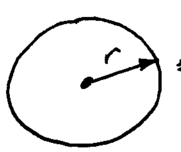
Harmonic Spreading Waves

Velocity on Surface moving sinusoidally ($e^{i\omega t}$)



$\bar{u} = U_0 e^{i\omega t}$

IN 3-D Spherical solution is of the form:

$$P(\bar{x}, t) = \frac{F(r - c_0 t)}{r} \qquad \left(\bar{x} = [r, \theta, \phi]\right)$$

$$\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \to 0\right)$$

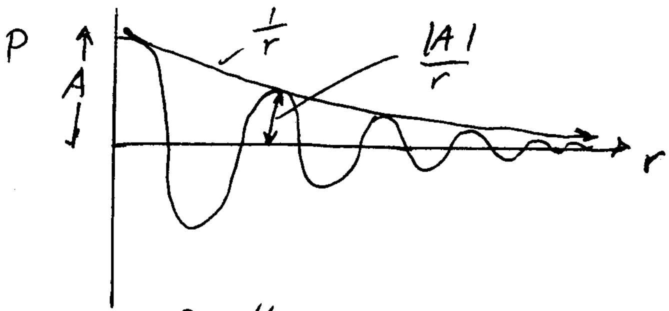
Acoustic wave eqn in spherical coordinates:

$$\boxed{\nabla^2 p - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rP) = 0}$$

then $\nabla \to \frac{\partial}{\partial r}$

So ~~[illegible]~~ ~~(not often used in sonar)~~
The harmonic waves are:

$$P(\bar{x}, t) = \frac{A e^{i\omega t - ikr}}{r} \qquad (\bar{x} = [r])$$



~~consider~~ Recall Acoustic Momentum Eqn $\quad -\frac{\partial P}{\partial r} = \rho \frac{\partial \bar{u}}{\partial t}$

"Relates motion of a fluid to the motion of the driver"

$\bar{u} = u_r$ (only radial)

$u = U e^{i\omega t}$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial t} U e^{i\omega t}$$

$$= i\omega U e^{i\omega t}$$

$$= i\omega \bar{u}$$

$$\frac{\partial \bar{u}}{\partial t} = i\omega u_r$$

∇ in spherical coordinate system

$$\nabla = u_r \frac{\partial}{\partial r} + u_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + u_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

For a pulsating sphere
u, p only vary radially
∇ reduces to
$\nabla = u_r \frac{\partial}{\partial r}$ where u_r = unit vector

So

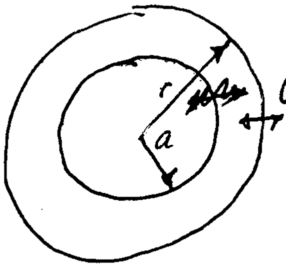
$$-\frac{\partial}{\partial r}(P) = i\omega\rho_0 u_r$$

$$\left(P = \tilde{P}\, e^{i\omega t}\right)$$

space phase   Time

$$\boxed{-\frac{\partial\tilde{P}}{\partial r}\, e^{i\omega t} = i\omega\rho_0 u_r}$$

Acoustic Momentum Equation for a Pulsating WAVE



$$U = U_0 e^{i\omega t}$$

$$\tilde{P}e^{i\omega t} = \frac{A\, e^{i\omega t - ikr}}{r}$$

Now if we assume: $\left.\begin{array}{l} a \ll \lambda \\ k = \frac{2\pi}{\lambda} \end{array}\right\}$ $\boxed{ka \ll 1}$

The vibrations are Tiny.

$$-\frac{\partial\tilde{P}}{\partial r}\, e^{i\omega t} = i\omega\rho_0 u_r \quad \text{where } u_r = U_0\, e^{i\omega t}$$

Now with the condition $r = a$ (at surface)

$$\left[-\frac{\partial\tilde{P}}{\partial r}\, e^{i\omega t}\right]_{r=a} = \left[i\omega\rho_0 U_0\, e^{i\omega t}\right]_{r=a}$$

So what is $\frac{\partial\tilde{P}e^{i\omega t}}{\partial r}$?

$$\tilde{P}e^{i\omega t} = \frac{A e^{i\omega t - ikr}}{r}$$

Now recall: $\frac{\partial}{\partial x}\left(\frac{A}{x}\right) = \frac{1}{x}\frac{dA}{dx} - \frac{1}{x^2}A$

from This we get

$$\frac{\partial \tilde{P} e^{i\omega t}}{\partial r} = \frac{1}{r} \cdot \left(-ikA e^{i\omega t - ikr}\right) - \frac{1}{r^2} A e^{i\omega t - ikr}$$

so

$$\frac{\partial \tilde{P} e^{i\omega t}}{\partial r} = -\frac{A e^{i\omega t - ikr}}{r} \left(\frac{1}{r} + ik\right)$$

or

$$\frac{\partial \tilde{P} e^{i\omega t}}{\partial r} = -\frac{A e^{i\omega t - ikr}}{r^2} \left(1 + ikr\right)$$

Now putting this into the Acoustic Momentum Eqn with $r = a$

$$+\frac{A e^{i\omega t - ika}}{a^2} \left(1 + ika\right) = i\omega \rho_0 U_0 e^{i\omega t}$$

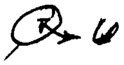
but since $ka \ll 1$

$$+\frac{A e^{i\omega t}}{a^2} = i\omega \rho_0 U_0 e^{i\omega t}$$

$$\boxed{A = +i\, \omega \rho_0 U_0 a^2}$$ for a pulsating sphere

Definitions:

Volume Velocity: $Q = \int_s U\,ds$



s - sphere

sphere: $S = 4\pi a^2$

$Q = Us$ (for a pulsating sphere)

$$\boxed{Q = 4\pi a^2 U}$$

$$A = \frac{i\omega\rho Q}{4\pi}$$

or

Acoustic Pressure field around a pulsating sphere

$$\boxed{\tilde{P}e^{i\omega t} = \frac{i\omega\rho Q}{4\pi}\frac{e^{i\omega t - ikr}}{r}}$$

$(ka \ll 1)$

$$P(t) = RP\left(\tilde{P}e^{i\omega t}\right)$$

Summary:

$$\boxed{\tilde{P}e^{i\omega t} = \frac{i\omega\rho Q}{4\pi r}e^{i\omega t - ikr}}$$

Acoustic pressure field around a pulsating sphere

$$P(t) = RP\left(\tilde{P}e^{i\omega t}\right) = |\tilde{P}|\cos(\omega T)$$

ω = frequency

ρ = density

r = radius

t = time

$k = \frac{\omega}{c}$ wave #

Q = Volume velocity

$Q = \int_s U\,ds_{sphere} = 4\pi a^2 U$

Assume $ka \ll 1$

a = radius of sphere

Particle Velocity around a pulsating sphere

Use Acoustic wave equation

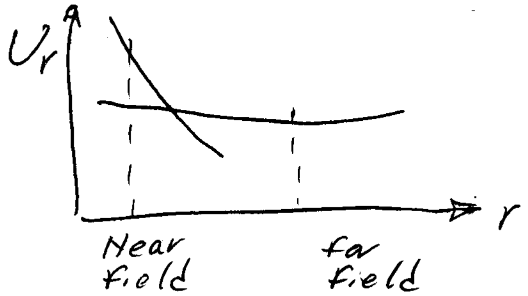
$$U_r = -\frac{1}{i\omega\rho}\frac{\partial \tilde{p}}{\partial r}e^{i\omega t}$$

we know that

$$\frac{\partial \tilde{p}}{\partial r}e^{i\omega t} = -\frac{Ae^{i\omega t - ikr}}{r^2}(1+ikr)$$

Therefore $U_r = +\frac{1}{i\omega\rho}\frac{Ae^{i\omega t - ikr}}{r^2}(1+ikr)$

(1) ⇒ IF we look at the case of $kr \ll 1$ (near field)



$$U_r \rightarrow +\frac{1}{i\omega\rho}\frac{Ae^{i\omega t - ikr}}{r^2}(1)$$

so $|U_r| \propto \frac{1}{r^2}$

(2) ⇒ consider $kr \gg 1$ (far field)

$$U_r \rightarrow +\frac{1}{i\omega\rho}\frac{Ae^{i\omega t - ikr}}{r^{\cancel{2}}}(ikr)$$

so $|U_r| \propto \frac{1}{r}$

FAR FIELD

$$U_r \simeq \frac{\cancel{i} k A e^{i\omega t - ikr}}{\cancel{i} \omega \rho_0 r} \quad \tilde{p} e^{i\omega t}$$

so $$\boxed{U_r \simeq \frac{\tilde{p} e^{i\omega t}}{\rho_0 c_0}}$$ where $\tilde{U} = \frac{\tilde{p}}{\rho_0 c_0}$

$$U_r = \tilde{U} e^{i\omega t}$$

$\rho_0 c_0$ = specific Acoustic impedence Units: RAYLS ($Pa \frac{s}{m}$)

$\rho_0 c_0 = 415$ Rayl (air)

$\rho_0 c_0 = 1.54 (10)^6$ Rayl (seawater)

near field - near to a source $kr < 1$

far field - far from a source