Mathematics - Fundamentals

Vectors - physical quantities (direction & magnitude)

$$\vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$$

The magnitude of a vector

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Vector Operations (addition - subtraction)

$$\vec{A} \pm \vec{B} = \hat{i}(A_x \pm B_x) + \hat{j}(A_y \pm B_y) + \hat{k}(A_z \pm B_z)$$

Vector multiplication (scalar)

$$\vec{A} \cdot \vec{B} = A_i B_i = A_x B_x + A_y B_y + A_z B_z = AB \cos(A,B)$$

vector multiplication (vectors)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = i(A_y B_z - A_z B_y + \ldots$$

$$|\vec{A} \times \vec{B}| = AB \sin(A,B)$$

Direction

$$\vec{B} \times \vec{A} = -[\vec{A} \times \vec{B}]$$

derivative of a vector

$$\frac{d\vec{A}}{ds} = \hat{i}\frac{dA_x}{ds} + \hat{j}\frac{dA_y}{ds} + \hat{k}\frac{dA_z}{ds}$$

derivative of scalar & vector products

$$\frac{d(\vec{A}\cdot\vec{B})}{ds} = A\cdot\frac{d\vec{B}}{ds} + \vec{B}\cdot\frac{d\vec{A}}{ds}$$

$$\frac{d[\vec{A}\times\vec{B}]}{ds} = \left[\vec{A}\times\frac{d\vec{B}}{ds}\right] + \left[\frac{d\vec{A}}{ds}\times\vec{B}\right]$$

Vector Operators

gradient of a scalar is a vector having the magnitude and direction of the greatest space rate of change of the scalar

$$\text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

the components of the gradient are the rates of change in each direction. ∇ is commonly used to represent the gradient vector differential operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Pronounced del

also known as the divergence operator

the <u>divergence</u> of a vector is a scalar obtained by taking the scalar product of the gradient operator and the vector:

$$\text{div}\,\vec{A} = \underset{\text{del}}{\nabla} \cdot \vec{A} = \overset{?}{\frac{\partial A_i}{\partial x_i}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

It represents the net outward flow of a quantity from a differential volume.

the <u>curl</u> of a field vector is a vector giving the magnitude and direction of its rotation.

cross product of gradient operator and vector

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \ldots$$

since the curl of a vector is a measure of its rotation, vector fields having zero curl are termed irrotational fields

Scalar Potentials

many fluid flows, in acoustics as well as in fluid mechanics are irrotational.

when the curl of a vector $= 0$

it is possible to define that vector quantity in terms of the gradient of ~~the~~ a scalar potential

$$\vec{A} = \pm\,\text{grad}\,\phi \qquad \text{generally (negative)}$$

In many instances the differential equation defining a scalar potential is of second order involving the divergence of the gradient of the potential. this 2nd order differential is called the <u>Laplacian</u>, Represented by ∇^2

$$\nabla^2 \phi \overset{def}{=} \text{div grad } \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial x_i^2}$$

the Laplacian operator plays a center role in equations of acoustics

Spherical Symmetry

Spherical coordinates involve a radial unit vector $\hat{r}$ and 2 angular coordinates

$$\vec{A} = \hat{r} A_r(r)$$

$$\nabla \phi \overset{def}{=} \frac{\partial \phi}{\partial r}$$

$$\nabla \cdot \vec{A} \overset{def}{=} \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} = \frac{\partial A_r}{\partial r} + \frac{2}{r} A_r$$

$$\nabla^2 \phi \overset{def}{=} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \frac{1}{r} \frac{\partial^2 (r \phi)}{\partial r^2}$$

Line integrals of a function are carried out between 2 pts along a specified path

$$\int_A^B F\, ds$$

when Quantity is a Vector

$$\int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (F_x dx + F_y dy + F_z dz) = \int_A^B F_i ds_i$$

When rotation is zero and a scalar potential exists, the integral is independent of path and dependent only on its end points

$$\int_A^B \nabla \phi \cdot \vec{ds} = \int_A^B \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \int_A^B d\phi = \phi_B - \phi_A$$

Surface integral of a vector function is the integral of its normal component over the surface

$$\int_S \vec{F} \cdot \vec{dS} = \int_S (\vec{F} \cdot \hat{n})\, dS = \int_S F_n\, dS$$

the surface integral gives the flux of the quantity through the surface and is a scalar quantity

<u>Volume integrals</u> are used to sum a quantity over a specific volume.

eg. the mass within a volume is given by the volume integral of density

$$m = \int_V \rho \, dV$$

<u>Complex Quantities</u>

A complex number — real, imaginary
or as a magnitude & phase ∡ or argument

$$\underline{A} = A_1 + iA_2 = Ae^{i\theta}$$

complex amplitude expresses the phase angle as well as the magnitude of a rotating complex vector

where the magnitude A is

$$A = |A| = \sqrt{A_1^2 + A_2^2}$$

and angle/argument θ

$$\theta = \tan^{-1} \frac{A_2}{A_1}$$

A_1 = real
A_2 = imaginary

<u>complex conjugate</u> of a complex # has same amplitude but negative argument

$$\underline{A}^* = A_1 - iA_2 = Ae^{-i\theta}$$

alternative expression $A = \sqrt{\underline{A} \cdot \underline{A}^*}$

real parts $RP(\underline{A})$ $Re(\underline{A})$

i is a 90° rotational operator

$$i = \sqrt{-1} = e^{i\left(\frac{\pi}{2}\right)}$$

$ii \Rightarrow$ 180° rotation or π radians

$$\underline{A} = A_1 + iA_2 = A\cos\theta + iA\sin\theta = Ae^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Definitions

ξ = particle displacement from the equilibrium position

$$\xi = \xi_x \hat{x} + \xi_y \hat{y} + \xi_z \hat{z}$$

$\bar{u}$ = Particle velocity

$$\bar{u} = \frac{\partial \xi}{\partial t} = u_x \hat{x} + u_y \hat{y} + u_z \hat{z}$$

ρ = instantaneous density at any point

ρ_0 = constant equilibrium density of the fluid

s = condensation at any point $s = \frac{(\rho - \rho_0)}{\rho_0}$

P = instantaneous pressure at any point

P_0 = constant equilibrium pressure in the fluid

$\mathbb{P}$ = excess pressure or acoustic pressure at any point $\mathbb{P} = P - P_0$

c = Phase speed of the wave