Arithmetic vs Geometric Mean: Error Analysis

When considering model errors, particularly in statistical modelling or data analysis, selecting the incorrect mean (Arithmetic Mean (AM) instead of Geometric Mean (GM) or vice versa) can lead to biased results. This bias can be quantified by the difference between the AM and GM, which is:

$$AM - GM = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2}.$$

Proof Using the AM-GM-Inequality for Two Numbers

For two non-negative real numbers a and b, the AM-GM-Inequality can be proven as follows:

$$AM \geq GM.$$

$$\left(\frac{a+b}{2}\right) \geq \sqrt{ab}.$$

Square both sides to remove the square root:

$$\left(\frac{a+b}{2}\right)^2 \geq ab.$$

Expanding the left side:

$$\frac{a^2 + 2ab + b^2}{4} \geq ab.$$

Multiply both sides by 4 to clear the fraction:

$a^2 + 2ab + b^2 \geq 4ab,$

$a^2 + 2ab + b^2 - 4ab \geq 0,$

$a^2 + b^2 + (2ab - 4ab) \geq 0,$

$a^2 + b^2 - 2ab \geq 0,$

$a^2 - 2ab + b^2 \geq 0.$

Notice that

$a^2 - 2ab + b^2 \geq 0.$

is a **perfect square trinomial**, thus:

$(a - b)^2 \geq 0.$

The expression for the difference between the Arithmetic Mean (AM) and the Geometric Mean (GM) for two numbers $a \neq b$ can be connected to the binomial theorems.

$AM - GM = \left(\frac{a+b}{2}\right) - \sqrt{ab} = ?$

Expressing both terms under a common denominator, yields:

$\frac{(a+b)}{2} - \frac{2\sqrt{ab}}{2} = \frac{a - 2\sqrt{ab} + b}{2}.$

Hence,

$AM - GM = \left(\frac{a+b}{2}\right) - \sqrt{ab} = \frac{a - 2\sqrt{ab} + b}{2}.$

Notice that the numerator can be factored as a perfect square or the second binomial formula:

$a - 2\sqrt{ab} + b = \left(\sqrt{a} - \sqrt{b}\right)^2.$

Therefore,

$AM - GM = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2}. \quad \blacksquare$

<u>Conclusion:</u>

The expression $AM - GM = \frac{(\sqrt{a} - \sqrt{b})^2}{2}$ shows that the difference between the arithmetic mean and the geometric mean is always non-negative and is zero only when $a = b$.

The way it seems to me, based on a few attempts, we can also get

$$AM = \frac{(\sqrt{a} - \sqrt{b})^2}{2} + GM$$

and

$$GM = AM - \frac{(\sqrt{a} - \sqrt{b})^2}{2}.$$

Example:

Imagine a scenario where you're averaging growth rates:

- Suppose you have growth rates: $a = 1.5, b = 0.6$.
- The AM would be $\frac{1.5 + 0.6}{2} = 1.05$, suggesting growth.
- The GM would be $\sqrt[2]{1.5 \cdot 0.6} = \sqrt{0.9} \approx 0.95$, indicating a contraction.

Here, the difference is:

$AM - GM = 1.05 - 0.95 \approx 0.1$.

This bias of 0.1 could be relevant depending on the application, especially in financial models, where compounding effects are crucial.

As we have seen,

$$AM - GM = \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2}$$

holds, hence we can compare and get

$$\frac{(\sqrt{a} - \sqrt{b})^2}{2} = \frac{(\sqrt{1.5} - \sqrt{0.6})^2}{2} = 0.1013 \approx 0.1.$$