \tag{5}$$

$$-\frac{\partial^2 p}{\partial z^2} = \frac{\partial^2}{\partial t\,\partial z}(\rho w) \tag{6}$$

Adding equations (4), (5), (6) yields

$$-\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \tag{7}$$

or

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\partial^2 \rho}{\partial t^2} \tag{8}$$

Now $\rho = \rho_0(1+s)$ and $p = Bs$. Then $\dfrac{\partial^2 \rho}{\partial t^2} = \rho_0 \dfrac{\partial^2 s}{\partial t^2}$, $\dfrac{\partial^2 s}{\partial t^2} = \dfrac{1}{B}\dfrac{\partial^2 p}{\partial t^2}$ where ρ_0 is the static density, s is the condensation, and B is bulk modulus. Equation (8) can be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\rho_0}{B}\frac{\partial^2 p}{\partial t^2} \tag{9}$$

Equation (9) is then the three-dimensional wave equation with acoustic pressure p as the variable.

3.2. Obtain solutions for the general two-dimensional wave equation in rectangular coordinates.

The general two-dimensional wave equation in rectangular coordinates is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

where p is the acoustic pressure and c the speed of sound.

(a) As in the case of the one-dimensional wave equation, we can write the solution in progressive waves form as

$$p(x,y,t) = f(mx+ny-ct) + g(mx+ny+ct), \qquad m^2+n^2 = 1 \tag{2}$$

which represents waves of the same shape moving in opposite directions along x and y axes with velocity c. This can be verified by differentiating equation (2) and substituting into (1).

(b) Let us next look for solutions in standing waves form which is represented by $p = X(x)\,Y(y)\,T(t)$ where X, Y, and T are functions of x, y and t respectively. Substitute this expression for p into (1) to obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} \tag{3}$$

Since the right-hand side of (3) is a function of t alone, and the left-hand side a function of x and y, each side must be equal to the same constant. Let this constant be $-p^2$. This leads to the following two differential equations:

$$\frac{d^2T}{dt^2} + c^2p^2T = 0 \tag{4}$$