Please write your first and last name here:

Name ______________________________

Instructions:

- This practice exam is meant to give you an idea of the types of questions that will be asked on Exam 2
- This exam is not exhaustive, please refer to homeworks 5 and 6 for other types of questions over the material
- For Exam 2, plan for 3 questions over module 3 material and 2 questions over module 4 material

1. A continuous random variable, X, has the following probability density function (pdf).

$$f_X(x) = \begin{cases} \frac{4}{3}(1 - x^3) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What are the two properties that a pdf must have to be valid? Verify that they are met for this function. (3 points)

(b) Give $\mathbb{P}(X \leq 0.50)$? (3 points)

(c) What is $\mathbb{E}(X)$? (3 points)

(d) What is $Var(X)$? (3 points)

2. Suppose a continuous random variable X has the following probability density function:

$$f_X(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 1 & 1 < x \leq 1.5 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write down the cumulative distribution function (CDF) in functional form. (Make sure to cover **all** cases). (5 points)

(b) Use your CDF from part (a) to find $\mathbb{P}(0.5 < X \leq 1.25)$. (Write an expression in terms of F_X first, then solve). (3 points)

3. Suppose for a particular video game, the *average time* between deaths for a certain player is 15 minutes. We will model a random variable, X, as the time between deaths for this player. Thus, $X \sim \text{Expo}(\lambda)$.

   (a) What is the value of λ? (use minutes as the units) (2 points)

   (b) The player starts a new game. What is the probability their first death comes after 10 mins of play? (3 points)

   (c) The player starts a new game. Given they have stayed alive for 15 minutes, what is the probability their first death comes before 20 mins? (3 points)

   (d) The player starts a new game. Give the time (in minutes) such that there is only a 10% chance that they will survive past that time before their first death. (3 points)

4. Suppose the weight of Chipotle burritos follows a normal distribution with mean of 450 grams, and variance of 100 grams^2. Define a random variable to be the weight of a randomly chosen burrito.

   (a) What is the probability that a Chipotle burrito weighs less than 445 grams? (3 points)

   (b) 20% of Chipotle burritos weigh more than what weight? (3 points)

   For the questions below, suppose you are catering a party of 30 people using Chipotle.

   (c) Give the (approximate) distribution for the *total* weight of 30 burritos. State the name and give the parameter value(s). (3 points)

   (d) Give the (approximate) distribution for the *average* weight of 30 burritos. State the name and give the parameter value(s). (3 points)

   (e) What is the (approximate) probability that the *average* weight of 30 burritos is less than 445 grams. (4 points)

5. A fast food worker (perhaps a Chipotle worker?) has two job tasks. They either spend their work day preparing food or working the register and notice the following Markov dependence. If they are preparing food one day, they will be working the register the next day with probability 0.35. If they are working the register one day, they will be working the register the next day with probability 0.85. Let 1 = preparing food and 2 = working the register. Use a Markov chain model for the following.

   (a) What is the 1-step transition matrix? (5 points)

   (b) The worker is assigned to prepare food on Monday. What is the probability they will be working the register on Wednesday (5 points)

   (c) Give the steady state distribution of the worker's tasks. In other words, in the long run what proportion of days will the worker be preparing food or working the register? (**Give both**) (4 points)

6. A Markov chain with the states 1, 2, 3 has transition probability matrix:

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.3 & a & 0.7 \\ 0.4 & 0.3 & b \\ c & 0.2 & 0.7 \end{pmatrix} \end{array}$$

(a) Give the values of a, b, and c that make the above a valid transition probability matrix. (3 points)

(b) The 2-step transition probability matrix is

$$P^{(2)} = \begin{pmatrix} 0.16 & 0.14 & 0.70 \\ 0.27 & 0.15 & 0.58 \\ 0.18 & 0.20 & 0.62 \end{pmatrix}$$

If the initial distribution of states is $P_0 = (0.3,\ 0.7,\ 0)$, calculate the distribution of the states after 2 transitions. (3 points)

(c) Will this Markov chain have a steady state distribution? Justify your answer. (2 points)

7. Customers comes to a restaurant at a rate of 5 customers per hour. We assume customer arrivals follows a homogenous Poisson process.

   (a) What is the probability of more than 2 customer arrivals in a period of one hour? (2 points)

   (b) What is the probability of more than 4 customer arrivals in a period of 2 hours? (2 points)

   (c) What is the expected value and the variance of inter-arrival times? (2 points)

   (d) Compute the probability that the next customer does not arrive during the next 30 minutes? (2 points)

   (e) Compute the probability that the time till the third customer arrives exceeds 40 minutes? (3 points)