Acoustic Wave Equation

Properties of a Vibrating Fluid

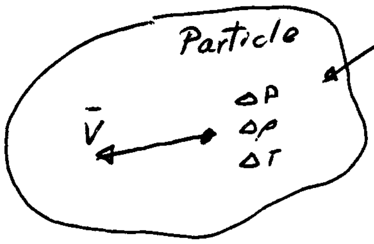
Assumption / conditions

- Restoring force is due to pressure alone
- constant amplitude (no decay)
- Fluid is homogenous
- Fluid is isotropic
- stationary
- inviscid
- Small pertubations

1) General Fluid properties
2) continuity equations
3) Eulers equation
4) Linearize
5) AWE – Acoustic WAVE Momentum Equation
   AME

1 - General Fluid Properties

speed of Propagation



P_0 Pressure

ρ_0 Density

T_0 Temperature

$\Delta P \ll P_0 \qquad \Delta \rho \ll \rho_0 \qquad \Delta T \ll T_0$

see next page

~~CONDENSATION~~ ~~thick~~ ~~this~~

CONDENSATION $\quad s \hat{=} \dfrac{\rho - \rho_0}{\rho_0} \quad (\rho = \rho_0 + \Delta\rho)$

ρ_0 = ambient density (density in absence of acoustic wave)

or $s \ll 1$

PERFECT GAS. ADIABATIC PROCESS (no heat transfer)

$$\frac{P}{P_0} \hat{=} \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

γ = ratio of specific heat

$\gamma = 1.4$ in Air at 1 bar and 20°C

$$\frac{P_0 + \Delta P}{P_0} = \left(\frac{\rho_0 + \Delta\rho}{\rho_0}\right)^{\gamma}$$

$$1 + \frac{\Delta P}{P_0} = \left(1 + \frac{\Delta\rho}{\rho_0}\right)^{\gamma}$$

$$1 + \frac{\Delta P}{P_0} = (1 + s)^{\gamma}$$

TAYLOR EXPANSION

$$(1+x)^m = 1 + mx\left(+ \frac{m^2 x}{2!} + \ldots\right) \quad \text{if } x \ll 1$$

$$P = P_0 + \left(\frac{\partial P}{\partial \rho}\right)_{\rho_0}(\rho - \rho_0) + \frac{1}{2}\left(\frac{\partial^2 P}{\partial \rho^2}\right)_{\rho_0}(\rho - \rho_0)^2 + \ldots$$

$$1 + \frac{\Delta P}{P_0} = 1 + \gamma s$$

(if fluctuations are small only the lowest order term in $(\rho - \rho_0)$ need be retained
$P - P_0 = B\left(\frac{\rho - \rho_0}{\rho_0}\right)$)

$$\frac{\Delta P}{P_0} = \gamma s$$

Define B = ~~Bulk~~ adiabatic modulus $= P_0 \gamma$

i.e. changes in thermal energy are negligible

$$\boxed{\Delta P = Bs}$$

Condensation is a dimensionless quantity
& in most acoustical problems much less than 1

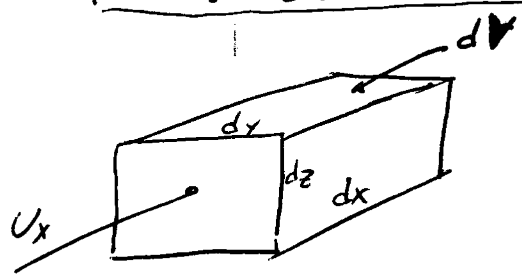
= ~~~~ change of density from its initial ~~state~~
value ρ_0 with the acoustic field absent.

2) Continuity Equation

(wave eqn in 3-D)

1^{st} law of the conservation of mass

1) We cannot create or destroy matte
2) Matter that goes into a box and does not come out must still be there
3) Matter that is in a sealed box must always be there



Newton's 2nd law of motion - The force necessary to move matter must be equal to the rate of change of its momentum

3rd law - existence of an equation of state (a thermodynamic relation between sets of variables such as pressure & density)

X - Direction

Mass Flow Rate in =

$$\rho U_x \, dy \, dz$$

Mass Flow Rate out =

$$\left(\rho U_x + \frac{\partial}{\partial x}(\rho U_x)\, dx\right) dy\, dz$$

change

NeT CHANGE = IN - OUT
in (X-DIR)

$$X \quad = -\frac{\partial}{\partial x}(\rho U_x)\, dx\, dy\, dz$$

$$Y \quad = -\frac{\partial}{\partial y}(\rho U_y)\, dx\, dy\, dz \qquad (dV = dx\, dy\, dz)$$

$$Z \quad = -\frac{\partial}{\partial z}(\rho U_z)\, dx\, dy\, dz$$

NeT MASS FLOW RATE IN ALL DIRECTIONS

$$\left[-\frac{\partial}{\partial x}(\rho U_x) - \frac{\partial}{\partial y}(\rho U_y) - \frac{\partial}{\partial z}(\rho U_z)\right] dV$$

$$= -\nabla\cdot(\rho \bar{U})\, dV$$

(= any change caused by a change in ρ)

$\nabla = \text{div}$

$$-\nabla\cdot(\rho\bar{U})\, dV = \frac{\partial \rho}{dt}\, dV$$

so since the net influx must equal the rate of increase we obtain

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \bar{U}) = 0}$$

Continuity equation

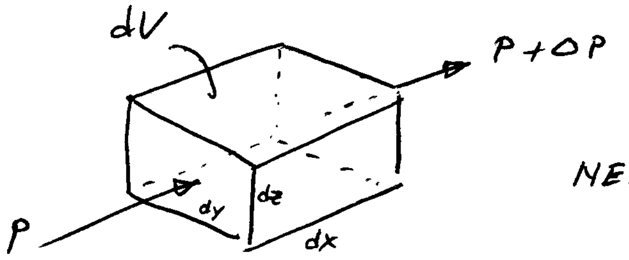
ρ = density

$\bar{U}$ = particle velocity

$$\cancel{\frac{\partial \rho}{\partial t} + \nabla(\rho \bar{v}) = 0}$$

$$\underbrace{\frac{\partial \rho}{\partial t}\, dV}_{\substack{\text{mass change} \\ \text{per unit time}}} + \underbrace{\nabla(\rho \bar{U})\, dV}_{\substack{\text{Flow in/out} \\ \text{per unit time}}} = 0$$

3- Eulers Equation



NET FORCE (x-Dir)

$$F_x = -(P+\Delta P)\, dy\, dz + P\, dy\, dz$$

$$= -\Delta P\, dy\, dz$$

$$= -\frac{\partial P}{\partial x}\, dx\, dy\, dz$$

$$= -\frac{\partial P}{\partial x}\, dV$$

Total Force $\quad \bar{F} = \left(-\hat{i}\frac{\partial P}{\partial x} - \hat{j}\frac{\partial P}{\partial y} - \hat{k}\frac{\partial P}{\partial z}\right) dV$

$$\bar{F} = -\nabla P\, dV$$

use "F=ma" $\quad -\nabla P\, dV = \rho\, dV\, \bar{a}$

these are all "u" not μ

$\vec{a}$ can be written more succinctly $\quad \bar{a} \doteq \frac{\partial \bar{u}}{\partial t} + (\bar{U}\cdot\nabla)\,\bar{U} = \frac{\partial \bar{u}}{\partial t} + U_x \frac{\partial \bar{u}}{\partial x} + U_y \frac{\partial \bar{u}}{\partial y} + U_z \frac{\partial \bar{u}}{\partial z}$

Euler's Equation $\quad \boxed{-\nabla P = \rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\,\bar{u}\right)}$

since mass dm of the element is $\rho\, dV$, substitution into $d\vec{F} = a\, dm$ gives

4 Linearize

1) $\Delta P = \mathcal{B}s$

2) $$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\bar{u})$$

3) $$\rho = \rho_0(1+s)$$

4) $$-\nabla P = \rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right)$$

Substitute 3 into 2

$$\frac{\partial(\rho_0+\rho_0 s)}{\partial t} = -\nabla_0(\rho_0\bar{u}+\rho_0 s\bar{u})$$

$$\rho_0\frac{\partial s}{\partial t} = \rho_0\left(-\nabla\cdot\bar{u} - \underbrace{\cancel{\nabla\cdot(s\bar{u})}}_{\approx 0}\right)$$

$$\boxed{\frac{\partial s}{\partial t} = -\overset{\text{del}}{\nabla}\cdot\bar{u}}$$

linearized continuity equation

ASSUMPTION

$$(\bar{u}\cdot\nabla)\bar{u} << \frac{\partial u}{\partial t}$$

changes in Space << changes in time (small vibrations)

so (4) becomes:

$$-\nabla P = \rho\frac{\partial \bar{u}}{\partial t}$$

$$-\nabla P = \rho_0\frac{\partial \bar{u}}{\partial t} + \underbrace{\cancel{\rho_0 s\frac{\partial u}{\partial t}}}_{\approx 0}$$

$$\boxed{-\nabla P = \rho_0\frac{\partial \bar{u}}{\partial t}}$$

Acoustic Momentum Equation

CONSIDER

$$\Delta P = P - P_0$$

$$\nabla(\Delta P) = \nabla P - \cancel{\nabla P_0}^{\,0} \quad (P_0 \text{ is constant})$$

$$\boxed{\nabla(\Delta P) = \nabla P}$$

simplified Eulers eqn

$$\frac{\partial(\Delta P)}{\partial t} = \frac{\partial P}{\partial t} - \frac{\partial(P_0)}{\partial t}$$

$$\frac{\partial(\Delta P)}{\partial t} = \frac{\partial P}{\partial t}$$

Next Rearange eqn 1

$$S = \frac{\Delta P}{B}$$

$$\frac{\partial\left(\frac{\Delta P}{B}\right)}{\partial t} = -\nabla \cdot \bar{u}$$

$$\frac{1}{B}\left(\frac{\partial(\Delta P)}{\partial t}\right) = -\nabla \cdot \bar{u}$$

$$\frac{1}{B}\frac{\partial P}{\partial t} = -\nabla \cdot \bar{u}$$

$$\frac{d(a)}{dt} \Rightarrow \frac{1}{B}\frac{\partial^2 P}{\partial t^2} = -\frac{\partial}{\partial t}(\nabla \cdot \bar{u})$$

$$\nabla(4) \Rightarrow -\nabla^2 P = -\rho_0 \nabla \frac{\partial}{\partial t}(\bar{u})$$

$$-\frac{\nabla^2 P}{\rho_0} = -\frac{1}{B}\frac{\partial^2 P}{\partial t^2}$$

$$\frac{1}{B}\frac{\partial^2 P}{\partial t^2} - \frac{\nabla^2 P}{\rho_0} = 0$$

{ this is the Linearized, lossless wave equation for the propagation of sound in Fluids

Define $c = \sqrt{\frac{B}{\rho_0}}$

c = speed of sound

$$\boxed{\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0}$$

Acoustic Wave Equation

PS 104 5,12

this is the relation between pressure and speed of sound as a function of space and time

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This equation can be written as

Pg 105 Handout

$$\boxed{p = \rho_0 c^2 s}$$

because p and s are proportional, the condensation satisfies the wave equation.

since the density ρ and the condensation are linearly related, the instantaneous density also satisfies the wave equation.

since the curl of the gradient of a function must vanish $\nabla \times \nabla f = 0$

from $\boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p}$ the particle velocity must be irrotational $\nabla \times \vec{u} = 0$

$\therefore$ can be expressed as the gradient of a scalar function Φ

$$\boxed{\vec{u} = \nabla \Phi} \qquad \Phi = \text{velocity potential}$$

meaning of this result is that the acoustical excitation of an inviscid fluid involves no rotation

(there are no effects such as boundary layers, shear waves or turbulence)

substituting: $\rho_0 \frac{\partial}{\partial t} \nabla \Phi = -\nabla p$

or $\nabla \left(\rho_0 \frac{\partial \Phi}{\partial t} + p \right) = 0$

if no acoustic excitation $\boxed{p = -\rho_0 \frac{\partial \Phi}{\partial t}}$

integrating with respect to time will show that Φ also satisfies the wave eqn.

The wave equation for condensation fluctuations is:

$$\nabla^2 s = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$ (Acoustic wave eqn for condensation)

Acoustic wave eqn for pressure fluctuations

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

Acoustic wave eqn for velocity fluctuation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$ u = particle velocity